ADVANCED METHODS IN DERIVATIVES PRICING

with application to volatility modelling

M5MF6

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Contents

Notations and standard definitions

The notations below will be used throughout the notes. We also wish to emphasize some common notational mistakes.

$\mathbb{N}$	strictly positive integer numbers $\{1, 2, \dots\}$
$\mathbb{R}^*$	non-zero real numbers: $\mathbb{R} \setminus \{0\}$
$\mathbb{R}_+$	non-negative real numbers: $[0, +\infty)$
$\mathbb{R}_+^*$	strictly positive real numbers: $(0, +\infty)$
$\mathcal{M}_{n,d}(\mathbb{R})$	space of $n \times d$ matrices with entries valued in $\mathbb{R}$
$\mathcal{S}_n(\mathbb{R})$	space of symmetric $n \times n$ matrices with real entries
$\mathcal{N}$	cumulative distribution function of the standard Gaussian distribution
$X = (X_t)_{t \geq 0} \neq X_t$	a process evolving in time, as opposed to X_t, which represents the (random) value of the process X at time t
$\mathbb{1}_{\{x \in A\}}$	indicator function equal to 1 if $x \in A$ and zero otherwise
$x \wedge y$	$\min(x, y)$
$x \vee y$	$\max(x, y)$
a.s.	almost surely
$(x - y)_+$	$\max(0, x - y)$
$\delta_x(\cdot)$	Dirac measure at x
$C_c(\mathbb{R})$	space of real continuous functions with compact support

Chapter 1

Option pricing: from super-replication to FTAP

1.1 Zoology of stochastic analysis

In these notes, we shall follow a utilitarian approach, and only introduce the tools we need when we need them. Some of them, being core to everything else, shall be introduced right away. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ will denote the ambient (given) probability space, where Ω is the sample space of possible outcomes, $\mathcal{F}$ the set of events, and $\mathbb{P}$ a probability, namely a map from $\mathcal{F}$ to $[0,1]$. In this introductory part, we shall let $\mathcal{T}$ denote a (time) index set, which can be either countable ($\mathcal{T} = \{t_1, t_2, \ldots\}$) or uncountable ($\mathcal{T} = \mathbb{R}_+$). We recall that a σ-field on Ω is a non-empty collection of subsets of Ω, closed under countable unions and intersections, and closed under complementation. A filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is defined as a non-decreasing (in the sense of inclusion) family of σ-fields in $\mathcal{F}$; we shall say that a process $X = (X_t)_{t\in\mathcal{T}}$ is adapted to the filtration if $X_t \in \mathcal{F}_t$ for each $t \in \mathcal{T}$. The following definition is standard in the stochastic analysis / mathematical finance literature, and will always be taken for granted in these lecture notes.

Definition 1.1.1. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual hypotheses if

- $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$;
- the filtration is right-continuous: for any $t \in \mathcal{T}$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_u$.

1.1.1 Stopping times

We introduce here stopping times, which are measurable random variables, and which will be defining ingredients of local martingales, as we shall see later.

Definition 1.1.2. A random variable τ taking values in $\mathcal{T} \cup \{\sup \mathcal{T}\}$ is called a stopping time (resp. optional time) if the event $\{\tau \le t\}$ (resp. $\{\tau < t\}$) belongs to $\mathcal{F}_t$ for all $t \in \mathcal{T}$.

Consider for instance $\mathcal{T} = [0, \infty)$, a continuous, adapted, real-valued process X (a standard Brownian motion for example) and fix a point $x \in \mathbb{R}$. Then the first hitting time of the level x, $\tau := \inf\{t \ge 0 : X_t = x\}$, is a stopping time. However, the last hitting time of the level x, $\tau := \sup\{t \ge 0 : X_t = x\}$, is not a stopping time. Intuitively, the information available at time t is not sufficient to determine whether the process will reach the level x at some point in the future. The following properties of optional and stopping times are left as an exercise:

Proposition 1.1.3. *The following assertions hold:*

(i) Let τ be a random variable in $\mathcal{T} \cup \{\sup \mathcal{T}\}$; it is a stopping time if and only if $\{\tau > t\} \in \mathcal{F}_t$ for all $t \in \mathcal{T}$;

(ii) every stopping time is optional;

(iii) an optional time is a stopping time if the filtration is right-continuous;

(iv) if τ is an optional time and t_0 a strictly positive constant, then $\tau + t_0$ is a stopping time;

(v) if τ_1 and τ_2 are stopping times, then so are $\tau_1 \wedge \tau_2$, $\tau_1 \vee \tau_2$ and $\tau_1 + \tau_2$;

Proof. Statement (i) follows directly since $\{\tau > t\} = \{\tau \le t\}^c$. Regarding (ii), let τ be a stopping time, and assume that $\mathcal{T} = \mathbb{N}$. Then the event $\{\tau < t\}$ is equal to $\{\tau \le t - 1\} \in \mathcal{F}_{t-1} \subset \mathcal{F}_t$. Consider now the general case $\mathcal{T} = [0, \infty)$ and fix some $t \in \mathcal{T}$. Let $(t_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence in $\mathcal{T}$ converging to t as n increases, so that

$$\{\tau < t\} = \bigcup_{n \ge 1} \{\tau \le t_n\} \in \mathcal{F}_t.$$

Indeed, for each $n \in \mathbb{N}$, the event $\{\tau \le t_n\}$ belongs to $\mathcal{F}_{t_n}$, which is itself a subset of $\mathcal{F}_t$. Consider now (iii): assume that the filtration is right-continuous and τ a random time. For any $t \in \mathcal{T}$, consider a strictly decreasing sequence $(t_n)_{n \in \mathbb{N}}$ converging to t; for any $m > 0$, we can write

$$\{\tau \le t\} = \bigcap_{n \ge m} \{\tau < t_n\};$$

now, for any $n \ge m$, $\{\tau < t_n\} \in \mathcal{F}_{t_n} \subset \mathcal{F}_{t_m}$. Therefore $\{\tau \le t_n\} \in \mathcal{F}_{t^+} = \mathcal{F}_t$. We leave (iv) and (v) as exercises. □

For every $(\mathcal{F}_t)_{t \in \mathcal{T}}$-stopping time τ, we can associate the σ-field

$$\mathcal{F}_\tau := \{B \in \mathcal{F} : B \cap \{\tau \le t\} \in \mathcal{F}_t, t \in \mathcal{T}\}.$$

1.1.2 Non-negative local martingales

We now consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$, for some index set $\mathcal{T}$. All the processes in this section shall be consider as taking values in $\mathbb{R}^d$, for some $d \in \mathbb{N}$.

Definition 1.1.4. An $\mathbb{R}^d$-valued, adapted process S on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ is called a martingale (respectively supermartingale, submartingale) if, for all $t \in \mathcal{T}$, $\mathbb{E}(|S_t|)$ is finite (or $S \in L^1(\mathbb{P})$) and $\mathbb{E}(S_t|\mathcal{F}_u) = S_u$ (resp. $\mathbb{E}(S_t|\mathcal{F}_u) \leq S_u$, $\mathbb{E}(S_t|\mathcal{F}_u) \geq S_u$) for all $u \in [0,t] \cap \mathcal{T}$.

Example 1.1.5. Let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion, $(\mathcal{F}_t^W)_{t\geq 0}$ its natural filtration, i.e. $\mathcal{F}_t := \{\sigma(W_s) : s \geq t\}$, and $\mathcal{T} = [0, \infty)$.

- W is an $(\mathcal{F}_t)$-martingale;
- the processes $(W_t^3 - 3tW_t)_{t\geq 0}$ and $(W_t^4 - 6tW_t^2 + 3t^2)_{t\geq 0}$ are true $(\mathcal{F}_t)$-martingales.
- the solution to the stochastic differential equation $\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t$, starting at $S_0 > 0$, corresponds to the Black-Scholes model (see Section 2.1.1), which is the canonical model in mathematical finance. Then, for any $0 \leq u \leq t$, $S_t = S_u \exp\left(-\frac{1}{2}\sigma^2(t-u) + \sigma(W_t - W_u)\right)$, and the process S is clearly a true martingale.

Unless otherwise specified, all processes here will be adapted to the filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$. The following lemma shows how (sub/super) martingale properties are preserved under transformations:

Lemma 1.1.6. *Let S be an $\mathcal{F}$-martingale and $f : \mathbb{R}^d \to \mathbb{R}$ a convex function such that $f(S)$ is integrable (i.e. $\int_{\mathbb{R}^d} |f(s)|\mathbb{P}(S \in \mathrm{d}s) < \infty$), then $f(S)$ is a submartingale.*

Useful examples in mathematical finance of such convex functions are $f(x) \equiv x^2$, $f(x) \equiv x_+$.

Proof. The proof follows directly from Jensen's inequality: for any $t \in \mathcal{T}$ and $u \leq t$,

$$f(S_u) = f(\mathbb{E}(S_t|\mathcal{F}_u)) \leq \mathbb{E}(f(S_t)|\mathcal{F}_u), \qquad \text{almost surely.}$$

□

Remark 1.1.7. One could naturally wonder whether the converse holds, namely whether any submartingale can be generated from a true martingale via a convex function. This is not true in general, but some results hold in particular cases: evey non-negative submartingale is the absolute value of some martingale [72], see also [8] and [124].

Definition 1.1.8. A process S on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ is a local martingale if there exists a sequence of $(\mathcal{F}_t)_{t\in\mathcal{T}}$-stopping times $(\tau_n)_{n\geq 0}$ satisfying $\lim_{n\uparrow\infty} \tau_n = \sup \mathcal{T}$ almost surely, and such that the stopped process $(S_t^{\tau_n})_{t\in\mathcal{T}} := (S_{t\wedge\tau_n})_{t\in\mathcal{T}}$ is a true martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$, for any $n \geq 0$. Such a sequence of stopping times is called a localising sequence for the process S.

The following space of processes is of primary importance in stochastic analysis, and we will make use of it repeatedly. Of particular interest will be the cases $n = 1$ and $n = 2$.

Definition 1.1.9. For any $n \in \mathbb{N}$, the space $\mathrm{L}^n_{\mathrm{loc}}(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ denotes the space of progressively measurable (with respect to $(\mathcal{F}_t)_{t\in\mathcal{T}}$) processes φ such that

$$\mathbb{P}\left(\int_0^t |\varphi_u|^n \mathrm{d}u < \infty\right) = 1, \qquad \text{for all } t \in \mathcal{T},$$

while $\mathrm{L}^n(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ denotes the space of progressively measurable processes φ such that

$$\mathbb{E}\left(\int_0^\infty |\varphi_u|^n \mathrm{d}u < \infty\right) < \infty.$$

The following proposition allows us to construct large classes of (local) martingales:

Proposition 1.1.10. *For a given Brownian motion W, define pathwise the process $(X_t)_{t\geq 0}$ by*

$$X_t := \int_0^t u_s \mathrm{d}W_s, \qquad \textit{for all } t \geq 0.$$

- *If $u \in \mathrm{L}^2(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, then X is a continuous square integrable martingale;*
- *if $u \in \mathrm{L}^2_{\mathrm{loc}}(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, then X is a continuous local martingale.*

The theory of local martingales is very profound, and we refer the avid reader to the excellent monograph [95]. True martingales are of course local martingales (take $\tau_n = \sup \mathcal{T}$ for all $n \in \mathbb{N}$), but the converse is not true in general. For instance (as first indicated in [93]), if W is a three-dimensional Brownian motion not starting at the origin, then the inverse Bessel process defined by $(\|W_t\|^{-1})_{t\geq 0}$ is a strict local martingale (e.g. a local martingale, but not a true martingale). However, the following always holds:

Lemma 1.1.11. *Every bounded local martingale (from below) is a true martingale.*

Proof. Let X be a bounded local martingale, and $(\tau_n)_{n\in\mathbb{N}}$ a localising sequence of stopping times for X. Pointwise in $\omega \in \Omega$, it is clear that the sequence $(X_t^{\tau_n}(\omega))_{n\in\mathbb{N}}$ converges to $X_t(\omega)$ for any $t \in \mathcal{T}$. By dominated convergence, we obtain

$$\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}\left(\lim_{n\uparrow\infty} X_t^{\tau_n}|\mathcal{F}_u\right) = \lim_{n\uparrow\infty} \mathbb{E}\left(X_t^{\tau_n}|\mathcal{F}_u\right) = \lim_{n\uparrow\infty} X_u^{\tau_n} = X_u,$$

for any $0 \leq u \leq t$, and the lemma follows. □

We shall see some implications on option prices of the difference between strict local martingales and true martingales in Chapter 2.

Proposition 1.1.12. *A non-negative local martingale S is a super-martingale and $\mathbb{E}(S_t|\mathcal{F}_u)$ is finite for all $0 \leq u \in [0, t] \cap \mathcal{T}$.*

Proof. For any localising sequence $(\tau_n)_{n\geq 0}$ for S, the proposition follows from Fatou's lemma (Appendix A.1.3):

$$\mathbb{E}_u(S_t) = \mathbb{E}_u(\liminf_{n\uparrow\infty} S_t^{\tau_n}) \leq \liminf_{n\uparrow\infty} \mathbb{E}_u(S_t^{\tau_n}) = \liminf_{n\uparrow\infty} S_u = S_u.$$

□

In particular, the proposition implies that any non-negative local martingale is a supermartingale. This will be the exact framework of Chapter 2. A blind application of the reverse Fatou lemma would yield

$$\mathbb{E}(S_t|\mathcal{F}_u) = \mathbb{E}\left(\limsup_{n\uparrow\infty} S_t^{\tau_n}\middle|\mathcal{F}_u\right) \geq \limsup_{n\uparrow\infty} \mathbb{E}\left(S_t^{\tau_n}|\mathcal{F}_u\right) = \liminf_{n\uparrow\infty} S_u = S_u,$$

which, in combination with Proposition 1.1.12, would imply that a local martingale is always a martingale. There is obviously a contradiction here, which comes from the fact the the reverse Fatou lemma does not apply, since the sequence of functions $f_n(s) \equiv s$ is not bounded above by an integrable function.

Remark 1.1.13. This property in particular implies (see [86, Chapter III, Lemma 3.6]) that, for any continuous non-negative martingale S, if there exists $t^* > 0$ such that $\mathbb{P}(S_{t^*} = 0) > 0$, if $S_{t^*} = 0$, then $S_t = 0$ almost surely for all $t \geq t^*$. This seemingly trivial property is fundamental when considering discretisation schemes for stochastic differential equations. As a motivating example, consider the CEV (Constant Elasticity of Volatility) process, defined as the unique strong solution, starting at $S_0 = 1$, to $\mathrm{d}S_t = \sigma S_t^{1+\beta}\mathrm{d}W_t$, where W is a standard Brownian motion, $\sigma > 0$ and $\beta \in \mathbb{R}$. We refer the reader to [30, Chapter 6.4] for full details. The process $(S_t)_{t\geq 0}$ is a local martingale, and is a true martingale if and only if $\beta \leq 0$. Note that when $\beta = 0$, this is nothing else than the Black-Scholes stochastic differential equation, and S_t is strictly positive almost surely for all $t \geq 0$. Computations involving Bessel processes show that, for any $t \geq 0$, $\mathbb{P}(S_t = 0) > 0$ if and only if $\beta \in [-1/2, 0)$. Consider now a simple Euler scheme for the CEV dynamics, along a given partition $0 = t_0 < t_1 < \cdots < t_n = T$, as $S_{t_0} = 1$ and, for any $i = 0, \ldots, n-1$,

$$S_{t_{i+1}} = S_{t_i} + \sigma S_{t_i}^{1+\beta} \tilde{n}\sqrt{t_{i+1} - t_i},$$

where $\tilde{n}$ is a Gaussian random variable with mean zero and unit variance. Suppose that along some simulated path, there exists $i = 0, \ldots, n-1$ such that $S_{t_i} > 0$ and $S_{t_{i+1}} < 0$. Then, in order for the simulated path to be a true approximation of the original one (the solution of the SDE), the only possibility is to set $S_{t_{i+1}}$ to zero and leave it there until time T. Note that, economically speaking, this absorption property also makes sense, as required by no-arbitrage arguments.

1.1.3 Brackets, uniform integrability and time changes

Uniform integrability is a key property in probability theory, and controls how much the tail of the distribution of random variables accounts for the expectation.

Definition 1.1.14. A family $\mathcal{X}$ of random variables is said to be uniformly integrable if

$$\lim_{K\uparrow\infty}\left(\sup_{X\in\mathcal{X}}\mathbb{E}\left(|X|\mathbb{1}_{|X|\geq K}\right)\right)=0.$$

The following result, the proof of which is omitted, provides an easy-to-check characterisation:

Proposition 1.1.15. *The family $\mathcal{X}$ is uniformly integrable if and only if there exists a Borel function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\lim_{x\uparrow\infty}x^{-1}\phi(x)=\infty$ for which $\sup_{X\in\mathcal{X}}\mathbb{E}\left(\phi(X)\right)$ is finite.*

Theorem 1.1.16. *Let M be continuous local martingale starting from zero. Then there exists a unique continuous increasing process $\langle M\rangle$, called the quadratic variation process, null at zero such that $M^2-\langle M\rangle$ is a continuous local martingale. If M and N are two continuous local martingales starting from the origin, then there exists a unique unique continuous process $\langle M,N\rangle$, null at zero, called the bracket process, such that $MN-\langle M,N\rangle$ is a (continuous) local martingale.*

Remark 1.1.17. The bracket process satisfies the polsarisation identity

$$\langle M,N\rangle_t=\frac{1}{2}\left(\langle M+N\rangle_t-\langle M\rangle_t-\langle N\rangle_t\right).$$

Example 1.1.18.

- For a standard Brownian motion W, $\langle W\rangle_t=t$ for all $t\geq 0$;
- $\langle\int_0^{\cdot}\varphi(s)\mathrm{d}W_s\rangle_t=\int_0^t\varphi(s)^2\mathrm{d}s$ for $\varphi\in\mathrm{L}^2_{\mathrm{loc}}(\Omega,(\mathcal{F}_t)_{t\geq0},\mathbb{P})$.

Remark 1.1.19. Other versions exist, with stronger assumptions, and hence more precise results. In particular, if M is a continuous square-integrable martingale starting from zero, then there exists a unique continuous increasing process $\langle M\rangle$ null at zero such that $M^2-\langle M\rangle$ is a uniformly integrable martingale. See for example [134, Theorem 30.1, Chapter IV, Section 5].

Remark 1.1.20. A related process, the square bracket process $[M]$, exists in the literature, and is defined as $[M]_t := X_t^2-X_0^2-2\int_0^t X_{s^-}\mathrm{d}X_s$; its construction follows from the Doob-Meyer decomposition (see Theorem 1.5.6 below). In the case where the process M is continuous, the two brackets $\langle\cdot\rangle$ and $[\cdot]$ however coincide.

The last technical tool we shall need in order to move on is the technique of time change. We state the following fundamental result without proof:

Theorem 1.1.21 (Dubins-Schwarz)**.** *Every continuous local martingale $(M_t)_{t\geq0}$ can be written as a time-changed Brownian motion $(W_{\langle M\rangle_t})_{t\geq0}$.*

In particular, for any Brownian motion W and any independent, non-negative, càdlàg process $(\sigma_t)_{t\geq0}$, the continuous local martingale $M_\cdot:=\int_0^\cdot\sigma_s\mathrm{d}W_s$ can be written as

$$M_t=W_{\int_0^t\sigma_s^2\mathrm{d}W_s},\qquad\text{for all } t\geq 0.$$

This theorem, and the refined version by Monroe [118], have been used extensively in mathematical finance. One motivation, as detailed in [5], is that trading time and real time are not synchronised, namely that more trading activities take place during the day, and none during the night.

1.1.4 A brief introduction to Itô processes and stochastic calculus

In Chapter 3.1 below, we shall discuss in detail existence, uniqueness and properties of stochastic differential equations. The objective of this section here is to quickly go through the main ingredients of Itô's theory, as we shall need it in the next pages. Again, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a given filtered probability space, supporting an $\mathbb{R}^d$-valued Brownian motion W.

Definition 1.1.22. An $\mathbb{R}^n$-valued Itô process $(X_t)_{t\geq 0}$ is a stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \cdot \mathrm{d}W_s, \qquad t \in [0,T], \tag{1.1.1}$$

where X_0 is an $\mathcal{F}_0$-measurable random variable, $\mu \in \mathbb{L}^1_{\mathrm{loc}}(\mathbb{R}^n \to \mathbb{R}^n)$ and $\sigma \in \mathbb{L}^2_{\mathrm{loc}}(\mathbb{R}^n \to \mathcal{M}_{n,d}(\mathbb{R}))$.

The differential notation $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \cdot \mathrm{d}W_t$, with $X_0 \in \mathbb{R}^n$ is a useful shortcut, even though the noise term $\mathrm{d}W_t$ should be handled with care.

Corollary 1.1.23. *The quadratic variation of an Itô process of the form* (1.1.1) *is given by*

$$\langle X^i, X^j \rangle_t = \int_0^t (\sigma_s \cdot \sigma_s^\top)_{ij} \mathrm{d}s =: \int_0^t C_s^{ij} \mathrm{d}s.$$

For any $t \in [0,T]$, the matrix $C_t \in \mathcal{M}_n(\mathbb{R})$ represents the covariance matrix of the n-dimensional random variable X_t.

Theorem 1.1.24 (Itô's formula)**.** *Let X be an $\mathbb{R}^n$-valued Itô process of the form* (1.1.1) *and let $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R})$. Then the following formula holds almost surely for every $t \geq 0$:*

$$\begin{aligned} f(t, X_t) &= f_0 + \int_0^t \partial_s f_s \mathrm{d}s + \int_0^t \langle \mathrm{d}X_s, \nabla f_s \rangle + \frac{1}{2}\int_0^t (\mathrm{d}X_s)^\top \cdot \Delta f_s \cdot \mathrm{d}X_s \\ &= f(X_0) + \int_0^t \left\{ \partial_s f_s + \langle \mu_s, \nabla f_s \rangle + \frac{1}{2}\mathrm{Tr}\left(\sigma_s^\top \cdot \Delta f_s \cdot \sigma_s\right)\right\} \mathrm{d}s + \int_0^t \langle \sigma_s \cdot \mathrm{d}W_s, \nabla f_s \rangle, \end{aligned}$$

where we used the short-hand notation $f_s \equiv f(s, X_s)$.

Here, the gradient operator ∇ only acts on the space variable, and is a vector in $\mathbb{R}^n$: $\nabla f(\cdot, \mathrm{x}) = (\partial_{x_i} f(\cdot, \mathrm{x}))_{i=1,\dots,n}$, and the angle bracket $\langle \cdot, \cdot \rangle$ is simply the Euclidean product in $\mathbb{R}^n$.

Remark 1.1.25. In coordinates, we can re-write Itô's formula as

$$f(t, X_t) = f_0 + \int_0^t \partial_s f_s \mathrm{d}s + \int_0^t \left\{ \sum_{i=1}^n \mu_s^i \partial_{x_i} f_s + \frac{1}{2}\sum_{i,j=1}^n C_s^{ij} \partial_{x_i x_j} f_s \right\} \mathrm{d}s + \sum_{i=1}^n \sum_{j=1}^d \sigma_s^{ij} \partial_{x_i} f_s \mathrm{d}W_s^j.$$

Remark 1.1.26. In the case $X = W$ and $f \in \mathcal{C}^2(\mathbb{R}^d)$, Itô's formula reads, in differential form:

$$\mathrm{d}f(W_t) = \langle \mathrm{d}W_t, \nabla f(W_t)\rangle + \frac{1}{2}\Delta f(W_t)\mathrm{d}t.$$

The following corollary is left as a straightforward exercise.

Corollary 1.1.27. *For two $\mathbb{R}^n$-valued Itô processes X and Y, the following product rule holds:*

$$X_t \cdot Y_t = X_0 Y_0 + \int_0^t X_s \cdot \mathrm{d}Y_s + \int_0^t Y_s \cdot \mathrm{d}X_s + \int_0^t \mathrm{d}X_s \cdot \mathrm{d}Y_s.$$

Note that rearranging the terms in the corollary yields

$$\int_0^t X_s \cdot \mathrm{d}Y_s = (X_t \cdot Y_t - X_0 Y_0) - \int_0^t Y_s \cdot \mathrm{d}X_s - \int_0^t \mathrm{d}X_s \cdot \mathrm{d}Y_s,$$

which can be seen as a (stochastic) integration by parts formula.

Example 1.1.28. Prove the following representations:

(i) Show that the process Y defined by $Y_t := t^2 W_t^3$ satisfies

$$Y_t = \int_0^t \left(\frac{2Y_s}{s} + 3t^{4/3} Y_s^{1/3}\right) \mathrm{d}s + 3\int_0^t (sY_s)^{2/3} \mathrm{d}W_s.$$

(ii) For any $\alpha, \sigma \in \mathbb{R}$, show that $X_t := \mathrm{e}^{-\alpha t}\left(X_0 + \sigma \int_0^t \mathrm{e}^{\alpha s}\mathrm{d}W_s\right)$ satisfies

$$X_t = X_0 - \alpha \int_0^t X_s \mathrm{d}s + \sigma W_t.$$

(iii) Consider the couple $(X, Y)_t := (\cos(W_t), \sin(W_t))$. Show that it satisfies

$$\begin{aligned} X_t &= 1 - \frac{1}{2}\int_0^t X_s \mathrm{d}s - \int_0^t Y_s \mathrm{d}W_s, \\ Y_t &= -\frac{1}{2}\int_0^t Y_s \mathrm{d}s + \int_0^t X_s \mathrm{d}W_s. \end{aligned}$$

1.2 Fundamental probabilistic results for finance

We introduce in this section two key results from stochastic analysis that are used extensively in mathematical finance, and in the rest of these notes: the martingale representation theorem and Girsanov's theorem.

1.2.1 Martingale representation theorem and hedging

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is the standard filtration generated by a Brownian motion W, i.e. for any $t \geq 0$, $\mathcal{F}_t = \sigma\left(W_s, 0 \leq s \leq t\right)$. Unless otherwise stated, $T > 0$ will be a fixed (finite) time horizon. We shall need the following space of admissible integrands (in order for the Itô integral to make sense):

Definition 1.2.1. We let $\mathcal{V}$ denote the space of functions f from $[0,\infty)\times\Omega\to\mathbb{R}$ that satisfy the following conditions:

- the map $(t,\omega)\to f(t,\omega)$ is $\mathcal{B}\times\mathcal{F}$-measurable, with $\mathcal{B}$ being the Borel sigma-algebra on $[0,\infty)$;
- for any $t\geq 0$, the function $f(t,\cdot)$ is $\mathcal{F}_t$-adapted;
- $\mathbb{E}\left(\int_0^T f(t,\omega)^2\mathrm{d}t\right)$ is finite.

Theorem 1.2.2 (Martingale Representation Theorem)**.** *Let $(M_t)_{t\geq 0}$ be a $(\mathcal{F}_t)$-adapted squared integrable martingale with respect to $\mathbb{P}$. Then there exists a unique process $(\varphi_t)_{t\geq 0}\in\mathcal{V}$ such that*

$$M_t = \mathbb{E}(M_0) + \int_0^t \varphi_s \mathrm{d}W_s \quad \textit{almost surely for any } t\geq 0. \tag{1.2.1}$$

Remark 1.2.3.

- We have stated here the theorem in dimension one. A similar result holds in any (finite) dimension, and we leave this extension to the keen reader.
- The theorem has a converse result, namely that the Itô integral $(\int_0^t \varphi_s \mathrm{d}W_s)_{t\geq 0}$ is a $(\mathcal{F}_t)$-martingale whenever φ is adapted and square integrable.
- The financial consequence of this result is that the only source of uncertainty comes from the Brownian motion, which can be removed by hedging.
- The theorem only asserts existence of a process φ. Explicit representations thereof can be determined using Malliavin calculus, but this falls outside the scope of these lectures.

The proof follows from the following lemmas. Recall that a subset $\mathcal{D}$ of $\mathcal{F}$ is dense if for every X in $\mathcal{F}$, then $\mathcal{D}\cap B\neq\emptyset$ for every neighbourhood B of X.

Lemma 1.2.4. *Theorem 1.2.2 holds if the representation* (1.2.1) *holds for any random variable in some dense subset of $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$.*

For some integer n, and some sequence $0\leq t_1\leq\cdots\leq t_n\leq T$, we consider the subset $\mathcal{D}_T$ of $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ consisting of all the random variables of the form $h(W_{t_1},\ldots,W_{t_n})$, where h is some bounded continuous function from $\mathbb{R}^n$ to $\mathbb{R}$.

Lemma 1.2.5. *The space $\mathcal{D}_T$ is dense in $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$.*

Lemma 1.2.6. *Let X and Y be two random variables on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$ such that Y is $\mathcal{G}$-measurable and X independent of $\mathcal{G}$. Then, for any measurable function $f:\mathbb{R}^{n+m}\to\mathbb{R}$ such that $\mathbb{E}|f(X,Y)|$ is finite, the equality*

$$\mathbb{E}\left[f(X,Y)|\mathcal{G}\right] = b(Y),$$

holds, where $b(y) := \int_{\mathbb{R}^n} f(x,y)\mathbb{P}(X\in\mathrm{d}x)$.

Proof of Lemma 1.2.4. Let X be a random variable in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, and consider a sequence $(X_n)_{n\geq 1}$ in a dense subset $\mathcal{D} \subset L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, converging to X in L^2. By assumption, there exists a sequence of adapted and square integrable functions (g_n) such that,

$$X_n = \mathbb{E}(X_n) + \int_0^T g_n(s)\mathrm{d}W_s, \qquad \text{for any } n \geq 1.$$

Itô's isometry applied to the centered random variables $\widetilde{X}_n := X_n - \mathbb{E}(X_n)$ yields

$$\mathbb{E}\left(\widetilde{X}_n - \widetilde{X}_m\right)^2 = \mathbb{E}\int_0^T [g_n(s) - g_m(s)]^2 \mathrm{d}s.$$

Being convergent, $(\widetilde{X}_n)_{n\geq 1}$ is a Cauchy sequence, and hence the sequence $(g_n)_{n\geq 1}$ is convergent in $L^2(\omega \times [0,T], \mathrm{d}\mathbb{P} \times \mathrm{d}t)$, which is a complete space by the Riesz-Fischer theorem. Therefore, there exists a limiting function g such that the expectation $\mathbb{E}\int_0^T (g_n(s) - g(s))^2$ tends to zero as n becomes large, and the representation (1.2.1) holds with this very function g. □

Proof of Lemma 1.2.5. Consider a sequence $(t_i)_{i\geq 1}$, which forms a dense subset of the closed interval $[0,T]$ and define the increasing sequence $(\mathcal{F}_i)_{i\geq 1}$ as the sigma-algebras generated by $\{W_{t_j}(\cdot)\}_{1\leq j\leq i}$, for each $i \geq 1$. The martingale convergence theorem (Theorem A.4.2) implies that, for any $\varphi \in L^2(\mathcal{F}_T, \mathbb{P})$, the pointwise limit $\varphi = \mathbb{E}(\varphi|\mathcal{F}_T) = \lim_{i\uparrow\infty} \mathbb{E}(\varphi|\mathcal{F}_i)$ holds $\mathbb{P}$-almost everywhere and in $L^2(\mathcal{F}_T, \mathbb{P})$. Doob-Dynkyn Lemma (Lemma A.4.1) then yields $\mathbb{E}(\varphi|\mathcal{F}_i) = \varphi_i(W_{t_1}, \ldots, W_{t_i})$ for some Borel measurable function $\varphi_i : \mathbb{R}^i \to \mathbb{R}$, which can be approximated in $L^2(\mathcal{F}_T, \mathbb{P})$ by smooth and bounded functions $h_i(W_{t_1}, \ldots, W_{t_i})$, and the lemma follows. □

Proof of Lemma 1.2.6. The statement of the lemma is clearly equivalent to showing that the equality $\mathbb{E}[f(X,Y)Z] = \mathbb{E}[Zb(Y)]$ holds for any $\mathcal{G}$-measurable random variable Z. Let then μ_{XYZ} denote the law of the triplet (X, Y, Z) (taking values in $\mathbb{R}^{n+m+1}$). By independence, we clearly have $\mu_{XYZ}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = \mu_X(\mathrm{d}x)\mu_{YZ}(\mathrm{d}y, \mathrm{d}z)$, so that

$$\begin{aligned}\mathbb{E}[f(X,Y)Z] &= \int_{\mathbb{R}^{n+m+1}} zf(x,y)\mu_{XYZ}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = \int_{\mathbb{R}^{m+1}} z\left(\int_{\mathbb{R}^n} f(x,y)\mu_X(\mathrm{d}x)\right)\mu_{YZ}(\mathrm{d}y, \mathrm{d}z) \\ &= \int_{\mathbb{R}^{n+1}} zb(y)\mu_{YZ}(\mathrm{d}y, \mathrm{d}z) = \mathbb{E}[Zb(Y)].\end{aligned}$$

□

We now move on to the proof of the Martingale Representation theorem for Brownian motion:

Proof of Theorem 1.2.2. It is clear from the lemmas above that it is sufficient to show that the random variable $h(W_{t_1}, \ldots, W_{t_n})$ has the representation property (1.2.1) for any $n \geq 1$ and $h : \mathbb{R}^n \to \mathbb{R}$ continuous and bounded. We prove the case $n = 2$, but the proof extends easily to the general case. Let $n(\cdot; \mathfrak{m}, \sigma^2)$ denote the density of a Gaussian random variable with mean $\mathfrak{m}$ and variance σ^2:

$$n(z; \mathfrak{m}, \sigma^2) := \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(z-\mathfrak{m})^2}{2\sigma^2}\right\}, \qquad \text{for all } z \in \mathbb{R},$$

and introduce the function $v : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$ as

$$v(t, x, y) = \int_{\mathbb{R}} h(x, z) n(z; y, t_2 - t) \mathrm{d}z, \qquad \text{for all } t \in [0, t_2).$$

One can immediately check that it satisfies the partial differential equation

$$\frac{\partial v}{\partial t}(t, x, y) + \frac{1}{2} \frac{\partial^2 v}{\partial y^2}(t, x, y) = 0,$$

for all $t \in [0, t_2)$, $x, y \in \mathbb{R}$, with boundary condition $v(t_2, x, y) = h(x, y)$. Itô's formula then yields

$$\begin{aligned} h(W_{t_1}, W_{t_2}) = v(t_2, W_{t_1}, W_{t_2}) &= v(t_1, W_{t_1}, W_{t_1}) + \int_{t_1}^{t_2} \frac{\partial v}{\partial y}(s, W_{t_1}, W_s) \, \mathrm{d}W_s \\ &= \mathbb{E}\left[h(W_{t_1}, W_{t_2}) | \mathcal{F}_{t_1}\right] + \int_{t_1}^{t_2} \frac{\partial v}{\partial y}(s, W_{t_1}, W_s) \, \mathrm{d}W_s, \end{aligned}$$

by Lemma 1.2.6, since $\mathbb{E}\left[h(W_{t_1}, W_{t_2}) | \mathcal{F}_{t_1}\right] = \int_{\mathbb{R}} h(W_{t_1}, z) \mathbb{P}(W_{t_2} \in \mathrm{d}z)$. Consider now the function $\mathfrak{v} : [0, t_1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$\mathfrak{v}(t, x) = \begin{cases} v(t_1, x, x), & \text{if } t = t_1, \\ \displaystyle\int_{\mathbb{R}} v(t_1, z, z) n(z; x, t_1 - t) \mathrm{d}z, & \text{if } t \in (0, t_1), \end{cases}$$

Note in particular that the equality $\partial_t \mathfrak{v}(t, x) + \frac{1}{2} \partial_{xx} \mathfrak{v}(t, x) = 0$ holds for all $t \in [0, t_1)$ and $x \in \mathbb{R}$. Itô's formula then yields

$$\mathfrak{v}(t_1, W_{t_1}) = v(t_1, W_{t_1}, W_{t_1}) = \mathfrak{v}(0, 0) + \int_0^{t_1} \frac{\partial \mathfrak{v}}{\partial y}(s, W_s) \mathrm{d}W_s,$$

and hence

$$\begin{aligned} h(W_{t_1}, W_{t_2}) &= v(t_1, W_{t_1}, W_{t_1}) + \int_{t_1}^{t_2} \frac{\partial v}{\partial y}(s, W_{t_1}, W_s) \, \mathrm{d}W_s \\ &= \mathfrak{v}(t_1, W_{t_1}) + \int_{t_1}^{t_2} \frac{\partial v}{\partial y}(s, W_{t_1}, W_s) \, \mathrm{d}W_s \\ &= \mathfrak{v}(0, 0) + \int_0^{t_1} \frac{\partial \mathfrak{v}}{\partial y}(s, W_s) \mathrm{d}W_s + \int_{t_1}^{t_2} \frac{\partial v}{\partial y}(s, W_{t_1}, W_s) \, \mathrm{d}W_s \\ &= \mathfrak{v}(0, 0) + \int_0^{t_2} \psi(s) \mathrm{d}W_s, \end{aligned}$$

with the obvious definition

$$\psi(s) := \begin{cases} \partial_y \mathfrak{v}(s, W_s), & \text{if } s < t_1, \\ \partial_y v(s, W_{t_1}, W_s), & \text{if } t_1 \leq s < t_2. \end{cases}$$

The theorem then follows since $\mathfrak{v}(0, 0) = \mathbb{E}\left[h(W_{t_1}, W_{t_2}\right]$. □

Example 1.2.7. Fix some time horizon $T > 0$ and consider the $(\mathcal{F}_t)$-martingale M defined pathwise by $M_t := \mathbb{E}\left(\mathrm{e}^{W_T} | \mathcal{F}_t\right)$. Now,

$$M_t = \mathbb{E}\left(\mathrm{e}^{W_T - W_t} \mathrm{e}^{W_t} | \mathcal{F}_t\right) = \mathrm{e}^{W_t} \mathbb{E}\left(\mathrm{e}^{W_T - W_t}\right) = \exp\left(W_t + \frac{1}{2}(T - t)\right) =: f(t, W_t).$$

On the other hand, Itô's lemma yields

$$\begin{aligned} f(t, W_t) &= f(0, W_0) + \int_0^t \partial_w f(u, W_u)\mathrm{d}W_u + \int_0^t \partial_u f(u, W_u)\mathrm{d}u + \frac{1}{2}\int_0^t \partial_{ww} f(u, W_u)\mathrm{d}\langle W, W\rangle_u \\ &= f(0, W_0) + \int_0^t M_u \mathrm{d}W_u - \frac{1}{2}\int_0^t M_u \mathrm{d}u + \frac{1}{2}\int_0^t M_u \mathrm{d}u = M_0 + \int_0^t M_u \mathrm{d}W_u, \end{aligned}$$

which corresponds to the representation (1.2.1). Note that the filtration generated by M is the same as that from the Brownian motion W.

1.2.2 Change of measure and Girsanov Theorem

The second fundamental theorem in mathematical finance is the Girsanov[1] Theorem, which allows to change probability measures, and often allows for neat simplifications.

Change of measure

We first introduce the concept of change of measure, which shall be used later in a dynamic version to state Girsanov Theorem.

Theorem 1.2.8. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* Z *a non-negative (*$\mathbb{P}$*-almost surely) random variable satisfying* $\mathbb{E}(Z) = 1$. *Define the probability measure* $\widetilde{\mathbb{P}}$ *by*

$$\widetilde{\mathbb{P}}(A) := \int_A Z(\omega)\mathrm{d}\mathbb{P}(\omega), \qquad \text{for any } A \in \mathcal{F}.$$

Then the following hold:

- $\widetilde{\mathbb{P}}$ *is a probability measure;*
- *if* X *is a non-negative random variable then* $\widetilde{\mathbb{E}}(X) = \mathbb{E}(XZ)$*;*
- *if* $Z > 0$ $\mathbb{P}$*-almost surely, then* $\mathbb{E}(Y) = \widetilde{\mathbb{E}}(Y/Z)$ *for any non-negative random variable* Y.

Proof. We only prove the first statement in the theorem, the other two being straightforward. In order to prove that $\widetilde{\mathbb{P}}$ is a probability measure, we need to show that $\widetilde{\mathbb{P}}(\Omega) = 1$ and that it is countably additive. By definition, $\widetilde{\mathbb{P}}(\Omega) = \int_\Omega Z(\omega)\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}(Z) = 1$. Now, let $(A_k)_{k\geq 1}$ be a sequence of disjoint sets, and set, for any $n \geq 0$,

$$B_n := \bigcup_{k=1}^n A_k.$$

[1] Igor Vladimirovich Girsanov (1934-1967) was a Russian mathematician, who first worked in the group led by Eugene B. Dynkyn at Moscow State University, and later became an advocate of quantitative methods in mathematical economics, and of applications of optimal control to chemistry. He died of a hiking accident at the age of 33.

Since (B_n) is an increasing sequence, then $\mathbb{1}_{B_1} \leq \mathbb{1}_{B_2} \leq \dots$, $B_\infty = \cup_{k\geq 1} A_k$, and the monotone convergence theorem implies

$$\begin{aligned}
\widetilde{\mathbb{P}}(B_\infty) &= \int_\Omega \mathbb{1}_{B_\infty}(\omega) Z(\omega) \mathrm{d}\mathbb{P}(\omega) = \lim_{n\uparrow\infty} \int_\Omega \mathbb{1}_{B_n}(\omega) Z(\omega) \mathrm{d}\mathbb{P}(\omega) \\
&= \lim_{n\uparrow\infty} \int_\Omega \sum_{k=1}^{n} \mathbb{1}_{A_k}(\omega) Z(\omega) \mathrm{d}\mathbb{P}(\omega) = \lim_{n\uparrow\infty} \sum_{k=1}^{n} \int_\Omega \mathbb{1}_{A_k}(\omega) Z(\omega) \mathrm{d}\mathbb{P}(\omega) \\
&= \lim_{n\uparrow\infty} \sum_{k=1}^{n} \widetilde{\mathbb{P}}(A_k) = \sum_{k=1}^{\infty} \widetilde{\mathbb{P}}(A_k),
\end{aligned}$$

which proves the statement. □

Exercise 1.2.9. Let X be a centred Gaussian random variable with unit variance under the probability $\mathbb{P}$, and, for $\theta > 0$, define $Y := X + \theta$. Using the variable

$$Z := \exp\left\{-\theta X - \frac{\theta^2}{2}\right\},$$

and introducing the probability $\mathbb{Q}$ defined via its Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := Z$, show that Y is a centred Gaussian random variable with unit variance under $\mathbb{Q}$.

Solution. *Define the random variable* $Z := \exp\left\{-\theta X - \frac{1}{2}\theta^2\right\}$*, and a new probability measure* $\widetilde{\mathbb{P}}$ *as in Theorem 1.2.8, which is well defined since* Z *is strictly positive almost surely with unit expectation. For any* $y \in \mathbb{R}$*, we can now write (considering the set* $A = \{\omega \in \Omega : Y(\omega) \leq y\} \in \mathcal{F}$*)*

$$\begin{aligned}
\mathbb{P}(Y \leq y) &= \int_{\{\omega\in\Omega: Y(\omega)\leq y\}} Z(\omega) \mathrm{d}\mathbb{P}(\omega) = \int_\Omega \mathbb{1}_{\{Y(\omega)\leq y\}}(\omega) Z(\omega) \mathrm{d}\mathbb{P}(\omega) \\
&= \int_\Omega \mathbb{1}_{\{X(\omega)\leq y-\theta\}}(\omega) \exp\left\{-\theta X(\omega) - \frac{\theta^2}{2}\right\} \mathrm{d}\mathbb{P}(\omega) \\
&= \int_{\mathbb{R}} \mathbb{1}_{\{x\leq y-\theta\}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\theta x - \frac{\theta^2}{2}\right\} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \\
&= \int_{-\infty}^{y-\theta} \frac{1}{\sqrt{2\pi}} \exp\left\{-\theta x - \frac{\theta^2}{2}\right\} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \int_{-\infty}^{y-\theta} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x.
\end{aligned}$$

Girsanov Theorem

We present the general multi-dimensional version of Girsanov theorem below, but will restrict the proof–mainly for notational convenience–to the one-dimensional case.

Theorem 1.2.10 (Girsanov Theorem)**.** *Consider an* n*-dimensional Brownian motion* W *defined on some filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$*. Fix a time horizon* $T > 0$*, introduce the* n*-dimensional adapted process* $\Theta := (\Theta^1, \dots, \Theta^n)$*, and define*

$$Z_t := \exp\left\{-\int_0^t \Theta_u \cdot \mathrm{d}W_u - \frac{1}{2}\int_0^t \|\Theta_u\|^2 \mathrm{d}u\right\} \qquad \textit{and} \qquad \widetilde{W}_t := W_t + \int_0^t \Theta_u \mathrm{d}u.$$

If $\mathbb{E}\left(\int_0^T \|\Theta_u\|^2 Z_u^2 \mathrm{d}u\right)$ *is finite, then* $\mathbb{E}(Z_T) = Z_0 = 1$ *and* $\widetilde{W}$ *is a* $\widetilde{\mathbb{P}}$*-Brownian motion, where*

$$\widetilde{\mathbb{P}}(A) := \int_A Z_T(\omega) \mathrm{d}\mathbb{P}(\omega), \quad \textit{for any } A \in \mathcal{F}.$$

The other way of writing the new measure, which we shall always do here, is

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} := Z_T.$$

Remark 1.2.11. Novikov's condition [121] ensures that a given process is a martingale: let X is a real-valued adapted process on some filtered probability space supporting a Brownian motion W; if $\mathbb{E}\exp\left(\frac{1}{2}\int_0^t |X_s|^2\mathrm{d}s\right)$ is finite, then the Doléans-Dade exponential

$$\mathcal{E}\left(\int_0^\cdot X_s\mathrm{d}W_s\right) := \exp\left(\int_0^\cdot X_s\mathrm{d}W_s - \frac{1}{2}\int_0^\cdot X_s^2\mathrm{d}s\right)$$

is a true martingale. Other conditions exist in the literature, in particular Kazamaki's condition, but Novikov's criterion is the most widely used. Recent (technical) developments can be found in the works of Mijatović and Urusov [117] and of Ruf [138].

Before proving the theorem, we need a few tools.

Theorem 1.2.12 (Lévy's characterisation of Brownian motion)**.** *Let $(M_t)_{t\geq 0}$ be a continuous $(\mathcal{F}_t)_{t\geq 0}$-martingale such that $M_0 = 0$ and $\langle M\rangle_t = t$ for all $t \geq 0$, then M is a Brownian motion.*

Now, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if $\mathcal{Z}$ is a non-negative random variable with expectation equal to one, then one can define a new probability measure $\widetilde{\mathbb{P}}$ via

$$\widetilde{\mathbb{P}}(A) := \int_A \mathcal{Z}(\omega)\mathrm{d}\mathbb{P}(\omega), \qquad \text{for all } A \in \mathcal{F}. \tag{1.2.2}$$

Adding a filtration $(\mathcal{F}_t)_{t\geq 0}$ on $[0, T]$, we can define a process $(Z_t)_{t\geq 0}$ as $Z_t := \mathbb{E}(\mathcal{Z}|\mathcal{F}_t)$ for all $t \in [0, T]$. This process is called the Radon-Nikodým derivatives process and it clearly a $\mathbb{P}$-martingale. The following lemma then holds (see for instance [144, Lemma 5.2.2]):

Lemma 1.2.13. *For any $0 \leq u \leq t \leq T$, and any $\mathcal{F}_t$-measurable random variable Y,*

$$Z_u\widetilde{\mathbb{E}}\left(Y|\mathcal{F}_u\right) = \mathbb{E}\left(YZ_t|\mathcal{F}_u\right).$$

Proof of Theorem 1.2.10. We only prove Girsanov theorem in dimension one, the general case being analogous, albeit with more involved notations. From Lévy characterisation theorem (Theorem 1.2.12), since the process $\widetilde{W}$ starts at zero and has quadratic variation equal to t, all is left to prove is that it is a martingale under $\widetilde{\mathbb{P}}$. It is easy to see, using Itô's lemma, that $\mathrm{d}Z_t = -\Theta_t Z_t\mathrm{d}W_t$, so that

$$Z_t = Z_0 - \int_0^t \Theta_u Z_u\mathrm{d}W_u,$$

and Z is a martingale with $\mathbb{E}(Z_T) = \mathbb{E}(Z_0) = 1$. Now, the Itô product rule (Corollary 1.1.27) and simple manipulations yield

$$\mathrm{d}\left(\widetilde{W}_t Z_t\right) = \left(1 - \Theta_t\widetilde{W}_t\right) Z_t\mathrm{d}W_t,$$

so that $\widetilde{W}Z$ is also a martingale under $\mathbb{P}$, which implies that

$$\widetilde{\mathbb{E}}\left(\widetilde{W}|\mathcal{F}_u\right) = Z_u^{-1}\mathbb{E}\left(\widetilde{W}_t Z_t|\mathcal{F}_u\right) = Z_u^{-1}\widetilde{W}_u Z_u = \widetilde{W}_u,$$

by Lemma 1.2.13, and the theorem follows. □

1.3 The super-replication paradigm and FTAP

The route we choose to follow in this first approach to mathematical finance may sound unorthodox at first sight, starting with a more abstract framework about sub/super-replication, before venturing back into the realm of more 'classical' mathematical finance and option pricing. We believe, however, that this logic, borrowed from [77], follows the practical intuition more closely. Our aim is to answer the following question: what is the correct price the buyer or seller of an option should pay?

