Optimal make-take fees for market making regulation

O. El Euch, T. Mastrolia, M. Rosenbaum and N. Touzi

Ecole Polytechnique

11 September 2018

Table of contents

1. Introduction
2. The model
3. Solving the market maker problem
4. Solving the exchange problem

Table of contents

1. Introduction
2. The model
3. Solving the market maker problem
4. Solving the exchange problem

Introduction

Exchanges in competition

- With the fragmentation of financial markets, exchanges are nowadays in competition.
- Traditional international exchanges are now challenged by alternative trading venues.
- Consequently, they have to find innovative ways to attract liquidity on their platforms.
- A possible solution : using a make-taker fees system, that is charging in an asymmetric way liquidity provision and liquidity consumption.

Introduction

A controversial topic

- Make-take fees policies are seen as a major facilitating factor to the emergence of a new type of market makers aiming at collecting fee rebates : the high frequency traders.
- As stated by the Securities and Exchanges commission : "Highly automated exchange systems and liquidity rebates have helped establish a business model for a new type of professional liquidity provider that is distinct from the more traditional exchange specialist and over-the-counter market maker."

HFT market makers

The concern with high frequency traders becoming the new liquidity providers is two-fold.

- Their presence implies that slower traders no longer have access to the limit order book, or only in unfavorable situations when high frequency traders do not wish to support liquidity.
- They tend to leave the market in time of stress.

Introduction

Our aim

- Providing a quantitative and operational answer to the question of relevant make-take fees.
- We take the position of an exchange (or of the regulator) wishing to attract liquidity. The exchange is looking for the best make-take fees policy to offer to market makers in order to maximize its utility.
- In other words, it aims at designing an optimal contract with the (unique) market marker to create an incentive to increase liquidity.
- Principal/agent type approach : the wealth of the principal (exchange) depends on the agent's (market maker) effort (essentially his spread), but the principal cannot directly control the effort.

Table of contents

1. Introduction
2. The model
3. Solving the market maker problem
4. Solving the exchange problem

The market maker

Market maker's controls

- The market maker has a view on the efficient price (midprice) of the asset

$$S_t = S_0 + \sigma W_t,$$

where σ is the price volatility.

- He fixes the ask and bid prices

$$P_t^a = S_t + \delta_t^a, \quad P_t^b = S_t - \delta_t^b.$$

The order flow

Arrival of market orders

- We model the arrival of buy (resp. sell) market orders by a point process $(N_t^a)_{t\geq 0}$ (resp. $(N_t^b)_{t\geq 0}$) with intensity $(\lambda_t^a)_{t\geq 0}$ (resp. $(\lambda_t^b)_{t\geq 0}$).
- The inventory of the market maker $Q_t = N_t^b - N_t^a$.
- We consider a threshold inventory $\bar{q}$ above which the market maker stops quoting on the ask or bid side.
- From financial economics arguments :

$$\lambda_t^a = \lambda(\delta_t^a)1_{\{Q_t > -\bar{q}\}}, \quad \lambda_t^b = \lambda(\delta_t^b)1_{\{Q_t < \bar{q}\}}.$$

where $\lambda(x) = Ae^{-k(x+c)/\sigma}$.

Martingale processes

Equivalent probabilities

The market maker controls the spread $\delta = (\delta^a, \delta^b)$. We define the associated probability $\mathbb{P}^\delta$ such that

$$\widetilde{N}_t^{a,\delta} = N_t^a - \int_0^t \lambda(\delta_s^a) 1_{\{Q_s > -\bar{q}\}} ds$$

and

$$\widetilde{N}_t^{b,\delta} = N_t^b - \int_0^t \lambda(\delta_s^b) 1_{\{Q_s < \bar{q}\}} ds$$

are martingales.

The market maker viewpoint

The profit and loss of the market maker

- We consider a final time horizon $T > 0$.
- The cash flow of the market maker

$$X_t^\delta = \int_0^t P_u^a dN_u^a - \int_0^t P_u^b dN_u^b.$$

- The inventory risk of the market maker is $Q_t S_t$.
- For a given contract ξ given by the exchange, seen as an $\mathcal{F}_T$ measurable random variable, the market maker chooses his spread δ by maximizing his utility.

