

ON THE GENERIC PART OF THE COHOMOLOGY OF NON-COMPACT UNITARY SHIMURA VARIETIES

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ABSTRACT. We prove that the generic part of the mod ℓ cohomology of Shimura varieties associated to quasi-split unitary groups of even dimension is concentrated above the middle degree, extending previous work [CS17] to a non-compact case. The result applies even to Eisenstein cohomology classes coming from the locally symmetric space of the general linear group, and has been used in [ACC⁺18] to get good control on these classes and deduce potential automorphy theorems without any self-duality hypothesis.

Our main geometric result is a computation of the fibers of the Hodge–Tate period map on compactified Shimura varieties, in terms of similarly compactified Igusa varieties.

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1. INTRODUCTION

In this paper, we aim to control the torsion in the cohomology of some non-compact unitary Shimura varieties, extending previous results [CS17] to the non-compact case. Before explaining our results, let us explain our motivation.

The original Taylor–Wiles method for proving automorphy lifting theorems, used for example in proving the modularity of elliptic curves over \mathbb{Q} , is restricted to settings where a certain numerical criterion holds: these are roughly the settings where the objects on the automorphic side arise from the middle degree cohomology of a Shimura variety. Recently, Calegari–Geraghty [CG18] outlined a strategy for extending the Taylor–Wiles method to GL_n over a general CM field F . Their method requires a detailed understanding of the cohomology of the locally symmetric spaces associated with GL_n/F . Part of their insight was to realize the central role played by torsion classes in the cohomology of these locally symmetric spaces, and another part of their insight was to reinterpret the failure of the Taylor–Wiles numerical criterion in terms of invariants q_0, l_0 seen on the automorphic side.

More precisely, for a connected reductive group G/\mathbb{Q} , let $l_0 := \mathrm{rk}(G(\mathbb{R})) - \mathrm{rk}(K_\infty) - \mathrm{rk}(A_\infty)$ and $q_0 := \frac{1}{2}(d - l_0)$.¹ The Calegari–Geraghty method works for GL_n over an arbitrary number field F as long as the following prerequisites are in place:

- (1) The construction of Galois representations associated to classes in the cohomology with \mathbb{Z}_ℓ -coefficients of the locally symmetric space for GL_n/F .
- (2) Local-global compatibility for these Galois representations at all primes of F , including at primes above ℓ .
- (3) A folklore conjecture that predicts that, under an appropriate non-degeneracy condition, the cohomology with \mathbb{Z}_ℓ -coefficients of the locally symmetric space for GL_n/F vanishes outside the range of degrees $[q_0, q_0 + l_0]$.

From now on, assume that F is a CM field. Then the first problem was solved in [Sch15], strengthening previous work [HLTT16] that applies to \mathbb{Q}_ℓ -coefficients. (See also [NT16] and [CGH⁺18] for some further refinements.)

The remaining two problems were within reach with \mathbb{Q}_ℓ -coefficients: see [Var14] for local-global compatibility and [ACC⁺18, Theorem 2.4.9], which builds on [Fra98], for vanishing results. However, both problems remained largely open for torsion classes. In the second problem, local-global compatibility at primes dividing ℓ is particularly subtle, because the construction of Galois representations in [Sch15] uses congruences to automorphic forms with arbitrarily deep level at ℓ . The third

¹Here $K_\infty \subseteq G(\mathbb{R})$ is a maximal compact subgroup, A_∞ is the identity component of the \mathbb{R} -points of the maximal \mathbb{Q} -split torus in the center of G , and d is the dimension over \mathbb{R} of the symmetric space for G .

problem is known only in low-dimensional cases, such as arithmetic hyperbolic three-manifolds (arising from GL_2 over an imaginary quadratic field F).

When the locally symmetric spaces for the group G arise from Shimura varieties, $l_0 = 0$ and q_0 is equal to the complex dimension of the Shimura variety. Originally, we were trying to understand the third problem, by approaching it in the easier case of Shimura varieties first; this is a result we obtained in [CS17] for compact unitary Shimura varieties. Our hope was that a suitable adaptation of these results to the case of non-compact unitary Shimura varieties, whose compactification contains the locally symmetric spaces for GL_n/F , could give new information in that situation. This turns out to be the case: however, so far not about the third problem, but about the second problem.

This has been taken up in [ACC⁺18], where the main result of the present paper is used to obtain local-global compatibility in the ordinary and (many) Fontaine–Laffaille cases. The third problem was overcome in the particular setting of [ACC⁺18] via an alternative argument that reduces it to the case of \mathbb{Q}_ℓ -coefficients, allowing us to implement the Calegari–Geraghty method unconditionally in arbitrary dimensions, and prove the meromorphic continuation of the L -function and the Sato–Tate conjecture for elliptic curves over CM fields (among other results).

In conclusion, the results of the present paper are tailored to give interesting information about GL_n/F . For this reason, we restrict attention to the following specific case at hand, although our methods should extend to somewhat more general Shimura varieties.

1.0.1. *Locally symmetric spaces.* Fix a CM field F and an integer $n \geq 1$. We assume that F contains an imaginary quadratic field $F_0 \subset F$; if we let $F^+ \subset F$ be the maximal totally real subfield, we have $F = F^+ \cdot F_0$. Let $V = F^{2n}$ be equipped with the skew-hermitian form

$$\langle (x_1, \dots, x_{2n}), (y_1, \dots, y_{2n}) \rangle = \sum_{i=1}^n (x_i \bar{y}_{2n+1-i} - x_{2n+1-i} \bar{y}_i)$$

where \bar{y} denotes the complex conjugate of y , and consider the associated alternating form

$$(\cdot, \cdot) : V \times V \rightarrow \mathbb{Q} : (x, y) = \mathrm{tr}_{F/\mathbb{Q}} \langle x, y \rangle .$$

Then V admits \mathcal{O}_F -lattices $L \subset V$ that are self-dual with respect to (\cdot, \cdot) ; fixing one, we get an alternating perfect pairing

$$(\cdot, \cdot) : L \times L \rightarrow \mathbb{Z} .$$

Let G be the algebraic group over \mathbb{Z} defined by

$$G(R) := \{g \in \mathrm{GL}_{\mathcal{O}_F}(L)(R) \times \mathbb{G}_m(R) \mid (gv, gw) = (v, w) \ \forall v, w \in L\} .$$

The generic fibre of G is then a quasi-split unitary group.²

Over \mathbb{R} , we have $G_{\mathbb{R}} = \mathrm{U}(n, n)^{[F^+:\mathbb{Q}]}$. The associated symmetric space $X = G(\mathbb{R})/K_\infty$ for $G(\mathbb{R})$ is given by

$$X = \prod_{\tau:F^+ \hookrightarrow \mathbb{R}} X_{\tau,+}$$

²In the main text, we will denote by G a unitary similitude group. The translation is explained in §2.1.

where $X_{\tau,+}$ is the space of positive definite n -dimensional subspaces in the split hermitian space $V \otimes_{F^+, \tau} \mathbb{R} \cong \mathbb{C}^{2n}$.

For any neat compact open subgroup $K \subset G(\mathbb{A}_f)$, we are interested in the double quotient

$$X_K = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f) / K) .$$

As detailed below, this has the structure of a complex manifold of complex dimension $d = [F^+ : \mathbb{Q}]n^2$. The Borel–Serre compactification of X_K includes strata related to the locally symmetric spaces for GL_n / F ; this explains our interest in X_K .

Assume that $K = \prod_p K_p$ is a product of compact open subgroups $K_p \subset G(\mathbb{Q}_p)$, and fix a finite set of primes S containing all primes that ramify in F or at which $K_p \neq G(\mathbb{Z}_p)$. The abstract unramified Hecke algebra

$$\mathbb{T} = \mathbb{T}^S = \bigotimes_{p \notin S, p \in \mathrm{Spl}_{F_0/\mathbb{Q}}} \mathbb{Z}[G(\mathbb{Q}_p) // G(\mathbb{Z}_p)]$$

acts naturally on $H^i(X_K, A)$ and $H_c^i(X_K, A)$ for any coefficient module A ; here we restrict to the primes $\mathrm{Spl}_{F_0/\mathbb{Q}}$ of \mathbb{Q} that split in F_0 . (We remark that including fewer Hecke operators makes our main theorem stronger.)

If v is a prime of F that divides a prime $p \notin S$ that splits in the imaginary quadratic field F_0 , then we get a lift $\mathfrak{p}|p$ of p in F_0 , and an isomorphism

$$G(\mathbb{Q}_p) \cong \prod_{w|\mathfrak{p}} \mathrm{GL}_{2n}(F_w) = \mathrm{GL}_{2n}(F_v) \times \prod_{w|\mathfrak{p}, w \neq v} \mathrm{GL}_{2n}(F_w) .$$

Here w runs over primes of F dividing \mathfrak{p} . For $i = 1, \dots, 2n$, we let

$$T_{i,v} \in \mathbb{Z}[G(\mathbb{Q}_p) // G(\mathbb{Z}_p)]$$

be the Hecke operator given by the double coset of

$$\mathrm{GL}_{2n}(\mathcal{O}_{F_v}) \mathrm{diag}(\underbrace{\varpi_v, \dots, \varpi_v}_i, 1, \dots, 1) \mathrm{GL}_{2n}(\mathcal{O}_{F_v}) \times \prod_{w|\mathfrak{p}, w \neq v} \mathrm{GL}_{2n}(\mathcal{O}_{F_w})$$

inside $G(\mathbb{Q}_p)$.

Now fix a prime ℓ and let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal occurring in the support of $H^*(X_K, \mathbb{F}_\ell)$ (i.e. a system of Hecke eigenvalues occurring in the cohomology of X_K), and fix an embedding $\mathbb{T}/\mathfrak{m} \rightarrow \overline{\mathbb{F}}_\ell$. Enlarge S to include ℓ . It follows from [Sch15, Theorem 4.3.1] and [ACC⁺18, Theorem 2.3.3] (which relies on [Shi14]) that there exists a continuous semisimple $2n$ -dimensional Galois representation

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_{2n}(\overline{\mathbb{F}}_\ell)$$

unramified at all places not dividing a prime of S and such that for every prime v of F as above, the characteristic polynomial of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is the reduction of

$$X^{2n} - T_{1,v} X^{2n-1} + \dots + (-1)^i q_v^{i(i-1)/2} T_{i,v} X^{2n-i} + \dots + q_v^{n(2n-1)} T_{2n,v} .$$

modulo \mathfrak{m} , where q_v is the cardinality of the residue field at v .

Our main theorem is the following.

Theorem 1.1. *Assume the following conditions.*

- (i) $F^+ \neq \mathbb{Q}$;
- (ii) $\bar{\rho}_{\mathfrak{m}}$ is of length at most 2;

- (iii) there is a prime $p \neq \ell$ that splits completely in F and such that for all primes $v|p$ of F , the representation $\bar{\rho}_{\mathfrak{m}}$ is unramified at v with eigenvalues $\{\alpha_{1,v}, \dots, \alpha_{2n,v}\}$ of $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_v) \in \text{GL}_{2n}(\bar{\mathbb{F}}_\ell)$ satisfying $\alpha_{i,v} \neq p\alpha_{j,v}$ for all $i \neq j$.

Then

- (1) If $H^i(X_K, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$ then $i \geq d$.
- (2) If $H_c^i(X_K, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$ then $i \leq d$.

Remark 1.2. Assumption (i) simplifies some trace formula computations (essentially, it implies that certain boundary terms vanish). Roughly, the point is that the geometric side of the trace formula simplifies whenever the test function at two auxiliary places is of a simple form, and here we would like to use two infinite places. We would expect that one could remove this assumption with additional work.

Remark 1.3. We use assumption (ii) to ensure that only one boundary stratum (the one of interest, giving rise to GL_n/F) can possibly contribute to the cohomology localized at \mathfrak{m} . This is required to ensure a tight relation between cohomology and compactly supported cohomology after localization at \mathfrak{m} , and seems to be a critical assumption. In order to ensure this tight relation, we also need the existence of Galois representations associated to torsion classes in the cohomology of locally symmetric spaces for GL_m/F for $m \leq n$, as proved in [Sch15, Theorem 5.4.3].

Remark 1.4. Assumption (iii) is slightly weaker than a corresponding assumption in [CS17, Theorem 1.1], which would in addition ask that $\alpha_{i,v} \neq \alpha_{j,v}$. As observed by Koshikawa, this extra assumption is in fact superfluous: The critical [CS17, Lemma 6.2.2] holds (with very minor modifications) in this more general setup, cf. proof of Corollary 5.1.3 below, so in fact [CS17, Theorem 1.1] holds in this more general setup. (The condition $\alpha_i \neq \alpha_j$ was also not necessary in Boyer's work, [Boy19].)

Remark 1.5. The theorem implies formally that the same conclusion holds for $H^i(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ and $H_c^i(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$, and in addition that $H^d(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ is torsion-free. There is an excision long exact sequence relating $H_c^i(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$, $H^i(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ (which agrees with the cohomology of the Borel–Serre compactification of X_K) and the cohomology of the boundary $H^i(\partial X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$. The cohomology of the boundary is related to the cohomology of GL_n/F , and can contribute in many different degrees, both below and above the middle degree. By the long exact sequence, this cohomology has to be split between H^i and H_c^i . The theorem asserts that this is done in the most transparent way:³ Below the middle degree, everything maps into H_c^i ; above middle degree, everything comes from H^i ; and one has an exact sequence

$$0 \rightarrow H^{d-1}(\partial X_K, \mathbb{Z}_\ell)_{\mathfrak{m}} \rightarrow H_c^d(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}} \rightarrow H^d(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}} \rightarrow H^d(\partial X_K, \mathbb{Z}_\ell)_{\mathfrak{m}} \rightarrow 0$$

around the middle degree. In particular, the torsion-free group $H^d(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ surjects onto the cohomology $H^d(\partial X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ of the boundary, yielding control over the various torsion classes there. Using the Hochschild–Serre spectral sequence, we can deduce the theorem and the same consequences also for non-trivial \mathbb{Z}_ℓ -coefficient systems on X_K , such as \mathcal{V}_λ , where λ is a highest weight for G . See [ACC⁺18] for applications.

³It surprises the second author that such a clean picture can possibly be true.

Remark 1.6. If \mathfrak{m} is non-Eisenstein, i.e. if $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible, the theorem and the excision long exact sequence with \mathbb{F}_{ℓ} -coefficients imply that

$$H_c^i(X_K, \mathbb{F}_{\ell})_{\mathfrak{m}} \xrightarrow{\sim} H^i(X_K, \mathbb{F}_{\ell})_{\mathfrak{m}}$$

and that these are non-zero only in degree $i = d$. Indeed, if \mathfrak{m} is non-Eisenstein, then $H^i(\partial X_K, \mathbb{F}_{\ell})_{\mathfrak{m}} = 0$, which follows from the proof of [ACC⁺18, Theorem 2.4.2]. This also implies the same result with \mathbb{Z}_{ℓ} -coefficients and that $H_c^d(X_K, \mathbb{Z}_{\ell})_{\mathfrak{m}} \xrightarrow{\sim} H^d(X_K, \mathbb{Z}_{\ell})_{\mathfrak{m}}$ is torsion-free. This matches the folklore conjecture stated in (3) above, in the case of the locally symmetric space X_K .

1.6.1. *Shimura varieties.* The space X_K can be identified, Hecke-equivariantly, with a union of connected components of $\mathcal{S}_{K(N)}(\mathbb{C})$, for a Shimura variety $\mathcal{S}_{K(N)}$ of PEL type over \mathbb{Z} ; see § 2.1 for the precise relationship. As in [CS17], we will prove the theorem by using the p -adic geometry of $\mathcal{S}_{K(N)}$ for a prime p as guaranteed by Assumption (iii). More precisely, we will use the Shimura variety with infinite level at p regarded as a perfectoid space, and analyze the fibers of the Hodge–Tate period map in terms of Igusa varieties, including compactifications.

Our geometric results hold for any imaginary CM field F (not necessarily containing an imaginary quadratic field). Let Δ_F be the discriminant of F . From now on, set $K := K(N)$ to be a principal congruence subgroup in the unitary similitude group corresponding to G , for some integer $N \geq 3$. We can define an algebraic variety \mathcal{S}_K over $\mathbb{Z}[\frac{1}{N\Delta_F}]$ whose S -valued points parametrize quadruples $(A, \iota, \lambda, \eta)$ where A is an abelian scheme of dimension $[F : \mathbb{Q}]n$ over S with an action $\iota : \mathcal{O}_F \rightarrow \text{End}(A)$ such that $\text{Lie } A$ is free of rank n over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$, $\lambda : A \cong A^{\vee}$ is a principal polarization of A whose associated Rosati involution is compatible with complex conjugation on \mathcal{O}_F via ι , and η is an isomorphism $A[N] \cong L/NL$ compatible with the \mathcal{O}_F -action and polarization. Moreover, \mathcal{S}_K admits a minimal compactification \mathcal{S}_K^* and a toroidal compactification $\mathcal{S}_K^{\text{tor}}$, the latter depending on the choice of a certain family of cone decompositions Σ , as usual.

Now fix a prime p that is unramified in F and prime to N , and an algebraically closed field k of characteristic p . Moreover, fix a p -divisible group \mathbb{X} over k , with an \mathcal{O}_F -action ι and a principal polarization λ , satisfying the same assumptions as the abelian variety above. Then the subset

$$\{x \in \mathcal{S}_K \times k \mid A[p^{\infty}] \times k(\bar{x}) \cong \mathbb{X} \times_k k(\bar{x})\}$$

of all points x such that $A[p^{\infty}]$ is isomorphic to \mathbb{X} , compatibly with the extra structures, defines a leaf $\mathcal{C}^{\mathbb{X}} \subset \mathcal{S}_K \times k$, a locally closed smooth subscheme. In [LS18a], Lan–Stroh prove that the leaf $\mathcal{C}^{\mathbb{X}}$ is *well-positioned*, which implies that one can define partial minimal and toroidal compactifications $\mathcal{C}^{\mathbb{X},*}$, $\mathcal{C}^{\mathbb{X},\text{tor}}$, which have many of the same properties as the ambient Shimura varieties.

Over the leaf $\mathcal{C}^{\mathbb{X}}$, we can look at the scheme parametrizing isomorphisms $A[p^{\infty}] \cong \mathbb{X}$ compatible with extra structures. This defines an $\underline{\text{Aut}}(\mathbb{X})$ -torsor

$$\mathfrak{Jg}^{\mathbb{X}} \rightarrow \mathcal{C}^{\mathbb{X}},$$

where $\mathfrak{Jg}^{\mathbb{X}}$ is a perfect scheme. Note that in general when \mathbb{X} is not isoclinic, $\underline{\text{Aut}}(\mathbb{X})$ is a highly non-reduced group scheme. Its group of connected components is the profinite group $\Gamma_{\mathbb{X}} := \underline{\text{Aut}}(\mathbb{X})(k)$. Then the map to the perfection

$$\mathfrak{Jg}^{\mathbb{X}} \rightarrow \mathcal{C}_{\text{perf}}^{\mathbb{X}}$$

is a $\Gamma_{\mathbb{X}}$ -torsor. One important property of $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ is that it depends on \mathbb{X} only up to isogeny, i.e. an isogeny $\phi : \mathbb{X} \rightarrow \mathbb{X}'$ (compatible with the extra structures) induces an isomorphism $\mathfrak{I}\mathfrak{g}^{\mathbb{X}} \cong \mathfrak{I}\mathfrak{g}^{\mathbb{X}'}$.

Our first result is that there are good partial toroidal compactifications of Igusa varieties.

Proposition 1.7. *The $\Gamma_{\mathbb{X}}$ -torsor $\mathfrak{I}\mathfrak{g}^{\mathbb{X}} \rightarrow \mathcal{C}_{\text{perf}}^{\mathbb{X}}$ extends uniquely to a $\Gamma_{\mathbb{X}}$ -torsor*

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}} \rightarrow \mathcal{C}_{\text{perf}}^{\mathbb{X},\text{tor}}.$$

Moreover, one can describe the local structure of $\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}}$ at the toroidal boundary as in the case of Shimura varieties, cf. e.g. Theorem 3.2.13. As a consequence, we prove that an isogeny $\phi : \mathbb{X} \rightarrow \mathbb{X}'$ that induces isomorphisms on étale quotients induces an isomorphism $\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}} \cong \mathfrak{I}\mathfrak{g}^{\mathbb{X}',\text{tor}}$.⁴

One can also define partial minimal compactifications of $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*}$, for example via normalization. We can describe the boundary components of the minimal compactification explicitly in terms of *Igusa cusp labels*, cf. Theorem 3.3.15. Another important result is that partial minimal compactifications are affine:

Proposition 1.8. *For any \mathbb{X} , the leaf $\mathcal{C}^{\mathbb{X},*}$ and the Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*}$ are affine.*

To prove this, we first prove it for a special leaf in a given isogeny class (one that is contained in a fundamental Ekedahl–Oort stratum). Such leaves exist in general by a result of Nie, [Nie15], and in their case we appeal to a result of Boxer, [Box15], that partial minimal compactifications of Ekedahl–Oort strata are affine. (The latter result was also proved independently by Goldring–Koskivirta [GK19].) Then we deduce the general case by using isogenies and the invariance of the partial toroidal compactifications and minimal compactifications under isogenies.

Remark 1.9. When \mathbb{X} is completely slope divisible the Igusa variety and its compactifications admit non-perfect versions that are sometimes useful to keep the situation more geometric. In particular, one can naturally define versions with finite level structure. For this reason, some of the analysis is carried out only for completely slope divisible \mathbb{X} . As any \mathbb{X} is isogenous to a completely slope divisible \mathbb{X} this results in no essential loss of generality.

Our main geometric result is a description of the fibres of the Hodge–Tate period map on both minimal and toroidal compactifications. Let

$$\mathcal{S}_{K(p^\infty N)}^* = \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Q}_p}^*$$

and similarly

$$\mathcal{S}_{K(p^\infty N)}^{\text{tor}} = \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{\text{tor}},$$

as “ p -adic analytic spaces”.⁵ There is a Hodge–Tate period map

$$\pi_{\text{HT}}^* : \mathcal{S}_{K(p^\infty N)}^* \rightarrow \mathcal{F}\ell$$

⁴The condition on étale quotients is required to ensure that the choice of cone decomposition does not cause trouble.

⁵Technically, as diamonds, [Sch17]. It is known that both limits are representable by perfectoid spaces, but we will not need to know this.

where the flag variety \mathcal{Fl} parametrizes totally isotropic F -stable subspaces of V ; by pre-composition with the projection from the toroidal to the minimal compactification, we also get a map

$$\pi_{\text{HT}}^{\text{tor}} : \mathcal{S}_{K(p^\infty N)}^{\text{tor}} \rightarrow \mathcal{Fl}.$$

Let C be some complete algebraically closed nonarchimedean extension of \mathbb{Q}_p , and let $x \in \mathcal{Fl}(C)$ be a point. By [SW13], this is equivalent to a p -divisible group X over \mathcal{O}_C with extra structure as above, and an isomorphism $T_p(X) \cong L$. The special fibre \mathbb{X} of X gives rise to $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ with its partial minimal and toroidal compactification $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*}$ and $\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}}$. As these are perfect schemes, they admit canonical lifts to p -adic formal schemes over \mathcal{O}_C ; let us simply denote these by a subscript \mathcal{O}_C , and then their generic fibres, which are perfectoid spaces over C , by a subscript C .

Theorem 1.10. *There are canonical maps*

$$\begin{aligned} \mathfrak{I}\mathfrak{g}_C^{\mathbb{X},\text{tor}} &\rightarrow (\pi_{\text{HT}}^{\text{tor}})^{-1}(x), \\ \mathfrak{I}\mathfrak{g}_C^{\mathbb{X},*} &\rightarrow (\pi_{\text{HT}}^*)^{-1}(x). \end{aligned}$$

They are open immersions of perfectoid spaces with the same rank-1-points; in fact, the target is the canonical compactification of the source, in the sense of [Sch17, Proposition 18.6].

In particular, the cohomology of the fibres of the Hodge–Tate period map agrees with the cohomology of Igusa varieties, even on compactifications. This extends the result of [CS17] to the non-compact case.

1.10.1. Organization. Let us now summarize the contents of the different sections, highlighting also some further results not mentioned so far.

In Section 2, we set up the basic formalism of our Shimura varieties and recall some foundational results, especially about the construction of Igusa varieties. At the end of this section, we also explain the main argument, referring to the key results of the later sections.

In Section 3, we construct the compactifications of Igusa varieties and prove basic results about their geometry, as stated above.

In Section 4, we describe the fibres of the compactified Hodge–Tate period maps $\pi_{\text{HT}}^{\text{tor}}$ and π_{HT}^* in terms of compactified Igusa varieties. We use toroidal compactifications first to write down an explicit map in terms of boundary strata (even integrally). Once we have a map, checking that it is an isomorphism can be done on geometric points by general properties of perfectoid spaces (or even diamonds). The case of minimal compactifications follows rather formally by using that both sides are affine. Finally, we also use these results to deduce a semiperversity result for $R\pi_{\text{HT}*}^{\circ} \mathbb{F}_\ell$.

In Section 5, we compute the cohomology of the Igusa varieties Ig^b , as a virtual \mathbb{Q}_ℓ -representation. This builds on the previous work of Shin, [Shi09], [Shi10], [Shi11]; a similar analysis also appeared in [CS17]. The present situation requires us to understand more precisely the boundary terms. It is here that we have to assume that $F^+ \neq \mathbb{Q}$, which ensures that G admits no *cuspidal subgroups*, cf. Definition 5.5.2. Essentially, this ensures that we have a special test function at at least two places (the infinite places) which simplifies the geometric of the side trace formula, and allows us to compare it to Shin’s trace formula for Igusa varieties. If

$F^+ = \mathbb{Q}$, then the work of Morel, [Mor10], shows that the situation is more complicated even in the case of Shimura varieties, and one can expect a clean answer only for intersection cohomology.

In Section 6, we analyze the cohomology of the boundary of the Igusa varieties in terms of the Igusa varieties for smaller unitary groups and the cohomology of general linear groups. What we prove is essentially a version for Igusa varieties of Pink’s formula [Pin92]. It is actually somewhat tricky to show that these computations are Hecke-equivariant, and we use some new ideas involving adic spaces to get a transparent argument. As an application, we also construct Galois representations associated to mod ℓ systems of Hecke eigenvalues that occur in the cohomology of Igusa varieties (in the setting of Theorem 1.1).

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2. QUASISPLIT UNITARY SHIMURA VARIETIES

In this section, we define the relevant Shimura varieties associated to quasi-split unitary similitude groups and prove some basic related results. Our notation in this section will be slightly different from the notation we used in the introduction.

2.1. Shimura varieties. As in the introduction, fix an imaginary CM field F with totally real subfield $F^+ \subset F$, and an integer $n \geq 1$. Consider the F -vector space $V = F^{2n}$ with the skew-hermitian form

$$\langle (x_1, \dots, x_{2n}), (y_1, \dots, y_{2n}) \rangle = \sum_{i=1}^n (x_i \bar{y}_{2n+1-i} - x_{2n+1-i} \bar{y}_i)$$

and consider the induced alternating form

$$(\cdot, \cdot) : V \times V \rightarrow \mathbb{Q} : (x, y) = \mathrm{tr}_{F/\mathbb{Q}} \langle x, y \rangle.$$

Then V admits self-dual \mathcal{O}_F -lattices $L \subset V$ with respect to the pairing (\cdot, \cdot) : For example, if $I \subset F$ is the inverse different, then $L = \mathcal{O}_F^n \oplus \bar{I}^n$ is such a self-dual lattice. Fix any self-dual \mathcal{O}_F -lattice $L \subset V$, and define the group G over \mathbb{Z} by

$$G(R) = \{(g, c) \in \mathrm{GL}_{\mathcal{O}_F}(L)(R) \times \mathbb{G}_m(R) \mid (gv, gw) = c(v, w) \ \forall v, w \in L\},$$

which is a unitary similitude group. We let

$$X = \prod_{\tau: F^+ \hookrightarrow \mathbb{R}} X_{\tau,+} \sqcup \prod_{\tau: F^+ \hookrightarrow \mathbb{R}} X_{\tau,-}$$

be the symmetric space for $G(\mathbb{R})$, where $X_{\tau,+}$ (resp. $X_{\tau,-}$) is the space of positive (resp. negative) definite n -dimensional subspaces of $V \otimes_{F^+} \mathbb{R} \cong \mathbb{C}^{2n}$. For any neat compact open subgroup $K \subset G(\mathbb{A}_f)$, we consider the double quotient

$$X_K = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f) / K).$$

To compare with Theorem 1.1, we will also need to consider the corresponding unitary group G_0 over \mathbb{Z} :

$$G_0(R) = \{g \in \mathrm{GL}_{\mathcal{O}_F}(L)(R) \mid (gv, gw) = (v, w) \ \forall v, w \in L\}.$$

Then G_0 is naturally a subgroup of G (and is the group denoted by G in the introduction). We also consider $X^0 = \prod_{\tau: F^+ \hookrightarrow \mathbb{R}} X_{\tau,+}$, which is the symmetric space for $G_0(\mathbb{R})$ and, for a neat compact open subgroup $K_0 \subset G_0(\mathbb{A}_f)$, the corresponding double quotient $X_{K_0}^0$ (this is the space denoted X_K in the introduction).

Moreover, for a prime v of F above a prime away from S and that splits in F_0 , let $T_{i,v}^0$ be the double coset operator

$$\mathrm{GL}_{2n}(\mathcal{O}_{F_v}) \mathrm{diag}(\underbrace{\varpi_v, \dots, \varpi_v}_i, 1, \dots, 1) \mathrm{GL}_{2n}(\mathcal{O}_{F_v}) \times \prod_{w \mid \mathfrak{p}, w \neq v} \mathrm{GL}_{2n}(\mathcal{O}_{F_w})$$

for G_0 , and $T_{i,v}$ be the double coset operator

$$\mathrm{GL}_{2n}(\mathcal{O}_{F_v}) \mathrm{diag}(\underbrace{\varpi_v, \dots, \varpi_v}_i, 1, \dots, 1) \mathrm{GL}_{2n}(\mathcal{O}_{F_v}) \times \prod_{w \mid \mathfrak{p}, w \neq v} \mathrm{GL}_{2n}(\mathcal{O}_{F_w}) \times \mathbb{Z}_p^\times$$

for G . The inclusion $G_0 \hookrightarrow G$ induces a restriction map $\mathbb{T}^S \rightarrow \mathbb{T}^{0,S}$ on the level of Hecke algebras that identifies $T_{i,v}$ with $T_{i,v}^0$.

Lemma 2.1.1. *Assume that $K_0 = K \cap G_0(\mathbb{A}_f)$. The inclusion $G_0 \hookrightarrow G$ induces a natural map $X_{K_0}^0 \rightarrow X_K$, which is an open and closed immersion. The induced map on cohomology and compactly supported cohomology is Hecke-equivariant for the restriction map $\mathbb{T}^S \rightarrow \mathbb{T}^{0,S}$.*

Proof. We have a short exact sequence of algebraic groups over \mathbb{Z} :

$$1 \rightarrow G_0 \rightarrow G \xrightarrow{c} \mathbb{G}_m \rightarrow 1.$$

The inclusion $G_0 \hookrightarrow G$ identifies X^0 with a connected component of X . The locally symmetric space X_K is a disjoint union of quotients of X by congruence subgroups of $G(\mathbb{Q})$; the analogous statement also holds true for $X_{K_0}^0$. To prove the lemma, it is enough to check that the natural map $X_{K_0}^0 \rightarrow X_K$ is injective. Assume that we have pairs $(h_i, g_i) \in X^0 \times G_0(\mathbb{A}_f)$ for $i = 1, 2$ and elements $\gamma \in G(\mathbb{Q})$ and $k \in K$ such that $(h_1, g_1) = \gamma(h_2, g_2)k$. From $g_1 = \gamma g_2 k$, we deduce that

$$c(\gamma) = c(k)^{-1} \in \mathbb{Q}^\times \cap \left(\prod_p \mathbb{Z}_p^\times \right) = \{\pm 1\}.$$

On the other hand, $h_1 = \gamma h_2$ implies that $c(\gamma) > 0$. We deduce that $\gamma \in G_0(\mathbb{Q})$, $k \in K_0$. The last statement is clear. \square

It will be convenient to know that G satisfies the Hasse principle:

Proposition 2.1.2. *The group G satisfies the Hasse principle, i.e. the map*

$$H^1(\mathbb{Q}, G) \rightarrow \prod_v H^1(\mathbb{Q}_v, G)$$

is injective, where v runs over all places of F . Equivalently, if $(V', (\cdot, \cdot)')$ is any other $2n$ -dimensional F -vector space with an alternating form $(\cdot, \cdot)'$ such that $(xv, w)' = (v, \bar{x}w)'$ for all $v, w \in V'$ and $x \in F$, such that there are isomorphisms $(V_v, (\cdot, \cdot)_v) \cong (V'_v, (\cdot, \cdot)'_v)$, identifying the forms up to a scalar, after base change to any local field \mathbb{Q}_v of \mathbb{Q} , then there is an isomorphism $(V, (\cdot, \cdot)) \cong (V', (\cdot, \cdot)')$, identifying the forms up to a scalar.

Proof. (cf. [Kot92, Section 7]) The derived group G_{der} of G is simply connected, so $H^1(\mathbb{Q}, G_{\text{der}}) = 0$. It follows that it suffices to see that $D = G/G_{\text{der}}$ satisfies the Hasse principle. One can identify D with the subtorus of $\text{Res}_{F/\mathbb{Q}}\mathbb{G}_m \times \mathbb{G}_m$ of all pairs (z, t) such that $\text{Nm}_{F/F^+}(z) = t^{2n}$. Via $(z, t) \mapsto (z/t^n, t)$, this is isomorphic to $\text{Res}_{F^+/\mathbb{Q}}T \times \mathbb{G}_m$, where $T = \ker(\text{Nm} : \text{Res}_{F/F^+}\mathbb{G}_m \rightarrow \mathbb{G}_m)$. Both of these tori satisfy the Hasse principle. \square

We can now interpret X_K as the \mathbb{C} -points of a moduli scheme of abelian varieties with certain extra structures. As we will have much occasion to consider similar kinds of extra structure, let us fix some terminology for this paper.

Definition 2.1.3.

- (1) *Let S be a scheme over $\mathbb{Z}[\frac{1}{\Delta_F}]$. An abelian variety with G -structure over S is a triple (A, ι, λ) where A is an abelian scheme of dimension $[F : \mathbb{Q}]n$ over S , $\iota : \mathcal{O}_F \rightarrow \text{End}(A)$ is an \mathcal{O}_F -action such that $\text{Lie } A$ is free of rank n over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$, and $\lambda : A \cong A^\vee$ is a principal polarization on A whose Rosati involution is compatible with complex conjugation on \mathcal{O}_F via ι .*

- (2) Let p be a prime that is unramified in F and let S be a scheme on which p is locally nilpotent. A p -divisible group with G -structure over S is a triple (X, ι, λ) where X is a p -divisible group of height $2[F : \mathbb{Q}]n$ and dimension $[F : \mathbb{Q}]n$, $\iota : \mathcal{O}_F \rightarrow \text{End}(X)$ is an \mathcal{O}_F -action such that $\text{Lie } X$ is free of rank n over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$, and $\lambda : X \cong X^\vee$ is a principal polarization on X whose Rosati involution is compatible with complex conjugation on \mathcal{O}_F via ι .