We fix a priori a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the historical probability measure. A market model is then defined as a pair process $(B_t, \mathrm{S}_t)_{t\geq 0}$ taking values in $(0,\infty)\times(0,\infty)^n$, for some $n\in\mathbb{N}$. The first component denotes the money-market account and satisfies the stochastic differential equation

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t, \qquad B_0 = 1, \tag{1.3.1}$$

where $(r_t)_{t\geq 0} \geq 0$ denotes the instantaneous risk-free rate process. Note that this ordinary differential equation can be solved in closed form as $B_t = \exp(\int_0^t r_s \mathrm{d}s)$. The remaining n-asset vector $\mathrm{S} = (S^1, \ldots, S^n)$ is the unique strong solution to the stochastic differential equation

$$\mathrm{dS}_t = \mathrm{b}(t, \mathrm{S}_t)\mathrm{d}t + \sigma(t, \mathrm{S}_t)\cdot \mathrm{dW}_t, \quad \mathrm{S}_0 \in (0,\infty)^n,$$

with W a d-dimensional standard Brownian motion, and where the drift $\mathrm{b} : \mathbb{R}_+\times(0,\infty)^n \to (0,\infty)^n$ and the diffusion $\sigma : \mathbb{R}_+\times(0,\infty)^n \to \mathcal{M}_{n,d}(\mathbb{R}_+^*)$ are assumed to be bounded and globally Lipschitz continuous (this in particular guarantees existence and uniqueness of the solution, as we shall show in Chapter 3). Consider now a portfolio $(\Pi_t)_{t\geq 0}$ consisting of (invested) cash and the n assets:

$$\Pi_t = \pi_t^0 B_t + \pi_t \cdot \mathrm{S}_t,$$

where $\pi = (\pi_1, \ldots, \pi^n)$ represents the vector of quantities of each asset in the portfolio, which can be readjusted as time passes.

Definition 1.3.1. The portfolio Π is self-financing if, for any $t\geq 0$, $\mathrm{d}\Pi_t = \pi_t^0 \mathrm{d}B_t + \pi_t\cdot \mathrm{dS}_t$; the variation of the portfolio only comes from the evolution of (B,S), and not from exogenous transfer (in or out) of money.

Assuming that the portfolio Π is self-financing and denoting by $D_{s,t} := B_s B_t^{-1}$ the discount factor between time s and time t (in particular, $D_{0,t} = B_t^{-1}$ satisfies $\mathrm{d}D_{0,t} = -r_t D_{0,t}\mathrm{d}t$), we have

$$\mathrm{d}\widetilde{\Pi}_t := \mathrm{d}(D_{0,t}\Pi_t) = \mathrm{d}\left(\pi_t^0 D_{0,t} B_t + \pi_t\cdot D_{0,t}\mathrm{S}_t\right) = \pi_t^0 \mathrm{d}\left(D_{0,t}B_t\right) + \mathrm{d}\left(\pi_t\cdot D_{0,t}\mathrm{S}_t\right) = \pi_t \cdot \mathrm{d}\widetilde{\mathrm{S}}_t,$$

where the $\widetilde{\ }$ notation denotes discounting ($\widetilde{\Pi}_t := D_{0,t}\Pi_t$), and hence, for any $t\geq 0$,

$$\widetilde{\Pi}_t = \widetilde{\Pi}_0 + \int_0^t \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u. \tag{1.3.2}$$

Consider an asset S with no drift ($\mathrm{b}(\cdot) \equiv 0$), and volatility coefficient σ_t, adapted to the filtration of the Brownian motion W. We can rewrite the previous equation as

$$\widetilde{\Pi}_t = \widetilde{\Pi}_0 + \int_0^t (D_{0,u}\pi_u \cdot \sigma_u) \cdot \mathrm{dW}_u,$$

which is strongly reminiscent of the Martingale Representation Theorem 1.2.2 with $\varphi_t = D_{0,t}\pi_t \cdot \sigma_t$.

Remark 1.3.2. Let Z denote the stochastic process defined pathwise as $Z_t := \pi_t^0 B_t$, for all $t \geq 0$. Then, for any $t \geq 0$,

$$\begin{aligned}\Pi_t = \pi_t^0 B_t + \pi_t \cdot S_t = Z_t + \pi_t \cdot S_t = \Pi_0 + \int_0^t \mathrm{d}\Pi_u &= \Pi_0 + \int_0^t \pi_u^0 \mathrm{d}B_u + \int_0^t \pi_u \cdot \mathrm{d}S_u \\ &= \Pi_0 + \int_0^t r_u Z_u \mathrm{d}u + \int_0^t \pi_u \cdot \mathrm{d}S_u,\end{aligned}$$

so that $\mathrm{d}Z_t = \mathrm{d}\overline{S}_t + r_t Z_t \mathrm{d}t$, with $\overline{S}_t := \int_0^t \pi_u \cdot \mathrm{d}S_u - \pi_t \cdot S_t$, which we can solve (Example 1.1.28(ii)) as $D_{0,t}Z_t = \pi_0^0 + \int_0^t D_{0,u}\mathrm{d}\overline{S}_u$, or equivalently, using integration by parts (Corollary 1.1.27),

$$\begin{aligned}D_{0,t}Z_t = \pi_t^0 = \pi_0^0 + \int_0^t D_{0,u}\mathrm{d}\overline{S}_u &= \pi_0^0 + D_{0,t}\overline{S}_t - D_{0,0}\overline{S}_0 - \int_0^t \overline{S}_u \mathrm{d}D_{0,u}, \\ &= \Pi_0 + D_{0,t}\overline{S}_t + \int_0^t r_u D_{0,u}\overline{S}_u \mathrm{d}u. \end{aligned} \tag{1.3.3}$$

This implies that the weight π^0 is an Itô process that can be chosen such that Π is self-financing.

Definition 1.3.3. A portfolio $(\pi_t^0, \pi_t^1, \ldots, \pi_t^n)_{t\in[0,T]}$ is said to be admissible over the horizon $[0,T]$ if, for any $t \in [0,T]$, Π_t is bounded from below $\mathbb{P}$-almost surely, e.g. there exists $M \in \mathbb{R}$ such that

$$\mathbb{P}\left(\inf_{0\leq t\leq T} \Pi_t \geq M\right) = 1. \tag{1.3.4}$$

We shall denote by $\mathcal{A}$ the space of admissible portfolios.

Definition 1.3.4. A self-financing admissible portfolio is called an arbitrage if

$$\Pi_0 = 0 \quad \text{and} \quad \Pi_T \geq 0, \quad \mathbb{P}\text{-almost surely}, \quad \text{and } \mathbb{P}(\Pi_T > 0) > 0.$$

In plain words, an arbitrage occurs if it is possible to obtain a strictly positive gain out of a strategy with zero initial cost. Intuitively, absence of arbitrage in a market is related to the notion of equilibrium, and mathematically, this is stated in terms of conditions on admissible portfolios. One could wonder about the necessity of Condition (1.3.4) in Definition 1.3.3. This condition imposes a limit accepted by creditors. As the following example shows, one cannot do without it:

Example 1.3.5. Within a fixed time horizon $[0,T]$, consider a market without interest rate ($B_t = 1$ for all $t \in [0,T]$) and consisting of one asset S satisfying $\mathrm{d}S_t = \mathrm{d}W_t$, with $S_0 = 0$. Define now the process Y as $Y_t := \int_0^t (T-s)^{-1/2}\mathrm{d}W_s$. By time-change techniques (Theorem 1.1.21), there exists a Brownian motion $\widehat{W}$ such that $Y_t = \widehat{W}_{\langle Y\rangle_t}$, where

$$\langle Y\rangle_t = \int_0^t \frac{\mathrm{d}s}{(T-s)} = -\log(T-t), \qquad \text{for all } t \in [0,T).$$

Fix a constant $M > 0$, and define the stopping times $\tau_M := \inf\{t \geq 0 : \widehat{W}_t = M\}$ and $\tau_M^Y := \inf\{t \geq 0 : Y_t = M\}$. As τ_M is finite almost surely (Lemma 1.5.9 below) and equal to $-\log(T-\tau_M^Y)$, so that $\tau_M^Y < T$ almost surely. We now introduce the weight process

$$\pi_t^1 := \begin{cases} (T-t)^{-1/2}, & \text{if } t \in [0, \tau_M^Y), \\ 0, & \text{if } t \in [\tau_M^Y, T], \end{cases}$$

and let π^0 be given by (1.3.3) such that the corresponding portfolio Π has zero value at inception:

$$\pi_t^0 = \overline{S}_t = \int_0^t \pi_u^1 \mathrm{d}S_u - \pi_t^1 S_t, \qquad \text{for } t \in [0,T].$$

Indeed, $\Pi_0 = \pi_0^0 + \pi_0^1 S_0 = 0$ if and only it $\pi_0^0 = -\pi_0^1$. The value of the portfolio therefore reads

$$\Pi_t = \int_0^t \pi_u^1 \mathrm{d}S_u = \int_0^{t\wedge\tau_M^Y} \frac{\mathrm{d}W_u}{\sqrt{T-u}},$$

and in particular, at maturity T, the portfolio is worth

$$\Pi_T = \int_0^{\tau_M^Y} \frac{\mathrm{d}W_u}{\sqrt{T-u}} = \widehat{W}_{\tau_M^Y} = M \text{ almost surely.}$$

This implies that, starting from zero initial value, the porfolio thus constructed can reach any strictly positive value almost surely, leading to an arbitrage. Note, however, that, for any $t \in [0,T]$, Π_t is not bounded below almost surely, thus violating Condition (1.3.4).

The following lemma, despite its simplicity, is a core result for option pricing.

Lemma 1.3.6. *Assume that there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the discounted asset prices $\widetilde{\mathrm{S}}$ are $\mathbb{Q}$-local martingales. Then the market does not admit arbitrage.*

For two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, $\mathbb{P}$ is said to be absolutely continuous with respect to $\mathbb{Q}$, which is denoted by $\mathbb{P} \ll \mathbb{Q}$, if for any $A \in \mathcal{F}$ such that $\mathbb{Q}(A) = 0$, then $\mathbb{P}(A) = 0$. In particular, if $\mathbb{P} \ll \mathbb{Q}$, then there exists a random variable $Z \in L^1(\mathrm{d}\mathbb{P})$ such that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = Z$ and $\mathbb{E}^{\mathbb{P}}(Z) = 1$. The two probability measures are said to be equivalent, and we write $\mathbb{P} \sim \mathbb{Q}$, if $\mathbb{P} \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P}$. In that case, they have the same null sets and $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q} = (\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})^{-1}$. This analysis is reminiscent of Girsanov's theorem (Theorem 1.2.10).

Proof of Lemma 1.3.6. Assume there exists an arbitrageable strategy $(\pi^0, \ldots, \pi^n)$. By construction, the corresponding discounted portfolio is worth (see (1.3.2)), at time t, $\widetilde{\Pi}_t = \widetilde{\Pi}_0 + \int_0^t \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u$, and hence is a $\mathbb{Q}$-local martingale. Being bounded below (by zero), it is a $\mathbb{Q}$-supermartingale (by Proposition 1.1.12), so that $\mathbb{E}^{\mathbb{Q}}(\widetilde{\Pi}_T) \leq \Pi_0 = 0$. However, since $\widetilde{\Pi}_T \geq 0$ $\mathbb{P}$-almost surely, then it is also non-negative $\mathbb{Q}$-almost surely. Since its expectation is null, then $\widetilde{\Pi}_T = 0$ $\mathbb{Q}$-almost surely, hence $\mathbb{P}$-almost surely, which yields a contradiction. □

Definition 1.3.7. Any measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\widetilde{\mathrm{S}}$ is a $\mathbb{Q}$-local martingale is called an Equivalent Local Martingale Measure (ELMM), or a risk-neutral measure.

Remark 1.3.8. In the classical Black-Scholes framework, see also below in Section 1.2, there exists a unique probability measure under which the discounted stock price ($n = 1$) is a true martingale. However, there may be markets, as will be the case for stochastic volatility models for instance, where an infinity number of such probability measures exist.

The last result we wish to state in this framework is a necessary and sufficient condition ensuring that a market has no arbitrage. It also makes (absence of) arbitrage easier to check than the general conditions above.

Theorem 1.3.9.

(i) Assume that there exists a process $(u_t)_{t\in[0,T]} \in \mathcal{V}$ *(Definition 1.2.1) such that*

$$\sigma(t, \mathrm{S}_t)u_t = \mathrm{b}(t, \mathrm{S}_t) - r_t\mathrm{S}_t \text{ almost surely for all } t \in [0, T], \tag{1.3.5}$$

and such that $\mathbb{E}\left(\exp\left\{\frac{1}{2}\int_0^T u_t^2 \mathrm{d}t\right\}\right)$ *is finite. Then the market does not admit any arbitrage.*

(ii) If the market has no arbitrage, there exists an $(\mathcal{F}_t)$*-adapted process* (u_t) *such that* (1.3.5) *holds.*

Proof. We only prove the sufficient condition (i) and assume for simplicity that $r_t = 0$ almost surely for all $t \geq 0$. The probability measure $\mathbb{Q}$ on $\mathcal{F}_T$, defined via the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{-\int_0^T u_t \mathrm{d}W_t - \frac{1}{2}\int_0^T u_t^2 \mathrm{d}t\right\},$$

is equivalent to $\mathbb{P}$ and Girsanov's theorem (Theorem 1.2.10) ensures that the process $\widetilde{\mathrm{W}}$ defined as

$$\widetilde{\mathrm{W}}_t := \int_0^t u_s \mathrm{d}s + \mathrm{W}_t,$$

is a $\mathbb{Q}$-Brownian motion, and furthermore $\mathrm{dS}_t = \sigma(t, \mathrm{S}_t) \cdot \mathrm{d}\widetilde{\mathrm{W}}_t$. Therefore S is a local $\mathbb{Q}$-martingale and the theorem follows from Lemma 1.3.6. □

Example 1.3.10.

- Consider the market given by $B_t = 1$ almost surely for all $t \geq 0$, and

$$\mathrm{dS}_t = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mathrm{dW}_t.$$

  Show that the market does not admit arbitrage.

- Consider now the given by $B_t = 1$ almost surely for all $t \geq 0$, and

$$\mathrm{dS}_t = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathrm{dW}_t.$$

  Show that the market admits an arbitrage and explain what happens if one considers the strategy $\pi_t \equiv (1, 1)$.

The above framework guarantees that market participants will be able to buy or sell derivatives at a 'fair' price. That said, it is not clear what this price should be and, intuitively, there is no reason why a buyer should be willing to pay the same price as a seller[2]. Assume that, at time t, the buyer buys at price $\mathfrak{p}$, a European option with maturity T and payoff P_T (which may depend on all the assets), and hedges it until maturity. At maturity, using (1.3.2), the discounted value of its portfolio is worth

$$\widetilde{\Pi}_T = D_{0,T}\left(-D_{t,T}^{-1}\mathfrak{p}\right) + \int_t^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u + D_{0,T}P_T = -D_{0,t}\mathfrak{p} + \int_t^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u + D_{0,T}P_T. \qquad (1.3.6)$$

This yields the following natural definitions of the buyers and the seller's price:

Definition 1.3.11. At time t, the buyer's ($\mathcal{B}_t$) and the seller's ($\mathcal{S}_t$) prices are defined by

$$\mathcal{B}_t(P_T) := \sup\left\{\mathfrak{p} \in \mathcal{F}_t : \exists \pi \in \mathcal{A} : \widetilde{\Pi}_T = -D_{0,t}\mathfrak{p} + \int_t^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u + D_{0,T}P_T \geq 0, \mathbb{P}\text{-a.s}\right\},$$
$$\mathcal{S}_t(P_T) := \inf\left\{\mathfrak{p} \in \mathcal{F}_t : \exists \pi \in \mathcal{A} : \widetilde{\Pi}_T = D_{0,t}\mathfrak{p} + \int_t^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u - D_{0,T}P_T \geq 0, \mathbb{P}\text{-a.s}\right\},$$

namely the buyer's price (resp. seller's price) is the largest (resp. smallest) initial amount to pay in order to obtain a non-negative value of the portfolio at maturity.

The following theorem is a key result to determine bounds for these prices.

Theorem 1.3.12. *If there exists a local martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, then*

$$\mathcal{B}_t(P_T) \leq \mathbb{E}^{\mathbb{Q}}\left(D_{t,T}P_T|\mathcal{F}_t\right) \leq \mathcal{S}_t(P_T).$$

Proof. Using the definition of the buyer's price (Definition 1.5.1), there exists an admissible portfolio π such that

$$-D_{0,t}\mathfrak{p} + \int_t^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u + D_{0,T}P_T \geq 0, \quad \mathbb{P}\text{-a.s.}$$

By construction, the integral is a local martingale, which is bounded below (since the portfolio is admissible), and therefore is a supermartingale, starting from zero at time zero, and hence, taking $\mathbb{Q}$-expectations conditional on $\mathcal{F}_t$, we obtain

$$D_{0,t}\mathfrak{p} \leq \mathbb{E}^{\mathbb{Q}}(D_{0,T}P_T|\mathcal{F}_t),$$

or $\mathfrak{p} \leq \mathbb{E}^{\mathbb{Q}}(D_{t,T}P_T|\mathcal{F}_t)$. Taking the supremum over all $\mathfrak{p} \in \mathcal{F}_t$ yields one inequality in the theorem; the proof for the seller's price is analogous. □

This result provides arbitrage-free bounds for the price of any European option. However, from a practical point of view, it might seem limited in the sense that (i) the equivalent local

[2] these are not mere theoretical considerations, and can be clearly observed through limit order books and bid-ask spreads.

martingale measure $\mathbb{Q}$ may not be unique, and (ii) the range of admissible prices might be very wide. Uniqueness (or absence thereof) of $\mathbb{Q}$ is related to the notion of complete market, in which these bounds collapse to a single arbitrage-free price.

Definition 1.3.13. A payoff P_T is said to be attainable if there exists a self-financing admissible portfolio $\pi \in \mathcal{A}$ and a real number $\mathfrak{p}$ such that

(a) $\mathfrak{p} + \int_0^T \pi_t \mathrm{d}\widetilde{\mathrm{S}}_t - D_{0,T} P_T = 0$, $\mathbb{P}$-almost surely;

(b) the process $\left(\int_0^t \pi_t \mathrm{d}\widetilde{\mathrm{S}}_t\right)_{t\geq 0}$ is a true $\mathbb{Q}$-martingale with $\mathbb{Q} \sim \mathbb{P}$.

Note as a comparison to (a), that Equation (1.3.6) represents the value of the discounted portfolios, from the seller's point of view.

Definition 1.3.14. A market is said to be complete if every payoff is attainable.

Theorem 1.3.15. *In a complete market, the double equality* $\mathcal{B}_t(P_T) = \mathbb{E}^{\mathbb{Q}}\left(D_{t,T} P_T | \mathcal{F}_t\right) = \mathcal{S}_t(P_T)$ *holds for any equivalent local martingale measure* $\mathbb{Q}$.

Proof. Since the market is complete, there exist $\widetilde{\mathfrak{p}} \in \mathbb{R}$ and an admissible portfolio π such that

$$D_{0,T} P_T = \widetilde{\mathfrak{p}} + \int_0^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u, \quad \mathbb{P}\text{-almost surely},$$

and therefore

$$D_{0,T} P_T = D_{0,t} Z_t + \int_t^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u, \quad \mathbb{Q}\text{-almost surely},$$

where $Z_t := D_{0,t}^{-1}\left(\tilde{\mathfrak{p}} + \int_0^t \pi_u \mathrm{d}\widetilde{\mathrm{S}}_u\right)$ is an $\mathcal{F}_t$-measurable random variable for any $t \in [0,T]$. The integral is a $\mathbb{Q}$-martingale, so that, taking $\mathbb{Q}$-expectations (conditional on $\mathcal{F}_t$) on both sides yields $Z_t = \mathbb{E}^{\mathbb{Q}}(D_{t,T} P_T | \mathcal{F}_t)$, and, identifying $Z_t = \widetilde{\mathfrak{p}}$, $\mathcal{S}_t(P_T) \leq \mathbb{E}^{\mathbb{Q}}(D_{t,T} P_T | \mathcal{F}_t)$ and the theorem follows from the arbitrage-free bounds in Theorem 1.3.12. The same arguments apply to the buyer's price, and the theorem follows. □

The following theorem, the proof of which unfortunately[3] falls outside the scope of these notes, provides a new characterisation of the buyer's and the seller's prices in terms of the equivalent local martingale measures:

Theorem 1.3.16. *Let* $\mathcal{M}_{\mathbb{P}}^{e}$ *denotes the set of* $\mathbb{P}$*-equivalent local martingale measures. Then*

$$\mathcal{B}_t(P_T) = \inf_{\mathbb{Q} \in \mathcal{M}_{\mathbb{P}}^{e}} \mathbb{E}^{\mathbb{Q}}\left(D_{t,T} P_T | \mathcal{F}_t\right) \qquad \text{and} \qquad \mathcal{S}_t(P_T) = \sup_{\mathbb{Q} \in \mathcal{M}_{\mathbb{P}}^{e}} \mathbb{E}^{\mathbb{Q}}\left(D_{t,T} P_T | \mathcal{F}_t\right),$$

The proof of the following corollary, however, is within reach and is left as a tedious exercise:

[3]The expression 'unfortunately' here refers to the fact that its proof is a beautiful, yet tedious, exercise in convex duality. The hard-working interested reader can consult [132] for full details.

Corollary 1.3.17. *A market is complete if and only if there exists a unique equivalent local martingale measure.*

Similarly to Theorem 1.3.9, we provide–without proof–an easy to check criterion for market completeness.

Theorem 1.3.18. *The market is complete if and only if σ admits a left inverse almost surely.*

Note in particular that if $\sigma(\cdot)$ is invertible (almost surely), then clearly the process u in Theorem 1.3.9 is well defined, and hence the market is free of arbitrage. The converse does not hold in general, though.

Example 1.3.19.

- Show that the market characterised by $B_t \equiv 1$ and $\mathrm{dS}_t = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \mathrm{d}(W_t^{(1)}, W_t^{(2)})$ is complete.
- The market given by $B_t \equiv 1$ and $\mathrm{dS}_t = \mathrm{d}W_t^{(1)} + \mathrm{d}W_t^{(2)}$ is incomplete.

In layman terms, the completeness of the market is due to the fact that the number of Brownian motions driving the system, quantifying the randomness, is equal to the number of assets. The Black-Scholes model is the obvious example of a complete model, as is Dupire's local volatility model, which we will study in Section 4.1. Stochastic volatility models, on the other hand, are classical examples of incomplete market models (Section 4.2), since the system is driven by two Brownian motions, but only one asset (the stock price) is tradable.

In a complete market, Corollary 1.3.17 ensures the existence of a unique equivalent local martingale measure $\mathbb{Q}$ such that the unique arbitrage-free price reads $\mathcal{B}_t(P_T) = \mathcal{S}_t(P_T) = \mathbb{E}^{\mathbb{Q}}\left(D_{t,T}P_T|\mathcal{F}_t\right)$ (see Theorem 1.3.16). As an example, in the Black-Scholes model, there is one source of randomness only (one Brownian motion), driving a single asset. The market is then complete. We shall see later some examples of incomplete markets, in particular stochastic volatility models, where the number of Brownian motions driving the system is larger than the number of traded assets.

1.3.1 Overture on optimal transport problems and model-free hedging

1.4 Application to the Black-Scholes model

We now assume that a probability $\mathbb{P}$ is given, under which the stock price process is the (unique strong) solution to the following stochastic differential equation:

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t), \quad S_0 > 0, \tag{1.4.1}$$

where σ is a strictly positive real number, and W a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We are interested in evaluating, at inception, a derivative on S, with payoff H_T at maturity $T > 0$. The (random) payoff H_T is assumed to be $\mathcal{F}_T$-measurable and square integrable. Let now Υ denote the following set of strategies:

$$\Upsilon := \left\{ (\pi_t)_{t\in[0,T]} \text{ adapted } : \int_0^T \pi_t^2 \mathrm{d}t < \infty \text{ almost surely} \right\}, \tag{1.4.2}$$

and, for each $\pi \in \Upsilon$, let Π denote the solution to the SDE $\mathrm{d}\Pi_t = \pi_t \mathrm{d}S_t + D_{0,t} (\Pi_t - \pi_t S_t) \mathrm{d}B_t$, starting at $\Pi_0 = x_0$. Note that Υ is nothing else than the almost sure version of $\mathcal{V}$ from Definition 1.2.1. In the context of Section 1.3, the process Π corresponds to the portfolio associated to the strategies π and $\pi_t^0 = D_{0,t}(X_t - \pi_t S_t)$, which is clearly self-financing. The discounted process $\widetilde{\Pi} := (D_{0,t}\Pi_t)_{t\geq 0}$ satisfies the SDE

$$\mathrm{d}\widetilde{\Pi}_t = \pi_t D_{0,t} S_t \left((\mu - r)\mathrm{d}t + \sigma \mathrm{d}W_t\right) =: \sigma \pi_t \widetilde{S}_t \mathrm{d}\widetilde{W}_t,$$

where $\widetilde{S}$ is the discounted stock price process and $\widetilde{W}$ a $\widetilde{\mathbb{P}}$-Brownian motion. The probability $\widetilde{\mathbb{P}}$ here is defined via the Girsanov transformation

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(-\frac{1}{2}\gamma^2 T + \gamma W_T\right),$$

and $\widetilde{W}_t := W_t - \gamma t$ for $t \in [0, T]$, with $\gamma := (r - \mu)/\sigma$. Under $\widetilde{\mathbb{P}}$, the stock price process S satisfies $\mathrm{d}S_t = S_t(r\mathrm{d}t + \sigma \mathrm{d}\widetilde{W}_t)$, and hence $\widetilde{\mathbb{P}}$ is a risk-neutral probability measure. The Martingale Representation Theorem implies that there exists an adapted square integrable process $\widetilde{\phi}$ such that

$$D_{0,T} H_T = \mathbb{E}^{\widetilde{\mathbb{P}}}\left(D_{0,T} H_T\right) + \int_0^T \widetilde{\phi}_t \mathrm{d}\widetilde{W}_t.$$

Letting $\pi_t = \widetilde{\phi}_t/(\sigma D_{0,t} S_t)$ and π_t^0 as above, and $x_0 = \mathbb{E}^{\widetilde{\mathbb{P}}}(D_{0,T} H_T)$, the trading strategy (π, π^0, x_0) generates a portfolio Π such that $\Pi_T = H_T$ almost surely, i.e. a replicating portfolio for the contingent claim H_T, and therefore the option value at inception is equal to x_0.

Theorem 1.4.1. *If there exists a strictly positive constant A^π such that $\Pi_t \geq -A^\pi$ almost surely for all $t \in [0, T]$, then the set Υ does not contain any arbitrage opportunity.*

Proof. Suppose there exists $\pi \in \Upsilon$ and that $x_0 = 0$. The discounted process $\widetilde{\Pi}$ is a $\widetilde{\mathbb{P}}$-local martingale bounded below, and therefore the process $\widetilde{\Pi} + A^\pi$ is a supermartingale, and so is $\widetilde{\Pi}$; in particular, for any $t \geq 0$,

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left(\widetilde{\Pi}_t\right) \leq \mathbb{E}^{\widetilde{\mathbb{P}}}\left(\widetilde{\Pi}_0\right) = 0.$$

If $\Pi_t \geq 0$ $\mathbb{P}$-almost surely for any $t \geq 0$ and $\mathbb{P}(\Pi_t > 0) > 0$ then $\mathbb{E}^{\widetilde{\mathbb{P}}}(\widetilde{\Pi}_t) > 0$ since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, which yields a contradiction. □

We can now derive the celebrated Black-Scholes formula:

Theorem 1.4.2. *Under the risk-neutral probability $\widetilde{\mathbb{P}}$, the European Call price with strike K and maturity T, with payoff $H_T := (S_T - K)_+$ is worth, at any time $t \in [0, T]$,*

$$C_{\mathrm{BS}}(S_0, K, t, T, \sigma) := \mathbb{E}^{\widetilde{\mathbb{P}}}\left[D_{t,T}(S_T - K)_+ | \mathcal{F}_t\right] = S_t \mathcal{N}(d_+) - K D_{t,T} \mathcal{N}(d_-),$$

where $d_\pm := \frac{1}{\sigma\sqrt{T-t}} \log\left(\frac{S_t}{D_{t,T}K}\right) \pm \frac{1}{2}\sigma\sqrt{T-t}$.

Corollary 1.4.3. *For a European Put price, we have the following:*

$$P_{\mathrm{BS}}(S_0, K, t, T, \sigma) := \mathbb{E}^{\widetilde{\mathbb{P}}}\left[D_{t,T}(K - S_T)_+ | \mathcal{F}_t\right] = K D_{t,T} \mathcal{N}(-d_-) - S_t \mathcal{N}(-d_+),$$

Proof. Under $\widetilde{\mathbb{P}}$, the stock price satisfies $\mathrm{d}S_t = S_t(r\mathrm{d}t + \sigma \mathrm{d}W_t)$; hence Itô's formula yields

$$S_T = S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(W_T - W_t)\right\}, \qquad \text{for any } 0 \le t \le T,$$

and the theorem follows by direct integration of the Gaussian random variable $(W_T - W_t)$. The proof for the Put price is analogous. □

We now introduce some notations to simplify computations. Rewrite Theorem 1.4.2 as

$$C_{\mathrm{BS}}(S_0, K, t, T, \sigma) = \mathbb{E}\left[D_{t,T}(S_T - K)_+ | \mathcal{F}_t\right] = S_t \mathrm{BS}\left(\log\left(\frac{K D_{t,T}}{S_t}\right), \sigma^2(T - t)\right),$$

for any $t \in [0, T]$, where the function $\mathrm{BS} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$\mathrm{BS}(k, v) := \begin{cases} \mathcal{N}(d_+(k, v)) - \mathrm{e}^k \mathcal{N}(d_-(k, v)), & \text{if } v > 0, \\ \left(1 - \mathrm{e}^k\right)_+, & \text{if } v = 0, \end{cases} \tag{1.4.3}$$

with $d_+(k, v) := -k/\sqrt{v} \pm \sqrt{v}/2$, where $\mathcal{N}$ denotes the cumulative distribution function of the Gaussian distribution. The Black-Scholes model has independent and stationary increments, so that the price of the Call option only depends on time through the remaining time to maturity $T - t$; note that it also only depends on the volatility through the total variance: $v := \sigma^2(T - t)$. Therefore, from now on, we shall prefer the notation $C_{\mathrm{BS}}(k, T - t, \sigma)$, or even $C_{\mathrm{BS}}(k, v)$, to the over-parameterised $C_{\mathrm{BS}}(S_t, K, t, T, \sigma)$, where $k := S_t/(D_{t,T}K)$ is the log-forward moneyness.

Robustness of the Black-Scholes strategy

The way we proved the Black-Scholes formula in Theorem 1.4.2, via the martingale representation theorem, does not follow the original proof of the authors [20]. They assumed that there exists a smooth function $C : \mathbb{R}_+^2 \to \mathbb{R}$ describing the value of the option, and such that $C(t, s)$ converges to the option payoff $h(\cdot)$ as t approaches the maturity T. This function satisfies the Black-Scholes partial differential equation

$$\left(\partial_t + rS\partial_S + \frac{\sigma^2}{2} S^2 \partial_{SS} - r\right) C(t, S) = 0, \tag{1.4.4}$$

for all $t \in [0,T)$ and $S \geq 0$, with boundary condition $C(T,S) = h(S)$. In the Black-Scholes framework, assuming that we believe that σ is the true volatility, then we hedge ourselves by buying/selling an amount $\partial_S C(t,S)$ of the stock price. This is called delta hedging. Suppose now that the true dynamics of the underlying stock price are in fact given by

$$\mathrm{d}S_t = S_t \left(\alpha_t \mathrm{d}t + \beta_t \mathrm{d}W_t\right),$$

where α and β are two adapted processes. As seen in the previous section, we can construct a self-financing replicating portfolio Π satisfying

$$\mathrm{d}\Pi_t = \partial_S C(t,S_t)\mathrm{d}S_t + \left(\Pi_t - \partial_S C(t,S_t)S_t\right) r\mathrm{d}t.$$

Now, using Itô's formula, the Call price function satisfies

$$\mathrm{d}C(t,S_t) = \partial_S C(t,S_t)\mathrm{d}S_t + \left(\partial_t + \frac{\beta_t^2 S_t^2}{2}\partial_{SS}\right) C(t,S_t)\mathrm{d}t.$$

The hedging error $\mathcal{E} := \Pi - C$ then satisfies

$$\begin{aligned}
\mathrm{d}\mathcal{E}_t &= r\Pi_t - \left(\partial_t + \frac{\beta_t^2 S_t^2}{2}\partial_{SS} + rS_t\partial_S\right) C(t,S_t)\mathrm{d}t \\
&= r\Pi_t - \left(\frac{(\sigma^2 - \beta_t^2)S_t^2}{2}\partial_{SS} - r\right) C(t,S_t)\mathrm{d}t, \\
&= r\mathcal{E}_t - \frac{1}{2}(\sigma^2 - \beta_t^2)S_t^2 \partial_{SS} C(t,S_t)\mathrm{d}t,
\end{aligned}$$

where, in the second line, we used the fact that the Call price satisfies the Black-Scholes PDE (1.4.4). An application of Itô's formula yields, at maturity,

$$\mathcal{E}_T = \frac{1}{2}\int_0^T \mathrm{e}^{r(T-s)} S_t^2 \Gamma_t^2 \left(\sigma^2 - \beta_t^2\right) \mathrm{d}t.$$

This formula is fundamental for hedging, and indicates that the error in the hedging strategy is due to (i) the under/overestimation of the actual volatility (the sign of $\widehat{\sigma}^2 - \xi_t^2$), and (ii) the amount of (positive) convexity Γ_t of the option price with respect to the underlying. Taking derivatives of the Call option price with respect to the parameters yields the so-called 'Greeks', which, as seen above, are fundamental tools for hedging purposes:

$$\left\{\begin{array}{ll}
\partial_k \mathrm{BS}(k,v) & = -\mathrm{e}^k \mathcal{N}(d_-(k,v)), \\
\partial_{kk} \mathrm{BS}(k,v) & = -\mathrm{e}^k \left[\mathcal{N}(d_-(k,v)) - \dfrac{n(d_-(k,v))}{\sqrt{v}}\right] = \partial_k \mathrm{BS}(k,v) + \dfrac{\mathrm{e}^k n(d_-(k,v))}{\sqrt{v}}, \\
\partial_v \mathrm{BS}(k,v) & = \dfrac{n(d_+(k,v))}{2\sqrt{v}}, \\
\partial_{vv} \mathrm{BS}(k,v) & = \dfrac{n(d_+(k,v))}{16v^{5/2}}\left(4k^2 - v^2 - 4v\right), \\
\partial_{kv} \mathrm{BS}(k,v) & = -n(d_+(k,v))\partial_v d_-(k,v) = -\dfrac{n(d_+(k,v))}{4}\dfrac{2k-v}{v^{3/2}},
\end{array}\right. \tag{1.4.5}$$

where $n \equiv \mathcal{N}'$ denotes the Gaussian density. The proof of the equalities (1.4.5) is left as an exercise.

1.5 Beyond Vanilla options: a probabilistic approach

1.5.1 American options: stopping the Brownian motion

We follow here the notations and framework of Section 1.3. We consider an American option with exercise price P_t at time t, written on the underlying stock price S. For such an option, the buyer (seller) can exercise the option at any time during the life of the contract, not only at maturity, as is the case for European options. Mathematically, we can then define, in a very intuitive way, and similarly to Definition 1.5.1, the buyer's and seller's price:

Definition 1.5.1. At time t, the buyer's and the seller's prices ($\mathcal{B}_t$ and $\mathcal{S}_t$) are defined by

$$\begin{aligned}
\mathcal{B}_t(P) &:= \sup\left\{\mathfrak{p} \in \mathcal{F}_t : \exists(\pi,\tau) \in \mathcal{A}\times\mathcal{T}_{t,T} : \quad \widetilde{\Pi}_\tau = -D_{0,t}\mathfrak{p} + \int_t^\tau \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u + D_{0,\tau}P_\tau \geq 0, \mathbb{P}\text{-a.s}\right\},\\
\mathcal{S}_t(P) &:= \inf\left\{\mathfrak{p} \in \mathcal{F}_t : \exists\pi \in \mathcal{A} : \forall \mathfrak{t} \in [t,T], \quad \widetilde{\Pi}_\tau = D_{0,t}\mathfrak{p} + \int_t^{\mathfrak{t}} \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u - D_{0,\mathfrak{t}}P_{\mathfrak{t}} \geq 0, \mathbb{P}\text{-a.s}\right\},\\
&= \inf\left\{\mathfrak{p} \in \mathcal{F}_t : \exists\pi \in \mathcal{A} : \forall \tau \in \mathcal{T}_{t,T}, \quad \widetilde{\Pi}_\tau = D_{0,t}\mathfrak{p} + \int_t^\tau \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u - D_{0,\tau}P_\tau \geq 0, \mathbb{P}\text{-a.s}\right\}.
\end{aligned}$$

Namely the buyer's price is the largest initial amount to pay in order to obtain a non-negative value of the portfolio at some point between time t and the maturity of the contract. The seller, however, does not decide when the option is exercised, and hence has to hedge himself at any time between inception t and maturity T. Here P denotes the payoff, which can be gained at any time between t and T, so that it does not have any T-dependence, and $\mathcal{T}_{t,T}$ represent the set of stopping times with values in the closed interval $[t,T]$. The following theorem is similar to the European case and characterises the price of the American option: Before stating the main valuation theorem, let us introduce several tools.

Definition 1.5.2. The process $\mathcal{S}$ defined pathwise by

$$\mathcal{S}_t := D_{0,t}\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}^{\mathbb{Q}}\left[D_{t,\tau}P_\tau|\mathcal{F}_t\right]$$

is called the Snell envelope (corresponding to the American option with payoff P) under $\mathbb{Q}$.

In general, the Snell[4] envelope of a stochastic process is the smallest supermartingale dominating it. In discrete time, an explicit construction (by recursion) is available, see for example [59, Chapter 6].

Lemma 1.5.3. *$\mathcal{S}$ is a $\mathbb{Q}$-supermartingale.*

Theorem 1.5.4. *Assume that there exists an equivalent local martingale measure $\mathbb{Q}$. Then*

$$\mathcal{B}_t(P) \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}^{\mathbb{Q}}\left(D_{t,\tau}P_\tau|\mathcal{F}_t\right) \leq \mathcal{S}_t(P);$$

if furthermore the market is complete, then all the inequalities become equalities, and the value of the American option at time $t \in [0,T]$ is given by $D_{0,t}^{-1}\mathcal{S}_t$.

[4]James Laurie Snell (1925-2011), American mathematician, published this result in 1952, see [146].

Part of the proof of the theorem is analogous to the European case studied above. However, the seller's price needs some more work, and requires the so-called Doob-Meyer decomposition. The following theorem is a cornerstone in stochastic analysis, and was proved, albeit in a slightly less general version, by Meyer [115, 116]. Its proof is clearly outside the scope of these lectures, but, in order to satiate the reader's appetite for curiosity, we shall include a statement and a proof of the discrete-time version, originally proposed by Doob.

Theorem 1.5.5 (Doob Decomposition)**.** *Let $(\Omega, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ be some discrete probability space, and $(Z_n)_{n\in\mathbb{N}}$ a discrete-time $(\mathcal{F}_n)$-adapted process starting at zero such that $\mathbb{E}(Z_n)$ is finite for all $n \in \mathbb{N}$. Then there exist a $\mathbb{P}$-martingale $(M_n)_{n\in\mathbb{N}}$ and a previsible process $(A_n)_{n\in\mathbb{N}}$ ($A_n \in \mathcal{F}_{n-1}$ for all $n \geq 1$) such that the decomposition $Z_n = Z_0 + M_n + A_n$ holds uniquely for all $n \in \mathbb{N}$. In particular, Z is a submartingale if and only if A is increasing almost surely.*

Proof. Assume that the Doob decomposition holds, then for any $n \geq 1$,

$$\mathbb{E}(Z_n - Z_{n-1}|\mathcal{F}_{n-1}) = \mathbb{E}(M_n - M_{n-1}|\mathcal{F}_{n-1}) + \mathbb{E}(A_n - A_{n-1}|\mathcal{F}_{n-1}) = A_n - A_{n-1},$$

therefore $A_n = \sum_{i=1}^n \mathbb{E}(Z_i - Z_{i-1}|\mathcal{F}_{i-1})$ almost surely. We can use this definition for the process A, and the theorem follows. The submartingale consequence is clear from this very definition. □

Theorem 1.5.6 (Doob-Meyer Decomposition)**.** *For any càdlàg supermartingale Z, there exist a local martingale M and a predictable increasing process A starting from zero such that the decomposition $Z = Z_0 + M - A$ holds uniquely. In particular, if $\lim_{t\uparrow\infty} \mathbb{E}(Z_t) > -\infty$, then A_∞ has finite expectation.*

Remark 1.5.7. A continuous-time stochastic process X is said to be predictable if it is measurable with respect to the σ-algebra generated by all left-continuous adapted processes. In particular, every continuous-time left-continuous adapted process is predictable.

Proof of Theorem 1.5.4. Fix some $\mathcal{F}_t$-measurable random variable $\mathfrak{p}$ and assume the existence of a stopping time $\tau \in \mathcal{T}_{t,T}$ and of an admissible portfolio $\pi \in \mathcal{A}$ such that, $\mathbb{P}$-almost surely,

$$-D_{0,t}\mathfrak{p} + \int_t^\tau \pi_u \cdot \mathrm{d}S_u + D_{0,\tau}P_\tau \geq 0.$$

The integral term is clearly a $\mathbb{Q}$-supermartingale, and hence $\mathbb{E}^{\mathbb{Q}}\left[\int_t^\tau \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u|\mathcal{F}_t\right] \leq 0$, so that $\mathfrak{p} \leq \mathbb{E}^{\mathbb{Q}}\left[D_{t,\tau}P_\tau|\mathcal{F}_t\right]$. Taking the supremum over all $\tau \in \mathcal{T}_{t,T}$ and using the fact that this is valid for any $\mathfrak{p} \in \mathcal{F}_t$ concludes the first assertion of the theorem.

If the market is complete, then for any stopping time $\tau \in \mathcal{T}_{t,T}$, the payoff $\Psi_T := D_{\tau,T}^{-1}P_\tau$ is attainable, i.e. there exists a real number $\tilde{\mathfrak{p}}$ and an admissible portfolio $\pi \in \mathcal{A}$ such that $(\int_0^t \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale with

$$-\tilde{\mathfrak{p}} + \int_0^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u + D_{0,T}\Psi_T = 0 = -\tilde{\mathfrak{p}} + \int_0^t \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u + \int_t^T \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u + D_{0,\tau}P_\tau. \qquad (1.5.1)$$

Taking $\mathbb{Q}$-expectations conditional on $\mathcal{F}_t$ then yields $\tilde{\mathfrak{p}} - \int_0^t \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u = \mathbb{E}^{\mathbb{Q}}\left[D_{0,\tau}P_\tau|\mathcal{F}_t\right]$. From the definition of the buyer's price and the right-hand side of (1.5.1), letting $\mathfrak{q} := \tilde{\mathfrak{p}} - \int_0^t \pi_u \cdot \mathrm{d}S_u$, and identifying $\mathfrak{q} = D_{0,t}\mathfrak{p}$, we see that $\mathcal{B}_t(P) \geq \mathfrak{p} = D_{0,t}^{-1}\mathfrak{q} = \mathbb{E}^{\mathbb{Q}}\left[D_{t,\tau}P_\tau|\mathcal{F}_t\right]$, and therefore, from the first part of the theorem, the equality holds. Similar arguments follow for the seller's price.

The last part of the proof, for the seller's price, is slightly more technical, as one still needs to show that, letting $\mathfrak{p} := \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}^{\mathbb{Q}}\left[D_{t,\tau}P_\tau|\mathcal{F}_t\right]$, there exists a super-replicating admissible portfolio π such that

$$D_{0,t}\mathfrak{p} + \int_t^{\mathfrak{t}} \pi_u \cdot \mathrm{d}\widetilde{\mathrm{S}}_u \geq D_{0,\mathfrak{t}}P_{\mathfrak{t}}, \qquad \text{for all } \mathfrak{t} \in [t,T].$$

Using Lemma 1.5.3, the process $\mathcal{S}$ defined pathwise by $\mathcal{S}_t := D_{0,t}^{-1}\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}^{\mathbb{Q}}\left[D_{t,\tau}P_\tau|\mathcal{F}_t\right]$ is a $\mathbb{Q}$-supermartingale, so that Doob-Meyer's Decomposition (Theorem 1.5.6) yields the existence of a $\mathbb{Q}$-local martingale M with $M_t = D_{0,t}\mathfrak{p}$ and a non-decreasing process A starting at zero such that $\mathcal{S} = M - A$. Furthermore, the Martingale Representation Theorem 1.2.2 implies that there exists an $\mathcal{F}$-adapted process ζ such that

$$M_{\mathfrak{t}} = D_{0,t}\mathfrak{p} + \int_t^{\mathfrak{t}} \zeta_u \mathrm{d}W_u^{(1)}.$$

The market being complete, we can rewrite this as

$$M_{\mathfrak{t}} = D_{0,t}\mathfrak{p} + \int_t^{\mathfrak{t}} \pi_u \mathrm{d}\widetilde{\mathrm{S}}_u = \mathcal{S}_{\mathfrak{t}} + A_{\mathfrak{t}} \geq \mathcal{S}_{\mathfrak{t}} \geq D_{0,\mathfrak{t}}P_{\mathfrak{t}},$$

and the theorem follows. □

A special case: American Call option in the Black-Scholes framework

We now specialise the above results to the Black-Scholes framework, and deduce some interesting properties of American options, in particular of an American Call option, for which the exercise price is equal to $P_t = (S_t - K)_+$, for some strike $K > 0$. In the Black-Scholes model, the stock price process is the unique strong solution to the stochastic differential equation $\mathrm{d}S_t = (r-q)S_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t$, starting at $S_0 > 0$, where r and q are respectively the interest rate and (continuous) dividend yield. The process W is a standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is a given risk-neutral measure. Let $C_a(k,T,\sigma)$ and $P_a(k,T,\sigma)$ denote the American Call and Put option prices. From Theorem 1.5.4, we can write

$$C_a(k,T,\sigma) = \sup_{\tau\in\mathcal{T}_T} C_{\mathrm{BS}}(k,\tau,\sigma),$$
$$P_a(k,T,\sigma) = \sup_{\tau\in\mathcal{T}_T} P_{\mathrm{BS}}(k,\tau,\sigma).$$

Proposition 1.5.8. *The American Put-Call symmetry $C_a(S_0,K,T,r,q) = P_a(K,S_0,T,q,r)$ holds.*

Proof. The proof follows from a simple application of Girsanov's theorem applied to the new probability measure $\mathbb{Q}$ defined via

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right).$$

Under this new probability measure, Girsanov's theorem implied that the process B defined by $B_t := W_t - \sigma t$ is a standard $\mathbb{Q}$-Brownian motion. Therefore,

$$\begin{aligned}
\sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}}\left[\mathrm{e}^{-r\tau}(S_\tau - K)_+\right] &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}}\left[\mathrm{e}^{-r\tau}\left(S_0\mathrm{e}^{(r-q)\tau}\exp\left\{-\frac{1}{2}\sigma^2\tau + \sigma W_\tau\right\} - K\right)_+\right] \\
&= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_T}\mathrm{e}^{-r\tau}\left(S_0\mathrm{e}^{(r-q)\tau}\exp\left\{-\frac{1}{2}\sigma^2\tau + \sigma W_\tau\right\} - K\right)_+\right] \\
&= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-q\tau}\left(S_0 - K\exp\left\{(q-r)\tau + \frac{1}{2}\sigma^2\tau - \sigma W_t\right\}\right)_+\right] \\
&= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-q\tau}\left(S_0 - K\exp\left\{(q-r)\tau - \frac{1}{2}\sigma^2\tau - \sigma B_t\right\}\right)_+\right] \\
&= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-q\tau}\left(S_0 - K\exp\left\{(q-r)\tau - \frac{1}{2}\sigma^2\tau + \sigma B_t\right\}\right)_+\right]
\end{aligned}$$

where in the last line, we used the symmetry (in distribution) of the Brownian motion. □

1.5.2 Barrier options

A barrier option is a European option which depends on the path of the underlying stock price between inception and the maturity T of the contract. For a given underlying stock price S, we shall denote by $\underline{S}$ and $\overline{S}$ respectively the running infimum and running supremum of the process:

$$\underline{S}_t := \inf_{u\in[0,t]} S_u \qquad \text{and} \qquad \overline{S}_t := \sup_{u\in[0,t]} S_u. \tag{1.5.2}$$

A knock-out barrier Call option has the following payoff at maturity:

$$(S_T - K)_+ \mathbb{1}_{\{\underline{S}_T \geq \underline{K}\}} \mathbb{1}_{\{\overline{S}_T \leq \overline{K}\}},$$

i.e. it has the same payoff as a standard European option as long as the stock price has remained within the interval $(\underline{K}, \overline{K})$ during the life of the contract. Different barrier options exist in practice, namely *knock-in* options, where the payoff is only exercised if the stock price has hit a barrier, and so on. We shall not list all the possible combinations here, but refer the reader to the many books and papers on this. It is clear, however, that the structure of the solution will be similar in all cases. For general processes, closed-form expressions are not available, and one has to resort to numerical methods. We shall get back to this point later in Chapter 3. In the Black-Scholes case, however, one is able to compute such closed-form representations.