The market maker optimization problem

The market maker problem

Under the exchange incentive policy ξ, the market maker solves now

$$V_{MM}(\xi) = \sup_{\delta} \mathbb{E}^{\delta}\left[-\exp\left(-\gamma(X_T^{\delta} + Q_T S_T + \xi)\right)\right].$$

- We obtain an optimal response given by $\hat{\delta}_t(\xi) = (\hat{\delta}_t^a(\xi), \hat{\delta}_t^b(\xi))$.
- We will only consider contracts such that $V_{MM}(\xi)$ is above a threshold utility value R :

$$\mathcal{C} = \{\xi \ \mathcal{F}_T\text{-measurable such that } V_{MM}(\xi) > R\}$$

  $+$ integrability conditions.
- For $\xi = 0$, well studied problem since Avellaneda and Stoikov.

The exchange viewpoint

We assume that the exchange

- Earns $c > 0$ for each market order occurring in its platform.
- Pays the incentive policy ξ to the market maker.

The profit and loss of the exchange is

$$c(N_T^a + N_T^b) - \xi.$$

The exchange problem

The exchange designs the contract ξ by solving

$$V_E = \sup_{\xi \in \mathcal{C}} \mathbb{E}^{\hat{\delta}(\xi)} \left[-\exp\left(-\eta (c(N_T^a + N_T^b) - \xi) \right) \right],$$

where η is the risk aversion of the exchange.

Table of contents

1. Introduction
2. The model
3. Solving the market maker problem
4. Solving the exchange problem

Solving the market maker problem for a given contract

Dynamic programming principle

- We fix ξ and compute the best response of the market maker.
- Let τ be a stopping time with values in $[t, T]$ and $\mu \in \mathcal{A}_\tau$, where $\mathcal{A}_\tau$ denotes the restriction of the set of admissible controls $\mathcal{A}$ to controls on $[\tau, T]$.
- Let $J_T(\tau, \mu) = \mathbb{E}_\tau^\mu \left[-e^{-\gamma \int_\tau^T (\mu_u^a dN_u^a + \mu_u^b dN_u^b + Q_u dS_u)} e^{-\gamma \xi} \right]$ and

$$V_\tau = \operatorname*{ess\,sup}_{\mu \in \mathcal{A}_\tau} J_T(\tau, \mu).$$

- Dynamic programming principle :

$$V_t = \operatorname*{ess\,sup}_{\delta \in \mathcal{A}} \mathbb{E}_t^\delta \Big[-e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + Q_u dS_u)} V_\tau \Big].$$

Solving the market maker problem for a given contract

A convenient super-martingale

- Let
$$U_t^\delta = V_t \, e^{-\gamma \int_0^t \delta_u^a dN_u^a + \delta_u^b dN_u^b + Q_u dS_u}.$$
- $U_0^\delta = V_0$ and
$$U_T^\delta = -e^{-\gamma \left(\int_0^T \delta_u^a dN_u^a + \delta_u^b dN_u^b + Q_u dS_u + \xi \right)}.$$
- From the DPP, we get that U_t^δ is a $\mathbb{P}^\delta-$super-martingale. We want to find the optimal controls (δ^a, δ^b) turning it into a martingale.
- To do so we find a suitable representation of U_t^δ.

Solving the market maker problem for a given contract

Doob-Meyer and martingale representation

- Doob-Meyer : $U_t^\delta = M_t^\delta - A_t^\delta$, where M^δ is a $\mathbb{P}^\delta-$martingale and A^δ is an integrable non-decreasing predictable process starting at zero.
- Martingale representation theorem : There exists a predictable process $\widetilde{Z}^\delta = (\widetilde{Z}^{\delta,S}, \widetilde{Z}^{\delta,a}, \widetilde{Z}^{\delta,b})$ such that M_t^δ can be represented as

$$V_0 + \int_0^t \widetilde{Z}_r^\delta . d\chi_r - \int_0^t \widetilde{Z}_r^{\delta,a} \lambda(\delta_r^a) 1_{\{Q_r > -\bar{q}\}} dr - \int_0^t \widetilde{Z}_r^{\delta,b} \lambda(\delta_r^b) 1_{\{Q_r < \bar{q}\}} dr,$$

with $\chi = (S, N^a, N^b)$.