Let $N \geq 3$ and let

$$K = K(N) = \{g \in G(\widehat{\mathbb{Z}}) \mid g \equiv 1 \pmod{N}\}$$

be a principal congruence subgroup, automatically neat. In the following definition, we do not invert N ; we do not claim any nice geometric properties of the resulting scheme at primes dividing N .

Definition 2.1.4. Let $\mathcal{S}_K^{\text{pre}}$ over $\text{Spec } \mathbb{Z}[\frac{1}{\Delta_F}]$ parametrize over a test scheme S an abelian variety with G -structure $\underline{A} = (A, \iota, \lambda)$ together with an \mathcal{O}_F -linear map $L/N \rightarrow A[N]$ and a primitive N -th root of unity $\zeta_N \in \mu_N(\mathcal{O}_S)$ such that the diagram

$$\begin{array}{ccc} L/N \times L/N & \xrightarrow{(\cdot, \cdot)} & \mathbb{Z}/N \\ \downarrow & & \downarrow \zeta_N \\ A[N] \times A[N] & \longrightarrow & \mu_N \end{array}$$

commutes, where the lower map is the Weil pairing on $A[N]$ induced by λ , and such that the map $L/N \rightarrow A[N]$ extends to similar \mathcal{O}_F -linear maps $L/\Delta_F^m N \rightarrow A[\Delta_F^m N]$ compatible with the pairing for all $m \geq 1$.

It is standard that $\mathcal{S}_K^{\text{pre}}$ is a Deligne–Mumford stack and that $\mathcal{S}_K^{\text{pre}} \times \mathbb{Z}[\frac{1}{\Delta_F N}]$ is representable, as it is relatively representable over the Siegel modular variety of principal level $N \geq 3$. In fact, even if p divides N , $\mathcal{S}_K^{\text{pre}} \times \mathbb{Z}_{(p)}$ is representable as soon as the part of N prime to p is at least 3; we will always be in such a situation. We will need to normalize $\mathcal{S}_K^{\text{pre}}$:

Definition 2.1.5. Let \mathcal{S}_K be the normalization of $\mathcal{S}_K^{\text{pre}}$ in $\mathcal{S}_K^{\text{pre}} \times \mathbb{Z}[\frac{1}{\Delta_F N}]$.

The reason for the final condition on lifting the level structure will become apparent in the following proof:

Proposition 2.1.6. There is a natural isomorphism of manifolds

$$X_K \cong \mathcal{S}_K(\mathbb{C}).$$

Proof. If $\underline{A} = (A, \iota, \lambda)$ is an abelian variety with G -structure over \mathbb{C} , then the first homology $L' = H_1(A, \mathbb{Z})$ is a finite projective \mathcal{O}_F -module of rank $2n$ equipped with a perfect alternating form $(\cdot, \cdot)' : L' \times L' \rightarrow \mathbb{Z}$ (up to sign). We want to see that when \underline{A} comes from \mathcal{S}_K , there is an F -linear isomorphism $L_{\mathbb{Q}} \cong L'_{\mathbb{Q}}$ compatible with the forms up to scalar. By the Hasse principle (Proposition 2.1.2), it is enough to show that such isomorphisms exist locally. At the archimedean places, both forms are of the same signature by the condition on the dimension of the Lie algebra. At the primes that are unramified in F , both lattices are self-dual, which determines the isomorphism class of the form. Finally, at the ramified primes we use the final condition in Definition 2.1.4.

Now the choice of an isomorphism $L_{\mathbb{Q}} \cong L'_{\mathbb{Q}}$ compatible with the form, up to scalar, gives a $G(\mathbb{Q})$ -torsor over $S_K(\mathbb{C})$. We want to identify this with $X \times G(\mathbb{A}_f)/K$. The element of X comes from Hodge theory – the (positive or negative) definite n -dimensional $F \otimes_{\mathbb{Q}} \mathbb{C}$ -subspace is the Hodge filtration – and the element of $G(\mathbb{A}_f)/K$ is exactly the given level structure. Note again that at ramified primes, we need to use the assumption that the isomorphism lifts to ensure that the isomorphism modulo N comes from an element of $G(\mathbb{A}_f)$. \square

2.2. p -divisible groups. Next, we give some results about p -divisible groups with G -structure.

Let $(\mathbb{X}, \iota, \lambda)$ be a p -divisible group with G -structure over an algebraically closed field k of characteristic p . We get a filtration

$$\mathbb{X}^{\mu} \subset \mathbb{X}^{\circ} \subset \mathbb{X}$$

into the multiplicative and connected part. This filtration is \mathcal{O}_F -stable and symplectic with respect to λ . We let

$$\mathbb{X}^{(0,1)} = \mathbb{X}^{\circ}/\mathbb{X}^{\mu}, \mathbb{X}^{\acute{e}t} = \mathbb{X}/\mathbb{X}^{\circ}$$

be the graded pieces for the filtration. As we work over a perfect field, this filtration splits uniquely, so

$$\mathbb{X} \cong \mathbb{X}^{\mu} \oplus \mathbb{X}^{(0,1)} \oplus \mathbb{X}^{\acute{e}t}$$

under which λ decomposes similarly into a direct sum. In particular, up to isomorphism, \mathbb{X} is determined by $\mathbb{X}^{(0,1)}$ with its \mathcal{O}_F -action and principal polarization and the finite projective $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -module $T_p(\mathbb{X}^{\acute{e}t})$. Slightly more generally, we have the following observation.⁶

Proposition 2.2.1. *Assume that*

$$\mathbb{Z}_{-2} \subset \mathbb{Z}_{-1} \subset \mathbb{X}$$

is an \mathcal{O}_F -stable filtration by sub- p -divisible groups such that \mathbb{Z}_{-2} is multiplicative, $\mathbb{X}/\mathbb{Z}_{-1}$ is étale, and the polarization identifies \mathbb{Z}_{-2} with $\mathbb{X}/\mathbb{Z}_{-1}$. Then there is an \mathcal{O}_F -linear splitting

$$\mathbb{X} \cong \mathbb{Z}_{-2} \oplus \mathbb{Z}_{-1}/\mathbb{Z}_{-2} \oplus \mathbb{X}/\mathbb{Z}_{-1}$$

under which λ decomposes similarly into a direct sum.

Proof. We can decompose into multiplicative, biconnected, and étale parts, which gives us only two splitting problems, on the étale and on the multiplicative parts. We only have to arrange that they are dual, so we can make an arbitrary choice at one side and then arrange the other one to be dual. \square

2.2.2. Construction of Igusa covers. Let k be an algebraically closed field of characteristic p and let \mathbb{X}/k be an isoclinic p -divisible group equipped with extra structures of EL type or of PEL type. For every $m \in \mathbb{Z}_{\geq 1}$, let Γ_m denote the group of automorphisms of $\mathbb{X}[p^m]$ which commute with the extra structures and which lift to automorphisms of $\mathbb{X}[p^{m'}]$ for every $m' \geq m$. We let $\Gamma_{m,k}$ denote the finite étale group scheme over k corresponding to Γ_m . The following is a consequence of [CS17, Corollary 4.1.10].

⁶The motivation for the result are the Igusa cusp labels introduced later; this also explains the notation.

Proposition 2.2.3. *The finite étale group scheme $\Gamma_{m,k}$ represents the functor*

$$\mathcal{F}_{\Gamma_m} : \text{Spec } k - \text{Schemes} \rightarrow \text{Sets}$$

for which $\mathcal{F}_{\Gamma_m}(\mathcal{T})$ is the set of \mathcal{T} -automorphisms of $\mathbb{X}_{\mathcal{T}}[p^m]$ which commute with the extra structures and which lift (fppf locally on \mathcal{T}) to \mathcal{T} -automorphisms of $\mathbb{X}_{\mathcal{T}}[p^{m'}]$ for all $m' \geq m$. In fact, for any m there is some $m' \geq m$ so that it is enough to ask for a lifting to $\mathbb{X}_{\mathcal{T}}[p^{m'}]$ (fppf locally on \mathcal{T}).

Proof. We consider the functor \mathcal{F}_{H_m} sending a $\text{Spec } k$ -scheme \mathcal{T} to the set of \mathcal{T} -endomorphisms of $\mathbb{X}_{\mathcal{T}}[p^m]$ which lift (fppf locally on \mathcal{T}) to \mathcal{T} -endomorphisms of $\mathbb{X}_{\mathcal{T}}[p^{m'}]$ for all $m' \geq m$ (we do not require these to be invertible or to commute with the extra structures). Then \mathcal{F}_{Γ_m} is a subfunctor of \mathcal{F}_{H_m} . It is enough to see that \mathcal{F}_{H_m} is representable by a finite étale scheme $H_{m,k}$. This is proved in the second part of [CS17, Corollary 4.1.10], which applies because \mathbb{X} is isoclinic.

To see that one m' is enough, use that everything is finitely presented. \square

Let X/k be a seminormal scheme. Assume that \mathcal{G}/X is a p -divisible group equipped with extra structures of EL type or of PEL type as in [CS17, Section 4.2], such that for any point $x \in X$ and for any geometric point \bar{x} above x there exist isomorphisms

$$\mathcal{G} \times_X k(\bar{x}) \xrightarrow{\sim} \mathbb{X} \times_k k(\bar{x})$$

compatible with the extra structures. The following result is proved in [Man05] in the special case when X is an Oort central leaf in a PEL-type Shimura variety, but the result holds more generally and we will need to appeal to the general result repeatedly later on.

Theorem 2.2.4. *Let $m \in \mathbb{Z}_{\geq 1}$. Consider the functor from X -schemes to sets which sends a scheme \mathcal{T}/X to the set of isomorphisms*

$$\rho_m : \mathcal{G}[p^m] \times_X \mathcal{T} \xrightarrow{\sim} \mathbb{X}[p^m] \times_k \mathcal{T}$$

compatible with the extra structures and which lift (fppf locally on \mathcal{T}) to isomorphisms $\rho_{m'}$ for all $m' \geq m$ (or just for one sufficiently large m' as in the previous proposition). This functor is representable by a Γ_m -torsor

$$J_m(\mathcal{G}/X) \rightarrow X.$$

Proof. We may assume that $X = \text{Spec } R$ is affine. We first make a reduction to the case that R is perfect, so assume that the result holds true when R is perfect. Our goal is to show that after some faithfully flat cover, \mathcal{G} is isomorphic to \mathbb{X} ; this clearly implies the theorem by faithfully flat descent. Assuming the result for X_{perf} , we get for any $m \geq 1$ a Γ_m -torsor $J_m(\mathcal{G} \times_X X_{\text{perf}}/X_{\text{perf}}) \rightarrow X_{\text{perf}}$. As $X \rightarrow X_{\text{perf}}$ is a universal homeomorphism, this Γ_m -torsor descends uniquely to a Γ_m -torsor $J'_m \rightarrow X$. Let $J' = \varprojlim_m J'_m$, which is faithfully flat over X . We claim that the isomorphism $\mathcal{G} \cong \mathbb{X}$ that exists tautologically over $J' \times_X X_{\text{perf}} = \varprojlim_m J_m(\mathcal{G} \times_X X_{\text{perf}}/X_{\text{perf}})$ is in fact defined over J' . Note that this isomorphism is given by a series of isomorphisms between finite locally free group schemes $\mathcal{G}[p^m] \cong \mathbb{X}[p^m]$, so this amounts to a countable system of elements of R_{perf} being elements of R . As R is seminormal, an element $f \in R_{\text{perf}}$ lies in R if and only if for all $x \in X = \text{Spec } R$, the element $f(x) \in k(x)_{\text{perf}}$ lies in $k(x)$, cf. [Swa80, Theorem 2.6]. Thus, we can reduce to the case that R is a field. In that

case $R \rightarrow R_{\text{perf}}$ is faithfully flat, so the theorem follows by faithfully flat descent for R ; and then we also see the claim, as necessarily $J_m(\mathcal{G}/X) = J'_m$.

We therefore assume that X is perfect. We further want to reduce to the case that X is strictly henselian local. For this, consider for any m the scheme $J''_m(\mathcal{G}/X) \rightarrow X$ parametrizing isomorphisms $\mathcal{G}[p^m] \cong \mathbb{X}[p^m]$ compatible with extra structures; note that this functor is representable by a finitely presented affine scheme over X . (For this, we think of group scheme homomorphisms $\mathcal{G}[p^m] \times_X \mathcal{T} \rightarrow \mathbb{X}[p^m] \times_k \mathcal{T}$ as maps of Hopf algebras over \mathcal{T} .) Let $J''_m(\mathcal{G}/X) = \text{Spec } A_m$. For $m' \geq m$, we get maps $A_m \rightarrow A_{m'}$; let \bar{A}_m denote the image of A_m in $A_{m'}$ where m' is chosen large enough as in the previous proposition. Then the formation of \bar{A}_m commutes with flat base change, and \bar{A}_m is still a finitely presented R -algebra, as it only depends on $\mathcal{G}[p^{m'}]$ which is finitely presented. It is enough to show that \bar{A}_m is faithfully finite flat over R : Indeed, if this holds for all m , then $\varinjlim_m A_m = \varinjlim_m \bar{A}_m$ is faithfully finite flat over R , and we can by flat base change reduce to the case that $\mathcal{G} \cong \mathbb{X}$, in which case the claim is clear. But checking that \bar{A}_m is faithfully finite flat can be done over the strictly henselian local rings of R , so from now on we assume $X = \text{Spec } R$, with R a strictly henselian local ring that is also perfect. It is then enough to find an isomorphism $\mathcal{G} \xrightarrow{\sim} \mathbb{X} \times_{\kappa} R$ compatible with extra structures.

From [CS17, Lemma 4.3.15], we see that we can construct an isogeny $\mathcal{G} \rightarrow \mathbb{X} \times_{\kappa} R$, a priori not compatible with extra structures. This induces a morphism from $\text{Spec } R$ to the reduced special fiber $\overline{\mathcal{M}}_{\mathbb{X}}^{0,d}$ of a truncated Rapoport–Zink space for \mathbb{X} . Lemma 2.2.5 below shows that the subset of points of $\overline{\mathcal{M}}_{\mathbb{X}}^{0,d}$ over which the p -divisible group parametrised by the Rapoport–Zink space is isomorphic to \mathbb{X} is finite. As $\text{Spec } R$ is connected, the map from $\text{Spec } R$ to the Rapoport–Zink space factors through one such (closed) point. Modifying the isogeny, we can then find an isomorphism $\rho : \mathcal{G} \xrightarrow{\sim} \mathbb{X} \times_{\kappa} R$, a priori not compatible with extra structure. But [CS17, Lemma 4.3.15] ensures that it must automatically be compatible with the extra structure. \square

We used the following general lemma: Let k be an algebraically closed field of characteristic p and let \mathbb{Y}/k be a p -divisible group. Let $\mathcal{M}_{\mathbb{Y}}$ be the Rapoport–Zink space for \mathbb{Y} parametrizing quasi-isogenies from \mathbb{Y} to varying p -divisible groups. Consider the truncated Rapoport–Zink space $\mathcal{M}_{\mathbb{Y}}^{0,d}$, which parametrises isogenies with kernel contained in the p^d -torsion subgroup. Let $\overline{\mathcal{M}}_{\mathbb{Y}}^{0,d}$ be its reduced special fiber. The following lemma is a slight generalisation of [Man04, Lemma 3.4], which applies to the special case of an isoclinic, completely slope divisible p -divisible group.

Lemma 2.2.5. *Let \mathcal{H} be the universal p -divisible group over $\overline{\mathcal{M}}_{\mathbb{Y}}^{0,d}$. The subset*

$$Z := \{x \in \overline{\mathcal{M}}_{\mathbb{Y}}^{0,d} \mid \mathcal{H} \times k(\bar{x}) \cong \mathbb{Y} \times_k k(\bar{x})\} \subseteq \overline{\mathcal{M}}_{\mathbb{Y}}^{0,d}$$

consists of finitely many reduced points.

Proof. We know that the subset Z is constructible by [Oor04, Corollary 2.5]. To show that it is finite, it is enough to see that all points are defined over k . Consider the scheme $\tilde{Z} \rightarrow \overline{\mathcal{M}}_{\mathbb{Y}}^{0,d}$ parametrizing isomorphisms $\mathcal{H} \cong \mathbb{Y}$; this surjects onto Z . Then for any algebraically closed field k' over k , $\tilde{Z}(k')$ is the set of self-isogenies $\mathbb{Y}_{k'} \rightarrow \mathbb{Y}_{k'}$ of degree $\leq d$. But $\text{End}(\mathbb{Y}) = \text{End}(\mathbb{Y}_{k'})$, so we see that $\tilde{Z}(k') = \tilde{Z}(k)$, and so it follows that all points of Z are k -valued, and hence Z is finite. \square

Note that in the previous lemma, we did not ask that \mathbb{Y} be isoclinic. In fact, we will now use it to prove a variant of Theorem 2.2.4 for non-isoclinic groups.

Let \mathbb{X}/k be a p -divisible group with extra structure of EL or PEL type. Let $\Gamma = \varprojlim_m \Gamma_m = \text{Aut}(\mathbb{X})$ be the profinite group of automorphisms of \mathbb{X}/k compatible with the extra structure. Let X/k be a perfect scheme, and assume that \mathcal{G} is a p -divisible group over X with the same kind of extra structure such that for all geometric points \bar{x} of X , there is an isomorphism $\mathcal{G} \times k(\bar{x}) \cong \mathbb{X} \times_k k(\bar{x})$ compatible with extra structures.

Proposition 2.2.6. *The functor on perfect X -schemes \mathcal{T} parametrizing isomorphisms $\mathcal{G} \times_X \mathcal{T} \cong \mathbb{X} \times_k \mathcal{T}$ is representable by a Γ -torsor $J(\mathcal{G}/X) \rightarrow X$.*

Proof. First one verifies the assertion when \mathcal{G} is constant, for which one uses that if R is a strictly henselian perfect ring then all automorphisms of \mathbb{X}_R are constant, cf. [CS17, Lemma 4.3.15]. In general, the functor is evidently representable by a scheme affine over X ; it is enough to show that it is faithfully flat, as one can then by faithfully flat descent reduce to the case that \mathcal{G} is constant. Now faithful flatness can be checked on strictly henselian local rings, so we can assume that $X = \text{Spec } R$ where R is a strictly henselian perfect ring. In this case, the argument from Theorem 2.2.4 shows that \mathcal{G} is already constant, so the result follows. \square

A variant is the following. Let $\underline{\text{Aut}}(\mathbb{X})$ denote the group scheme of automorphisms of \mathbb{X} compatible with extra structures. This is in general highly non-reduced, with the non-reduced structure related to the failure of \mathbb{X} to be isoclinic (see [CS17, Corollary 4.1.11]). Let X/k be a regular scheme, and assume as above that \mathcal{G} is a p -divisible group over X with the same kind of extra structure such that for all geometric points \bar{x} of X , there is an isomorphism $\mathcal{G} \times k(\bar{x}) \cong \mathbb{X} \times_k k(\bar{x})$ compatible with extra structures.

Proposition 2.2.7. *The functor on all X -schemes \mathcal{T} parametrizing isomorphisms $\mathcal{G} \times_X \mathcal{T} \cong \mathbb{X} \times_k \mathcal{T}$ is representable by an $\underline{\text{Aut}}(\mathbb{X})$ -torsor over X .*

Proof. It is clearly a quasi-torsor, so it is enough to find a section over a faithfully flat cover. Proposition 2.2.6 gives a section over the faithfully flat $J(\mathcal{G} \times_X X_{\text{perf}}/X_{\text{perf}}) \rightarrow X_{\text{perf}}$, and $X_{\text{perf}} \rightarrow X$ is faithfully flat as X is regular. \square

2.3. Igusa varieties. Using the results of the previous subsection, we will now define the Igusa varieties related to the Shimura variety \mathcal{S}_K . We assume that $K = K(N)$ is a principal level with $N \geq 3$, and we fix a prime p that is unramified in F and prime to N . Let k be an algebraically closed field of characteristic p , and let \mathbb{X}/k be a p -divisible group with G -structure.

Definition 2.3.1. *The central leaf⁷ corresponding to \mathbb{X} is the subset of $\mathcal{S}_{K,k} := \mathcal{S}_K \times_{\mathbb{F}_p} k$ where the fibers of the p -divisible group $\mathcal{A}[p^\infty]$ at all geometric points are isomorphic to \mathbb{X} :*

$$\mathcal{C}^{\mathbb{X}} := \{x \in \mathcal{S}_{K,k} \mid \mathcal{A}[p^\infty] \times k(\bar{x}) \cong \mathbb{X} \times_k k(\bar{x})\}.$$

(The isomorphisms are understood to be isomorphisms of p -divisible groups with G -structure.)

⁷In [Oor04], Oort calls these objects central leaves to distinguish them from so-called isogeny leaves. We will only consider central leaves in this paper, so we will simply call these leaves.

By the argument in [Man05, Proposition 1], this is a locally closed subset of $\mathcal{S}_{K,k}$ and becomes a smooth subscheme when endowed with the induced reduced structure.

The results of the previous section imply the following result.

Corollary 2.3.2. *The scheme*

$$\mathfrak{Jg}^{\mathbb{X}} \rightarrow \mathcal{C}^{\mathbb{X}} \subset \mathcal{S}_{K,k}$$

parametrizing isomorphisms $\mathcal{A}[p^\infty] \cong \mathbb{X}$ of p -divisible groups with G -structure is representable by an $\underline{\text{Aut}}(\mathbb{X})$ -torsor over $\mathcal{C}^{\mathbb{X}}$.

Proof. Apply Proposition 2.2.7. \square

Moreover, $\mathfrak{Jg}^{\mathbb{X}}$ can be reinterpreted in terms of a moduli space of abelian varieties with G -structures up to p -power isogeny, and isomorphisms of $\mathcal{A}[p^\infty]$ with \mathbb{X} -action up to p -power isogeny, cf. [CS17, Lemma 4.3.4]. This shows in particular that $\mathfrak{Jg}^{\mathbb{X}}$ is perfect, and that an isogeny $\phi : \mathbb{X} \rightarrow \mathbb{X}'$ (compatible with extra structures) induces an isomorphism $\mathfrak{Jg}^{\mathbb{X}} \cong \mathfrak{Jg}^{\mathbb{X}'}$. In particular, we get pro-finite correspondences

$$\mathcal{C}^{\mathbb{X}} \leftarrow \mathfrak{Jg}^{\mathbb{X}} \cong \mathfrak{Jg}^{\mathbb{X}'} \rightarrow \mathcal{C}^{\mathbb{X}'}$$

between different leaves in the same isogeny class.

Now we wish to obtain a variant of Igusa varieties that works at finite level. For this, we work with completely slope divisible p -divisible groups:

Definition 2.3.3. *Let $\mathcal{T}/\text{Spec } \mathbb{F}_p$ be a scheme and \mathcal{G}/\mathcal{T} a p -divisible group. Let $\text{Frob}_{\mathcal{G}}$ denote the Frobenius morphism relative to \mathcal{T} .*

- (1) *\mathcal{G} is isoclinic and slope divisible of slope $\lambda \in \mathbb{Q}_{\geq 0}$ if one can write $\lambda = \frac{r}{s}$ so that the quasi-isogeny*

$$p^{-r}\text{Frob}_{\mathcal{G}}^s : \mathcal{G} \rightarrow \mathcal{G}^{(p^s)}$$

is an isomorphism.

- (2) *\mathcal{G} is slope divisible with respect to $\lambda \in \mathbb{Q}_{\geq 0}$ if one can write $\lambda = \frac{r}{s}$ so that the quasi-isogeny*

$$p^{-r}\text{Frob}_{\mathcal{G}}^s : \mathcal{G} \rightarrow \mathcal{G}^{(p^s)}$$

is an isogeny.

- (3) *\mathcal{G} is completely slope divisible if it has a filtration by closed immersions of p -divisible groups*

$$0 = \mathcal{G}_{<0} \subset \mathcal{G}_{<1} \subset \cdots \subset \mathcal{G}_{<r} = \mathcal{G}$$

such that, for each i , $\mathcal{G}_{<i}/\mathcal{G}_{<i-1}$ is isoclinic and slope divisible of slope λ_i and $\mathcal{G}_{<i}$ is slope divisible with respect to λ_i , where λ_i is a strictly decreasing sequence of rational numbers.

We note that such slope decompositions are unique if they exist (as when \mathcal{G} is isoclinic and slope divisible of slope λ and \mathcal{G}' is slope divisible with respect to $\lambda' > \lambda$, then there are no maps $\mathcal{G}' \rightarrow \mathcal{G}$) and stable under base change. In particular, the property of being completely slope divisible is fpqc local.

We will use repeatedly the following basic result on completely slope divisible p -divisible groups.

Lemma 2.3.4. *Let $\mathcal{T}/\mathrm{Spec}\mathbb{F}_p$ be a connected regular scheme and \mathcal{G}/\mathcal{T} a p -divisible group. Let η be the generic point of \mathcal{T} , and $\bar{\eta}$ a geometric point above η . If $\mathcal{G}_{\bar{\eta}}$ is completely slope divisible, then so is \mathcal{G} .*

Proof. This is shown in the first part of the proof of [Zin01, Theorem 7]. \square

Choose a p -divisible group with G -structure \mathbb{X} over an algebraically closed field k of characteristic p . Assume that \mathbb{X} is completely slope divisible (this is a condition without extra structures). Note that a p -divisible group over $\bar{\mathbb{F}}_p$ is completely slope divisible if and only if it is a direct sum of its isoclinic pieces, which are defined over a finite field, cf. [OZ02, Corollary 1.5]. One can check that such a choice of \mathbb{X} exists in any isogeny class.

We have

$$\mathbb{X} = \bigoplus_{i=1}^r \mathbb{X}_i,$$

where the \mathbb{X}_i are isoclinic p -divisible groups of strictly decreasing slopes $\lambda_i \in [0, 1]$. The polarization λ on \mathbb{X} induces isomorphisms $\lambda^i : \mathbb{X}_i \rightarrow (\mathbb{X}_j)^\vee$ for all i, j with $\lambda_i + \lambda_j = 1$ which satisfy $(\lambda^i)^\vee = -\lambda^j$. Let $\mathcal{G} = \mathcal{G}_{\mathbb{X}}$ be the p -divisible group of the universal abelian variety $\mathcal{A}/\mathcal{S}_K$ restricted to $\mathcal{C}^{\mathbb{X}}$. Then \mathcal{G} is completely slope divisible, since it is geometrically fiberwise constant. (Note that $\mathcal{C}^{\mathbb{X}}$ is a regular scheme and the fibers of \mathcal{G} over every generic point of $\mathcal{C}^{\mathbb{X}}$ are completely slope divisible, so Lemma 2.3.4 implies the result.) It has a slope filtration

$$0 \subset \mathcal{G}_{\leq 1} \subset \cdots \subset \mathcal{G}_{\leq r} = \mathcal{G}$$

for which the graded pieces $\mathcal{G}_i := \mathcal{G}_{\leq i} / \mathcal{G}_{\leq i-1}$ are isoclinic of slope λ_i . The $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -action on \mathcal{G} respects this filtration, so that each \mathcal{G}_i is endowed with an $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -action. Moreover, the polarization on \mathcal{G} induces isomorphisms $l^i : \mathcal{G}_i \rightarrow (\mathcal{G}_j)^\vee$ for all i, j with $\lambda_i + \lambda_j = 1$, which satisfy $(l^i)^\vee = -l^j$.

Definition 2.3.5. *The (pro-)Igusa variety is the map*

$$\mathrm{Ig}^{\mathbb{X}} \rightarrow \mathcal{C}^{\mathbb{X}}$$

which over a $\mathcal{C}^{\mathbb{X}}$ -scheme \mathcal{T} parametrizes tuples $(\rho_i)_{i=1}^r$ of isomorphisms

$$\rho_i : \mathcal{G}_i \times_{\mathcal{C}^{\mathbb{X}}} \mathcal{T} \xrightarrow{\sim} \mathbb{X}_i \times_k \mathcal{T}$$

which are compatible with the $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -actions on \mathcal{G}_i and \mathbb{X}_i and commute with the polarizations on \mathcal{G} and \mathbb{X} up to an element of $\mathbb{Z}_p^\times(\mathcal{T})$ that is independent of i .

Remark 2.3.6. We can also define $\mathrm{Ig}_m^{\mathbb{X}}$ for any $m \geq 0$ as the moduli space of isomorphisms on $\mathcal{C}^{\mathbb{X}}$ -schemes \mathcal{T}

$$\rho_{i,m} : \mathcal{G}_i[p^m] \times_{\mathcal{C}^{\mathbb{X}}} \mathcal{T} \xrightarrow{\sim} \mathbb{X}_i[p^m] \times_k \mathcal{T}$$

which (fppf locally on \mathcal{T}) lift to arbitrary $m' \geq m$ and which respect the extra structures. Explicitly, this means that the isomorphisms $\rho_{i,m}$ commute with the

⁸This element of $\mathbb{Z}_p^\times(\mathcal{T})$ can be identified with an automorphism of the multiplicative p -divisible group $\mu_{p^\infty, \mathcal{T}}$.

$\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -actions on $\mathcal{G}_i[p^m]$ and $\mathbb{X}_i[p^m]$ and for i, j with $\lambda_i + \lambda_j = 1$ they fit into a diagram of isomorphisms

$$\begin{array}{ccc} \mathcal{G}_i[p^m] \times_{\mathcal{C}^{\mathbb{X}}} \mathcal{T} & \xrightarrow{\rho^{i,m}} & \mathbb{X}_i[p^m] \times_k \mathcal{T} \\ \downarrow l^i & & \downarrow \lambda^i \\ (\mathcal{G}_j)^\vee[p^m] \times_{\mathcal{C}^{\mathbb{X}}} \mathcal{T} & \xleftarrow[\rho_{j,m}^\vee]{} & (\mathbb{X}_j)^\vee[p^m] \times_k \mathcal{T} \end{array} \quad ,$$

that commutes up to an element of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ that is independent of i . By Theorem 2.2.4, $\text{Ig}_m^{\mathbb{X}}$ is a finite étale cover of $\mathcal{C}^{\mathbb{X}}$, Galois with Galois group $\Gamma_{m,\mathbb{X}}$, the group of automorphisms of $\mathbb{X}[p^m]$ that lift to isomorphisms of \mathbb{X} compatible with extra structure.

In particular, passing to the inverse limit, we see that

$$\text{Ig}^{\mathbb{X}} \rightarrow \mathcal{C}^{\mathbb{X}}$$

is a pro-finite étale cover with Galois group $\Gamma_{\mathbb{X}} = \text{Aut}(\mathbb{X})$.

Remark 2.3.7. The scheme $\mathfrak{Jg}^{\mathbb{X}}$ maps naturally to $\text{Ig}^{\mathbb{X}}$ (as is evident from the moduli description), and then even to its perfection. The resulting map $\mathfrak{Jg}^{\mathbb{X}} \rightarrow \text{Ig}_{\text{perf}}^{\mathbb{X}}$ is an isomorphism, as the slope filtration splits uniquely over a perfect base, cf. [CS17, Proposition 4.3.8], or simply because it is a map between $\Gamma_{\mathbb{X}}$ -torsors over $\mathcal{C}_{\text{perf}}^{\mathbb{X}}$.

2.4. Serre–Tate theory. Our goal in this subsection is to prove a version of Serre–Tate theory for semi-abelian schemes that are globally extensions of abelian schemes by tori. We follow Drinfeld’s original proof, cf. [Kat81], but see also [And03]. Note that we do not assume that we are working over a Noetherian base.

Theorem 2.4.1. *Let $S' \twoheadrightarrow S$ be a surjection of rings in which p is nilpotent, with nilpotent kernel $I \subset S'$.*

- (1) *The functor $\mathcal{G}_{S'} \mapsto \mathcal{G}_S := \mathcal{G}_{S'} \times_{S'} S$ from p -divisible groups up to isogeny over S' to p -divisible groups up to isogeny over S is an equivalence of categories.*
- (2) *The functor $A_{S'} \mapsto A_S := A_{S'} \times_{S'} S$ from abelian varieties up to p -power isogeny over S' to abelian varieties up to p -power isogeny over S is an equivalence of categories.*
- (3) *We now consider the category $\mathcal{R}_{S'}$ of semi-abelian schemes $A_{S'}$ which are globally over S' an extension*

$$0 \rightarrow T_{S'} \rightarrow A_{S'} \rightarrow B_{S'} \rightarrow 0$$

of an abelian scheme $B_{S'}$ by a split torus $T_{S'}$ (necessarily of constant rank over S'), with morphisms in $\mathcal{R}_{S'}$ given by $\text{Hom}(A_{S'}, A'_{S'})[1/p]$. Then the functor $\mathcal{R}_{S'} \rightarrow \mathcal{R}_S$ given by $A_{S'} \mapsto A_S := A_{S'} \times_{S'} S$ is an equivalence of categories.

Proof. Consider first the case of p -divisible groups. We prove that the functor $\mathcal{G}_{S'} \mapsto \mathcal{G}_S$ is faithful. Let $\mathcal{G}_{S'}$ be a p -divisible group over S' . Then its formal completion along its identity section $\widehat{\mathcal{G}}_{S'}$ is a formal Lie group. Let T' be an affine S' -scheme and set $T := T' \times_{S'} S$. Then

$$\ker(\mathcal{G}_{S'}(T') \rightarrow \mathcal{G}_S(T)) \xrightarrow{\sim} \ker(\widehat{\mathcal{G}}_{S'}(T') \rightarrow \widehat{\mathcal{G}}_S(T))$$

is killed by p^n for some sufficiently large positive integer n that depends only on the degree of nilpotence of the ideal $(p, I) \subset S'$. This follows from [Mes72, Ch. II, §4].

Now let $\mathcal{G}_{i,S'}$ be p -divisible groups over S' for $i = 1, 2$ and assume $f : \mathcal{G}_{1,S'} \rightarrow \mathcal{G}_{2,S'}$ is a map that reduces to 0 on S . Then for all T' , the image of $f(T')$ lies in $\ker(\mathcal{G}_{2,S'}(T') \rightarrow \mathcal{G}_{2,S}(T))$ which is killed by p^n . It follows that $p^n f = 0$, which implies that $f = 0$ as Hom's between p -divisible groups are torsion-free.