Let us therefore assume that the stock price process is the unique strong solution to the stochastic differential equation $\mathrm{d}S_t = S_t\left(r\mathrm{d}t + \sigma\mathrm{d}W_t\right)$, starting at $S_0 = S_0 > 0$, and consider the case of an up-and-out Call option with payoff

$$\Pi_T := (S_T - K)_+ \mathbb{1}_{\{\overline{S}_T \leq B\}},$$

for some knock-out level B. Obviously if $K \geq B$, the option is worthless, so we shall assume otherwise from now on. Itô's formula applied to the Black-Scholes SDE yields

$$S_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} = S_0 \exp\left(\sigma W_t^\alpha\right),$$

where $W_t^\alpha := \alpha t + W_t$ defines a Brownian motion with drift, with $\alpha := \frac{r}{\sigma} - \frac{\sigma}{2}$. In particular, for any $t \geq 0$, $\overline{S}_t = S_0 \exp\left(\sigma \overline{W}_t^\alpha\right)$. Therefore the payoff of the knock-out Call option reads

$$\Pi_T = (S_T - K)_+ \mathbb{1}_{\{\overline{W}_T^\alpha \leq b\}} = \left(S_0 \mathrm{e}^{\sigma W_T^\alpha} - K\right) \mathbb{1}_{\{\overline{W}_T^\alpha \leq b, W_T^\alpha \geq k\}}, \tag{1.5.3}$$

with $b := \log(B/S_0)/\sigma$ and $k := \log(K/S_0)/\sigma$. It therefore suffices to compute the joint density of a (drifted) Brownian motion and its running supremum. We start with the zero-drift case, and, for any $w \in \mathbb{R}$, denote τ_w the first hitting time of the Brownian motion at level w:

$$\tau_w := \inf\{t \geq 0 : W_t = w\} = \begin{cases} \inf\{t \geq 0 : W_t \leq w\}, & \text{if } w > 0, \\ \inf\{t \geq 0 : W_t \geq w\}, & \text{if } w < 0, \\ 0, & \text{if } w = 0. \end{cases} \tag{1.5.4}$$

The second equalities follow immediately from the continuity of the paths of the Brownian motion.

Lemma 1.5.9. *For any $w \in \mathbb{R}$, τ_w is finite almost surely.*

Proof. Assume that $w > 0$. By continuity arguments, if the Brownian motion W hits some level $\widetilde{w} \geq w$ almost surely, then it will also hit w almost surely. Since $\limsup_{t\uparrow\infty} W_t = \infty$, the result follows. The other cases are analogous. □

We can now state the following reflection property for the standard Brownian motion:

Proposition 1.5.10. *For every $a > 0$, $y \geq 0$, $\mathbb{P}(\overline{W}_t \geq a, W_t \leq a - y) = \mathbb{P}(W_t \geq a + y)$.*

Proof. By the total law of probability, we can write

$$\begin{aligned}
\mathbb{P}(W_t \geq a + y) &= \mathbb{P}(W_t \geq a + y, \overline{W}_t \geq a) + \mathbb{P}(W_t > a + y, \overline{W}_t < a) \\
&= \mathbb{P}(W_t \geq a + y, \overline{W}_t \geq a) \\
&= \mathbb{P}(W_{\tau_a + (t - \tau_a)} - a \geq y, \overline{W}_t \geq a) \\
&= \mathbb{P}(W_{\tau_a + (t - \tau_a)} - a \geq y | \overline{W}_t \geq a)\mathbb{P}(\overline{W}_t \geq a) \\
&= \mathbb{P}(W_{\tau_a + (t - \tau_a)} - a \leq -y | \overline{W}_t \geq a)\mathbb{P}(\overline{W}_t \geq a) \\
&= \mathbb{P}(W_t \leq a - y, \overline{W}_t \geq a),
\end{aligned}$$

where we used the symmetry of the Brownian motion 'restarted' at level a at time τ_a. □

Setting $m = a$ and $w = a - y$, we can rewrite the reflection property as

$$\mathbb{P}(W_t \leq w, \tau_m \leq t) = \mathbb{P}(W_t \leq w, \overline{W}_t \geq m) = \mathbb{P}(W_t \geq 2m - w). \tag{1.5.5}$$

In passing, one can derive the distribution of the first hitting time:

Exercise 1.5.11. For any $m \neq 0$,

$$\mathbb{P}(\tau_m \leq t) = \sqrt{\frac{2}{\pi}} \int_{|m|/\sqrt{t}}^{\infty} \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z.$$

Lemma 1.5.12. *For any $t \geq 0$, the joint density of W_t and $\overline{W}_t$ is equal to*

$$f(w,m) := \partial_{w,m}\mathbb{P}\left(W_t \leq w, \overline{W}_t \leq m\right) = \frac{2(2m-w)}{t\sqrt{2\pi t}} \exp\left\{\frac{(2m-w)^2}{2t}\right\}, \qquad \text{for } 0 \leq m, w \in (-\infty, m).$$

Proof. From the reflection property (1.5.5), we can write

$$\int_m^{\infty}\int_{-\infty}^{w} f(x,y)\mathrm{d}x\mathrm{d}y = \mathbb{P}(W_t \leq w, \overline{W}_t \geq m) = \mathbb{P}(W_t \geq 2m-w) = \frac{1}{\sqrt{2\pi t}}\int_{2m-w}^{\infty} \exp\left(-\frac{z^2}{2t}\right)\mathrm{d}z.$$

Differentiating on both sides with respect to m and w, the claim follows from the computation

$$-f(w,m) = -\frac{2(2m-w)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2m-w)^2}{2t}\right\}.$$

□

The idea now is to use Girsanov theorem to compute the transition density of the Brownian motion with drift W^α from that of the one without drift.

Theorem 1.5.13. *For any $\alpha \in \mathbb{R}$ and $t \geq 0$, the joint density of W_t^α and $\overline{W}_t^\alpha$ reads*

$$\partial_{w,m}\mathbb{P}\left(W_t^\alpha \leq w, \overline{W}_t^\alpha \leq m\right) = \frac{2(2m-w)}{t\sqrt{2\pi t}} \exp\left\{\alpha w - \frac{\alpha^2 t}{2} - \frac{(2m-w)^2}{2t}\right\}, \quad \text{for } 0 \leq m, w \in (-\infty, m).$$

Proof. The process Z^α defined pathwise by

$$Z_t^\alpha := \exp\left(-\alpha W_t - \frac{1}{2}\alpha^2 t\right) = \exp\left(-\alpha W_t^\alpha + \frac{1}{2}\alpha^2 t\right)$$

is a $\mathbb{P}$-martingale, and therefore we can define a new probability measure $\mathbb{P}^\alpha$ via $\frac{\mathrm{d}\mathbb{P}^\alpha}{\mathrm{d}\mathbb{P}} = Z_T^\alpha$ on $\mathcal{F}_T$ such that W^α is a standard $\mathbb{P}^\alpha$-Brownian motion. Therefore

$$\begin{aligned}
\mathbb{P}\left(W_t^\alpha \leq w, \overline{W}_t^\alpha \leq m\right) &= \mathbb{E}\left(\mathbb{1}_{\{W_t^\alpha \leq w, \overline{W}_t^\alpha \leq m\}}\right) = \mathbb{E}^{\mathbb{P}^\alpha}\left(\frac{1}{Z_T^\alpha}\mathbb{1}_{\{W_t^\alpha \leq w, \overline{W}_t^\alpha \leq m\}}\right) \\
&= \mathbb{E}^{\mathbb{P}^\alpha}\left[\exp\left(\alpha W_t^\alpha - \frac{1}{2}\alpha^2 t\right)\mathbb{1}_{\{W_t^\alpha \leq w, \overline{W}_t^\alpha \leq m\}}\right] \\
&= \int_{-\infty}^{m}\int_{-\infty}^{w} \exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right) f(x,y)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

where $f(\cdot,\cdot)$ again is the joint density of a standard Brownian motion and its running supremum. The theorem then follows from Lemma 1.5.12. □

From no-arbitrage arguments, using the payoff (1.5.3) and Theorem 1.5.13, we deduce, after simple, yet long and tedious, computations, the price at inception of the Knock-out Call option:

Proposition 1.5.14. *At inception of the contract, the Knock-out Call option is worth*

$$\begin{aligned}
\Pi_0 &= \mathrm{e}^{-rT}\mathbb{E}(\Pi_T) \\
&= S_0\left\{\mathcal{N}\left(\delta_+\left(\frac{S_0}{K}\right)\right) - \mathcal{N}\left(\delta_+\left(\frac{S_0}{B}\right)\right)\right\} - \mathrm{e}^{-rT}K\left\{\mathcal{N}\left(\delta_-\left(\frac{S_0}{K}\right)\right) - \mathcal{N}\left(\delta_-\left(\frac{S_0}{B}\right)\right)\right\} \\
&- B\left(\frac{S_0}{B}\right)^{-2r/\sigma^2}\left\{\mathcal{N}\left(\delta_+\left(\frac{B^2}{S_0K}\right)\right) - \mathcal{N}\left(\delta_+\left(\frac{B}{S_0}\right)\right)\right\} \\
&+ \mathrm{e}^{-rT}K\left(\frac{S_0}{B}\right)^{1-2r/\sigma^2}\left\{\mathcal{N}\left(\delta_-\left(\frac{B^2}{KS_0}\right)\right) - \mathcal{N}\left(\delta_-\left(\frac{B}{S_0}\right)\right)\right\},
\end{aligned}$$

where $\delta_\pm(x) := \frac{1}{\sigma\sqrt{T}}\left\{\log(x) + \left(r \pm \frac{1}{2}\sigma^2\right)T\right\}$.

1.5.3 Forward-start options

We consider here a new type of European option, namely forward-start calls and puts. Let $(S_t)_{t\geq 0}$ be the stock price process starting at some strictly positive value S_0. By no-arbitrage arguments, the forward-start Call option C^f and Put option P^f with strike $K \geq 0$, forward-start date $T \geq 0$ and maturity $T + \tau \geq T$ are worth at inception

$$C^f(K,T,\tau) := \mathrm{e}^{-r(T+\tau)}\mathbb{E}_0\left[\left(\frac{S_{T+\tau}}{S_T} - K\right)_+\right] \quad \text{and} \quad P^f(K,T,\tau) := \mathrm{e}^{-r(T+\tau)}\mathbb{E}_0\left[\left(K - \frac{S_{T+\tau}}{S_T}\right)_+\right].$$

Clearly, it reduces to a standard European Call option whenever $T = 0$. In the Black-Scholes model, under the risk-neutral probability measure $\mathbb{P}$, the stock price process is the unique strong solution to the SDE $\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma\mathrm{d}W_t$, so that

$$\begin{aligned}
\mathrm{BS}^f(K,T,\tau,\sigma) &:= \mathrm{e}^{-r(T+\tau)}\mathbb{E}_0\left[\left(\frac{S_{T+\tau}}{S_T} - K\right)_+\right] \\
&= \mathrm{e}^{-r(T+\tau)}\mathbb{E}_0\left[\left(\mathrm{e}^{(r-\sigma^2/2)\tau+\sigma W_\tau} - K\right)_+\right] = \mathrm{e}^{-rT}\mathrm{BS}(S_0 = 1, K, \tau, \sigma),
\end{aligned}$$

where we use the fact that the Brownian increments are stationary. In fact, this result holds for any model with stationary increments, in particular any exponential Lévy process. We can therefore define the forward implied volatility similarly to the standard implied volatility:

Definition 1.5.15. For any $T > 0$, $\tau, K \geq 0$, the forward implied volatility $\sigma_{T,\tau}(K)$ is the unique non-negative solution to the equation $C^f(K,T,\tau) = \mathrm{BS}^f(K,T,\tau,\sigma_{T,\tau}(K))$.

1.5.4 Variance and volatility swaps

Variance swaps are swaps written on the realised variance over a given period of time, i.e. have a terminal payoff of the form $N\left(\sigma_{\mathrm{R}}^2 - K\right)$, where N is the notional (in some given currency), K is the strike of the contract (expressed in units of variance) and σ_{R}^2 represents the realised variance of S over the life of the contract. Being a swap, the strike is chosen such that the contract has no value at inception, i.e. $K = \mathbb{E}_0\left(\sigma^2_{\text{Realised}}\right)$. The realised variance is calculated as follows:

$$\sigma^2_{\mathrm{R}} := A\sum_{i=1}^{n}\frac{1}{n}\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2,$$

where A is an annualisation factor (usually 252 working days), $t_0 < t_1 < \ldots < t_n$ are sampling dates, specified in the contract, and $(S_t)_{t\geq 0}$ denotes the stock price process under consideration. In practice however, pricing of variance swaps, namely computing the expectation of the realised variance, is performed by approximating the discrete sampling above by its continuous version. More precisely take a partition $0 = t_0 < t_1 < \ldots < t_n = T$, then the following limit holds in probability:

$$\lim_{n\uparrow+\infty}\sum_{i=1}^{n}\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 = \langle\log(S),\log(S)\rangle_T \tag{1.5.6}$$

Computing the quadratic variation on the right is an easier exercise (from a stochastic calculus point of view) and generally yields to simple closed-form expressions. Note however that the convergence in (1.5.6) is in probability and hence does not guarantee convergence of the expectations. We refer the interested reader to [91] for more details on this issue.

Assume now that the stock price satisfies $\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma_t\mathrm{d}W_t$, with $S_0 > 0$, where $(\sigma_t)_{t\geq 0}$ is adapted to the Brownian filtration. Applying Itô's formula to the smooth function (on $\mathbb{R}^*_+$) $s \mapsto \log(s)$, we obtain $\mathrm{d}\log(S_t) = \left(r - \frac{1}{2}\sigma_t^2\right)\mathrm{d}t + \sigma_t\mathrm{d}W_t$, so that

$$\frac{1}{T}\int_0^T\sigma_t^2\mathrm{d}t = \frac{2}{T}\int_0^T\left(\frac{\mathrm{d}S_t}{S_t} - \mathrm{d}\log(S_t)\right) = \frac{2}{T}\int_0^T\frac{\mathrm{d}S_t}{S_t} - \frac{2}{T}\log\left(\frac{S_T}{S_0}\right). \tag{1.5.7}$$

Note that the first term on the right-hand side corresponds to a rebalanced hedge of the stock while the second term represents a so-called log-contract. Apply now the replication formula (C.0.1) to the function $f \equiv \log$ with $F = S^*$:

$$\log\left(\frac{S_T}{S^*}\right) = \frac{S_T - S^*}{S^*} - \int_0^{S^*}\frac{(K-S)_+}{K^2}\mathrm{d}K - \int_{S^*}^{\infty}\frac{(S-K)_+}{K^2}\mathrm{d}K.$$

Taking expectations on both sides yields

$$\mathbb{E}\log\left(\frac{S_T}{S^*}\right) = \frac{S_0\mathrm{e}^{rT}}{S^*} - 1 - \mathrm{e}^{rT}\int_0^{S^*}\frac{P(K,T)}{K^2}\mathrm{d}K - \mathrm{e}^{rT}\int_{S^*}^{\infty}\frac{C(K,T)}{K^2}\mathrm{d}K, \tag{1.5.8}$$

where $C(K,T)$ and $P(K,T)$ are European Call and Put options written on S with strike K and maturity T. Therefore, combining (1.5.8) and (1.5.7), we obtain

$$\mathbb{E}\left(\frac{1}{T}\int_0^T\sigma_t^2\mathrm{d}t\right) = \frac{2}{T}\left[rT - \left(\frac{S_0\mathrm{e}^{rT}}{S^*} - 1\right) - \log\left(\frac{S^*}{S_0}\right) + \mathrm{e}^{rT}\int_0^{S^*}\frac{P(K,T)}{K^2}\mathrm{d}K + \mathrm{e}^{rT}\int_{S^*}^{\infty}\frac{C(K,T)}{K^2}\mathrm{d}K\right]$$

Take for example $S^* = S_0\mathrm{e}^{rT}$, namely the forward price, then the fair strike of the variance swap reads

$$\mathbb{E}\left(\frac{1}{T}\int_0^T\sigma_t^2\mathrm{d}t\right) = \frac{2\mathrm{e}^{rT}}{T}\int_0^{S^*}\frac{P(K,T)}{K^2}\mathrm{d}K + \frac{2\mathrm{e}^{rT}}{T}\int_{S^*}^{\infty}\frac{C(K,T)}{K^2}\mathrm{d}K. \tag{1.5.9}$$

Therefore the fair strike of a (continuously monitored) variance swap can be computed all the European Call and Put options for all possible strikes for a fixed maturity.

Remark 1.5.16. Note that we have assumed here that the stock price is a strictly positive martingale. If, at time T, S_T has a strictly positive mass at the origin, then by Lemma 2.1.8, we have $\lim_{K\downarrow 0}\frac{P(K,T)}{K}=\mathbb{P}(S_T=0)$, so that the first integral in (1.5.9) is infinite, and so is the fair value of the variance swap.

Let us now try to understand the precise meaning of of the weighting factor $1/K^2$. Consider the portfolio $\Pi:=\int_0^\infty \rho(K)O(K)\mathrm{d}K$, where $O(K)$ represents either a Call or a Put option, and where ρ is a weighting scheme to be determined. In the Black-Scholes model, consider the total variance $\nu:=\sigma^2 T$, and the derivative of the Call (or the Put) with respect to ν reads

$$\partial_\nu O(K)=\frac{S_0}{2\sqrt{\nu}}n(d_+(K/S_0)).$$

Therefore

$$\partial_\nu \Pi=\int_0^\infty \rho(K)\frac{S_0}{2\sqrt{\nu}}n(d_+(K/S_0))\mathrm{d}K=\int_0^\infty \rho(xS_0)\frac{S_0^2}{2\sqrt{\nu}}n(d_+(x))\mathrm{d}x,$$

where we used the change of variable $x=K/S_0$. Furthermore

$$\partial_{S_0}\left(\partial_\nu \Pi\right)=\frac{1}{2\sqrt{\nu}}\int_0^\infty\left[2\rho(xS_0)+xS_0\rho'(xS_0)\right]S_0 n(d_+(x))\mathrm{d}x.$$

The Vega $\partial_\nu\Pi$ of the portfolio is then insensitive to the movement of the stock price if and only if $2\rho(xS_0)+xS_0\rho'(xS_0)=0$. Solving the ordinary differential equation yields $\rho(K)=1/K^2$.

Chapter 2

Martingale theory and implied volatility

2.1 Existence of implied volatility

In this section, we shall endeavour to answer the following question: given a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and a non-negative local martingale $(S_t)_{t\geq 0}$, how does one define the implied volatility? The blunt answer is 'the volatility parameter to plug into the Black-Scholes formula in order to recover a given or observed European Call (or Put) option price' (adapted from [126]). Let us try, though, to propose a more rigorous definition, and study the properties of this object. In order to do so, we first start by recalling the basics of the Black-Scholes model, which shall also serve to fix the notations.

2.1.1 Preliminaries: general properties of Call option prices

Before moving on to the implied volatility, let us first consider some general properties of Call option prices in a local martingale model.

Proposition 2.1.1. *Let S be a non-negative local martingale and let $0 \leq t \leq T$. Define the map $C_t : \mathbb{R}_+ \to \mathbb{R}$ by $C_t(K) := \mathbb{E}_t(S_T - K)_+$. Then the following properties hold:*

(i) C_t is convex and non-increasing on $\mathbb{R}_+$;

(ii) $\lim_{K\uparrow\infty} C_t(K) = 0$ and $\lim_{K\downarrow 0} C_t(K) = C_t(0)$;

(iii) for any $K \geq 0$, $(C_t(0) - K)_+ \leq C_t(K) \leq C_t(0)$;

(iv) on $\mathbb{R}_+$, $\partial_K^+ C_t$ exists, is right-continuous, non-decreasing and satisfies $-1 \leq \partial_K^+ C_t(\cdot) \leq 0$;

(v) for any $K > 0$, we have $-1 \leq [C_t(K) - C_t(0)]/K \leq \partial_K^+ C_t(K)$.

Proof. Statement (i) follows by linearity of the expectation operator and the convexity of the map $K \mapsto (x-K)_+$. The large-and small-K limits in (ii) follow from Lebesgue's Dominated Convergence Theorem and Proposition 1.1.12:

$$\lim_{K\uparrow\infty} \mathbb{E}_t(S_T-K)_+ = \mathbb{E}_t\left[\lim_{K\uparrow\infty}(S_T-K)_+\right] = 0,$$
$$\lim_{K\downarrow 0} \mathbb{E}_t(S_T-K)_+ = \mathbb{E}_t\left[\lim_{K\downarrow 0}(S_T-K)_+\right] = \mathbb{E}_t(S_T).$$

The inequalities in (iii) are consequences of the following: for any $x, K \geq 0$, we have $(x-K)_+ \leq x_+ = x$, and hence $\mathbb{E}_t(S_T-K)_+ \leq \mathbb{E}_t(S_T) \leq S_t$ by Proposition 1.1.12; the lower bound in (iii) follows from Jensen's inequality for convex functions. Regarding (iv), we know from (iii) that the map C_t is right-continuous at the origin, and hence statement (iv) follows from simple analytic properties of extensions (from $\mathbb{R}_+^*$ to $\mathbb{R}_+$) of convex functions. □

The following lemma provides a link between true martingales and bounds for European Call option prices.

Lemma 2.1.2. *If S is a non-negative $\mathcal{F}$-adapted process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, then the following statements are equivalent:*

(i) S is a true martingale;

(ii) S is integrable and the inequality $(S_u-K)_+ \leq \mathbb{E}_u(S_t-K)_+ \leq S_u$ holds almost surely for all $0 \leq u \leq t$ and $K \geq 0$;

(iii) S is a non-negative local martingale with the same inequalities as in (ii).

Proof. This lemma can be proved in the following way: assume that S is a true martingale (i), then it is integrable and for any $0 \leq u \leq t$ and $K \geq 0$, since the map $K \mapsto \mathbb{E}_u(S_t-K)_+$ is convex and strictly decreasing, Jensen's inequality implies $(S_u-K)_+ = (\mathbb{E}_u(S_t)-K)_+ \leq \mathbb{E}_u(S_t-K)_+ \leq \mathbb{E}_u(S_t)_+ = S_u$. Clearly also S is a non-negative local martingale and so (iii) holds. Now assume (ii). The lemma then follows by a direct application of dominated convergence:

$$S_u = \lim_{K\downarrow 0}(S_u-K)_+ \leq \lim_{K\downarrow 0}\mathbb{E}_u(S_t-K)_+ \leq \mathbb{E}_u \lim_{K\downarrow 0}(S_t-K)_+ = \mathbb{E}_u(S_t)_+ \leq S_u.$$

□

2.1.2 Characterisation of European option price functions

Under absence of arbitrage and assuming that the underlying stock price process does not pay any dividends, recall the (European) Put-Call parity:

$$C_t(T,K) - P_t(T,K) = S_t - KB(t,T), \tag{2.1.1}$$

where $B(t,T)$ represents the price at time t of a zero-coupon bond paying one unit at time T. From this equality, the following bounds for the Call and the Put are immediate

$$\begin{aligned}(S_t - K)_+ \leq (S_t - KB(t,T))_+ \leq C_t(T,K) \leq S_t, \\ (KB(t,T) - S_t)_+ \leq P_t(T,K) \leq KB(t,T).\end{aligned}$$

Furthermore, a simple Call-spread arbitrage argument (buy a Call with strike K_1 and sell a Call with strike K_2) shows that for any $t \leq T$, the Call price is a decreasing function of the strike. Calls and Puts are convex function of the strike. This property follows directly from the so-called butterfly strategy: buy a Call with strike K_1, buy one with strike K_2 and sell two Calls with strike $(K_1 + K_2)/2$, where $K_1 < K_2$. The calendar spread strategy (for a given strike, buy a Call with maturity T_2 and sell one with maturity $T1 < T_2$) implies that Calls are increasing functions of the (remaining) maturity. This is not necessarily true, however, for European Put options. In the case of dividends , the Put-Call parity (2.1.1) does not hold any more since one does not need to invest the amount $S_t - KB(t,T)$ at time t in order to obtain the difference between the Call and the Put at maturity. Suppose for instance that the stock price pays fixed dividends $D_1, \ldots, D_n$ during the period $[t,T]$, then the Put-Call parity becomes

$$C_t(K,T) - P_t(K,T) = S_t - \sum_{i=1}^{n} D_i B(t,t_i) - KB(t,T).$$

In the case of a continuous dividend yield, say $q > 0$ (i.e. the stock price pays continuously $qS_t \mathrm{d}t$), the the Put-Call parity becomes

$$C_t(K,T) - P_t(K,T) = S_t \mathrm{e}^{-q(T-t)} - KB(t,T).$$

Suppose now that one observes some function C. It is natural to wonder (i) if it an actual Call price function, and (ii) if there exists some (martingale) process generating these prices. In light of Proposition 2.1.1, we obtain the following theorem, in the case of true martingales:

Theorem 2.1.3. *For $s > 0$, assume that there exists a map $C : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ such that*

1. $C(\cdot, T)$ *is convex and non-increasing;*
2. $C(K, \cdot)$ *is non-decreasing;*
3. $\lim_{K\uparrow\infty} C(K, \cdot) = 0$;
4. $(s - K)_+ \leq C(K,T) \leq s$;
5. $C(K,0) = (s - K)_+$;

then C can be continuously extended to $\mathbb{R}_+ \times \mathbb{R}_+$ with $\lim_{K\downarrow 0} C(K,T) = s$. Furthermore, there exists a non-negative Markov martingale $(S_t)_{t\geq 0}$ such that $C(K,T) = \mathbb{E}\left[(S_T - K)_+ | S_0 = s\right]$.

Proof. The proof of the extension of the function C is fairly straightforward. Proving that there exists a Markov martingale satisfying the required properties is a more tricky exercise. In Lemma 2.1.6 below, we first prove that for each $T > 0$, the map $C(\cdot, T)$ characterises a probability measure μ_T on $\mathbb{R}_+$. We are thus left to prove, with the help of Theorem 2.1.5 that the family $\mathcal{M} := (\mu_T)_{T\geq 0}$ is in balayage order. Fix two maturities $0 \leq T_1 < T_2 < \infty$; by monotonicity, we can write

$$\int_{\mathbb{R}} (z - K)_+ \mu_{T_1}(\mathrm{d}z) = C(K, T_1) \leq C(K, T_2) = \int_{\mathbb{R}} (z - K)_+ \mu_{T_2}(\mathrm{d}z),$$

for all strikes $K \geq 0$, and the theorem follows from Theorem 2.1.5, noting that a submartingale with constant finite expectation is a true martingale. □

Theorem 2.1.5 below is the key tool in order to prove Theorem 2.1.3, but requires first the notion of balayage order for a family of probability measures.

Definition 2.1.4. Two measures μ, ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are said to be in balayage order (or in convex order), and we write $\mu \preceq \nu$, if $\int_{\mathbb{R}} f(x)\mu(\mathrm{d}x) \leq \int_{\mathbb{R}} f(x)\nu(\mathrm{d}x)$, for all real convex functions f on $\mathbb{R}$.

Theorem 2.1.5 (Kellerer [97])**.** *Let $\mathcal{M} = (\mu_t)_{t\geq 0}$ be a family of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite expectation for each $t \geq 0$. If $\mathcal{M}$ is in balayage order, then there exists a Markov submartingale with marginals μ_t at time t, for all $t \geq 0$.*

For notational simplicity, we shall from now on assume that there is no interest rate, so that the risk-free bond is constant and equal to one.

Lemma 2.1.6. *For any $T \geq 0$, there exists a unique measure μ on $\mathbb{R}_+$ such that $C(K, T) = \int (x - K)_+ \mu(\mathrm{d}x)$. In particular $\int x\mu(\mathrm{d}x) = s$.*

Proof. Since the function $C(\cdot, T)$ is convex and decreasing, there exists a right-continuous decreasing function[1] $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $C(K, T) = C(0, T) - \int_0^K f(x)\mathrm{d}x = s - \int_0^K f(x)\mathrm{d}x$, and therefore $-f(K) = \partial_K^+ C(K, T)$. By linearity of the expectation operator, taking the limit as K tends to infinity yields $\int_0^\infty f(x)\mathrm{d}x = s$, so that necessarily $\lim_{x\uparrow\infty} f(x) = 0$. We can thus define a measure μ on $\mathbb{R}_+$ through the identity $\mu(x, \infty) := f(x)$, for any $x \geq 0$. Applying Fubini's theorem then yields

$$\int_0^\infty f(x)\mathrm{d}x = \int_0^\infty \int_x^\infty \mu(\mathrm{d}z)\mathrm{d}x = \int_0^\infty \left(\int_0^z \mathrm{d}x \right) \mu(\mathrm{d}z) = \int_{(0,\infty)} z\mu(\mathrm{d}z) = s,$$

and

$$\begin{aligned} C(K, T) &= s - \int_0^K \int_x^\infty \mu(\mathrm{d}z)\mathrm{d}x = s - \int_0^\infty \left(\int_0^K \mathbb{1}_{\{z>x\}}\mathrm{d}x \right) \mu(\mathrm{d}z) \\ &= \int_0^\infty \left(\int_0^z - \int_0^K \mathbb{1}_{\{z>x\}} \right) \mathrm{d}x\mu(\mathrm{d}z) = \int_0^\infty (z - K)_+ \mu(\mathrm{d}z). \end{aligned}$$

□

[1] this is a standard result in convex analysis, originally proved by Stolz [148]. Please consult [120] for elementary properties of (real) convex functions

Remark 2.1.7. Define the family of functions $\widetilde{C}_t(K,T) := S_t - \mathbb{E}[S_T \wedge K|\mathcal{F}_t]$. For any K, T, the process $(\widetilde{C}_t(K,T))_{t\geq 0}$ is a non-negative local martingale. It is not however necessarily a true martingale. Indeed,

$$\mathbb{E}[\widetilde{C}_T(K,T)|\mathcal{F}_t] = \mathbb{E}[(S_T-K)_+|\mathcal{F}_t] = \mathbb{E}[S_T - S_T \wedge K|\mathcal{F}_t] \leq S_t - \mathbb{E}[S_T \wedge K|\mathcal{F}_t] = \widetilde{C}_t(K,T),$$

and the equality holds if and only if S is a true martingale. In that case, we also have $\widetilde{C}_t(K,T) = \mathbb{E}_t(S_T-K)_+$. More details on related issues will be provided later.

Let us now come back to option prices, fix a maturity T, a true martingale S, and recall that $C(K,T)$ and $P(K,T)$ denote respectively Call and Put prices with strike K and maturity T, at time zero. It is clear that there is a deep link between Call and Put prices and the distribution of the stock price at time T. The following lemma provides more precise details:

Lemma 2.1.8. *For every $K>0$, one has*

$$\partial_K^+ P(K,T) = \mathbb{P}(S_T \leq K), \qquad \partial_K^- P(K,T) = \mathbb{P}(S_T < K),$$

$$\partial_K^+ C(K,T) = -\mathbb{P}(S_T > K), \quad \partial_K^- C(K,T) = -\mathbb{P}(S_T \geq K).$$

In particular,

$$\lim_{K\downarrow 0} \frac{P(K,T)}{K} = \mathbb{P}(S_T = 0).$$

Proof. The first part of the lemma is immediate and is left as an exercise; the second line follows from the first by Call-Put parity. Using the equality $P(K,T) = \int_0^K \partial_K^+ P(L,T)\mathrm{d}L$ (by convexity of the Put option price) and $\lim_{K\downarrow 0} \partial_K^+ P(K,T) = \mathbb{P}(S_T = 0)$, the limit then follows from $\lim\limits_{K\downarrow 0} K^{-1}\int_0^K \partial_K^+ P(L,T)\mathrm{d}L = \mathbb{P}(S_T = 0)$. □

Remark 2.1.9. The last equality in the proposition above in particular implies that, at least in theory, the observed small-strike European Put option prices provide information on whether the underlying stock price can default or not.

Option prices in strict local martingale models

Let us consider the double inequality in Theorem 2.1.3(iv), and let S be a non-negative strict local martingale (i.e. a non-negative supermartingale which is not a martingale). In particular, we know that $\mathbb{E}(S_T|\mathcal{F}_0) < S_0$, so that the European Call price, defined as the map $C(K,T) \mapsto \mathbb{E}((S_T-K)_+|\mathcal{F}_0)$ is not in the no-arbitrage bounds $[(S_0-K)_+, S_0]$ at the point $K=0$. One therefore needs to modify the definition of the European Call option price to account for this loss of martingality. This loss has been dubbed 'bubble' in the mathematical finance literature, and we shall appeal to the excellent review by Alex Cox and David Hobson [36] on the topic:

Definition 2.1.10. The price process has a bubble if it is a strict local martingale under the risk-neutral measure.

Examples of bubbles in history are legion: the Dutch tulip mania in the seventeenth century, the South Sea Bubble around 1720, the Roaring Twenties, the Internet (Dot-Com) bubble, or more recently the financial crash involving Lehman Brothers (see [112] for more details). In Economics terms, a bubble occurs when the traded value of an asset deviates from its intrinsic value. Mathematically, one can create bubbles simply out of stochastic models. Let $T > 0$, and consider the stochastic differential equation

$$\mathrm{d}S_t = \frac{S_t}{\sqrt{T-t}}\mathrm{d}W_t, \qquad S_0 = s > 0,$$

where W is a standard Brownian motion. One can show that $S_t = s\exp\left(B_{A_t} - \frac{1}{2}A_t\right)$, where B is another standard Brownian motion and $A_t \equiv -\log(1 - t/T)$. The process S is a true martingale over the interval $[0, T)$, but clearly $S_T = 0$ almost surely. This example shows a simple case of a strict local martingale, where of course the inequalities in Theorem 2.1.3(iv) break down.

We shall consider the following running example, for which closed-form computations are possible: let S be the unique strong solution to $\mathrm{d}S_t = S_t^2\mathrm{d}W_t$, $S_0 = s > 0$. Lewis [105] (among others) showed that the process is a strict local martingale. It actually represents the reciprocal of the radial part of a three-dimensional Brownian motion, and is furthermore bounded in L^2. In particular, the following formulae hold:

$$\begin{aligned}\mathbb{P}(S_t \in \mathrm{d}z) &= \frac{s}{z^3}\frac{\mathrm{d}z}{\sqrt{2\pi t}}\left\{\exp\left(-\frac{(1/z - 1/s)^2}{2t}\right) - \exp\left(-\frac{(1/z + 1/s)^2}{2t}\right)\right\},\\ \mathbb{E}(S_t|S_0 = s) &= s\left(1 - 2\mathcal{N}\left(-\frac{1}{s\sqrt{t}}\right)\right),\end{aligned}$$

and the martingale defect is precisely quantified from the second equation. Let us now try to provide a valid definition of a European option in this strict local martingale framework. Consider as before a given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and a complete market with a traded continuous asset price process S. We are interested in evaluating the price, today, of a European option with payoff, at some future time $T > 0$, $H(S_T)$.

Definition 2.1.11. An admissible wealth process is a self-financing process $(Z_t)_{t\geq 0}$ of the form $Z_t = Z_0 + \int_0^t \theta_u \mathrm{d}S_u$, where θ is predictable[2], S-integrable, such that $\lim_{n\uparrow\infty} n\mathbb{P}\left(\inf_{t\in[0,T]} Z_t < -n\right) = 0$.

We can now introduce the following definition:

Definition 2.1.12. The fair price of a financial instrument is the smallest initial capital required to finance an admissible super-replicating wealth process.

[2]Recall that θ is said to be predictable if it is measurable with respect to the σ-algebra generated by all left-continuous adapted processes.

Armed with this definition, Cox and Hobson [36] proved the following representation:

Theorem 2.1.13. *The fair price at inception of a European option with payoff $H(S_T)$ at time T is equal to $\mathbb{E}(H(S_T)|\mathcal{F}_0)$.*

When $H(x) \equiv (x-K)_+$ (resp. $H(x) \equiv (K-x)_+$), the corresponding value of the Call option (resp. Put option) shall be denoted by $C(K,T)$ (resp. $P(K,T)$). With this result, and keeping in mind the Put-Call parity relation in true martingale models, let us consider the following result:

Theorem 2.1.14. *The local martingale S has a bubble if and only if any of the following holds:*

(i) S *is a strict supermartingale;*

(ii) $\mathbb{E}(S_t|\mathcal{F}_0) < S_0$;

(iii) $C(K,T) - P(K,T) < S_0 - K$, *for some* K;

(iv) $\limsup_{n\uparrow\infty} n\mathbb{P}\left(\sup_{t\in[0,T]} S_t > n\right) > 0$.

Note that Condition (iii) in particular implies that Put-Call parity breaks down for some strike, while Condition (iv) means that the stock price is unbounded above almost surely.

Proof. Note that the following decomposition always holds:

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K.$$

This decomposition, together with Lebesgue Dominated convergence and basic properties of supermartingales yield (i)-(ii)-(iii). Statement (iv) is more technical to prove and we refer the interested reader to [7]. □

Exercise 2.1.15. Consider the first example above.

In the CEV example above, $\mathrm{d}S_t = S_t^2 \mathrm{d}W_t$, $S_0 > 0$, the European Call option price has the closed-form representation:

$$\begin{aligned}\mathbb{E}(S_T - K)_+ =& S_0\Big(\mathcal{N}(\kappa-\delta) - \mathcal{N}(-\delta) + \mathcal{N}(\delta) - \mathcal{N}(\kappa+\delta)\Big) \\ &- K\left(\mathcal{N}(\kappa+\delta) - \mathcal{N}(\delta-\kappa) + \frac{n(\kappa+\delta)-n(\kappa-\delta)}{\delta}\right),\end{aligned} \tag{2.1.2}$$

where $\delta := 1/(S_0\sqrt{T})$ and $\kappa := 1/(K\sqrt{T})$.

Exercise 2.1.16. From (2.1.2), compute the limit of the Call option price as the initial stock price tends to infinity, and comment.

2.1.3 Implied volatility

We consider now a market model where the stock price S is a non-negative process on some probability space adapted to a given filtration $(\mathcal{F}_t)_{t\geq 0}$. We further assume the existence of an equivalent local martingale measure $\mathbb{P}$ under which S is a non-negative local martingale (namely a supermartingale). In this market, we assume the existence of a family of European Call option prices $(C_t(K,T))_{t,K,T}$. We begin with a rigorous definition of the implied volatility, via the following proposition:

Proposition 2.1.17. *Let $s \geq 0$ and C be a map from $(0,\infty) \times [0,\infty)$ to $\mathbb{R}$ such that for any $K > 0$ and $T \geq 0$, the inequalities $(s-K)_+ \leq C(K,T) \leq s$ hold. Then the equation $C(K,T) = C_{\mathrm{BS}}(K,T,\sigma)$ has a unique non-negative solution, which is called the implied volatility.*

Proof. In the Black-Scholes model starting at $S_0 = s$, for any $K > 0$ and $T \geq 0$, the map $\sigma \mapsto C_{\mathrm{BS}}(K,T,\sigma)$ is strictly increasing, tends to $(s-K)_+$ as σ tends to zero and to s when σ tends to infinity. □

Remark 2.1.18. From Proposition 2.1.1 and Theorem 2.1.3, as well as Theorem 2.1.14, the bounds in the proposition above, assumed to hold for all $K \geq 0$, are equivalent to saying that the stock price process is a true martingale. By Put-Call parity, the proposition could be stated using Put options, replacing the double inequality there by $(K-s)_+ \leq P(K,T) \leq K$, for all $K, T \geq 0$.

Example 2.1.19. See the IPython notebook for an example based on (a particular case of) the CEV model, using Formula (2.1.2).

Notation 2.1.20. Let $\Sigma_t(k,T)$ denote the implied volatility at log-moneyness $k := \log(KB(t,T)/S_t)$ and maturity T, computing at time $t \in [0,T]$. We shall from now onwards denote the total implied variance $V_t(k,T) \equiv \Sigma_t(k,T)^2(T-t)$. We may write $V(k,T)$ in place of $V_0(k,T)$ whenever $t = 0$.

Let us first look at some basic properties of the implied volatility.

Proposition 2.1.21. *The distribution of S_T is fully characterised by the function $k \mapsto V(k,T)$.*

Proof. By definition of the implied volatility, knowledge of the function V is equivalent to knowledge of the Call price function C. But Fubini's theorem implies $\partial_k^+ C(k,T) = -\mathrm{e}^k \mathbb{P}(S_T > \mathrm{e}^k)$, from which the proposition follows. □

Proposition 2.1.22. *Let $T \geq 0$ and $[k_-, k_+]$ be the smallest interval containing the essential support of $\log(S_T)$, with possibly $k_\pm = \pm\infty$. Then $V(k,T) > 0$ if and only if $k \in (k_-, k_+)$.*

Proof. Note first that since S is a $\mathbb{P}$-martingale, $\mathbb{E}(S_T) = S_0 = 1$, and necessarily $k_- \leq 0 \leq k_+$. Let us now prove the identity $\mathrm{BS}(-k, V(k,T)) = \mathbb{E}(1 - S_T \mathrm{e}^{-k})_+$. The Put-Call parity reads $\mathrm{BS}(-k, v) = 1 - \mathrm{e}^{-k} + \mathrm{e}^{-k}\mathrm{BS}(k,v)$, which implies

$$\mathrm{BS}(-k, V(k,T)) = 1 - \mathrm{e}^{-k} + \mathrm{e}^{-k}\mathbb{E}(S_T - \mathrm{e}^k)_+ = \mathbb{E}(1 - S_T \mathrm{e}^{-k})_+. \tag{2.1.3}$$

Consider the case where $k \geq 0$. This identity yields that $V(k,T) = 0$ if and only if $\mathbb{E}(S_T - \mathrm{e}^k)_+ = (1-\mathrm{e}^k)_+$, which is clearly equal to zero, so that we have $V(k,T) = 0$ if and only if $S_T \leq \mathrm{e}^k$ almost surely, e.g. $k \geq k_+$. Similarly, assume that $k \leq 0$. Identity (2.1.3) also implies that $V(k,T) = 0$ if and only if $\mathbb{E}(1 - S_T\mathrm{e}^{-k})_+ = (1 - \mathrm{e}^{-k})_+$, from which we obtain $k \leq k_-$, and the proposition follows. □

From Lemma 2.1.2 and Proposition 2.1.17 above, it is then clear that the implied volatility is uniquely well-defined for any given true martingale. We now investigate the case of strict (non-negative) local martingale, i.e. local martingale which are not true martingales. As mentioned before, in the case of strict local martingales, the Put price is well defined (since it is bounded above as a function of the stock price), but the Call price is not. Should one define the latter via Put-Call parity, the implied volatility is clearly well defined for all strikes, since the computed Call price then lies within the no-arbitrage bounds from Proposition 2.1.17. Suppose now that the Call price is defined via expectation of the final payoff, then the following holds:

Lemma 2.1.23. *Assume that S is a strict (non-negative) local martingale. If $T > 0$, then there exists some strike $K^* \in [S_0 - \mathbb{E}_0(S_T), S_0]$ such that the implied volatility is ill-defined for all strikes in the interval $[0, K^*)$.*

Proof. The strict local martingale property implies that for any $t, \tau > 0$, $\mathbb{E}_t(S_{t+\tau}) < S_t$. Since the map $K \mapsto C_t(K) := \mathbb{E}_t(S_T - K)_+$ is decreasing, continuous, non-negative on the whole positive real line, and $C_t(K) \leq \mathbb{E}_t(S_T) < S_t$, there exists $K^* \in [S_t - \mathbb{E}_t(S_T), S_t]$ such that $C_t(K^*) = S_t - K^*$. Therefore, for any $\hat{K} \geq K^*$, we have $(S_t - \hat{K})_+ \leq C_t(\hat{K}) \leq S_t$, which follows from the fact that C_t is non-negative with $-1 \leq \partial_K^+ C_t(K) \leq 0$. □

Remark 2.1.24. For strict local martingales, the implied volatility is well defined for all $K \geq S_t$.

In Theorem 2.1.3, we exhibited necessary properties for a given two-dimensional map to define a genuine Call price function, arising as the conditional expectation of the payoff of a European Call option written on a true martingale. We now translate these conditions into conditions on a given map to define a proper implied volatility surface.