Solving the market maker problem for a given contract

Reducing the class of contracts

- Let Y be the process defined by $V_t = -e^{-\gamma Y_t}$.
- $Y_T = \xi$ and using Ito's formula together with the previous result and the martingale property of U_t for the optimal controls we get

$$dY_t = Z_t^a dN_t^a + Z_t^b dN_t^b + Z_t^S dS_t - H(Z_t, Q_t)dt,$$

  for an explicit function H and where the Z^i do not depend on δ.
- Any contract ξ can be (uniquely) represented under the preceding form ! We can restrict ourselves to such contracts.
- Natural financial interpretation of the contracts :
  - The exchange rewards the market maker by Z^a (resp. Z^b) for each buy (resp.sell) market order.
  - The exchange participates to the market/inventory risk of the market maker by taking $-Z^S$ of his share.
  - The market maker pays a continuous coupon $H(Z_t, Q_t)dt$.

Solving the market maker problem for a given contract

New super-martingale representation

- In term of this new representation, we obtain

$$U_t^\delta = M_t^\delta + \gamma \int_0^t U_u^\delta \big(H(Z_u, q_u) - h(\delta_u, Z_u, q_u)\big)\, du,$$

where h is explicit and

$$H(z, q) = \sup_{|\delta^a| \vee |\delta^b| \leq \delta_\infty} h(\delta, z, q).$$

- The process U_t^δ becomes a martingale if and only if δ is chosen as the maximizer of h.

Solving the problem for a specific contract

Optimal quotes

Let ξ be an admissible contract. The unique optimal spread of the market maker is given by

$$\hat{\delta}_t^a(\xi) = -Z_t^a + \frac{1}{\gamma}\log(1+\frac{\sigma\gamma}{k}), \quad \hat{\delta}_t^b(\xi) = -Z_t^b + \frac{1}{\gamma}\log(1+\frac{\sigma\gamma}{k}).$$

Table of contents

1 Introduction

2 The model

3 Solving the market maker problem

4 Solving the exchange problem

Solving the exchange problem

By representing any contract $\xi = Y_T^{Y_0^\xi, Z^\xi}$, the exchange problem

$$V_E = \sup_{\xi \in \mathcal{C}} \mathbb{E}^{\hat{\delta}(\xi)} \left[-\exp\left(-\eta(c(N_T^a + N_T^b) - \xi) \right) \right]$$

is equivalent to

$$V_E = \sup_{Z} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0,Z})} \big[-\exp\big(-\eta(c(N_T^a + N_T^b) - Y_T^{Y_0,Z})\big)\big]$$

Reduction to a classical control problem

$$V_E = \sup_{Z} \mathbb{E}^{\hat{\delta}(Y_T^{0,Z})} \Big[-\exp\Big(-\eta \int_0^T (c - Z_t^a)dN_t^a + (c - Z_t^a)dN_t^b - Z_t^S dS_t + H(Z_t, Q_t)dt \Big) \Big]$$

HJB of the exchange problem

Reduction to a HJB equation

- $V_E = v(0, Q_0)$ with

$$\partial_t v(t,Q) + \sup_z h_E(Q, v(t,Q), v(t,Q+1), v(t,Q-1), z) = 0$$

  and $v(T,q) = -1$, where the function h_E is explicit.

- The optimal control $Z^\star$ is obtained so that $Z_t^\star$ is solution of the maximization problem of

$$z \mapsto h_E(Q_t, v(t,Q_t), v(t,Q_t+1), v(t,Q_t-1), z).$$

  It is explicit in terms of the parameters of h_E.