We now prove that the functor for p -divisible groups is full. Let $f \in \text{Hom}(\mathcal{G}_{1,S}, \mathcal{G}_{2,S})$. We want to show that there exists a lifting $\tilde{g} \in \text{Hom}(\mathcal{G}_{1,S'}, \mathcal{G}_{2,S'})$ of $p^n f$, where n is the sufficiently large positive integer chosen above. Let $x \in \mathcal{G}_{1,S'}(T')$ with image $\bar{x} \in \mathcal{G}_{1,S}(T)$. Since $\mathcal{G}_{2,S'}$ is formally smooth over S' , there exists a lift $\tilde{f}(\bar{x}) \in \mathcal{G}_{2,S'}(T')$ of $f(\bar{x})$. Since p^n kills

$$\ker(\mathcal{G}_{2,S'}(T') \rightarrow \mathcal{G}_{2,S}(T)),$$

the element $\tilde{g}(x) := p^n \tilde{f}(\bar{x})$ is well-defined and gives rise to an element

$$\tilde{g} \in \text{Hom}(\mathcal{G}_{1,S'}(T'), \mathcal{G}_{2,S'}(T'))$$

lifting $p^n f$.

Essential surjectivity in the case of p -divisible groups follows from [Ill85].

We now consider the case of abelian schemes. The full faithfulness is proved in the same way as for p -divisible groups. The key point is that $\ker(\widehat{A}_{S'}(T') \rightarrow \widehat{A}_S(T))$ is the same as $\ker(\widehat{A}_{S'}(T') \rightarrow \widehat{A}_S(T))$, where \widehat{A} is the formal completion of A along its identity section and is a formal Lie group. Essential surjectivity for abelian schemes is standard (the obstruction theory of abelian schemes is unobstructed).

We now consider the case of semi-abelian schemes. The full faithfulness is also proved in the same way as for p -divisible groups: note that $A_{S'}$ is still smooth and therefore formally smooth. Moreover, $\ker(A_{S'}(T') \rightarrow A_S(T))$ can still be identified with $\ker(\widehat{A}_{S'}(T') \rightarrow \widehat{A}_S(T))$, where \widehat{A} is the formal completion of A along its identity section and is a formal Lie group. For essential surjectivity, note that we can first lift the base abelian scheme B ; afterwards, lifting the semiabelian scheme is equivalent to lifting a section of the dual abelian scheme, which is possible by (formal) smoothness. \square

Theorem 2.4.2. *Let $S' \twoheadrightarrow S$ be a surjection of rings in which p is nilpotent, with nilpotent kernel $I \subset S'$.*

- (1) *Consider the category of triples $(A_S, \mathcal{G}_{S'}, \rho)$, where A_S is an abelian scheme over S , $\mathcal{G}_{S'}$ is a p -divisible group over S' and*

$$\rho : A_S[p^\infty] \xrightarrow{\sim} \mathcal{G}_{S'} \times_{S'} S$$

is an isomorphism. The functor $A_{S'} \mapsto (A_{S'} \times_{S'} S, A_{S'}[p^\infty], \text{id})$ induces an equivalence of categories between the category of abelian schemes over S' and the category of triples defined above.

- (2) *Consider the category of triples $(A_S, \mathcal{G}_{S'}, \rho)$, where A_S is an extension*

$$1 \rightarrow T_S \rightarrow A_S \rightarrow B_S \rightarrow 1$$

of an abelian scheme B_S by a split torus T_S over S (necessarily of constant rank over S), $\mathcal{G}_{S'}$ is a p -divisible group over S' and

$$\rho : A_S[p^\infty] \xrightarrow{\sim} \mathcal{G}_S := \mathcal{G}_{S'} \times_{S'} S$$

is an isomorphism. We let morphisms between these triples be compatible morphisms of triples. The functor $A_{S'} \mapsto (A_{S'} \times_{S'} S, A_{S'}[p^\infty], \text{id})$ induces an equivalence of categories between the category of extensions

$$1 \rightarrow T_{S'} \rightarrow A_{S'} \rightarrow B_{S'} \rightarrow 1$$

over S' and the category of triples defined above.

Proof. Consider first the case of abelian schemes. We prove that the functor is fully faithful. The faithfulness follows from the fact that we have an injection

$$\text{Hom}(A_{1,S'}, A_{2,S'}) \hookrightarrow \text{Hom}(A_{1,S}, A_{2,S}),$$

as proved in Theorem 2.4.1. To prove that the functor is also full, let $f : A_{1,S} \rightarrow A_{2,S}$ and $f^\infty : \mathcal{G}_{1,S'} \rightarrow \mathcal{G}_{2,S'}$ be compatible with the isomorphisms ρ_1, ρ_2 . We can lift $p^n f$ to a morphism $\tilde{g} : A_{1,S'} \rightarrow A_{2,S'}$, for n as in the proof of Theorem 2.4.1. In order to obtain a lift $\tilde{f} : A_{1,S'} \rightarrow A_{2,S'}$ of f , we need to show that \tilde{g} is divisible by p^n , in other words that $A_{1,S'}[p^n] \subseteq \ker \tilde{g}$. This can be checked on the level of p -divisible groups, where it is automatic that $p^n f^\infty$ is divisible by p^n .

For essential surjectivity, take a triple $(A_S, \mathcal{G}_{S'}, \rho)$ and pick any abelian scheme $A'_{S'}$ over S' with an isogeny $f : A'_{S'} \rightarrow A_S$, which exists by Theorem 2.4.1. We get an isogeny $\rho f^\infty : A'_{S'}[p^\infty] \rightarrow \mathcal{G}_S$. Up to multiplying f by p^n , we may assume that this lifts (necessarily uniquely) to an isogeny $\tilde{\rho} : A'_{S'}[p^\infty] \rightarrow \mathcal{G}_{S'}$. Let $K = \ker \tilde{\rho} \subset A'_{S'}$, which is a finite flat group scheme. Replacing $A'_{S'}$ by $A'_{S'}/K$, we get the result.

Consider now the case of semi-abelian schemes. The fact that the functor is fully faithful is proved in the same way as above. In the proof of essential surjectivity, the only thing to make sure is that the quotient $A'_{S'}/K$ is indeed a semi-abelian scheme of the desired form. But note that this quotient is a flat algebraic space over S' whose base change to S is A_S . This implies that it is a scheme, smooth over S , with all fibers being semi-abelian schemes, i.e. it is a semi-abelian scheme. Moreover, by the rigidity of multiplicative groups as in [SGA70, Exposé IX Théorème 3.6 bis], the torus $T_S \subset A_S$ deforms uniquely to a torus $T_{S'} \subset A_{S'}$, and then necessarily the quotient is an abelian scheme (as it deforms an abelian scheme), showing that $A_{S'}$ is still an extension of an abelian variety by a split torus $T_{S'}$. \square

2.5. Compactifications. In this section, we recall the minimal and toroidal compactifications of the Shimura varieties under consideration.

2.5.1. Degenerations. Before explaining the precise combinatorics of the compactifications, let us quickly describe the C -valued points of \mathcal{S}_K where C is some complete algebraically closed nonarchimedean field with ring of integers \mathcal{O}_C , such that $\Delta_F^{-1} \in \mathcal{O}_C$.

Let us recall first the structure of principally polarized abelian varieties (A, λ) over C . By the semistable degeneration theorem, there is a unique semiabelian scheme \mathcal{A} over \mathcal{O}_C with generic fibre A . The completion $\widehat{\mathcal{A}}$ of \mathcal{A} along its special fibre sits in an exact sequence

$$0 \rightarrow \widehat{T} \rightarrow \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{B}} \rightarrow 0$$

where \widehat{T} is the completion of a torus T over \mathcal{O}_C , and $\widehat{\mathcal{B}}$ is the completion of an abelian variety \mathcal{B} over \mathcal{O}_C . In fact, this short exact sequence algebraizes uniquely to a short exact sequence

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0,$$

where \mathcal{G} is the so-called Raynaud extension. The polarization induces a line bundle \mathcal{L} on $A \times A$; its pullback to the generic fibre of $\widehat{\mathcal{G}} \times \widehat{\mathcal{G}}$ arises from a line bundle on the generic fibre of $\mathcal{B} \times \mathcal{B}$, inducing a principal polarization of the generic fibre of \mathcal{B} , and thus on \mathcal{B} itself by properness. Thus, \mathcal{B} carries a canonical principal polarization $\lambda_{\mathcal{B}} : \mathcal{B} \cong \mathcal{B}^{\vee}$.

Let X be the cocharacter group of T (as \mathcal{O}_C is strictly henselian, this is simply a finite free abelian group). Then the Raynaud extension is given by a map $X \rightarrow \mathcal{B}^{\vee}$. Using the principal polarization of \mathcal{B} , this gives a map $f_0 : X \rightarrow \mathcal{B}$. On the other hand, the map $\widehat{\mathcal{G}} \rightarrow \widehat{\mathcal{A}}$ extends to a map between the rigid-analytic generic fibres of \mathcal{G} and \mathcal{A} ; this is a covering map, and identifies A as a rigid-analytic variety with a quotient \mathcal{G}_{η}/Y for some discrete subgroup $Y \subset \mathcal{G}(C)$. The principal polarization in fact induces an identification $Y \cong X$ under which the composite $Y \rightarrow \mathcal{G}(C) \rightarrow \mathcal{B}(C)$ is identified with the given map $f_0 : X \rightarrow \mathcal{B}(\mathcal{O}_C) = \mathcal{B}(C)$. The lift $f : X \rightarrow \mathcal{G}(C)$ of $f_0 : X \rightarrow \mathcal{B}(C)$ is equivalent to a section of the pullback of the Poincaré bundle over $\mathcal{B} \times \mathcal{B}^{\vee} \cong \mathcal{B} \times \mathcal{B}$ to $X \times X$; this section has to be symmetric (recalling that the Poincaré bundle is symmetric). Moreover, for all $x \in X$, the section at (x, x) must be topologically nilpotent (this condition is independent of a local integral trivialization of the Poincaré bundle).

Conversely, given a symmetric lift $f : X \rightarrow \mathcal{G}(C)$ of the given map $X \rightarrow \mathcal{B}(C)$ satisfying this topological nilpotence condition, one can form the quotient \mathcal{G}_{η}/X to get a principally polarized abelian variety A over C .

If (A, λ) carries in addition an \mathcal{O}_F -action, all objects involved will carry an \mathcal{O}_F -action; note that X will then automatically be a finite projective \mathcal{O}_F -module. Regarding level structures, we note that for any integer N invertible in C , the N -torsion $A[N] = A(C)[N]$ of A carries a canonical filtration

$$0 \subset T(C)[N] \subset \mathcal{G}(C)[N] \subset A(C)[N]$$

with associated graded

$$\begin{aligned} \text{Gr}_{-2} &= T(C)[N], \text{Gr}_{-1} = \mathcal{G}(C)[N]/T(C)[N] = B(C)[N], \\ \text{Gr}_0 &= A(C)[N]/\mathcal{G}(C)[N] = X/NX. \end{aligned}$$

Moreover, $T(C)[N] = G(C)[N]^{\perp}$, and $G(C)[N] = T(C)[N]^{\perp}$, under the Weil pairing on $A(C)[N]$. Thus, any level structure $\eta : A(C)[N] \cong L/N$ will induce a filtration $Z_N = \{Z_{N,-2} \subset Z_{N,-1} \subset L/N\}$ such that $Z_{N,-2} = Z_{N,-1}^{\perp}$ and $Z_{N,-1} = Z_{N,-2}^{\perp}$ (and the filtration lifts to a similar filtration modulo M for all $N|M$). Let $\text{Gr}_i^{Z_N}$ denote the associated graded for $i = -2, -1, 0$. Let us pick an \mathcal{O}_F -linear splitting of this sequence, so

$$(2.5.1) \quad L/NL = Z_{N,-2} \oplus Z_{N,-1}/Z_{N,-2} \oplus (L/N)/Z_{N,-1} = \bigoplus_{i=-2}^0 \text{Gr}_i^{Z_N}.$$

In that case, level structures $\eta : A(C)[N] \cong L/N$ inducing the given filtration Z_N are in bijection with the following data:

- (1) \mathcal{O}_F -linear isomorphisms $\eta_{-2} : T(C)[N] \cong \text{Gr}_{-2}^{Z_N}$, $\eta_{-1} : B(C)[N] \cong \text{Gr}_{-1}^{Z_N}$, $\eta_0 : X/NX \cong \text{Gr}_0^{Z_N}$ such that η_{-2} and η_0 are dual under the canonical pairing, and η_{-1} is compatible with the Weil pairing;
- (2) an \mathcal{O}_F -linear splitting of the short exact sequence

$$0 \rightarrow T(C)[N] \rightarrow \mathcal{G}(C)[N] \rightarrow B(C)[N] \rightarrow 0.$$

This amounts to giving an extension of the given map $f_0 : X \rightarrow B(C)$ to a map $\frac{1}{N}X \rightarrow B(C)$.

- (3) An \mathcal{O}_F -linear extension of the map $f : X \rightarrow \mathcal{G}(C)$ to a map $\frac{1}{N}X \rightarrow \mathcal{G}(C)$ whose composition with $\mathcal{G}(C) \rightarrow B(C)$ gives the extension in (2), which amounts to a splitting $X/NX \rightarrow A(C)[N]$,

subject to the condition that the induced isomorphism

$$(2.5.2) \quad L/N \cong \bigoplus_{i=-2}^0 \mathrm{Gr}_i^{Z_N} \cong T(C)[N] \oplus B(C)[N] \oplus X/NX \cong A(C)[N]$$

is compatible with the Weil pairing, and that this data lifts to similar data modulo M for all $N|M$. In (2.5.2), the first isomorphism is the splitting chosen in (2.5.1), the second is induced by the η_i for $i = -2, -1, 0$, and the third is induced by the splittings in (2) and (3).

2.5.2. *Cusp labels.* In our situation (using critically that the alternating form (\cdot, \cdot) on L is perfect), we can define a cusp label to be a pair (Z, X) where

- (1) The filtration

$$Z = \{Z_{-2} \subset Z_{-1} \subset L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}\}$$

is \mathcal{O}_F -stable and symplectic in the sense that $Z_{-1} = Z_{-2}^\perp$ and $Z_{-2} = Z_{-1}^\perp$ with respect to the perfect alternating form (\cdot, \cdot) on L .

- (2) X is a finite projective \mathcal{O}_F -module with an \mathcal{O}_F -linear isomorphism

$$X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \rightarrow \mathrm{Gr}_0^Z = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} / Z_{-1}.$$

In the following, we write $\mathrm{Gr}_{-2}^Z = Z_{-2}$, $\mathrm{Gr}_{-1}^Z = Z_{-1}/Z_{-2}$ and $\mathrm{Gr}_0^Z = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} / Z_{-1}$. Then Gr_{-2}^Z and Gr_0^Z are in natural perfect duality, and Gr_{-1}^Z admits a natural perfect pairing. Note also that in (2), one could equivalently ask for an isomorphism $X^\vee \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \cong Z_{-2}$, where $X^\vee = \mathrm{Hom}(X, \mathbb{Z})$ is the dual of X .

There is an action of $G(\mathbb{A}_f)$ on cusp labels (Z, X) . Indeed, cusp labels are in bijection with pairs $(Z_{\mathbb{Q}}, X_{\mathbb{Q}})$ defined similarly but on the rational level, and there is an obvious action on the latter.

Definition 2.5.3. *For $K \subset G(\mathbb{A}_f)$ a compact open subgroup, a cusp label at level K is a K -orbit of cusp labels (Z, X) . If $H \subset G(\mathbb{A}_f)$ is any closed subgroup, a cusp label at level H is a compatible collection of cusp labels at level K for all open compact subgroups $K \subset G(\mathbb{A}_f)$ containing H .*

In case $K = K(N)$ as above, we note that the cusp labels at level K can be identified with pairs (Z_N, X) where Z is an \mathcal{O}_F -stable filtration

$$Z_{N,-2} \subset Z_{N,-1} \subset L/N$$

that admits a lift to a filtration Z as above (in particular, $Z_{N,-2} = Z_{N,-1}^\perp$ and $Z_{N,-1} = Z_{N,-2}^\perp$), and X is a finite projective \mathcal{O}_F -module with an \mathcal{O}_F -linear isomorphism $X/N \cong \mathrm{Gr}_0^{Z_N}$.

Given a cusp label $Z = (Z_N, X)$ at principal level $K = K(N)$, we define the group

$$\Gamma_Z = \{g \in \mathrm{GL}_{\mathcal{O}_F}(X) \mid g \equiv 1 \pmod{N}\}.$$

Moreover, we define a smaller Shimura variety \mathcal{S}_Z associated to Z as follows, via its moduli problem. Let r be the \mathcal{O}_F -rank of X . Then $\mathcal{S}_Z^{\mathrm{pre}}/\mathbb{Z}[\frac{1}{\Delta_F}]$ parametrizes over a test scheme S quadruples $(B, \iota, \lambda, \eta)$ where

- (1) B is an abelian scheme of dimension $[F : \mathbb{Q}](n - r)$;
- (2) $\iota : \mathcal{O}_F \rightarrow \text{End}(B)$ is an action making $\text{Lie } B$ free of rank $n - r$ over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$;
- (3) $\lambda : B \cong B^\vee$ is a principal polarization whose Rosati involution is compatible with complex conjugation on \mathcal{O}_F along ι ;
- (4) $\eta : \text{Gr}_{-1}^{Z_N} \rightarrow B[N]$ is a map compatible with ι and such that the diagram

$$\begin{array}{ccc} \text{Gr}_{-1}^{Z_N} \times \text{Gr}_{-1}^{Z_N} & \longrightarrow & \mathbb{Z}/N\mathbb{Z} \\ \downarrow & & \downarrow \zeta_N \\ B[N] \times B[N] & \longrightarrow & \mu_N \end{array}$$

commutes for some primitive N -th root of unity $\zeta_N \in \mathcal{O}_S$; here, the upper map is the natural perfect pairing on $\text{Gr}_{-1}^{Z_N}$ and the lower map is the Weil pairing induced by λ . Moreover, we demand that étale locally there is a lifting to such a level structure modulo $N\Delta_F^m$ for all $m \geq 0$.

Let \mathcal{S}_Z be the normalization of $\mathcal{S}_Z^{\text{pre}}$ in $\mathcal{S}_Z^{\text{pre}} \times \mathbb{Z}[\frac{1}{\Delta_F N}]$. Note that this is again one of the unitary Shimura varieties of our setup, with n replaced by $n - r$. We will make use of this observation for some inductive arguments.

2.5.4. Cone decompositions. Each cusp label $Z = (Z_N, X)$ at level $K(N)$ determines an \mathbb{R} -vector space M_Z of symmetric pairings

$$(\cdot, \cdot) : X \otimes \mathbb{R} \times X \otimes \mathbb{R} \rightarrow \mathbb{R}$$

such that $(av, w) = (v, \bar{a}w)$ for $v, w \in X \otimes \mathbb{R}$, $a \in F$; these are equivalent to Hermitian pairings

$$\langle \cdot, \cdot \rangle : X \otimes \mathbb{R} \times X \otimes \mathbb{R} \rightarrow F \otimes \mathbb{R}.$$

We consider the cones $P_Z^+ \subset P_Z \subset M_Z$, where P_Z^+ is the cone of positive definite Hermitian pairings in M_Z and P_Z is the cone of positive semidefinite Hermitian pairings with F -rational radicals⁹. There is an action of Γ_Z on M_Z and both P_Z^+ and P_Z are stable under this action.

Remark 2.5.5. In our case we can identify M_Z with $[F^+ : \mathbb{Q}]$ copies of the space of Hermitian matrices in $M_r(\mathbb{C})$, where r is the \mathcal{O}_F -rank of X . The subspace P_Z^+ is then obtained by taking $[F^+ : \mathbb{Q}]$ copies of the space of positive definite Hermitian matrices in $M_r(\mathbb{C})$, but P_Z does not decompose into a product.

The cusp label Z also determines the \mathbb{Z} -lattice $\mathbb{S}_Z \subset M_Z^\vee$ that is the image of $\frac{1}{N}(X \otimes X)$ under the natural map

$$\frac{1}{N}(X \otimes X) \rightarrow M_Z^\vee$$

sending $v \otimes w \in X \otimes X$ to the map taking the alternating form $(\cdot, \cdot) \in M_Z$ to (v, w) . Then \mathbb{S}_Z is stable under the action of Γ_Z on M_Z^\vee . A *rational polyhedral cone* $\sigma \subset M_Z$ is a subset of the form

$$\sigma = \mathbb{R}_{>0}v_1 + \dots + \mathbb{R}_{>0}v_r$$

⁹In general, one would impose the condition of admissible radicals, but in our case the order \mathcal{O}_F is maximal and having admissible radicals is equivalent to having rational radicals, cf. [Lan13, Remark 6.2.5.5].

for $v_1, \dots, v_r \in \mathbb{S}_Z^\vee$. (In particular, $\{0\}$ is a rational polyhedral cone because it is obtained from the empty sum.) A rational polyhedral cone $\sigma \subset M_Z$ is *non-degenerate* if the closure $\bar{\sigma}$ of σ does not contain any non-trivial \mathbb{R} -vector subspace of M_Z . A rational polyhedral cone $\sigma \subset M_Z$ is *smooth* if it is of the form

$$\sigma = \mathbb{R}_{>0}v_1 + \dots + \mathbb{R}_{>0}v_r$$

for $v_1, \dots, v_r \in \mathbb{S}_Z^\vee$ which extend to a basis of \mathbb{S}_Z^\vee . A rational polyhedral cone τ is a *face* of a rational polyhedral cone σ if there exists a linear functional $\lambda : M_Z \rightarrow \mathbb{R}$ with $\lambda(\sigma) \subset \mathbb{R}_{\geq 0}$ and $\bar{\tau} = \bar{\sigma} \cap \lambda^{-1}(0)$. (This definition implies that σ is always a face of itself.)

A rational polyhedral cone $\sigma \subset M_Z$ determines semigroups

$$\begin{aligned} \sigma^\vee &= \{l \in \mathbb{S}_Z \mid l(v) \geq 0, \forall v \in \sigma\}, \\ \sigma_0^\vee &= \{l \in \mathbb{S}_Z \mid l(v) > 0, \forall v \in \sigma\}, \\ \sigma^\perp &= \{l \in \mathbb{S}_Z \mid l(v) = 0, \forall v \in \sigma\}. \end{aligned}$$

A Γ_Z -*admissible rational polyhedral cone decomposition* is a set Σ_Z of non-degenerate rational polyhedral cones such that

- (1) The cones in Σ_Z are disjoint and $P_Z = \cup_{\sigma \in \Sigma_Z} \sigma$.
- (2) For each $\sigma \in \Sigma_Z$ and each face τ of σ we have $\tau \in \Sigma_Z$.
- (3) The set Σ_Z is invariant under Γ_Z and the set of orbits Σ_Z/Γ_Z is finite.

Such a Σ_Z has a Γ_Z -stable subset Σ_Z^+ forming a rational polyhedral cone decomposition of P_Z^+ . We call Σ_Z *smooth* if each cone $\sigma \in \Sigma_Z$ is smooth.

Let us write $Z \leq Z'$ if $Z = (Z_N, X), Z' = (Z_{N'}, X')$ are cusp labels at level K that admit lifts $(Z, X), (Z', X')$ such that $Z_{-2} \subset Z'_{-2}$ and $X^\vee \subset (X')^\vee$. In that case, fixing such a lift, the injection $X^\vee \rightarrow (X')^\vee$ induces a surjection $X' \rightarrow X$ and then inclusions $M_{Z'} \subset M_Z$ and $P_{Z'} \subset P_Z$. We say that two admissible rational polyhedral cone decompositions Σ_Z and $\Sigma_{Z'}$ are *compatible* if for each $\sigma \in \Sigma_{Z'}$ we also have $\sigma \in \Sigma_Z$ (via the inclusion $M_{Z'} \subset M_Z$). A *compatible family* of cone decompositions at level K is a collection $\Sigma = \{\Sigma_Z\}$ of Γ_Z -admissible rational polyhedral cone decompositions for each cusp label Z at level K that are pairwise compatible.

Remark 2.5.6. We can and do assume that each Σ is smooth and projective, in the sense of [Lan13, Definition 7.3.1.1], and that for each cusp label Z and $\sigma \in \Sigma_Z^+$, its stabilizer Γ_σ in Γ_Z is trivial. The fact that compatible families of cone decompositions satisfying these properties exist follows from [Lan13, Proposition 7.3.1.4]. Moreover, we note that such a cone decomposition at some principal level $K(N)$ induces a cone decomposition with the same properties at principal level $K(M)$ for any $N|M$. For the following, we fix some smooth projective cone decomposition Σ (satisfying the assumption that all stabilizers Γ_σ are trivial) at some auxiliary principal level $K(N)$ in the beginning, and pull it back to any other principal level considered.

2.5.7. Compactifications. Under the above assumptions, in particular $K = K(N)$, and away from the primes dividing N , the Shimura variety \mathcal{S}_K has good toroidal and minimal compactifications $\mathcal{S}_{K,\Sigma}^{\text{tor}}$ and \mathcal{S}_K^* , whose properties we summarize below. Actually, we will define a naive extension of the toroidal compactification to unramified primes dividing N .

Theorem 2.5.8. *There exists a flat, projective, normal scheme $\mathcal{S}_K^*/\mathrm{Spec}\mathbb{Z}[\frac{1}{N\Delta_F}]$ together with a dense open embedding*

$$j : \mathcal{S}_K[\frac{1}{N}] \hookrightarrow \mathcal{S}_K^*,$$

satisfying the following additional properties:

- (1) *For each cusp label Z at level K , there is a canonical locally closed immersion $\mathcal{S}_Z[\frac{1}{N}] \hookrightarrow \mathcal{S}_K^*$, where \mathcal{S}_Z is the Shimura variety defined above.*
- (2) *The incidence relations among strata are determined by the partial order relation on cusp labels: $\mathcal{S}_Z[\frac{1}{N}]$ lies in the closure of $\mathcal{S}_{Z'}[\frac{1}{N}]$ if and only if $Z \leq Z'$. Moreover, this property is preserved under pullback to the special fiber.*
- (3) *If $K_1 = K(N_1), K_2 = K(N_2) \subset G(\widehat{\mathbb{Z}})$ with $N_1, N_2 \geq 3$ and $g \in G(\mathbb{A}_f)$ satisfies $gK_1g^{-1} \subset K_2$ then there is a finite surjective morphism*

$$[g] : \mathcal{S}_{K_1}^*[\frac{1}{N_1N_2}] \rightarrow \mathcal{S}_{K_2}^*[\frac{1}{N_1N_2}]$$

extending the usual morphism $[g] : \mathcal{S}_{K_1}[\frac{1}{N_1N_2}] \rightarrow \mathcal{S}_{K_2}[\frac{1}{N_1N_2}]$.

Proof. This summarizes the main results in [Lan13, §7]. There is one subtlety, namely the identification of the boundary stratum \mathcal{S}_Z . See [Lan13, Definition 5.4.2.6] and [LS18a, Remark 4.3.3] for an explanation of why the description of the boundary stratum simplifies in our case, when the level $K = K(N)$ is principal, when $p \nmid N\Delta_F$, and when [Lan13, Condition 1.4.3.10] is satisfied. \square

In the following statement, we assume that for all primes p unramified in F , the part of N prime to p is still ≥ 3 . (Otherwise, we would have to talk about Deligne–Mumford stacks.)

Theorem 2.5.9. *Given a choice Σ of a compatible family of cone decompositions as in §2.5.4 (and Remark 2.5.6), there exists a projective scheme $\mathcal{S}_{K,\Sigma}^{\mathrm{tor}}/\mathrm{Spec}\mathbb{Z}[\frac{1}{\Delta_F}]$, together with an open dense embedding $j_\Sigma^{\mathrm{tor}} : \mathcal{S}_K \hookrightarrow \mathcal{S}_{K,\Sigma}^{\mathrm{tor}}$, satisfying the following additional properties:*

- (1) *After inverting N , the scheme $\mathcal{S}_{K,\Sigma}^{\mathrm{tor}}[\frac{1}{N}]$ is smooth.*
- (2) *There is a set-theoretic decomposition*

$$\mathcal{S}_{K,\Sigma}^{\mathrm{tor}} = \bigsqcup_Z \mathcal{S}_{K,\Sigma,Z}^{\mathrm{tor}},$$

where the disjoint union runs over the set of cusp labels at level K where each $\mathcal{S}_{K,\Sigma,Z}^{\mathrm{tor}}$ is reduced, flat over \mathbb{Z} , and locally closed.

- (3) *Let $\widehat{\mathcal{S}}_{K,\Sigma,Z}^{\mathrm{tor}}$ denote the formal completion of $\mathcal{S}_{K,\Sigma}^{\mathrm{tor}}$ along $\mathcal{S}_{K,\Sigma,Z}^{\mathrm{tor}}$. Moreover, pick a splitting*

$$L/N \cong \mathrm{Gr}_{-2}^{Z_N} \oplus \mathrm{Gr}_{-1}^{Z_N} \oplus \mathrm{Gr}_0^{Z_N}$$

of the filtration Z_N . Then we have the following canonical description of the formal completion $\widehat{\mathcal{S}}_{K,\Sigma,Z}^{\mathrm{tor}}$. There exists an abelian scheme

$$C_Z \rightarrow \mathcal{S}_Z$$

together with a compatible action of Γ_Z . There exists a torsor under the split torus with character group \mathbb{S}_Z ,

$$\Xi_Z \rightarrow C_Z,$$

with a compatible action of Γ_Z . We have

$$\Xi_Z = \underline{\text{Spec}}_{\mathcal{O}_{C_Z}} \bigoplus_{l \in \mathbb{S}_Z} \Psi_Z(l),$$

where for each $l \in \mathbb{S}_Z$, $\Psi_Z(l)$ is a line bundle on C_Z and for each $l, l' \in \mathbb{S}_Z$ there is an isomorphism $\Psi_Z(l) \otimes \Psi_Z(l') \cong \Psi_Z(l + l')$, giving $\bigoplus_{l \in \mathbb{S}_Z} \Psi_Z(l)$ the structure of a sheaf of \mathcal{O}_{C_Z} -algebras. The choice of Σ_Z determines a Γ_Z -equivariant relative torus embedding

$$\partial_{Z, \Sigma_Z} : \Xi_Z \hookrightarrow \Xi_{Z, \Sigma_Z}.$$

We let $\mathfrak{X}_{Z, \Sigma_Z}$ denote the formal completion of Ξ_{Z, Σ_Z} along the boundary ∂_{Z, Σ_Z} of the relative torus embedding. We may form the quotient $\mathfrak{X}_{Z, \Sigma_Z} / \Gamma_Z$ as a formal scheme and we have a canonical isomorphism of formal schemes

$$\widehat{\mathcal{F}}_{K, \Sigma, Z}^{\text{tor}} \simeq \mathfrak{X}_{Z, \Sigma_Z} / \Gamma_Z.$$

Moreover, there exists a semi-abelian scheme \mathcal{A} over $\mathcal{S}_K^{\text{tor}}$ endowed with an $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -action and a Raynaud extension \mathcal{G}_Z over C_Z also endowed with an $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -action such that we have a canonical $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -equivariant isomorphism of the formal completion of \mathcal{A} with \mathcal{G}_Z over $\widehat{\mathcal{F}}_{K, \Sigma, Z}^{\text{tor}} \simeq \mathfrak{X}_{Z, \Sigma_Z} / \Gamma_Z$.

(4) There is a stratification

$$\mathcal{S}_{K, \Sigma}^{\text{tor}} = \bigsqcup_{(Z, [\sigma])} \mathcal{S}_{K, \Sigma, (Z, [\sigma])}^{\text{tor}},$$

where Z runs over cusp labels at level K and $[\sigma]$ runs over Γ_Z -orbits in Σ_Z^+ . If $(Z, [\sigma])$ and $(Z', [\sigma'])$ are two such pairs, then $\mathcal{S}_{K, \Sigma, (Z, [\sigma])}^{\text{tor}}$ lies in the closure of $\mathcal{S}_{K, \Sigma, (Z', [\sigma'])}^{\text{tor}}$ if and only if $Z \leq Z'$ and there are representatives σ, σ' such that via the inclusion $M_Z \subseteq M_{Z'}$, σ' is a face of σ . These incidence relations among strata are preserved under pullback to fibers.

(5) Fix a representative σ of an orbit $[\sigma] \in \Sigma_Z^+ / \Gamma_Z$. We have the relatively affine toroidal embedding

$$\Xi_Z \hookrightarrow \Xi(\sigma) := \underline{\text{Spec}}_{\mathcal{O}_{C_Z}} \bigoplus_{l \in \sigma^\vee} \Psi_Z(l).$$

The scheme $\Xi(\sigma)$ has a closed subscheme Ξ_σ defined by the ideal sheaf $\underline{\text{Spec}}_{\mathcal{O}_{C_Z}} \bigoplus_{l \in \sigma_0^\vee} \Psi_Z(l)$ (so naturally isomorphic to $\underline{\text{Spec}}_{\mathcal{O}_{C_Z}} \bigoplus_{l \in \sigma^\perp} \Psi_Z(l)$).

Let \mathfrak{X}_σ denote the formal completion of $\Xi(\sigma)$ along Ξ_σ and $\widehat{\mathcal{F}}_{K, \Sigma, (Z, [\sigma])}^{\text{tor}}$ denote the formal completion of $\mathcal{S}_{K, \Sigma}^{\text{tor}}$ along $\mathcal{S}_{K, \Sigma, (Z, [\sigma])}^{\text{tor}}$. Then there is a canonical isomorphism

$$\widehat{\mathcal{F}}_{K, \Sigma, (Z, [\sigma])}^{\text{tor}} \simeq \mathfrak{X}_\sigma.$$

(6) After inverting N , there is a projective morphism

$$\pi_{K, \Sigma} : \mathcal{S}_{K, \Sigma}^{\text{tor}} \left[\frac{1}{N} \right] \rightarrow \mathcal{S}_K^*$$

which is the identity on the open subscheme $\mathcal{S}_K \left[\frac{1}{N} \right]$. For each cusp label Z we have

$$\pi_{K, \Sigma}^{-1}(\mathcal{S}_Z) = \mathcal{S}_{K, \Sigma, Z}^{\text{tor}} \left[\frac{1}{N} \right]$$

set-theoretically and $\pi_{K, \Sigma}^* \mathcal{O}_{\mathcal{S}_K^*} \left[\frac{1}{N} \right] = \mathcal{O}_{\mathcal{S}_K^*}$. This final equality holds on any fiber.

Proof. Away from the primes dividing N , this summarizes results in [Lan13, §6,§7]. See [LS18a, Remark 2.1.8] for an explanation why $C_Z \rightarrow \mathcal{S}_Z$ is an abelian scheme in our particular case rather than an abelian scheme torsor over a finite étale cover of \mathcal{S}_Z .

More precisely, let us give a construction of the Raynaud extension in part (2). Let r be the \mathcal{O}_F -rank of X . Then \mathcal{S}_Z parametrizes on S -valued points quadruples $(B, \iota, \lambda, \eta)$ where B is an abelian scheme of dimension $[F : \mathbb{Q}](n - r)$, $\iota : \mathcal{O}_F \rightarrow \text{End}(B)$ is an \mathcal{O}_F -action making $\text{Lie } B$ free of rank $n - r$ over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$, $\lambda : B \cong B^\vee$ is a principal polarization whose Rosati involution is compatible with complex conjugation on \mathcal{O}_F via ι , and $\eta : B[N] \cong \text{Gr}_{-1}^{Z_N}$ is an isomorphism that étale locally lifts to an isomorphism modulo M for all $N|M$.

