PROBLEM CLASS: CONSTRUCTING THE SSVI LOCAL VOLATILITY

ANTOINE JACQUIER

The starting point is a set of European Call option prices $C(K_i, T_j)_{i=1,\ldots,n,j=1,\ldots,m}$, which we assume are free of arbitrage. Here $(K_i)_{i=1,\ldots,n}$ denotes the set of observed strikes and $(T_j)_{j=1,\ldots,m}$ the set of observed maturities. We further assume for simplicity that interest rates are null. Note that we considered the same observed strikes for all maturities. This is not necessarily true in practice, but simplifies the implementation from a numerical point of view. We now consider the SSVI parameterisation of the total implied variance smile:

$$w(x,t) := \frac{\theta_t}{2}\left\{1 + \rho\varphi(\theta_t)x + \sqrt{(\varphi(\theta_t)x + \rho)^2 + 1 - \rho^2}\right\},$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Here, as usual, x represents the log-(forward) moneyness, and t the time horizon. The parameters of the model can be understood as follows:

- ρ is an asymmetry parameter and lie in $[-1, 1]$;
- $(\theta_t)_{t\geq 0}$ represents the at-the-money term structure; indeed for any $t \geq 0$, $\theta_t = w(0, t)$;
- $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function representing the backbone structure. We shall consider
$$\varphi(\theta) := \frac{1}{\gamma\theta}\left(1 - \frac{1 - \mathrm{e}^{-\gamma\theta}}{\gamma\theta}\right),$$
for some parameter $\gamma > 0$.

We now wish to implement and discuss the following calibration procedure:

(i) We first calibrate the term structure of $(\theta_t)_{t\geq 0}$; plotting the at-the-money total variances suggest to consider a parameterisation of the form $\theta_t = \alpha_0 t + \alpha_1 \left(1 - \mathrm{e}^{-\alpha_2 t}\right)$. Calibrate $\alpha_0, \alpha_1, \alpha_2$.

(ii) We now move on to the rest of the calibration: For each maturity T_j $(j = 1, \ldots, m)$, calibrate the parameters γ and ρ, thus obtaining two vectors $(\gamma_j)_{j=1,\ldots,m}$ and $(\rho_j)_{j=1,\ldots,m}$.

(iii) Observing that γ is roughly constant for all maturities, we fix it as such.

(iv) The observed term structure for ρ suggests the following parameterisation:
$$\rho(t) = \max\left\{\min\left\{\arctan(p_0 t + p_1) + p_2, 1\right\}, -1\right\},$$
so that $\rho(\cdot) \in [-1, 1]$. Given the calibrated vector $(\rho_j)_{j=1,\ldots,m}$, determine the optimal vector (p_0, p_1, p_2);

(v) Compare the original volatility surface with the calibrated SSVI surface.

(vi) For each maturity, plot the probability density function corresponding to the calibrated SSVI volatility surface, and check for arbitrage.

(vii) Given the function SSVI, compute the corresponding local volatility.

(viii) For the maturity $T \approx 2.47$, compare the original implied volatility data with the new one generated by Monte Carlo using the calibrated local volatility surface.

Department of Mathematics, Imperial College London
E-mail address: a.jacquier@imperial.ac.uk

Date: February 24, 2017.