Reduction of the HJB equation

Reduction to a linear equation

If we take $u(t,Q) = (-v(t,Q))^{-\frac{k}{\sigma\eta}}$, we get

$$\begin{cases} \partial_t u(t,Q) + C_1 Q^2 - C_2 (u(t,Q-1)1_{\{Q>-\bar{q}\}} + u(t,Q+1)1_{\{Q<\bar{q}\}}) = 0, \\ u(T,Q) = 1, \end{cases}$$

with C_1 and C_2 are positive explicit constants.

- Guarantees the existence and uniqueness of v.
- Easy numerical computation of u and v.

Optimal contract

Theorem

The contract $\xi^\star$ that solves the exchange problem is given by

$$\xi^\star = Y^\star + \int_0^T Z_t^{a,\star} dN_t^a + Z_t^{b,\star} dN_t^b + Z_t^{S,*} dS_t - H(Z_t^\star, Q_t)dt,$$

with

$$Z_t^{a\star} = -\frac{\sigma}{k} \log\left(\frac{u(t, Q_t)}{u(t, Q_t - 1)}\right) + \hat{c},$$

$$Z_t^{b\star} = -\frac{\sigma}{k} \log\left(\frac{u(t, Q_t)}{u(t, Q_t + 1)}\right) + \hat{c}, \quad Z_t^{S\star} = -\frac{\gamma}{\eta + \gamma} Q_t.$$

with $\hat{c} = c + \frac{1}{\eta} \log\left(1 - \frac{\sigma^2 \gamma \eta}{(k+\sigma\gamma)(k+\sigma\eta)}\right)$.

Comments on the optimal contract

Discussion

- The quantities

$$-\log\left(\frac{u(t,Q_t)}{u(t,Q_t-1)}\right) \text{ and } -\log\left(\frac{u(t,Q_t)}{u(T,Q_t+1)}\right)$$

  are roughly proportional respectively to Q_t and $-Q_t$.
- Thus, when the inventory is highly positive, the exchange provides incentives to the market-maker so that it attracts buy market orders and tries to dissuade him to accept more sell market orders, and conversely for a negative inventory.

Comments on the optimal contract

Discussion

- The integral

$$\int_0^T Z_u^{S\star} dS_u$$

can be understood as a risk sharing term.
- Indeed, $\int_0^t Q_u dS_u$ corresponds to the price driven component of the inventory risk $Q_t S_t$. Hence in the optimal contract, the exchange supports part of this risk so that the market maker maintains reasonable quotes despite some inventory.
- The proportion of risk handled by the platform is $\frac{\gamma}{\gamma+\gamma_p}$

Comments on the optimal contract

Discussion

- We see that when acting optimally, the exchange transfers the totality of the taker fee c to the market maker. It is neutral to the value of c (its optimal utility function does not depend on c).
- However, c plays an important role in the optimal spread offered by the market maker which is approximately given by

$$-2c - \frac{2}{\gamma_p}\log\left(1 - \frac{\sigma^2\gamma\gamma_p}{(k+\sigma\gamma)(k+\sigma\gamma_p)}\right) + \frac{2}{\gamma}\log(1+\frac{\sigma\gamma}{k}).$$

- The exchange may fix in practice the transaction cost c so that the spread is close to one tick by setting

$$c \approx -\frac{1}{2}\text{Tick} - \frac{1}{\gamma_p}\log\left(1 - \frac{\sigma^2\gamma\gamma_p}{(k+\sigma\gamma)(k+\sigma\gamma_p)}\right) + \frac{1}{\gamma}\log(1+\frac{\sigma\gamma}{k}).$$

- For $\sigma\gamma/k$ small enough, $c \approx \frac{\sigma}{k} - \frac{1}{2}\text{Tick}$.