Let T be the torus with character group X . Over \mathcal{S}_Z , the abelian scheme $C_Z \rightarrow \mathcal{S}_Z$ parametrizes the following data:

- (1) An \mathcal{O}_F -linear extension

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0 .$$

- (2) An \mathcal{O}_F -linear splitting of the short exact sequence

$$0 \rightarrow T[N] \rightarrow \mathcal{G}[N] \rightarrow \mathcal{B}[N] \rightarrow 0 .$$

We note that the first piece of data is equivalent to an \mathcal{O}_F -linear map $X \rightarrow \mathcal{B}^\vee$, and then the second piece of data is equivalent to lifting this to a map $\frac{1}{N}X \rightarrow \mathcal{B}^\vee$. As $\frac{1}{N}X$ is a finite projective \mathcal{O}_F -module, one sees that

$$C_Z = \underline{\text{Hom}}_{\mathcal{O}_F}(\frac{1}{N}X, \mathcal{B}^\vee)$$

defines an abelian scheme over \mathcal{S}_Z .

Identifying \mathcal{B}^\vee with \mathcal{B} via λ , we get a map $f_0 : \frac{1}{N}X \rightarrow \mathcal{B}$ over C_Z . The \mathbb{S}_Z -torsor $\Xi_Z \rightarrow \mathcal{S}_Z$ parametrizes “polarizable” lifts $f : \frac{1}{N}X \rightarrow \mathcal{G}$ of f_0 . Via the theory of degenerations of abelian varieties, one can after formal completion along the boundary of a corresponding torus compactification construct the quotient

$$\mathcal{A} = \mathcal{G}/X$$

over $\mathfrak{X}_{Z, \Sigma_Z}$, as a semiabelian scheme equipped with \mathcal{O}_F -action and a principal polarization away from the boundary. From the construction, $\mathcal{A}[N]$ admits, away from the boundary, a filtration with graded pieces $T[N]$, $\mathcal{B}[N]$ and X/NX . In fact, the construction makes this filtration split. The similar decomposition

$$L/N \cong \text{Gr}_{-2}^{Z_N} \oplus \text{Gr}_{-1}^{Z_N} \oplus \text{Gr}_0^{Z_N}$$

induces a level- N -structure on the generic fibre of \mathcal{A} .

At primes dividing N , these models have been constructed by Lan, [Lan16]. Let $N = p^m N'$ and assume that still $N' \geq 3$. Then one can construct the desired toroidal compactification relatively over the toroidal compactification at level N' . On boundary charts, the extra data parametrized is an \mathcal{O}_F -linear splitting of the short exact sequence

$$0 \rightarrow T[p^m] \rightarrow \mathcal{G}[p^m] \rightarrow \mathcal{B}[p^m] \rightarrow 0$$

(which is incorporated into the abelian scheme $C_Z \rightarrow \mathcal{S}_Z$) and a polarizable lift of $f_0 : \frac{1}{N}X \rightarrow \mathcal{B}$ to $f : \frac{1}{N}X \rightarrow \mathcal{G}$. Both of these structures extend the corresponding

structures present away from characteristic p . Away from the toroidal boundary, the given data induce an \mathcal{O}_F -linear map

$$L/p^m \cong \mathrm{Gr}_{-2}^{Z_{p^m}} \oplus \mathrm{Gr}_{-1}^{Z_{p^m}} \oplus \mathrm{Gr}_0^{Z_{p^m}} \rightarrow T[p^m] \oplus \mathcal{B}[p^m] \oplus X/p^m X \rightarrow \mathcal{A}[p^m]$$

giving the desired level- p^m -structure. \square

In the following, we will fix a choice of Σ as in Remark 2.5.6 and leave its choice implicit; in particular, we will simply write $\mathcal{S}_K^{\mathrm{tor}} := \mathcal{S}_{K,\Sigma}^{\mathrm{tor}}$ for principal level $K = K(N)$.

2.6. Perfectoid Shimura varieties. In this section, we recall what is known about perfectoid Shimura varieties in our setup. Fix a prime p that can actually be arbitrary for this section (i.e., even be ramified in F). We can take the adic spaces over \mathbb{Q}_p associated with $\mathcal{S}_{K(N)}^*$ and $\mathcal{S}_{K(N)}^{\mathrm{tor}}$; let us denote these simply by a subscript \mathbb{Q}_p . We would like to take the inverse limit over levels $K(p^m N)$ as $m \rightarrow \infty$. Unfortunately, inverse limits do not exist in the category of adic spaces, but they do in the category of diamonds. As we are mainly interested in étale cohomology, which is entirely functorial in diamonds, [Sch17], this is good enough for our purposes.

As above, we fix a cone decomposition Σ as in Remark 2.5.6.

Definition 2.6.1. *Let*

$$\begin{aligned} \mathcal{S}_{K(p^\infty N)}^* &= \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{*, \diamond} \\ \mathcal{S}_{K(p^\infty N)}^{\mathrm{tor}} &= \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{\mathrm{tor}, \diamond} \end{aligned}$$

in the category of diamonds over $\mathrm{Spd}\mathbb{Q}_p$.

As both are inverse limits of diamonds associated to qcqs analytic adic spaces, they are spatial diamonds. We have the following theorem. We will not need the part on toroidal compactifications in this paper, but we record it for reassurance.

Theorem 2.6.2. *The diamonds $\mathcal{S}_{K(p^\infty N)}^*$ and, for a cofinal choice of cone decompositions Σ , $\mathcal{S}_{K(p^\infty N)}^{\mathrm{tor}}$ are representable by perfectoid spaces.*

Proof. In [Sch15, Theorem 4.1.1], certain spaces $\mathcal{S}_{K(p^m N)}^*$ finite under $\mathcal{S}_{K(p^m N)}^*$ were defined such that

$$\mathcal{S}_{K(p^\infty N)}^* = \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{*, \diamond}$$

is representable by a perfectoid space. Now [BS19, Theorem 1.16 (1)] ensures that $\mathcal{S}_{K(p^\infty N)}^*$ is itself representable by a perfectoid space.

In the case of $\mathcal{S}_{K(p^\infty N)}^{\mathrm{tor}}$, the result follows from [PS16, Théorème 0.4], at least for cone decompositions that are compatible with cone decompositions on the Siegel moduli space, as the induced map of toroidal compactifications is a closed immersion at high enough level (here, we are using that Zariski closed subsets of perfectoid spaces are themselves perfectoid by [Sch15, Section 2.2]). \square

We note that for any locally spatial diamond D over $\mathrm{Spd}\mathbb{Q}_p$, there is a well-defined étale site $D_{\mathrm{ét}}$ and a sheaf \mathcal{O}_D^+/p on $D_{\mathrm{ét}}$ whose pullback to any quasi-pro-étale perfectoid space $X \rightarrow D$ agrees with \mathcal{O}_X^+/p . Indeed, it is clear that one can define a quasi-pro-étale sheaf this way, but this comes via pullback from the étale site when

X is a perfectoid space; thus, [Sch17, Theorem 14.12 (ii)] applies. If $D = X^\diamond$ for an analytic adic space X over \mathbb{Q}_p , then $D_{\text{ét}} \cong X_{\text{ét}}$ and $\mathcal{O}_D^+/p \cong \mathcal{O}_X^+/p$ naturally (as this can be checked after pullback to a pro-étale perfectoid cover). For a cofiltered inverse limit of spatial diamonds D_i over $\text{Spd } \mathbb{Q}_p$ along qcqs transition maps, the inverse limit $D = \varprojlim_i D_i$ is a locally spatial diamond by [Sch17, Lemma 11.22], and denoting by $\pi_i : D \rightarrow D_i$ the projections, we have

$$\mathcal{O}_D^+/p = \varinjlim_i \pi_i^* \mathcal{O}_{D_i}^+/p.$$

This can be proved by identifying the stalks at geometric points of D ; these factor over geometric points of D_i , and then one is reduced to the case that $D_i = \text{Spa}(C_i, C_i^+)$ are all adic spectra of complete algebraically closed field C_i over \mathbb{Q}_p with open and bounded valuation subrings $C_i^+ \subset C_i$. Then $D = \text{Spa}(C, C^+)$ where C^+ is the p -adic completion of $\varinjlim_i C_i^+$ is of the same form, and indeed $C^+/p = \varinjlim_i C_i^+/p$.

We see in particular that we have such an identification of étale \mathcal{O}^+/p -sheaves in the situation of Definition 2.6.1.

Remark 2.6.3. In [Sch15, Theorem 4.1.1], a stronger result is proved, namely that the map

$$\varinjlim_m \mathcal{O}_{\mathcal{S}_{K(p^m N), \mathbb{Q}_p}^*} \rightarrow \mathcal{O}_{\mathcal{S}_{K(p^\infty N)}^*}$$

has dense image when evaluated on some (explicit) affinoid cover of the limit. A similar result is known for the toroidal compactification by [PS16, Théorème 0.4], but for the minimal compactification it is still open (it was claimed by the second author in the original version of [SW19, Lecture X], but the argument was incorrect).

Inside $\mathcal{S}_{K(N), \mathbb{Q}_p}^*$, we have the locus $\mathcal{S}_{K(N), \mathbb{Q}_p}^\circ \subset \mathcal{S}_{K(N), \mathbb{Q}_p}$, contained in the adic space associated to the scheme $\mathcal{S}_{K(N)} \times \text{Spec } \mathbb{Q}_p$, where the universal abelian variety has good reduction. This is a Hecke-equivariant quasicompact open subspace. If p is a prime of good reduction, this can also be defined as the adic generic fibre of the p -adic completion $\mathcal{S}_{K(N), \mathbb{Z}_p}$ of $\mathcal{S}_{K(N)}$. We also let

$$\mathcal{S}_{K(p^\infty N)}^\circ := \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{\circ, \diamond}$$

and note that it is itself a perfectoid space; this case follows directly from [Sch15, Theorem 4.1.1] (and so we even have the stronger assertion from Remark 2.6.3).

We note that the cohomology of the good reduction locus captures the whole cohomology of the Shimura variety by results of Lan-Stroh, [LS18b]:¹⁰

Proposition 2.6.4. *Let C be a complete algebraically closed extension of \mathbb{Q}_p . For any $N \geq 3$ (not necessarily prime to p), the natural Hecke-equivariant map*

$$H^i(\mathcal{S}_{K(N), \overline{\mathbb{Q}}}, \mathbb{F}_\ell) \rightarrow H^i(\mathcal{S}_{K(N), C}^\circ, \mathbb{F}_\ell)$$

is an isomorphism.

¹⁰We will actually only use the result when N is prime to p , in which case it is a standard consequence of the existence of a compactification with a relative normal crossing boundary divisor, here given by the toroidal compactification.

Proof. We apply the results of Lan-Stroh, [LS18b], in the case (Nm) of normal integral models defined by normalization over the Siegel moduli problem; let $\mathfrak{S}_{K(N), \mathbb{Z}_p}$ be such a model. Then [LS18b, Corollary 5.20] shows that

$$H^i(\mathcal{S}_{K(N), \overline{\mathbb{Q}}}, \mathbb{F}_\ell) \cong H^i(\mathfrak{S}_{K(N), \overline{\mathbb{F}}_p}, R\psi\mathbb{F}_\ell)$$

using the nearby cycles $R\psi\mathbb{F}_\ell$. (This would be clear for proper varieties, and Lan-Stroh prove that even in case of bad reduction one can understand the boundary well enough to justify the assertion.) On the other hand the natural map

$$H^i(\mathcal{S}_{K(N), C}^\circ, \mathbb{F}_\ell) \rightarrow H^i(\mathfrak{S}_{K(N), k}, R\psi\mathbb{F}_\ell)$$

is always an isomorphism, where k is the residue field, by the comparison of nearby cycles in the algebraic and adic setting, [Hub96, Theorem 3.5.13]. We conclude by invariance of nearby cycles and cohomology under the extension of algebraically closed fields $k/\overline{\mathbb{F}}_p$. \square

2.7. The Hodge–Tate period morphism. Let $\mathcal{F}\ell$ be the adic space over $\mathrm{Spa} \mathbb{Q}_p$ associated to the flag variety Fl parametrizing totally isotropic F -linear subspaces of V .

Theorem 2.7.1. *There exists a $G(\mathbb{A}_f)$ -equivariant Hodge–Tate period morphism¹¹ fitting in the commutative diagram*

$$\begin{array}{ccc} j : \mathcal{S}_{K(p^\infty N)}^\circ & \hookrightarrow & \mathcal{S}_{K(p^\infty N)}^* \\ & \searrow \pi_{\mathrm{HT}}^\circ & \downarrow \pi_{\mathrm{HT}}^* \\ & & \mathcal{F}\ell. \end{array}$$

In particular, by projection to the minimal compactification we also get a Hodge–Tate period morphism

$$\pi_{\mathrm{HT}}^{\mathrm{tor}} : \mathcal{S}_{K(p^\infty N)}^{\mathrm{tor}} \rightarrow \mathcal{F}\ell.$$

Proof. This relies on [Sch15, Theorem 4.1.1] and [CS17, Theorem 2.1.3], though the precise statement regarding π_{HT}^* requires the extra argument in [CGH⁺18, Theorem 3.3.1]. \square

In [CS17], we also described the fibers of the Hodge–Tate period morphism π_{HT}° on the good reduction locus. More precisely, assume from now on again that p is unramified in F and let C be a complete algebraically closed nonarchimedean field over \mathbb{Q}_p with ring of integers \mathcal{O}_C and residue field k , and with a section $k \rightarrow \mathcal{O}_C/p$. Take a point $x \in \mathcal{F}\ell(C)$. By [SW13, Theorem B], the point x corresponds to a p -divisible group $\mathbb{X}_{\mathcal{O}_C}$ with G -structure over \mathcal{O}_C together with an isomorphism $T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ compatible with G -structures. Associated to the special fibre \mathbb{X}_k , we get the perfect Igusa variety

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X}_k}$$

defined above, where the level prime to p is given by the part of N prime to p . As it is perfect a scheme, it admits a canonical lift to $W(k)$, and thus to \mathcal{O}_C ; let us denote by subscript C the adic generic fibre, which is a perfectoid space over C .

¹¹The morphisms are equivariant for the natural action of $G(\mathbb{Q}_p)$ on $\mathcal{F}\ell_{G, \mu}$ and for the trivial action of $G(\mathbb{A}_f^p)$ on $\mathcal{F}\ell_{G, \mu}$.

Theorem 2.7.2. *There is a canonical open immersion*

$$\mathfrak{I}\mathfrak{g}_C^{\mathbb{X}_k} \hookrightarrow (\pi_{\text{HT}}^\circ)^{-1}(x)$$

whose image contains all points of rank 1. In particular, for a prime $\ell \neq p$, there is a canonical Hecke-equivariant isomorphism

$$(R(\pi_{\text{HT}}^\circ)_* \mathbb{F}_\ell)_x \cong R\Gamma(\mathfrak{I}\mathfrak{g}^{\mathbb{X}_k}, \mathbb{F}_\ell).$$

Proof. This is [CS17, Theorem 4.4.4]. \square

We also recall the Newton stratification of $\mathcal{F}\ell$. Let $B(G_{\mathbb{Q}_p})$ be the Kottwitz set attached to the group $G_{\mathbb{Q}_p}$; see [Kot85] and [RR96] for more details on the Kottwitz set. This is equipped with a partial order called the Bruhat order, which we denote by \leq . Let μ^{-1} be a dominant representative of the inverse of μ . Let $B(G_{\mathbb{Q}_p}, \mu^{-1})$ denote the set of μ^{-1} -admissible elements in $B(G_{\mathbb{Q}_p})$, cf. [CS17, Definition 3.1.2].

For any $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$, let d_b be the dimension of the Igusa variety Ig^b ; this is known to be given explicitly by $\langle 2\rho, \nu_b \rangle$, cf. e.g. [Ham15]. In particular, $d_{b'} \geq d_b$ whenever $b' \geq b$.

Recall from above that for a complete algebraically closed nonarchimedean extension C/\mathbb{Q}_p and a point $x \in \mathcal{F}\ell(C)$, we have a p -divisible group $\mathbb{X}_{\mathcal{O}_C}$ with G -structure over \mathcal{O}_C . In particular, the special fiber \mathbb{X}_k defines a p -divisible group with G -structure, and this is classified up to isogeny by an element $b = b(x) \in B(G_{\mathbb{Q}_p}, \mu^{-1})$. The following theorem asserts that this defines a reasonable stratification of $\mathcal{F}\ell$:

Theorem 2.7.3 (§3 of [CS17]). *There exists a stratification*

$$\mathcal{F}\ell = \bigsqcup_{b \in B(G_{\mathbb{Q}_p}, \mu^{-1})} \mathcal{F}\ell^b$$

with locally closed partially proper strata such that $x \in \mathcal{F}\ell(C)$ lies in $\mathcal{F}\ell^b(C)$ if and only if $b(x) = b$. The dimension of $\mathcal{F}\ell^b$ (i.e., the Krull dimension of the locally spectral space $|\mathcal{F}\ell^b|$) is given by $d - d_b$.

Moreover, the strata

$$\mathcal{F}\ell^{\geq b} = \bigsqcup_{b' \geq b} \mathcal{F}\ell^{b'}$$

are closed.

Since the reflex field of the Shimura datum is \mathbb{Q} , the largest element of $B(G_{\mathbb{Q}_p}, \mu^{-1})$ is the ordinary one, cf. [Wed99, Theorem 1.6.3]. We have $\mathcal{F}\ell^{\text{ord}} = \mathcal{F}\ell(\mathbb{Q}_p)$ (see, for example, [CGH⁺18, Proposition 3.3.8]) and in particular this stratum is 0-dimensional.

2.8. The main argument. Let us now give the proof of Theorem 1.1. It uses the following inputs. Assume again that p is unramified in F and fix some level $N \geq 3$ prime to p and cone decomposition Σ as in Remark 2.5.6. For any $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$, we have the associated Igusa variety $\text{Ig}^b = \text{Ig}_{K^p(N)}^{\mathbb{X}}$ associated to some choice of completely slope divisible \mathbb{X} in the isogeny class given by b . In Section 3, we define a partial minimal compactification $j : \text{Ig}^b \hookrightarrow \text{Ig}^{b,*}$, and set

$$H_{c-\partial}^i(\text{Ig}^b, \mathbb{F}_\ell) = H^i(\text{Ig}^{b,*}, j_! \mathbb{F}_\ell).$$

The first result we need is that $\text{Ig}^{b,*}$ is affine:

Theorem 2.8.1. *The partial minimal compactification $\text{Ig}^{b,*}$ is affine.*

Proof. This is Lemma 3.3.8. \square

In particular, we get the following result about cohomology:

Proposition 2.8.2. *For any $\ell \neq p$, the cohomology group*

$$H_{c-\partial}^i(\mathrm{Ig}^b, \mathbb{F}_\ell)$$

is nonzero only for $i \leq d_b = \dim \mathrm{Ig}^b$.

Proof. This is a direct consequence of Artin vanishing and the affineness of $\mathrm{Ig}^{b,*}$. \square

The next result we need is the following semiperversity result.

Theorem 2.8.3. *Let C be a complete algebraically closed extension of \mathbb{Q}_p with ring of integers \mathcal{O}_C and residue field k . Consider the Hodge–Tate period map*

$$\pi_{\mathrm{HT}}^\circ : \mathcal{S}_{K(p^\infty N), C}^\circ \rightarrow \mathcal{F}l_C.$$

There is a cofinal system of formal models $\mathfrak{F}l$ of $\mathcal{F}l_C$ over \mathcal{O}_C such that

$$R\psi(R(\pi_{\mathrm{HT}}^\circ)_*\mathbb{F}_\ell) \in {}^pD^{\geq d}(\mathfrak{F}l_k, \mathbb{F}_\ell).$$

Proof. This is Theorem 4.6.1. It proceeds by first proving an analogous result for toroidal compactifications, for which one uses an identification of the fibers of the Hodge–Tate period map with partial toroidal compactifications of Igusa varieties. \square

Now fix a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ containing ℓ , where the finite set S of places of \mathbb{Q} contains ∞ and all primes dividing $p\ell N\Delta_F$. Pick a b with d_b minimal such that $H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m}$ is nonzero; recall that d_b is the dimension of Ig^b . (If several b achieve the same value of d_b , pick any of them.) The previous theorem implies the following result that proves a bound going in the other direction than Proposition 2.8.2:

Lemma 2.8.4. *If $H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m} \neq 0$ then $i \geq d_b$.*

Remark 2.8.5. One could prove a variant of Theorem 2.8.3 for $R\pi_{\mathrm{HT}*}^*(j_!\mathbb{F}_\ell)$, where j denotes the Zariski open immersion of the open Shimura variety into its minimal compactification, where one would get an object in ${}^pD^{\leq d}$ at the end. However, the result obtained this way would only amount to Proposition 2.8.2 (definitely for the largest b as in this lemma; for deeper strata there is some nontrivial translation to do between our notion of (semi)perversity in terms of nearby cycles and a more naive version one could hope to define directly on the flag variety).

Proof. Consider the complex of étale sheaves $A = (R(\pi_{\mathrm{HT}}^\circ)_*\mathbb{F}_\ell)_\mathfrak{m}$ on $\mathcal{F}l_C$. By assumption and Theorem 2.7.2 (that also applies to higher rank points; in fact qcqs pushforwards of overconvergent sheaves are overconvergent), we know that it is concentrated on $\bigsqcup_{b', d_{b'} \geq d_b} \mathcal{F}l^{b'}$, which is of dimension $\leq d - d_b$. In particular, for any formal model $\mathfrak{F}l$ of $\mathcal{F}l_C$, the nearby cycles $R\psi A \in D(\mathfrak{F}l_k, \mathbb{F}_\ell)$ are concentrated on a closed subscheme of dimension $\leq d - d_b$: The closure of a subset of the generic fibre of dimension δ is of dimension at most δ , as the specialization map (from the generic fibre to the special fibre) is specializing. Choosing formal models as in Theorem 2.8.3, we note that $R\psi A \in {}^pD^{\geq d}(\mathfrak{F}l_k, \mathbb{F}_\ell)$ as localization is a filtered colimit and thus preserves ${}^pD^{\geq d}$. Together, we see that the stalks of $R\psi A$ at all points of dimension $d - d_b$ are concentrated in degrees $\geq d_b$. Now choose a rank 1 point of $\mathcal{F}l$ of dimension $d - d_b$ and compute the stalk of A at this point as a

filtered colimit of the stalks of $R\psi A$ at its specialization (which will define a point of dimension $d - d_b$ for fine enough formal models, cf. [CS17, Corollary 6.1.4, end of proof]), over a cofinal system of formal models. This gives the desired result.¹² \square

We need two more results about the cohomology of Igusa varieties. These rely on the trace formula for which we have to assume that F contains an imaginary quadratic field F_0 , that $F^+ \neq \mathbb{Q}$, and we have to fix a character $\varpi : A_{F_0}^\times / F_0^\times \rightarrow \mathbb{C}^\times$ such that $\varpi|_{\mathbb{A}^\times / \mathbb{Q}^\times}$ is the quadratic character corresponding to the extension F_0 . We include all primes above which ϖ is ramified into S . We assume moreover that the level $N \geq 3$ is only divisible by primes in $S \setminus \{p\}$, and is divisible by some specific sufficiently large N_0 with these properties, cf. Remark 5.4.5.

Theorem 2.8.6. *Assume that $F^+ \neq \mathbb{Q}$, that p is split in the imaginary quadratic field $F_0 \subset F$, and that $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$ is such that*

$$H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m}$$

is nonzero for exactly one i . Then there exists a continuous semisimple Galois representation

$$\bar{\rho}_\mathfrak{m} : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_{2n}(\bar{\mathbb{F}}_\ell)$$

such that for all primes v dividing a rational prime $q \notin S$ that splits in F_0 , the characteristic polynomial of $\bar{\rho}_\mathfrak{m}$ is given by

$$X^{2n} - T_{1,v}X^{2n-1} + \dots + (-1)^i q_v^{i(i-1)/2} T_{i,v} X^{2n-i} + \dots + q_v^{n(2n-1)} T_{2n,v}$$

with notation as in the introduction. Moreover, if p is totally split in F and $\bar{\rho}_\mathfrak{m}$ is unramified at all places v dividing p , such that the Frobenius eigenvalues $\{\alpha_{1,v}, \dots, \alpha_{2n,v}\}$ of $\bar{\rho}_\mathfrak{m}(\mathrm{Frob}_v)$ satisfy $\alpha_{i,v} \neq p\alpha_{j,v}$ for all $i \neq j$, then b is ordinary.

Proof. This is Corollary 5.1.3. It is proved by computing $H^*(\mathrm{Ig}^b, \bar{\mathbb{Q}}_\ell)$, as a virtual representation, in terms of automorphic representations. \square

To get concentration in one degree, we need to understand the cohomology of the boundary of Igusa varieties in order to play off upper bounds on $H_{c-\partial}^i$ with lower bounds on H^i . This is achieved in Section 6 and gives the following result:

Theorem 2.8.7. *Assume that $F^+ \neq \mathbb{Q}$, that p is split in the imaginary quadratic field $F_0 \subset F$, and that the map*

$$H_{c-\partial}^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m} \rightarrow H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m}$$

is not an isomorphism for some i . Then there exists a continuous semisimple Galois representation

$$\bar{\rho}_\mathfrak{m} : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_{2n}(\bar{\mathbb{F}}_\ell)$$

such that for all primes v dividing a rational prime $q \notin S$ that splits in F_0 , the characteristic polynomial of $\bar{\rho}_\mathfrak{m}$ is given by

$$X^{2n} - T_{1,v}X^{2n-1} + \dots + (-1)^i q_v^{i(i-1)/2} T_{i,v} X^{2n-i} + \dots + q_v^{n(2n-1)} T_{2n,v}.$$

Moreover, if b is not ordinary, then the length of $\bar{\rho}_\mathfrak{m}$ is at least 3.

Proof. This is proved in Section 6. \square

¹²When writing [CS17], the authors wanted their perverse sheaves to be constructible, not realizing that the theory works well without constructibility. This explains the circumlocutions involving the quotient by K_p in [CS17].

Proof of Theorem 1.1. Now assume all of our hypotheses: that $F = F_0 \cdot F^+$ with $F^+ \neq \mathbb{Q}$, that p is totally split in F , and that \mathfrak{m} is so that $\bar{\rho}_{\mathfrak{m}}$ is unramified and generic at all places dividing p , and of length at most 2.

Pick b as in Lemma 2.8.4. If b is not ordinary, then the map

$$H_c^i(\mathrm{Ig}^b, \mathbb{F}_{\ell})_{\mathfrak{m}} \rightarrow H^i(\mathrm{Ig}^b, \mathbb{F}_{\ell})_{\mathfrak{m}}$$

is an isomorphism by Theorem 2.8.7. Combining Lemma 2.8.4 with Proposition 2.8.2, we see that both sides are nonzero only for $i = d_b$. But then Theorem 2.8.6 gives a contradiction unless b is ordinary.

It follows that $H^i(\mathrm{Ig}^b, \mathbb{F}_{\ell})_{\mathfrak{m}}$ can be nonzero only if b is ordinary. By Lemma 2.8.4, this shows that also in this case $H^i(\mathrm{Ig}^b, \mathbb{F}_{\ell})_{\mathfrak{m}} \neq 0$ only for $i \geq d = d_b$. By Theorem 2.7.2, this shows that $(R(\pi_{\mathrm{HT}}^{\circ})_* \mathbb{F}_{\ell})_{\mathfrak{m}}$ is concentrated on the ordinary locus $\mathcal{F}\ell(\mathbb{Q}_p)$ and in degrees $\geq d$. We see that

$$H^i(\mathcal{S}_{K(p^{\infty}N), C}^{\circ}, \mathbb{F}_{\ell})_{\mathfrak{m}} \cong H^i(\mathcal{F}\ell, (R(\pi_{\mathrm{HT}}^{\circ})_* \mathbb{F}_{\ell})_{\mathfrak{m}})$$

is concentrated in degrees $\geq d$. By a Hochschild-Serre spectral sequence, this implies that

$$H^i(\mathcal{S}_{K(N), C}^{\circ}, \mathbb{F}_{\ell})_{\mathfrak{m}}$$

is concentrated in degrees $\geq d$. By Proposition 2.6.4, we see that

$$H^i(S_{K(N)}(\mathbb{C}), \mathbb{F}_{\ell})_{\mathfrak{m}} \cong H^i(\mathcal{S}_{K(N), \bar{\mathbb{Q}}}, \mathbb{F}_{\ell})_{\mathfrak{m}} \cong H^i(\mathcal{S}_{K(N), C}^{\circ}, \mathbb{F}_{\ell})_{\mathfrak{m}}$$

is concentrated in degrees $\geq d$, which is what we wanted to prove (by Lemma 2.1.1 and Proposition 2.1.6).

Finally, the case of $H_c^i(X_K, \mathbb{F}_{\ell})_{\mathfrak{m}}$ follows by Poincaré duality applied to the “dual” system of Hecke eigenvalues. More precisely, let $\iota : \mathbb{T}^S \rightarrow \mathbb{T}^S$ be the (anti-)involution that sends the double coset operator $[K^S g K^S]$ to $[K^S g^{-1} K]$. Set $\mathfrak{m}^{\vee} := \iota(\mathfrak{m})$. Then Poincaré duality, cf. e.g. [ACC⁺18, Proposition 2.2.12 and Corollary 2.2.13], provides an isomorphism between $H_c^i(X_K, \mathbb{F}_{\ell})_{\mathfrak{m}}$ and the dual of $H^{2d-i}(X_K, \mathbb{F}_{\ell})_{\mathfrak{m}^{\vee}}$. We only need to check that the conditions of the theorem are also satisfied for \mathfrak{m}^{\vee} .

By explicitly computing the characteristic polynomial of $\bar{\rho}_{\mathfrak{m}^{\vee}}(\mathrm{Frob}_v)$ for any prime $v \notin S$ of F in terms of Hecke operators, we deduce the relationship

$$\bar{\rho}_{\mathfrak{m}^{\vee}} \xrightarrow{\sim} \bar{\rho}_{\mathfrak{m}}^{\vee} |\mathrm{Art}_F^{-1}|^{1-2n},$$

where the global Artin reciprocity map Art_F is normalized to take uniformizers to geometric Frobenius elements. Note that $\bar{\rho}_{\mathfrak{m}^{\vee}}$ still has length at most 2, and, when restricted to $\mathrm{Gal}(\bar{F}_v/F_v)$ for any prime $v|p$ of F , is still unramified. Moreover, if $\bar{\rho}_{\mathfrak{m}}$ has Frob_v -eigenvalues equal to $\alpha_{i,v}$ for $i = 1, \dots, 2n$, then $\bar{\rho}_{\mathfrak{m}^{\vee}}$ has Frob_v -eigenvalues equal to $p^{2n-1} \alpha_{i,v}^{-1}$ for $i = 1, \dots, 2n$. This implies that \mathfrak{m}^{\vee} satisfies our genericity hypothesis at all $v|p$. \square

3. COMPACTIFICATIONS OF IGUSA VARIETIES

In this section, we construct the partial minimal and toroidal compactifications of Igusa varieties and establish their main geometric properties. Throughout this section, we fix: a prime p unramified in F , an integer $N \geq 3$ prime to p , and a cone decomposition Σ as in Remark 2.5.6 at principal level $K(N)$. We abbreviate $K = K(N)$ and $\mathcal{S} = \mathcal{S}_K = \mathcal{S}_{K(N)}$.

3.1. Well-positioned subsets. In this section, we recall the notion of *well-positioned* subsets introduced by [Box15, LS18a] and the fact that Oort central leaves are well-positioned subsets of the special fiber of \mathcal{S} . Fix an algebraically closed field k of characteristic p .

To state the condition of being well-positioned, we note that if C is some complete algebraically closed nonarchimedean field, with ring of integers \mathcal{O}_C , and $x = (A, \iota, \lambda, \eta) \in \mathcal{S}(C)$, then the associated Raynaud extension

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

over \mathcal{O}_C endows the abelian scheme \mathcal{B} over \mathcal{O}_C with a principal polarization, an \mathcal{O}_F -action, and a level- N -structure, defining a point of $\mathcal{S}_Z(\mathcal{O}_C)$, where Z is the cusp into which x degenerates, and in particular a point $\pi(x) \in \mathcal{S}_Z(C)$.

Definition 3.1.1. *A locally closed subset $Y \subseteq \mathcal{S}_k = \mathcal{S} \times k$ is well-positioned if there exists a family $Y^\natural = \{Y_Z^\natural\}$ indexed by the cusp labels Z at level K such that*

- (1) Y_Z^\natural is a locally closed subset of $\mathcal{S}_{Z,k}$.
- (2) For any C over k and $x = (A, \iota, \lambda, \eta) \in \mathcal{S}_k(C)$ as above, degenerating into a cusp Z , the point x lies in Y if and only if the associated point $\pi(x) = (\mathcal{B}_C, \dots) \in \mathcal{S}_{Z,k}(C)$ lies in Y_Z^\natural .

Remark 3.1.2. Let us verify that this agrees with [LS18a, Definition 2.2.1] in our case. In what follows, we will need to work with an open cover $\mathfrak{X}_\sigma^\circ$ of the formal scheme $\mathfrak{X}_{Z,\Sigma_Z}$, indexed by elements $\sigma \in \Sigma_Z^+$. The formal scheme $\mathfrak{X}_\sigma^\circ$ is obtained by taking the formal completion along a larger closed subscheme of Ξ_{Z,Σ_Z} than Ξ_σ , namely

$$\Xi(\sigma)^+ := \bigcup_{\tau \in \Sigma_Z^+, \bar{\tau} \subset \bar{\sigma}} \Xi_\tau.$$

(The reason for considering $\Xi(\sigma)^+$ is the following: when $\bar{\tau} \subset \bar{\sigma}$ we have $\sigma^\vee \subset \tau^\vee$, which defines an open embedding $\Xi(\tau) \subset \Xi(\sigma)$. The closed subscheme $\Xi(\sigma)^+$ is precisely the set-theoretic intersection of the boundary ∂_{Z,Σ_Z} with $\Xi(\sigma)$.) See [LS18a, Proposition 2.1.3] for properties of this formal scheme; the key property is that there is a canonical isomorphism

$$(3.1.1) \quad \widehat{\mathcal{S}}_{\bigcup_{\tau \in \Sigma_Z^+, \bar{\tau} \subset \bar{\sigma}} (Z, [\tau])}^{\text{tor}} \simeq \mathfrak{X}_\sigma^\circ.$$

For any affine open formal subscheme $\text{Spf } R$ of $\mathfrak{X}_\sigma^\circ$, we obtain canonical morphisms $W := \text{Spec } R \rightarrow \Xi(\sigma)$ and $W \rightarrow \mathcal{S}^{\text{tor}}$ (induced by the isomorphism (3.1.1)). Then the two stratifications of W induced by the stratifications of $\Xi(\sigma)$ and \mathcal{S}^{tor} coincide. In particular, the preimages of \mathcal{S} and Ξ_Z coincide and we denote this open subscheme by W^0 . Now [LS18a, Definition 2.1.1] asks that for any affine open $\text{Spf } R \subset \mathfrak{X}_\sigma^\circ \subset \mathfrak{X}_{Z,\Sigma_Z}$ as above, we have an identification

$$Y \times_{\mathcal{S}} W^0 = Y_Z^\natural \times_{\mathcal{S}_Z} W^0,$$

where the morphism $W^0 \rightarrow \mathcal{S}_Z$ is obtained by composing the induced morphism $W^0 \rightarrow \Xi_Z$ with the canonical morphism $\Xi_Z \rightarrow \mathcal{S}_Z$. But this identification can be checked on points, where it amounts precisely to the condition we stated.