Theorem 2.1.25. *If the two-dimensional map $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies*

(i) $w(\cdot, T)$ is of class $\mathcal{C}^2$;

(ii) $w(k,T) > 0$ for all $(k,T) \in \mathbb{R} \times \mathbb{R}_+$;

(iii) $w(k, \cdot)$ is non-decreasing;

(iv) for all $(k,T) \in \mathbb{R} \times \mathbb{R}_+$,

$$g(k,T) := \left(\left(1 - \frac{k\partial_k w}{2w}\right)^2 - \frac{(\partial_k w)^2}{4}\left(\frac{1}{4} + \frac{1}{w}\right) + \frac{\partial_{kk} w}{2} \right)\Bigg|_{(k,T)} \geq 0; \tag{2.1.4}$$

(v) $w(k,0) = 0$ *for all* $k \in \mathbb{R}$*;*

(vi) $\lim_{k\uparrow\infty} d_+(k, w(k,T)) = -\infty$*;*

then the corresponding Call price surface defined by $(k,T) \mapsto C_{\mathrm{BS}}(k,T,w(k,T))$ *satisfies the assumptions of Theorem 2.1.3.*

Remark 2.1.26. Let p_T denote the probability density function of the log stock price at time T (the maturity). Then, by twice differentiating the Call price function, we obtain

$$\begin{aligned} p_T(k) &= \mathrm{e}^k \left. \frac{\partial^2 C(K,T)}{\partial K^2}\right|_{K=S_0\mathrm{e}^k} = \mathrm{e}^k \left. \frac{\partial^2 C_{\mathrm{BS}}(K, \sqrt{w(\log(K),T)})}{\partial K^2}\right|_{K=S_0\mathrm{e}^k} \\ &= \frac{g(k,T)}{\sqrt{2\pi w(k,T)}} \exp\left(-\frac{d_-(k,w(k,T))^2}{2}\right) = g(k,T) n\left(d_-(k,w(k,T))\right). \end{aligned}$$

Proof. In view of Theorem 2.1.3, it is clear that is it enough to check that items (i)-(v) are satisfied for the Call option surface $\mathbb{R} \times \mathbb{R}_+ \ni (k,T) \mapsto C_{\mathrm{BS}}(k, w(k,T))$. Since the option price is convex and of class $\mathcal{C}^2$, we can write $\partial_{KK} C(K,T) \ge 0$ for all $(K,T) \in \mathbb{R}_+ \times \mathbb{R}_+$. Let us write w' and w'' for the first and second derivatives of w with respect to k. Using the fact that $\partial_K k = 1/K$, we obtain $\partial_K C = \partial_K k \cdot D_k \mathrm{BS} = \frac{1}{K}\left[\partial_k + w'\partial_w\right]\mathrm{BS}$ and

$$\begin{aligned} \partial_{KK} C &= -\frac{1}{K^2}\left[\partial_k + w'\partial_w\right]\mathrm{BS} \\ &+ \frac{1}{K}\left\{\frac{1}{K}\left(\partial_{kk} + w'\partial_{kw}\right) + \frac{1}{K}\left[w''\partial_w + w'\left(\partial_{kw} + w'\partial_{ww}\right)\right]\right\}\mathrm{BS} \\ &= \frac{1}{K^2}\Big\{ - \partial_k - w'\partial_w + \partial_{kk} + w'\partial_{kw} + w''\partial_w + w'\left(\partial_{kw} + w'\partial_{ww}\right)\Big\}\mathrm{BS} \\ &= \frac{1}{K^2}\Big\{\partial_{kk} - \partial_k + \left(w'' - w'\right)\partial_w + 2w'\partial_{kw} + (w')^2\partial_{ww}\Big\}\mathrm{BS} \\ &= \frac{1}{K^2}\left\{\frac{\mathrm{e}^k n(d_-(k,w))}{\sqrt{w}} + \left(w'' - w'\right)\frac{n(d_+(k,w))}{2\sqrt{w}} - (w')\frac{n(d_+(k,w))}{2}\frac{2k-w}{w^{3/2}}\right. \\ &\left. + (w')^2\frac{n(d_+(k,w))}{16 w^{5/2}}\left(4k^2 - w^2 - 4w\right)\right\} \\ &= \frac{n(d_+(k,w))}{K^2\sqrt{w}}\left\{1 + \frac{w''}{2} - \frac{w'}{2} - (w')\frac{2k-w}{2w} + (w')^2\frac{4k^2 - w^2 - 4w}{16w^2}\right\}, \end{aligned}$$

using the Black-Scholes Greeks in (1.4.5), as well as the simple identities $n'(z) = -zn(z)$ and $n(d_+(k,w)) = \mathrm{e}^k n(d_-(k,w))$. The term outside the bracket is clearly strictly positive for all $(K,T) \in \mathbb{R}_+ \times \mathbb{R}_+$, so that the convexity condition on the Call price reads off from the bracket being non-negative, which is precisely item (iv) in the theorem. □

Remark 2.1.27. There are some interesting symmetries appearing with the functions d_- and d_+. Recall that $d_\pm(k,w) \equiv -k/\sqrt{w} \pm \sqrt{w}/2$. Then for any real positive function w, we have

$$\lim_{k\uparrow\infty} d_-(k, w(k)) = -\infty \qquad \text{and} \qquad \lim_{k\downarrow-\infty} d_+(k, w(k)) = +\infty.$$

Indeed, the arithmetic mean-geometric inequality reads $-d_-(k,w(k)) = \frac{k}{\sqrt{w(k)}} + \frac{\sqrt{w(k)}}{2} \ge \sqrt{2k}$, when $k > 0$, which implies the first limit, and the second one follows using $d_+(k,w(k)) = \frac{-k}{\sqrt{w(k)}} + \frac{1}{2}\sqrt{w(k)} \ge \sqrt{-2k}$, when $k < 0$.

Arbitrage with American options

We assume here that the stock price does not give dividends. In the case of American options, the holder of the option has the right to exercise it at any time before the maturity T, so that clearly the price of an American option is always at least worth its European counterpart, the difference between the two being called the 'early exercise premium'. In the case of a Call, it can be shown (exercise) that both prices are equal, since it is never optimal to exercise before maturity. This is no longer true in the case of the Put, for which the bound

$$P^A(K,T) - P^E(K,T) \leq K\left(1 - B(0,T)\right),$$

holds, under the assumption of deterministic interest rates, where P^A denotes the American Put price value at time zero. It is standard that Put-Call parity is violated in the American case. However, it is easy to show (exercise) that

- for any fixed maturity, the American Put is an increasing and convex function of the strike;
- for any fixed strike, the American Put is a non-decreasing function of the maturity.

2.1.4 A new look at variance swaps

In Section 1.5.4, we showed that variance swaps could be replicated exactly using European Call and Put options. We show here how to recast this result in terms of the implied volatility.

Proposition 2.1.28. *For a given maturity $T > 0$, let w denote the implied total variance. Then the fair strike of the variance swap reads*

$$\mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_t^2 \mathrm{d}t\right) = \frac{1}{T}\int_{-\infty}^{+\infty} n(z)\widetilde{\sigma}^2(z)\mathrm{d}z,$$

where $\widetilde{\sigma}(z) := \sqrt{w\left(d_-^{-1}(z)\right)}$.

Proof. Let $F_T := S_0\mathrm{e}^{rT}$ denote the forward price, and recall the price of the Call option:

$$C = \mathrm{e}^{-rT}F_T\left\{\mathcal{N}(d_+(x)) - \mathrm{e}^x\mathcal{N}(d_-(x)\right\},$$

where $x := \log(K/F_T)$ is the log forward moneyness, $d_\pm(x) := -x/\sqrt{w} \pm \sqrt{w}/2$, and where w stands for the total variance. Differentiating the Call price function with respect to the strike K, we obtain

$$\begin{aligned}
\mathrm{e}^{rT}\partial_K C = \frac{\mathrm{e}^{rT}}{K}\partial_x C &= \mathrm{e}^{-x}\left\{n(d_+(x))d_+'(x) - \mathrm{e}^x\mathcal{N}(d_-(x)) - \mathrm{e}^x n(d_-(x))d_-'(x)\right\}\\
&= \left(\mathrm{e}^{-x}\left[d_-'(x) + \partial_x\sqrt{w(x)}\right]n(d_+(x)) - n(d_-(x))d_-'(x)\right) - \mathcal{N}(d_-(x))\\
&= n(d_-(x))\partial_x\sqrt{w(x)} - \mathcal{N}(d_-(x)) = n(d_-(x))\sigma'(x) - \mathcal{N}(d_-(x)),
\end{aligned}$$

where we use the fact that $d'_+(x) = d'_-(x) + \partial_x\sqrt{w(x)}$ and $\mathrm{e}^{-x}n(d_+(x)) = n(d_-(x))$. We also use the simplified notations $d'_\pm(x) := \partial_x d_\pm(x)$ and $\sigma'(x) := \partial_x\sqrt{w(x)}$. Likewise,

$$\mathrm{e}^{rT}\partial_K P = n(d_-(x))\sigma'(x) + \mathcal{N}(-d_-(x)).$$

Let us define

$$I = \mathrm{e}^{rT}\int_0^{F_T}\frac{P(K,T)}{K^2}\mathrm{d}K + \mathrm{e}^{rT}\int_{F_T}^{\infty}\frac{C(K,T)}{K^2}\mathrm{d}K,$$

i.e. a rescaled version of the fair price of (1.5.9). An integration by parts yields

$$I = \mathrm{e}^{rT}\left(\int_0^{F_T}\frac{\partial_K P(K,T)}{K}\mathrm{d}K + \int_{F_T}^{\infty}\frac{\partial_K C(K,T)}{K}\mathrm{d}K - \left.\frac{P(K)}{K}\right|_0^{F_T} - \left.\frac{C(K)}{K}\right|_{F_T}^{+\infty}\right).$$

By Put-Call parity, we have $C(F_T,T) = P(F_T,T)$. Furthermore since the stock price is strictly positive, we have $\lim_{K\downarrow 0} P(K,T)/K = 0$, and hence the boundary terms above vanish. Using the derivatives of the Calls and Puts derived above and changing the variable from K to x, we obtain

$$\begin{aligned}
I &= \int_{-\infty}^{0}\left[n(d_-(x))\sigma'(x) + \mathcal{N}(-d_-(x))\right]\mathrm{d}x + \int_0^{\infty}\left[n(d_-(x))\sigma'(x) - \mathcal{N}(d_-(x))\right]\mathrm{d}x \\
&= \int_0^{+\infty} n(d_-(x))\sigma'(x)\mathrm{d}x + \int_{-\infty}^{0}\mathcal{N}(-d_-(x))\mathrm{d}x + \int_{-\infty}^{0} n(d_-(x))\sigma'(x) - \int_0^{+\infty}\mathcal{N}(d_-(x))\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} n(d_-(x))\sigma'(x)\mathrm{d}x + \left\{\int_{-\infty}^{+\infty} xd'_-(x)n(-d_-(x))\mathrm{d}x + x\mathcal{N}(-d_-(x))\Big|_{-\infty}^{0} - x\mathcal{N}(d_-(x))\Big|_0^{+\infty}\right\}
\end{aligned}$$

From Roger Lee's moment formula, it is then sufficient that there exists $\varepsilon > 0$ such that $\mathbb{E}(S_T^{1+\varepsilon})$ and $\mathbb{E}(S_T^{-\varepsilon})$ are both finite in order for the two boundary terms above to vanish. If we now integrate by parts the first integral we obtain

$$\begin{aligned}
I &= \int_{-\infty}^{+\infty} d'_-(x)d_-(x)n(d_-(x))\sigma(x)\mathrm{d}x - \sigma(x)n(d_-(x))|_{\mathbb{R}} + \int_{-\infty}^{+\infty} xd'_-(x)n(-d_-(x))\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} d'_-(x)d_-(x)n(d_-(x))\sigma(x)\mathrm{d}x + \int_{-\infty}^{+\infty} xd'_-(x)n(d_-(x))\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} d'_-(x)n(d_-(x))\left[d_-(x)\sigma(x) + x\right]\mathrm{d}x \\
&= -\int_{-\infty}^{+\infty} d'_-(x)n(d_-(x))\frac{\sigma^2(x)}{2}\mathrm{d}x
\end{aligned}$$

since the boundary terms vanish, where we used the fact that the function n is symmetric, and the last line follows from the definition of d_-. With the change of variables $z := d_-(x)$, the integral becomes

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} n(z)\widetilde{\sigma}^2(z)\mathrm{d}z,$$

with $\widetilde{\sigma}(z) := \sqrt{w\left(d_-^{-1}(z)\right)}$. □

2.2 No-arbitrage properties of the implied volatility surface

In this section, we shall endeavour to determine the properties of the implied volatility surface in a model-independent martingale framework. In Theorem 2.1.25 above, we found sufficient conditions on a given volatility surface that allowed for the generated option prices to be valid (in the sense of Theorem 2.1.3) We first start with the definition of static arbitrage for a given volatility surface in the following way, which is consistent with the framework of Theorem 2.1.3.

Definition 2.2.1. A volatility surface is free of static arbitrage if and only if the following conditions hold: (i) it is free of calendar spread arbitrage; (ii) each time slice is free of buttery arbitrage.

In particular, absence of buttery arbitrage ensures the existence of a (non-negative) probability density, and absence of calendar spread arbitrage implies monotonicity of option prices with respect to the maturity. In light of Theorem 2.1.25, butterfly arbitrage is guaranteed as soon as Conditions (iv)-(v) are satisfied, whereas calendar spread corresponds to Condition (iii).

2.2.1 Slope of the implied volatility

We shall again fix some $t \geq 0$, and recall that $[k_-, k_+]$ denotes the smallest interval containing the essential support of $\log(S_t)$, possibly with $k_\pm = \pm\infty$. The following proposition derives lower and upper bounds for the derivative of the total implied variance.

Proposition 2.2.2. *The right-derivative $\partial_k^+ V(k,t)$ (resp. left derivative $\partial_k^- V(k,t)$) exists for all $k \neq k_-$ (resp. for all $k \neq k_+$) and*

$$\begin{aligned} \partial_k^- V(k,t) \leq \partial_k^+ V(k,t) \leq \frac{4V(k,t)}{V(k,t)+2k} \leq 4, \qquad & \textit{for all } k \geq 0, \\ -4 \leq -\frac{4V(k,t)}{V(k,t)-2k} \leq \partial_k^- V(k,t) \leq \partial_k^+ V(k,t), \quad & \textit{for all } k \leq 0. \end{aligned}$$

Proof. The proposition is obvious whenever k is not in the essential support of $\log(S_T)$. From the implicit equation defining the implied variance $C(k,t) = \mathrm{BS}(k, V(k,t))$, we obtain by differentiation $\partial_k^+ C(k,t) = \partial_k^+ V(k,t) \partial_V \mathrm{BS}(k, V(k,t)) + \partial_k^+ \mathrm{BS}(k, V(k,t))$, and hence

$$\begin{aligned} \partial_k^+ V(k,t) = \frac{\partial_k^+ C(k,t) - \partial_k \mathrm{BS}(k, V(k,t))}{\partial_V \mathrm{BS}(k, V(k,t))} &= 2\sqrt{V(k,t)} \frac{\mathcal{N}(d_-(k, V(k,t))) - \mathbb{P}(S_t > \mathrm{e}^k)}{n(d_-(k, V(k,t)))} \\ &\geq 2\sqrt{V(k,t)} \frac{\mathcal{N}(d_-(k, V(k,t))) - \mathbb{P}(S_t \geq \mathrm{e}^k)}{n(d_-(k, V(k,t)))} \\ &= \partial_k^- V(k,t), \end{aligned}$$

where we used the Black-Scholes Greeks (1.4.5) as well as Proposition 2.1.8. Using the standard bounds on the Gaussian Mills ratio $\mathcal{N}(-x)/n(-x) < x^{-1}$, for $x > 0$, we obtain

$$\partial_k^+ V(k,t) < -\frac{2\sqrt{V(k,t)}}{d_-(k, V(k,t))} = \frac{4V(k,t)}{V(k,t)+2k} = \frac{4}{1 + 2k/V(k,t)} \leq 4,$$

for any $k \in [0, k_+)$. Differentiating the identity $\mathrm{BS}(-k, V(k,t)) = \mathbb{E}(1 - S_t \mathrm{e}^{-k})_+$ (proved in Proposition 2.1.22) yields $-\partial_k^- \mathrm{BS}(-k, V(k,t)) + \partial_k^- V(k,t) \partial_V \mathrm{BS}(-k, V(k,t)) = \mathrm{e}^{-k} \mathbb{E}(S_T \mathbb{1}_{S_t < \mathrm{e}^k})$, which also reads

$$\begin{aligned}\partial_k^- V(k,t) &= \frac{\mathrm{e}^{-k}\mathbb{E}(S_t \mathbf{1}_{S_t<\mathrm{e}^k}) + \partial_k^- \mathrm{BS}(-k, V(k,t))}{\partial_V \mathrm{BS}(-k, V(k,t))} \\ &= 2\sqrt{V(k,t)}\frac{\mathbb{E}(S_t \mathbf{1}_{S_t<\mathrm{e}^k}) - \mathcal{N}(-d_+(k, V(k,t)))}{n(-d_+(k, V(k,t)))} \\ &> -\frac{2\sqrt{V(k,t)}}{d_+(k, V(k,t))} = -\frac{4V(k,t)}{V(k,t) - 2k} \geq -4,\end{aligned}$$

whenever $k \in (k_-, 0]$. □

Remark 2.2.3. Recall the standard arithmetic / geometric mean inequality $x + y \geq 2\sqrt{xy}$, for any two non-negative real numbers. This in particular implies

$$\begin{aligned}\partial_k^+ V(k,t) &< \frac{4V(k,t)}{V(k,t) + 2k} \leq \sqrt{\frac{2V(k,t)}{k}}, \quad \text{for } k \geq 0, \\ \partial_k^- V(k,t) &> -\frac{4V(k,t)}{V(k,t) - 2k} \geq \sqrt{\frac{2V(k,t)}{|k|}}, \quad \text{for } k \leq 0,\end{aligned}$$

and hence, by integration, we finally obtain $\sqrt{V(k,t)} \leq \sqrt{V(0,t)} + \sqrt{2k}$ when $k \geq 0$ and $\sqrt{V(k,t)} \leq \sqrt{V(0,t)} + \sqrt{-2k}$ when $k \leq 0$.

2.2.2 Time asymptotics

Consider a stock price process $(S_t)_{t\geq 0}$, assumed to be a non-negative martingale on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The core of this section is the following result (recall Notations 2.1.20 for $V(\cdot)$ and $\Sigma(\cdot)$):

Theorem 2.2.4. *For any $M > 0$, the following holds:*

$$\lim_{t\uparrow\infty} \sup_{k\in[-M,M]} \left| \Sigma(k,t) - \left(-\frac{8}{t} \log \mathbb{E}(1 \wedge S_t) \right)^{1/2} \right| = 0.$$

Before proving the theorem, we need a few preliminary elementary results.

Lemma 2.2.5. *Let S be a true martingale starting at $S_0 = 1$ on a given filtered probability space. Then the following are equivalent as t tends to infinity:*

- *S_t converges to zero in distribution;*
- *S_t converges to zero almost surely;*
- *$\mathbb{E}(S_t - \mathrm{e}^k)_+$ converges to 1 from below for all $k \in \mathbb{R}$;*
- *for any $k \in \mathbb{R}$, $V(k,t)$ tends to infinity as t tends to infinity.*

Recall the following fundamental theorem:

Theorem 2.2.6 (Martingale Convergence Theorem)**.** *If X is a non-negative supermartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, then the limit $X_\infty := \lim_{t\uparrow\infty} X_t$ exists $\mathbb{P}$-almost everywhere and $X_\infty \in L^1(\mathbb{P})$.*

Proof. The martingale convergence theorem yields the existence of a random variable $S_\infty \in L^1(\mathbb{P})$ to which S_t converges almost surely, so that the first two items are clearly equivalent. The remainder of the lemma follows directly from the identity $\mathbb{E}(S_t \wedge \mathrm{e}^k) = 1 - \mathbb{E}(S_t - \mathrm{e}^k)_+$ and the Black-Scholes formula. □

The following corollary is an immediate consequence of the Lemma 2.2.5, and will be used in the proof of Theorem 2.2.4.

Corollary 2.2.7. *If $\mathbb{P}\left(\lim_{t\uparrow\infty} S_t > 0\right) > 0$ then $\lim_{t\uparrow\infty} \Sigma(k,t) = 0$ for all $k \in \mathbb{R}$.*

Proof of Theorem 2.2.4. Clearly on the set $\{\mathbb{P}(\lim_{t\uparrow\infty} S_t = 0) < 1\}$, the lemma holds since in that case $\lim_{t\uparrow\infty} \Sigma(k,t) = 0$ by Corollary 2.2.7 and $\lim_{t\uparrow\infty} \mathbb{E}(1 \wedge S_t) > 0$. Assume now that S_t converges to zero almost surely as t tends to infinity, so that, for any fixed k, $V(k,t)$ diverges to infinity as t becomes large by Lemma 2.2.5. Let us define the function $\psi(x) \equiv 1 - x(1 - \mathcal{N}(x))/n(x)$, where we recall that $n(\cdot)$ and $\mathcal{N}(\cdot)$ denote respectively the Gaussian pdf and cdf. The bounds $0 \leq \psi(x) \leq (1+x^2)^{-1}$ can be proved by a simple integration by parts, and therefore, for any $v > 2k$, we can write

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^k \wedge S_t) = 1 - \mathbb{E}(S_t - \mathrm{e}^k)_+ &= 1 - \mathcal{N}(d_+) + \mathrm{e}^k \mathcal{N}(d_-) = 1 - \mathcal{N}(d_+) + \mathrm{e}^k \left[1 - \mathcal{N}(-d_-)\right] \\
&= 1 - \mathcal{N}(d_+) + \frac{n(d_+)}{n(d_-)} \left[1 - \mathcal{N}(-d_-)\right] \\
&= n(d_+) \left\{ \frac{1 - \psi(d_+)}{d_+} + \frac{1 - \psi(-d_-)}{-d_-} \right\} \\
&= n(d_+) \left\{ \frac{v^{3/2}}{v^2/4 - k^2} - \frac{\psi(d_+)}{d_+} - \frac{\psi(-d_-)}{-d_-} \right\}.
\end{aligned}$$

Fix now some $M > 0$, then $\inf_{|k|\leq M} V(k,t)$ tends to infinity as t tends to infinity. Indeed, from Lemma 2.2.5, V converges pointwise to infinity, so that there exists $t^* > 0$ for which both $\mathbb{P}(S_{t^*} < \mathrm{e}^{-M})$ and $\mathbb{P}(S_{t^*} > \mathrm{e}^{-M})$ are strictly positive (since the martingale cannot be bounded). Therefore, for any $t > t^*$, the function $V(\cdot, t)^{-1}$ is positive, continuous on $[-M, M]$ and converges pointwise monotonically to zero, so that Dini's theorem proves the claim[3]. Now, fix some $M > 0$; for t large enough, $V(k,t)$ is greater than $2M$, and hence, for any $k \in [-M, M]$,

$$\frac{\psi(d_+)}{d_+} \leq 2d_+^{-3} \leq cv^{-3/2} \qquad \text{and} \qquad \frac{\psi(-d_-)}{-d_-} \leq cv^{-3/2},$$

where c is a strictly positive constant that can change from line to line. Indeed, it is easy to see that $\psi(x) \leq 2/x^2$ for any $x \neq 0$, and, on $[2M, \infty)$, the map $v \mapsto v^{3/2} d_+(v)^{-3}$ is strictly decreasing

[3] Recall that Dini's theorem states that a monotone sequence of functions converging on a compact space also converges uniformly.

and continuous from $(2M)^{3/2}d_+(2M)^{-3}$ down to 8, in particular is bounded above by some strictly positive constant. Therefore

$$-8\log\mathbb{E}(1\wedge S_t) = \frac{(v-2k)^2}{v} + 4\log(v) + \delta(v) + c = v + \eta(v),$$

where δ a function decreasing to zero at infinity, and $|\eta(v)| \leq A + B\log(v)$ for t large enough, for some real (positive) A and B. This therefore proves

$$\lim_{t\uparrow\infty}\sup_{k\in[-M,M]}\left|\Sigma(k,t) - \left(-\frac{8}{t}\log\mathbb{E}(S_t\wedge \mathrm{e}^k)\right)^{1/2}\right| = 0.$$

Applying the double inequality $1\wedge(a/b) \leq \frac{x\wedge a}{x\wedge b} \leq 1\vee(a/b)$ to $x = S_t$, $a = 1$ and $b = \mathrm{e}^k$ concludes the proof of the theorem. □

Remark 2.2.8. Consider the Black-Scholes model:

$$S_t = S_0\exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right) = S_0\exp\left\{\left(-\frac{1}{2}\sigma^2 + \sigma\frac{W_t}{t}\right)t\right\},$$

which clearly converges to zero as t tends to zero as soon as $\sigma > 0$. The stock price process is a non-negative supermartingale, hence as an almost sure limit $S_\infty = 0$ as t tends to infinity. This provides a simple example to Corollary 2.2.7. Consider now the case where, for each $t > 0$, S is Black-Scholes (with $\sigma = 1$) with probability 1/2 and is equal to 1 with probability 1/2. Then clearly $(S_t)_{t>0}$ converges to 1/2 almost surely as t tends to infinity, and the implied volatility converges to zero. We refer the interested reader to the paper by David Hobson [83] for examples of stochastic volatility models in which S_∞ might or might not tend to zero.

We finish this section with the following result, due to Chris Rogers and Mike Tehranchi [133]:

Theorem 2.2.9. *If S_τ is strictly positive almost surely for all $\tau \geq 0$, then for any $k_1, k_2 \in \mathbb{R}$ and any $0 \leq s \leq t$,*

$$\limsup_{\tau\uparrow\infty}\left(\Sigma_t(k_1,\tau) - \Sigma_s(k_2,\tau)\right) = 0.$$

Proof. Without loss of generality, let $s = 0$. Define the process $(M_t(\tau))_{t\geq 0}$ by $M_t(\tau) := \mathbb{E}_t(1\wedge S_\tau)$, so that clearly $(M_t(\tau)/M_0(\tau))_{t\geq 0}$ is a martingale. Applying Lemma 2.2.10 to $M_t(\tau)$, we obtain

$$\limsup_{\tau\uparrow\infty}\left\{-\frac{8}{\tau}\log(M_t(\tau)) + \frac{8}{\tau}\log(M_0(\tau))\right\} \geq 0,$$

and therefore, Theorem 2.2.4 implies that $\limsup_{\tau\uparrow\infty}\{\Sigma_t(k_1,\tau-t) - \Sigma_t(k_2,\tau-t)\} \geq 0$ for any $k_1, k_2 \in \mathbb{R}$. Since the map $\tau \mapsto V_\cdot(\cdot,\tau)$ is increasing under absence of arbitrage, then

$$\Sigma_t(k_1,\tau) \geq \sqrt{\frac{\tau - t}{\tau}}\Sigma_t(k_1,\tau),$$

and the theorem follows. □

Lemma 2.2.10 (see [84]). *If $(X_n)_{n\geq 0}$ is a family of non-negative random variables with finite mean, then* $\liminf_{n\uparrow\infty} X_n^{1/n} \leq \liminf_{n\uparrow\infty}\mathbb{E}(X_n)^{1/n}$.

2.2.3 Wing properties

Let us first start with some crude asymptotic behaviour for the implied volatility when the strike becomes very large or very small: We first start with the following simple model-independent result, which gives us the leading behaviour of the total implied variance for large (log) strikes:

Proposition 2.2.11. *For any $t > 0$, the following equalities hold:*

$$\lim_{k\uparrow\infty}\left(\sqrt{V(k,t)} - \sqrt{2k}\right) = -\infty, \tag{2.2.1}$$

$$\lim_{k\downarrow-\infty}\left(\sqrt{V(k,t)} - \sqrt{-2k}\right) = \mathcal{N}^{-1}\left(\mathbb{P}[(S_t = 0)]\right). \tag{2.2.2}$$

Note that in the case where the stock price process is a strictly positive martingale, then $\mathbb{P}(S_t = 0) = 0$ and the right-hand side of (2.2.2) is equal to $-\infty$.

Proof. Let us first prove (2.2.1). For $k > 0$, the arithmetic-geometric inequality reads $k/\sqrt{v} + \sqrt{v}/2 \geq \sqrt{2k}$, whenever $v > 0$. Therefore

$$\mathrm{e}^k\mathcal{N}(d_-(k, V(k,t))) = \mathrm{e}^k\mathcal{N}\left(-\frac{k}{\sqrt{V(k,t)}} - \frac{\sqrt{V(k,t)}}{2}\right) \leq \mathrm{e}^k\mathcal{N}\left(-\sqrt{2k}\right),$$

which converges to zero (see also a similar computation in the proof of Lemma 2.2.13 using L'Hopital's rule). Since the Call price tends to zero as k tends to infinity, we therefore deduce that $\mathcal{N}(d_+(k, V(k,t)))$ converges to zero as well, so that $d_+(k, V(k,t)) = -k/\sqrt{V(k,t)} + \frac{1}{2}\sqrt{V(k,t)}$ converges to $-\infty$, and (2.2.1) follows.

We now prove (2.2.2), and recall that $\lim_{K\downarrow 0}\frac{P(K,t)}{K} = \mathbb{P}(S_t = 0)$. Let us prove the following claim:

$$\begin{aligned}
&\lim_{k\downarrow-\infty} d_-(k, V(k,t)) = +\infty, && \text{if } \mathbb{P}(S_t = 0) = 0;\\
&\lim_{k\downarrow-\infty} d_-(k, V(k,t)) = -\mathcal{N}^{-1}[\mathbb{P}(S_t = 0)], && \text{if } \mathbb{P}(S_t = 0) > 0.
\end{aligned} \tag{2.2.3}$$

Recall the identity $\mathrm{BS}(-k, V(k,t)) = \mathbb{E}(1 - S_t\mathrm{e}^{-k})_+$. Since $\mathbb{E}(1 - S_t\mathrm{e}^{-k})^+ = \mathbb{P}(S_t = 0) + \mathbb{E}\left[(1 - S_t\mathrm{e}^{-k})^+\mathbb{1}_{\{S_t>0\}}\right]$, then $\mathrm{BS}(-k, V(k,t))$ tends to $\mathbb{P}(S_t = 0)$ as k tends to $-\infty$ by dominated convergence. Therefore, for every $k < 0$,

$$\begin{aligned}
\mathcal{N}(d_+(-k, V(k,t))) &= \mathrm{BS}(-k, V(k,t)) + \mathrm{e}^{-k}\mathcal{N}(d_-(-k, V(k,t)))\\
&\leq \mathrm{BS}(-k, V(k,t)) + \mathrm{e}^{-k}\mathcal{N}(-\sqrt{2|k|}),
\end{aligned} \tag{2.2.4}$$

and the right-hand side converges to $\mathbb{P}(S_t = 0)$ as k tends to $-\infty$. The second line above follows from the arithmetic-geometric inequality $d_-(-k, V(k,t)) = -\frac{|k|}{\sqrt{V(k,t)}} - \frac{\sqrt{V(k,t)}}{2} \leq -\sqrt{2|k|}$ (see also Remark 2.2.3) and the fact that $\mathrm{e}^{z^2/2}\mathcal{N}(-z)$ tends to zero for large z. The claim (2.2.3) then follows from the identity $d_+(-k, V(k,t)) = -d_-(k, V(k,t))$.

Let now $\mathrm{p} := \mathcal{N}^{-1}(\mathbb{P}(S_t = 0))$, and assume first that $\mathrm{p} = -\infty$. The estimate (2.2.4) implies that for every $M > 0$ we have $d_+(-k, V(k,t)) = \frac{k}{\sqrt{V(k,t)}} + \frac{1}{2}\sqrt{V(k,t)} < -M$ for k small enough, or yet $\sqrt{V(k,t)} < -M + \sqrt{M^2 + 2|k|}$. Therefore,

$$\limsup_{k\downarrow-\infty}\left(\sqrt{V(k,t)} - \sqrt{2|k|}\right) < -M + \limsup_{k\downarrow-\infty}(\sqrt{M^2+2|k|} - \sqrt{2|k|}) = -M$$

for every $M > 0$, which proves (2.2.2).

Now assume $\mathrm{p} > -\infty$. Then for fixed $\varepsilon > 0$, we have $\mathrm{p} - \varepsilon < d_+(-k, V(k,t)) < \mathrm{p} + \varepsilon$ for k small enough. It follows that:

$$\mathrm{p} - \varepsilon + \sqrt{2|k|} < \sqrt{V(k,t)} = d_+(-k, V(k,t)) - d_-(-k, V(k,t)) < \mathrm{p} + \varepsilon + \sqrt{(\mathrm{p}+\varepsilon)^2 + 2|k|}.$$

The lower bound again follows from the arithmetic-geometric inequality for d_-, and the upper bound from the identity $d_-(-k,v)^2 = d_+(-k,v)^2 + 2|k|$. Hence $\lim_{k\downarrow-\infty}(\sqrt{V(k,t)} - \sqrt{2|k|}) = \mathrm{p}$, and (2.2.2) is proved. □

Roger Lee's moment formula

Roger Lee's moment formula establishes a precise link between the tails (small and large strike) of the implied volatility smile and the tail behaviour of the stock price process. The underlying process $(S_t)_{t\geq 0}$ is assumed to be a true non-negative martingale with respect to a given filtered probability space. Let $V(\cdot, t)$ denote the total implied variance of the underlying stock price at maturity t, and define the function $V_\beta(k) \equiv \beta|k|$.

Theorem 2.2.12 (Lee's Moment Formula [102]). *Fix some time $t \geq 0$. Let $p^* := \sup\{p \geq 0 : \mathbb{E}(S_t^{1+p}) < \infty\}$ and $\beta_R := \limsup_{k\uparrow\infty}(V(k,t)/k)$ Then*

$$p^* = \frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2} \qquad \text{or} \qquad \beta_R = 2 - 4\left(\sqrt{p^*(1+p^*)} - p^*\right),$$

and $\beta_R \in [0,2]$. Similarly, for low strikes, let $q^* := \sup\{q \geq 0 : \mathbb{E}(S_t^{-q}) < \infty\}$ and $\beta_L := \limsup_{k\downarrow-\infty}(V(k,t)/|k|)$ Then

$$q^* = \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2} \qquad \text{or} \qquad \beta_L = 2 - 4\left(\sqrt{q^*(1+q^*)} - q^*\right),$$

and $\beta_L \in [0,2]$.

The proof of this theorem requires a few tools.

Lemma 2.2.13. *There exists $k^* > 0$ such that $V(k,t) < V_2(k)$ for all $k > k^*$.*

Proof. We consider the case $k > 0$. Since the Black-Scholes Call price is an increasing function of the volatility, the lemma follows directly from the inequality $\mathrm{BS}(k, V(k,t)) < \mathrm{BS}(k, V_2(k))$ for k large enough. Note that, since the stock price is in L^1, dominated convergence yields

$$\lim_{k\uparrow\infty}\mathrm{BS}(k, V(k,t)) = \lim_{k\uparrow\infty}\mathbb{E}\left(S_t - \mathrm{e}^k\right)_+ = \mathbb{E}\left[\lim_{k\uparrow\infty}\left(S_t - \mathrm{e}^k\right)_+\right] = 0.$$

Now,

$$\begin{aligned}
\lim_{k\uparrow\infty}\mathrm{BS}(k,V_2(k)) &= \lim_{k\uparrow\infty}\left[\mathcal{N}\left(-\frac{k}{\sqrt{V_2(k)}}+\frac{\sqrt{V_2(k)}}{2}\right)-\mathrm{e}^k\mathcal{N}\left(-\frac{k}{\sqrt{V_2(k)}}-\frac{\sqrt{V_2(k)}}{2}\right)\right]\\
&= \lim_{k\uparrow\infty}\left[\mathcal{N}(0)-\mathrm{e}^k\mathcal{N}(-\sqrt{2k})\right]=\frac{1}{2}-\lim_{k\uparrow\infty}\mathrm{e}^k\mathcal{N}(-\sqrt{2k}).
\end{aligned}$$

As k tends to infinity, the second term is ambiguous, but L'Hôpital's rule yields

$$\frac{\partial_k(\mathcal{N}(-\sqrt{2k}))}{\partial_k(\mathrm{e}^{-k})}=\frac{n(-\sqrt{2k})\mathrm{e}^k}{\sqrt{2k}},$$

which clearly converges to zero as k tends to infinity and hence $\mathrm{BS}(k,V_2(k))$ tends to $1/2$. Since the Call price associated with the implied variance $V(\cdot,t)$ tends to zero as k tends to infinity (see Theorem 2.1.3(iii)), the lemma follows. Note in particular that the function V_2 is not a valid implied volatility surface. □

For each $p>0$, the following moment inequality holds for all $k\in\mathbb{R}$:

$$C(k,t)\leq\frac{\mathbb{E}(S_t^{p+1})}{p+1}\left(\frac{p}{p+1}\right)^p\mathrm{e}^{-pk}. \tag{2.2.5}$$

Indeed, the inequality $S-\mathrm{e}^k\leq\frac{S^{p+1}}{p+1}\left(\frac{p}{p+1}\right)^p\mathrm{e}^{-pk}$ is a simple analysis exercise, and letting $S=S_t$, the claim follows by taking expectation on both sides. As a corollary of the moment inequality, if $\mathbb{E}(S_t^{p+1})$ is finite for some $p>0$ (recall that, S being a martingale, it is clearly finite for $p=0$), then $C(k,t)=\mathcal{O}(\mathrm{e}^{-pk})$ as k tends to infinity. The last technical result we shall need is the following lemma, describing the large-strike behaviour of Call prices:

Lemma 2.2.14. *Define the functions f_-,f_+ by $f_\pm(z):=z^{-1}+z/4\pm1$. For any $\alpha>0$, $\beta\in(0,2]$,*

$$\lim_{k\uparrow\infty}\frac{\mathrm{e}^{-\alpha k}}{\mathrm{BS}(k,V_\beta(k))}=\begin{cases}0, & \text{if } \alpha>f_-(\beta)/2,\\ +\infty, & \text{if } \alpha\leq f_-(\beta)/2.\end{cases}$$

Proof. The asymptotic $\mathcal{N}(-z)\sim\mathrm{e}^{-z^2/2}/(z\sqrt{2\pi})$ holds for the Gaussian cumulative distribution function as z tends to infinity, so that, for any $\beta\in(0,2]$, as k tends to infinity,

$$\begin{aligned}
\mathrm{BS}(k,V_\beta(k)) &= \mathcal{N}\left(-\sqrt{f_-(\beta)k}\right)-\mathrm{e}^k\mathcal{N}\left(-\sqrt{f_+(\beta)k}\right)\\
&\sim\frac{1}{\sqrt{2\pi}}\left(\frac{\exp\left(\frac{1}{2}kf_-(\beta)\right)}{\sqrt{f_-(\beta)k}}-\frac{\mathrm{e}^k\exp\left(\frac{1}{2}kf_-(\beta)\right)}{\sqrt{f_-(\beta)k}}\right)\\
&=\frac{\exp\left(\frac{1}{2}kf_-(\beta)\right)}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{f_-(\beta)k}}-\frac{1}{\sqrt{f_+(\beta)k}}\right)=\gamma\frac{\exp\left(\frac{1}{2}kf_-(\beta)\right)}{\sqrt{k}},
\end{aligned}$$

for some $\gamma>0$, where we used the identity $f_+(\beta)=f_-(\beta)+2$, and therefore

$$\lim_{k\uparrow\infty}\mathrm{BS}(k,V_\beta(k))=\lim_{k\uparrow\infty}k^{-1/2}\exp\left(\frac{1}{2}kf_-(\beta)\right).$$

The lemma then follows immediately. □

We can now move on to the proof of Theorem 2.2.12.

Proof of Theorem 2.2.12. We start with the right wing of the smile. Since $V(k,t) \in (0,2k)$ (for large k, by Lemma 2.2.13), clearly $\beta_R \in [0,2]$ and we only need to show that $p^* = f_-(\beta_R)/2$.

We first prove that $p^* \leq f_-(\beta_R)/2$. The function f_- is strictly decreasing from $(0,2)$ to $(0,\infty)$, and therefore it suffices to show that for any $\beta \in (0,2)$ such that $f_-(\beta)/2 < p^*$, we have $\beta_R \leq \beta$. Choose $p \in (f_-(\beta)/2, p^*)$; then the corollary of (2.2.5) yields

$$\frac{\mathrm{BS}(k, V(k,t))}{\mathrm{BS}(k, V_\beta(k))} = \frac{\mathcal{O}(\mathrm{e}^{-pk})}{\mathrm{BS}(k, V_\beta(k))},$$

which clearly tends to zero as k tends to infinity by Lemma 2.2.14. Therefore, there exists $k^* > 0$ such that for all $k > k^*$, $\mathrm{BS}(k, V(k,t)) \leq \mathrm{BS}(k, V_\beta(k))$, and hence $V(k,t) \leq V_\beta(k)$, so that β is an eventual upper bound and thus $\beta_R \leq \beta$.

We now prove that $p^* \geq f_-(\beta_R)/2$; it is clear that it is enough to show that $\mathbb{E}(S_t^{1+p})$ is finite for all $p \in (0, f_-(\beta)/2)$. Choose now β such that $\gamma := f_-(\beta)/2 \in (p, f_-(\beta_R)/2)$, so that, for k sufficiently large,

$$\frac{\mathrm{BS}(k, V(k,t))}{\mathrm{e}^{-\gamma k}} \leq \frac{\mathrm{BS}(k, V_\beta(k))}{\mathrm{e}^{-\gamma k}},$$

which tends to zero as k tends to infinity, and hence $C(k,t) \leq \mathrm{e}^{-\gamma k}$ for k large enough. Then, for any $K^* > 0$,

$$\mathbb{E}\left(S_t^{1+p}\right) = p(p+1)\int_0^\infty K^{p-1}C(K)\mathrm{d}K \leq p(p+1)\left[\int_0^{K^*} K^{p-1}C(K)\mathrm{d}K + \int_{K^*}^\infty K^{p-1-\gamma}\mathrm{d}K\right].$$

Now, it is easy to show that the two integrals on the right-hand side are finite:

$$\int_0^{K^*} K^{p-1}C(K)\mathrm{d}K \leq \int_0^{K^*} K^{p-1}\mathrm{d}K = \frac{(K^*)^p}{p} \qquad \text{and} \qquad \int_{K^*}^\infty K^{p-1-\gamma}\mathrm{d}K = \frac{(K^*)^{p-\gamma}}{\gamma - p},$$

which proves the statement.

The left wing of the smile can be proved by symmetry. Note first that, if $\mathbb{P}(S_t = 0) > 0$, then obviously $q^* = 0$ and hence $\beta_L = 2$. Suppose now that $\mathbb{P}(S_t = 0) = 0$. Then, since S is a true martingale, we can define a new probability measure $\mathbb{Q}$ via $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = S_t$ so that

$$\mathbb{E}^{\mathbb{P}}\left(\mathrm{e}^k - S_t\right)_+ = \mathbb{E}^{\mathbb{Q}}\left[S_t^{-1}\left(\mathrm{e}^k - S_t\right)_+\right] = \mathrm{e}^k\mathbb{E}^{\mathbb{Q}}\left(U_t - \mathrm{e}^{-k}\right)_+,$$

where $U_t := S_t^{-1}$. Therefore, the change of measure expresses the $k > 0$ Put price into a $-k$ Call price. Lee's left wing formula therefore follows immediately. □

Examples

Note that of course, p^* and q^* in general depend on the time t. The first example to look at is the Black-Scholes model, where the stock price follows—under the risk-neutral measure— $\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma\mathrm{d}W_t$, with $S_0 > 0$. We can then compute, for any $t \geq 0$,

$$\mathbb{E}(S_t^u) = S_0\mathbb{E}\exp\left(u\left(r - \frac{\sigma^2}{2}\right)t + u\sigma W_t\right) = S_0\exp\left(urt + \frac{\sigma^2 t}{2}u(u-1)\right),$$

which is well defined for any $u \in \mathbb{R}$. Therefore $p^* = \infty$ and $q^* = \infty$, and we deduce that $\beta_L = \beta_R = 0$, which is not surprising since the implied volatility in the Black-Scholes model is flat (i.e. does not depend on the strike).

Let us now have a look at some more advanced model, namely exponential Lévy processes. In the Kou model, the log-stock price process $X := \log(S)$ satisfies the following dynamics:

$$X_t = \gamma t + \sigma W_t + \sum_{n=1}^{N_t} Y_n,$$

where $\gamma \in \mathbb{R}$, $\sigma > 0$, W is a standard Brownian motion, N is a Poisson process with intensity $\lambda > 0$ and the $(Y_n)_n$ forms a family of independent random variables with common distribution

$$\mu(\mathrm{d}x) = p\lambda_+ \mathrm{e}^{-\lambda_+ x}\mathbb{1}_{\{x>0\}}\mathrm{d}x + (1-p)\lambda_- \mathrm{e}^{-\lambda_-|x|}\mathbb{1}_{\{x<0\}}\mathrm{d}x,$$

so that is S experiences both positive and negative jumps. Here we assume that λ_+ and λ_- are both strictly positive and $p \in [0,1]$. The constant γ is chosen so that the stock price process remains a true martingale. For any fixed $t \geq 0$, we can then compute

$$\mathbb{E}\left(\mathrm{e}^{uX_t}\right) = \exp\left(u\gamma t + \frac{\sigma^2 u^2}{2}t + \lambda t\left\{\mathbb{E}\left(\mathrm{e}^{uY_1}\right) - 1\right\}\right), \quad \text{with } \mathbb{E}\left(\mathrm{e}^{uY_1}\right) = p\frac{\lambda_+}{\lambda_+ - u} + (1-p)\frac{\lambda_-}{\lambda_- + u},$$

the latter expression being well defined only for $u \in (-\lambda_-, \lambda_+)$. Therefore $q^* = \lambda_-$ and $p^* = \lambda_+ - 1$. Note that since exponential Lévy processes have the property that there exists some function ϕ such that $\mathbb{E}\left(\mathrm{e}^{uX_t}\right) = \mathrm{e}^{\phi(u)t}$, the upper and lower moments p^* and q^* will never depend on t.

Example 2.2.15. See the IPython notebook for an example.

2.2.4 The SVI parameterisation

As mentioned previously, the implied volatility is defined as the unique non-negative solution of some highly non-linear equation. It is therefore not available in closed form in most models used in practice. In Sections 2.2.1, 2.2.2 and 2.2.3, we provided some general results on its behaviour, either in a fully model-free framework, or for some large classes of models. We consider here an alternative, not based on some stochastic dynamics, but on a given parameterisation. For a fixed maturity slice $T > 0$, consider the following parameterisation of the (square of) the implied volatility:

$$\sigma^2_{\mathrm{SVI}}(k) = a + b\left\{(k-m) + \rho\sqrt{(k-m)^2 + \xi^2}\right\}, \qquad \text{for all } k \in \mathbb{R}, \tag{2.2.6}$$

with $\rho \in [-1,1]$, $a, b, \xi \geq 0$, $m \in \mathbb{R}$.

Example 2.2.16. See the IPython notebook for an example.

Exercise 2.2.17. Using the results from this chapter, find necessary conditions on the parameters a, b, ρ, m, ξ ensuring absence of arbitrage.

Remark 2.2.18. As can be seen from the exercise, it is not at all easy—and virtually impossible—to find necessary and sufficient conditions on the SVI parameters ensuring that the resulting volatility surface is free of arbitrage. In [70], Gatheral and Jacquier determined sufficient and almost necessary conditions on an extended family of SVI-type surfaces (depending on both strike and maturity) preventing arbitrage.

Chapter 3

From SDEs to PDEs

3.1 Stochastic differential equations: existence and uniqueness

In this chapter, we shall investigate the existence and uniqueness of real-valued stochastic differential equations (SDEs). For a fixed time horizon $T > 0$, we define the functions $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \to \mathcal{M}_{n,m}(\mathbb{R})$ as the drift and diffusion coefficients (where m and n are two strictly positive integers). Unless otherwise stated, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ will denote a given filtered probability space, with the usual hypotheses, supporting a m-dimensional standard Brownian motion W. We will consider in this chapter the following equation, for any $t \in [0, T]$:

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s) \cdot \mathrm{d}W_s, \tag{3.1.1}$$

or, written in a differential form:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t) \cdot \mathrm{d}W_t, \quad X_0 \in \mathbb{R}^n. \tag{3.1.2}$$

Note that a solution to (3.1.1) or (3.1.2) depends on the smoothness of the coefficients and is relative to a given Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$.

Definition 3.1.1.

- A solution to (3.1.1) is an $(\mathcal{F}_t)$-adapted process such that b is locally integrable and σ is locally square integrable;
- the SDE (3.1.2) admits a weak solution if the latter depends on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and on the Brownian motion W;
- a strong solution to (3.1.2) does not depend on the choice of the driving Brownian motion.

In the weak sense, the Brownian motion and the probability space are part of the problem, whereas they are fixed and given in the strong framework. A strong solution is always a weak solution, but the converse does not hold in general. An example of this is the ‘Tanaka equation’ $\mathrm{d}X_t = \mathrm{sgn}(X_t)\mathrm{d}W_t$, starting at $X_0 = 0$. For a given $\mathbb{R}^n$-valued process X and any $0 \le t_0 \le t_1$, we shall denote by $p(t_0, t_1, \cdot, \cdot)$ its transition density, namely the unique non-negative function such that, for any continuous function h, the following holds:

$$\mathbb{E}\left[h(X_{t_1})|X_{t_0} = x\right] = \int_{\mathbb{R}^n} h(y)p(t_0, t_1; x, y)\mathrm{d}y, \qquad \text{for all } y \in \mathbb{R}^n.$$

Remark 3.1.2. We have assumed above that the starting point $X_0 = x_0$ of the SDE was fixed. This can be extended to the case where X_0 is an $\mathcal{F}_0$-measurable random variable. This is important, for instance, when considering forward-start options in stochastic volatility models.