Analyzing the effect of the exchange optimal incentive policy

The benefits of the incentive policy

- We can compute the spread, optimal contract, profit and losses of the market maker and exchange, order flows...
- We compare these quantities to the ones obtained in the case where $\xi = 0$.

$$T = 600s, \quad \sigma = 0.3\text{Tick}.s^{-1/2}, \quad A = 0.9s^{-1}, \quad k = 0.3s^{-1/2},$$
$$\bar{q} = 50 \text{ unities}, \quad \gamma = 0.01\text{Tick}^{-1}, \quad \eta = 1\text{Tick}^{-1}, \quad c = 0.5\text{Tick}.$$

Impact of the incentive policy on the spread

The optimal spread is given by $S_t^\star = \delta_t^{a\star} + \delta_t^{b\star}$ with

$$\delta_t^{i\star} = \delta_t^i(\xi^*) = -Z_t^{i\star} + \frac{1}{\gamma}\log\left(1 + \frac{\sigma\gamma}{k}\right), \quad i = a, b.$$

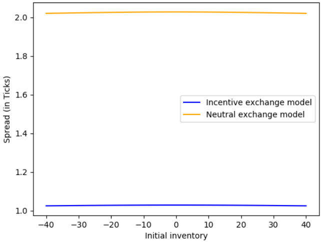

FIGURE: Optimal initial spread with/without the exchange incentive policy as a function of the initial inventory Q_0.

Impact of the incentive policy on the spread

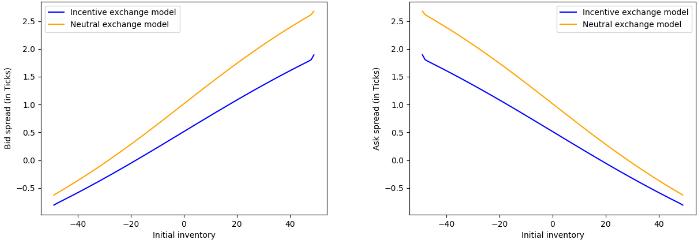

FIGURE: Optimal initial ask (left) and bid (right) spread component with/without the exchange incentive policy as a function of the initial inventory Q_0.

Impact of the volatility on the incentive policy

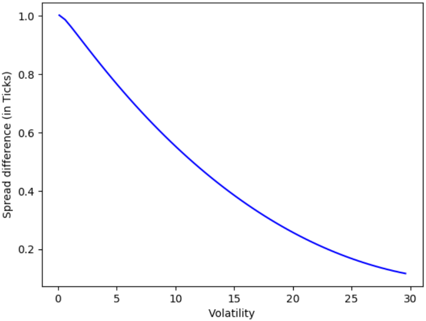

FIGURE: The initial optimal spread difference between both situations with/without incentive policy from the exchange toward the market maker as a decreasing function of the volatility σ.

Impact of the incentive policy on the market liquidity

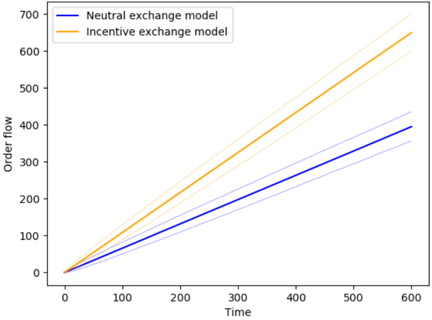

FIGURE: Average order flow on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).

Impact of the incentive policy on the market maker and exchange profit and loss

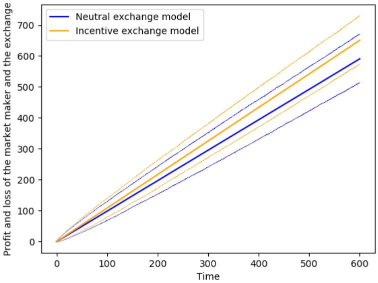

FIGURE: Average total P&L of the market maker maker and the exchange on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).

Impact of the incentive policy on trading costs

We consider that there is only one market taker who wants to buy a fixed quantity $Q_{final} = 200$ units. We compute the trading cost in both situations :

$$\int_0^T \delta_s^a dN_s^a.$$

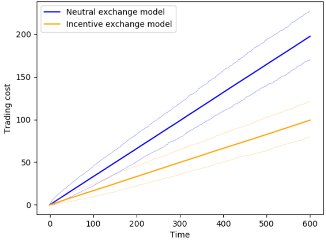

FIGURE: Average trading cost on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).

Conclusion

Benefits of the exchange incentive policy

- Smaller spreads.
- Better market liquidity.
- Increase of the profit and loss of the market maker and the exchange.
- Lower transaction costs.