In particular, the definition is independent of the choice of Σ , cf. also [LS18a, Lemma 2.2.2]. Remark 2.3.8 of [LS18a] shows that this notion is consistent (in our particular case, when there exist good integral models) with [Box15, Definition 3.4.1].

A well-positioned subset admits partial toroidal and minimal compactifications that satisfy many nice properties. Let $Y \subset \mathcal{S}_k$ be a well-positioned subset. Let \tilde{Y} be the closure of Y in \mathcal{S}_k , with complement Y_0 . Let \tilde{Y}^* and Y_0^* denote the closures of Y and Y_0 in \mathcal{S}_k^* . Define

$$Y^* := \tilde{Y}^* \setminus Y_0^*$$

and call this the *partial minimal compactification* of Y . Define Y^{tor} analogously by

$$Y^{\text{tor}} := \tilde{Y}^{\text{tor}} \setminus Y_0^{\text{tor}}$$

and call it the *partial toroidal compactification* of Y . See [LS18a, Theorem 2.3.3] for the first basic properties of these partial compactifications. We have an identification

$$Y^* \times_{\mathcal{S}_k^*} \mathcal{S}_{Z,k} = Y_Z^{\natural}$$

as subsets of $\mathcal{S}_{Z,k}$.

We need the following basic proposition.

Proposition 3.1.3. *Let $Y \subset Y' \subset \mathcal{S}_k$ be well-positioned locally closed subsets such that Y is closed in Y' . Then Y^* is a closed subset of Y'^* .*

Proof. The maps $W^0 \rightarrow \mathcal{S}_Z$, for varying W^0 , form an fpqc cover, and using this one can check that Y_Z^{\natural} must be a closed subset of $Y'_Z{}^{\natural}$ for all cusp labels Z . In particular, the last displayed formula now shows that Y^* is a subset of Y'^* . On the other hand, \tilde{Y}^* is a closed subset of \tilde{Y}'^* , and then

$$Y^* = \tilde{Y}^* \setminus Y_0^* \subset \tilde{Y}'^* \setminus Y_0^*$$

is a closed subset. As Y'^* is a subset of $\tilde{Y}'^* \setminus Y_0^*$ (as $Y_0 \subset Y'_0$), the result follows. \square

Now we want to apply these ideas to the case of central leaves. For this, we fix a p -divisible group \mathbb{X} with G -structures over k .

Proposition 3.1.4. *For any p -divisible group \mathbb{X} with extra structure over k , the associated Oort central leaf $\mathcal{C}^{\mathbb{X}} \subset \mathcal{S}_k$ is well-positioned. For a cusp label $Z = (Z_{N'}, X)$, the subset $(\mathcal{C}^{\mathbb{X}})_Z^{\natural}$ is either the central leaf $\mathcal{C}_Z^{\mathbb{X},Z}$ on $\mathcal{S}_{Z,k}$ associated with the unique p -divisible group \mathbb{X}_Z with extra structure that admits a decomposition*

$$\mathbb{X} \cong \text{Hom}(X, \mu_{p^\infty}) \oplus \mathbb{X}_Z \oplus X \otimes (\mathbb{Q}_p/\mathbb{Z}_p) ,$$

or empty if there is no such \mathbb{X}_Z .¹³

Proof. This is [LS18a, Proposition 3.4.2], but let us give the proof. Choose C as above and a point $(A, \iota, \lambda, \eta) \in \mathcal{S}_k(C)$. Let $0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$ be the Raynaud extension and X the cocharacter group of T .

We have two short exact sequences

$$0 \rightarrow \mathcal{G}_C[p^m] \rightarrow A_C[p^m] \rightarrow (X/p^m X)_C \rightarrow 0$$

¹³The uniqueness follows from the discussion around Proposition 2.2.1.

and

$$0 \rightarrow T_C[p^m] \rightarrow \mathcal{G}_C[p^m] \rightarrow \mathcal{B}_C[p^m] \rightarrow 0.$$

These two short exact sequences give a canonical \mathcal{O}_F -linear filtration on $A_C[p^m]$ which is symplectic with respect to the Weil pairing. Passing to the direct limit over m , we get a filtration on $A_C[p^\infty]$ with graded pieces $\mathrm{Hom}(X, \mu_{p^\infty})$, $\mathcal{B}_C[p^\infty]$, and $X \otimes (\mathbb{Q}_p/\mathbb{Z}_p)$.

By Proposition 2.2.1, this implies that there is an isomorphism

$$\mathbb{X}_C \cong \mathrm{Hom}(X, \mu_{p^\infty}) \oplus \mathcal{B}_C[p^\infty] \oplus X \otimes (\mathbb{Q}_p/\mathbb{Z}_p)$$

and, as observed there, the corresponding p -divisible group $\mathcal{B}_C[p^\infty] \cong (\mathbb{X}_Z)_C$ is then unique up to isomorphism. \square

We let $\mathcal{E}^{\mathbb{X}, \mathrm{tor}}$ and $\mathcal{E}^{\mathbb{X}, *}$ denote the partial toroidal and minimal compactifications of the leaf $\mathcal{E}^{\mathbb{X}}$. Let $\mathcal{E}_Z^{\mathbb{X}} := \mathcal{E}^{\mathbb{X}, *} \times_{\mathcal{S}^*} \mathcal{S}_Z$. By Proposition 3.1.4 and the remark above it, we can identify $\mathcal{E}_Z^{\mathbb{X}}$ with $\mathcal{E}_Z^{\mathbb{X}, Z}$.

Lemma 3.1.5. *The partial toroidal compactification $\mathcal{E}^{\mathbb{X}, \mathrm{tor}}$ is a smooth variety.*

Proof. Since we are working over the perfect field k , it is enough to show that $\mathcal{E}^{\mathbb{X}, \mathrm{tor}}$ is a regular scheme. This follows from the fact that $\mathcal{E}^{\mathbb{X}}$ is smooth, thus regular, and from [LS18a, Proposition 2.3.13]. \square

3.2. Partial toroidal compactifications of Igusa varieties. We continue in the same setup, in particular \mathbb{X} is a p -divisible group with G -structure over k , as before. Let \mathcal{A} denote the restriction of the semi-abelian scheme over $\mathcal{S}^{\mathrm{tor}}$ to $\mathcal{E}^{\mathbb{X}, \mathrm{tor}}$. Note that the group schemes $\mathcal{A}[p^m]$ for $m \in \mathbb{Z}_{\geq 1}$ are quasi-finite and flat, but not finite and flat. Therefore, the inductive system $\mathcal{A}[p^\infty]$ is not a p -divisible group. Nonetheless, we show below that the connected part $\mathcal{A}[p^\infty]^\circ$ of $\mathcal{A}[p^\infty]$ (which can be defined as the ind-scheme $\hat{\mathcal{A}}[p^\infty]$, where $\hat{\mathcal{A}}$ is the completion of \mathcal{A} along its identity section) is a p -divisible group.

Proposition 3.2.1. *The connected part $\mathcal{A}[p^\infty]^\circ$ of $\mathcal{A}[p^\infty]$ is a p -divisible group over $\mathcal{E}^{\mathbb{X}, \mathrm{tor}}$.*

Proof. We can check this on the completed strict local rings of $\mathcal{E}^{\mathbb{X}, \mathrm{tor}}$; more precisely, we restrict to $\mathrm{Spec} R$, where R is the completed strict local ring of $\mathcal{E}^{\mathbb{X}, \mathrm{tor}}$ at some point x that lies in a boundary stratum indexed by a cusp label Z .

In that case, there is the Raynaud extension

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

over $\mathrm{Spec} R$ and a map $\mathcal{G} \rightarrow \mathcal{A}$ over $\mathrm{Spf} R$. On formal completions at the identity, this gives an isomorphism $\hat{\mathcal{G}} \cong \hat{\mathcal{A}}$ that is in fact defined over $\mathrm{Spec} R$ (as modulo any power of the augmentation ideal, both schemes are finite over $\mathrm{Spf} R$, so one can apply formal GAGA). This induces an isomorphism $\mathcal{G}[p^\infty]^\circ \cong \mathcal{A}[p^\infty]^\circ$ over $\mathrm{Spec} R$. But $\mathcal{G}[p^\infty]$ is a p -divisible group, and its connected part has constant rank, as the abelian variety \mathcal{B} is given by a map from $\mathrm{Spec} R$ into the leaf $\mathcal{E}_Z^{\mathbb{X}, Z}$. Thus, $\mathcal{G}[p^\infty]^\circ$ is also a p -divisible group, as desired. \square

Clearly, $\mathcal{A}[p^\infty]^\circ$ carries an \mathcal{O}_F -action. At every geometric point \bar{x} of $\mathcal{E}^{\mathbb{X}, \mathrm{tor}}$, there is an \mathcal{O}_F -linear isomorphism

$$\mathcal{A}[p^\infty]^\circ \times_{\mathcal{E}^{\mathbb{X}, \mathrm{tor}}} k(\bar{x}) \cong \mathbb{X}^\circ \times_k k(\bar{x}).$$

Indeed, in the notation of the previous proof, this follows directly from $\mathcal{B}[p^\infty] \times_k k(\bar{x}) \cong \mathbb{X}_Z$ as

$$\mathbb{X}^\circ \cong \mathrm{Hom}(X, \mu_{p^\infty}) \oplus \mathbb{X}_Z^\circ.$$

In particular, we see that the multiplicative part $\mathcal{A}[p^\infty]^\mu$ is of constant rank, and thus defines a p -divisible group itself. Let

$$\mathcal{A}[p^\infty]^{(0,1)} = \mathcal{A}[p^\infty]^\circ / \mathcal{A}[p^\infty]^\mu$$

be the biconnected part.

Proposition 3.2.2. *There exists a polarization on $\mathcal{A}[p^\infty]^{(0,1)}$ extending the polarization that exists after restriction to $\mathcal{C}^\mathbb{X}$.*

Proof. Again, this can be checked after restriction to $\mathrm{Spec} R$ for the completed local rings of $\mathcal{C}^{\mathbb{X}, \mathrm{tor}}$. With notation as before, this identifies $\mathcal{A}[p^\infty]^{(0,1)}$ with the biconnected part of $\mathcal{B}[p^\infty]$, which has a natural principal polarization, by construction compatible with the polarization on the generic fibre of \mathcal{A} . \square

3.2.3. Partial toroidal compactifications of Igusa varieties in the completely slope divisible case. Assume now that $\mathbb{X} = \bigoplus_i \mathbb{X}_i$ is completely slope divisible. Let $\mathrm{Ig}_m^\mathbb{X}$ be the Igusa variety over $\mathcal{C}^\mathbb{X}$ of some finite level p^m as defined in Remark 2.3.6; recall that this is a finite étale cover of $\mathcal{C}^\mathbb{X}$, which is Galois with Galois group $\Gamma_{m, \mathbb{X}}$.

Theorem 3.2.4. *The finite étale cover $\mathrm{Ig}_m^\mathbb{X} \rightarrow \mathcal{C}^\mathbb{X}$ extends uniquely to a finite étale cover $\mathrm{Ig}_m^{\mathbb{X}, \mathrm{tor}} \rightarrow \mathcal{C}^{\mathbb{X}, \mathrm{tor}}$, Galois with group $\Gamma_{m, \mathbb{X}}$.*

In fact, we can be more precise, and give a moduli description of $\mathrm{Ig}_m^{\mathbb{X}, \mathrm{tor}}$. Note that $\mathcal{A}[p^\infty]^\circ$ is also completely slope divisible, so we obtain p -divisible groups $\mathcal{A}[p^\infty]_i$ equipped with extra structures of EL or of PEL type for all i such that $\lambda_i < 1$.

Definition 3.2.5. *An Igusa level p^m structure ρ_m on a $\mathcal{C}^{\mathbb{X}, \mathrm{tor}}$ -scheme \mathcal{T} consists of the following data:*

For each i such that $\lambda_i > 0$, an isomorphism

$$\rho_{i,m} : \mathcal{A}[p^\infty]_i[p^m] \times_{\mathcal{C}^{\mathbb{X}, \mathrm{tor}}} \mathcal{T} \xrightarrow{\sim} \mathbb{X}_i[p^m] \times_k \mathcal{T},$$

that commutes with the \mathcal{O}_F -action and lifts fppf locally to $p^{m'}$ -torsion for all $m' \geq m$, and an element of $(\mathbb{Z}/p^m\mathbb{Z})^\times(\mathcal{T})$ such that, for all i, j such that $\lambda_i, \lambda_j > 0$ and $\lambda_i + \lambda_j = 1$, the isomorphisms $\rho_{i,m}$ and $\rho_{j,m}$ commute with the polarizations up the given scalar in $(\mathbb{Z}/p^m\mathbb{Z})^\times(\mathcal{T})$.

Proof of Theorem 3.2.4. We prove that the moduli problem in Definition 3.2.5 is representable by a finite étale scheme over $\mathcal{C}^{\mathbb{X}, \mathrm{tor}}$. This follows from Theorem 2.2.4 applied to the isoclinic p -divisible groups $\mathcal{A}[p^\infty]_i$ for all i with $\lambda_i > \frac{1}{2}$ as p -divisible groups with EL structure, and the p -divisible group $\mathcal{A}[p^\infty]_i$ for i with $\lambda = \frac{1}{2}$ as p -divisible group with PEL structure. Indeed, the isomorphisms for i with $0 < \lambda_i < \frac{1}{2}$ are then formally determined. We now apply Theorem 2.2.4 to each graded piece with its induced extra structures and take the fiber product of the resulting finite étale covers of $\mathcal{C}^{\mathbb{X}, \mathrm{tor}}$.

It is clear that over the open part, this recovers $\mathrm{Ig}_m^\mathbb{X}$, again as the polarization determines the structure on the étale quotient from the structure on the multiplicative quotient. In particular, it defines a $\Gamma_{m, \mathbb{X}}$ -torsor, as desired. \square

For further use, we would like to give a preliminary description of the formal completions along boundary strata in $\mathrm{Ig}_m^{\mathbb{X},\mathrm{tor}}$ in terms of the toroidal boundary charts in Theorem 2.5.9.¹⁴ Let Z be a cusp label at level N . This determines a locally closed boundary stratum $\mathcal{C}_Z^{\mathbb{X},\mathrm{tor}} \subset \mathcal{C}^{\mathbb{X},\mathrm{tor}}$. Let $\widehat{\mathcal{C}}_Z^{\mathbb{X},\mathrm{tor}}$ denote the completion along this boundary stratum. Let $\widehat{\mathrm{Ig}}_{m,Z}^{\mathbb{X},\mathrm{tor}}$ denote the formal completion of $\mathrm{Ig}_m^{\mathbb{X},\mathrm{tor}}$ along the boundary stratum $\mathrm{Ig}_{m,Z}^{\mathbb{X},\mathrm{tor}}$ determined by the preimage of $\mathcal{C}_Z^{\mathbb{X},\mathrm{tor}}$. We have a finite étale map of formal schemes

$$\widehat{\mathrm{Ig}}_{m,Z}^{\mathbb{X},\mathrm{tor}} \rightarrow \widehat{\mathcal{C}}_Z^{\mathbb{X},\mathrm{tor}},$$

which is Galois, with Galois group $\Gamma_{m,\mathbb{X}}$.

Because $\mathcal{C}^{\mathbb{X}}$ is well-positioned, we can identify $\widehat{\mathcal{C}}_Z^{\mathbb{X},\mathrm{tor}}$ with the quotient by Γ_Z of the formal completion of

$$\mathcal{C}_{Z,\Sigma_Z}^{\mathbb{X}} := \Xi_{Z,\Sigma_Z} \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}}$$

along $\partial_{Z,\Sigma_Z} \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}}$, cf. [LS18a, Theorem 2.3.2 (5)]. We would like to give a similar description for $\widehat{\mathrm{Ig}}_{m,Z}^{\mathbb{X},\mathrm{tor}}$. By construction, over C_Z , we have a semi-abelian scheme \mathcal{G}_Z together with an \mathcal{O}_F -action. Here \mathcal{G}_Z is a Raynaud extension

$$0 \rightarrow T \rightarrow \mathcal{G}_Z \rightarrow \mathcal{B}_Z \rightarrow 0,$$

where T is the constant torus with character group X (recall that X is part of the torus argument that comes with the cusp label Z) and \mathcal{B}_Z is the pullback of the universal abelian scheme over \mathcal{S}_Z to C_Z . As observed in the proof of Proposition 3.2.1, over $C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}}$, the connected part $\mathcal{G}_Z[p^\infty]^\circ$ defines a p -divisible group itself (as the étale part has constant rank); let us write \mathcal{H}_Z for this p -divisible group with \mathcal{O}_F -action on $C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}}$. Again, it is completely slope divisible, so we get p -divisible groups $\mathcal{H}_{Z,i}$ of slope λ_i , equipped with polarizations as above.

With the obvious version of Definition 3.2.5, we can now define a $\Gamma_{m,\mathbb{X}}$ -torsor over $C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}}$, whose pullback to $\Xi_{Z,\Sigma_Z} \times_{\mathcal{S}_Z} C_Z^{\mathbb{X}}$ we denote by

$$\mathrm{Ig}_{Z,\Sigma_Z}^{\mathbb{X}} \rightarrow \Xi_{Z,\Sigma_Z}.$$

We also let $\partial \mathrm{Ig}_{Z,\Sigma_Z}^{\mathbb{X}}$ denote the corresponding finite étale cover of $\partial_{Z,\Sigma_Z} \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}}$. This is a closed subscheme of $\mathrm{Ig}_{Z,\Sigma_Z}^{\mathbb{X}}$. Finally, we let Y_{Z,Σ_Z} denote the formal completion of $\mathrm{Ig}_{Z,\Sigma_Z}^{\mathbb{X}}$ along $\partial \mathrm{Ig}_{Z,\Sigma_Z}^{\mathbb{X}}$.

Theorem 3.2.6. *With the same choice of a splitting of the filtration Z_N as in Theorem 2.5.9 (3), there is a canonical isomorphism of formal schemes*

$$\widehat{\mathrm{Ig}}_{m,Z}^{\mathbb{X},\mathrm{tor}} \xrightarrow{\sim} Y_{Z,\Sigma_Z}/\Gamma_Z.$$

Proof. This follows once we prove that there exists an isomorphism between the pullbacks of $\mathcal{A}[p^\infty]^\circ$ and \mathcal{H}_Z to $\widehat{\mathcal{C}}_Z^{\mathbb{X},\mathrm{tor}}$ that commutes with the extra structures.

We work Zariski locally on $\widehat{\mathcal{C}}_Z^{\mathbb{X},\mathrm{tor}}$. We cover $\widehat{\mathcal{C}}_Z^{\mathbb{X},\mathrm{tor}}$ by affine open formal subschemes $\mathrm{Spf} R$ which lift to affine opens in the formal completion of $\mathcal{C}_{Z,\Sigma_Z}^{\mathbb{X}}$ along the toroidal boundary stratum (because taking the quotient by Γ_Z is a local isomorphism) and which arise by formal completion from affine opens in $\mathcal{C}^{\mathbb{X},\mathrm{tor}}$ and $\mathcal{C}_{Z,\Sigma_Z}^{\mathbb{X}}$. We obtain induced flat maps of schemes

$$f_1 : \mathrm{Spec} R \rightarrow \mathcal{C}^{\mathbb{X},\mathrm{tor}} \text{ and } f_2 : \mathrm{Spec} R \rightarrow \mathcal{C}_{Z,\Sigma_Z}^{\mathbb{X}}.$$

¹⁴A different description will be given in Theorem 3.3.12.

We let A denote the pullback of the semi-abelian scheme \mathcal{A} along f_1 and \tilde{A} denote the pullback of the Raynaud extension \tilde{G}_Z along f_2 . As above, the completions along the identity $\hat{A} \cong \tilde{\hat{A}}$ are isomorphic over $\mathrm{Spf} R$, and thus over $\mathrm{Spec} R$.

Now

$$f_2^* \mathcal{H}_Z = \tilde{A}[p^\infty]^\circ \cong A[p^\infty]^\circ = f_1^* \mathcal{A}[p^\infty]^\circ,$$

as desired. \square

3.2.7. Partial toroidal compactifications of perfect Igusa varieties. In this subsection, we will repeat the previous constructions for perfect Igusa varieties, allowing now general \mathbb{X} (not necessarily completely slope divisible).

Let $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ be the perfect Igusa variety over $\mathcal{C}^{\mathbb{X}}$; this is a pro-finite étale cover of the perfection of $\mathcal{C}^{\mathbb{X}}$, Galois with group $\Gamma_{\mathbb{X}} = \mathrm{Aut}(\mathbb{X})$.

Theorem 3.2.8. *The pro-finite étale cover $\mathfrak{I}\mathfrak{g}^{\mathbb{X}} \rightarrow \mathcal{C}_{\mathrm{perf}}^{\mathbb{X}}$ extends uniquely to a pro-finite étale cover $\mathfrak{I}\mathfrak{g}^{\mathbb{X},\mathrm{tor}} \rightarrow \mathcal{C}_{\mathrm{perf}}^{\mathbb{X},\mathrm{tor}}$, Galois with group $\Gamma_{\mathbb{X}}$.*

We remark that if \mathbb{X} is completely slope divisible, then $\mathfrak{I}\mathfrak{g}^{\mathbb{X},\mathrm{tor}}$ is the perfection of $\varprojlim_m \mathrm{Ig}_m^{\mathbb{X},\mathrm{tor}}$ (by uniqueness).

Again, we can be more precise, and give a moduli description of $\mathfrak{I}\mathfrak{g}^{\mathbb{X},\mathrm{tor}}$:

Definition 3.2.9. *A perfect Igusa level structure ρ on a perfect $\mathcal{C}_{\Sigma}^{b,\mathrm{tor}}$ -scheme \mathcal{T} is given by an \mathcal{O}_F -linear isomorphism*

$$\rho : \mathcal{A}[p^\infty]^\circ \times_{\mathcal{C}^{\mathbb{X},\mathrm{tor}}} \mathcal{T} \xrightarrow{\sim} \mathbb{X}^\circ \times_k \mathcal{T}$$

and a scalar in $\mathbb{Z}_p^\times(\mathcal{T})$ such that the induced isomorphism

$$\rho_{(0,1)} : \mathcal{A}[p^\infty]^{(0,1)} \times_{\mathcal{C}^{\mathbb{X},\mathrm{tor}}} \mathcal{T} \xrightarrow{\sim} \mathbb{X}^{(0,1)} \times_k \mathcal{T}$$

obtained by quotienting by the multiplicative parts commutes with the polarizations up to the given element of $\mathbb{Z}_p^\times(\mathcal{T})$.

Proof of Theorem 3.2.8. Uniqueness is clear as the base is normal and the cover is given on an open dense subspace. The proof is now similar to the proof of Theorem 3.2.4, except that we refer to Proposition 2.2.6 instead of Theorem 2.2.4. \square

Remark 3.2.10. By Corollary 2.3.2, $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ is an fpqc $\mathrm{Aut}(\mathbb{X})$ -torsor over $\mathcal{C}^{\mathbb{X}}$ (before perfection). However, this no longer holds true over the boundary strata of the partial toroidal compactification. This phenomenon already occurs for the modular curves, in which case the precise structure at the boundary is described in [How18, §3].

Again, we can give a description of the formal completions along boundary strata. More precisely, we would like to describe the $\Gamma_{\mathbb{X}}$ -torsor

$$\widehat{\mathfrak{I}\mathfrak{g}}_Z^{\mathbb{X},\mathrm{tor}} \rightarrow \widehat{\mathcal{C}}_{Z,\mathrm{perf}}^{\mathbb{X},\mathrm{tor}}$$

with notation following the previous subsection.

We can identify $\widehat{\mathcal{C}}_{Z,\mathrm{perf}}^{\mathbb{X},\mathrm{tor}}$ with the quotient by Γ_Z of the formal completion of

$$\mathcal{C}_{Z,\Sigma_Z,\mathrm{perf}}^{\mathbb{X}} = (\Xi_{Z,\Sigma_Z} \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}})_{\mathrm{perf}}$$

along its toroidal boundary. We would like to give a similar description for $\widehat{\mathfrak{I}\mathfrak{g}}_Z^{\mathbb{X},\mathrm{tor}}$.

As above, we have the p -divisible group \mathcal{H}_Z over $C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}}$ (the connected part of the p -divisible group of the Raynaud extension). On perfect test schemes over $C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}}$, we can parametrize \mathcal{O}_F -linear isomorphisms

$$\mathcal{H}_Z \cong \mathbb{X}^\circ$$

together with a section of \mathbb{Z}_p^\times such that the induced isomorphism of biconnected parts is compatible with the polarization. By Proposition 2.2.6, this is representable by a $\Gamma_{\mathbb{X}}$ -torsor

$$C_Z^{\mathcal{J}\mathfrak{g},\mathbb{X}} \rightarrow (C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}})_{\text{perf}}.$$

Let us give an explicit description of $C_Z^{\mathcal{J}\mathfrak{g},\mathbb{X}}$.

Proposition 3.2.11. *The perfect scheme $C_Z^{\mathcal{J}\mathfrak{g},\mathbb{X}}$ parametrizes points $(\mathcal{B}, \iota, \lambda, \eta) \in \mathcal{S}_Z$ together with an extension*

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

by the split torus T with cocharacter group X , and an \mathcal{O}_F -linear embedding

$$\rho : \mathcal{G}[p^\infty] \hookrightarrow \mathbb{X}$$

that is compatible with the polarization in the following sense: The filtration $Z_{\mathbb{X}}$ of \mathbb{X} given by

$$T[p^\infty] \subset \mathcal{G}[p^\infty] \subset \mathbb{X}$$

is symplectic, and the induced isomorphism

$$\mathcal{B}[p^\infty] = \mathcal{G}[p^\infty]/T[p^\infty] \cong \text{Gr}_{-1}^{Z_{\mathbb{X}}}$$

is compatible with principal polarizations.

Proof. The description given above would only require an isomorphism $\mathcal{G}[p^\infty]^\circ = \mathcal{H}_Z \cong \mathbb{X}^\circ$. However, over a perfect base the étale part is a direct summand, which by duality with the multiplicative part has a natural map into the étale part of \mathbb{X} , inducing a canonical extension $\mathcal{G}[p^\infty] \hookrightarrow \mathbb{X}$. The result follows easily. \square

Now define $\Xi_{Z,\Sigma_Z}^{\mathcal{J}\mathfrak{g},\mathbb{X}}$ by the cartesian square

$$\begin{array}{ccc} \Xi_{Z,\Sigma_Z}^{\mathcal{J}\mathfrak{g},\mathbb{X}} & \longrightarrow & (\Xi_{Z,\Sigma_Z} \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}})_{\text{perf}} \\ \downarrow & & \downarrow \\ C_Z^{\mathcal{J}\mathfrak{g},\mathbb{X}} & \longrightarrow & (C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}})_{\text{perf}}. \end{array}$$

We would also like to understand this explicitly. The essential point is to understand the $\mathbb{S}_{Z,\text{perf}}$ -torsor

$$\Xi_Z^{\mathcal{J}\mathfrak{g},\mathbb{X}} \rightarrow C_Z^{\mathcal{J}\mathfrak{g},\mathbb{X}}$$

of which $\Xi_{Z,\Sigma_Z}^{\mathcal{J}\mathfrak{g},\mathbb{X}}$ is a partial compactification.

Proposition 3.2.12. *The $\mathbb{S}_{Z,\text{perf}}$ -torsor*

$$\Xi_Z^{\mathcal{J}\mathfrak{g},\mathbb{X}} \rightarrow C_Z^{\mathcal{J}\mathfrak{g},\mathbb{X}}$$

is the perfection of the \mathbb{S}_Z -torsor that parametrizes lifts of the map $f_0 : X \rightarrow \mathcal{B}^\vee \cong \mathcal{B}$ (given by the extension $0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$) to symmetric $f : X \rightarrow \mathcal{G}$.

Explicitly, this can be described as follows. Choose a symplectic splitting $\delta_{\mathbb{X}}$ of the filtration $Z_{\mathbb{X}}$ of \mathbb{X} over $C_Z^{\mathfrak{J}\mathfrak{g}, \mathbb{X}}$. Via the embedding $\rho : \mathcal{G}[p^\infty] \hookrightarrow \mathbb{X}$, this induces a splitting of

$$0 \rightarrow T[p^\infty] \rightarrow \mathcal{G}[p^\infty] \rightarrow \mathcal{B}[p^\infty] \rightarrow 0$$

and dually a lift of f_0 to a map $\tilde{f}_0 : X[\frac{1}{p}] \rightarrow \mathcal{B}^\vee \cong \mathcal{B}$. Then the $\mathbb{S}_{Z, \text{perf}}$ -torsor

$$\Xi_Z^{\mathfrak{J}\mathfrak{g}, \mathbb{X}} \rightarrow C_Z^{\mathfrak{J}\mathfrak{g}, \mathbb{X}}$$

parametrizes symmetric lifts $\tilde{f} : X[\frac{1}{p}] \rightarrow \mathcal{G}$ of \tilde{f}_0 .

Proof. The first description follows directly from the definition. For the second description, it is enough to observe that it does indeed define a lift of the given \mathbb{S}_Z -torsor to a $\mathbb{S}_{Z, \text{perf}}$ -torsor; by uniqueness, it must be the correct one. \square

Let $\mathfrak{Y}_{Z, \Sigma_Z}$ be the completion of $\Xi_{Z, \Sigma_Z}^{\mathfrak{J}\mathfrak{g}, \mathbb{X}}$ along its toroidal boundary.

Theorem 3.2.13. *With the same choice of a splitting of the filtration Z_N as in Theorem 2.5.9 (3), there is a canonical isomorphism of formal schemes*

$$\widehat{\mathfrak{J}\mathfrak{g}}_Z^{\mathbb{X}, \text{tor}} \xrightarrow{\sim} \mathfrak{Y}_{Z, \Sigma_Z} / \Gamma_Z.$$

Proof. The proof is the same as for Theorem 3.2.6. \square

A corollary of this description is the independence of $\mathfrak{J}\mathfrak{g}_Z^{\mathbb{X}, \text{tor}}$ from \mathbb{X} up to quasi-isogeny:

Corollary 3.2.14. *Let $\phi : \mathbb{X}_{b,1} \rightarrow \mathbb{X}_{b,2}$ be an isogeny between p -divisible groups with G -structure over k . Assume that ϕ induces an isomorphism on étale and multiplicative parts. Then the isomorphism*

$$\mathfrak{J}\mathfrak{g}^{\mathbb{X}_1} \cong \mathfrak{J}\mathfrak{g}^{\mathbb{X}_2}$$

induced by ϕ extends uniquely to an isomorphism

$$\mathfrak{J}\mathfrak{g}^{\mathbb{X}_1, \text{tor}} \cong \mathfrak{J}\mathfrak{g}^{\mathbb{X}_2, \text{tor}}.$$

Proof. For each cusp Z , we will produce an isomorphism

$$\widehat{\mathfrak{J}\mathfrak{g}}_{K^p, \Sigma, Z, 1}^{b, \text{tor}} \cong \widehat{\mathfrak{J}\mathfrak{g}}_{K^p, \Sigma, Z, 2}^{b, \text{tor}}.$$

For this, we use the description in Theorem 3.2.13. First, consider the $\Gamma_{\mathbb{X}}$ -torsor $C_Z^{\mathfrak{J}\mathfrak{g}, \mathbb{X}} \rightarrow (C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^{\mathbb{X}})_{\text{perf}}$. We use the description in Proposition 3.2.11. We have given the Raynaud extension

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

together with an embedding

$$\rho : \mathcal{G}[p^\infty] \hookrightarrow \mathbb{X},$$

and the isogeny $\phi : \mathbb{X} \rightarrow \mathbb{X}'$ whose kernel $\mathcal{K} \subset \mathbb{X}$ is contained in \mathbb{X}° and does not meet the multiplicative part. It follows that $\mathcal{G}' = \mathcal{G}/\rho^{-1}(\mathcal{K})$ defines another semiabelian scheme

$$0 \rightarrow T \rightarrow \mathcal{G}' \rightarrow \mathcal{B}' \rightarrow 0$$

(where $\mathcal{B}' = \mathcal{B}/\rho^{-1}(\mathcal{K})$, noting that the composite $\rho^{-1}(\mathcal{K}) \rightarrow \mathcal{G} \rightarrow \mathcal{B}$ is still a closed immersion as K does not meet the multiplicative part), together with an injection

$$\rho' : \mathcal{G}'[p^\infty] \hookrightarrow \mathbb{X}'.$$

It is clear that this induces an isomorphism

$$\phi_* : C_Z^{\mathcal{J}\mathfrak{g}, \mathbb{X}} \cong C_Z^{\mathcal{J}\mathfrak{g}, \mathbb{X}'}$$

We need to compare the perfect torus torsors. Note that the maps $f_0 : X \rightarrow \mathcal{B}$ and $f'_0 : X \rightarrow \mathcal{B}'$ satisfy that the composite $X \xrightarrow{f_0} \mathcal{B} \rightarrow \mathcal{B}'$ is $p^m f'_0$ where p^m is the degree of ϕ . In particular, the pullback of the natural \mathbb{S}_Z -torsor over $C_Z^{\mathcal{J}\mathfrak{g}, \mathbb{X}'}$ under ϕ_* is the pushout of the natural \mathbb{S}_Z -torsor over $C_Z^{\mathcal{J}\mathfrak{g}, \mathbb{X}}$ under $p^m : \mathbb{S}_Z \rightarrow \mathbb{S}_Z$. After perfection, we get a natural isomorphism, and as p^m is a scalar, it preserves the cone decomposition.

It is now easy to check that this isomorphism on completions at the boundary is compatible with the given one in the interior, which implies the desired extension. \square

3.3. Partial minimal compactifications of Igusa varieties. In this section, we will analyze the partial minimal compactifications of Igusa varieties, by using the geometry of the toroidal compactifications from the last subsection. As before, we fix an algebraically closed field k of characteristic p and a p -divisible group \mathbb{X} with G -structure as usual.

3.3.1. Partial minimal compactifications are affine. First, we observe that the partial minimal compactifications of leaves are affine.

Theorem 3.3.2. *The partial minimal compactification $\mathcal{C}^{\mathbb{X},*}$ is affine.*

Proof. First, we prove that there exists some \mathbb{X} in the given isogeny class for which the result is true. By a result of Nie, [Nie15, Proposition 1.5, Corollary 1.6], there is an Ekedahl–Oort stratum that is completely contained in the given Newton stratum. Taking \mathbb{X} so that $\mathbb{X}[p]$ defines such an Ekedahl–Oort stratum, the leaf $\mathcal{C}^{\mathbb{X}}$ is a closed subset of the corresponding Ekedahl–Oort stratum (as it is closed in its Newton stratum). By [Box15, Theorem C], the partial minimal compactifications of Ekedahl–Oort strata are affine. As mentioned above, the definitions of partial minimal compactifications of [LS18a] and [Box15] are equivalent in our situation. We conclude by Proposition 3.1.3.