Our definition of a diffusion follows Itô’s construction, i.e. as a (continuous) map of Brownian paths: $(X_t) = f(t, X_t, W_t)$. Following Rogers and Williams [134], though, the introduction of a Brownian motion is not necessary, and the following alternative definition is possible:

Definition 3.1.3. An $\mathbb{R}^n$-valued diffusion with drift b and covariance $a := \sigma^{\mathrm{T}}\sigma$ is a continuous semimartingale $X = (X_1, \ldots, X_n)$ (on some probability space satisfying the usual assumptions), such that $M_t^i := X_t^i - X_0^i - \int_0^t b^i(X_s)\mathrm{d}s$ is a continuous local martingale satisfying $[M^i, M^j]_t = \int_0^t a_{ij}(X_s)\mathrm{d}s$.

Of course, in the case where $(M_t)_{t\ge0}$ is a standard Brownian motion, this definition matches up with Definition 3.1.1. Let us state and prove the main result of this section, before looking at examples, counterexamples and curiosities.

Theorem 3.1.4. *Assume that the drift and diffusion coefficients are measurable and satisfy*

$$|b(t, x)| + |\sigma(t, x)| \le C_0(1 + |x|), \tag{3.1.3}$$

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \le C_1|x - y|, \tag{3.1.4}$$

for some constants $C_0, C_1 > 0$ *and all* $x \in \mathbb{R}^n$*,* $t \in [0, T]$*. If* X_0 *is independent of the* σ*-algebra* $\sigma(W_s, s \ge 0)$ *and* $\mathbb{E}(|X_0|^2)$ *is finite, then the stochastic differential equation* (3.1.2) *admits a unique continuous solution on* $[0, T]$*, adapted to the filtration* $(\mathcal{F}_t^{X_0})_{t\in[0,T]}$ *generated by* X_0 *and* (W_t)*, such that the expectation* $\mathbb{E}\left(\int_0^T |X_s|^2\mathrm{d}s\right)$ *is finite.*

Remark 3.1.5. The linear growth and Lipschitz assumptions on the drift and diffusions are natural. Consider indeed the ordinary differential equation $\mathrm{d}X_t = X_t^2\mathrm{d}t$, starting at $X_0 = 1$. The unique solution is $X_t = (1 - t)^{-1}$ for all $t \in [0, 1)$, but is not defined after the point $t = 1$. Note here that $b(x) \equiv x^2$ violates the linear growth condition.

Exercise 3.1.6 (Absence of uniqueness)**.** Consider the one-dimensional stochastic differential equation $\mathrm{d}X_t = 3X_t^{1/3}\mathrm{d}t + 3X_t^{2/3}\mathrm{d}W_t$, starting from $X_0 = 0$. Show that both W^3 and the null process are solutions. Compare this observation with the hypotheses of Theorem 3.1.4.

Exercise 3.1.7. Consider the one-dimensional stochastic differential equation $\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(X_t\mathrm{d}W_t)$, starting at $X_0 = x \in \mathbb{R}$.

1. If $b(t, x) \equiv \mu \in \mathbb{R}$ and $\sigma(x) \equiv \sigma > 0$, then $X_t = x + \mu t + \sigma W_t$;
2. If $b(t, x) \equiv \mu x$ and $\sigma(x) \equiv \sigma x$, then $X_t = x \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}$;
3. If $b(t, x) \equiv -\lambda x$ and $\sigma(x) \equiv \sigma > 0$, then $X_t = x\mathrm{e}^{-\lambda t} + \sigma \int_0^t \exp\left\{-\lambda(t-s)\right\}\mathrm{d}W_s$.

Compute the transition density $\mathbb{P}(X_t \in \mathrm{d}y|X_0 = x)$ of each of the above processes.

Remark 3.1.8. In Example 3.1.7 above, the first one corresponds to the standard arithmetic Brownian motion with drift (Bachelier model), the second one to the geometric Brownian motion (Black-Scholes model). the last one is called the Ornstein-Uhlenbeck model[1]. The model describes the velocity of a Brownian particle in a medium with friction. This is also the standard model for the dynamics of a spring.

Before proving Theorem 3.1.4, let us recall the following simple but useful lemma:

Lemma 3.1.9 (Gronwall[2])**.** *Let $\beta : [0, \infty) \to \mathbb{R}$ be a continuous, integrable, non-negative function, and $u, \alpha : [0, \infty) \to \mathbb{R}$ continuous functions such that $u(t) \le \alpha(t) + \int_0^t \beta(s)u(s)\mathrm{d}s$. Then*

$$u(t) \le \alpha(t) + \int_0^t u(s)\beta(s)\exp\left\{\int_s^t \beta(u)\mathrm{d}u\right\}\mathrm{d}s.$$

In particular, if α is non-decreasing, then

$$u(t) \le \alpha(t)\exp\left(\int_0^t \beta(s)\mathrm{d}s\right).$$

Proof. We only provide some hints. Consider the simple case where the functions α and β are constant. Define the functions v and w by $v(t) \equiv v(0) + \int_0^t u(s)\mathrm{d}s$, and $w(t) \equiv v(t)\exp(-\beta t)$; it is easy to show the inequality $v(t) \le \frac{\alpha}{\beta}\left(\mathrm{e}^{\beta t} - 1\right)$, from which the lemma follows. □

Proof of Theorem 3.1.4. Let us first prove uniqueness. To do so, consider two solutions of (3.1.1)

[1]Leonard Ornstein (1880-1941) and George Eugene Uhlenbeck (1900-1988) were Dutch physicists. They did not define the process via the stochastic differential equation though, since that very concept did not exist at the time.

[2]Thomas Hakon Grönwall (1877-1932) was a Swedish mathematician

(or (3.1.2)), $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$, starting from Y_0 and Z_0. We can compute, for any $t \geq 0$,

$$\begin{aligned}
\mathbb{E}\left(|Y_t - Z_t|^2\right) &= \mathbb{E}\left\{\left|Y_0 - Z_0 + \int_0^t [b(s,Y_s) - b(s,Z_s)]\,\mathrm{d}s + \int_0^t [\sigma(s,Y_s) - \sigma(s,Z_s)]\cdot \mathrm{d}W_s\right|^2\right\}\\
&\leq 3\mathbb{E}\left(|Y_0 - Z_0|^2\right) + 3\mathbb{E}\left[\left(\int_0^t [b(s,Y_s) - b(s,Z_s)]\,\mathrm{d}s\right)^2\right] + 3\mathbb{E}\left[\left(\int_0^t [\sigma(s,Y_s) - \sigma(s,Z_s)]\cdot \mathrm{d}W_s\right)^2\right]\\
&\leq 3\mathbb{E}\left(|Y_0 - Z_0|^2\right) + 3t\mathbb{E}\left(\int_0^t [b(s,Y_s) - b(s,Z_s)]^2\,\mathrm{d}s\right) + 3\mathbb{E}\left(\int_0^t [\sigma(s,Y_s) - \sigma(s,Z_s)]^2\cdot \mathrm{d}s\right)\\
&\leq 3\mathbb{E}\left(|Y_0 - Z_0|^2\right) + 3(1+t)C_1\mathbb{E}\left(\int_0^t (Y_s - Z_s)^2\,\mathrm{d}s\right),
\end{aligned}$$

where the second line follows from Cauchy-Schwartz, the third line from Itô's isometry and Cauchy-Schwartz, and the last one from the Lipschitz assumption (3.1.4). The function $u : t \mapsto \mathbb{E}|Y_t - Z_t|^2$ therefore satisfies the inequality $u(t) \leq \alpha + \beta \int_0^t u(s)\mathrm{d}s$, where $f \equiv 0$, $\alpha := 3\mathbb{E}\left(|Y_0 - Z_0|^2\right)$ and $\beta := 3(1+T)C_1$. Gronwall's Lemma 3.1.9 therefore yields the inequality

$$\mathbb{E}\left(|Y_t - Z_t|^2\right) \leq 3\mathbb{E}\left(|Y_0 - Z_0|^2\right)\exp\left\{3C_1(1+T)t\right\}.$$

When $Y_0 = Z_0$ almost surely, then $\mathbb{E}\left(|Y_t - Z_t|^2\right) = 0$ for all $t \in [0,T]$, and hence $Y_t = Z_t$ almost surely for all $t \in [0,T]$. Pathwise uniqueness then follows from the continuity of the map $t \mapsto |Y_t - Z_t|$.

To prove existence of the solution, we follow similar steps to the proof for standard ordinary differential equations, using Picard's iteration scheme: defining $Y_t^{(0)} := X_0$ almost surely, and then

$$Y_t^{(n+1)} := X_0 + \int_0^t b(s, Y_s^{(n)})\mathrm{d}s + \int_0^t \sigma(s, Y_s^{(n)})\cdot \mathrm{d}W_s, \tag{3.1.5}$$

almost surely for $n \geq 0, t \in [0,T]$, we can write, by a computation similar to the one above,

$$\mathbb{E}\left(\left|Y_t^{(n+1)} - Y_t^{(n)}\right|^2\right) \leq 3(1+T)D^2 \int_0^t \mathbb{E}\left(\left|Y_s^{(n+1)} - Y_s^{(n)}\right|^2\right)\mathrm{d}s$$

whenever $n \geq 1$, and $\mathbb{E}\left(|Y_t^{(1)} - Y_t^{(0)}|^2\right) \leq 2C^2t^2\left(1 + \mathbb{E}(|X_0|^2)\right) \leq Ct$. Induction therefore yields, for any $n \geq 0$.

$$\mathbb{E}\left(|Y_t^{(n+1)} - Y_t^{(n)}|^2\right) \leq \frac{C^{n+1}t^{n+1}}{(n+1)!}. \tag{3.1.6}$$

Now, we can write

$$\sup_{t\in[0,T]}\left|Y_t^{(n+1)} - Y_t^{(n)}\right| \leq \int_0^T \left|b(s, Y_s^{(n)}) - b(s, Y_s^{(n-1)})\right|\mathrm{d}s + \sup_{t\in[0,T]}\left|\int_0^t \left[\sigma(s, Y_s^{(n)}) - \sigma(s, Y_s^{(n-1)})\right]\mathrm{d}W_s\right|.$$

Doob's Martingale Inequality[3] together with Markov's inequality therefore imply

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]}\left|Y_t^{(n+1)}-Y_t^{(n)}\right|>2^{-n}\right)\\
&\leq \mathbb{P}\left(\left\{\int_0^T \left|b(s,Y_s^{(n)})-b(s,Y_s^{(n-1)})\right|\mathrm{d}s\right\}^2>2^{-2n-2}\right)\\
&\quad+\mathbb{P}\left(\sup_{t\in[0,T]}\left|\int_0^t\left[\sigma(s,Y_s^{(n)})-\sigma(s,Y_s^{(n-1)})\right]\mathrm{d}W_s\right|>2^{-2n-1}\right)\\
&\leq 2^{2(n+1)}T\int_0^T\mathbb{E}\left(\left|b(s,Y_s^{(n)})-b(s,Y_s^{(n-1)})\right|^2\right)\mathrm{d}s+2^{2(n+1)}\int_0^T\mathbb{E}\left(\left|\sigma(s,Y_s^{(n)})-\sigma(s,Y_s^{(n-1)})\right|^2\right)\mathrm{d}s\\
&\leq 2^{2(n+1)}C(1+T)\int_0^T\frac{c^n t^n}{n!}\mathrm{d}t\leq\frac{(4cT)^{n+1}}{(n+1)!}
\end{aligned}$$

if $c \geq C^2(1+T)$. Borel-Cantelli lemma therefore implies that

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left|Y_t^{(n+1)}-Y_t^{(n)}\right|>2^{-n}\text{ for infinitely many }n\right)=0,$$

which implies that the sequence $(Y_t^n(\omega))_{n\geq 0}$ is a uniformly Cauchy sequence, and therefore is uniformly convergent to, say $\widetilde{Y}$, for almost all $\omega\in\Omega$. The limit therefore exists and is continuous and adapted to $(\mathcal{F}_t^{X_0})_{t\in[0,T]}$. If λ is the Lebesgue measure on $[0,T]$, for any $m,n\geq 0$, we can then write

$$\begin{aligned}
\left\|Y_t^{(m)}-Y_t^{(n)}\right\|_{L^2(\lambda\times\mathbb{P})}&=\left\|\sum_{k=n}^{m-1}\left(Y_t^{(k+1)}-Y_t^{(k)}\right)\right\|_{L^2(\lambda\times\mathbb{P})}\leq\sum_{k=n}^{m-1}\left\|\left(Y_t^{(k+1)}-Y_t^{(k)}\right)\right\|_{L^2(\lambda\times\mathbb{P})}\\
&=\sum_{k=n}^{m-1}\left[\mathbb{E}\left(\int_0^T\left|Y_s^{(k+1)}-Y_s^{(k)}\right|^2\mathrm{d}s\right)\right]^{1/2}\\
&\leq\sum_{k=n}^{m-1}\left[\left(\int_0^T\frac{C^{k+1}s^{k+1}}{(k+1)!}\mathrm{d}s\right)\right]^{1/2}=\sum_{k=n}^{m-1}\left(\frac{C^{k+1}T^{k+2}}{(k+2)!}\right)^{1/2},
\end{aligned}$$

which clearly tends to zero as m and n tend to infinity, so that $(Y^{(n)})_{n\geq 0}$ is a Cauchy sequence in $L^2(\lambda\times\mathbb{P})$, and therefore a convergent sequence, the limit of which (in $L^2(\lambda\times\mathbb{P})$) is adapted to $\mathcal{F}^{X_0}$, and we denote it Y. Being convergent in L^2 implies almost everywhere (ω-pointwise) convergence along a subsequence, and hence $\widetilde{Y}=Y$. Let us now prove that Y satisfies (3.1.2). In (3.1.5), Hölder's inequality implies that $\int_0^t b(s,Y_s^{(n)})\mathrm{d}s$ converges to $\int_0^t b(s,Y_s)\mathrm{d}s$ in $L^2(\mathbb{P})$ as n tends to infinity, and Itô's isometry yields the convergence of $\int_0^t\sigma(s,Y_s^{(n)})\cdot\mathrm{d}W_s$ to $\int_0^t\sigma(s,Y_s)\cdot\mathrm{d}W_s$, also in $L^2(\mathbb{P})$, which concludes the proof. □

Exercise 3.1.10. Consider the stochastic differential equation $\mathrm{d}X_t=\kappa X_t(\theta-\log(X_t))\mathrm{d}t+\sigma X_t\mathrm{d}W_t$, starting at $X_0>0$, where W is a standard Brownian motion.

[3] The following result is due to J.L. Doob [45]: if M is a martingale with continuous paths, then,

$$\mathbb{P}\left(\sup_{t\in[0,T]}|M_t|\geq\lambda\right)\leq\lambda^{-p}\mathbb{E}\left(|M_T|^p\right),\qquad\text{for all }p\geq 1,T\geq 0,\lambda>0.$$

1. Show that the SDE admits a unique strong solution.

2. Prove that the solution reads

$$X_t = \exp\left\{\mathrm{e}^{-\kappa t}\log(x) + \left(\theta - \frac{\sigma^2}{2\kappa}\right)\left(1-\mathrm{e}^{\kappa t}\right) + \sigma\mathrm{e}^{-\kappa t}\int_0^t \mathrm{e}^{\kappa s}\mathrm{d}W_s\right\}.$$

3. Compute the expectation $\mathbb{E}(X_t)$.

Exercise 3.1.11 (Removing stochasticity – importance of the assumptions)**.**

1. Assume that the diffusion coefficient σ is null everywhere, and consider the ordinary differential equation $\mathrm{d}X_t = |X_t|^\alpha \mathrm{d}t$, starting from $X_0 = 0$.

   - If $\alpha \geq 1$, the equation has a unique solution. What is it?
   - However, when $\alpha \in (0,1)$, check that the family $(X_t^{(s)})_{t\in[0,T]}$ defined by

   $$X_t^{(s)} := \begin{cases} 0, & \text{if } t \in [0,s],\\ \left(\dfrac{t-s}{1-\alpha}\right)^{1/(1-\alpha)}, & \text{if } t \in [s,T],\end{cases}$$

   satisfies the equation. What do you conclude?

2. Consider the equation $X_t = x + \int_0^t X_s^2 \mathrm{d}s$ for some $x \neq 0$. Show that $X_t := x/(1-xt)$ is a solution (actually the only one). What happens at t approaches x^{-1}?

3.1.1 Properties of solutions of SDEs

We state the following result about estimates of moments of solutions to stochastic differential equations without proof, but refer the interested reader to [99, Theorem 4.5.4]. These types of results are fundamental in the construction of accurate simulation schemes (Euler or else) for SDEs.

3.1.2 Moment estimates

Proposition 3.1.12. *Consider the stochastic differential equation* (3.1.2)*, and the assumptions of Theorem 3.1.4. Assume further that* $\mathbb{E}(|X_0|^{2p})$ *is finite for some* $p \geq 1$*. Then there exists a strictly positive constant* C*, depending on* p, T, K *such that, for any* $0 \leq t_0 \leq t_1 \leq T$*,*

$$\mathbb{E}\left(\sup_{t_0\leq t\leq t_1} |X_t|^{2p}\right) \leq C\left(1+\mathbb{E}\left(|X_{t_0}|^{2p}\right)\right)\mathrm{e}^{C(t_1-t_0)}, \tag{3.1.7}$$

$$\mathbb{E}\left(\sup_{t_0\leq t\leq t_1} |X_t - X_{t_0}|^{2p}\right) \leq C\left(1+\mathbb{E}\left(|X_{t_0}|^{2p}\right)\right)(t_1-t_0)^{2p}. \tag{3.1.8}$$

3.1.3 Itô diffusions and the Markov property

Definition 3.1.13. An Itô diffusion is a stochastic process $X_{\cdot}(\omega)$ satisfying

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t) \cdot \mathrm{d}W_t, \qquad X_0 = x \in \mathbb{R}^n, \tag{3.1.9}$$

where the drift $b : \mathbb{R}^n \to \mathbb{R}^n$ and diffusion $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ coefficients are Lipschitz continuous, i.e. there exists $C > 0$ such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C|x - y|, \qquad \text{for any } x, y \in \mathbb{R}^n, t \geq 0,$$

and W is an $\mathbb{R}^m$-valued standard Brownian motion. The diffusion is time-homogeneous if the coefficients b and σ do not depend on time.

Remark 3.1.14. An Itô diffusion is in general not a martingale. However, as soon as the drift $b(\cdot)$ is null everywhere (almost surely), it is a local martingale. Indeed, suppose that $b(\cdot)$ is not everywhere null, and that X is a local martingale. Then, clearly, the process

$$\int_0^t b(u, X_u)\mathrm{d}u = X_t - X_0 - \int_0^t \sigma(u, X_u) \cdot \mathrm{d}W_u$$

is also a local martingale. Being a Lebesgue integral, however, it has bounded variation on any compact, which contradicts its local martingale behaviour.

Denote by $(X_t^{s,x})_{t \geq s}$, the solution to (3.1.9) starting at $X_s = x$ at some time $s \geq 0$. The time homogeneity property precisely means that, for any $t \geq 0$, the processes $(X_{t+h}^{t,x})$ and $(X_h^{0,x})_{h \geq 0}$ have the same law. The following theorem, the proof of which we shall skip, is fundamental:

Theorem 3.1.15 (Markov property for Itô diffusions)**.** *Let X be a time-homogeneous diffusion and $f : \mathbb{R}^n \to \mathbb{R}$ a Borel bounded function. Then, for any $s, t \geq 0$,*

$$\mathbb{E}\left(f(X_{t+s})|\mathcal{F}_t\right) = \mathbb{E}\left(f(X_{t+s})|X_t\right) \quad \textit{almost surely.}$$

Remark 3.1.16. Computing both sides for every bounded Borel function f is in general not very tractable. One can however use Dynkyn's lemma, which implies that it is enough to check the equality $\mathbb{E}(X_{t+s} \in A|\mathcal{F}_t) = \mathbb{E}(X_{t+s} \in A|X_t)$, for all $s, t \geq 0$ for every Borel set A.

Example 3.1.17. The Brownian motion has the Markov property.

The following tool is a key concept in the study of Itô diffusions, and will allow us to bridge the gap between stochastic differential equations and partial differential equations. In particular, as we shall see, it provides an alternative—and efficient—path to (numerically) compute expectations.

Definition 3.1.18. Let $(X_t)_{t \geq 0}$ be an Itô diffusion on $\mathbb{R}^n$. The infinitesimal generator $\mathcal{A}$ of X is the linear operator defined as

$$(\mathcal{A}f)(t, x) := \lim_{t \downarrow 0} \frac{\mathbb{E}(f(X_t)|X_0 = x) - f(x)}{t}.$$

We shall further denote $\mathcal{D}_{\mathcal{A}}$ the space of functions such that the limit exists for all $x \in \mathbb{R}^n$.

The following theorem characterises completely the infinitesimal generator of an Itô diffusion, and its proof is left as an exercise using Itô's formula.

Theorem 3.1.19. *Let X be an Itô diffusion as in Definition 3.1.13. Then $\mathcal{C}_c^2(\mathbb{R}^n) \subset \mathcal{D}_{\mathcal{A}}$ and*[4]

$$\mathcal{A}f = \langle b, \nabla f\rangle + \frac{1}{2}\langle \sigma, \sigma^\top \Delta f\rangle, \qquad \text{for any } f \in \mathcal{C}_c^2(\mathbb{R}^n).$$

Pointwise in $\mathbb{R}^n$, this can also be written, for any $f \in \mathcal{C}_c^2(\mathbb{R}^n), x \in \mathbb{R}^n, t \geq 0$, as

$$\mathcal{A}f(t,x) = \sum_{i=1}^n b(t,x_i)\partial_{x_i} f(t,x) + \frac{1}{2}\sum_{i,j=1}^n (\sigma(t,x)\sigma(t,x)^\top)_{i,j}\partial_{x_i,x_j} f(t,x).$$

Proof. Let us consider, for notational simplicity, the case where the function f does not depend on time. For $f \in \mathcal{C}_c^2(\mathbb{R}^n)$, Itô's formula yields

$$\begin{aligned}
\mathrm{d}f(X_t) &= \langle \mathrm{d}X_t, \nabla f(X_t)\rangle + \frac{1}{2}(\mathrm{d}X_t)^\top \cdot \Delta f(X_t) \cdot \mathrm{d}X_t \\
&= \langle b(X_t), \nabla f(X_t)\rangle\,\mathrm{d}t + \langle \sigma(X_t)\mathrm{d}W_t, \nabla f(X_t)\rangle\,\mathrm{d}t + \frac{1}{2}(\mathrm{d}X_t)^\top \cdot \Delta f(X_t) \cdot \mathrm{d}X_t \\
&= \sum_{i=1}^n \partial_{x_i} f(X_t)\mathrm{d}X_t^{(i)}\mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^n \partial_{x_i,x_j} f(X_t)\mathrm{d}X_s^{(i)}\mathrm{d}X_t^{(j)} \\
&= \sum_{i=1}^n \partial_{x_i} f(X_t)b_i(X_t)\mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^n \partial_{x_i,x_j} f(X_t)(\sigma(X_t)\mathrm{d}W_t)_i(\sigma(X_t)\mathrm{d}W_t)_j + \sum_{i=1}^n \partial_{x_i} f(X_t)(\sigma(X_t)\mathrm{d}W_t)_i \\
&= \sum_{i=1}^n \partial_{x_i} f(X_t)b_i(X_t)\mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^n (\sigma(X_t)\sigma(X_t)^\top)_{i,j}\partial_{x_i,x_j} f(X_t)\mathrm{d}t + \sum_{i,k=1}^n \partial_{x_i} f(X_t)\sigma_{ik}(X_t)\mathrm{d}W_t^{(k)}
\end{aligned}$$

Integrating both sides between 0 and t, and taking expectations yield

$$\mathbb{E}^x[f(X_t)] - f(x) = \mathbb{E}^x \int_0^t \left(\sum_{i=1}^n \partial_{x_i} f(X_s)b_i(X_s) + \frac{1}{2}\sum_{i,j=1}^n (\sigma(X_s)\sigma(X_s)^\top)_{i,j}\partial_{x_i,x_j} f(X_s)\right)\mathrm{d}s,$$

and the lemma follows. □

Exercise 3.1.20. Write the infinitesimal generator of the following Itô diffusions:

1. Black-Scholes model: $\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma X_t\mathrm{d}W_t$;
2. the n-dimensional Brownian motion (it is half of the Laplace operator on $\mathbb{R}^n$);
3. Ornstein-Uhlenbeck: $\mathrm{d}X_t = \kappa X_t\mathrm{d}t + \sigma\mathrm{d}W_t$;
4. $\mathrm{d}(X_t^1, X_t^2)^\top = (1, X_t^2)^\top\mathrm{d}t + (0, \mathrm{e}^{X_t^1})^\top\mathrm{d}W_t$, where W is a one-dimensional Brownian motion;

If τ is a stopping time with finite expectation, adapted to the filtration of the Brownian motion, then, by a localisation argument, the computation in the proof of Theorem 3.1.19 holds, and Dynkyn's formula [48] reads:

[4]Here, $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^n$, so that for any vector $\mathrm{a}, \mathrm{b} \in \mathbb{R}^n$, we have $\langle \mathrm{a}, \mathrm{b}\rangle = \mathrm{b}^\top\mathrm{a} = \sum_{i=1}^n a_i b_i$.

Theorem 3.1.21. *Let $f \in \mathcal{C}_0^2(\mathbb{R}^n)$ and X an $\mathbb{R}^n$-valued Itô diffusion. For any stopping time τ such that $\mathbb{E}(\tau|X_0)$ is finite,*

$$\mathbb{E}^x[f(\tau, X_\tau)] = f(x) + \mathbb{E}^x\left(\int_0^\tau \mathcal{A}f(s, X_s)\mathrm{d}s\right).$$

3.1.4 More results in the one-dimensional case

Existence and strong uniqueness of stochastic differential equations usually hold (see Theorem 3.1.4) under Lipschitz continuity of the coefficients. In the one-dimensional case, these assumptions can be relaxed. One underlying motivation is the study of the following two processes:

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \xi\sqrt{X_t}\mathrm{d}W_t, \qquad \text{CIR Process}, \tag{3.1.10}$$

$$\mathrm{d}X_t = \mu X_t\mathrm{d}t + \xi X_t^\alpha \mathrm{d}W_t, \qquad \text{CEV Process}. \tag{3.1.11}$$

The first one was originally proposed by Cox-Ingersoll-Ross model [37] describing the evolution of the short rate, and has been later implemented–and widely used since–for stochastic volatility modelling by Heston [81]. The second model is the Constant Elasticity of Variance, proposed by Cox [35], has been used for equity and commodities modelling, and is able to capture the leverage effect between the stock price and its instantaneous volatility. Existence of these processes rely on the notion of weak solutions, and we shall not delve into this here. The main uniqueness results, however, is the following:

Theorem 3.1.22 (Yamada-Watanabe [155])**.** *Let the two functions $b, \sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfy*

$$|b(t,x) - b(t,y)| \le K|x-y| \qquad \text{and} \qquad |\sigma(t,x) - \sigma(t,y)| \le h(|x-y|),$$

for all $t \ge 0$, $x, y \in \mathbb{R}$, where K is a strictly positive constant and $h : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function satisfying

$$h(0) = 0 \qquad \text{and} \qquad \int_{(0,\varepsilon)} \frac{\mathrm{d}u}{h(u)^2} = \infty,$$

for any $\varepsilon > 0$. Then the one-dimensional real stochastic differential equation $X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s$ admits a strong unique solution.

Example 3.1.23. Any function of the form $h(u) \equiv u^\alpha$ with $\alpha \ge 1/2$ satisfies the conditions. In particular, taking $h(u) \equiv \sqrt{u}$ ensures that the CIR process in (3.1.10) admits a unique strong solution.

Example 3.1.24. In Exercise 3.1.11, we saw that the deterministic ODE $\dot{X}_t = |X_t|^\alpha$ admitted an infinite number of solutions when $\alpha \in (0,1)$. In light of Theorem 3.1.22, however, the addition of a Brownian noise regularises the equation and ensures uniqueness of the solution, at least in the case $\alpha \in [1/2, 1)$.

The drift and diffusion coefficients of a (one-dimensional) stochastic differential equation have a physical meaning. However, it is not a priori obvious how to estimate the behaviour of the solution depending on them. The following result is a first step in this direction, and in particular allows to compare the solutions to two different SDEs.

Theorem 3.1.25. *Let $\mu_1, \mu_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous functions, and $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfy the assumptions of Theorem 3.1.22. Let X and Y be the (adapted) solutions to*

$$X_t = X_0 + \int_0^t \mu_1(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s,$$
$$Y_t = Y_0 + \int_0^t \mu_2(s, Y_s)\mathrm{d}s + \int_0^t \sigma(s, Y_s)\mathrm{d}W_s.$$

If $\mu_1(\cdot) \leq \mu_2(\cdot)$ and $X_0 \leq Y_0$ almost surely, then $X \leq Y$ almost surely.

3.2 The PDE counterpart

3.2.1 From Kolmogorov to Feynman-Kać; from SDEs to PDEs

In this section, the process $(X_t)_{t\geq 0}$ shall denote an Itô diffusion, in the sense of Definition 3.1.13, whose infinitesimal generator is given by Theorem 3.1.19, with domain $\mathcal{D}_{\mathcal{A}}$.

Theorem 3.2.1 (Kolmogorov's backward equation)**.** *Let f belong to $\mathcal{C}_0^2(\mathbb{R}^n)$. For any $t > 0$, the function $u(t, x) := \mathbb{E}(f(X_t)|X_0 = x)$ ($x \in \mathbb{R}^n$) belongs to $\mathcal{D}_{\mathcal{A}}$, and satisfies the equation*

$$\partial_t u(t, x) = (\mathcal{A}u)(t, x), \qquad \text{for any } (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

with boundary condition $u(0, x) \equiv f(x)$. Conversely, any $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ solution of the equation (together with the boundary condition) is equal to u.

Remark 3.2.2. A fundamental corollary is that, under the assumptions of the theorem, the process $(f(t, X_t))_{t\geq 0}$ is a local martingale if and only if the function f satisfies the backward Kolmogorov equation. The proof of this fact simply follows by applying Itô's Lemma to the function f.

Proof. Note that, since $f \in \mathcal{C}_0^2(\mathbb{R}^n)$, the function $u(\cdot, x)$ is differentiable for any x by Dynkyn's formula (Theorem 3.1.21). Let ψ be the function defined as $\psi(x) \equiv u(t, x)$ (for some fixed $t > 0$).

Since the map $t \mapsto u(t,\cdot)$ is differentiable, we can write, for any $t>0$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
(\mathcal{A}u)(t,x) = (\mathcal{A}\psi)(x) &:= \lim_{s\downarrow 0} \frac{\mathbb{E}^x[\psi(X_s)] - \psi(x)}{s} = \lim_{s\downarrow 0} \frac{\mathbb{E}^x[u(t,X_s)] - u(t,x)}{s} \\
&= \lim_{s\downarrow 0} \frac{\mathbb{E}^x\left[\mathbb{E}^{X_s}\left[f(X_t)\right]\right] - \mathbb{E}^x\left[f(X_t)\right]}{s}, \text{ by the law of iterated expectations} \\
&= \lim_{s\downarrow 0} \frac{\mathbb{E}^x\left[\mathbb{E}^{X_s}\left[f(X_t)\right] - f(X_t)\right]}{s} \\
&= \lim_{s\downarrow 0} \frac{\mathbb{E}^x\left[\mathbb{E}^x\left[f(X_{t+s})|\mathcal{F}_s\right] - f(X_t)\right]}{s} \\
&= \lim_{s\downarrow 0} \frac{\mathbb{E}^x\left[f(X_{t+s}) - f(X_t)\right]}{s} = \lim_{s\downarrow 0} \frac{u(t+s,x) - u(t,x)}{s} = \partial_t u(t,x).
\end{aligned}$$

In order to prove uniqueness, assume that there exists a function v satisfying $\widetilde{\mathcal{A}}v := (-\partial_t + \mathcal{A})v \equiv 0$, together with the appropriate boundary conditions. For any fixed $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, define then the extended process $\widetilde{X}$ pathwise by $\widetilde{X}_t := (s-t, X_t^{0,x})$ Since its generator is $\widetilde{\mathcal{A}}$, we can apply Dynkyn's formula (Theorem 3.1.21):

$$\mathbb{E}\left(v(\widetilde{X}_{t\wedge\tau})|\widetilde{X}_s = x\right) = v(s,x) + \mathbb{E}\left(\int_0^{t\wedge\tau} (\widetilde{\mathcal{A}}v)(\widetilde{X}_r)\mathrm{d}r|\widetilde{X}_s = x\right) = v(s,x),$$

since $\widetilde{\mathcal{A}}v \equiv 0$, and where τ represents the first exit time (and hence is a stopping time) from a large ball. Letting the radius of the ball tend to infinity, we obtain $v(s,x) = \mathbb{E}\left(v(\widetilde{X}_t)|\widetilde{X}_s = x\right)$, and hence

$$v(s,x) = \mathbb{E}\left(v(\widetilde{X}_s)|\widetilde{X}_s = x\right) = \mathbb{E}\left(v(0, X_s^{0,x})\right) = \mathbb{E}\left(f(X_s^{0,x})\right) = \mathbb{E}\left(f(X_s)|X_0 = x\right).$$

□

Remark 3.2.3. By reversing time, setting $\tau := T - t$, for some time horizon T, and $v(\tau,\cdot) \equiv u(t,\cdot)$, then $v(\tau,x) = \mathbb{E}(f(X_{T-\tau})|X_0 = x)$, and the Kolmogorov backward equation can be rewritten as

$$(\partial_\tau + \mathcal{A})\, v(\tau,x) = 0, \qquad \text{for all } (\tau,x) \in [0,T) \times \mathbb{R}^n,$$

with boundary condition $v(T,x) \equiv f(x)$, since the operator $\mathcal{A}$ does not act on the time variable.

Remark 3.2.4. The Kolmogorov backward equation in Theorem 3.2.1 can be slightly generalised as follows: if f belongs to $\mathcal{C}_0^2(\mathbb{R}^n)$ and $r : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function bounded below, then the function

$$v(t,x) := \mathbb{E}^x\left[\exp\left(-\int_0^t r(X_s)\mathrm{d}s\right) f(X_t)\right]$$

satisfies the equation

$$\partial_t v(t,x) = (\mathcal{A}v)(t,x) - r(x)v(t,x), \qquad \text{for all } (t,x) \in (0,\infty)\times \in \mathbb{R}^n,$$

with boundary condition $v(0,x) = f(x)$, for all $x \in \mathbb{R}^n$. And the converse holds as well.

Exercise 3.2.5. Consider the one-dimensional drifted Brownian motion $\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$, starting at $X_0 = x$, with $\mu \in \mathbb{R}$ and $\sigma > 0$. Denote its transition density from time zero to time t by $p(t; x, \cdot)$. From Exercise 3.1.7, we know that

$$p(t; x, y) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{[y - (x + \mu t)]^2}{2\sigma^2 t}\right), \qquad \text{for any } (t, x, y) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}.$$

Prove by differentiation that it satisfies the ordinary differential equation

$$\partial_t p(t; x, y) = \left(\mu\partial_x + \frac{1}{2}\sigma^2 \partial_{xx} p\right), \qquad \text{for any } (t, x, y) \in (0, \infty) \times \mathbb{R} \times \mathbb{R},$$

with boundary condition $p(0; \cdot, y) = \delta_y(\cdot)$. Note that the space derivatives are taken with respect to x (the backward variable), and not with respect to y.

The backward equation above admits a dual version, called the Kolmogorov forward equation. Recall that, if $\mathcal{A}$ is some (differential) operator, then its adjoint $\mathcal{A}^*$ is the unique operator satisfying

$$\langle \mathcal{A}f, g\rangle = \langle f, \mathcal{A}^* g\rangle, \qquad \text{for all } f \in \mathcal{C}_0^2, g \in \mathcal{C}^2.$$

In particular, if $\mathcal{A}$ is of the same form (say in one dimension) as in Theorem 3.1.19, then straightforward computations yield

$$(\mathcal{A}^* f)(y) = -\sum_{i=1}^n \partial_{y_i}(b_i f)(y) + \frac{1}{2}\sum_{i,j=1}^n \partial_{y_i, y_j}\left((\sigma\sigma^\top)_{i,j} f\right)(y). \qquad \text{for any } f \in C^2(\mathbb{R}^n), y \in \mathbb{R}^n.$$

Theorem 3.2.6 (Kolmogorov's forward equation)**.** *Let X be an $\mathbb{R}^n$-valued Itô process with infinitesimal generator $\mathcal{A}$. If, for any $t \geq 0$, the transition measure of X_t admits a density p_t, i.e. $\mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^n} f(y)p_t(x, y)\mathrm{d}y$, for any $f \in \mathcal{C}_0^2$, such that $y \mapsto p_t(x, y)$ is smooth. Then it satisfies the Kolmogorov forward equation*

$$\partial_t p_t(x, y) = \mathcal{A}^* p_t(x, y), \qquad \textit{for all } t > 0, x, y \in \mathbb{R}^n,$$

with boundary condition $p_0(x, y) \equiv \delta_x(y)$, where $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$.

3.2.2 Parabolic PDEs: existence, uniqueness and properties

See Guyon-Labordère: Lions' argument about parabolic PDEs

3.2.3 The PDE approach to path-dependent options

Asian options: chasing the average

Feynman-Kać's Theorem is clearly suitable to deriving a partial differential equation for the value function corresponding to a European option, with underlying following an Itô diffusion. Suppose that one wishes to evaluate the price of an Asian option, with payoff, at maturity

$$\Pi_T := \left(\frac{1}{T}\int_0^T S_t \mathrm{d}t - K\right)_+.$$

There is no closed-form expression in general for the price of this option, and it is not clear how Feynman-Kać applies here. Consider the Black-Scholes model $\mathrm{d}S_t = S_t\left(r\mathrm{d}t + \sigma \mathrm{d}W_t\right)$. Under the risk-neutral measure, the price of the Asian option, at inception, is worth $\mathbb{E}\left(\mathrm{e}^{-rT}\Pi_T\right)$. Let now $(Y_t)_{t\geq 0}$ denote the integrated stock price process $Y_t := \int_0^t S_u \mathrm{d}u$, which clearly satisfies the stochastic differential equation

$$\mathrm{d}Y_t = S_t \mathrm{d}t, \quad Y_0 = 0.$$

Now, the pair $(S_t, Y_t)_{t\geq 0}$ is a two-dimensional Markov process satisfying Itô SDEs, so that Feynman-Kać (with a bit of additional work) implies the following result:

Theorem 3.2.7. *The function*

$$u(t,x,y) := \mathbb{E}\left[\mathrm{e}^{-r(T-t)}\left(\frac{Y_T}{T} - K\right)_+ |\mathcal{F}_t\right].$$

satisfies the partial differential equation

$$\left(\partial_t + rx\partial_x + x\partial_y + \frac{1}{2}\sigma^2 x^2 \partial_{xx}\right) u(t,x,y) = ru(t,x,y), \qquad \text{for all } t \in [0,T), x \geq 0, y \in \mathbb{R},$$

with boundary conditions

$$\begin{aligned} u(t,0,y) &= \mathrm{e}^{-r(T-t)}\left(\frac{y}{T} - K\right)_+, & t \in [0,T), y \in \mathbb{R}, \\ \lim_{y \downarrow -\infty} u(t,x,y) &= 0, & t \in [0,T), x \geq 0, \\ u(T,x,y) &= \left(\frac{y}{T} - K\right)_+, & x \geq 0, y \in \mathbb{R}. \end{aligned}$$

3.3 Overture to non-linear PDEs and control theory

Let $T > 0$ be a fixed time horizon, and consider here the stochastic differential equation

$$\mathrm{d}X_t^\alpha = b(X_t^\alpha, \alpha_t)\mathrm{d}t + \sigma(X_t^\alpha, \alpha_t)\mathrm{d}W_t, \qquad X_0^\alpha = x \in \mathbb{R}^n, \tag{3.3.1}$$

where $(\alpha_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$-adapted control process valued in some domain $A \subset \mathbb{R}^m$, and W a d-dimensional standard Brownian motion. We shall assume that the coefficients $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathcal{M}_{n,d}(\mathbb{R})$ are uniformly Lipschitz, i.e. there exists a constant $C > 0$ such that

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq C|x-y|, \qquad \text{for all } x, y \in \mathbb{R}^n.a \in A.$$

We now define the set $\mathcal{A}$ of control processes as

$$\mathcal{A} := \left\{\alpha : (\mathcal{F}_t) - \text{adapted, such that } \mathbb{E}\left[\int_0^T \left(|b(0,\alpha_t)|^2 + |\sigma(0,\alpha_t)|^2\right)\mathrm{d}t\right] < \infty\right\}.$$

Restricting the controls in $\mathcal{A}$ ensures, together with the uniform Lipschitz assumption, that the controlled SDE (3.3.1) admits a unique strong solution. Let us now consider two functions f :

$[0,T] \times \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ such that either g is bounded below or there exists $C > 0$ for which $|g(x) \leq C(1+|x|^2)$ for all $x \in \mathbb{R}^n$; We finally introduce our gain/cost function

$$J(t,x,\alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{t,s}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right].$$

Our objective is finally to determine the associated value function

$$u(t,x) := \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x,\alpha), \qquad \text{for any } t \in [0,T], x \in \mathbb{R}^n, \tag{3.3.2}$$

where $\mathcal{A}(t,x)$ denotes the set of admissible controls:

$$\mathcal{A}(t,x) := \left\{\alpha \in \mathcal{A} : \mathbb{E}\int_t^T |f(s, X_s^{t,x}, \alpha_s)|\mathrm{d}s < \infty\right\}.$$

3.3.1 Bellman's principle and dynamic programming

Let $\mathcal{T}_{t,T}$ denote the set of stopping times valued in the closed interval $[t,T]$. The following result, known as dynamic programming principle, is key tool in (stochastic) control theory.

Theorem 3.3.1. *For any $(t,x) \in [0,T] \times \mathbb{R}^n$, the value function* (3.3.2) *satisfies*

$$\begin{aligned} u(t,x) &= \sup_{\alpha \in \mathcal{A}(t,x)} \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + u(\tau, X_\tau^{t,x})\right] \\ &= \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + u(\tau, X_\tau^{t,x})\right]. \end{aligned}$$

Proof. The proof consists of squeezing the value function between the sup sup and the sup inf terms. We first start with the upper bound. Fix an admissible control $\alpha \in \mathcal{A}(t,x)$. The Markovian stucture of the stochastic differential equation (3.3.1) implies that, for any stopping time $\tau \in \mathcal{T}_{t,T}$, the equality $X_s^{t,x} = X_s^{\tau, X_\tau^{t,x}}$ holds almost surely for all $s \geq \tau$. Applying the tower property for expectations, conditioning on $\mathcal{F}_\tau$, we obtain

$$\begin{aligned} J(t,x,\alpha) &= \mathbb{E}\left\{\mathbb{E}\left(\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\middle| \mathcal{F}_\tau\right)\right\} \\ &= \mathbb{E}\left\{\int_t^\tau f\left(s, X_s^{t,s}, \alpha_s\right)\mathrm{d}s + \mathbb{E}\left(\int_\tau^T f(s, X_s^{\tau, X_\tau^{t,x}}, \alpha_s)\mathrm{d}s + g\left(X_T^{\tau, X_\tau^{t,x}}\right)\middle| \mathcal{F}_\tau\right)\right\} \\ &= \mathbb{E}\left\{\int_t^\tau f\left(s, X_s^{t,s}, \alpha_s\right)\mathrm{d}s + J(\tau, X_\tau^{t,x}, \alpha)\right\}. \end{aligned}$$

By definition of the value function as a supremum over the control, $u(\cdot,\cdot) \geq J(\cdot,\cdot,\alpha)$, and, since τ is chosen arbitrarily in $\mathcal{T}_{t,T}$, we can write

$$\begin{aligned} J(t,x,\alpha) &\leq \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + u(\tau, X_\tau^{t,x})\right] \\ &\leq \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + u(\tau, X_\tau^{t,x})\right], \end{aligned}$$

and the upper bound in the second line of the theorem follows. We now prove the lower bound: fix some control $\alpha \in \mathcal{A}(t,x)$ and a stopping time $\tau \in \mathcal{T}_{t,T}$. The definition of the value function implies that for any $\varepsilon > 0$ and any $\omega \in \Omega$, there exists a control $\alpha(\varepsilon,\omega) \in \mathcal{A}(\tau(\omega), X^{t,x}_{\tau(\omega)}(\omega))$:

$$u(\tau, X^{t,x}_{\tau(\omega)}(\omega)) - \varepsilon \leq J\left(\tau(\omega), X^{t,x}_{\tau(\omega)}(\omega), \alpha^{\varepsilon,\omega}\right).$$

It can be shown that the process $\widetilde{\alpha}(\omega)$, defined as

$$\widetilde{\alpha}_s(\omega) := \alpha_s(\omega)\mathbb{1}_{\{s\in[0,\tau(\omega)]\}} + \alpha^{\varepsilon,\omega}_s(\omega)\mathbb{1}_{\{s\in[\tau(\omega),T]\}}$$

is progressively measurable[5] and belongs to $\mathcal{A}(t,x)$. Therefore

$$u(t,x) \geq J(t,x,\widetilde{\alpha}) = \mathbb{E}\left[\int_t^\tau f(s,X^{t,x}_s,\alpha_s)\mathrm{d}s + J(\tau,X^{t,x}_\tau,\alpha^\varepsilon)\right] \geq \mathbb{E}\left[\int_t^\tau f(s,X^{t,x}_s,\alpha_s)\mathrm{d}s + u(\tau,X^{t,x}_\tau)\right] - \varepsilon.$$

Since the control and the stopping time are arbitrary, the upper lower bound follows by taking the supremum on the left-hand side, which concludes the proof.

□

3.3.2 Hamilton-Jacobi-Bellman equation

The Hamilton-Jacobi-Bellman equation allows us to write the value function (3.3.2) as the solution of a non-linear partial differential equation. This can be thought of as a non-linear version of the Feynman-Kać formula. Before stating the main result, let us introduce the so-called Hamiltonian of the problem, namely the operator $\mathcal{H} : [0,T) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R})$ defined as

$$\mathcal{H}(t,x,\mathrm{p},\mathrm{z}) := \sup_{a\in A}\left\{b(x,a)\cdot\mathrm{p} + \frac{1}{2}\mathrm{Tr}(\sigma(x,a)\cdot\sigma(x,a)^\top\cdot\mathrm{z}) + f(t,x,a)\right\}.$$

Theorem 3.3.2. *The value function* (3.3.2) *satisfies the PDE*

$$-\partial_t u(t,x) - \mathcal{H}(t,x,\nabla u(t,x),\Delta u(t,x)) = 0, \qquad \textit{for all } (t,x) \in [0,T) \times \mathbb{R}^n, \tag{3.3.3}$$

with boundary condition $u(T,\cdot) = g(\cdot)$, where the derivatives ∇ and Δ are taken with respect to x.

