Patching and p-adic local Langlands

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The p-adic local Langlands correspondence is an exciting, recent generalization of the classical Langlands program. For $GL_2(\mathbb{Q}_p)$ it consists of functors between two-dimensional, continuous p-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and certain admissible unitary p-adic Banach space representations of $GL_2(\mathbb{Q}_p)$ [4, 9, 5]. The correspondence has several remarkable properties, namely it is compatible with deformations and reduction mod p, with the classical local Langlands correspondence via taking locally algebraic vectors, and with the global p-adic correspondence, i.e. with the completed cohomology of modular curves. These properties led to spectacular applications to the Fontaine-Mazur conjecture for GL_2 over $\mathbb{Q}$ [6, 8].

However, most techniques involved in the construction of the p-adic local Langlands correspondence seem to break down if one tries to move beyond $GL_2(\mathbb{Q}_p)$. For $GL_n(F)$, it is unclear even what the precise conjectures should be, though the best possible scenario would involve all three of the properties listed above. In this talk, I described the construction of a candidate for the p-adic local Langlands correspondence for $GL_n(F)$, where $F/\mathbb{Q}_p$ is a finite extension, using global techniques, specifically the Taylor-Wiles-Kisin patching method applied to completed cohomology [3].

More precisely, we associate to a continuous n-dimensional representation r of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/F)$ an admissible Banach space representation $V(r)$ of $GL_n(F)$, by p-adically interpolating completed cohomology for global definite unitary groups. The method involves working over an unrestricted local deformation ring of the residual $\bar{r}$, finding a global residual Galois representation which is automorphic and restricts to our chosen local representation $\bar{r}$, and gluing corresponding spaces of completed cohomology with varying tame level at so-called Taylor-Wiles primes. The output is a module M_∞ over $R_{\bar{r}}$, which also has an action of $GL_n(F)$ and whose fibers over closed points are admissible, unitary p-adic Banach spaces. We define $V(r)$ to be the fiber of M_∞ over the point of $R_{\bar{r}}$ corresponding to r.

We also show that, when r is de Rham, we can recover the compatibility with classical local Langlands $r \mapsto \pi_{\mathrm{sm}}(r)$ in many situations. More precisely, when r lies on an automorphic component of a local deformation ring, we can compute the locally algebraic vectors in $V(r)$ and show that they have the expected form $\pi_{\mathrm{sm}}(r) \otimes \pi_{\mathrm{alg}}(r)$. This involves first establishing an inertial local Langlands correspondence via the theory of types. The next step is to construct a map from an appropriate Bernstein center to a local deformation ring for a specific inertial type, a map which interpolates classical local Langlands. Finally, we appeal to the automorphy lifting theorems of [1] to guarantee that the locally algebraic vectors we obtain are non-zero. Our control over locally algebraic vectors allows us to prove many new cases of an admissible refinement of the Breuil-Schneider conjecture [2], concerning the existence of certain unitary completions.

Theorem 1. *Suppose that $p > 2$, that $r : G_F \to GL_n(\bar{\mathbb{Q}}_p)$ is de Rham of regular weight, and that r is generic. Suppose further that either*

(1) *$n = 2$, and r is potentially Barsotti–Tate, or*
(2) *$F/\mathbb{Q}_p$ is unramified and r is crystalline with Hodge–Tate weights in the extended Fontaine–Laffaille range, and $n \neq p$.*

Then $\pi_{\mathrm{sm}}(r) \otimes \pi_{\mathrm{alg}}(r)$ admits a nonzero unitary admissible Banach completion.

For example, when $F/\mathbb{Q}_p$ is unramified and p is large, Theorem 1 applies to all unramified principal series representations. Note that this existence is a purely local result, even though it is proved using global, automorphic methods.

Unfortunately, the Taylor-Wiles patching method involves gluing spaces of automorphic forms with varying tame level in a non-canonical way, using a sort of diagonal argument to ensure that their compatibility can always be achieved. Therefore, it is not at all clear that $r \mapsto V(r)$ is a purely local correspondence: it depends on the choice of global residual representation as well as on the choice of a compatible system of Taylor-Wiles primes. If there was a purely local correspondence satisfying all three properties listed in the beginning, then our construction would necessarily recover it. This is the case for $GL_2(\mathbb{Q}_p)$ and, in fact, the six of us are in the process of writing a paper elaborating on this and reproving many properties of the p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, without making use of Colmez's functors. Our arguments rely heavily instead on the ideas of [9], especially the use of projective envelopes. For $GL_n(F)$, the question of whether our construction is purely local seems quite hard.

However, there is forthcoming work of Scholze, who constructs a purely local functor in the opposite direction: from admissible unitary p-adic Banach space representations of $GL_n(F)$ to admissible representations of $D^\times \times W_F$, where D is a division algebra with center F and invariant $1/n$. His construction uses the cohomology of the Lubin-Tate tower, which is known to realize both classical local Langlands and the Jacquet-Langlands corespondence when $l \neq p$ [7]. This functor satisfies local-global compatibility, in the following sense: if the input is the completed cohomology for a definite unitary group G, split at p, then the output is the completed cohomology of a Shimura variety associated to an inner form J of G which is isomorphic to $D^\times$ at p. Just as one can patch completed cohomology for G, it is also possible to patch completed cohomology for J. Moreover, Scholze can even prove that if one uses our patched module M_∞ as the input for his functor, then the output is the patched object for J. A consequence of this is that, at the very least, it should be possible to recover the Galois representation r from the Banach space $V(r)$.

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