In general, we can find an isogeny $\phi : \mathbb{X}' \rightarrow \mathbb{X}$ from a p -divisible group \mathbb{X}' with G -structure for which the result holds, and we may assume that ϕ induces isomorphisms of étale and multiplicative parts. Using Corollary 3.2.14, we get a correspondence

$$\mathcal{C}^{\mathbb{X}',*} \xleftarrow{\pi_1^{\text{tor}}} \mathcal{J}\mathfrak{g}^{\mathbb{X},\text{tor}} \xrightarrow{\pi_2^{\text{tor}}} \mathcal{C}^{\mathbb{X},*}$$

which extends the diagram

$$\mathcal{C}^{\mathbb{X}'} \xleftarrow{\pi_1} \mathcal{J}\mathfrak{g}^{\mathbb{X}} \xrightarrow{\pi_2} \mathcal{C}^{\mathbb{X}}.$$

Approximating $\mathcal{J}\mathfrak{g}^{\mathbb{X},\text{tor}}$ by a scheme of finite type, the result now follows from Lemma 3.3.3 below, and induction to ensure that the boundary of $\mathcal{C}^{\mathbb{X},*}$ is affine. \square

Lemma 3.3.3. *Consider a diagram*

$$\begin{array}{ccccc} X_0 & \longleftarrow & C_0 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{\pi_1} & C & \xrightarrow{\pi_2} & Y \end{array}$$

of schemes, where the vertical arrows are closed immersions and $C_0 \subset C$ is topologically the preimage both of $X_0 \subset X$ and of $Y_0 \subset Y$. Assume that X , X_0 and Y_0 are affine, and that π_1 and π_2 are proper and surjective and finite away from C_0 . Then Y is affine.

Proof. Let $C \rightarrow C' \rightarrow Y$ be the Stein factorization of π_2 . Note that the map $C \rightarrow X$ factors uniquely over C' as $H^0(C', \mathcal{O}_{C'}) = H^0(C, \mathcal{O}_C)$ and X is affine. The resulting map $C' \rightarrow X$ is automatically proper and surjective (cf. e.g. [Sta19, Tag 03GN]). We can then replace C by C' .

In particular, we can assume that π_2 is finite and surjective. As affineness descends along finite surjections, it follows that it is enough to prove that C is affine. For this, it suffices to show that π_1 is finite, for which it suffices to prove that π_1 is quasi-finite. This is evident away from X_0 . On X_0 , the map $C_0 \rightarrow X_0$ is a proper map of affine schemes (as $C_0 \rightarrow Y_0$ is finite and Y_0 is affine), and thus finite. This gives the result. \square

In particular, if one defines the partial minimal compactification $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*}$ as the normalization of $\mathcal{C}^{\mathbb{X},*}$ in $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$, we have the following results.

Proposition 3.3.4. *The map $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*} \rightarrow \mathcal{C}^{\mathbb{X},*}$ is integral, and in particular $\mathfrak{I}\mathfrak{g}^{\mathbb{X},*}$ is affine. It agrees with the Stein factorization of*

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}} \rightarrow \mathcal{C}^{\mathbb{X},*}.$$

In particular,

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X},*} = H^0(\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}}, \mathcal{O}_{\mathfrak{I}\mathfrak{g}^{\mathbb{X},\text{tor}}}),$$

and a G -isogeny $\phi : \mathbb{X} \rightarrow \mathbb{X}'$ inducing isomorphisms of étale and multiplicative parts induces an isomorphism

$$\mathfrak{I}\mathfrak{g}^{\mathbb{X},*} \cong \mathfrak{I}\mathfrak{g}^{\mathbb{X}',*}.$$

Remark 3.3.5. One would expect that the final statement does not need ϕ to induce isomorphisms of étale and multiplicative parts.

Proof. All statements are immediate. \square

3.3.6. Partial minimal compactifications of Igusa varieties in the completely slope divisible case. From now on, we fix, for a given $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$, a p -divisible group with G -structure $\mathbb{X} = \mathbb{X}_b$ in the given isogeny class such that \mathbb{X}_b is completely slope divisible. As the perfect Igusa varieties are invariant under isogenies, it is for the rest enough to understand one choice of \mathbb{X}_b . Thus, by abuse of notation, we denote by $\text{Ig}_m^b := \text{Ig}_m^{\mathbb{X}_b}$ be the Igusa variety over $\mathcal{C}^b := \mathcal{C}^{\mathbb{X}_b}$ of level p^m ; recall that this is a finite étale cover of \mathcal{C}^b , which is Galois with Galois group $\Gamma_{m,b} = \Gamma_{m,\mathbb{X}_b}$.

Definition 3.3.7. *Define the partial minimal compactification $\text{Ig}^{b,*}$ of Ig^b to be the normalization of $\mathcal{C}^{b,*}$ in Ig^b .*

While we cannot in general expect $\text{Ig}^{b,*}$ to be finite étale over $\mathcal{C}^{b,*}$, the partial minimal compactifications of Igusa varieties satisfy the following basic properties that follow formally from the definition.

Lemma 3.3.8.

- (1) *The morphism $h^{b,*} : \text{Ig}^{b,*} \rightarrow \mathcal{C}^{b,*}$ is finite and surjective. In particular, $\text{Ig}^{b,*}$ is affine.*

- (2) The morphism $\mathrm{Ig}^b \hookrightarrow \mathrm{Ig}^{b,*}$ is the open immersion of a dense subset and $\mathrm{Ig}^{b,*}$ is normal.

Proof. This is clear. \square

3.3.9. *Igusa cusp labels.* In order to describe the boundary strata in $\mathrm{Ig}^{b,*}$ explicitly, we will define a notion of Igusa cusp label that mirrors the notion of cusp label introduced in § 2.5.2. Recall that we have fixed the completely slope divisible p -divisible group \mathbb{X}_b with G -structure.

Definition 3.3.10. An Igusa cusp label is a triple $\tilde{Z} = (Z_b, Z^p, X)$ where

- (1) Z_b is an \mathcal{O}_F -stable filtration of \mathbb{X}_b of the form

$$Z_{b,-2} \subset Z_{b,-1} \subset \mathbb{X}_b$$

such that $\mathrm{Gr}_{-2}^{Z_b} = Z_{b,-2}$ is multiplicative, $\mathrm{Gr}_0^{Z_b} = \mathbb{X}_b/Z_{b,-1}$ is étale, and the polarization identifies these as Cartier dual. In particular, it induces a principal polarization of

$$\mathrm{Gr}_{-1}^{Z_b} = Z_{b,-1}/Z_{b,-2}.$$

- (2) Z^p is an \mathcal{O}_F -stable symplectic filtration

$$Z_{-2}^p \subset Z_{-1}^p \subset L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p.$$

- (3) X is a finite projective \mathcal{O}_F -module equipped with isomorphisms $X \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathrm{Gr}_0^{Z_b}$, $X \otimes \widehat{\mathbb{Z}}^p \cong \mathrm{Gr}_0^{Z^p}$.

There is an action of $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ on Igusa cusp labels. If $K \subset J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ is a compact open subgroup, then an Igusa cusp label at level K is a K -orbit of Igusa cusp labels. For a general closed subgroup $H \subset J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$, an Igusa cusp label at level H is a compatible family of Igusa cusp labels at level K for all $K \supset H$.

If $K = \Gamma_b(p^m)K^p(N)$ where $\Gamma_b(p^m) = \ker(\Gamma_b \rightarrow \Gamma_{m,b})$ is the principal level p^m subgroup of Γ_b , one can identify Igusa cusp labels at level K with triples $\tilde{Z} = (Z_{m,b}, Z_N, X)$ where $Z = (Z_N, X)$ form a usual cusp label at level $K(N)$, and $Z_{m,b}$ is an \mathcal{O}_F -linear symplectic filtration of $\mathbb{X}_b[p^m]$ with an isomorphism

$$X/p^m \cong \mathrm{Gr}_0^{Z_{m,b}}.$$

(This forces in particular that this is an étale p -divisible group, and then the symplectic pairing forces $\mathrm{Gr}_{-2}^{Z_{m,b}}$ to be multiplicative. The filtration will automatically lift modulo higher powers of p .)

Given an Igusa cusp label \tilde{Z} at level $\Gamma_b(p^m)K^p(N)$, we define the group

$$\Gamma_{\tilde{Z}} := \{\gamma \in \mathrm{Aut}_{\mathcal{O}_F}(X) \mid \gamma \equiv 1 \pmod{p^m N}\}.$$

3.3.11. *Boundary strata in partial toroidal compactifications of Igusa varieties.* Consider a cusp label $Z = (Z_N, X)$ at level $K(N)$. We will now give a description of the formal completion $\widehat{\mathrm{Ig}}_{m,Z}^{b,\mathrm{tor}}$ of $\mathrm{Ig}_m^{b,\mathrm{tor}}$ along $\mathrm{Ig}_{m,Z}^{b,\mathrm{tor}}$ in terms of Igusa cusp labels at level $\Gamma_b(p^m)K^p(N)$.

Theorem 3.3.12.

(1) There exists a decomposition into open and closed formal subschemes

$$\widehat{\mathrm{I}}\mathfrak{g}_{m,Z}^{b,\mathrm{tor}} = \bigsqcup_{\tilde{Z}} \widehat{\mathrm{I}}\mathfrak{g}_{m,\tilde{Z}}^{b,\mathrm{tor}}$$

where \tilde{Z} runs over Igusa cusp labels at level $\Gamma_b(p^m)K^p(N)$ above Z .

(2) Let \tilde{Z} be an Igusa cusp label at level $\Gamma_b(p^m)K^p(N)$ above Z . Fix a symplectic \mathcal{O}_F -linear splitting

$$\mathbb{X}_{m,b} = \bigoplus_{i=-2}^0 \mathrm{Gr}_i^{Z_{m,b}},$$

cf. Proposition 2.2.1, and as usual a splitting of Z_N . The formal scheme $\widehat{\mathrm{I}}\mathfrak{g}_{m,\tilde{Z}}^{b,\mathrm{tor}}$ can be described as follows:

Consider the diagram

$$\begin{array}{ccc} \Xi_Z & \hookrightarrow & \Xi_{Z,\Sigma_Z} \\ \downarrow & \swarrow & \\ C_Z & & \\ \downarrow & & \\ \mathcal{S}_Z & & \end{array}$$

describing $\widehat{\mathcal{F}}_Z^{\mathrm{tor}}$. Then there is an abelian variety

$$C_{\tilde{Z}} \rightarrow \mathrm{Ig}_{Z,m}^b$$

over the level- p^m -Igusa variety over the leaf \mathcal{C}_Z^b of \mathcal{S}_Z , and a commutative diagram

$$\begin{array}{ccc} C_{\tilde{Z}} & \longrightarrow & C_Z \\ \downarrow & & \downarrow \\ \mathrm{Ig}_{Z,m}^b & \longrightarrow & \mathcal{S}_Z. \end{array}$$

Let $\Xi_{\tilde{Z},\Sigma_Z} \rightarrow C_{\tilde{Z}}$ be the pullback of $\Xi_{Z,\Sigma_Z} \rightarrow C_Z$. Then there is an action of $\Gamma_{\tilde{Z}}$ on all objects, and letting $\mathfrak{X}_{\tilde{Z},\Sigma_Z}$ be the completion of $\Xi_{\tilde{Z},\Sigma_Z}$ at its toroidal boundary, there is an isomorphism

$$\widehat{\mathrm{I}}\mathfrak{g}_{m,\tilde{Z}}^{b,\mathrm{tor}} \cong \mathfrak{X}_{\tilde{Z},\Sigma_Z} / \Gamma_{\tilde{Z}}.$$

Proof. For the first part, note that in the notation of Theorem 3.2.6, over $\widehat{\mathrm{I}}\mathfrak{g}_{m,Z}^{b,\mathrm{tor}}$ one parametrizes in particular isomorphisms $\mathcal{H}_Z^\mu[p^m] \cong \mathbb{X}_b^\mu[p^m]$. Here \mathcal{H}_Z is the connected part of the p -divisible group of the Raynaud extension. Inside its multiplicative part, there is the p -divisible group $T[p^\infty] \subset \mathcal{H}_Z^\mu$. In particular, we get a multiplicative subspace

$$T[p^m] \cong \mathcal{H}_Z^\mu[p^m] \cong \mathbb{X}_b^\mu[p^m] \subset \mathbb{X}_b[p^m].$$

This is necessarily locally constant, and together with its dual defines a symplectic filtration

$$T[p^m] = Z_{m,b,-2} \subset Z_{m,b,-1} \subset \mathbb{X}_b[p^m].$$

Together with the identification $T[p^m] = Z_{m,b,-2}$, this defines the Igusa cusp label \tilde{Z} above Z (as T is the torus with cocharacter group X , so $T[p^m] = \text{Hom}(X, \mu_{p^m})$, and the identification $T[p^m] \cong Z_{m,b,-2}$ is Cartier dual to an identification $\text{Gr}_0^{Z_{m,b}} \cong X/p^m$).

If we fix the cusp label \tilde{Z} together with a symplectic \mathcal{O}_F -linear splitting

$$\delta_{m,b} : \mathbb{X}_b[p^m] \cong \bigoplus_{i=-2}^0 \text{Gr}_i^{Z_{m,b}},$$

then the data of an Igusa level- p^m -structure can be described as follows. Recall that over $C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^b$ we have the Raynaud extension

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

where T is the split torus with cocharacter group X and \mathcal{B} lies in the leaf \mathcal{C}_Z^b of the smaller Shimura variety \mathcal{S}_Z . In particular, $\mathcal{B}[p^\infty]$ carries a slope filtration; we let \mathcal{B}_i denote the graded pieces, with slopes λ_i .

Definition 3.3.13. *An Igusa level- p^m -structure on a $C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^b$ -scheme \mathcal{T} that is compatible with the Igusa cusp label \tilde{Z} and $\delta_{m,b}$ consists of the following data:*

- (1) *An isomorphism*

$$\rho_{\mathcal{B}^\mu, m} : \mathcal{B}^\mu[p^m] \times_{C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^b} \mathcal{T} \xrightarrow{\sim} (\text{Gr}_{-1}^{Z_{m,b}})^\mu[p^m] \times_k \mathcal{T}$$

compatible with the \mathcal{O}_F -action on both sides.

- (2) *A splitting*

$$\delta_m : \mathcal{B}^\mu[p^m] \times_{C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^b} \mathcal{T} \rightarrow \mathcal{H}_Z^\mu[p^m] \times_{C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^b} \mathcal{T}$$

compatible with the \mathcal{O}_F -action on both sides.

- (3) *For each $\lambda_i \in (0, 1)$, an isomorphism*

$$\rho_{i,m} : \mathcal{B}_i[p^m] \times_{C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^b} \mathcal{T} \xrightarrow{\sim} \mathbb{X}_i[p^m] \times_k \mathcal{T},$$

that commutes with the \mathcal{O}_F -action, and commutes with the isomorphisms $\mathcal{B}_i[p^m] \xrightarrow{\sim} (\mathcal{B}_j[p^m])^\vee$ and $\mathbb{X}_i[p^m] \xrightarrow{\sim} (\mathbb{X}_j[p^m])^\vee$ induced by polarizations for $\lambda_i + \lambda_j = 1$ up to an element of $(\mathbb{Z}/p^m\mathbb{Z})^\times(\mathcal{T})$ that is independent of i . The isomorphisms $\rho_{i,m}$ are required to lift fppf locally to $p^{m'}$ torsion for any $m' \geq m$.

Note that the Igusa cusp label itself defines an isomorphism $X/p^m \cong \text{Gr}_0^{Z_{m,b}}$, and thus $T[p^m] \cong Z_{m,b,-2}$ by Cartier duality. It is easy to see that Igusa level- p^m -structures compatible with \tilde{Z} and $\delta_{m,b}$ are representable by a scheme

$$C_{\tilde{Z}} \rightarrow C_Z \times_{\mathcal{S}_Z} \mathcal{C}_Z^b$$

that we will identify in a second. Moreover, a direct comparison of moduli problems, and Theorem 3.2.6, shows that

$$\widehat{\text{Ig}}_{m,\tilde{Z}}^{b,\text{tor}}$$

can be described in terms of the pullback of $\Xi_{Z,\Sigma_Z} \rightarrow C_Z$ to $C_{\tilde{Z}}$ in the way described in the statement. (In Theorem 3.2.6, one was taking the quotient by Γ_Z whereas here it is $\Gamma_{\tilde{Z}}$; this difference is accounted for by the isomorphism $T[p^m] \cong Z_{m,b,-2}$ that is part of the cusp label.)

It remains to identify $C_{\tilde{Z}}$. We note that the data in parts (1) and (3) precisely define an Igusa level structure of level p^m on \mathcal{B} . This shows that there is a natural map

$$C_{\tilde{Z}} \rightarrow C_Z \times_{\mathcal{S}_Z} \mathrm{Ig}_{Z,m}^b,$$

where $\mathrm{Ig}_{Z,m}^b \rightarrow \mathcal{C}_Z^b$ denotes the Igusa level- p^m -covering. Recall that $C_Z \rightarrow \mathcal{S}_Z$ parametrizes \mathcal{O}_F -linear Raynaud extensions $0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$ with a splitting $\mathcal{B}[N] \rightarrow \mathcal{G}[N]$; equivalently, C_Z is given by $\underline{\mathrm{Hom}}_{\mathcal{O}_F}(\frac{1}{N}X, \mathcal{B}^\vee)$. Parametrizing data of type (2) in addition amounts to a splitting of the extension

$$0 \rightarrow T[p^m] \rightarrow \mathcal{G}^\mu[p^m] \rightarrow \mathcal{B}^\mu[p^m] \rightarrow 0.$$

This means that $C_{\tilde{Z}} \rightarrow \mathrm{Ig}_{Z,m}^b$ is given by $\underline{\mathrm{Hom}}_{\mathcal{O}_F}(\frac{1}{N}X, (\mathcal{B}/\mathcal{B}^\mu[p^m])^\vee)$, which is an abelian scheme. (We note that this explicit description also implies that the natural map $C_{\tilde{Z}} \rightarrow C_Z \times_{\mathcal{S}_Z} \mathrm{Ig}_{Z,m}^b$ is finite étale.) \square

3.3.14. *Boundary strata in partial minimal compactifications of Igusa varieties.* As an application, we can describe the boundary strata of $\mathrm{Ig}_m^{b,*}$ in terms of Igusa cusp labels at level $\Gamma_b(p^m)K^p(N)$.

Theorem 3.3.15. *We have a decomposition into locally closed strata*

$$\mathrm{Ig}_m^{b,*} = \bigsqcup_{\tilde{Z}} \mathrm{Ig}_{\tilde{Z}}^b,$$

where \tilde{Z} runs over Igusa cusp labels at level $\Gamma_b(p^m)K^p(N)$.

If \tilde{Z} lies over a cusp label Z of level $K(N)$, then $\mathrm{Ig}_{\tilde{Z}}^b \cong \mathrm{Ig}_{Z,m}^b \rightarrow \mathcal{C}_Z^b$ is isomorphic to the Igusa variety of level p^m over the Oort central leaf \mathcal{C}_Z^b in the special fiber of the boundary stratum \mathcal{S}_Z .

Proof. Observe that $\mathrm{Ig}_m^{b,*} \rightarrow \mathcal{C}^{b,*}$ is the Stein factorization of $\mathrm{Ig}_m^{b,\mathrm{tor}} \rightarrow \mathcal{C}^{b,*}$: since the toroidal compactification is smooth, its Stein factorization factors over the normalization of $\mathcal{C}^{b,*}$ in Ig_m^b , and then the map is an isomorphism by Zariski's main theorem. The description of the boundary strata of $\mathrm{Ig}_m^{b,*}$ now follows from this observation, from Theorem 3.3.12, and from the same arguments as in the case of Shimura varieties using Fourier–Jacobi series, cf. e.g. [Lan13, Section 7]. (In fact, our situation is even simpler than the one in *loc. cit.* because $\mathrm{Ig}_m^{b,*}$ is affine.) \square

4. THE HODGE–TATE PERIOD MORPHISM ON COMPACTIFICATIONS

4.1. Statements. As before, fix a prime p that is unramified in the CM field F , and fix an integer $N \geq 3$ prime to p and a cone decomposition Σ level $K(N)$ as in Remark 2.5.6. We consider the infinite-level minimal and toroidal compactifications

$$\begin{aligned} \mathcal{S}_{K(p^\infty N)}^* &:= \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{*, \diamond}, \\ \mathcal{S}_{K(p^\infty N)}^{\text{tor}} &:= \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{\text{tor}, \diamond} \end{aligned}$$

as diamonds. They admit the Hodge–Tate period map

$$\begin{aligned} \pi_{\text{HT}}^* : \mathcal{S}_{K(p^\infty N)}^* &\rightarrow \mathcal{F}\ell, \\ \pi_{\text{HT}}^{\text{tor}} : \mathcal{S}_{K(p^\infty N)}^{\text{tor}} &\rightarrow \mathcal{F}\ell, \end{aligned}$$

where the flag variety $\mathcal{F}\ell$ is the adic space (or diamond) over \mathbb{Q}_p parametrizing totally isotropic F -linear subspaces over V .

Our goal in this section is to describe the fibers of these maps. For this, let C be any complete algebraically closed nonarchimedean extension of \mathbb{Q}_p with ring of integers \mathcal{O}_C and residue field k . Let $x \in \mathcal{F}\ell(C)$ be a point of the flag variety, which by [SW13, Theorem B] gives rise to a p -divisible group $\mathbb{X}_{\mathcal{O}_C}$ with G -structure over \mathcal{O}_C , equipped with an isomorphism $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ compatible with G -structures. Associated to the special fibre \mathbb{X}_k , we get the perfect Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$ equipped with its partial minimal and toroidal compactification $\mathfrak{I}\mathfrak{g}^{\mathbb{X}, *}, \mathfrak{I}\mathfrak{g}^{\mathbb{X}, \text{tor}}$.

Being perfect schemes, they admit unique flat deformations to formal schemes over $W(k)$, which we can then base change to \mathcal{O}_C . Let us denote simply by a subscript C the generic fibres of these formal schemes, which are perfectoid spaces over $\text{Spa } C$.

The main theorem of this section is the following:

Theorem 4.1.1. *There are natural maps*

$$\begin{aligned} \mathfrak{I}\mathfrak{g}_C^{\mathbb{X}, *} &\rightarrow (\pi_{\text{HT}}^*)^{-1}(x), \\ \mathfrak{I}\mathfrak{g}_C^{\mathbb{X}, \text{tor}} &\rightarrow (\pi_{\text{HT}}^{\text{tor}})^{-1}(x). \end{aligned}$$

They are open immersions of perfectoid spaces with the same rank-1-points; in fact, the target is the canonical compactification of the source.

In particular, this implies the following corollary:

Corollary 4.1.2. *There are natural Hecke-equivariant isomorphisms*

$$\begin{aligned} R\Gamma(\mathfrak{I}\mathfrak{g}_C^{\mathbb{X}, *}, \mathbb{F}_\ell) &\cong (R(\pi_{\text{HT}}^*)_* \mathbb{F}_\ell)_x, \\ R\Gamma(\mathfrak{I}\mathfrak{g}_C^{\mathbb{X}, \text{tor}}, \mathbb{F}_\ell) &\cong (R(\pi_{\text{HT}}^{\text{tor}})_* \mathbb{F}_\ell)_x, \end{aligned}$$

Moreover, the stalks at higher-rank points agree with the stalks at the corresponding rank 1 points.

Proof. This follows from Theorem 4.1.1 and the arguments in [CS17, Section 4.4]. \square

We start by giving an explicit description of the Hodge–Tate period morphism on boundary strata, in terms of toroidal boundary charts. Next, we construct maps from $\mathfrak{I}\mathfrak{g}_C^{\mathbb{X}, \text{tor}}$ into the fibre of the Hodge–Tate period map, by describing the map on toroidal boundary charts. We check that it is (essentially) an isomorphism by

checking that it has the correct geometric points; by general properties of (qcqs) diamonds, this is enough to imply the result.

In the final part of this section, we deduce a semiperversity result for $R\pi_{\text{HT}*}\mathbb{F}_\ell$.

4.2. The Hodge–Tate period morphism on boundary strata. Our first goal is to describe the Hodge–Tate period map explicitly on the toroidal compactification. We have the naive integral models

$$\mathcal{S}_{K(p^m N), \mathbb{Z}_p}^{\text{tor}}$$

of $\mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{\text{tor}}$ from Theorem 2.5.9, regarded as p -adic formal schemes. We can take the inverse limit

$$\mathcal{S}_{K(p^\infty N), \mathbb{Z}_p}^{\text{tor}} = \varprojlim_m \mathcal{S}_{K(p^m N), \mathbb{Z}_p}^{\text{tor}}$$

as a p -adic formal scheme. Fix a cusp label Z of level $K(N)$. We can take the fibre product of formal schemes

$$\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Z}_p}^{\text{tor}} = \mathcal{S}_{K(p^\infty N), \mathbb{Z}_p}^{\text{tor}} \times_{\mathcal{S}_{K(N)}^{\text{tor}}} \widehat{\mathcal{F}}_{K(N), Z}^{\text{tor}}$$

with the formal completion at the boundary stratum corresponding to the cusp Z . The cusp Z defines a rational parabolic subgroup $P = P_r \subset G$ (up to conjugation) depending on the \mathcal{O}_F -rank r of the \mathcal{O}_F -lattice X implicit in the cusp label Z , and the projection to the cusp labels at p defines a natural map

$$\pi_1 : \widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Z}_p}^{\text{tor}} \rightarrow (G/P)(\mathbb{Q}_p) ;$$

here, $(G/P)(\mathbb{Q}_p)$ parametrizes symplectic \mathcal{O}_F -linear filtrations

$$\mathbb{Z}_{p^\infty, -2} \subset \mathbb{Z}_{p^\infty, -1} \subset L \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

where the $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -rank of $\mathbb{Z}_{p^\infty, -2}$ is r . In particular, over $\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Z}_p}^{\text{tor}}$, we have a natural local system of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -lattices $L_Z := \text{Gr}_{-1}^{\mathbb{Z}_{p^\infty}}$ with a perfect alternating form. On the other hand, there is a natural projection

$$\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Z}_p}^{\text{tor}} \rightarrow \mathcal{S}_{Z, K(N), \mathbb{Z}_p}$$

remembering only the base \mathcal{B} of the Raynaud extension, with its extra structure. Moreover, over $\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Z}_p}^{\text{tor}}$, we have compatible maps $L_Z/p^m \rightarrow \mathcal{B}[p^m]$ for varying m , inducing level structures on the generic fibre.

Let

$$\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}} = \widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Z}_p}^{\text{tor}} \times_{\text{Spa } \mathbb{Z}_p} \text{Spa } \mathbb{Q}_p$$

be the generic fibre, as a (pre-)adic space over \mathbb{Q}_p (where we regard any formal scheme as a (pre-)adic space via the natural fully faithful functor). Its associated diamond is an open subspace

$$(\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}})^\diamond \subset \mathcal{S}_{K(p^\infty N)}^{\text{tor}}$$

whose C -valued points are exactly those that specialize into the cusp Z . The Hodge–Tate filtration of \mathcal{B} now defines a totally isotropic subspace

$$\text{Lie } \mathcal{B}(1) \otimes \mathcal{O}_{(\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}})^\diamond} \hookrightarrow L_Z \otimes_{\mathbb{Z}_p} \mathcal{O}_{(\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}})^\diamond}.$$

Pulling this filtration back to

$$\mathbb{Z}_{p^\infty, -1} \otimes_{\mathbb{Z}_p} \mathcal{O}_{(\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}})^\diamond}$$

then defines a totally isotropic subspace of

$$L \otimes_{\mathbb{Z}} \mathcal{O}_{(\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}})^{\diamond}},$$

yielding a map

$$\pi_{\text{HT}, Z} : (\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}})^{\diamond} \rightarrow \mathcal{F}l.$$

Theorem 4.2.1. *The map $\pi_{\text{HT}, Z}$ constructed above agrees with the composite*

$$(\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}})^{\diamond} \rightarrow (\mathcal{S}_{K(p^\infty N), \mathbb{Q}_p}^{\text{tor}})^{\diamond} \xrightarrow{\pi_{\text{HT}}^{\text{tor}}} \mathcal{F}l.$$

Proof. For points away from the boundary, this follows from [Sch15, Proposition 3.3.1] (as the forgetful map to the Siegel modular variety is injective on flag varieties). We deduce the general case by continuity: The locus where the two maps agree is closed. If they do not agree, they differ on some quasicompact open subspace

$$U \subset |(\widehat{\mathcal{F}}_{K(p^\infty N), Z, \mathbb{Q}_p}^{\text{tor}})^{\diamond}| \subset |(\mathcal{S}_{K(p^\infty N)}^{\text{tor}})^{\diamond}|,$$

necessarily contained in the boundary. Note that U is necessarily the preimage of some quasicompact open subspace $U_m \subset |\mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{\text{tor}}|$ for m large enough, where $U \rightarrow U_m$ will be surjective, so that U_m is contained in the boundary. But there are no open subspaces of $\mathcal{S}_{K(p^m N), \mathbb{Q}_p}^{\text{tor}}$ contained in the boundary. \square

We can now also describe the restriction of π_{HT}^* to the boundary of the minimal compactification $\mathcal{S}_{K(p^\infty N)}^*$. We have a set-theoretic decomposition

$$\mathcal{S}_{K(p^\infty N)}^* = \bigsqcup_{\widehat{Z}} \mathcal{S}_{K(p^\infty N), \widehat{Z}}$$

in terms of cusp labels at level $K(p^\infty N)$ on the generic fiber, defined by taking the inverse limit of the corresponding decomposition on the level of schemes.

Corollary 4.2.2. *The restriction of $\pi_{\text{HT}}^* : \mathcal{S}_{K(p^\infty N)}^* \rightarrow \mathcal{F}l$ to $\mathcal{S}_{K(p^\infty N), \widehat{Z}}$ agrees with the composition*

$$\mathcal{S}_{K(p^\infty N), \widehat{Z}} \rightarrow \mathcal{F}l_{\widehat{Z}} \hookrightarrow \mathcal{F}l,$$

where the first morphism is the Hodge–Tate period morphism for the smaller Shimura variety $\mathcal{S}_{K(p^\infty N), \widehat{Z}}$.

4.3. Construction of the map. In this subsection, fix a complete algebraically closed nonarchimedean field C over \mathbb{Q}_p with ring of integers \mathcal{O}_C and residue field k , and fix a point $x \in \mathcal{F}l(C)$ of the flag variety. This corresponds to a pair $(\mathbb{X}_{\mathcal{O}_C}, \alpha)$ consisting of a p -divisible group $\mathbb{X}_{\mathcal{O}_C}$ with G -structure over \mathcal{O}_C and a trivialization $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ compatible with G -structures.

Note that the filtration $\mathbb{X}_{\mathcal{O}_C}^{\mu} \subset \mathbb{X}_{\mathcal{O}_C}^{\circ} \subset \mathbb{X}_{\mathcal{O}_C}$ induces by transport of structure via α a symplectic \mathcal{O}_F -linear filtration of $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$; we fix a symplectic splitting δ of this filtration. Transporting it back via α , this induces an \mathcal{O}_F -linear splitting

$$\delta_{\mathbb{X}_{\mathcal{O}_C}} : \mathbb{X}_{\mathcal{O}_C} \cong \mathbb{X}_{\mathcal{O}_C}^{\mu} \oplus \mathbb{X}_{\mathcal{O}_C}^{(0,1)} \oplus \mathbb{X}_{\mathcal{O}_C}^{\text{ét}}$$

under which the polarization similarly decomposes into a direct sum.

Let $\mathbb{X} = \mathbb{X}_k$ be the special fiber of $\mathbb{X}_{\mathcal{O}_C}$, which comes with an induced splitting $\delta_{\mathbb{X}}$, and fix a section $k \rightarrow \mathcal{O}_C/p$.

Proposition 4.3.1. *There exists $\epsilon \in \mathbb{Q}$, $1 \geq \epsilon > 0$ such that there exists an isomorphism*

$$\rho : \mathbb{X} \times_k \mathcal{O}_C/p^\epsilon \cong \mathbb{X}_{\mathcal{O}_C} \times_{\mathcal{O}_C} \mathcal{O}_C/p^\epsilon$$

of p -divisible groups with G -structures, lifting the identity. Moreover, we can arrange that $\delta_{\mathbb{X}_{\mathcal{O}_C}} \times_{\mathcal{O}_C} \mathcal{O}_C/p^\epsilon = \delta_{\mathbb{X}} \times_k \mathcal{O}_C/p^\epsilon$.

Proof. Consider the p -divisible group $X := \mathbb{X}_{\mathcal{O}_C} \times_{\mathcal{O}_C} \mathcal{O}_C/p$ equipped with the induced extra structures. Its associated Dieudonné module is a finite projective A_{crys} -module equipped with a Frobenius; let (M, ϕ) be the corresponding ϕ -module over $B_{\text{crys}}^+ = A_{\text{crys}}[\frac{1}{p}]$. Let (M_0, ϕ_0) be similarly the Dieudonné module associated with $\mathbb{X} \times_k \mathcal{O}_C/p$. By [SW13, Theorem A], it suffices to find an isomorphism $(M, \phi) \cong (M_0, \phi_0)$ of G -Dieudonné modules reducing to the identity on $W(k)[\frac{1}{p}]$ (indeed, then the associated quasi-isogeny will be an isomorphism over k , and thus over \mathcal{O}_C/p^ϵ for ϵ small enough). It is in fact enough to find any such isomorphism, as we can always change it by an automorphism of (M_0, ϕ_0) .

Now [Far15, Théorème 5.1] classifies G -Dieudonné modules over B_{crys}^+ , as they are equivalent to G -bundles on the Fargues-Fontaine curve, and in particular shows that they are determined up to isomorphism by their restriction to $W(k)[\frac{1}{p}]$, cf. [Far15, Théorème 5.6].

For the final statement, note that we can also apply the argument individually to all three summands (identifying the outer ones as Cartier duals) and then pass to the direct sum. \square

We consider the perfect toroidally compactified Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}, \text{tor}}$ associated with \mathbb{X} . By construction, this comes with a map

$$f^{\text{tor}} : \mathfrak{I}\mathfrak{g}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}^{\text{tor}}$$

which factors through the partial toroidal compactification $\mathcal{C}^{\mathbb{X}, \text{tor}}$ of the leaf corresponding to \mathbb{X} .