Proof. Fix a constant control $\alpha \in A$ and let $\tau = t + \varepsilon$. Since the value function is a supremum of expectations over all controls and stopping times, it is in particular greater than the expectations at τ and α:

$$u(t,x) \geq \mathbb{E}\left[\int_t^{t+\varepsilon} f(s,X^{t,x}_s,\alpha)\mathrm{d}s + u(t+\varepsilon,X^{t,x}_{t+\varepsilon})\right]. \tag{3.3.4}$$

If $u \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R}^n)$, Itô's formula yields

$$u(t+\varepsilon,X^{t,x}_{t+\varepsilon}) = u(t,x) + \int_t^{t+\varepsilon} \left(\partial_t u + \mathcal{L}^\alpha u\right)(s,X^{t,x}_s)\mathrm{d}s + \mathcal{Z},$$

[5] The term 'progressively measurable' and 'adapted' are not equivalent in general; however, when the process has at least left-or right-continuous paths, they denote the same property, see [128, Proposition 4.8] for the related technical details.

where $\mathcal{Z}$ is some local martingale (non-deterministic) term with zero expectation, and $\mathcal{L}^\alpha$ is the infinitesimal generator of (3.3.1) with constant control α:

$$\mathcal{L}^\alpha u = b(x,a)\cdot\nabla u + \frac{1}{2}\mathrm{Tr}(\sigma(x,a)\cdot\sigma(x,a)^\top\cdot\Delta u).$$

Inserting this into (3.3.4), dividing by ε, taking the limit as ε tends to zero and using the mean-value theorem implies that $\partial_t u(t,x) + \mathcal{L}^\alpha u(t,x) + f(t,x,\alpha) \leq 0$, and therefore

$$-\partial_t u(t,x) - \sup_{\alpha\in A}\{\mathcal{L}^\alpha u(t,x) + f(t,x,\alpha)\} \geq 0. \tag{3.3.5}$$

Assume now that α is an optimal control, i.e. one for which the supremum, in the definition of the value function, is attained. Note first that the dynamic programming principle in Theorem 3.3.1 is equivalent to

$$u(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + u(\tau, X_\tau^{t,x})\right].$$

Using this formulation with the optimal control and the corresponding solution to (3.3.1), we can use similar arguments as above to show that

$$-\partial_t u(t,x) - \mathcal{L}^{\alpha_t}u(t,x) + f(t,x,\alpha) = 0,$$

and therefore, together with (3.3.5), we obtain that the inequality (3.3.5) is in fact an equality for all $(t,x)\in[0,T)\times\mathbb{R}^n$ whenever the supremum is finite, and the theorem follows. □

We end this theoretical part with the so-called Verification theorem. Theorem 3.3.2 proved that the value function (3.3.2), defined as the maximal cost, solves a non-linear partial differential equation. We now show, albeit without proof, that, under some smoothness assumptions, the solution to the HJB PDE is in fact equal to the value function.

Theorem 3.3.3 (Verification Theorem). *Let $w \in \mathcal{C}^{1,2}([0,T)\times\mathbb{R}^n)\cap\mathcal{C}^0([0,T]\times\mathbb{R}^n)$ be such that there exists a constant $C>0$ for which $|w(t,x)| \leq C(1+|x|^2)$ for all $(t,x)\in[0,T]\times\mathbb{R}^n$.*

- *if $-\partial_t w(t,x) - \mathcal{H}(t,x,\nabla w,\Delta w) \geq 0$ for all $(t,x)\in[0,T)\times\mathbb{R}^n$ with boundary condition $w(T,\cdot)\geq g(\cdot)$, then $w\geq u$ on $[0,T]\times\mathbb{R}^n$.*
- *if $w(T,\cdot) = g(\cdot)$ and if there exists an A-valued measurable function α such that*

$$-\partial_t w(t,x) - \mathcal{H}(t,x,\nabla w,\Delta w) = -\partial_t w(t,x) - \mathcal{L}^{\alpha(t,x)}w(t,x) - f(t,x,\alpha(t,x)),$$

then the SDE (3.3.1) controlled by α admits a unique solution $\bar{X}^{t,x}$ starting at $X_t = x$, and $(\alpha(s,\bar{X}_s^{t,x})_{s\in[t,T]}\in\mathcal{A}(t,x)$. In that case, w is equal to the value function (3.3.2) and α is an optimal control.

3.3.3 Financial examples

American options

As seen in Section 1.5.1, and in particular Theorem 1.5.4, once an equivalent local martingale $\mathbb{Q}$ has been chosen, the price at time t of an American option expiring at T is given by $\sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}(f(S_\tau)|S_t = x)$, for some payoff function f.

The uncertain volatility model

In this model, which we will study later in more details, the underlying stock price follows Black-Scholes dynamics, apart from the fact that the volatility parameter σ is a process adapted to the ambient Brownian filtration, and allowed to moved between two bounds $\underline{\sigma}$ and $\overline{\sigma}$. Therefore the value of the option reads

$$\sup\left\{\mathbb{E}(g(S_T))|\mathcal{F}_t : (\sigma_t)_{t\geq 0}\mathcal{F} - \text{adapted, and } \sigma \in [\underline{\sigma}, \overline{\sigma}]\right\}.$$

Models with transaction costs

3.4 A rough introduction to PIDEs

All the models considered so far had continuous paths. However, many phenomena observed on financial markets, such as dividends, policy announcements, unexpected profit or loss, political events, induce a sudden discontinuity–upward or downward jump–in the dynamics of financial assets.

3.4.1 Preamble on semimartingales

Let $\mathbb{D}$ denote the space of right-continuous with left limit (càdlàg) processes. Similarly to stochastic integration with continuous integrators, we would like to make sense of an expression of the form $\int u_t \mathrm{d}S_t$, when $S \in \mathbb{D}$, for some (simple) predictable process u. This is however not trivial, and the right class of processes to consider is the class of semimartingales.

Definition 3.4.1. A process $X \in \mathbb{D}$ is called a semimartingale if there exist $(M_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$, both starting from zero, such that the decomposition $X_t = X_0 + M_t + Z_t$ holds almost surely for all $t \geq 0$, where M is a local martingale and Z has bounded variation.

Example 3.4.2.

- A Brownian motion is a semimartingale;
- every square integrable martingale in $\mathbb{D}$ is a semimartingale;
- every Lévy process is a semimartingale;

- every process of bounded variation is a semimartingale.

In the course of these notes, we shall not use this level of generality, and will restrict ourselves to a very tractable subset of semimartingales, namely (exponential) Lévy processes.

3.4.2 Introduction to Lévy processes

Lévy processes are a wide class of semimartingales with very tractable properties. They have been used extensively in the mathematical finance literature, and we shall introduce them here together with some of their properties. Again, the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is assumed to be granted a priori.

Definition 3.4.3. An $\mathbb{R}^n$-valued Lévy process X is an adapted process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $X_0 = 0$ almost surely satisfying the following properties:

(i) independent increments: for any $0 \leq s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s$;

(ii) stationary increments: for any $0 \leq s < t$, the random variables X_{t-s} and $(X_t - X_s)$ have the same distributions;

(iii) X is stochastically continuous, namely for all $\varepsilon > 0$ and $t \geq 0$, $\lim_{h\downarrow 0} \mathbb{P}\left(|X_{t+h} - X_t| \geq \varepsilon\right) = 0$.

The following theorem provides a useful and tractable representation for its characteristic function, and in particular highlights the singular time dependence of the distribution of the process.

Theorem 3.4.4. *If X is a Lévy process in $\mathbb{R}^n$, there exists a unique function $\phi_X \in \mathcal{C}(\mathbb{R}^n \to \mathbb{C})$, called the Lévy exponent, with $\phi_X(0) = 0$ such that*

$$\mathbb{E}\left(\mathrm{e}^{\mathtt{i}\langle\xi, X_t\rangle}\right) = \exp\left(t\phi_X(\xi)\right), \qquad \textit{for all } t \geq 0, \xi \in \mathbb{R}^n$$

Example 3.4.5 (Brownian motion)**.** The simplest example of a Lévy process is the (one-dimensional) Brownian motion with drift: $W_t^\mu = \mu t + \sigma W_t$. Clearly, for any $t \geq 0$, X_t is a Gaussian random variable with mean $X_0 + \mu t$ and variance $\sigma^2 t$; therefore

$$\mathbb{E}\left(\mathrm{e}^{\mathtt{i}\xi W_t^\mu}\right) = \exp\left(\mathtt{i}\xi\mu t\right)\mathbb{E}\left(\mathrm{e}^{\mathtt{i}\xi\sigma W_t}\right) = \exp\left(\mathtt{i}\mu\xi t - \frac{\sigma^2\xi^2 t}{2}\right).$$

Example 3.4.6 (Poisson process)**.** A Poisson process $(N_t)_{t\geq 0}$ is a counting process, in the sense that at time $t \geq 0$, N_t represents the number of events that have happened up to time t. Such a process has independent increments and is such that for each $t \geq 0$, the random variable N_t is Poisson distributed with parameter λt for $\lambda > 0$ (the intensity), i.e.

$$\mathbb{P}\left(N_t = n\right) = \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}, \quad \text{for any } n = 0, 1, \ldots$$

Its characteristic function can be computed as

$$\left(\mathrm{e}^{\mathtt{i}\xi N_t}\right) = \sum_{n\geq 0} \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}\mathrm{e}^{\mathtt{i}\xi n} = \exp\left(\lambda t\left(\mathrm{e}^{\mathtt{i}\xi} - 1\right)\right), \qquad \text{for any } \xi \in \mathbb{R}.$$

Note that the paths of a Poisson process are non-decreasing and discontinuous.

Example 3.4.7 (Compound Poisson processes)**.** A compound Poisson process $(\mathcal{J}_t)_{t\geq 0}$ is defined as $\mathcal{J}_t := \sum_{n=1}^{N_t} Z_n$, where N is a Poisson process with parameter λt and $(Z_k)_{k\geq 0}$ a family of independent and identically distributed random variables with common law F. Therefore, for any $\xi \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{\mathtt{i}\xi\mathcal{J}_t}\right) &= \mathbb{E}\left(\exp\left(\mathtt{i}\xi\sum_{n=1}^{N_t} Z_n\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\mathrm{e}^{\mathtt{i}\xi\sum_{n=1}^{m} Z_n}\right)\middle| N_t = m\right)\\ &= \sum_{m\geq 0}\mathbb{E}\left(\mathrm{e}^{\mathtt{i}\xi\sum_{n=1}^{m} Z_n}\right)\frac{(\lambda t)^m}{m!}\mathrm{e}^{-\lambda t} = \sum_{m\geq 0}\left(\int_{\mathbb{R}}\mathrm{e}^{\mathtt{i}\xi z}F(\mathrm{d}z)\right)^m\frac{(\lambda t)^m}{m!}\mathrm{e}^{-\lambda t}\\ &= \exp\left(\lambda t\int_{\mathbb{R}}\left(\mathrm{e}^{\mathtt{i}\xi z}-1\right)F(\mathrm{d}z)\right).\end{aligned}$$

From now on, for simplicity and in order to avoid technical difficulties, we shall restrict ourselves to Compound Poisson processes (CPP), namely to processes with the following representation:

$$\mathcal{J}_t := \sum_{n=1}^{N_t} Z_n, \tag{3.4.1}$$

where N is a Poisson process with intensity λ and the $(Z_k)_{k\geq 0}$ are iid random variables with common distribution η. We shall denote by $(\tau_n^{\mathcal{J}})_{n\geq 1}$ denote the sequence of jump times. The jump measure J of $\mathcal{J}$, defined as

$$J(I\times A) := \sum_{n\geq 1}\delta_I(\tau_n^{\mathcal{J}})\delta_A(Z_n), \qquad \text{for any } (I\times A)\in\mathcal{B}\left([0,\infty)\times\mathbb{R}^n\right),$$

counts the expected number of jumps of amplitude A occurring in the time period I.

Lemma 3.4.8. *For any $t>0$, $\mathbb{E}\left(J([0,t]\times A)\right) = t\lambda\eta(A)$, for any $A\subset\mathbb{R}^n$. In particular, $\mathbb{E}\left(J([0,t]\times A)\right) = t\mathbb{E}\left(J([0,1]\times A)\right)$.*

Proof. For $I=[0,t]$, the jump measure simplifies to $J([0,t]\times A) := \sum_{n=1}^{N_t}\delta_A(Z_n)$, so that

$$\begin{aligned}\mathbb{E}\left[J([0,t]\times A)\right] &= \mathbb{E}\left[\sum_{n=1}^{N_t}\delta_A(Z_n)\right] = \mathbb{E}\left[\mathbb{E}\left(\sum_{n=1}^{N_t}\delta_A(Z_n)\middle| N_t\right)\right]\\ &= \sum_{n\geq 1}\mathbb{E}\left(\sum_{k=1}^{n}\delta_A(Z_k)\mathbb{1}_{\{N_t=n\}}\right) = \sum_{n\geq 1}\mathbb{P}(N_t=n)\sum_{k=1}^{n}\mathbb{P}(Z_k\in A)\\ &= \sum_{n\geq 1}\frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}n\eta(A) = \lambda t\eta(A).\end{aligned}$$

□

Definition 3.4.9. The map $\nu: A\mapsto\mathbb{E}\left(J([0,1]\times A)\right)$ is a finite measure on $\mathcal{B}(\mathbb{R}^n)$ with $\nu(\mathbb{R}^n)=\lambda$, and is called the Lévy measure of the process $\mathcal{J}$.

With this definition of the Lévy measure, Lemma 3.4.8 can be rewritten as $\nu(A) = \lambda\eta(A)$ for any $A \subset \mathbb{R}^n$, and the characteristic function of a Compound Poisson process can be rewritten as

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\langle\xi,\mathcal{J}_t\rangle}\right) = \exp\left\{t\int_{\mathbb{R}^n}\left(\mathrm{e}^{\mathrm{i}\langle\xi,x\rangle}-1\right)\nu(\mathrm{d}x)\right\}, \qquad \text{for any } \xi \in \mathbb{R}^n.$$

The following theorem proves a pathwise decomposition of a compound Poisson process, and will be a crucial element in deriving a partial differential equation for pricing purposes.

Theorem 3.4.10 (Lévy-Itô decomposition for CPP)**.** *Let $\mathcal{J}$ be a $\mathbb{R}^n$-valued CPP of the form* (3.4.1) *with jump measure J and Lévy measure ν. Then, for any function $f : [0,\infty) \times \mathbb{R}^n$,*

$$\sum_{0<s\leq t, \Delta X_s \neq 0} f(s, \Delta X_s) = \int_0^t \int_{\mathbb{R}^n} f(s,x) J(\mathrm{d}x, \mathrm{d}s).$$

Let $\widetilde{J}(\mathrm{d}t, \mathrm{d}x) := J(\mathrm{d}t, \mathrm{d}x) - \mathrm{d}t\nu(\mathrm{d}x)$ denote the compensated jump measure. If furthermore the function f is integrable with respect to the product measure $\mathrm{d}s \otimes \nu$, then the process M defined as

$$M_t := \int_0^t \int_{\mathbb{R}^n} f(s,x)\widetilde{J}(\mathrm{d}x, \mathrm{d}s)$$

is a martingale with zero expectation.

Corollary 3.4.11. *The following expressions hold for the process $\mathcal{J}$ and its quadratic variation $\langle\mathcal{J}\rangle$:*

$$\begin{aligned} \mathcal{J}_t &= \int_0^t \int_{\mathbb{R}^n} xJ(\mathrm{d}x, \mathrm{d}s), \\ \langle\mathcal{J}\rangle_t &= \int_0^t \int_{\mathbb{R}^n} |x|^2 J(\mathrm{d}x, \mathrm{d}s) = \sum_{n=1}^{N_t} |Z_n|^2. \end{aligned} \tag{3.4.2}$$

Proof. The corollary follows from Theorem 3.4.10 applied to $f(x) \equiv x$ and $f(x) \equiv |x|^2$. □

Example 3.4.12.

- Merton model [114]: $\eta = \mathcal{N}(m, \delta^2)$, so that

$$\nu(\mathrm{d}x) = \frac{\lambda}{\delta\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\delta^2}\right)\mathrm{d}x;$$

  here the jumps are symmetric around a constant m.

- Kou model [101]: $\eta(\mathrm{d}x) = \left(p\lambda_+\mathrm{e}^{-\lambda_+ x}\mathbb{1}_{\{x>0\}} + (1-p)\lambda_-\mathrm{e}^{-\lambda_- x}\mathbb{1}_{\{x<0\}}\right)\mathrm{d}x$, with $\lambda_+ > 1$ and $\lambda_- > 0$; here the jumps are not symmetric.

Quadratic variation

In order to write down an Itô formula for Jump diffusion processes, we first need to extend the notion of quadratic variation to this class of discontinuous processes.

Definition 3.4.13. The quadratic variation process of a semimartingale $(X_t)_{t\geq 0}$ is defined as

$$\langle X\rangle_t := X_t^2 - 2\int_0^t X_{u^-}\mathrm{d}X_u, \qquad \text{for all } t\geq 0.$$

Example 3.4.14.

- The quadratic variation of a Brownian motion W is $\langle W\rangle_t = t$;
- For a Poisson process N, the quadratic variation is $\langle N\rangle_t = N_t$ (see Exercise sheet);
- For a Compound Poisson process $\mathcal{J}$, we have $\langle \mathcal{J}\rangle_t = \sigma^2 t + \int_0^t\int_{\mathbb{R}} x^2 J(\mathrm{d}s, \mathrm{d}x)$.

Itô formula for jump diffusions

We finish this section with an Itô formula for a one-dimensional jump diffusion model, namely the unique solution (under growth and smoothness conditions on the coefficients) to the equation

$$X_t = X_0 + \int_0^t \mu_s(X_{s^-})\mathrm{d}s + \int_0^t \sigma_s(X_{s^-})\mathrm{d}W_s + \mathcal{J}_t, \tag{3.4.3}$$

where $\mathcal{J}$ represents the discontinuous part of the process, which we assume to be a compound Poisson process as in (3.4.1). In differential form, we can rewrite (3.4.3) as

$$\mathrm{d}X_t = \mu_t(X_{t^-})\mathrm{d}t + \sigma_t(X_{t^-})\mathrm{d}W_t + \mathrm{d}\mathcal{J}_t, \qquad X_0\in\mathbb{R}. \tag{3.4.4}$$

We shall denote by $\langle X\rangle^c$ the quadratic variation of the continuous part of X (i.e. drift and Brownian motion). Let $\mathcal{A}^c := \mu\partial_x + \frac{1}{2}\sigma^2\partial_{xx}$ denote the infinitesimal generator of the continuous part of the jump diffusion X.

Theorem 3.4.15 (Itô formula for jump diffusions)**.** *Let X be the unique strong solution to* (3.4.3), *and let $f\in\mathcal{C}^{1,2}((0,\infty),\mathbb{R}^n)$. Then the infinitesimal generator of X reads*

$$\mathcal{A}f(t,x) = \mathcal{A}^c f(t,x) + \int_{\mathbb{R}}\Big[f(t,x+y) - f(t,x) - y\partial_x f(t,x)\Big]\nu(\mathrm{d}y), \tag{3.4.5}$$

and the following Itô formula holds

$$\begin{aligned}
f(t,X_t) &= f(0,X_0) + \sigma(t,X_{t^-})\partial_x f(t,X_{t^-})\mathrm{d}W_t + \int_0^t (\partial_s + \mathcal{A}) f(s,X_{s^-})\mathrm{d}s\\
&+ \int_0^t\int_{\mathbb{R}}\Big[f(s,X_{s^-}+y) - f(s,X_{s^-}) - y\partial_x f(s,X_{s^-})\Big]\widetilde{J}(\mathrm{d}s,\mathrm{d}y)\\
&= f(0,X_0) + \sigma(t,X_{t^-})\partial_x f(t,X_{t^-})\mathrm{d}W_t + \int_0^t (\partial_s + \mathcal{A}^c) f(s,X_{s^-})\mathrm{d}s\\
&+ \int_0^t\int_{\mathbb{R}}\Big[f(s,X_{s^-}+y) - f(s,X_{s^-}) - y\partial_x f(s,X_{s^-})\Big]J(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

3.4.3 PIDES for compound Poisson processes

We terminate this section with an option pricing formulation in the context of Lévy processes. This can be seen as an analogue (and is actually an extension) of the Feynman-Kać formula in Theorem 3.2.1. for Itô diffusions.

Definition 3.4.16. Let $\mathcal{A}$ be the infinitesimal generator in (3.4.5) and φ a bounded continuous function. A bounded function $u \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R}) \cap \mathcal{C}([0,T] \times \mathbb{R})$ is called a classical solution to the Cauchy problem for $(\mathcal{A} + \partial_t)$ with initial datum φ if

$$\begin{cases} (\mathcal{A} + \partial_t)u(t,x) &= u(t,x), \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}, \\ u(T,x) &= \varphi(x), \quad \text{for all } x \in \mathbb{R}. \end{cases} \tag{3.4.6}$$

Theorem 3.4.17 (Feynman-Kać for jumps)**.** *If a classical solution u exists for* (3.4.6) *such that $u, \partial_x u \in L^\infty((0,T) \times \mathbb{R})$, then $u(t,x) = \mathbb{E}\varphi(X_T^{t,x})$, for all $(t,x) \in [0,T) \times \mathbb{R}$.*

Chapter 4

Volatility modelling

We previously introduced the notion of implied volatility, and saw how it is characterised through absence of arbitrage and the Black-Scholes formula. This was done in a completely model-independent framework, Black-Scholes only being used as a quoting mechanism. By definition, the implied volatility in the Black-Scholes model is a strictly positive constant parameter, which does not depend on the strike or the maturity of the option. Therefore, the implied volatility is also constant. This is clearly unrealistic, and we shall now introduce several classes of models that exhibit non-constant implied volatility surfaces.

4.1 Local volatility

A local volatility model for the stock price process $(S_t)_{t\geq 0}$ is defined as follows:

$$\mathrm{d}S_t/S_t = \mu_t \mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t, \qquad S_0 > 0. \tag{4.1.1}$$

Again here W is a standard Brownian motion adapted to the given filtration, and r represents the risk-free interest rate assumed to be constant. Note now that the diffusion component $\sigma(\cdot)$ depends on both time and space, and that the market is complete, since there is a unique source of randomness. The drift μ_t is again assumed to be adapted. The same argument as in the standard Black-Scholes model applies, and the value (at time zero) of a European option with payoff $h(S_T)$ at some future time $T > 0$ satisfies the PDE

$$\partial_t C + r_t S \partial_S + \frac{1}{2}\sigma^2(t, S) S^2 \partial_{SS} C - r_t C = 0,$$

for all $t \in [0, T]$, $S \geq 0$, with boundary condition $C(S, T) = h(S)$. Recall that by Girsanov theorem, we can define a new probability measure $\mathbb{Q}$ by the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := \exp\left(-\int_0^t \frac{\mu_s - r_s}{\sigma_s}\mathrm{d}W_s - \int_0^t \frac{(\mu_s - r_s)^2}{2\sigma_s^2}\mathrm{d}s\right),$$

under which the stock price satisfies

$$\frac{\mathrm{d}S_t}{S_t} = r_t \mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t^{\mathbb{Q}}, \qquad S_0 > 0. \tag{4.1.2}$$

The probability $\mathbb{Q}$ is well defined if and only if the Radon-Nikodym derivative is a uniformly integrable martingale. Using Novikov's criterion, let $\gamma_t := (\mu_t - r_t)/\sigma_t$; then

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := \exp\left(-\int_0^t \gamma_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \gamma_s^2 \mathrm{d}s\right)$$

defines a true martingale if

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_s^2 \mathrm{d}s\right)\right] < \infty.$$

By absence of arbitrage, the option price therefore reads

$$C(S,t) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s \mathrm{d}s} h(S_T) | S_t = s\right).$$

From a practical point of view, assume that one consider the model (4.1.1) under some historical probability $\mathbb{P}$, and calibrate μ and σ to some historical data. Then, by construction of the Radon-Nikodym derivative, the new (risk-neutral) probability measure $\mathbb{Q}$ is fully characterised, so that nothing is left to determine. In particular this makes it impossible to calibrate it to currently quoted option prices. The standard way to deal with this is to consider directly the model under the (unique) risk-neutral measure $\mathbb{Q}$, satisfying the SDE (4.1.2).

4.1.1 Bruno Dupire's framework

Consider a stock price process S satisfying (under the risk-neutral measure) the stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma(S_t, t)\mathrm{d}W_t, \qquad S_0 > 0, \tag{4.1.3}$$

where W is a standard Brownian motion, $r \geq 0$ the instantaneous risk-free interest rate, and $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ the state-dependent diffusion coefficient. The question raised by Bruno Dupire (1994)[1] was whether given a smooth surface $(K,T) \mapsto C(K,T)$ of European Call prices, there exists such a function σ able to match these prices exactly. The answer turns out to be positive, as outlined below.

Definition 4.1.1. A function $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that the prices of Call / Put options for all strikes and maturities generated by model (4.1.3) correspond exactly to a given Vanilla price surface is called the local volatility.

Example 4.1.2. If the given option price surface (in strike and maturity) is flat, then the local volatility is simply equal to a constant, and (4.1.3) is nothing else than the standard Black-Scholes model.

[1] simultaneously, Derman and Kani came up with a similar concept, but in discrete time.

Theorem 4.1.3. *Assume that $\mathbb{E}(\int_0^t |S_u\sigma(u,S_u)|^2\mathrm{d}u)$ is finite for all $t \geq 0$, that S_t admits a continuous density p_t on $(0,\infty)$, and that the mapping σ is continuous on $(0,\infty)\times(0,\infty)$. Then the Call price function $C(K,T) := \mathrm{e}^{-rT}\mathbb{E}(S_T - K)_+$ satisfies the so-called Dupire Equation:*

$$\partial_T C(K,T) = \frac{\sigma^2(K,T)}{2}K^2\partial_{KK}C(K,T) - rK\partial_K C(K,T),$$

for all $(K,T)\in(0,\infty)\times(0,\infty)$, with boundary condition $C(K,0) = (S_0 - K)_+$.

Remark 4.1.4. For a given smooth surface $(K,T)\mapsto C(K,T)$, Dupire's equation implies that there exists a unique continuous function σ_{loc} defined by

$$\sigma_{\text{loc}}^2(K,T) := \frac{\partial_T C(K,T) + rK\partial_K C(K,T)}{\frac{1}{2}K^2\partial_{KK}C(K,T)}, \tag{4.1.4}$$

for all $(K,T)\in(0,\infty)\times(0,\infty)$, such that the solution to the stochastic differential equation $\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma(t,S_t)\mathrm{d}W_t$ exactly generates the European Call option prices $C(\cdot,T)$ for every maturity $T>0$.

Remark 4.1.5. The condition $\sigma(0,S_0)>0$ is a sufficient condition ensuring that the random variable S_t admits a density which is continuous with respect to the Lebesgue measure.

Remark 4.1.6. From Remark 2.1.26, the local volatility involves the density of the stock price. Given a model, it is not, however, guaranteed that such a density exists. Consider the example in [65], of the asymmetric Variance Gamma, in which the log stock price X_T, at time T, admits the following characteristic function:

$$\mathbb{E}\left(\mathrm{e}^{\mathtt{i}uX_T}\right) = \left(1 - \mathtt{i}\theta\nu u + \frac{\sigma^2\nu u^2}{2}\right)^{-T/\nu}, \qquad \text{for all } u\in\mathbb{R},$$

for some parameters $\nu,\sigma,\theta>0$. As u tends to infinity, a simple series expansion yields

$$\mathbb{E}\left(\mathrm{e}^{\mathtt{i}uX_T}\right) = \left(\frac{\sigma^2\nu}{2}\right)^{-T/\nu} u^{-2T/\nu} + \mathcal{O}\left(u^{-1-2T/\nu}\right).$$

Therefore, when $2T/\nu>1$, the characteristic function is integrable and hence X_T admits a continuous density. When $2T/\nu<1$, however, the density–given explicitly in [26, Page 82]–has a singularity at the origin, and therefore the Call price is not twice continuously differentiable.

Note that Dupire's framework applies to continuous Itô processes of the form (4.1.3). However, a lot of research has been devoted to processes with jumps, and it therefore makes sense to try and extend this definition to the discontinuous case. This is not that simple, though, and we refer the interested reader to [25, 65, 74].

Proof. Note first that the function $K\mapsto \mathrm{e}^{-rT}(S-K)_+$ is not twice differentiable, so that Itô's formula does not apply. We use here a smoothing argument to prove the Dupire equation. Let $f:\mathbb{R}\ni x\mapsto x_+$, and for any $\varepsilon>0$, define the $\mathcal{C}^2(\mathbb{R})$ function $f_\varepsilon:\mathbb{R}\to\mathbb{R}$ by

$$f_\varepsilon(x) := \frac{(x+\varepsilon/2)^2}{2\varepsilon}\mathbb{1}_{\{|x|\leq\varepsilon/2\}} + x\mathbb{1}_{\{x>\varepsilon/2\}}.$$

Note that $f_\varepsilon(x) = f(x)$ as soon as $|x| \geq \varepsilon/2$, that f_ε converges pointwise to f as ε tends to zero and that

$$f'_\varepsilon(x) = \frac{x+\varepsilon/2}{\varepsilon}\mathbb{1}_{\{|x|\leq\varepsilon/2\}} + \mathbb{1}_{\{x>\varepsilon/2\}} \qquad \text{and} \qquad f''_\varepsilon(x) = \frac{1}{\varepsilon}\mathbb{1}_{\{|x|\leq\varepsilon/2\}}.$$

For $\tau > 0$, applying Itô's formula to the function $(t, S_t) \mapsto \mathrm{e}^{-rt} f_\varepsilon(S_t - K)$ between T and $T+\tau$ yields:

$$\begin{aligned}\mathrm{e}^{-r(T+\tau)} f_\varepsilon(S_{T+\tau} - K) - \mathrm{e}^{-rT} f_\varepsilon(S_T - K) &= -r\int_T^{T+\tau} \mathrm{e}^{-rt} f_\varepsilon(S_t - K)\mathrm{d}t \\ &+ \int_T^{T+\tau} \mathrm{e}^{-rt}\left(f'_\varepsilon(S_t - K)\mathrm{d}S_t + \frac{1}{2}\sigma^2(S_t,t)S_t^2 f''_\varepsilon(S_t - K)\mathrm{d}t\right).\end{aligned}$$

Taking expectation (conditional on $\mathcal{F}_0$) on both sides yields

$$\begin{aligned}\mathrm{e}^{-r(T+\tau)}\mathbb{E} f_\varepsilon(S_{T+\tau} - K) - \mathrm{e}^{-rT}\mathbb{E} f_\varepsilon(S_T - K) &= -r\int_T^{T+\tau} \mathrm{e}^{-rt}\left(\mathbb{E} f_\varepsilon(S_t - K) - \mathbb{E} f'_\varepsilon(S_t - K)\right)\mathrm{d}t \\ &+ \frac{1}{2}\int_T^{T+\tau} \mathrm{e}^{-rt}\mathbb{E}\left(\sigma^2(S_t,t)S_t^2 f''_\varepsilon(S_t - K)\right)\mathrm{d}t,\end{aligned}$$

where we used the fact that $\mathbb{E}\left(f'_\varepsilon(S_t - K)\sigma(t, S_t)S_t\mathrm{d}W_t\right) = 0$. Note now that the last integral on the right-hand side of the above equality can be rewritten as

$$\frac{1}{2}\int_T^{T+\tau}\mathbb{E}\left(\sigma^2(t,S_t)S_t^2 f''_\varepsilon(S_t - K)\right)\mathrm{e}^{-rt}\mathrm{d}t = \frac{1}{2\varepsilon}\int_T^{T+\tau}\mathrm{e}^{-rt}\left(\int_{-\varepsilon/2}^{\varepsilon/2}\sigma^2(t, K+s)(K+s)^2 p_t(K+s)\mathrm{d}s\right)\mathrm{d}t.$$

Take now the limit as ε tends to zero from above in the equality, and we obtain

$$\begin{aligned}C(K, T+\tau) - C(K,T) &= \lim_{\varepsilon\downarrow 0}\left(\mathrm{e}^{-r(T+\tau)}\mathbb{E} f_\varepsilon(S_{T+\tau} - K) - \mathrm{e}^{-rT}\mathbb{E} f_\varepsilon(S_T - K)\right) \\ &= -r\int_T^{T+\tau}\left(C(K,t) + \mathrm{e}^{-rt}\mathbb{E}(S_t\mathbb{1}_{S_t\geq K})\right)\mathrm{d}t + \frac{1}{2}\int_T^{T+\tau}\mathrm{e}^{-rt}\sigma^2(K,t)K^2 p_t(K)\mathrm{d}t.\end{aligned}$$

We have here used the fact that, for a function g, with left derivative $F'_-(0)$ and right derivative $F'_+(0)$ at the origin, the following equality holds:

$$\lim_{\varepsilon\downarrow 0}\frac{F(\varepsilon/2) - F(-\varepsilon/2)}{\varepsilon} = \frac{F'_+(0) + F'_-(0)}{2}.$$

Therefore

$$\begin{aligned}C(K,T+\tau) - C(K,T) &= \frac{1}{2}\int_T^{T+\tau}\mathrm{e}^{-rt}\sigma^2(t,K)K^2 p_t(K)\mathrm{d}t - r\int_T^{T+\tau}\mathrm{e}^{-rt}\mathbb{E}\left((S_t - K)_+ + S_t\mathbb{1}_{\{S_t\geq K\}}\right)\mathrm{d}t \\ &= \frac{1}{2}\int_T^{T+\tau}\mathrm{e}^{-rt}\sigma^2(K,t)K^2 p_t(K)\mathrm{d}t + \int_T^{T+\tau}\mathrm{e}^{-rt}K\mathbb{P}\left((S_t \geq K\right)\mathrm{d}t.\end{aligned}$$

Since $\mathrm{e}^{-rt}\mathbb{P}(S_t \geq K) = -\partial_K C(K,t)$ and $\mathrm{e}^{-rt}p_t(K) = \partial_{KK}C(K,t)$, the theorem follows. □

Note that Dupire's original proof is different and relies on the fact that the density satisfies the forward Kolmogorov (or Fokker-Planck) equation. We outline his proof below.

$$\begin{aligned}C(K,T) = \mathrm{e}^{-rT}\mathbb{E}\left[(S_T - K)_+\right] &= \mathrm{e}^{-rT}\int_K^{+\infty}(s-K)p_T(s)\mathrm{d}s \\ &= \mathrm{e}^{-rT}\int_K^{+\infty}\left(p_T(s)\int_K^s\mathrm{d}y\right)\mathrm{d}s = \mathrm{e}^{-rT}\int_K^{+\infty}\left(\int_y^{+\infty}p_T(s)\mathrm{d}s\right)\mathrm{d}y,\end{aligned}$$

where we used Fubini's theorem to interchange the two integrals. Therefore

$$\partial_K C(K,T) = -\mathrm{e}^{-rT} \int_K^{+\infty} p_T(s)\mathrm{d}s = -\mathrm{e}^{-rT} \int_0^{+\infty} \mathbb{1}_{\{s>K\}} p_T(s)\mathrm{d}s.$$

and $\partial_{KK} C(K,T) = \mathrm{e}^{-rT} p_T(K)$. From this equality, we immediately deduce

$$\partial_T p_T(K) = \partial_T \left(\mathrm{e}^{rT} \partial_{KK} C(K,T)\right) = r\mathrm{e}^{rT} \partial_{KK} C(K,T) + \mathrm{e}^{rT} \partial_{KKT} C(K,T). \tag{4.1.5}$$

Since p_T satisfies the forward Kolmogorov equation

$$\partial_T p_T - \frac{1}{2}\partial_{KK}\left(K^2\sigma^2 p_T\right) + \partial_K\left(rKp_T\right) = 0,$$

where all the functions are evaluated at the point (K,T), Equation (4.1.5) then becomes

$$\frac{1}{2}\partial_{KK}\left(K^2\sigma^2\partial_{KK}C\right) - \partial_K\left(rK\partial_{KK}C\right) = r\partial_{KK}C + \partial_{TKK}C.$$

Rearranging this equation yields

$$\begin{aligned}
\partial_{TKK}C &= \frac{1}{2}\partial_{KK}\left(K^2\sigma^2\partial_{KK}C\right) - 2r\partial_{KK}C - rK\partial_{KKK}C \\
&= \partial_{KK}\left(\frac{\sigma^2K^2}{2}\partial_{KK}C - rK\partial_K C\right).
\end{aligned}$$

Integrating both sides twice with respect to the strike K yields

$$\partial_T C + rK\partial_K C = \frac{\sigma^2K^2}{2}\partial_{KK}C + \alpha_T K + \beta_T,$$

and the boundary conditions imposed by no-arbitrage conclude the proof.

4.1.2 Local volatility via local times

The proof of Theorem 4.1.3 above followed the original Dupire approach. We now present a different proof, using the theory of local times. Before doing so, though, we shall need a few results on local times for continuous semimartingales. Let us first prove the occupation time formula, which provides good intuition about the meaning of local times.

Proposition 4.1.7. *Let W be a standard Brownian motion on the real line. For any $x \in \mathbb{R}$, there exists an increasing family of local times $(L_t^x)_{t\geq 0}$ such that, for every bounded measurable function f, we have*

$$\int_0^t f(W_s)\mathrm{d}s = \int_{\mathbb{R}} L_t^x f(x)\mathrm{d}x, \qquad \textit{for any } t \geq 0. \tag{4.1.6}$$

Remark 4.1.8. This proposition implies the following:

(i) let A be a Borel subset of the real line, and let $f \equiv \mathbb{1}_A$. Then the occupation time for the Brownian motion reads

$$\int_0^t \mathbb{1}_{\{W_s\in A\}}\mathrm{d}s = \int_{\mathbb{R}} \mathbb{1}_A(x) L_t^x\mathrm{d}x = \int_A L_t^x\mathrm{d}x;$$

(ii) From (4.1.6), we can write

$$L_t^x = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[x-\varepsilon, x+\varepsilon]}(W_s)\mathrm{d}s.$$

Proof. Let f be a continuous function with compact support on $\mathbb{R}$, and denote

$$F(x) := \int_{-\infty}^x \mathrm{d}z \int_{-\infty}^z f(y)\mathrm{d}y = \int_{\mathbb{R}} (x-y)_+ f(y)\mathrm{d}y.$$

Clearly F belongs to $C^2(\mathbb{R})$ with $F'(x) = \int_{\mathbb{R}} f(y)\mathbb{1}_{\{x>y\}}\mathrm{d}y$ and $F''(x) = f(x)$, and hence combining Itô's formula and the stochastic version of Fubini yields

$$F(W_t) = \int_{\mathbb{R}} (W_t - y)_+ f(y)\mathrm{d}y = \int_{\mathbb{R}} (W_0 - y)_+ f(y)\mathrm{d}y + \int_{\mathbb{R}} \left(\int_0^t \mathbb{1}_{\{W_s > y\}}\mathrm{d}W_s \right) f(y)\mathrm{d}y + \frac{1}{2}\int_0^t f(W_s)\mathrm{d}s,$$

which we can rearrange as

$$\frac{1}{2}\int_0^t f(W_s)\mathrm{d}s = \int_{\mathbb{R}} \mathrm{d}y f(y) \left((W_t - y)_+ - (W_0 - y)_+ - \int_0^t \mathbb{1}_{\{W_s > y\}}\mathrm{d}W_s \right).$$

Setting

$$\frac{1}{2}L_t^y := (W_t - y)_+ - (W_0 - y)_+ - \int_0^t \mathbb{1}_{\{W_s > y\}}\mathrm{d}W_s$$

proves (4.1.6). From this formula, one can then easily deduce the expression in Remark 4.1.8(ii), which in particular shows that the family $(L_t^x)_{t\geq 0}$ is increasing, and the proposition follows. □

This definition of local times immediately yields Tanaka's formula for the Brownian motion:

Proposition 4.1.9 (Tanaka formula)**.** *Let W be a standard Brownian motion and L_t^x be its local time between 0 and t at the level x. Then,*

$$\begin{aligned}
(W_t - x)_+ &= (W_0 - x)_+ + \int_0^t \mathbb{1}_{\{W_s > x\}}\mathrm{d}W_s + \frac{1}{2}L_t^x, \\
(W_t - x)_- &= (W_0 - x)_- - \int_0^t \mathbb{1}_{\{W_s \leq x\}}\mathrm{d}W_s + \frac{1}{2}L_t^x, \\
|W_t - x| &= |W_0 - x| + \int_0^t \mathrm{sgn}(W_s - x)\mathrm{d}W_s + L_t^x,
\end{aligned}$$

with $\mathrm{sgn}(x) = 1$ *if* $x > 0$ *and* -1 *if* $x \leq 0$.

We can actually extend the definition and properties of local times to general semimartingales. In the following, we let X denote a one-dimensional continuous semimartingale.

Proposition 4.1.10. *For any $x \in \mathbb{R}$, there exists an increasing family of local times $(L_t^x)_{t\geq 0}$ such that, for every bounded measurable function f, we have*

$$\int_0^t f(X_s)\mathrm{d}\langle X, X\rangle_s = \int_{\mathbb{R}} L_t^x f(x)\mathrm{d}x, \qquad \textit{for any } t \geq 0. \tag{4.1.7}$$

As in the Brownian case, this proposition implies

$$L_t^x = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[x-\varepsilon, x+\varepsilon]}(X_s) \mathrm{d}\langle X, X\rangle_s.$$

This definition of local times immediately yields the Tanaka-Meyer formulae for continuous semimartingales:

Proposition 4.1.11 (Tanaka formula)**.** *Let X be a one-dimensional continuous semimartingale, and L_t^x its local time between 0 and t at the level x. Then,*

$$\begin{aligned}
(X_t - x)_+ &= (X_0 - x)_+ + \int_0^t \mathbb{1}_{\{X_s > x\}} \mathrm{d}X_s + \frac{1}{2} L_t^x, \\
(X_t - x)_- &= (X_0 - x)_- - \int_0^t \mathbb{1}_{\{X_s \leq x\}} \mathrm{d}X_s + \frac{1}{2} L_t^x, \\
|X_t - x| &= |X_0 - x| + \int_0^t \operatorname{sgn}(X_s - x) \mathrm{d}X_s + L_t^x,
\end{aligned}$$

with $\operatorname{sgn}(x) = 1$ *if* $x > 0$ *and* -1 *if* $x \leq 0$*.*

We are now able to give a rigorous proof of Theorem 4.1.3 using the machinery of local times.

Proof of Theorem 4.1.3. Tanaka-Meyer's formula applied to the semimartingale S reads

$$(S_T - K)_+ = (S_0 - K)_+ + \int_0^T \mathbb{1}_{\{S_u > K\}} \mathrm{d}S_u + \frac{1}{2} \int_0^T \mathrm{d}L_u^K(S),$$

and integration by parts yields

$$\mathrm{e}^{-rT}(S_T - K)_+ = (S_0 - K)_+ - r \int_0^T \mathrm{e}^{-ru}(S_u - K)_+ \mathrm{d}u + \int_0^T \mathrm{e}^{-ru} \mathbb{1}_{\{S_u > K\}} \mathrm{d}S_u + \frac{1}{2} \int_0^T \mathrm{e}^{-ru} \mathrm{d}L_u^K(S). \tag{4.1.8}$$

By definition of local times,

$$\mathbb{E}\left(\frac{1}{2} \int_0^T \mathrm{e}^{-ru} \mathrm{d}L_u^K(S)\right) = \int_0^T \mathrm{e}^{-ru} p_u(K) K^2 \sigma^2(K, u) \mathrm{d}u,$$

so that taking expectations (at time zero) on both sides of (4.1.8), we obtain

$$\begin{aligned}
C(K,T) = (S_0 - K)_+ &- r \int_0^T \mathrm{e}^{-ru} \mathbb{E}\left((S_u - K) \mathbb{1}_{\{S_u \geq K\}}\right) \mathrm{d}u + r \int_0^T \mathrm{e}^{-ru} \mathbb{E}\left(S_u \mathbb{1}_{\{S_u > K\}}\right) \mathrm{d}u \\
&+ \frac{1}{2} \mathbb{E}\left(\int_0^T \mathrm{e}^{-ru} \mathrm{d}L_u^K(S)\right) \\
&= (S_0 - K)_+ + rK \int_0^T \mathrm{e}^{-ru} \mathbb{P}(S_u > K) \mathrm{d}u + \frac{1}{2} \int_0^T \mathrm{e}^{-ru} p_u(K) K^2 \sigma^2(K, u) \mathrm{d}u.
\end{aligned}$$

Differentiating with respect to the maturity T implies

$$\partial_T C(K,T) = rK \mathrm{e}^{-rT} \mathbb{P}(S_T > K) + \frac{1}{2} \mathrm{e}^{-rT} p_T(K) K^2 \sigma^2(K, T),$$

and the theorem follows.

□

4.1.3 Implied volatility in local volatility models

We study here how the local volatility (defined in (4.1.4)) can be expressed in terms of the implied volatility. As before, we let $w(k,T) := \sigma^2(k,T)T$ denote the total implied variance at time T, where k represents the log-moneyness. The following theorem shows a one-to-one mapping between the Dupire local volatility and the implied volatility.

Theorem 4.1.12. *If the map $(k,T) \mapsto w(k,T)$ belongs to $\mathcal{C}^{2,1}(\mathbb{R}, \mathbb{R}_+^*)$, then*

$$\sigma_{\text{loc}}^2(k,T) = \frac{\partial_T w(k,T)}{g(k,T)}, \tag{4.1.9}$$

where the function g was defined in (2.1.4).

Proof. For ease of notation, we shall consider that interest rates are null, i.e. $r = 0$. For a given (observed or computed) Call price C, by definition of the implied volatility σ, the equality $C(K,T) = \text{BS}(K,T,\sigma(K,T))$ holds for all $(K,T) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. Changing the variables, we use the notations $\widetilde{C}(k,w) \equiv C(K,T)$ and the slight abuse $\text{BS}(k,w) \equiv \text{BS}(K,T,\sigma)$. In these new coordinates, the Dupire equation reads

$$\partial_T \widetilde{C}(k,T) = \frac{\sigma_{\text{loc}}^2(k,T)}{2}\left(\partial_{kk}\widetilde{C}(k,T) - \partial_k \widetilde{C}(k,T)\right),$$

for all $(k,T) \in \mathbb{R} \times \mathbb{R}_+^*$ with boundary condition $C(k,T)$ at maturity. The following identities between the Black-Scholes Greeks are straightforward to check using (1.4.5):

$$\begin{cases} \partial_{ww}\text{BS} &= \left(-\frac{1}{8} - \frac{1}{2w} + \frac{k^2}{w^2}\right)\partial_w \text{BS}, \\ \partial_{kw}\text{BS} &= \left(\frac{1}{2} - \frac{k}{w}\right)\partial_w \text{BS}, \\ \left(\partial_{kk} - \partial_k\right)\text{BS} &= 2\partial_w \text{BS}, \end{cases} \tag{4.1.10}$$

where of course BS and all its derivatives are evaluated at the point (k,w). Using the total rule of differentiation, we further have

$$\begin{cases} \partial_k \widetilde{C} = \partial_k \text{BS} + \partial_k w \cdot \partial_k \text{BS}, \\ \partial_{kk}\widetilde{C} = \partial_{kk}\text{BS} + 2\partial_k w \cdot \partial_{kw}\text{BS} + \left(\partial_k w\right)^2 \partial_{ww}\text{BS} + \partial_{kk} w \cdot \partial_w \text{BS}, \\ \partial_T \widetilde{C} = \partial_T \text{BS} + \partial_T w \cdot \partial_w \text{BS} = \partial_T w \cdot \partial_w \text{BS}. \end{cases} \tag{4.1.11}$$

Therefore, we can rewrite the Dupire equation as

$$\begin{aligned} \partial_w \text{BS} \cdot \partial_t w &= \frac{\sigma_{\text{loc}}^2(k,T)}{2}\left[\partial_{kk}\text{BS} + 2\partial_k w \cdot \partial_{kw}\text{BS} + (\partial_k w)^2 \partial_{ww}\text{BS} + \partial_{kk} w \cdot \partial_w \text{BS} - \partial_k \text{BS} - \partial_k w \cdot \partial_k \text{BS}\right] \\ &= \frac{\sigma_{\text{loc}}^2(k,T)}{2}\left[2 - \partial_k w + 2\left(\frac{1}{2} - \frac{k}{w}\right)\partial_k w + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{k^2}{w^2}\right)\left(\partial_k w\right)^2 + \partial_{kk} w\right]\partial_w \text{BS} \\ &= \frac{\sigma_{\text{loc}}^2(k,T)}{2}\left[1 - \frac{k}{w}\partial_k w + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{k^2}{w^2}\right)\left(\partial_k w\right)^2 + \frac{1}{2}\partial_{kk} w\right]\partial_w \text{BS}, \end{aligned}$$

and the theorem follows. □

The following remarks are in order here:

- Whenever the total variance is null, we know that we are outside the support of the stock price, so that the definition of the local volatility should obviously be taken as null as well.
- The conditions in Theorem 2.1.25 make even more sense now, as they ensure that the local volatility is well defined and strictly positive.
- For any time t, the implied volatility, as seen previously, provides information about the marginal law of the process at time t, but does not provide any information about transition probabilities between two different times. Therefore given an arbitrage-free implied volatility surface (or equivalently collection of all European Call and Put option prices), it is not clear how one can price more exotic options, such as path-dependent ones. Using (4.1.9), one can compute the equivalent local volatility, and run Monte Carlo simulations. Though very promising in theory, the local volatility map is in general difficult very sensitive to interpolation and extrapolation of the implied volatility.

Let us draw some immediate consequences of the relation (4.1.9). Suppose for instance that the total implied variance w does not depend on the strike. Then $g \equiv 1$ and the local variance reads $\sigma_{\text{loc}}^2(k,T) = \partial_T w(k,T)$, so that $w(k,T) = \int_0^T \sigma_{\text{loc}}^2(k,s)\mathrm{d}s$. From a financial perspective, this equality means that, in the absence of skew ($\partial_k w \equiv 0$), the total implied variance is the average of local variances. We can actually prove that for short maturity, the implied volatility is a harmonic average of the local volatility:

Proposition 4.1.13. *As the maturity T tends to zero, the implied volatility is the harmonic average of the local volatility:*

$$w(k,T) \approx k^2 T \left(\int_0^k \frac{\mathrm{d}y}{\sigma_{\text{loc}}(y,T)}\right)^{-2}.$$

Proof. Let $\Sigma(\cdot)$ denote the implied volatility, so that $w(k,T) \equiv \Sigma(k)^2 T$. Therefore $\partial_T w \equiv \Sigma^2 + 2T\Sigma\partial_T\Sigma$, and (4.1.9) reads

$$\begin{aligned}
\sigma_{\text{loc}}^2(k,T) &= \frac{\Sigma(k)^2 + 2T\Sigma(k)\partial_T\Sigma(k)}{\left(1 - \frac{k\partial_k w}{2w}\right)^2 - \frac{(\partial_k w)^2}{4}\left(\frac{1}{4} + \frac{1}{w}\right) + \frac{\partial_{kk} w}{2}} \\
&\approx \frac{\Sigma(k)^2}{\left(1 - \frac{k\partial_k w}{2w}\right)^2}, \quad \text{as } T \text{ tends to zero} \\
&= \left(\frac{\Sigma(k)}{1 - \frac{k\partial_k \Sigma^2}{2\Sigma^2}}\right)^2,
\end{aligned}$$

which is now a simple ordinary differential equation; solving it yields the required result. □

We terminate this theoretical introduction to local volatility by the following result, due to István Gyöngy, which provides the theoretical background of the existence of such a framework.

Theorem 4.1.14. *Let X be an Itô process on the real line satisfying the SDE $\mathrm{d}X_t = \alpha_t \mathrm{d}t + \beta_t \mathrm{d}W_t$, where W is a standard one-dimensional Brownian motion, and α and β two adapted processes. Then there exists a Markov process Y satisfying $\mathrm{d}Y_t = a(t, Y_t)\mathrm{d}t + b(t, Y_t)\mathrm{d}W_t$, such that X and Y have the same marginals. Furthermore, Y can be constructed as follows:*

$$a(t, y) = \mathbb{E}(\alpha_t | X_t = y) \qquad \text{and} \qquad b^2(t, y) = \mathbb{E}(\beta_t | X_t = y),$$

for all $t \geq 0$ and all y in the support of X_t.

If X is a given model, say a stochastic volatility model, then Y can be seen as the corresponding local volatility model, which will generate the exact same European (Call and Put) option prices for all strikes and maturities. That said, the two expectations describing $a(\cdot)$ and $b(\cdot)$ are in general difficult to compute.

4.1.4 A special example: the CEV model

The Constant Elasticity of Volatility (CEV) model, first proposed by J. Cox [35], is a local volatility model, in the sense that the stock price process is the unique solution

$$\mathrm{d}S_t = \sigma S_t^{1+\beta} \mathrm{d}W_t, \qquad S_0 > s_0. \tag{4.1.12}$$

The process $(S_t)_{t \geq 0}$ is a true martingale if and only if $\beta \leq 0$, see [92, Chapter 6.4] and Theorem 4.1.18 below. When $\beta = 0$, the SDE (4.1.12) reduces to the Black-Scholes SDE, and the stock price remains strictly positive almost surely for all $t \geq 0$. Let $\tau := \inf\{t \geq 0 : S_t = 0\}$ be the first time the process S hits the origin, and define a new process X by $X_t := S_t^{-2\beta}/(\sigma^2\beta^2)$ (pathwise) up to τ. Itô formula yields $\mathrm{d}X_t = \delta \mathrm{d}t + 2\sqrt{X_t}\mathrm{d}W_t$, with $X_0 = x_0 = s_0^{-2\beta}/(\sigma^2\beta^2) > 0$ and $\delta = 2 + 1/\beta$. The process X is a Bessel process with δ degrees of freedom (and index $\nu := \delta/2 - 1 = 1/(2\beta)$), and the Feller classification yields the following:

- If $\beta = 0$, S is a geometric Brownian motion started at some strictly positive value, and hence is always strictly positive almost surely..
- if $\delta \leq 0$, i.e. $\beta \in [-1/2, 0)$, the origin is an attainable and absorbing boundary. For every $t > 0$, the distribution μ_t of X_t on $[0, \infty)$ has a positive mass at zero and admits a density on the positive real line:

$$\mu_t(\mathrm{d}y) = \mathbb{P}(X_t = 0)\delta_0(\mathrm{d}y) + f_t(X_0, y)\mathrm{d}y,$$

with

$$f_t(x_0, y) = \frac{1}{2t}\left(\frac{y}{x_0}\right)^{\nu/2} \exp\left(-\frac{x_0 + y}{2t}\right) I_{-\nu}\left(\frac{\sqrt{x_0 y}}{t}\right), \qquad \text{for all } y > 0, \tag{4.1.13}$$

where $I_{-\nu}$ is the modified Bessel function of the first kind. Note that $\int_0^\infty f_t(X_0, y)\mathrm{d}y = \Gamma\left(-\nu, \frac{X_0}{2t}\right) < 1$, where Γ is the normalised lower incomplete Gamma function $\Gamma(v,z) := \Gamma(v)^{-1}\int_0^z u^{v-1}\mathrm{e}^{-u}\mathrm{d}u$, therefore

$$\mathbb{P}(X_t = 0) = 1 - \Gamma\left(-\nu, \frac{x_0}{2t}\right) > 0. \tag{4.1.14}$$

- If $\delta > 2$ ($\beta > 0$), the origin is not attainable and $\mathbb{P}(X_t = 0) = 0$ for all t, and

$$f_t(x_0, y) = \frac{1}{2t}\left(\frac{y}{x_0}\right)^{\nu/2} \exp\left(-\frac{x_0 + y}{2t}\right) I_\nu\left(\frac{\sqrt{x_0 y}}{t}\right), \qquad \text{for all } y > 0, \tag{4.1.15}$$

  which integrates to one.

- If $\delta \in (0,2)$ ($\beta < -1/2$), the origin is attainable, and can be specified as either absorbing or reflecting. In order to preserve the martingale property of the process, however, only absorption is possible (see [86, Chapter III, Lemma 3.6] for technical details about this). In that case, the density is given by (4.1.13), is norm defective (does not integrate to one) and mass at the origin is present and equal to (4.1.14). If reflection is specified, the process S is not a martingale any longer, its density is given by (4.1.15) and the mass at the zero is null.

We can recast these results in terms of the original process S. Let us define the functions φ and ψ by (note the sign differences)

$$\varphi(y) := -\frac{1}{\sigma^2\beta t} S_0^{1/2} y^{-2\beta-3/2} \exp\left(-\frac{S_0^{-2\beta} + y^{-2\beta}}{2\sigma^2\beta^2 t}\right) I_{-\nu}\left(\frac{S_0^{-\beta} y^{-\beta}}{\sigma^2\beta^2 t}\right),$$
$$\psi(y) := \frac{1}{\sigma^2\beta t} S_0^{1/2} y^{-2\beta-3/2} \exp\left(-\frac{S_0^{-2\beta} + y^{-2\beta}}{2\sigma^2\beta^2 t}\right) I_{\nu}\left(\frac{S_0^{-\beta} y^{-\beta}}{\sigma^2\beta^2 t}\right).$$

In the CEV model, the density has the following expression ([30, Section 6.4.1] and [22]):

$$\mathbb{P}(S_t \in \mathrm{d}y) = \begin{cases} \varphi(y)\mathrm{d}y, & \text{if } \beta \in [-1/2, 0) \text{ or } \beta < -\frac{1}{2} \text{ with aborption,} \\ \dfrac{\mathrm{d}y}{y\sigma\sqrt{2\pi t}} \exp\left(-\dfrac{1}{2}\dfrac{\left[\log\left(\frac{y}{s_0}\right) + \frac{1}{2}\sigma^2 t\right]^2}{\sigma^2 t}\right), & \text{if } \beta = 0, \\ \psi(y)\mathrm{d}y, & \text{if } \beta > 0 \text{ or } \beta < -\frac{1}{2} \text{ with reflection,} \end{cases}$$

and, in the case $\beta \in [-1/2, 0)$ or $\beta < -\frac{1}{2}$ with aborption specification,

$$\mathbb{P}(S_T = 0) = 1 - \Gamma\left(-\frac{1}{2\beta}, \frac{s_0^{-2\beta}}{2\sigma^2\beta^2 T}\right).$$

It can be proved that, as z tends to zero, we have $I_\alpha(z) \sim \Gamma(\alpha+1)^{-1}(z/2)^\alpha$ for positive α; since $-\nu = 1/(2|\beta|)$, the density behaves like $const \times y^{2|\beta|-1}$ as y tends to zero. Therefore the density of the stock price explodes at the origin when $\beta \in (-1/2, 0)$, and tends to a constant when $\beta = -1/2$, in contrast to the previous examples (where the density vanishes at the origin).

Remark 4.1.15. The modified Bessel functions of the first kind are the two solutions I_ν and K_ν of the Bessel equation with parameter ν:

$$z^2 f''(z) + zf'(z) - (z^2 + \nu^2)f(z) = 0.$$

They can be written explicitly as

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n\geq 0} \frac{z^{2n}}{2^{2n} n! \Gamma(\nu + n + 1)} \qquad \text{and} \qquad K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)},$$

where we recall the definition of the Gamma function $\Gamma(z) := \int_0^\infty x^{a-1} \mathrm{e}^{-x} \mathrm{d}x$, for any $z \geq 0$.

Remark 4.1.16. European Put option prices maturing at time $T \geq 0$ and with strike $K \geq 0$ are worth at inception

$$P(K, T) = \mathbb{E}[(K - S_T)_+] = K\mathbb{P}(S_T = 0) + \int_{(0,+\infty)} (K - s)^+ f_{S_T}(s)\mathrm{d}s.$$

Using perturbation expansions, Hagan et al. proved in [78] that the corresponding implied volatility reads as follows:

$$\sigma(K, T) = \frac{2^\beta \sigma}{(s_0 + K)^{-\beta}} \left\{ 1 - \frac{(3 + \beta)\beta}{6} \left(\frac{s_0 - K}{s_0 + K} \right)^2 + \frac{2^{2\beta} \beta^2 \sigma^2 T}{24(s_0 + K)^{-2\beta}} + \mathcal{O}(T^2) \right\}. \tag{4.1.16}$$

Exercise 4.1.17. Plot the approximation (4.1.16) for the implied volatility, as well as the corresponding density, and discuss any arbitrage that may occur.

Theorem 4.1.18. *If $\beta \in (-1, 0]$, the process S is a square integrable martingale.*

Proof. We only need to show here that $\mathbb{E}\left(\int_0^T S_t^{2(1+\beta)} \mathrm{d}t\right)$ is finite for all $T > 0$. By a localisation argument, let us define the hitting time at level n: $\tau_n := \inf\{t \geq 0 : S_t \geq n\}$. Clearly, $(S_{t\wedge\tau_n})_{t\geq 0}$ is a square-integrable martingale for each $n \geq 0$, and Itô's lemma implies

$$\mathbb{E}(S_{T\wedge\tau_n}^2) = s_0^2 + \sigma^2 \mathbb{E}\left(\int_0^{T\wedge\tau_n} S_t^{2(1+\beta)} \mathrm{d}t\right) \leq \sigma^2 \mathbb{E}\left(\int_0^{T\wedge\tau_n} \left(1 + S_t^2\right) \mathrm{d}t\right) \leq \sigma^2 \mathbb{E}\left(\int_0^T \left(1 + S_{t\wedge\tau_n}^2\right) \mathrm{d}t\right).$$

Recall now Gronwall's lemma: if a function u satisfies $u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)\mathrm{d}s$ and α is increasing, then $u(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s)\mathrm{d}s\right)$. Therefore, with $\alpha(t) \equiv \sigma^2 t$ and $\beta(t) \equiv \sigma^2$,

$$\sigma^2 \mathbb{E}\left(\int_0^{T\wedge\tau_n} S_t^{2(1+\beta)} \mathrm{d}t\right) = \mathbb{E}(S_{T\wedge\tau_n}^2) - s_0^2 \leq \sigma^2 T \mathrm{e}^{\sigma^2 T} - s_0^2,$$

and the result follows by the monotone convergence theorem. □

4.2 Stochastic volatility models

It is by now clear that the assumption of constant volatility in the Black-Scholes model is not suitable to account for the observed implied volatility. One is therefore led to consider a more refined approach. We shall consider here a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and we start with a time-dependent, though deterministic, volatility:

$$\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma_t \mathrm{d}W_t, \qquad S_0 > 0, \tag{4.2.1}$$

where again r is the risk-free interest rate and W a $(\mathcal{F}_t)_{t\geq 0}$-adapted standard standard Brownian motion. By Itô's formula and integration, we immediately obtain

$$\log(S_t) = \log(S_0) + rt - \frac{1}{2}\int_0^t \sigma_s^2 \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s,$$

for any $t \geq 0$. Denoting $\sigma_{[0,t]}^2 := t^{-1}\int_0^t \sigma_s^2 \mathrm{d}s$, we observe that $\log(S_t/S_0)$ is a Gaussian random variable with mean $\left(r - \frac{1}{2}\sigma_{[0,t]}^2\right)t$ and variance $\sigma_{[0,t]}^2 t$. Therefore the price of a European Call option with maturity $T > 0$ and log-strike x in (4.2.1) is equal to $C_t^{\mathrm{BS}}(x, \sigma_{[0,t]})$. For any given maturity, this clearly yields a flat implied volatility smile, again unable to cope with the skew observed on the market.

4.2.1 Pricing PDE and market price of volatility risk

Consider a general stochastic volatility model (S, V) satisfying the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}S_t/S_t &= \mu_t \mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, & S_0 > 0,\\ \mathrm{d}V_t &= \alpha(S_t, V_t, t)\mathrm{d}t + \beta(S_t, V_t, t)\sqrt{V_t}\mathrm{d}Z_t, & V_0 > 0,\\ \mathrm{d}\langle W, Z\rangle_t &= \rho \mathrm{d}t, \end{aligned} \tag{4.2.2}$$

where the correlation parameter ρ lies in $(-1, 1)$, W and Z are two Brownian motions, and the coefficients α and β are such that a unique strong solution exists. In the Black-Scholes model, there is only one source of randomness, so that one is able to replicate a given option with a dynamic rebalancing of stocks. Here, we have two sources of randomness, so that the same argument clearly does not apply. Consider therefore a portfolio $\Pi := C - \Delta S - \gamma\Psi$, consisting of a given option C, some amount Δ of the stock price and some quantity γ of another product Ψ which depends on the variance V. Itô formula therefore implies

$$\mathrm{d}\Pi_t = (\mathcal{L}C - \gamma\mathcal{L}\Psi)\,\mathrm{d}t + (\partial_s C - \gamma\partial_s\Psi - \Delta)\,\mathrm{d}S_t + (\partial_v C - \gamma\partial_v\Psi)\,\mathrm{d}V_t, \tag{4.2.3}$$

where we define the differential operator

$$\mathcal{L} := \partial_t + \frac{1}{2}V_t S_t^2 \partial_{ss} + \rho\beta(S_t, V_t, t)V_t S_t \partial_{sv} + \frac{1}{2}V_t\beta^2(S_t, V_t, t)\partial_{vv}.$$

The only sources of randomness appear in the $\mathrm{d}St$ and the $\mathrm{d}V_t$ terms in (4.2.3). The portfolio Π is therefore instantaneously risk-free if and only if $(\partial_s C - \gamma\partial_s\Psi - \Delta) = (\partial_v C - \gamma\partial_v\Psi) = 0$, e.g.

$$\Delta = \partial_s C - \gamma\partial_s\Psi \qquad \text{and} \qquad \gamma = \frac{\partial_v C}{\partial_v \Psi}. \tag{4.2.4}$$

Under these two conditions, the risk-freeness of the portfolio is then equivalent to

$$\mathrm{d}\Pi_t = r\Pi_t \mathrm{d}t = r\left(C - \Delta S_t - \gamma\Psi\right)\mathrm{d}t = \left(\mathcal{L}C - \gamma\mathcal{L}\Psi\right)\mathrm{d}t,$$

where the last equality follows from (4.2.3). We can rewrite this as

$$\frac{(\mathcal{L} - r + rS\partial_s)C}{\partial_v C} = \frac{(\mathcal{L} - r + rS\partial_s)\Psi}{\partial_v \Psi}. \tag{4.2.5}$$

The only way the equality (4.2.5) can hold is that both sides are equal to some function $-\Phi$ which depends on S, V and t. Without loss of generality, assume that this function has the form $\Phi(s,v,t) := \alpha(s,v,t) - \phi(s,v,t)\beta(s,v,t)\sqrt{v}$. The quantity ϕ is called the market price of volatility risk. To understand why, consider a delta-hedged portfolio $\widetilde{\Pi} := C - \Delta S$, where $\Delta := \partial_s C$. Itô's lemma yields $\mathrm{d}\widetilde{\Pi}_t = \mathcal{L}C\mathrm{d}t + (\partial_s C - \Delta)\,\mathrm{d}S_t + \partial_v C \mathrm{d}V_t$, and the portfolio is instantaneously risk free if and only if

$$\begin{aligned}
0 = \mathrm{d}\widetilde{\Pi}_t - r\widetilde{\Pi}_t\mathrm{d}t &= \mathcal{L}C\mathrm{d}t + (\partial_s C - \Delta)\,\mathrm{d}S_t + \partial_v C\mathrm{d}V_t - r\left(C - S\partial_s C\right)\mathrm{d}t \\
&= \left(-\Phi(S_t, V_t, t)\partial_v C + rC - rS_t\partial_s C\right)\mathrm{d}t + \partial_v C\mathrm{d}V_t - r\left(C - S\partial_s C\right)\mathrm{d}t \\
&= -\Phi(S_t, V_t, t)\partial_v C\mathrm{d}t + \partial_v C\mathrm{d}V_t \\
&= -\left[\alpha(s,v,t) - \phi(s,v,t)\beta(s,v,t)\sqrt{v}\right]\partial_v C + \partial_v C\mathrm{d}V_t \\
&= \beta(S_t, V_t, t)\sqrt{V_t}\partial_v C\left(\phi(S_t, V_t, t)\mathrm{d}t + \mathrm{d}Z_t\right),
\end{aligned}$$

where, in the fourth line, we used the representation of Φ, and, in the third line, the stochastic differential equation (4.2.2). Therefore, ϕ represents the extra return per unit of volatility risk $\mathrm{d}Z_t$. Had we started with a drift equal to $\widetilde{\alpha} \equiv \alpha - \beta\phi\sqrt{V_t}$ in (4.2.2), we would have obtained the same result but without any price of volatility risk, which explains why this new drift is called the risk-neutral drift.

4.2.2 Arbitrage and Equivalent Local Martingale Measures

Before looking at the properties of stochastic volatility models, their pricing equations, and the implied volatility smile they generate, let us step back temporarily and wonder about arbitrage. We saw, in the Black-Scholes model, with the help of the Girsanov transform, how to switch from the historical measure to the risk-neutral one, essentially by changing the drift of the stock price process. In the case of stochastic volatility models, there are two sources of randomness—the two driving Brownian motions—and, unless these are fully (un)correlated, this implies market incompleteness, and the equivalent local martingale measure, if it exists, may not be unique any longer. Consider the stochastic differential equations (4.2.2). We are looking for two 'market prices of risk' processes, $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that the new probability measure $\mathbb{Q}$ can be written as

$$L_t := \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{-\int_0^t \gamma_1(s)\mathrm{d}Z_s - \int_0^t \gamma_2(s)\mathrm{d}Z_s^{\perp} - \frac{1}{2}\int_0^t \left(\gamma_1(s)^2 + \gamma_2(s)^2\right)\mathrm{d}s\right\} \tag{4.2.6}$$

and where $Z^{\perp}$ is a Brownian motion independent of Z so that the Brownian motion W can be written as $W = \rho Z + \overline{\rho} Z^{\perp}$, with $\overline{\rho} := \sqrt{1-\rho^2}$. As proved by Freddy Delbaen and Walter Schachermayer (see their recent monograph [41] for a gentle introduction as well as a collection of the original papers), existence of a local martingale measure is equivalent to 'No Free Lunch with vanishing risk'. We discuss below, through two examples, whether arbitrage, in this sense, can be taken for granted or not.

Stein-Stein model

The Stein-Stein model corresponds to the following stochastic differential equation for the stock price process:

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t, && S_0 > 0, \\ \mathrm{d}\sigma_t &= \kappa(\theta - \sigma_t)\mathrm{d}t + \xi \mathrm{d}Z_t, && \sigma_0 > 0, \end{aligned}$$

where $\kappa, \theta, \xi > 0$, and where the two Brownian motions are uncorrelated. Using Girsanov transform, let us define a new probability measure $\mathbb{Q}$ and a new Brownian motion $W^* := W + \int_0^{\cdot} \gamma_2(s)\mathrm{d}s$ (in the context of (4.2.6), this precisely corresponds to the $\rho = 0$ case). The SDE for the stock price then reads

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \mathrm{d}t + \sigma_t \Big(\mathrm{d}W_t - \gamma_2(t)\mathrm{d}t\Big) = \Big(\mu_t - \gamma_2(t)\sigma_t\Big)\mathrm{d}t + \sigma_t \mathrm{d}W_t.$$

In order to satisfy absence of arbitrage, let r_t denote the instantaneous risk-free rate, then one needs to set $\mu_t - \gamma_1(t)\sigma_t = r_t$, e.g.

$$\gamma_1(t) = \frac{\mu_t - r_t}{\sigma_t}.$$

However, for any $t \geq 0$, the support of the random variable σ_t is the whole real line, and hence $\gamma_1(t)$ may be undefined, meaning that no Equivalent Local Martingale Measure exists in general, leading to immediate arbitrage.

Heston model

Let us now consider the Heston model, given by the following set of stochastic differential equations:

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= \mu_t \mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, && S_0 > 0, \\ \mathrm{d}V_t &= \kappa\left(\theta - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, && V_0 = v_0 > 0, \\ \mathrm{d}\langle W, Z\rangle_t &= \rho \mathrm{d}t, \end{aligned}$$

where $\kappa, \theta, \xi > 0$, $\rho \in (-1, 1)$. Let us now introduce two new Brownian motions $Z^* := Z + \int_0^{\cdot} \gamma_1(s)\mathrm{d}s$ and $Z^{\perp^*} := Z^{\perp} + \int_0^{\cdot} \gamma_2(s)\mathrm{d}s$, and set $W^* := \rho Z^* + \overline{\rho} Z^{\perp^*}$, so that the SDE for the stock price becomes

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \mathrm{d}t + \sqrt{V_t}\left(\rho \mathrm{d}Z_t + \overline{\rho}\mathrm{d}Z_t^{\perp}\right) = \Big(\mu_t - \left(\rho\gamma_1(t) + \overline{\rho}\gamma_2(t)\right)\sqrt{V_t}\Big)\,\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^*.$$

In order to ensure that the discounted stock price process is a local martingale, we therefore need to set $\mu_t - r_t = (\rho\gamma_1(t) + \overline{\rho}\gamma_2(t))\sqrt{V_t}$. Note that the Feller condition (see below in Section 4.2.3) ensures that the process V remains strictly positive almost surely. Heston, in his seminal paper [81], proposes the restrication $\gamma_1(t) = \lambda\sqrt{V_t}$, for some $\lambda > 0$. Under $\mathbb{Q}$, the stochastic differential equation satisfied by the variance process then reads

$$\begin{aligned} \mathrm{d}V_t &= \kappa(\theta - V_t)\,\mathrm{d}t + \xi\sqrt{V_t}\,(\mathrm{d}Z_t^* - \gamma_1(t)\mathrm{d}t)\,,\\ &= (\kappa\theta - (\kappa + \lambda\xi)\,V_t)\,\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t^*,\\ &= \widetilde{\kappa}\left(\widetilde{\theta} - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t^*, \end{aligned} \tag{4.2.7}$$

where we set $\widetilde{\kappa} := \kappa + \lambda\xi$ and $\widetilde{\theta} := \kappa\theta/\widetilde{\kappa}$. Regarding the second market price of risk, we obtain

$$\gamma_2(t) = \frac{1}{\overline{\rho}}\left(\frac{\mu_t - r_t}{\sqrt{V_t}} - \rho\lambda\sqrt{V_t}\right).$$

In order for the Equivalent Local martingale Measure (ELMM) to exist, we need the process L defined in (4.2.6) to be a true martingale, e.g. $\mathbb{E}(L_t) = 1$ for all $t \geq 0$. Note that we can decompose it as $L = L^1L^2$, where, for any $t \geq 0$,

$$\begin{aligned} L_t^1 &:= \exp\left\{-\int_0^t \gamma_1(s)\mathrm{d}Z_s - \frac{1}{2}\int_0^t \gamma_1^2(s)\mathrm{d}s\right\},\\ L_t^2 &:= \exp\left\{-\int_0^t \gamma_2(s)\mathrm{d}Z_s^\perp - \frac{1}{2}\int_0^t \gamma_2^2(s)\mathrm{d}s\right\}. \end{aligned}$$

Since the variance process and the Brownian motion $Z^\perp$ are independent, then clearly $\mathbb{E}(L_t^2) = 1$ and $\mathbb{E}(L_t) = \mathbb{E}(L_t^1)$ for all $t \geq 0$. The proof of the existence of the ELMM rests on the following results, which we shall prove later, albeit under a slightly different form:

Lemma 4.2.1. *For any $\alpha \geq 0$, $\beta \geq -\kappa/(2\xi^2)$, $\mathbb{E}\left(\exp\left\{-\alpha V_t - \beta\int_0^t V_s\mathrm{d}s\right\}\right)$ is finite for all $t \geq 0$.*

As a corollary, taking $\alpha = 0$, any $\beta \in [-\kappa/(2\xi^2), 0]$, the lemma, combined with Novikov's condition implies that for any $|\lambda| \leq \kappa/\xi$, $\mathbb{E}(L_t^1) = 1$, and an ELMM exists.

Proposition 4.2.2. *For any $\lambda > 0$, $t \geq 0$, $\mathbb{E}(L_t^1) = 1$.*

Proof. Rearranging the stochastic differential equation satisfied by the variance process, we can write, for any $t \geq 0$,

$$\lambda\int_0^t \sqrt{V_s}\mathrm{d}Z_s = \frac{\lambda}{\xi}\left(V_t - V_0 + \kappa\theta\int_0^t \mathrm{d}s - \kappa\int_0^t V_s\mathrm{d}s\right),$$

so that

$$\mathbb{E}(L_t^1) = \exp\left\{\frac{\lambda}{\xi}(V_0 + \kappa\theta t)\right\}\mathbb{E}\left(\exp\left\{-\frac{\lambda V_t}{\xi} - \left(\frac{\lambda\kappa}{\xi} + \frac{\lambda^2}{2}\right)\int_0^t V_s\mathrm{d}s\right\}\right),$$

and the result follows from Lemma 4.2.1 with $\alpha = \lambda/\xi$ and $\beta = \frac{\lambda\kappa}{\xi} + \frac{\lambda^2}{2}$. □

Note that we have only determined here whether there exist (at least) an Equivalent Local Martingale Measure. The underlying stock price is a non-negative martingale, and hence a supermartingale, but is not necessarily a true martingale under this ELMM.

Proposition 4.2.3. *Assume that an equivalent local martingale measure $\mathbb{Q}$ exists. Then the discounted stock price is a $\mathbb{Q}$-martingale if $\kappa + \lambda\xi \geq \rho\xi$.*

Proof. Let $\mathbb{Q}$ be such an equivalent local martingale measure, and recall that, under $\mathbb{Q}$, the variance process satisfies the stochastic differential equation (4.2.7). It is clear that it is enough to prove that $\mathbb{E}^{\mathbb{Q}}(\mathrm{e}^{-rt}S_t) = 1$ for all $t \geq 0$. The SDE for the stock price process can be rewritten as

$$S_t = S_0 \exp\left\{rt - \frac{1}{2}\int_0^t V_s \mathrm{d}s + \rho\int_0^t \sqrt{V_s}\mathrm{d}Z_s^* + \overline{\rho}\int_0^t \sqrt{V_s}\mathrm{d}Z_s^{\perp^*}\right\}.$$

Conditioning on the filtration generated by the Brownian motion $Z^\perp$, we obtain

$$\mathbb{E}(S_t) = S_0 \mathrm{e}^{rt}\exp\left\{-\frac{\rho^2}{2}\int_0^t V_s \mathrm{d}s + \rho\int_0^t \sqrt{V_s}\mathrm{d}Z_s^*\right\},$$

and the proposition follows directly from the analysis above. □

4.2.3 The Heston model

We now introduce the Heston stochastic volatility model, which is among the most widely used models in mathematical finance. We shall explain later some of the reasons for this success, but let us first describe some properties. The SDE satisfied by the log stock process $X := \log(S)$ reads

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t \mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, && X_0 = x_0 \in \mathbb{R}, \\ \mathrm{d}V_t &= \kappa\left(\theta - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, && V_0 = v_0 > 0, \\ \mathrm{d}\langle W, Z\rangle_t &= \rho \mathrm{d}t, \end{aligned} \tag{4.2.8}$$

where $\kappa, \theta, \xi > 0$, $\rho \in [-1, 1]$. For ease of notation, we shall write $X \sim \mathrm{H}(\kappa, \theta, \rho, \xi, v_0)$. We have assumed here that interest rates are null, and fixed a priori a given equivalent local martingale measure under which the (discounted) stock price is a true martingale. The Feller SDE for the variance process has a unique strong solution by the Yamada-Watanabe conditions (see [95, Proposition 2.13, page 291] or Theorem 3.1.22 in Chapter 3). The X process is a stochastic integral of the V process and is therefore well-defined. The Feller condition $2\kappa\theta > \xi^2$ ensures that the origin is an unattainable boundary for the variance process. If this condition is violated, then the origin is an attainable, regular and reflecting boundary (see [96, Chapter 15] for the classification of boundary points of one-dimensional diffusions). For any $0 \leq s \leq t$, we can compute the conditional (at time s) expectation and variance of the variance process at time t:

$$\begin{aligned} \mathbb{E}_s(V_t) &= V_s + \mathbb{E}_s\left(\int_s^t \mathrm{d}V_u\right) = V_s + \mathbb{E}_s\left(\int_s^t \left[\kappa(\theta - V_u)\mathrm{d}u + \xi\sqrt{V_u}\mathrm{d}Z_u\right]\right) \\ &= V_s + \kappa\theta(t-s) - \kappa\mathbb{E}_s\left(\int_s^t V_u \mathrm{d}u\right). \end{aligned}$$

Differentiating this equation in time, we obtain

$$\frac{\mathrm{d}\mathbb{E}_s(V_t)}{\mathrm{d}t} = \kappa\theta - \kappa\mathbb{E}_s(V_t),$$

which is a simple ordinary differential equation with boundary condition $\mathbb{E}_s(V_s) = V_s$, so that

$$\mathbb{E}_s(V_t) = \theta + (V_s - \theta)\mathrm{e}^{-\kappa(t-s)}.$$

In particular, note that $\lim_{t\uparrow\infty} \mathbb{E}_s(V_t) = \theta$, so that θ represents the long-term mean of the variance. Regarding the conditional variance, we first apply Itô's formula to obtain

$$\mathrm{d}V_t^2 = 2V_t\mathrm{d}V_t + \mathrm{d}\langle V, V\rangle_t = \kappa\left(\widetilde{\theta} - V_t\right)V_t\mathrm{d}t + \xi V_t^{3/2}\mathrm{d}Z_t,$$

where we set $\widetilde{\theta} := 2\theta + \xi^2/\kappa$. Taking conditional expectation, we obtain

$$\begin{aligned}\mathbb{E}_s(V_t^2) &= V_s^2 + \mathbb{E}_s\left(\int_s^t \mathrm{d}V_u^2\right) = V_s^2 + \mathbb{E}_s\left(\int_s^t \left[\kappa\left(\widetilde{\theta} - V_u\right)V_u\mathrm{d}u + \xi V_u^{3/2}\mathrm{d}Z_u\right]\right)\\ &= V_s^2 + \kappa\widetilde{\theta}\mathbb{E}_s\left(\int_s^t V_u\mathrm{d}u\right) - \kappa\mathbb{E}_s\left(\int_s^t V_u^2\mathrm{d}u\right).\end{aligned}$$

Differentiating in time, we obtain

$$\frac{\mathrm{d}\mathbb{E}_s(V_t^2)}{\mathrm{d}t} = \kappa\widetilde{\theta}\mathbb{E}_s(V_u) - \kappa\mathbb{E}_s(V_u^2),$$

which is again a simple ordinary differential equation whose explicit solution reads

$$\mathbb{V}_s(V_t) = \frac{\xi^2 V_s}{\kappa}\left(\mathrm{e}^{-\kappa(t-s)} - \mathrm{e}^{-2\kappa(t-s)}\right) + \frac{\theta\xi^2}{2\kappa}\left(1 - \mathrm{e}^{-\kappa(t-s)}\right)^2.$$

In the Heston model, the stock price is a Doléans-Dade exponential $S_\cdot = S_0\mathcal{E}(V \circ W)_\cdot$, where

$$\mathcal{E}(V \circ\ W)_t := \exp\left(\int_0^t \sqrt{V_s}\mathrm{d}W_s - \frac{1}{2}\int_0^t V_s\mathrm{d}s\right).$$

Before using the model for modelling purposes, we first need to check whether S is a true martingale. It is a positive local martingale, hence a supermartingale, so we only need to show that $\mathbb{E}(S_t) = 1$ for all $t \geq 0$. The Novikov condition ensures that this is the case (see [95, Section 3.5.D]):

Proposition 4.2.4. *Let M be a $(\mathcal{F}_t)$-continuous local martingale and $Z_t := \exp\left(M_t - \frac{1}{2}\langle M, M\rangle_t\right)$, for all $t \geq 0$. If $\mathbb{E}\left(\exp\left(\frac{1}{2}\langle M, M\rangle_t\right)\right)$ is finite, then $\mathbb{E}(Z_t) = 1$ for all $t \geq 0$.*

In the Heston case, we therefore simply need to show that $\mathbb{E}\left(\exp(\frac{1}{2}\int_0^t V_s\mathrm{d}s)\right)$ is finite for all $t \geq 0$; in fact, the following theorem immediately follows from Proposition 4.2.3 setting $\lambda = 0$.

Theorem 4.2.5. *In the Heston model, the stock price is a true martingale.*

The theorem follows from the following representation of the characteristic function of the process: for any $t \geq 0$, define the characteristic function of X_t by $\phi_t(u) := \mathbb{E}\left(\mathrm{e}^{\mathrm{i}uX_t}\right)$, $u \in \mathbb{R}$. Then the following holds:

Proposition 4.2.6. *For any real number u, $\phi_t(u) = \exp\left(\mathtt{i}ux_0 + C(u,t) + D(u,t)v_0\right)$, where*

$$
\begin{aligned}
C(u,t) &:= \frac{\kappa\theta}{\xi^2}\left((\chi_{\mathtt{i}u} + \gamma_{\mathtt{i}u})\,t - 2\log\left(\frac{\zeta_{\mathtt{i}u}\mathrm{e}^{\gamma_{\mathtt{i}u}t} - 1}{\zeta_{\mathtt{i}u} - 1}\right)\right),\\
D(u,t) &:= \frac{\chi_{\mathtt{i}u} + \gamma_{\mathtt{i}u}}{\xi^2}\left(\frac{\mathrm{e}^{\gamma_{\mathtt{i}u}t} - 1}{\zeta_{\mathtt{i}u}\mathrm{e}^{\gamma_{\mathtt{i}u}t} - 1}\right),\\
\gamma_u &:= \sqrt{\chi_u^2 - \xi^2 u(u-1)}, \qquad \zeta_u := \frac{\chi_u + \gamma_u}{\chi_u - \gamma_u}, \qquad \chi_u := \kappa - \rho\xi u.
\end{aligned}
$$

Remark 4.2.7. This representation of the characteristic function is special since its logarithm is a linear function of the state variable x_0 and v_0. This turns out to be more than a simple mathematical curiosity, and falls into the realm of so-called affine models, as introduced by Duffie, Filipovic and Schachermayer [47]. This constitutes a large class of (multi-dimensional) Markov models (with or without jumps), for which the characteristic function can be computed via a system of (generalised) Riccati equations. Martin Keller-Ressel [98] studied the behaviour of a subclass of these, namely the affine stochastic volatility models, which are essentially the Heston model with additional state-dependent and state-independent jumps, both in the stock price and in the variance dynamics.

Proof. For any $\xi \in \mathbb{R}$, $0 \le t < T$, $x \in \mathbb{R}$, $v > 0$, define the function $\psi(t,x,v) \equiv \mathbb{E}\left(\mathrm{e}^{\mathtt{i}uX_T} | X_t = x, V_t = v\right)$. Then Feynman-Kać lemma yields

$$
\left(\frac{v}{2}\partial_{xx} + \rho\xi v\partial_{xv} + \frac{\xi^2 v}{2}\partial_{vv} - \frac{v}{2}\partial_x + \kappa(\theta - v)\partial_v + \partial_t\right)\psi = 0, \tag{4.2.9}
$$

with boundary conditions $\psi(T,x,v) = \mathrm{e}^{\mathtt{i}ux}$. Let us consider a solution of the form $\psi(t,x,v) \equiv \Phi(\tau,v)\mathrm{e}^{\mathtt{i}ux}$, with $\tau := T - t$. Equation (4.2.9) then becomes

$$
\left(\frac{\xi^2 v}{2}\partial_{vv} + (\mathtt{i}u\rho\xi v + \kappa(\theta - v))\,\partial_v - \frac{v}{2}\left(u^2 + \mathtt{i}u\right) - \partial_\tau\right)\Phi = 0,
$$

with boundary condition $\Phi(0,v) = 1$. Assuming that there exist two functions C and D such that $\Phi(\tau,v) \equiv \exp\left(C(\tau) + vD(\tau)\right)$, we obtain the following system of Riccati [129][2] equations:

$$
\begin{cases}
\dot{D}(\tau) &= \dfrac{\xi^2}{2}D(\tau)^2 + (\mathtt{i}u\rho\xi - \kappa)D(\tau) - \dfrac{u^2 + \mathtt{i}u}{2},\\
\dot{C}(\tau) &= \kappa\theta D(\tau),
\end{cases}
$$

with boundary conditions $C(0) = D(0) = 0$. It turns out that this system admits a closed-form solution, which proves the proposition. □

A special case: the Heston-Nandi model

On Equity markets, the observed implied volatility smile is essentially decreasing, with very little uplift on the large-strike side. Heston and Nandi [82] therefore suggested to consider a (discrete

[2] The original paper by Jacopo Riccati (1724), in latin, and its English translation are available at http://www.17centurymaths.com/contents/euler/rictr.pdf

time version of) the anticorrelated version of the Heston model, i.e. considering $\rho = -1$. In this case, it is easy to show that the stock price is bounded above almost surely:

Proposition 4.2.8. *For any $t \geq 0$, $X_t \leq x_0 + \frac{1}{\xi}(V_0 + \kappa\theta t)$ almost surely.*

Remark 4.2.9. As a corollary, the implied volatility is null above the upper bound (see Proposition 2.1.22 in Chapter 2).

Proof. Since the correlation parameter is equal to -1, we can rewrite the Heston stochastic differential equations (4.2.8) as

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t \mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, & X_0 = x_0 \in \mathbb{R}, \\ \mathrm{d}V_t &= \kappa\left(\theta - V_t\right)\mathrm{d}t - \xi\sqrt{V_t}\mathrm{d}W_t, & V_0 = v_0 > 0, \end{aligned}$$

so that, for any $t \geq 0$,

$$X_t = x_0 - \frac{1}{2}\int_0^t V_s \mathrm{d}s + \frac{1}{\xi}\int_0^t (\kappa(\theta - V_s)\mathrm{d}s - \mathrm{d}V_s) = x_0 + \frac{V_0 + \kappa\theta t}{\xi} - \int_0^t \left(\frac{1}{2} + \frac{\kappa}{\xi}\right) V_s \mathrm{d}s - \frac{V_t}{\xi}.$$

Since V_t is non-negative almost surely, the proposition follows. □

Multivariate Heston model

A straightforward extension of the Heston model consists of adding several layers of volatility processes, as follows: let $n \in \mathbb{N}$, and consider the unique strong solution to the following SDE:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}\sum_{i=1}^n V_t^{(i)}\mathrm{d}t + \sum_{i=1}^n \sqrt{V_t^{(i)}}\mathrm{d}W_t^{(i)}, & X_0 = x_0 \in \mathbb{R}, \\ \mathrm{d}V_t^{(i)} &= \kappa_i\left(\theta_i - V_t^{(i)}\right)\mathrm{d}t - \xi_i\sqrt{V_t^{(i)}}\mathrm{d}Z_t^{(i)}, i = 1, \dots, n & V_0^{(i)} = v_0^{(i)} > 0, \\ \mathrm{d}\left\langle W^{(i)}, Z^{(j)}\right\rangle_t &= \rho_i \mathbb{1}_{\{i=j\}}\mathrm{d}t, & i, j = 1, \dots, n, \\ \mathrm{d}\left\langle W^{(i)}, W^{(j)}\right\rangle_t &= \mathrm{d}\left\langle Z^{(i)}, Z^{(j)}\right\rangle_t = 0, & i, j = 1, \dots, n, \end{aligned} \tag{4.2.10}$$

where $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_n) \in (0, \infty)^n$, $\boldsymbol{\rho} \in (-1, 1)^n$, $\boldsymbol{\theta} \in (0, \infty)^n$, $\boldsymbol{\xi} \in (0, \infty)^n$, $\boldsymbol{v_0} \in (0, \infty)^n$. Because of the special structure of the correlation matrix of the Brownian motions, it can be shown that the characteristic function $\mathbb{E}(\mathrm{e}iuX_t)$ can be represented as the product of standard Heston models:

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}uX_t}\right) = \prod_{i=1}^n \mathbb{E}\left(\mathrm{e}^{\mathrm{i}uX_t^{(i)}}\right),$$

for any $t \geq 0$, $u \in \mathbb{R}$, where $X^{(i)} \sim \mathrm{H}(\kappa_i, \theta_i, \rho_i, \xi_i, v_0^{(i)})$, with initial condition $X_0^{(i)} = x_0$.

Uncorrelated displaced Heston model

The uncorrelated displaced Heston stochastic volatility model is a slight modification of the uncorrelated Heston model, where the stock price process satisfies the following SDE:

$$\begin{aligned} \mathrm{d}S_t &= \left(\beta S_t + (1-\beta)S_0\right)\sqrt{V_t}\mathrm{d}W_t, & S_0 > 0, \\ \mathrm{d}V_t &= \kappa\left(\theta - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, & V_0 > 0, \end{aligned}$$

where the two Brownian motions W and Z are uncorrelated, and where $\beta \in [0,1]$ represents the displacement parameter. Pricing under this model is equivalent to pricing under a rescaled Heston model. Indeed, let $\widetilde{S}_t := \beta S_t + (1-\beta)S_0$ for all $t \geq 0$; then $\log(\widetilde{S}) \sim \mathrm{H}(\kappa, \beta^2\theta, \rho, \beta\xi, \beta^2 v_0)$. Therefore, the price of a European Call on S with strike e^x reads

$$\mathbb{E}\left(S_t - \mathrm{e}^x\right)_+ = \frac{1}{\beta}\mathbb{E}\left(\widetilde{S}_t - (1-\beta)S_0 - \beta\mathrm{e}^x\right)_+ = \frac{1}{\beta}\mathbb{E}\left(\widetilde{S}_t - \mathrm{e}^{\widetilde{x}}\right)_+,$$

where $\widetilde{x} := \log\left((\beta-1)S_0 - \beta\mathrm{e}^x\right)$.

4.2.4 Other popular stochastic volatility models

The 3/2 model

Consider the following stochastic volatility model for the logarithmic stock price process $(X_t)_{t\geq 0}$,

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, && X_0 = x_0 > 0, \\ \mathrm{d}V_t &= \kappa V_t\left(\theta - V_t\right)\mathrm{d}t + \xi V_t^{3/2}\mathrm{d}Z_t, && V_0 = v_0 > 0, \\ \mathrm{d}\left\langle W, Z\right\rangle_t &= \rho\mathrm{d}t, \end{aligned} \tag{4.2.11}$$

with $\kappa > 0$, $\theta > 0$, $\xi > 0$, $|\rho| < 1$. Let us first assume that the variance process $(V_t)_{t\geq 0}$ never hits the origin almost surely, and define its inverse $Z_t := V_t^{-1}$ for all $t \geq 0$. By Itô's lemma, we obtain $\mathrm{d}Z_t = \left(\kappa + \xi^2 - \kappa\theta Z_t\right)\mathrm{d}t - \xi Z_t^{1/2}\mathrm{d}W_t$, with $Z_0 = v_0^{-1}$. Define now $\widetilde{\kappa} := \kappa\theta$ and $\widetilde{\theta} := \dfrac{\kappa+\xi^2}{\kappa\theta}$. Then the process $(Z_t)_{t\geq 0}$ is equal in law to the Feller diffusion defining the variance in the Heston model. The Feller condition, $\kappa + \frac{1}{2}\xi^2 \geq 0$, is always satisfied when $\kappa > 0$. Define now the functions

$$\begin{aligned} \mu_u &:= 1 + \frac{2\widetilde{\gamma}_u}{\xi^2}, & \beta_t &:= \frac{v_0}{\kappa\theta}\left(\mathrm{e}^{\kappa\theta t} - 1\right), & & \\ \alpha_u &:= \frac{\widetilde{\gamma}_u - \widetilde{\chi}_u}{\xi^2}, & \widetilde{\chi}_u &:= \chi(u) + \frac{1}{2}\xi^2, & \widetilde{\gamma}_u &:= \left(\widetilde{\chi}_u^2 - \xi^2 u\left(u-1\right)\right)^{1/2}, \end{aligned} \tag{4.2.12}$$

with χ defined in Proposition 4.2.6. Recall the Kummer—confluent hypergeometric—function:

$$\mathrm{M}\left(\alpha, \mu, z\right) := \sum_{n\geq 0} \frac{(\alpha)_n}{(\mu)_n}\frac{z^n}{n!},$$

where the Pochhammer symbol is defined by $(\alpha)_0 = 1$ and $(\alpha)_n := \alpha\left(\alpha+1\right)\cdots\left(\alpha+n-1\right)$, $n \geq 1$.

Remark 4.2.10. If $\Re(\mu) > 0$ and $\Re(\alpha) > 0$ then M admits the following representation:

$$\mathrm{M}(\alpha, \mu, \mathtt{i}z) = \mathbb{E}\left(\mathrm{e}^{\mathtt{i}zY}\right) = \frac{\Gamma(\mu)}{\Gamma(\alpha)\Gamma(\mu-\alpha)}\int_0^1 \mathrm{e}^{\mathtt{i}zx}x^{\alpha-1}(1-x)^{\mu-\alpha-1}\mathrm{d}x,$$

where Y is a Beta-distributed random variable with parameters α and $\mu - \alpha$.

The following lemma can be found in [27].

Lemma 4.2.11. *For any $t \geq 0$,*

$$\mathbb{E}\left(\mathrm{e}^{uX_t}\right) = \mathrm{e}^{ux_0}\frac{\Gamma\left(\mu_u - \alpha_u\right)}{\Gamma\left(\mu_u\right)}\left(\frac{2}{\beta_t\xi^2}\right)^{\alpha_u}\mathrm{M}\left(\alpha_u, \mu_u, \frac{-2}{\beta_t\xi^2}\right),$$

for all $u \in \mathbb{R}$ such that the right-hand side is a well-defined real number.