We can base change everything from k to \mathcal{O}_C/p^ϵ and express everything in terms of $\mathbb{X}_{\mathcal{O}_C} \times_{\mathcal{O}_C} \mathcal{O}_C/p^\epsilon$ using ρ . Our goal in this section is to lift the morphism

$$f_\epsilon^{\text{tor}} : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p^\epsilon}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{\mathcal{O}_C/p^\epsilon}^{\text{tor}}$$

to a morphism of p -adic formal schemes. We first define the formal schemes: we set

$$\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}} := W(\mathfrak{I}\mathfrak{g}^{\mathbb{X}, \text{tor}}) \times_{W(k)} \mathcal{O}_C,$$

where $W(\mathfrak{I}\mathfrak{g}^{\mathbb{X}, \text{tor}})$ is the p -adic formal scheme over $\text{Spf } W(k)$ obtained by taking an affine cover of $\mathfrak{I}\mathfrak{g}^{\mathbb{X}, \text{tor}}$, applying Witt vectors, and gluing. This is also the unique flat formal lift of $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p^\epsilon}^{\mathbb{X}, \text{tor}}$ to \mathcal{O}_C . On the other hand, we have the p -adic formal scheme $\mathcal{S}_{\mathcal{O}_C}^{\text{tor}}$, and we want to construct a map

$$f_{\mathcal{O}_C}^{\text{tor}} : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{\mathcal{O}_C}^{\text{tor}},$$

and in fact a lift to

$$g^{\text{tor}} : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C}^{\text{tor}}.$$

Let us recall the construction away from the boundary: Over $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p^\epsilon}^{\mathbb{X}}$, we have an abelian scheme $\mathcal{A}_{\mathcal{O}_C/p^\epsilon}$ with extra structures together with an isomorphism of p -divisible groups with G -structures

$$(4.3.1) \quad \mathcal{A}_{\mathcal{O}_C/p^\epsilon}[p^\infty] \xrightarrow{\sim} \mathbb{X}_{\mathcal{O}_C} \times_{\mathcal{O}_C} \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p^\epsilon}^{\mathbb{X}}.$$

By Serre-Tate theory, the abelian scheme $\mathcal{A}_{\mathcal{O}_C/p^\epsilon}$ with G -structure lifts uniquely to an abelian scheme $\mathcal{A}_{\mathcal{O}_C}$ with G -structure over $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}}$ equipped with an isomorphism

$$\mathcal{A}_{\mathcal{O}_C}[p^\infty] \xrightarrow{\sim} \mathbb{X}_{\mathcal{O}_C} \times_{\mathcal{O}_C} \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}}$$

of p -divisible groups with G -structures. This induces the morphism of formal schemes

$$f : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}} \rightarrow \mathcal{S}_{\mathcal{O}_C},$$

such that $\mathcal{A}_{\mathcal{O}_C}$ can be identified with the pullback under f of the universal abelian scheme over $\mathcal{S}_{\mathcal{O}_C}$. Now the composition of $(T_p$ applied to) the isomorphism $\mathcal{A}_{\mathcal{O}_C}[p^\infty] \cong \mathbb{X}_{\mathcal{O}_C}$ with $\alpha : T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ gives a full level structure on the generic fibre of $\mathcal{A}_{\mathcal{O}_C}$, and thus (via the definition of integral models via normalization and the normality of Igusa varieties) a lift to a map

$$g : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C}.$$

Our goal now is to extend this to toroidal compactifications.

Theorem 4.3.2. *The morphism g constructed above extends to a morphism of p -adic formal schemes*

$$g^{\text{tor}} : \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C}^{\mathbb{X}, \text{tor}} \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C}^{\text{tor}}.$$

Proof. We will first construct analogues of the morphism g for boundary strata, then glue. Fix a cusp label $Z = (Z_N, X)$ at level N .

First, we recall the description of

$$\widehat{\mathfrak{I}\mathfrak{g}}_Z^{\mathbb{X}, \text{tor}}$$

from Theorem 3.2.13. By Proposition 3.2.11, over $C_Z^{\mathfrak{J}\mathfrak{g}, \mathbb{X}}$, we have a Raynaud extension

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

together with an \mathcal{O}_F -linear embedding

$$\rho : \mathcal{G}[p^\infty] \hookrightarrow \mathbb{X}$$

with étale quotient, and that is compatible with polarizations.

Now we lift these structures to \mathcal{O}_C . By base change, everything exists over \mathcal{O}_C/p^ϵ . The injection $\mathcal{G}[p^\infty] \hookrightarrow \mathbb{X}_b$ deforms uniquely to an injection $\mathcal{G}[p^\infty]_{\mathcal{O}_C} \hookrightarrow \mathbb{X}_{\mathcal{O}_C}$ as the quotient is étale. By Serre-Tate theory, this induces a unique deformation of the Raynaud extension compatibly with the deformation of $\mathcal{G}[p^\infty]$, yielding a Raynaud extension

$$0 \rightarrow T \rightarrow \mathcal{G}_{\mathcal{O}_C} \rightarrow \mathcal{B}_{\mathcal{O}_C} \rightarrow 0$$

over $C_{Z, \mathcal{O}_C}^{\mathfrak{J}\mathfrak{g}, \mathbb{X}}$ with an injection $\rho_{\mathcal{O}_C} : \mathcal{G}_{\mathcal{O}_C}[p^\infty] \hookrightarrow \mathbb{X}_{\mathcal{O}_C}$, which is \mathcal{O}_F -linear and compatible with polarizations after taking the quotient by the isotropic subspace $T[p^\infty]$.

We also need to describe the lift of the $\mathbb{S}_{Z, \text{perf}}$ -torsor

$$\Xi_Z^{\mathfrak{J}\mathfrak{g}, \mathbb{X}} \rightarrow C_Z^{\mathfrak{J}\mathfrak{g}, \mathbb{X}}.$$

To do this, we fix a symplectic \mathcal{O}_F -linear splitting $\delta_{\mathbb{X}_{\mathcal{O}_C}, Z}$ of the filtration $Z_{\mathbb{X}_{\mathcal{O}_C}}$ of $\mathbb{X}_{\mathcal{O}_C}$ over $C_{Z, \mathcal{O}_C}^{\mathfrak{J}\mathfrak{g}, \mathbb{X}}$ given by

$$T[p^\infty] \subset \mathcal{G}_{\mathcal{O}_C}[p^\infty] \subset \mathbb{X}_{\mathcal{O}_C},$$

compatible with the splitting $\delta_{\mathbb{X}_{\mathcal{O}_C}}$ chosen above. Equivalently, we only have to choose a splitting on étale parts (it will induce a similar splitting on multiplicative parts, which together with the splitting $\delta_{\mathbb{X}}$ induces the desired splitting). Note that we automatically have

$$\delta_{\mathbb{X}_{\mathcal{O}_C}, Z} \times_{\mathcal{O}_C} \mathcal{O}_C/p^\epsilon = (\delta_{\mathbb{X}_{\mathcal{O}_C}, Z} \times_{\mathcal{O}_C} k) \times_k \mathcal{O}_C/p^\epsilon$$

as this holds for $\delta_{\mathbb{X}_{\mathcal{O}_C}}$ by Proposition 4.3.1, and splittings of the étale parts are rigid.

Using the splitting $\delta_{\mathbb{X}_{\mathcal{O}_C}, Z}$ and $\rho_{\mathcal{O}_C}$, we get a splitting of

$$0 \rightarrow T[p^\infty] \rightarrow \mathcal{G}_{\mathcal{O}_C}[p^\infty] \rightarrow \mathcal{B}_{\mathcal{O}_C}[p^\infty] \rightarrow 0$$

and thus a map $\tilde{f}_0 : X[\frac{1}{p}] \rightarrow \mathcal{B}_{\mathcal{O}_C}$. Then the lift

$$\Xi_{Z, \mathcal{O}_C}^{\mathcal{J}\mathfrak{g}, \mathbb{X}} \rightarrow C_{Z, \mathcal{O}_C}^{\mathcal{J}\mathfrak{g}, \mathbb{X}}$$

of the $\mathbb{S}_{Z, \text{perf}}$ -torsor

$$\Xi_Z^{\mathcal{J}\mathfrak{g}, \mathbb{X}} \rightarrow C_Z^{\mathcal{J}\mathfrak{g}, \mathbb{X}}$$

parametrizes symmetric lifts $\tilde{f} : X[\frac{1}{p}] \rightarrow \mathcal{G}_{\mathcal{O}_C}$ of \tilde{f}_0 . Indeed, this describes a flat lift of this relatively perfect scheme, and over \mathcal{O}_C/p^ϵ , it agrees with the description from Proposition 3.2.12, using critically that $\delta_{\mathbb{X}_{\mathcal{O}_C}}$ is constant modulo \mathcal{O}_C/p^ϵ .

Now fix any $m \geq 0$. Note that $\widehat{\mathcal{J}\mathfrak{g}}_Z^{b, \text{tor}}$ decomposes into open and closed subsets $\widehat{\mathcal{J}\mathfrak{g}}_{\tilde{Z}}^{b, \text{tor}}$ according to Igusa cusp labels \tilde{Z} of level $\Gamma_b(p^m)K^p(N)$; concretely, this is given by the symplectic \mathcal{O}_F -stable filtration $T[p^m] \subset Z_{b, -1}[p^m] \subset \mathbb{X}[p^m]$ above. Note that $\mathbb{X}_{\mathcal{O}_C}$ and the isomorphism $\mathbb{X}_{\mathcal{O}_C}[p^m](C) \cong L/p^m$ induce a map from Igusa cusp labels to usual cusp labels. For an Igusa cusp label \tilde{Z} of level $\Gamma_b(p^m)K^p(N)$, let by abuse of notation \tilde{Z} also denote the corresponding usual cusp label of level $K(p^m N)$. We want to construct maps

$$\widehat{\mathcal{J}\mathfrak{g}}_{\tilde{Z}}^{b, \text{tor}} \rightarrow \widehat{\mathcal{S}}_{K(p^m N), \tilde{Z}}^{\text{tor}}.$$

For this, we want to use the explicit description of the right-hand side, and so choose a symplectic splitting $L/p^m \cong \bigoplus_{i=-2}^0 \text{Gr}_i^{Z_{p^m}}$ compatible with splitting δ chosen in the beginning of this section. (Also choose a similar splitting on L/N ; we will in the following ignore the discussion of level structures away from p as they are identical in the two setups.) By transfer of structure along $\mathbb{X}_{\mathcal{O}_C}[p^m](C) \cong L/p^m$, this induces a splitting $\text{Gr}_{-1}^{Z_{m,b}} \rightarrow Z_{m,b,-1}$, which then induces a splitting of

$$0 \rightarrow T[p^m] \rightarrow \mathcal{G}_{\mathcal{O}_C}[p^m] \rightarrow \mathcal{B}_{\mathcal{O}_C}[p^m] \rightarrow 0.$$

This gives a map to the abelian variety $C_{\tilde{Z}}$ in the description of $\widehat{\mathcal{S}}_{K(p^m N), \tilde{Z}}^{\text{tor}}$. Moreover, this splitting also induces a lift of the map $f_0 : X \rightarrow \mathcal{B}_{\mathcal{O}_C}$ corresponding to the Raynaud extension to a map $f_m : \frac{1}{p^m} X \rightarrow \mathcal{B}_{\mathcal{O}_C}$. By the description of the torus torsor

$$\Xi_{Z, \mathcal{O}_C}^{\mathcal{J}\mathfrak{g}, \mathbb{X}} \rightarrow C_{Z, \mathcal{O}_C}^{\mathcal{J}\mathfrak{g}, \mathbb{X}}$$

above, we get a map from $\Xi_{Z, \mathcal{O}_C}^{\mathcal{J}\mathfrak{g}, \mathbb{X}}$ to the torsor of lifts of f_m to $\mathcal{G}_{\mathcal{O}_C}$ over $C_{\tilde{Z}}$, and then also a map of torus compactifications. The given structures also evidently define a pre-level structure on $\mathcal{B}_{\mathcal{O}_C}[p^m]$ (as in the definition of $\mathcal{S}_K^{\text{pre}}$) but then by

normality of $\widehat{\mathfrak{I}\mathfrak{g}}_{Z, \mathcal{O}_C}^{b, \text{tor}}$ a level structure. In the inverse limit, these structures define a map

$$\widehat{\mathfrak{I}\mathfrak{g}}_{Z, \mathcal{O}_C}^{b, \text{tor}} \rightarrow \widehat{\mathcal{S}}_{K(p^\infty N), Z, \mathcal{O}_C}^{\text{tor}},$$

as desired.

It remains to show that these maps on toroidal boundary charts glue. We argue modulo any power p^M of p . Take any small enough

$$\text{Spf } R \subset \widehat{\mathfrak{I}\mathfrak{g}}_{Z, \mathcal{O}_C/p^M}^{b, \text{tor}}$$

and let $U_R \subset \text{Spec } R$ be the complement of the boundary. We get two induced maps $U_R \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C/p^M}$: One via the map on toroidal boundary charts, the other by mapping $U_R \rightarrow \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p^M}^b$ to the open Igusa variety first, and then using the map g above. We need to see that the two maps agree. Note that the first map also factors naturally over $\mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p^M}^b \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C/p^M}$: Indeed, the abelian variety A over U_R will come with an isomorphism $A[p^\infty]^\circ \cong \mathbb{X}_{\mathcal{O}_C}^\circ$ which is \mathcal{O}_F -linear and compatible with polarizations after taking the quotient by the multiplicative part; moreover, the chosen splitting δ (and thus $\delta_{\mathbb{X}_{\mathcal{O}_C}}$) induces an \mathcal{O}_F -linear symplectic splitting

$$A[p^\infty] \cong A[p^\infty]^\mu \oplus A[p^\infty]^{(0,1)} \oplus A[p^\infty]^{\text{ét}}.$$

Now this comes with the desired isomorphism to $\mathbb{X}_{\mathcal{O}_C}$, giving the lift $U_R \rightarrow \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p^M}^b$ of the first map $U_R \rightarrow \mathcal{S}_{K(p^\infty N), \mathcal{O}_C/p^M}$. At this point, we have to show that two maps $U_R \rightarrow \mathfrak{I}\mathfrak{g}_{\mathcal{O}_C/p^M}^b$ agree, where we note that all schemes are flat lifts of schemes that are relatively perfect over \mathcal{O}_C/p . As thus the cotangent complex vanishes, it suffices to see that these maps agree over \mathcal{O}_C/p^ϵ . But in that case, everything is simply the base change from k , so we get the desired result. \square

4.4. The fibers of the toroidal compactification.

Theorem 4.4.1. *The morphism g^{tor} constructed in Theorem 4.3.2 induces a map of diamonds*

$$\mathfrak{I}\mathfrak{g}_C^{b, \text{tor}} \xrightarrow{\sim} (\pi_{\text{HT}}^{\text{tor}})^{-1}(x)$$

that is an open immersion with the same rank 1 points. As the target is partially proper over $\text{Spa}(C, \mathcal{O}_C)$, this implies that it is the canonical compactification of $\mathfrak{I}\mathfrak{g}_C^{b, \text{tor}}$, and itself affinoid perfectoid.

Proof. By Lemma 4.4.2 below, it is enough to check that it induces a bijection on (C, \mathcal{O}_C) -valued points (for all possibly larger C). Note that on both sides, the C -valued points decompose according to Igusa cusp labels of level $K^p(N)$ (using Theorem 4.2.1 for the right-hand side) via looking at the induced k -valued point, and the map is by construction compatible with this decomposition. Thus, fix an Igusa cusp label \tilde{Z} of level $K^p(N)$. In particular, we fix a symplectic \mathcal{O}_F -stable filtration

$$Z_{b, -2} \subset Z_{b, -1} \subset \mathbb{X}_{\mathcal{O}_C}$$

with $Z_{b, -2} \cong \text{Hom}(X, \mu_{p^\infty})$ a trivialized multiplicative group. Moreover, we fix an \mathcal{O}_F -linear symplectic splitting δ_b of this filtration. Let

$$Z_{p^\infty, -2} \subset Z_{p^\infty, -1} \subset L \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

be the induced filtration (with induced splitting), using $T_p(\mathbb{X}_{\mathcal{O}_C}) \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

On the left-hand side, we parametrize now in addition the following data:¹⁵

- (1) A principally polarized abelian variety \mathcal{B} over \mathcal{O}_C with \mathcal{O}_F -action as usual.
- (2) An \mathcal{O}_F -linear extension

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

where T is a torus with cocharacter group X . This induces in particular an \mathcal{O}_F -linear map $f_0 : X \rightarrow \mathcal{B}$.

- (3) An isomorphism

$$\mathcal{G}[p^\infty] \cong Z_{b,-1}$$

compatible with the $T[p^\infty] = Z_{b,-2}$. By the splitting δ_b , this induces an \mathcal{O}_F -linear splitting of

$$0 \rightarrow T[p^\infty] \rightarrow \mathcal{G}[p^\infty] \rightarrow \mathcal{B}[p^\infty] \rightarrow 0$$

and in particular f_0 extends to a map $\tilde{f}_0 : X[\frac{1}{p}] \rightarrow \mathcal{B}$.

- (4) Away from the boundary, a lift of \tilde{f}_0 to an \mathcal{O}_F -map $\tilde{f} : X[\frac{1}{p}] \rightarrow \mathcal{G}(C)$ that is symmetric and positive. In general, we have a torsor \mathcal{P} over \mathcal{O}_C for the torus $\mathbb{S}_{\bar{Z}}$ with cocharacter group generated by $\frac{1}{p^\infty}X \otimes X$, parametrizing symmetric lifts of \tilde{f}_0 to $X[\frac{1}{p}] \rightarrow \mathcal{G}$, and the cone decomposition Σ_Z gives rise to an embedding $\mathcal{P} \subset \mathcal{P}_{\Sigma_Z}$. Then the final datum is an \mathcal{O}_C -point of \mathcal{P}_{Σ_Z} whose special fibre lies in the boundary.

On the right-hand side, we parametrize the following data:

- (1') A principally polarized abelian variety \mathcal{B} over \mathcal{O}_C with \mathcal{O}_F -action as usual, and an \mathcal{O}_F -linear symplectic isomorphism

$$T_p(\mathcal{B}) \cong \mathrm{Gr}_{-1}^{Z_{p^\infty}},$$

that induces an isomorphism

$$\mathcal{B}[p^\infty] \cong \mathrm{Gr}_{-1}^{Z_b},$$

by Theorem 4.2.1 (and [SW13, Theorem B]).

- (2') An \mathcal{O}_F -linear extension

$$0 \rightarrow T \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

where T is a torus with cocharacter group X . This induces in particular an \mathcal{O}_F -linear map $f_0 : X \rightarrow \mathcal{B}$.

- (3') An \mathcal{O}_F -linear splitting of

$$0 \rightarrow T[p^\infty] \rightarrow \mathcal{G}[p^\infty] \rightarrow \mathcal{B}[p^\infty] \rightarrow 0;$$

in particular, this induces a lift of f_0 to $\tilde{f}_0 : X[\frac{1}{p}] \rightarrow \mathcal{B}$.

- (4') Away from the boundary, a lift of \tilde{f}_0 to a symmetric positive $\tilde{f} : X[\frac{1}{p}] \rightarrow \mathcal{G}(C)$. In general, we have a torsor \mathcal{P}' over \mathcal{O}_C for the torus $\mathbb{S}_{\bar{Z}}$ with cocharacter group generated by $\frac{1}{p^\infty}X \otimes X$, parametrizing symmetric lifts of \tilde{f}_0 to $X[\frac{1}{p}] \rightarrow \mathcal{G}$, and the cone decomposition Σ_Z gives rise to an embedding $\mathcal{P}' \subset \mathcal{P}'_{\Sigma_Z}$. Then the final datum is an \mathcal{O}_C -point of \mathcal{P}'_{Σ_Z} whose special fibre lies in the boundary.

¹⁵We ignore the level- N -structures away from p .

It is clear that (2) and (2') as well as (4) and (4') correspond. But (3) is equivalent to a splitting of $0 \rightarrow T[p^\infty] \rightarrow \mathcal{G}[p^\infty] \rightarrow \mathcal{B}[p^\infty] \rightarrow 0$ plus an isomorphism $\mathcal{B}[p^\infty] \cong \mathrm{Gr}_{-1}^{\mathbb{Z}_b}$, i.e. (3') and the extra part of (1'), as desired. \square

Lemma 4.4.2. *Let $f : X \rightarrow Y$ be a map from a quasicompact separated perfectoid space X to a proper diamond Y over $\mathrm{Spd} C$ for some complete algebraically closed extension C of \mathbb{Q}_p . Assume that for all complete algebraically closed C'/C , the map $X(C', \mathcal{O}_{C'}) \rightarrow Y(C', \mathcal{O}_{C'})$ is a bijection. Then f induces an isomorphism $\overline{X} \rightarrow Y$ where \overline{X} is the canonical compactification of X over $\mathrm{Spd} C$, in the sense of [Sch17, Proposition 18.6]. In particular, if $X/\mathrm{Spd} C$ is compactifiable, then f is an open immersion.*

Proof. We may replace X by \overline{X} and assume that $X/\mathrm{Spd} C$ is also proper. Then f induces bijections on (C', C'^+) -valued points for general $C'^+ \subset C'$ containing \mathcal{O}_C (as these agree with the $(C', \mathcal{O}_{C'})$ -valued points by properness). Now the result follows from [Sch17, Lemma 11.11]. \square

4.5. The fibers of the minimal compactification.

Theorem 4.5.1. *The morphism f^{tor} induces an open immersion*

$$f^* : \mathfrak{I}\mathfrak{g}_C^{b,*} \xrightarrow{\sim} (\pi_{\mathrm{HT}}^*)^{-1}(x).$$

of affinoid perfectoid spaces with the same rank 1 points.

Proof. The fact that $\mathfrak{I}\mathfrak{g}_C^{b,*}$ is affinoid perfectoid follows from Lemma 3.3.8. By [Sch15, Theorem 4.1.1], we know that $(\pi_{\mathrm{HT}}^*)^{-1}(x)$ is an affinoid perfectoid space. By [BS19, Theorem 1.16], we see that $(\pi_{\mathrm{HT}}^*)^{-1}(x)$ is affinoid perfectoid as well. The theorem follows from Theorem 4.4.1 after taking global sections, and noting the results of Lemma 4.5.2 below. \square

Lemma 4.5.2.

- (1) *The canonical morphism $\mathfrak{I}\mathfrak{g}_C^{b,\mathrm{tor}} \rightarrow \mathfrak{I}\mathfrak{g}_C^{b,*}$ induces an isomorphism on global sections.*
- (2) *The canonical morphism $\mathcal{F}^{\mathrm{tor}} := (\pi_{\mathrm{HT}}^{\mathrm{tor}})^{-1}(x) \rightarrow \mathcal{F}^* := (\pi_{\mathrm{HT}}^*)^{-1}(x)$ induces an isomorphism on global sections.*

Proof. The first part is clear (we know it over k , thus over \mathcal{O}_C/p^ϵ by base change, then we can lift to \mathcal{O}_C , and invert p).

For the second part, let $\pi : \mathcal{S}_{K(p^\infty N)}^{b,\mathrm{tor}} \rightarrow \mathcal{S}_{K(p^\infty N)}^{b,*}$ denote the canonical projection. It is enough to prove that the natural map of sheaves on $(\mathcal{F}^*)_{\acute{\mathrm{e}}\mathrm{t}}$

$$(4.5.1) \quad \mathcal{O}_{\mathcal{F}^*}^{+a}/p^n \rightarrow \pi_* \mathcal{O}_{\mathcal{F}^{\mathrm{tor}}}^{+a}/p^n$$

is an almost isomorphism. The result in the statement of the lemma follows from this by passing to the inverse limit over n , inverting p , and taking global sections.

We now claim that it is enough to prove that the natural map of sheaves on $(\mathcal{S}_{K(p^\infty N), C}^{b,*})_{\acute{\mathrm{e}}\mathrm{t}}$

$$(4.5.2) \quad \mathcal{O}_{\mathcal{S}_{K(p^\infty N), C}^{b,*}}^{+a}/p^n \rightarrow \pi_* \mathcal{O}_{\mathcal{S}_{K(p^\infty N), C}^{b,\mathrm{tor}}}^{+a}/p^n$$

is an almost isomorphism. Since the pullback functor ν^* from the étale site to the quasi-pro-étale site is fully faithful, cf. [Sch17, Proposition 14.8], it is enough to consider the pullbacks of the sheaves to the corresponding quasi-pro-étale sites.

The map $x \rightarrow \mathcal{F}l_C$ is quasi-pro-étale, since the point x can be identified with $\varprojlim \mathcal{U}$, where the limit runs over a cofinal set of quasi-compact open neighbourhoods \mathcal{U} of x in $\mathcal{F}l_{G,\mu}$. This implies that the immersions $\iota^{\text{tor}} : \mathcal{F}^{\text{tor}} \rightarrow \mathcal{S}_{K(p^\infty N),C}^{\text{tor}}$ and $\iota : \mathcal{F}^* \rightarrow \mathcal{S}_{K(p^\infty N),C}^{b,*}$ are quasi-pro-étale as well. Therefore, $(\mathcal{F}^{\text{tor}})_{\text{qproét}}$ is a slice category of $(\mathcal{S}_{K(p^\infty N)}^{b,\text{tor}})_{\text{qproét}}$ and we can identify the sheaves

$$(\iota^{\text{tor}})^*(\mathcal{O}_{\mathcal{S}_{K(p^\infty N)}^{\text{tor}}}^+ / p^n) = \mathcal{O}_{\mathcal{F}^{\text{tor}}}^+ / p^n$$

on $(\mathcal{F}^{\text{tor}})_{\text{qproét}}$. Similarly, we can identify the sheaves

$$\iota^*(\mathcal{O}_{\mathcal{S}_{K(p^\infty N),C}^+} / p^n) = \mathcal{O}_{\mathcal{F}^*}^+ / p^n$$

on $(\mathcal{F}^*)_{\text{qproét}}$. By the base change result in [Sch17, Corollary 16.9], we obtain a natural isomorphism of sheaves on $(\mathcal{F}^*)_{\text{qproét}}$.

$$\iota^* \circ \pi_* \mathcal{O}_{\mathcal{S}_{K(p^\infty N)}^{\text{tor}}}^+ / p^n \xrightarrow{\sim} \pi_* \circ (\iota^{\text{tor}})^* \mathcal{O}_{\mathcal{S}_{K(p^\infty N)}^{\text{tor}}}^+ / p^n.$$

We conclude that (4.5.2) implies (4.5.1).

It remains to prove the almost isomorphism (4.5.2). At each finite level $K(p^m N)$, we have a natural isomorphism $\mathbb{Z}/p^n \mathbb{Z} \xrightarrow{\sim} \pi_{K(p^m N),*}(\mathbb{Z}/p^n \mathbb{Z})$ of sheaves on $(\mathcal{S}_{K(p^m N)}^{b,*})_{\text{ét}}$, by base change, e.g. [Sch17, Corollary 16.10], and by the fact that the fibers of the Stein factorisation are geometrically connected. We tensor with $\mathcal{O}_{\mathcal{S}_{K(p^m N)}^{b,*}}^+ / p^n$ and apply the relative primitive comparison isomorphism in the form of [Sch13, Theorem 3.13]; this gives an almost isomorphism

$$\mathcal{O}_{\mathcal{S}_{K(p^m N)}^{b,*}}^{+a} / p^n \xrightarrow{\sim} \pi_{K(p^m N),*} \mathcal{O}_{\mathcal{S}_{K(p^m N)}^{\text{tor}}}^{+a} / p^n.$$

We conclude by taking a direct limit over the finite levels $K(p^m N)$. □

4.6. Semiperversity. Let C be any complete algebraically closed extension of \mathbb{Q}_p with ring of integers \mathcal{O}_C and residue field k .

Theorem 4.6.1. *Consider*

$$R\pi_{\text{HT}*}^{\circ} \mathbb{F}_\ell \in D(\mathcal{F}l_C, \mathbb{F}_\ell).$$

There is a cofinal system of formal models $\mathfrak{F}l$ of $\mathcal{F}l$ over \mathcal{O}_C such that the nearby cycles

$$R\psi(R\pi_{\text{HT}*}^{\circ} \mathbb{F}_\ell) \in {}^p D^{\geq d}(\mathfrak{F}l_k, \mathbb{F}_\ell)$$

are semiperverse, where $d = [F^+ : \mathbb{Q}]n^2$ is the dimension of the Shimura variety.

For the proof, we need the following lemma about passage to ℓ^∞ -level structures. Let

$$j_{N\ell^\infty} : \mathcal{S}_{K(N\ell^\infty),\mathbb{Q}} \rightarrow \mathcal{S}_{K(N\ell^\infty),\mathbb{Q}}^{\text{tor}}$$

denote the open immersion, where the inverse limit over levels $N\ell^m$ is taken in the category of schemes.

Lemma 4.6.2. *The natural map*

$$\mathbb{F}_\ell \rightarrow Rj_{N\ell^\infty,*} \mathbb{F}_\ell$$

is an isomorphism.

Proof. This is a standard consequence of the structure of toroidal compactifications, see for example [Pin92, §2.7]. For completeness, we sketch the argument: At each finite level, the boundary of the toroidal compactification is a divisor with normal crossings, so one can compute $Rj_{N\ell^m, *} \mathbb{F}_\ell$ explicitly: The $R^1 j_{N\ell^m, *} \mathbb{F}_\ell$ is at each point freely generated by the divisors passing through this point, and the higher $R^i j_{N\ell^m, *} \mathbb{F}_\ell$ are wedge powers. Going up the ℓ^∞ -tower, the boundary divisors get more and more ramified with transition maps of degree divisible by ℓ , so the transition maps are zero on $R^i j_{N\ell^m, *} \mathbb{F}_\ell$ for $i > 0$, giving the desired result in the colimit over m . \square

Similarly:

Lemma 4.6.3. *For any Igusa variety $\mathfrak{I}\mathfrak{g}^{\mathbb{X}}$, the restriction map*

$$H^i(\mathfrak{I}\mathfrak{g}_{K(N\ell^\infty)}^{\mathbb{X}, \text{tor}}, \mathbb{F}_\ell) \rightarrow H^i(\mathfrak{I}\mathfrak{g}_{K(N\ell^\infty)}^{\mathbb{X}}, \mathbb{F}_\ell)$$

is an isomorphism.

Proof. One can assume that \mathbb{X} is completely slope divisible and reduce to the similar assertion for the non-perfect schemes

$$\text{Ig}_{m, K(N\ell^\infty)}^{\mathbb{X}} \subset \text{Ig}_{m, K(N\ell^\infty)}^{\mathbb{X}, \text{tor}}$$

that have the same local structure as Shimura varieties, so the same argument as for Shimura varieties works. \square

Proof of Theorem 4.6.1. First, we note that by the Hochschild–Serre spectral sequence, it is enough to show that

$$R\psi(R\pi_{\text{HT}, \ell^\infty, *}^\circ \mathbb{F}_\ell) \in {}^p D^{\geq d}(\mathfrak{F}\mathfrak{I}_k, \mathbb{F}_\ell)$$

where

$$\pi_{\text{HT}, \ell^\infty}^\circ : \mathcal{S}_{K(p^\infty N\ell^\infty), C}^\circ \rightarrow \mathcal{F}\ell_C$$

is the version with ℓ^∞ -level structure. Combining Lemma 4.6.3, Theorem 2.7.2 and Corollary 4.1.2, we see that the natural map

$$R\pi_{\text{HT}, \ell^\infty, *}^{\text{tor}} \mathbb{F}_\ell \rightarrow R\pi_{\text{HT}, \ell^\infty, *}^\circ \mathbb{F}_\ell$$

is an isomorphism. From now on, we will work with the left-hand side.

We may arrange, using [Sch15, Theorem 4.1.1], that there is an integral model $\mathfrak{G}_{K(p^\infty N), \mathcal{O}_C}^*$ of $\mathcal{S}_{K(p^\infty N), C}^*$ that is an inverse limit of integral models $\mathfrak{G}_{K(p^m N), \mathcal{O}_C}^*$ of $\mathcal{S}_{K(p^m N), C}^*$ for sufficiently large m , and an extension of π_{HT}^* to an affine map

$$\pi_{\text{HT}}^{*, \text{int}} : \mathfrak{G}_{K(p^\infty N), \mathcal{O}_C}^* \rightarrow \mathfrak{F}\mathfrak{I}.$$

The map on the special fibre factors over a map of special fibers

$$f_m : \mathfrak{G}_{K(p^m N), k}^* \rightarrow \mathfrak{F}\mathfrak{I}_k$$

for sufficiently large m . Now f_m is still affine, and is also partially proper (as π_{HT}^* is, cf. proof of [CS17, Proposition 6.1.3]), which implies that it is finite. In particular, pushforward along f_m preserves ${}^p D^{\geq d}$.

As taking nearby cycles commutes with pushforwards, we find that it is enough to see that if

$$g_m^{\infty, \infty} : \mathcal{S}_{K(p^\infty N\ell^\infty), C}^{\text{tor}} \rightarrow \mathcal{S}_{K(p^m N), C}^*$$

denotes the projection, then

$$R\psi(Rg_{m*}^{\infty, \infty} \mathbb{F}_\ell) \in {}^p D^{\geq d}(\mathfrak{S}_{K(p^m N), k}^*, \mathbb{F}_\ell).$$

If we let for any $m_1 \geq m$, $m_2 \geq 0$

$$g_m^{m_1, m_2} : \mathcal{S}_{K(p^{m_1} N \ell^{m_2}), C}^{\text{tor}} \rightarrow \mathcal{S}_{K(p^m N), C}^*$$

be the corresponding projection, and

$$j^{m_1, m_2} : \mathcal{S}_{K(p^{m_1} N \ell^{m_2}), C} \hookrightarrow \mathcal{S}_{K(p^{m_1} N \ell^{m_2}), C}^{\text{tor}}$$

the open immersion (of analytifications of smooth schemes over C), then on the one hand

$$Rg_{m*}^{\infty, \infty} \mathbb{F}_\ell = \varinjlim_{m_1, m_2} Rg_{m*}^{m_1, m_2} \mathbb{F}_\ell,$$

while on the other hand for any fixed m_1 , the natural map

$$\varinjlim_{m_2} Rg_{m*}^{m_1, m_2} \mathbb{F}_\ell \rightarrow \varinjlim_{m_2} Rg_{m*}^{m_1, m_2} Rj_*^{m_1, m_2} \mathbb{F}_\ell$$

is an isomorphism. For the latter claim, it suffices to check the analogous assertion on schemes (as all operations are between analytifications of schemes of finite type, where they agree with the operations on the scheme level), and then it follows from $Rj_*^{m_1, \infty} \mathbb{F}_\ell = \mathbb{F}_\ell$ in the setting of schemes, which is Lemma 4.6.2.