The Schöbel-Zhu model

Introduced in [142], the Schöbel-Zhu stochastic volatility model is an extension to non zero spot-volatility correlation of the Stein & Stein [147] model in which the logarithmic spot price process $(X_t)_{t\geq 0}$ satisfies the following system of SDEs

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}\sigma_t^2\mathrm{d}t + \sigma_t\mathrm{d}W_t, && X_0 = x_0 \in \mathbb{R}, \\ \mathrm{d}\sigma_t &= \kappa\left(\theta - \sigma_t\right)\mathrm{d}t + \xi\mathrm{d}Z_t, && \sigma_0 > 0, \\ \mathrm{d}\left\langle W, Z\right\rangle_t &= \rho\mathrm{d}t, \end{aligned} \tag{4.2.13}$$

where κ, θ and ξ are strictly positive real numbers, $\rho \in [-1, 1]$ and W, Z are two correlated standard Brownian motions. The volatility process $(\sigma_t)_{t\geq 0}$ is Gaussian and hence the SDE is well defined. The process $(X_t)_{t\geq 0}$ is simply the integrated volatility process and hence is well defined as well. The cumulant generating function $\Lambda_{\mathrm{SZ}}(\cdot, t)$ of the process $(X_t)_{t\geq 0}$ reads (see [94])

$$\Lambda_{\mathrm{SZ}}(u, t) = \Lambda_{\widehat{\mathrm{H}}}(u, t) + A(u, t) + B(u, t)\sigma_0, \tag{4.2.14}$$

where $\widehat{\mathrm{H}}\left(\kappa, \theta, \rho, \xi, v_0\right) := \mathrm{H}\left(2\kappa, \frac{\xi^2}{2\kappa}, \rho, 2\xi, v_0\right)$. Furthermore

$$\begin{aligned} B(u,t) &:= \frac{\kappa\theta}{\xi^2}\frac{\chi_u - \gamma_u}{\gamma_u}\frac{\left(1 - \exp\left(-\gamma_u t\right)\right)^2}{1 - \zeta_u \exp\left(-2\gamma_u t\right)}, \\ A(u,t) &:= \frac{\kappa^2\theta^2}{2\gamma_u^3\xi^2}\left(\chi_u - \gamma_u\right)\left(\chi_u\left(\gamma_u t - 2\right) + \gamma_u\left(\gamma_u t - 1\right) + 2\mathrm{e}^{-\gamma_u t}\frac{2\chi_u + \frac{\gamma_u^2 - 2\chi_u^2}{\chi_u + \gamma_u}\mathrm{e}^{-\gamma_u t}}{1 - \zeta_u \mathrm{e}^{-2\gamma_u t}}\right) \end{aligned}$$

where the functions χ, γ, ζ are the same as in the Heston model (Proposition 4.2.6). It is clear from the form of the Laplace transform that this model is not affine in the same sense as the Heston model, but is affine on the extended state space $\left(X_t, \sigma_t^2, \sigma_t\right)_{t\geq 0}$.

The ExpOU model

In the ExpOU model the variance is driven by the exponential of an Ornstein-Uhlenbeck process. It was first introduced by Scott [143] without correlation between the stock price and its instantaneous variance. The generalised version of it—with correlation—satisfies the following system of SDEs:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}\mathrm{e}^{Y_t}\mathrm{d}t + \sqrt{\mathrm{e}^{Y_t}}\mathrm{d}Wt, && X_0 = 0, \\ \mathrm{d}Y_t &= \kappa(\theta - Y_t)\mathrm{d}t + \omega\xi\mathrm{d}Z_t, && Y_0 = \theta, \\ \mathrm{d}\left\langle W, Z\right\rangle_t &= \rho\mathrm{d}t, \end{aligned} \tag{4.2.15}$$

with $\kappa > 0$, $\theta \in \mathbb{R}$, and $\omega > 0$. However, the characteristic function is not available in closed form, and hence the only pricing methods are numerical simulation (Monte Carlo) and finite differences.

Linear approximation of the ExpOU model

In [21], the authors give a closed-form expression for the characteristic function, under the risk-neutral measure, of a linear approximation of the ExpOU process (4.2.15):

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}\left(2L_t - 1 + \mathcal{M}_t\right)\mathrm{d}t + L_t \mathrm{d}Wt, && X_0 = 0, \\ \mathrm{d}L_t &= \kappa\left(\gamma - L_t\right)\mathrm{d}t + \xi \mathrm{d}Z_t, && L_0 = l_0, \\ \mathrm{d}\langle W, Z\rangle_t &= \rho \mathrm{d}t, \end{aligned} \tag{4.2.16}$$

where $\mathcal{M}_t$ is a deterministic factor ensuring that the stock price process is a true martingale and the other parameters are as in (4.2.15). The characteristic function is given in [21, Equation 2.8].

4.2.5 Fractional stochastic volatility models

One of the key assumptions among all the stochastic (volatility) models we have seen above is that the drivers are standard Brownian motions. As irregular as the paths might look like, Brownian motion is a Markov process, and hence has no memory. It has long been observed, by the econometrics community, that the instantaneous volatility exhibits long memory, which is not compatible with the Brownian framework (see for example the discrete-time literature on ARCH, GARCH, FIGARCH,....). We shall briefly show here how to relax the Brownian assumption in order to allow for more general drivers.

Fractional Brownian motion

Definition 4.2.12. A fractional Brownian motion $(W_t^H)_{t\geq 0}$ with Hurst exponent $H \in (0,1)$ is a Gaussian process with covariance function

$$R_H(s,t) := \mathbb{E}(W_t^H W_s^H) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

This process was introduced by Kolmogorov [100] and originally studied by Mandelbrot and Van Ness [113]. The Hurst exponent got its name from the climatologist Hurst [85], who used it the yearly water run-offs of the Nile river. It possesses the following interesting properties:

- there exists a version of the fractional Brownian motion with continuous trajectories.
- The paths of the fBm are Hölder continuous of order $H - \varepsilon$, for any $\varepsilon > 0$.
- Self-similarity: For any $\alpha > 0$, $t \geq 0$, the random variables $\alpha^{-H} W_{\alpha t}^H$ and W_t^H have the same distributions.
- Stationary increments: for any $0 \leq s \leq t$, the increment $W_t^H - W_s^H$ is Gaussian with mean zero and variance $|t-s|^{2H}$.

- When $H = 1/2$, the covariance function reads $R_H(s,t) \equiv \min(s,t)$, so that W^H is a standard Brownian motion. However, whenever $H \neq 1/2$, the increments are not independent.

- The sequence $\{W_n^H - W_{n-1}^H, n \geq 1\}$ is a Gaussian stationary sequence with variance equal to one and covariance

$$\begin{aligned}\rho_H(n) := \mathbb{E}\left[W_1^H\left(W_n^H - W_{n-1}^H\right)\right] &= \mathbb{E}\left(W_1^H W_n^H\right) - \mathbb{E}\left(W_1^H W_{n-1}^H\right) = R_H(1,n) - R_H(1,n-1)\\ &= \frac{1}{2}\left(n^{2H} - |n-1|^{2H} - (n-1)^{2H} + |n-2|^{2H}\right)\\ &= H(2H-1)n^{2H-2} + \mathcal{O}\left(n^{2H-3}\right), \quad \text{as } n \text{ tends to infinity,}\end{aligned}$$

which converges to zero. In particular, if $H > 1/2$, the series $\sum_{n\geq 1}\rho_H(n)$ diverges to infinity, and the sequence $\{W_n^H - W_{n-1}^H, n \geq 1\}$ is said to have long-range dependence. On the contrary, when $H < 1/2$, the series converges absolutely, and the sequence has short-range dependence.

- The fractional Brownian motion is the only self-similar Gaussian process with stationary increments.

Rough volatility

In [31], Comte and Renault proposed the following fractional stochastic volatility model:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}\mathrm{e}^{2\sigma_t}\mathrm{d}t + \mathrm{e}^{\sigma_t}\mathrm{d}B_t, & X_0 = 0,\\ \mathrm{d}\sigma_t &= \kappa(\theta - \sigma_t)\mathrm{d}t + \xi \mathrm{d}W_t^H, & \sigma_0 > 0, \end{aligned} \tag{4.2.17}$$

where B is a standard Brownian motion and W^H a fractional Brownian motion with Hurst exponent H. The two Brownian motions can exhibit a correlation structure. Instead of following this path, one could start with the simple Black-Scholes model, where the driving Brownian motion is replaced by a fractional one. However, as proved by Rogers [131], the process W^H (and hence the solution to the fractional SDE $\mathrm{d}S_t = S_t\mathrm{d}W_t^H$) is not a semimartingale whenever $H \neq 1/2$. As discussed in Chapter 1, the whole pricing framework, developed by Delbaen and Schachermayer [41] is based on the semimartingale assumption. In fact, when the latter fails, no-arbitrage theory essentially breaks down, thus giving no hope for a fractional version of Black-Scholes. Here, in the fractional stochastic volatility model, the stock price remains a semimartingale, and no-arbitrage theory carries over. Left aside for a decade, these models have recently been dug out from their temporary graves, and Gatheral, Jaisson and Rosenbaum [71] have calibrated the model to the S&P 500, showing that the Hurst parameter should be close to 0.11. This indicates extremely rough paths for the volatility process, much more irregular than those of standard stochastic volatility models, and short memory. The main drawback, at least for now, of these rough models, is the actual pricing side, for which numerical methods are currently not efficient enough.

4.2.6 The Uncertain Volatility Model

We have so far assumed that the instantaneous volatility of the stock price process was either stochastic or depended directly on the asset price. In 1995, Avellaneda, Levy and Paras [6] and Lyons [111] suggested a different route, later refined by Denis and Martini [44]. Consider the diffusion, on a given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$,

$$\mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t, \qquad S_0 = 1, \tag{4.2.18}$$

where W is a standard $\mathcal{F}$-adapted Brownian motion. The stochastic process $(\sigma_t)_{t\geq 0}$ is assumed to be progressively measurable and valued in some interval $\mathcal{A} = [\underline{\sigma}, \overline{\sigma}] \subset [0, \infty]$. Let now $T \geq 0$ be a fixed time horizon, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a payoff with at most linear growth, and define the following value function:

$$V(t,s) := \sup_{\sigma \in \mathcal{A}} \mathbb{E}\left[\varphi(S_T) | S_t = s\right], \qquad \text{for all } t \in [0,T], s \geq 0. \tag{4.2.19}$$

From the point of view of an option seller, $V(\cdot)$ corresponds to a worst-case scenario, and clearly super-replicates the European option with payoff φ. Since S is a non-negative supermartingale, the value function V inherits the growth property of φ and is locally bounded. This super-replicating problem can be seen as a stochastic control problem, where the Hamiltonian reads

$$\mathcal{H}(s,M) := \sup_{\alpha \in [\underline{\alpha}, \overline{\alpha}]} \left\{ \frac{1}{2}\alpha^2 s^2 M \right\}, \qquad \text{for all } (s,M) \in (0,\infty) \times \mathbb{R}.$$

We follow here the excellent book on stochastic control with financial applications by Pham [123], and in particular Section 4.6 therein. Consider first the case where the upper bound $\overline{\sigma}$ is finite. In that case, the Hamiltonian is finite everywhere, and clearly $\mathcal{H}(s,M) = \frac{1}{2}\widehat{\sigma}(M)s^2 M$, where $\widehat{\sigma}(M) := \overline{\sigma}\mathbb{1}_{\{M\geq 0\}} + \underline{\sigma}\mathbb{1}_{\{M<0\}}$, and the following holds.

Theorem 4.2.13. *If $\overline{\sigma}$ is finite, then the value function V is continuous on $[0,T) \times (0,\infty)$ and is the unique viscosity solution with linear growth of the Black-Scholes-Barenblatt equation*

$$\partial_t V(s,t) + \frac{1}{2}\widehat{\sigma}^2\left(\partial_{ss}V(s,t)\right) s^2 \partial_{ss} V(s,t) = 0, \qquad \text{on } (t,s) \in [0,T) \times (0,\infty), \tag{4.2.20}$$

with boundary condition $V(s,T) = \varphi(s)$ for all $s > 0$.

Remark 4.2.14. When $\underline{\sigma}$ is strictly positive, existence and uniqueness of a smooth solution to (4.2.20) is guaranteed under the Cauchy boundary condition (see for instance [39] or [58] for general results on viscosity solutions of partial differential equations).

Let us now consider the case where $\overline{\sigma}$ is infinite, so that the instantaneous volatility is unbounded above. In that case, the Hamiltonian of the control problem reads

$$\mathcal{H}(s,M) = \begin{cases} \frac{1}{2}\underline{\sigma}^2 s^2 M, & \text{if } M \leq 0, \\ +\infty, & \text{if } M > 0. \end{cases}$$

The value function defined in (4.2.19) is then the viscosity solution to the following variational inequality:

$$\max\left\{\partial_t V + \frac{1}{2}\underline{\sigma}^2 s^2 \partial_{ss} V, \partial_{ss} V\right\} 0, \qquad \text{for } (t,s) \in [0,T) \times (0,\infty),$$

together with the boundary condition $V(T^-, s) \equiv \widetilde{\varphi}(s)$ on $(0,\infty)$, where $\widetilde{\varphi}$ is the smallest function dominating φ and satisfying $\partial_{ss}\widetilde{\varphi}(s) \geq 0$ on $(0,\infty)$ in the viscosity sense.

Theorem 4.2.15. *If $\overline{\sigma} = \infty$, then the value function satisfies*

$$V(t,s) = \mathbb{E}\left[\widetilde{\varphi}\left(\widetilde{S}_T\right) | \widetilde{S}_t = s\right],$$

for all $(t,s) \in [0,T] \times (0,\infty)$, where $\widetilde{S}$ is the unique strong solution to the stochastic differential equation $\mathrm{d}\widetilde{S}_t = \underline{\sigma}\widetilde{S}_t \mathrm{d}W_t$, starting at $\widetilde{S}_0 > 0$.

4.2.7 The 'new' generation

Local-stochastic volatility models

Variance curve models

In a series of papers, Lorenzo Bergomi proposed the following model for the dynamics of the log stock price process:

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= -\frac{1}{2}\xi_t(t) + \sqrt{\xi_t(t)}\mathrm{d}Z_t, \qquad X_0^\varepsilon = 0, \\ \mathrm{d}\xi_t(u) &= \varepsilon\lambda(t,u,\xi_t) \cdot \mathrm{d}W_t, \end{aligned} \tag{4.2.21}$$

where we normalised the initial value of the stock price and assumed no interest rate or dividend; W a d-dimensional Brownian motion, and $(Z_t)_{t\geq 0}$ is a standard Brownian motion, which may be correlated with the components of W. The forward variance curve $\xi_\cdot(\cdot)$ is defined as

$$\xi_t(u) := \mathbb{E}\left[\xi_u(u)|\mathcal{F}_t\right].$$

As ε becomes small, Bergomi and Guyon showed that the price of a European option on $\exp(X^\varepsilon)$ had the following expansion:

$$P = \left\{1 + \frac{\varepsilon}{2}C^{x,\xi}\partial_{xx}\left(\partial_x - 1\right) + \varepsilon^2\left[\frac{1}{8}C^{\xi\xi}\left(\partial_x - 1\right)^2 + \frac{1}{8}\left(C^{x\xi}\right)^2\partial_x^4\left(\partial_x - 1\right)^2 + \frac{1}{2}C^{\mu}\partial_x^3\left(\partial_x - 1\right)\right]\right\}\mathrm{BS}.$$

The covariance functions are given by

$$\begin{aligned} C^{x\xi} &:= \int_0^T \mathrm{d}t \int_t^T \frac{\mathbb{E}\left(\mathrm{d}X_t \mathrm{d}\xi_t(u)\right)}{\mathrm{d}t}\mathrm{d}u, \\ C^{\xi\xi} &:= \int_0^T \mathrm{d}t \int_t^T \mathrm{d}s \int_t^T \frac{\mathbb{E}\left(\mathrm{d}\xi_t(s)\mathrm{d}\xi_t(u)\right)}{\mathrm{d}t}\mathrm{d}u, \\ C^{\mu} &:= \int_0^T \mathrm{d}t \int_t^T \frac{\mathbb{E}\left(\mathrm{d}X_t \mathrm{d}\xi_t(u)\right)}{\mathrm{d}t}\frac{\partial C_t^{x\xi}}{\partial\xi_t(u)}\mathrm{d}u. \end{aligned}$$

Application to the Heston model

Introducing jumps

4.3 Hedging: which volatility to choose?

We have so far studied different notions of volatility. We would now like to revisit the standard Black-Scholes Delta hedging theory in light of the following question: suppose an investor buys an option V_i (with maturity $T > 0$) today and delta-hedges it with some stock. The Delta hedge quantity is computed via the Delta of the option, using the Black-Scholes formula, with some volatility parameter. However which volatility parameter should one choose for the hedge? Clearly, the option is bought at the implied volatility Σ. We assume that the stock price process $(S_t)_{t\geq 0}$ is the unique strong solution to the following stochastic differential equation:

$$\mathrm{d}S_t/S_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, \qquad S_0 > 0,$$

where σ represents the actual realised volatility, and μ the actual realised drift. Let us first assume that the hedge is performed using the realised volatility σ. At inception of the contract, the investor buys the option, sells a quantity Δ_R of the stock, having a cash position worth $-V_i + \Delta_R S$, where we denote by Δ_R (resp. Δ_i) the Black-Scholes Delta computed with the realised volatility σ (resp. with the implied volatility Σ). Over a short period of time, the dynamics of the P&L reads

$$\mathrm{d}P\&L = \mathrm{d}V_i - \Delta_R \mathrm{d}S - r\left(V_i - \Delta_R S\right)\mathrm{d}t, \tag{4.3.1}$$

where r represents the risk-free interest rate over the period. Note that, had the investor bought the option at the volatility price σ, then $\mathrm{d}P\&L$ would be null, namely $\mathrm{d}V_R - \Delta_R \mathrm{d}S - r\left(V_R - \Delta_R S\right)\mathrm{d}t$. Plugging this expression in the P&L equation, we obtain

$$\mathrm{d}P\&L = \mathrm{d}V_i - \mathrm{d}V_R - r(V_i - V_R)\mathrm{d}t = \mathrm{e}^{rt}\mathrm{d}\left[\mathrm{e}^{-rt}(V_i - V_R)\right],$$

so that the present value of the P&L reads

$$PV(P\&L) = \int_0^T \mathrm{e}^{-rt}\mathrm{d}P\&L = V_R - V_i. \tag{4.3.2}$$

This in particular implies that the final P&L is known and deterministic as soon as we know the realised volatility σ.

Let us now consider a more dynamic version of this result. Itô's lemma reads

$$\begin{aligned}
\mathrm{d}P\&L &= \left(\theta_i \mathrm{d}t + \Delta_i \mathrm{d}S_t + \frac{1}{2}\sigma^2\Gamma_i S_t^2 \mathrm{d}t\right) - \Delta_R \mathrm{d}S_t - r(V_i - \Delta_R S_t)\mathrm{d}t \\
&= \left(\theta_i + \frac{1}{2}\sigma^2\Gamma_i^2 S_t^2\right)\mathrm{d}t + (\Delta_i - \Delta_R)\mathrm{d}S_t - r(V_i - \Delta_R S_t)\mathrm{d}t \\
&= \frac{1}{2}\Gamma_i^2 S_t^2\left(\sigma^2 - \Sigma^2\right)\mathrm{d}t + (\Delta_i - \Delta_R)\left[(\mu - r)S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t\right],
\end{aligned}$$

where θ_i and Γ_i respectively represent the Theta and the Gamma of the option computed in the Black-Scholes model with the implied volatility Σ. The third line follows from the Black-Scholes equation $\theta_i = -\frac{1}{2}\Sigma^2 S_t^2 \Gamma_i + rV_i - rS_t\Delta_i$. Note that, even though (Equation (4.3.2)) the final P&L is deterministic, its differential increments are random.

Let us now assume, in this dynamic framework, that the hedging is performed using the implied volatility Σ rather than the realised volatility σ. The dynamic equation (4.3.1) then reads $\mathrm{d}P\&L = \mathrm{d}V_i - \Delta_i \mathrm{d}S - r\left(V_i - \Delta_i S\right)\mathrm{d}t$. Applying Itô's formula again, we obtain

$$\begin{aligned}\mathrm{d}P\&L &= \left(\theta_i \mathrm{d}t + \Delta_i \mathrm{d}S_t + \frac{1}{2}\sigma^2\Gamma_i S_t^2 \mathrm{d}t\right) - \Delta_i \mathrm{d}S_t - r(V_i - \Delta_i S_t)\mathrm{d}t \\ &= \frac{1}{2}\Gamma_i S_t^2\left(\sigma^2 - \Sigma^2\right)\mathrm{d}t,\end{aligned}$$

where we, again, used the Black-Scholes partial differential equation satisfied by V_i. Therefore the present value of the P&L is worth

$$PV(P\&L) = \frac{1}{2}\int_0^T \mathrm{e}^{-rt}\Gamma_i S_t^2\left(\sigma^2 - \Sigma^2\right)\mathrm{d}t.$$

4.3.1 Application to volatility arbitrage

See for example the report and slides of the Mid-term project 'Volatility Arbitrage in Delta Hedging' (available on the course webpage).

4.4 Put-Call symmetry

We are interested here in the symmetry arising between European Put options and European Call options. Note that this is different from the standard Put-Call parity. We will in particular see how this symmetry property informs us about the properties of the implied volatility smile.

4.4.1 Black-Scholes

Consider the Black-Scholes model $\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma\mathrm{d}W_t$ with $S_0 > 0$. Since the discounted process $(\mathrm{e}^{-rt}S_t)_{t\geq 0}$ is a true martingale, then we can define—via the Radon-Nikodym derivative—a new probability measure $\mathbb{P}^*$ by

$$\left.\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := \frac{S_t}{S_0\mathrm{e}^{rt}} = \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right).$$

Using the Girsanov change of measure and the fact that $W_t^* := W_t - \sigma t$ is a standard Brownian motion under $\mathbb{P}^*$, we obtain that $\mathrm{e}^{2rt}\frac{S_0^2}{S_t} = S_0\mathrm{e}^{-\sigma W_t^* + (r - \frac{1}{2}\sigma^2)t}$ has the same law under $\mathbb{P}^*$ as S_t

under $\mathbb{P}$. Therefore

$$\begin{aligned}\mathrm{BS}(K,T,\sigma) := \mathrm{e}^{-rT}\mathbb{E}(S_T-K)_+ &= \mathrm{e}^{-rT}\frac{K}{S_0\mathrm{e}^{rT}}\mathbb{E}\left[\frac{S_T}{S_0\mathrm{e}^{rT}}\left(\frac{(S_0\mathrm{e}^{rT})^2}{K}-\frac{(S_0\mathrm{e}^{rT})^2}{S_T}\right)_+\right]\\ &= \mathrm{e}^{-rT}\frac{K}{S_0\mathrm{e}^{rT}}\mathbb{E}^*\left(\frac{(S_0\mathrm{e}^{rT})^2}{K}-S_0\mathrm{e}^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T^*}\right)_+\\ &= \frac{K}{S_0\mathrm{e}^{rT}}P_{\mathrm{BS}}\left(\frac{(S_0\mathrm{e}^{rT})^2}{K},T,\sigma\right).\end{aligned}$$

Using the change of coordinates $x := \log(K/S_0) - rT$, the above expression reduces to

$$\widetilde{C}_{\mathrm{BS}}(x,T,\sigma) = \mathrm{e}^x\widetilde{P}_{\mathrm{BS}}(-x,T,\sigma). \tag{4.4.1}$$

4.4.2 Stochastic volatility models

We now look at the implications of the above symmetry in the framework of stochastic volatility models. Let us therefore assume that the discounted stock price $(\mathrm{e}^{-rt}S_t)_{t\geq 0}$ is a true martingale under a given risk-neutral measure $\mathbb{P}$. This process is no longer restricted to the Black-Scholes framework and follows the dynamics

$$\begin{aligned} \mathrm{d}S_t/S_t &= r\mathrm{d}t + \xi_t\mathrm{d}W_t, & S_0>0,\\ \mathrm{d}\xi_t &= b(t,X_t)\mathrm{d}t + a(t,X_t)\mathrm{d}B_t, & \xi_0>0,\\ \mathrm{d}\left\langle W,B\right\rangle_t &= \rho\mathrm{d}t, \end{aligned} \tag{4.4.2}$$

where W and B are two correlated Brownian motions ($\rho\in(-1,1)$), and the coefficients b and a are such that a unique strong solution exists. A similar analysis holds, but the process $(S_t^*)_{t\geq 0}$ defined pathwise by $S_t^* := \mathrm{e}^{2rt}S_0^2/S_t$, which is a true martingale under $\mathbb{P}^*$ does not necessarily have the same law as S under $\mathbb{P}$. Still, the equality $\widetilde{C}(x,T) = \mathrm{e}^x\widetilde{P}^*(-x,T)$ holds. Since the implied volatility $\sigma_T(x)$ satisfies the equation $\widetilde{C}(x,T) = \widetilde{C}_{\mathrm{BS}}(x,T,\sigma_T(x))$, using (4.4.1), we also obtain the equality

$$\widetilde{P}^*\left(\frac{S_0^2\mathrm{e}^{2rT}}{K},T\right) = \widetilde{P}_{\mathrm{BS}}\left(\frac{S_0^2\mathrm{e}^{2rT}}{K},T,\sigma_T(K)\right).$$

The Put-Call parity therefore implies

$$\widetilde{C}^*\left(\frac{S_0^2\mathrm{e}^{2rT}}{K},T\right) = \widetilde{C}_{\mathrm{BS}}\left(\frac{S_0^2\mathrm{e}^{2rT}}{K},T,\sigma_T(K)\right),$$

and hence $\sigma_T(K)$ is also the implied volatility corresponding to a (European) Call option with strike $\frac{S_0^2\mathrm{e}^{2rT}}{K}$ in the transformed model under $\mathbb{P}^*$: $\widetilde{\sigma}(x,T) = \widetilde{\sigma}^*(-x,T)$.

This analysis, developed in [127], is particularly revealing when one consider the case of an uncorrelated stochastic volatility model, where $\rho = 0$. Indeed, in that case, the law of the stochastic volatility process ξ remains unchanged under $\mathbb{P}^*$, so that the law of S^* under $\mathbb{P}^*$ is the same as that of S under $\mathbb{P}$. In particular this implies that $\widetilde{\sigma}(x,T) = \widetilde{\sigma}(-x,T)$ for any $x\in\mathbb{R}$ and $T\geq 0$, i.e. the smile is symmetric.

4.5 Link with the Skorokhod embedding problem

Consider the diffusion process $\mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t$, starting at $S_0 > 0$, on some given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\sigma_t)_{t\geq 0}$ is $\mathcal{F}$-adapted. Denote by $V_t := \langle S\rangle_t$ the quadratic variation process of S and let $M_t := S_t - S_0$. We assume that σ is such that S is a true $\mathbb{P}$-martingale, so that the Dambis-Dubins-Schwartz theorem implies that the process $(B_t)_{t\geq 0}$ defined pathwise by

$$B_t := M_{\inf\{u:\langle M\rangle_u \geq t\}}$$

is a Brownian motion such that $M_t = B_{\langle B\rangle_t} = B_{\langle S\rangle_t}$ and hence $S_t = S_0 + B_{\langle M\rangle_t}$.

Consider now an option written on both the stock price and the realised variance, with pay-off $\Phi(S_T, V_T)$ at some maturity $T > 0$. Given that no assumption has been made no the process $(\sigma_t)_{t\geq 0}$, it is in general not possible to determine an exact price; however, bounds can be determined as follows.

$$\sup_{\sigma} \mathbb{E}\Phi(S_T, V_T) = \sup_{\sigma} \mathbb{E}\Phi\left(\langle S\rangle_T, S_0 + B_{\langle M\rangle_t}\right) = \sup_{\tau, \mathcal{F}^B-\text{stopping time}} \mathbb{E}\Phi\left(\tau, S_0 + B_\tau\right).$$

A similar statement obviously holds for the infimum. However, these bounds are usually too large to be useful (in the case of the variance swap, for instance, they yield the interval $(0, \infty)$). In order to refine this approach, one can add market constraints, namely that, at maturity T, the law of the underlying $S_T - S_0$ is known, say μ_T. Note that this law is fully determined by the knowledge of all European Call and Put options maturing at T. The new problem then reads

$$\begin{aligned} &\sup_{\tau, \mathcal{F}^B-\text{stopping time}} \mathbb{E}\Phi(\tau, S_0 + B_\tau), \\ &\text{subject to } B_\tau \sim \mu_\tau. \end{aligned} \tag{4.5.1}$$

This is reminiscent of the original Skorokhod Embedding Problem:

Given a measure μ such that $\int x\mu(\mathrm{d}x) = 0$ and $\int x^2\mu(\mathrm{d}x) < \infty$, find a stopping time τ with $\mathbb{E}(\tau)\infty$, such that $B_\tau \sim \mu$, where B is a standard Brownian motion.

This was first proposed by Skorokhod in 1964 (see [145] for an English translation), and many solutions, with different properties, have been proposed since. We refer the reader to the excellent survey paper [122] for details about these solutions. One of the simplest solutions is that of Hall [79]: let U and V denote two random variables with joint law ρ given by

$$\rho(\mathrm{d}u, \mathrm{d}v) \equiv \frac{|u| + v}{\alpha}\mu(\mathrm{d}u)\mu(\mathrm{d}v)\mathbb{1}_{\{u<0\}}\mathbb{1}_{\{v\geq 0\}},$$

where $\alpha := \int_0^\infty x\mu(\mathrm{d}x)$ is well defined by assumption. We now show that the stopping time $\tau := \inf\{t \geq 0 : B_t \notin (U, V)\}$ satisfies the Skorokhod embedding problem. For $u < 0$, we have

$$\mathbb{P}(B_\tau \in \mathrm{d}u) = \int_{[0,\infty)} \mathbb{P}(U \in \mathrm{d}u, V \in \mathrm{d}v)\mathbb{P}(B_\tau \in \mathrm{d}u | U \in \mathrm{d}u, V \in \mathrm{d}v) = \int_{[0,\infty)} \rho(\mathrm{d}u, \mathrm{d}v)\frac{v}{|u| + v} = \mu(\mathrm{d}u).$$

Furthermore,

$$\mathbb{E}(\tau) = \mathbb{E}[\mathbb{E}(\tau(U,V)] = \int_{-\infty}^{0}\int_{0}^{\infty}\rho(\mathrm{d}u,\mathrm{d}v)\left(\frac{u^2 v}{|u|+v} + \frac{v^2 u}{|u|+v}\right) = \int_{\mathbb{R}} u^2\mu(\mathrm{d}u),$$

which is finite by assumption; we have used here the martingale stopping theorem applied to the martingale process $(B_t^2 - t)_{t\geq 0}$. Note that this solution satisfies indeed Skorokhod embedding problem, but may not solve our optimisation problem (4.5.1).

Rost [137] proposed a different solution, with nicer optimality properties.

Appendix A

Miscellaneous tools

A.1 Essentials of probability theory

We provide here a brief overview of standard results in probability theory and convergence of random variables needed in these lecture notes. The reader is invited to consult [153] for instance for a more thorough treatment of the subject.

A.1.1 PDF, CDF and characteristic functions

In the following, $(\Omega, \mathcal{F}, \mathbb{P})$ shall denote a probability space and X a random variable defined on it. We define the cumulative distribution function $F : \mathbb{R} \to [0, 1]$ of S by

$$F(x) := \mathbb{P}(X \leq x), \qquad \text{for all } x \in \mathbb{R}.$$

The function F is increasing and right-continuous and satisfies the identities $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. If the function F is absolutely continuous, then the random variable X has a probability density function $f : \mathbb{R} \to \mathbb{R}_+$ defined by $f(x) = F'(x)$, for all real number x. Note that this in particular implies the equality $F(x) = \int_{-\infty}^{x} f(u)\mathrm{d}u$. Recall that a function $F : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$ is said to be *absolutely continuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that the implication

$$\sum_n |b_n - a_n| < \delta \qquad \Longrightarrow \qquad \sum_n |F(b_n) - F(a_n)| < \delta$$

holds for any sequence of pairwise disjoint intervals $(a_n, b_n) \subset \mathcal{D}$. Define now the characteristic function $\phi : \mathbb{R} \to \mathbb{C}$ of the random variable X by

$$\phi(u) := \mathbb{E}\left(\mathrm{e}^{\mathrm{i}uX}\right).$$

Note that it is well defined for all real number u and that we always have $|\phi(u)| \leq 1$. Extending it to the complex plane ($u \in \mathbb{C}$) is more subtle and shall be dealt with in Chapter **??**, along with some properties of characteristic functions.

A.1.2 Gaussian distribution

A random variable X is said to have a Gaussian distribution (or Normal distribution) with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$, and we write $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ if and only if its density reads

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\mu)^2\right), \qquad \text{for all } x \in \mathbb{R}.$$

For such a random variable, the following identities are obvious:

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}uX}\right) = \exp\left(\mathrm{i}\mu u - \frac{1}{2}u^2\sigma^2\right), \qquad \text{and} \qquad \mathbb{E}\left(\mathrm{e}^{uX}\right) = \exp\left(\mu u + \frac{1}{2}u^2\sigma^2\right),$$

for all $u \in \mathbb{R}$. The first quantity is the characteristic function whereas the second one is the Laplace transform or the random variable. If $X \in \mathcal{N}\left(\mu, \sigma^2\right)$, then the random variable $Y := \exp(X)$ is said to be lognormal and

$$\mathbb{E}(Y) = \exp\left(\mu + \frac{1}{2}\sigma^2\right) \qquad \text{and} \qquad \mathbb{E}\left(Y^2\right) = \exp\left(2\mu + 2\sigma^2\right).$$

A.1.3 Miscellaneous tools

Lemma A.1.1 (Fatou's lemma in analysis)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions on a given measure space (S, Σ, μ), and define the pointwise limit $f(x) := \liminf_{n\uparrow\infty} f_n(x)$ for all $x \in S$. Then f is also measurable and*

$$\int_S f(x)\mu(\mathrm{d}x) \leq \liminf_{n\uparrow\infty} \int_S f_n(x)\mu(\mathrm{d}x).$$

Lemma A.1.2 (Fatou's lemma in probability)**.** *For a given family of non-negative random variables $(X_n)_{n\in\mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, the following inequality holds, for any sub σ-algebra $\mathcal{G} \subset \mathcal{F}$:*

$$\mathbb{E}\left(\liminf_{n\uparrow\infty} X_n \middle| \mathcal{G}\right) \leq \liminf_{n\uparrow\infty} \mathbb{E}\left(X_n|\mathcal{G}\right).$$

Lemma A.1.3 (Reverse Fatou's lemma)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions on a given measure space (S, Σ, μ). If there exists an integrable function g such that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in S$, then*

$$\limsup_{n\uparrow\infty} \int_S f_n(x)\mu(\mathrm{d}x) \leq \int_S \limsup_{n\uparrow\infty} f_n(x)\mu(\mathrm{d}x).$$

Lemma A.1.4 (Dominated convergence)**.** *Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences of measurable functions on a given measure space (S, Σ, μ), such that $|f_n| \leq g_n$. Assume that there exist two measurable functions f and g such that, for all $x \in S$,*

$$\lim_{n\uparrow\infty} f_n(x) = f(x), \qquad \lim_{n\uparrow\infty} g_n(x) = g(x), \qquad \lim_{n\uparrow\infty} \int_S g_n(x)\mu(\mathrm{d}x) = \int_S g(x)\mu(\mathrm{d}x),$$

then $\lim_{n\uparrow\infty} \int_S f_n(x)\mu(\mathrm{d}x) = \int_S f(x)\mu(\mathrm{d}x).$

A.2 Convergence of random variables

We recall here the different types of convergence for family of random variables $(X_n)_{n\geq 1}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall denote $F_n : \mathbb{R} \to [0,1]$ the corresponding cumulative distribution functions and $f_n : \mathbb{R} \to \mathbb{R}_+$ their densities whenever they exist. We start with a definition of convergence for functions, which we shall use repeatedly.

Definition A.2.1. Let $(h_n)_{n\geq 1}$ be a family of functions from $\mathbb{R}$ to $\mathbb{R}$. We say that the family converges pointwise to a function $h : \mathbb{R} \to \mathbb{R}$ if and only if the equality $\lim_{n\uparrow\infty} h_n(x) = h(x)$ holds for all real number x.

Convergence in distribution

This is the weakest form of convergence, and is the one appearing in the central limit theorem.

Definition A.2.2. The family $(X_n)_{n\geq 1}$ converges in distribution—or weakly or in law—to a random variable X if and only if the family $(F_n)_{n\geq 1}$ converges pointwise to a function $F : \mathbb{R} \to [0,1]$, i.e. the equality

$$\lim_{n\uparrow\infty} F_n(x) = F(x),$$

holds for all real number x where F is continuous. Furthermore, the function F is the CDF of the random variable X.

Example A.2.3. Consider the family $(X_n)_{n\geq 1}$ such that each X_n is uniformly distributed on the interval $\left[0, n^{-1}\right]$. We then have $F_n(x) = nx\mathbb{1}_{\{x\in[0,1/n]\}} + \mathbb{1}_{\{x\geq 1/n\}}$. It is clear that the family of random variable converges weakly to the degenerate random variable $X = 0$. However, for any $n \geq 1$, we have $F_n(0) = 0$ and $F(0) = 1$. The function F is not continuous at 1, but the definition still holds.

Example A.2.4. Weak convergence does not imply convergence of the densities, even when they exist. Consider the family such that $f_n(x) = \Big(1 - \cos(2\pi nx)\Big)\mathbb{1}_{\{x\in(0,1)\}}$.

Even though convergence in law is a weak form of convergence, it has a number of fundamental consequences for applications. We list them here without proof and refer the interested reader to [19] for details

Corollary A.2.5. *Assume that the family $(X_n)_{n\geq 1}$ converges weakly to the random variable X. Then the following statements hold*

- $\lim_{n\uparrow\infty} \mathbb{E}(h(X_n)) = \mathbb{E}(h(X))$ *for all bounded and continuous function h.*
- $\lim_{n\uparrow\infty} \mathbb{E}(h(X_n)) = \mathbb{E}(h(X))$ *for all Lipschitz function h.*
- $\lim_{n\uparrow\infty} \mathbb{P}(X_n \in A) = \mathbb{P}(X \in A)$ *for all continuity sets A of X.*

- *(Continous mapping theorem). The sequence $(h(X_n))_{n\geq 1}$ converges in law to $h(X)$ for every continuous function h.*

The following theorem shall be of fundamental importance in many applications, and we therefore state it separately.

Theorem A.2.6 (Lévy's continuity theorem)**.** *The family $(X_n)_{n\geq 1}$ converges weakly to the random variable X if and only if the sequence of characteristic functions ϕ_n converges pointwise to the characteristic function ϕ of X and ϕ is is continuous at the origin.*

Convergence in probability

Definition A.2.7. The family $(X_n)_{n\geq 1}$ converges in probability to the random variable X if, for all $\varepsilon > 0$, we have

$$\lim_{n\uparrow\infty} \mathbb{P}\left(|X_n - X| \geq \varepsilon\right) = 0.$$

Remark A.2.8. The continuous mapping theorem still holds under this form of convergence.

Almost sure convergence

This form of convergence is the strongest form of convergence and can be seen as an analogue for random variables of the pointwise convergence for functions.

Definition A.2.9. The family $(X_n)_{n\geq 1}$ converges almost surely to the random variable X if

$$\mathbb{P}\left(\lim_{n\uparrow\infty} X_n = X\right) = 1.$$

Convergence in mean

Definition A.2.10. Let $r \in \mathbb{N}^*$. The family $(X_n)_{n\geq 1}$ converges in the L^r norm to the random variable X if the r-th absolute moments of X_n and X exist for all $n \geq 1$ and if

$$\lim_{n\uparrow\infty} \mathbb{E}\left(|X_n - X|^r\right) = 0.$$

Properties

- Almost sure convergence implies convergence in probability.
- Convergence in probability implies weak convergence.
- Convergence in the L^r norm implies convergence in probability.
- For any $r \geq s \geq 1$, convergence in the L^r norm implies convergence in the L^s norm.

A.2.1 Central limit theorem and Berry-Esséen inequality

Let $(X_i)_{i=1\ldots,n}$ form a sequence of independent and identically distributed random variables with finite mean μ and finite variance $\sigma^2 > 0$, and define the sequences of random variables $(\overline{X}_n)_{n\geq 1}$ and $(Z_n)_{n\geq 1}$ by

$$\overline{X}_n := \sum_{i=1}^{n} X_i \qquad \text{and} \qquad Z_n := \frac{\overline{X}_n - n\mu}{\sigma\sqrt{n}}, \qquad \text{for each } n \geq 1. \tag{A.2.1}$$

Recall now the central limit theorem:

Theorem A.2.11 (Central limit theorem)**.** *The family $(Z_n)_{n\geq 1}$ converges in distribution to a Gaussian distribution with zero mean and unit variance. In particular for any $a < b$, we have $\lim_{n\uparrow\infty} \mathbb{P}(Z_n \in [a,b]) = \mathcal{N}(b) - \mathcal{N}(a)$.*

The central limit theorem provides information about the limiting behaviour of the probabilities, but does not tell anything aboug the rate of convergence or the error made when approximating the Gaussian distribution by the distribution of Z_n for $n \geq 1$ fixed. The following theorem, proved by Berry [17] and Esséen [52] gives such estimates

Theorem A.2.12. *Assume that $\mathbb{E}\left(|X|^3\right) < \infty$. Then there exists a strictly positive universal (i.e. independent of n) constant C such that*

$$\sup_x \left|\mathbb{P}(Z_n \leq x) - \mathcal{N}(x)\right| \leq \frac{C\rho}{\sqrt{n}},$$

where $\rho := \mathbb{E}\left(\frac{|X_1 - \mu|^3}{\sigma^3}\right)$.

A.3 Uniformly integrable random variables

Definition A.3.1. The family of random variables $(X_n)_{n\in\mathbb{N}}$ is said to be uniformly integrable if

$$\lim_{K\uparrow\infty} \sup_n \mathbb{E}\left(|X_n| \middle| |X_n| > K\right) = 0.$$

The motivation underlying this notion can be seen through the following example: consider a random walk $(S_n)_{n\geq 0}$ with $S_0 = 1$, $S_{n+1} = S_n + \xi_n$, where $(\xi_n)_n$ is a family of independent Bernoulli trials taking values in $\{-1, 1\}$ with equal probability. Define now the stopping time $\tau := \inf\{n : S_n = 0\}$ and the family $(X_n)_n$ by $X_n := S_{\tau\wedge n}$. Then $(X_n)_{n\geq 0}$ is a non-negative martingale which converges almost surely to a finite limit X_∞; in particular, $\mathbb{P}(X_\infty = 0) = 1$. However, $\mathbb{E}(X_n) = \mathbb{E}(X_0) = 1$ for all $n \geq 0$, so that the family $(X_n)_{n\geq 0}$ cannot converge in L^1. Uniform integrability turns out to be the precise notion needed to ensure such convergence, as outlined in the following theorem.

Theorem A.3.2. *If the family $(X_n)_{n\geq 0}$ converges in probability to the random variable X_∞, then the following are equivalent:*

(i) *the family* $(X_n)_n$ *is uniformly integrable;*

(ii) $(X_n)_n$ *converges to* X_∞ *in* L^1*;*

(iii) $\lim_{n\uparrow\infty} \mathbb{E}|X_n| = \mathbb{E}|X_\infty| < \infty$.

A.4 Other stochastic analysis results

Lemma A.4.1 (Doob-Dynkyn Lemma)**.** *For any two random variables* $X, Y : \Omega \to \mathbb{R}^n$*,* Y *is* $\sigma(X)$*-measurable if and only if there exists a Borel measurable function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $y = f(X)$.

Theorem A.4.2 ((Version of the) Martingale Convergence Theorem)**.** *If* $X \in L^1(\mathbb{P})$ *and* $(\mathcal{G}_n)_{n\geq 1}$ *an increasing family of* σ*-algebra in* $\mathcal{F}$*, then*

$$\lim_{n\uparrow\infty} \mathbb{E}\left(X|\mathcal{G}_n\right) = \mathbb{E}\left(X|\bigvee_{n\geq 1}\mathcal{G}_n\right), \qquad \mathbb{P}\text{-almost everywhere and in } L^1(\mathbb{P}).$$

Appendix B

Kolmogorov equations

Consider a one-dimensional diffusion $(X_t)_{t\geq 0}$ defined on a given probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and satisfying the stochastic differential equation $\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t$, where $X_0 = x \in \mathbb{R}$ and W is a standard $(\mathcal{F}_t)_{t\geq 0}$-adapted Brownian motion. The coefficients b and σ are such that a unique strong solution exists. For any $0 \leq s \leq t$ and any $(x, y) \in \mathbb{R}^2$, we let its probability density function $p(s, x; t, y)$ be defined as

$$\mathbb{P}_{s,x}(X_t \in \mathrm{d}y) = p(s, x; t, y)\mathrm{d}y.$$

Then p satisfies the backward Kolmogorov equation

$$\begin{cases} \partial_s p + \frac{1}{2}\sigma^2(s, x)\partial_{xx}p + b(s, x)\partial_x p = 0, \\ \lim\limits_{s\uparrow t} p(s, x; t, y)\mathrm{d}y = \delta_x(\mathrm{d}y). \end{cases} \tag{B.0.1}$$

and the forward Kolmogorov equation

$$\begin{cases} \partial_t p - \frac{1}{2}\partial_{yy}\left(\sigma^2(t, y)p\right) + \partial_y\left(b(t, y)p\right) = 0, \\ \lim\limits_{s\uparrow t} p(s, x; t, y)\mathrm{d}y = \delta_x(\mathrm{d}y). \end{cases} \tag{B.0.2}$$

Appendix C

Spanning European payoffs

Consider a $\mathcal{C}^2$ function $f : \mathbb{R} \to \mathbb{R}$ and some constant $F \geq 0$. By the fundamental theorem of calculus, we have

$$
\begin{aligned}
f(S) &= f(F) + \mathbb{1}_{\{S>F\}} \int_F^S f'(u)\mathrm{d}u \;-\; \mathbb{1}_{\{S<F\}} \int_S^F f'(u)\mathrm{d}u \\
&= f(F) + \mathbb{1}_{\{S>F\}} \int_F^S \left[f'(F) + \int_F^u f''(v)\mathrm{d}v\right] \mathrm{d}u - \mathbb{1}_{\{S<F\}} \int_S^F \left[f'(F) - \int_u^F f''(v)\mathrm{d}v\right] \mathrm{d}u \\
&= f(F) + f'(F)(S-F) + \mathbb{1}_{\{S>F\}} \int_F^S \int_v^S f''(v)\mathrm{d}u\mathrm{d}v \;+\; \mathbb{1}_{\{S<F\}} \int_S^F \int_S^v f''(v)\mathrm{d}v\mathrm{d}u \\
&= f(F) + f'(F)(S-F) + \mathbb{1}_{\{S>F\}} \int_F^S f''(v)(S-v)\mathrm{d}v \;+\; \mathbb{1}_{\{S<F\}} \int_S^F f''(v)(v-S)\mathrm{d}v \\
&= f(F) + f'(F)(S-F) + \mathbb{1}_{\{S>F\}} \int_F^\infty f''(v)(S-v)_+\mathrm{d}v \;+\; \mathbb{1}_{\{S<F\}} \int_0^F f''(v)(v-S)_+\mathrm{d}v
\end{aligned}
\tag{C.0.1}
$$

The following two cases are of particular financial importance:

- if $F = 0$, then the expression above reduces to

  $$f(S) = f(0) + Sf'(0) + \int_F^\infty f''(v)(S-v)_+\mathrm{d}v,$$

  which means that the option with payoff $f(S)$ can be replicated by $f(0)$ invested in bonds, $f'(0)$ shares and an infinite strip of call options, each with strike v and in quantity $f''(v)$;

- if $F = S_0$, then the formula above reads

  $$f(S) = [f(S_0) - S_0 f'(S_0)] + Sf'(S_0) + \mathbb{1}_{\{S>S_0\}} \int_{S_0}^\infty f''(v)(S-v)_+\mathrm{d}v + \mathbb{1}_{\{S<S_0\}} \int_0^{S_0} f''(v)(v-S)_+\mathrm{d}v\mathrm{d}u,$$

  so that the option with payoff $f(S)$ can be replicated with bonds, stocks and European Calls and Puts.

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