Finally, it suffices to see that for all m_1, m_2 ,

$$R\psi(Rg_{m*}^{m_1, m_2} Rj_*^{m_1, m_2} \mathbb{F}_\ell) \in {}^p D^{\geq d}(\mathfrak{S}_{K(p^m N), k}^*, \mathbb{F}_\ell).$$

But the sheaf $Rg_{m*}^{m_1, m_2} Rj_*^{m_1, m_2} \mathbb{F}_\ell$ is simply the pushforward of \mathbb{F}_ℓ under the open immersion $\mathcal{S}_{K(p^{m_1} N \ell^{m_2}), C} \hookrightarrow \mathcal{S}_{K(p^m N), C}^*$, which can be computed in the same way on the level of schemes, and lies in ${}^p D^{\geq d}$ in the scheme setting. Now the result follows from t -exactness of nearby cycles for the perverse t -structure, cf. [Ill94] (and compatibility of nearby cycles between schemes and adic spaces, [Hub96, Theorem 3.5.13]). \square

5. THE COHOMOLOGY OF IGUSA VARIETIES AND AUTOMORPHIC REPRESENTATIONS

In this section, we analyze the $\overline{\mathbb{Q}}_\ell$ -cohomology of Igusa varieties as a virtual representation, and describe it in terms of automorphic representations. As an application, we can prove the existence of associated Galois representations when the cohomology is concentrated in one degree.

The methods and results of this section are parallel to those of Section 5 of [CS17], but we will replace the conditions imposed there on the ramification of primes in F with the condition that the level be sufficiently small at “bad places”. In [Shi10], Sug Woo Shin derived a formula for the alternating sum of the compactly supported cohomology of Igusa varieties as a sum of stable orbital integrals for G and its elliptic endoscopic groups. If F is not imaginary quadratic and the level N is sufficiently divisible, we can reinterpret this formula as the geometric side of the twisted trace formula. We then compare it to the spectral side of the twisted trace formula to compute it in terms of virtual representations of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$.

5.1. Statements. As usual, we fix a prime p that is unramified in F . In this section, we make the following further assumptions:

- (1) The CM field F contains an imaginary quadratic field F_0 in which p is split;
- (2) The totally real subfield F^+ is not equal to \mathbb{Q} .

We also fix a set S of places of \mathbb{Q} containing ∞ and all primes dividing $p\ell\Delta_F$. For technical reasons, we also fix a character $\varpi : \mathbb{A}_{F_0}^\times/F_0^\times \rightarrow \mathbb{C}^\times$ extending the quadratic character of $\mathbb{A}^\times/\mathbb{Q}^\times \rightarrow \{\pm 1\}$ corresponding to F_0 (via the embedding $\mathbb{A}^\times/\mathbb{Q}^\times \hookrightarrow \mathbb{A}_{F_0}^\times/F_0^\times$), and add to S all primes above which ϖ is ramified.

Throughout this section we also fix an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$.

Let $\text{Spl}_{F_0/\mathbb{Q}}$ be the set of primes that split in F_0 , and let

$$\mathbb{T}^S = \prod_{q \notin S, q \in \text{Spl}_{F_0/\mathbb{Q}}} \mathbb{Z}[G(\mathbb{Q}_q)//G(\mathbb{Z}_q)]$$

be the associated spherical Hecke algebra. If v is a place of F dividing a prime $q \notin S$ that splits in F_0 , then v determines a prime $\mathfrak{q}|q$ of F_0 , and

$$G(\mathbb{Q}_q) = \text{GL}_{2n}(F_v) \times \prod_{w|q, w \neq v} \text{GL}_{2n}(F_w) \times \mathbb{Q}_q^\times.$$

For any $i = 1, \dots, 2n$, we define $T_{i,v} \in \mathbb{T}^S$ as the Hecke operator corresponding to the $G(\mathbb{Z}_q)$ -double coset

$$\text{GL}_{2n}(\mathcal{O}_{F_v}) \text{diag}(\underbrace{\varpi_v, \dots, \varpi_v}_i, 1, \dots, 1) \text{GL}_{2n}(\mathcal{O}_{F_v}) \times \prod_{w|p, w \neq v} \text{GL}_{2n}(\mathcal{O}_{F_w}) \times \mathbb{Z}_q^\times$$

inside $G(\mathbb{Q}_q)$.

Fix $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$ and fix a p -divisible group $\mathbb{X} = \mathbb{X}_b$ with G -structure over $\overline{\mathbb{F}}_p$ in the isogeny class given by b ; we assume that \mathbb{X} is completely slope divisible. Note that if v is a prime of F dividing p , inducing a prime $\mathfrak{p}|p$ of F_0 , then we have a decomposition

$$J_b(\mathbb{Q}_p) \cong J_{b_v}(F_v) \times \prod_{w|p, w \neq v} J_{b_w}(F_w) \times \mathbb{Q}_p^\times$$

where $b_v \in B(\mathrm{GL}_{2n, F_v})$ and $b_w \in B(\mathrm{GL}_{2n, F_w})$ are the elements induced by b . If π is an irreducible representation of $J_b(\mathbb{Q}_p)$, we will write

$$\pi = \pi_v \otimes \bigotimes_{w|p, w \neq v} \pi_w \otimes \chi$$

for the associated decomposition. Note that $J_{b_v}(F_v)$ is an inner form of a Levi subgroup of $\mathrm{GL}_{2n}(F_v)$. As such, any irreducible smooth $\overline{\mathbb{Q}}_\ell$ representation π_v of $J_{b_v}(F_v)$ has a semisimple L -parameter

$$\sigma_{\pi_v} : W_{F_v} \rightarrow \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell),$$

using the results of Badulescu, [Bad07], on the Jacquet–Langlands correspondence for all inner forms of general linear groups.

Remark 5.1.1. Recall that the semisimple Langlands parameter associated to a Frobenius semisimple Weil–Deligne representation

$$W_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_m(\mathbb{C})$$

is the restriction along the embedding

$$W_{F_v} \rightarrow W_{F_v} \times \mathrm{SL}_2(\mathbb{C}) : w \mapsto (w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$$

where $|\cdot| : W_{F_v} \rightarrow \mathbb{R}_{>0}$ is the map sending a Frobenius to q_v . This has the property that nontrivial extensions can only exist between irreducible smooth representations with the same semisimple Langlands parameter; in fact the semisimple Langlands parameter gives exactly a point of the Bernstein center. Also, it is compatible with normalized parabolic induction.

We fix a level $N \geq 3$ that is only divisible by primes in $S_f^p := S_{\mathrm{fin}} \setminus \{p\}$. The goal of this section is to describe the virtual representation

$$[H_c(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)] := \sum_i (-1)^i [\varinjlim_m H_c^i(\mathrm{Ig}_{m, K(N)}^b, \overline{\mathbb{Q}}_\ell)]$$

of $J_b(\mathbb{Q}_p) \times \mathbb{T}^S$, under an additional assumption on N . As a virtual representation of $J_b(\mathbb{Q}_p) \times \mathbb{T}^S$, we can write

$$[H_c(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)] = \sum_{j \in J} n_j \pi_j \otimes \psi_j$$

over some countable index set J , where each π_j is an irreducible smooth representation of $J_b(\mathbb{Q}_p)$, $\psi_j : \mathbb{T}^S \rightarrow \overline{\mathbb{Q}}_\ell$ is a character, and we assume that all $n_j \neq 0$ and all $\pi_j \otimes \psi_j$ are pairwise distinct.

The goal of this section is to prove the following theorem.

Theorem 5.1.2. *Assume that the level N is divisible by some $N_0 \geq 3$ as in Remark 5.4.5; in particular, N_0 is only divisible by primes in $S_f^p := S_{\mathrm{fin}} \setminus \{p\}$.*

For each $j \in J$, there is a continuous semisimple Galois representation

$$\rho_j : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$$

that is almost everywhere unramified, and for all primes v of F dividing a prime $q \notin S$ that splits in F_0 , the representation ρ_j is unramified at v , and the characteristic polynomial of $\rho_j(\mathrm{Frob}_v)$ is given by

$$X^{2n} - \psi_j(T_{1,v})X^{2n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} \psi_j(T_{i,v})X^{2n-i} + \cdots + q_v^{n(2n-1)} \psi_j(T_{2n,v}),$$

where q_v is the cardinality of the residue field at v .

Moreover, for all primes v dividing p , the semisimple Langlands parameter of $\pi_{j,v}|\cdot|^{\frac{1}{2}-n}$ is given by the semisimple Langlands parameter induced by $\rho_j|_{\text{Gal}(\overline{F}_v/F_v)}$.

In particular, we get the following corollary.

Corollary 5.1.3. *Let N be divisible by N_0 as above. Assume that $\mathfrak{m} \subset \mathbb{T}^S$ is a maximal ideal such that*

$$H^i(\text{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$$

in exactly one degree. Then there exists a continuous semisimple Galois representation

$$\overline{\rho}_{\mathfrak{m}} : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_{2n}(\overline{\mathbb{F}}_\ell)$$

such that for all primes v of F dividing a prime $q \notin S$ that splits in F_0 , the representation $\overline{\rho}_{\mathfrak{m}}$ is unramified at v , and the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\text{Frob}_v)$ is given by the reduction of

$$X^{2n} - T_{1,v}X^{2n-1} + \dots + (-1)^i q_v^{i(i-1)/2} T_{i,v} X^{2n-i} + \dots + q_v^{n(2n-1)} T_{2n,v}$$

modulo \mathfrak{m} , where q_v is the cardinality of the residue field at v .

Moreover, if p splits completely in F and for all $v|p$, the representation $\overline{\rho}_{\mathfrak{m}}$ is unramified with the eigenvalues $\{\alpha_{1,v}, \dots, \alpha_{2n,v}\}$ of $\overline{\rho}_{\mathfrak{m}}(\text{Frob}_v)$ satisfying $\alpha_{i,v} \neq p\alpha_{j,v}$ for all $i \neq j$, then b is ordinary.

Proof. Recall the involution $\iota : \mathbb{T}^S \rightarrow \mathbb{T}^S$, as in [ACC⁺18, §2.2.11] and set $\mathfrak{m}^\vee := \iota(\mathfrak{m})$. By the assumption of concentration in one degree and Poincaré duality, it follows that $H_c^i(\text{Ig}^b, \mathbb{Z}_\ell)_{\mathfrak{m}^\vee}$ is also concentrated in one degree and torsion-free, so there must be some $j \in J$ such that $\psi_j : \mathbb{T}^S \rightarrow \mathbb{Q}_\ell$ reduces to \mathfrak{m}^\vee . Then we can apply Theorem 5.1.2, define $\overline{\rho}_{\mathfrak{m}^\vee}$ as the semisimplification of the reduction of ρ_j modulo ℓ , and set

$$\overline{\rho}_{\mathfrak{m}} := \overline{\rho}_{\mathfrak{m}^\vee} |\text{Art}_F^{-1}|^{1-2n}.$$

For the final statement, assume that b is not ordinary. Then J_b is a nontrivial inner form of a Levi subgroup of $G_{\mathbb{Q}_p}$. In particular, there is some prime v dividing p such that J_{b_v} is a nontrivial inner form of a Levi subgroup of $\text{GL}_{2n, F_v} = \text{GL}_{2n, \mathbb{Q}_p}$. It follows that there are no irreducible smooth representations π_v of $J_{b_v}(F_v)$ such that σ_{π_v} is a direct sum of characters $\chi_1 \oplus \dots \oplus \chi_{2n}$ such that χ_i/χ_j is not the cyclotomic character for any $i \neq j$. (Indeed, such L -parameters correspond to generic principal series representations which do not transfer to any nontrivial inner form, cf. [CS17, Lemma 5.4.3].)

Now we distinguish two cases. If $p \not\equiv 1 \pmod{\ell}$, then we claim that ρ_j must be unramified at v ; necessarily no two Frobenius eigenvalues can have ratio p (as they do not have ratio p modulo ℓ by assumption), leading to the desired contradiction. To show that $\rho_j|_{\text{Gal}(\overline{F}_v/F_v)}$ is unramified, look at the deformation theory of the unramified representation $\overline{\rho}_j|_{\text{Gal}(\overline{F}_v/F_v)}$ that is a direct sum of unramified characters with eigenvalues $\alpha_{1,v}, \dots, \alpha_{2n,v}$ and check, using the condition $\alpha_{i,v} \neq p\alpha_{j,v}$, that the obstructions, deformations, and automorphisms are the same whether regarded as representations of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ or $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$.¹⁶

On the other hand, if $p \equiv 1 \pmod{\ell}$, then the similar deformation theoretic argument implies that $\rho_j|_{\text{Gal}(\overline{F}_v/F_v)}$ is a direct sum of characters, and again their ratio cannot be the cyclotomic character as this does not happen modulo ℓ , cf. [CS17, Lemma 6.2.2]. \square

¹⁶We thank Koshikawa for pointing out this argument.

5.2. Unitary groups and endoscopic triples. If $\vec{n} = (n_i)_{i=1}^s$ is a vector with entries positive integers, one can define a quasi-split group $G_{\vec{n}}$ over \mathbb{Q} as in [Shi11, Section 3.1]. Define $\mathrm{GL}_{\vec{n}} := \prod_{i=1}^s \mathrm{GL}_{n_i}$ and let $i_{\vec{n}} : \mathrm{GL}_{\vec{n}} \hookrightarrow \mathrm{GL}_{(\sum_i n_i)}$ be the natural embedding. Let

$$\Phi_{\vec{n}} := i_{\vec{n}}(\Phi_{n_1}, \dots, \Phi_{n_s}),$$

where Φ_m is the matrix in GL_m with entries $(\Phi_m)_{ij} = (-1)^{i+1} \delta_{i, m+1-j}$. Then $G_{\vec{n}}$ is the algebraic group over \mathbb{Q} sending a \mathbb{Q} -algebra R to

$$G_{\vec{n}}(R) = \{(\lambda, g) \in R^\times \times \mathrm{GL}_{\vec{n}}(F \otimes_{\mathbb{Q}} R) \mid g \cdot \Phi_{\vec{n}} \cdot {}^t g^c = \lambda \Phi_{\vec{n}}\}.$$

Since $G_{\mathbb{Q}}$ is quasi-split, we have $G_{\mathbb{Q}} \cong G_{2n}$.

We identify the Langlands dual group of $G_{\vec{n}}$ with

$$\widehat{G}_{\vec{n}} = \mathbb{C}^\times \times \prod_{\sigma \in \Phi^+} \mathrm{GL}_{\vec{n}}(\mathbb{C}),$$

where Φ^+ is the set of embeddings $F \hookrightarrow \mathbb{C}$ above a fixed embedding $\tau_0 : F_0 \hookrightarrow \mathbb{C}$. Let $\mathcal{E}^{\mathrm{ell}}(G)$ be the set of isomorphism classes of elliptic endoscopic triples for G . By [Shi11, Section 3.2], a set of representatives for these isomorphism classes is given by

$$\{(G_{2n}, s_{2n}, \eta_{2n})\} \cup \{(G_{n_1, n_2}, s_{n_1, n_2}, \eta_{n_1, n_2}) \mid n_1 + n_2 = 2n, n_1 \geq n_2 \geq 0\},$$

where (n_1, n_2) may be excluded if all of n_1, n_2 and $[F^+ : \mathbb{Q}]$ are odd numbers (see condition 7.4.3 of [Kot84]). Here, $s_{2n} = 1 \in \widehat{G}_n$, $s_{n_1, n_2} = (1, (I_{n_1}, -I_{n_2})) \in \widehat{G}_{n_1, n_2}$, $\eta_{2n} : \widehat{G}_{2n} \rightarrow \widehat{G}_{2n}$ is the identity map and $\eta_{n_1, n_2} : \widehat{G}_{n_1, n_2} \rightarrow \widehat{G}_{2n}$ is the natural embedding induced by $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \hookrightarrow \mathrm{GL}_{2n}$.

We can extend the map $\eta_{n_1, n_2} : \widehat{G}_{n_1, n_2} \rightarrow \widehat{G}_{2n}$ on Langlands dual groups to an L -morphism of L -groups $\tilde{\eta}_{n_1, n_2} : {}^L G_{n_1, n_2} \rightarrow {}^L G_{2n}$. Then the L -group ${}^L G_{\vec{n}} := \widehat{G}_{\vec{n}} \rtimes W_{\mathbb{Q}}$ is the semidirect product defined by

$$w(\lambda, (g_\sigma)_\sigma)w^{-1} = (\lambda, (g_{w^{-1}\sigma})_\sigma), \text{ if } w \in W_{F_0}$$

and

$$w(\lambda, (g_\sigma)_\sigma)w^{-1} = \left(\lambda \prod_{\sigma \in \Phi^+} \det g_\sigma, (\Phi_{\vec{n}} \cdot {}^t g_{c w^{-1} \sigma}^{-1} \Phi_{\vec{n}}^{-1})_\sigma \right) \text{ if } w \notin W_{F_0}.$$

We will use the same formula for $\tilde{\eta}_{n_1, n_2}$ as in Section 3.2 of [Shi11], but we recall the precise definition here because it will be important when we attach Galois representations to systems of Hecke eigenvalues. As in the beginning of this section, we fix $\varpi : \mathbb{A}_{F_0}^\times / F_0^\times \rightarrow \mathbb{C}^\times$ such that $\varpi|_{\mathbb{A}^\times / \mathbb{Q}^\times}$ is the quadratic character corresponding to F_0 . Using the map Art_{F_0} , we can also view ϖ as a character $W_{F_0} \rightarrow \mathbb{C}^\times$. Set $\epsilon : \mathbb{Z} \rightarrow \{0, 1\}$ to be the unique map such that $\epsilon(m) \cong m \pmod{2}$. Let w^* be a fixed element in $W_{\mathbb{Q}} \setminus W_{F_0}$. We extend η_{n_1, n_2} to an L -morphism $\tilde{\eta}_{n_1, n_2}$ by

$$w \in W_{F_0} \mapsto \left(\varpi(w)^{-N(n_1, n_2)}, \left(\begin{array}{cc} \varpi(w)^{\epsilon(n_1) \cdot I_{n_1}} & 0 \\ 0 & \varpi(w)^{\epsilon(n_2) \cdot I_{n_2}} \end{array} \right)_{\sigma \in \Phi^+} \right) \rtimes w$$

$$w^* \mapsto (a_{n_1, n_2}, (\Phi_{n_1, n_2} \Phi_{2n}^{-1})_{\sigma \in \Phi^+}) \rtimes w^*,$$

where $N(n_1, n_2) = [F^+ : \mathbb{Q}](n_1 \epsilon(n_1) + n_2 \epsilon(n_2)) / 2 \in \mathbb{Z}$ and a_{n_1, n_2} is a square root of $(-1)^{N(n_1, n_2)}$. Using the definition of the L -groups ${}^L G_{\vec{n}}$ one can check that $\tilde{\eta}_{n_1, n_2}$ is indeed an L -morphism. We remark that this is an adaptation to unitary similitude groups of the L -morphism defined in Section 1.2 of [Rog92].

5.3. A stable trace formula. Let $\phi \in C_c^\infty(J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p))$ be a function of the form

$$\phi = \phi_p \cdot \phi_{S_f^p} \cdot \phi^S$$

where $\phi_p \in C_c^\infty(J_b(\mathbb{Q}_p))$,

$$\phi_{S_f^p} \in C_c^\infty(G(\mathbb{A}_{S_f^p}))$$

is simply the characteristic function of the principal congruence subgroup $K(N)$ where N is sufficiently divisible as defined later, and

$$\phi^S \in \mathcal{H}^{\text{ur}}(G(\mathbb{A}^S))$$

is in the unramified Hecke algebra. We assume moreover that ϕ is acceptable in the sense of Definition 6.2 of [Shi09].

For $G_{\bar{n}} \in \mathcal{E}^{\text{ell}}(G)$ as in Section 5.2, we can define the transfer of ϕ to $\phi^{\bar{n}} \in C_c^\infty(G_{\bar{n}}(\mathbb{A}_f))$ as follows.

Definition 5.3.1. Let $\phi^{\bar{n}} := \phi_p^{\bar{n}} \cdot \phi_{S_f^p}^{\bar{n}} \cdot \phi^{\bar{n},S} \cdot \phi_\infty^{\bar{n}}$, where:

- (1) $\phi_p^{\bar{n}}$ is constructed in Section 6 of [Shi10] and is the same function used in Sections 5.2 and 5.4 of [CS17].
- (2) $\phi_{S_f^p}^{\bar{n}}$ is the Langlands-Shelstad transfer of $\phi_{S_f^p}$.
- (3) $\phi^{\bar{n},S} \in \mathcal{G}^{\text{ur}}(G_{\bar{n}}(\mathbb{A}^S))$ is given by $\tilde{\eta}_{\bar{n}}^*(\phi^S)$, where

$$\tilde{\eta}_{\bar{n}}^* : \mathcal{H}^{\text{ur}}(G(\mathbb{A}^S)) \rightarrow \mathcal{H}^{\text{ur}}(G_{\bar{n}}(\mathbb{A}^S))$$

is induced by the L -morphism $\tilde{\eta}_{\bar{n}}$ defined in Section 5.2.

- (4) $\phi_\infty^{\bar{n}}$ is a linear combination of Euler-Poincaré functions, the same as in Section 5.2 of [CS17].

The following is the main result of [Shi10].

Theorem 5.3.2. If $\phi \in C_c^\infty(J_b(\mathbb{Q}_p) \times G(\mathbb{A}_{S_f^p})) \otimes \mathcal{H}^{\text{ur}}(G(\mathbb{A}^S))$ is as above, then

$$\text{tr} \left(\phi | H_c(\text{Ig}^b, \overline{\mathbb{Q}}_\ell) \right) = \sum_{G_{\bar{n}}} \iota(G, G_{\bar{n}}) ST_e^{G_{\bar{n}}}(\phi^{\bar{n}}),$$

where $G_{\bar{n}}$ runs over the set of representatives of $\mathcal{E}^{\text{ell}}(G)$ and $\phi^{\bar{n}}$ is obtained from ϕ as in Definition 5.3.1.

5.4. Base change and compatibility of transfers. Let $\mathbb{G}_{\bar{n}} := \text{Res}_{F_0/\mathbb{Q}}(G_{\bar{n}} \times_{\mathbb{Q}} F_0)$. The Langlands dual group $\widehat{\mathbb{G}}_{\bar{n}}$ can be identified with

$$\widehat{\mathbb{G}}_{\bar{n}} = \mathbb{C}^\times \times \mathbb{C}^\times \times \prod_{\sigma \in \Phi} \text{GL}_{\bar{n}}(\mathbb{C}),$$

where Φ is the set of embeddings $F \hookrightarrow \mathbb{C}$. We can describe the L -group ${}^L\mathbb{G}_{\bar{n}}$ as the semidirect product $\widehat{G}_{\bar{n}} \rtimes W_{\mathbb{Q}}$, where

$$w(\lambda_+, \lambda_-, (g_\sigma)_{\sigma \in \Phi})w^{-1} = (\lambda_+, \lambda_-, (g_{w^{-1}\sigma})_{\sigma \in \Phi}) \text{ if } w \in W_{F_0}$$

and if $w \notin W_{F_0}$

$$w(\lambda_+, \lambda_-, (g_\sigma)_{\sigma \in \Phi})w^{-1} = \left(\lambda_- \prod_{\sigma \in \Phi \setminus \Phi^+} \det g_\sigma, \lambda_+ \prod_{\sigma \in \Phi^+} \det g_\sigma, (\Phi_{\bar{n}}^{-t} g_{cw^{-1}\sigma}^{-1} \Phi_{\bar{n}}^{-1})_{\sigma \in \Phi} \right).$$

One can define L -morphisms $BC_{\bar{n}} : L G_{\bar{n}} \rightarrow L \mathbb{G}_{\bar{n}}$ and $\tilde{\zeta}_{n_1, n_2} : L \mathbb{G}_{n_1, n_2} \rightarrow L \mathbb{G}_{2n}$ and there is a commutative diagram of L -morphisms

$$(5.4.1) \quad \begin{array}{ccc} L G_{n_1, n_2} & \xrightarrow{\tilde{\eta}_{n_1, n_2}} & L G_{2n} \\ \downarrow BC_{n_1, n_2} & & \downarrow BC_{2n} \\ L \mathbb{G}_{n_1, n_2} & \xrightarrow{\tilde{\zeta}_{n_1, n_2}} & L \mathbb{G}_{2n}. \end{array}$$

We give the precise definition of $\tilde{\zeta}_{n_1, n_2}$, as it will be important for explicitly defining the transfer of functions and representations at unramified places, which in turn will determine the Galois representations. On the level of dual groups $\widehat{\mathbb{G}}_{\bar{n}}$, the morphism is induced by the embedding $\mathrm{GL}_{\bar{n}} \hookrightarrow \mathrm{GL}_{2n}$. We extend this to an L -morphism by

$$w \in W_{F_0} \mapsto \left(\varpi(w)^{-N(n_1, n_2)}, \varpi(w)^{-N(n_1, n_2)}, \begin{pmatrix} \varpi(w)^{\epsilon(n_1) \cdot I_{n_1}} & 0 \\ 0 & \varpi(w)^{\epsilon(n_2) \cdot I_{n_2}} \end{pmatrix}_{\sigma \in \Phi} \right) \rtimes w^*$$

$$w^* \mapsto (a_{n_1, n_2}, a_{n_1, n_2}, (\Phi_{n_1, n_2} \Phi_{2n}^{-1})_{\sigma \in \Phi}) \rtimes w^*.$$

The L -morphism $BC_{\bar{n}}$ is defined by

$$(\lambda, (g_\sigma)_{\sigma \in \Phi^+}) \rtimes w \mapsto (\lambda, \lambda, (g_\sigma)_{\sigma \in \Phi^+}, (g_{c\sigma})_{\sigma \in \Phi \setminus \Phi^+}).$$

One can check the commutativity of the diagram (5.4.1) by a direct computation.

In this section, we review the associated base change for the groups $G_{\bar{n}}$ and $\mathbb{G}_{\bar{n}}$ as well as the twisted trace formula. Recall that we have fixed a finite set of bad places S . Everything is identical with Section 5.3 of [CS17], except for the base change transfer of functions at primes in S_f^p .

More precisely, if v is a finite place of \mathbb{Q} such that $v \notin S$, then the dual map to the L -morphism $BC_{\bar{n}}$ defines the transfer

$$BC_{\bar{n}}^* : \mathcal{H}^{\mathrm{ur}}(\mathbb{G}_{\bar{n}}(\mathbb{Q}_v)) \rightarrow \mathcal{H}^{\mathrm{ur}}(G_{\bar{n}}(\mathbb{Q}_v)),$$

(Case 1) of Section 4.2 of [Shi11]. We remark that, if $v \notin S$ then we can define a transfer corresponding to the L -morphism $\tilde{\zeta}_{\bar{n}}$ via

$$\tilde{\zeta}_{\bar{n}}^* : \mathcal{H}^{\mathrm{ur}}(\mathbb{G}_{2n}(\mathbb{Q}_v)) \rightarrow \mathcal{H}^{\mathrm{ur}}(\mathbb{G}_{\bar{n}}(\mathbb{Q}_v)).$$

Moreover the diagram 5.4.1 gives rise to a commutative diagram of transfers

$$(5.4.2) \quad \begin{array}{ccc} \mathcal{H}^{\mathrm{ur}}(\mathbb{G}_{2n}(\mathbb{Q}_v)) & \xrightarrow{\tilde{\zeta}_{n_1, n_2}^*} & \mathcal{H}^{\mathrm{ur}}(\mathbb{G}_{n_1, n_2}(\mathbb{Q}_v)) \\ \downarrow BC_{\bar{n}}^* & & \downarrow BC_{n_1, n_2}^* \\ \mathcal{H}^{\mathrm{ur}}(G_{2n}(\mathbb{Q}_v)) & \xrightarrow{\tilde{\eta}_{n_1, n_2}^*} & \mathcal{H}^{\mathrm{ur}}(G_{n_1, n_2}(\mathbb{Q}_v)) \end{array}$$

We note that we can also transfer unramified representations and we have the corresponding commutative diagram

$$(5.4.3) \quad \begin{array}{ccc} \mathrm{Rep}^{\mathrm{ur}}(G_{n_1, n_2}(\mathbb{Q}_v)) & \xrightarrow{\tilde{\eta}_{n_1, n_2, *}^*} & \mathrm{Rep}^{\mathrm{ur}}(G_{2n}(\mathbb{Q}_v)) \\ \downarrow BC_{n_1, n_2, *} & & \downarrow BC_{2n, *} \\ \mathrm{Rep}^{\mathrm{ur}}(\mathbb{G}_{n_1, n_2}(\mathbb{Q}_v)) & \xrightarrow{\tilde{\zeta}_{n_1, n_2, *}^*} & \mathrm{Rep}^{\mathrm{ur}}(\mathbb{G}_{2n}(\mathbb{Q}_v)) \end{array}$$

Both commutative diagrams come from equation (4.18) of [Shi11]. Moreover, transfer of functions and transfer of representations are, in some sense, inverse operations, so that we have

$$\mathrm{tr} \pi(\tilde{\eta}_{n_1, n_2}^*(\phi_v)) = \mathrm{tr} \tilde{\eta}_{n_1, n_2, *}(\pi)(\phi_v)$$

and similarly for $BC_{\bar{n}}$ and $\tilde{\zeta}_{\bar{n}}$.

Remark 5.4.1. We give explicit formulas for $\tilde{\zeta}_{\bar{n}}^*$ and for $\tilde{\zeta}_{\bar{n}, *}$, using the fact that both \mathbb{G}_{2n} and \mathbb{G}_{n_1, n_2} are products of (restrictions of scalars of) general linear groups. Let Q_{n_1, n_2} be a parabolic subgroup of \mathbb{G}_{2n} with Levi subgroup \mathbb{G}_{n_1, n_2} . Recall that we have chosen a character $\varpi : \mathbb{A}_{F_0}^\times / F_0^\times \rightarrow \mathbb{C}^\times$. Let χ_ϖ be the character $\mathbb{G}_{n_1, n_2}(\mathbb{A}) \rightarrow \mathbb{C}^\times$ defined by

$$(\lambda, g_1, g_2) \in \mathbb{A}_{F_0}^\times \times \mathrm{GL}_{n_1}(\mathbb{A}_F) \times \mathrm{GL}_{n_2}(\mathbb{A}_F) \mapsto \varpi \left(\lambda^{-N(n_1, n_2)} \prod_{i=1}^2 N_{F/F_0}(\det(g_i))^{e(n_i)} \right).$$

Given a place v of \mathbb{Q} and a function $f_v \in C_c^\infty(\mathbb{G}_{2n}(\mathbb{Q}_v))$, we let $f_v^{Q_{n_1, n_2}}$ be the constant term of f_v along Q_{n_1, n_2} and we set $\tilde{\zeta}_{n_1, n_2}^*(f_v) = f_v^{Q_{n_1, n_2}} \cdot \chi_{\varpi, v}$. Given $\Pi_v \in \mathrm{Rep}(\mathbb{G}_{n_1, n_2}(\mathbb{Q}_v))$, define $\tilde{\zeta}_{n_1, n_2, *}(\Pi_v) := \mathrm{n} - \mathrm{Ind}_{Q_{n_1, n_2}}^{\mathbb{G}_{2n}}(\Pi_v \otimes \chi_{\varpi, v})$. The compatibility $\mathrm{tr} \Pi_v(\tilde{\zeta}_{n_1, n_2}^*(f_v)) = \mathrm{tr} \tilde{\zeta}_{n_1, n_2, *}(\Pi_v)(f_v)$ follows from Lemma 3.3 of [Shi11].

If $v = p$, then it splits in F_0 and Section 4.2 of [Shi11], (Case 2), constructs a BC -transfer $\phi_p \in C_c^\infty(G_{\bar{n}}(\mathbb{Q}_p))$ of $f_p \in C_c^\infty(\mathbb{G}_{\bar{n}}(\mathbb{Q}_p))$. Moreover, in this case one can check directly that $BC_{\bar{n}}^*$ is surjective.

At ∞ , the transfer is defined in Section 4.3 of [Shi11]. Let ξ be an irreducible algebraic representation of $(G_{\bar{n}})_{\mathbb{C}}$, giving rise to the representation Ξ of $(\mathbb{G}_{\bar{n}})_{\mathbb{C}}$ which is just $\Xi := \xi \otimes \xi$. Recall that $\phi_{G_{\bar{n}}, \xi}$ is the Euler-Poincaré function for ξ . Associated to Ξ , Labesse defined a twisted analogue of the Euler-Poincaré function, a Lefschetz function $f_{\mathbb{G}_{\bar{n}}, \Xi}$ [Lab91]. The discussion on page 24 of [Shi11] implies that $f_{\mathbb{G}_{\bar{n}}, \Xi}$ and $\phi_{G_{\bar{n}}, \xi}$ are BC -matching functions.

At places $v \in S_f^p$, we will be less explicit. We simply note that Theorem 3.3.1 of [Lab99] guarantees the existence of a base-change transfer: given $f_v^{\bar{n}} \in C_c^\infty(\mathbb{G}_{\bar{n}}(\mathbb{Q}_v))$, there exists $\phi_v^{\bar{n}} \in C_c^\infty(G_{\bar{n}}(\mathbb{Q}_v))$ such that $f_v^{\bar{n}}$ and $\phi_v^{\bar{n}}$ have matching orbital integrals. We will be more interested in *inverse transfer*: finding $f_v^{\bar{n}}$ given $\phi_v^{\bar{n}}$ such that the two are BC -matching functions. For this, we will follow the strategy used in Section 8 of [Mor10]. Proposition 3.3.2 of [Lab99], recalled below, gives a sufficient condition for a function $\phi_v^{\bar{n}}$ to be a BC -transfer of some $f_v^{\bar{n}}$.

Proposition 5.4.2. *Let $\phi^{\bar{n}} \in C_c^\infty(G_{\bar{n}}(\mathbb{Q}_v))$ be such that the stable orbital integral of $\phi^{\bar{n}}$, $SO_\gamma(\phi^{\bar{n}})$, is equal to 0 for every $\gamma \in G_{\bar{n}}(\mathbb{Q}_v)$ that is not a norm. Then there exists $f^{\bar{n}} \in C_c^\infty(\mathbb{G}_{\bar{n}}(\mathbb{Q}_v))$ such that $\phi^{\bar{n}}$ is a BC -transfer of $f^{\bar{n}}$.*

The following is an analogue of Lemma 8.3.6 of [Mor10] to the case where F is a general imaginary CM field rather than an imaginary quadratic field.

Lemma 5.4.3. *Let $D_{\bar{n}} := G_{\bar{n}}/G_{\bar{n}}^{\mathrm{der}}$. Then a semisimple element of $G_{\bar{n}}(\mathbb{Q}_v)$ is a norm if and only if its image in $D_{\bar{n}}(\mathbb{Q}_v)$ is a norm.*

Proof. This follows in the same way as Lemma 8.3.6 of [Mor10]. We give more details for completeness. If v splits in F_0 , then every element of $G_{\bar{n}}(\mathbb{Q}_v)$ is a norm and the same holds for $D_{\bar{n}}(\mathbb{Q}_v)$, so the lemma is immediate in this case. Assume that v is inert or ramified in F_0 and let w be the place of F_0 above v . For every Levi

subgroup M of $G_{\bar{n}}$, set $D_M := M/M^{\text{der}}$. For a semisimple element $\gamma \in G_{\bar{n}}(\mathbb{Q}_v)$, let M be a Levi subgroup of $G_{\bar{n}}$ such that $\gamma \in M(\mathbb{Q}_v)$ and γ is elliptic in M . Then Proposition 2.5.3 of [Lab99] says that γ is a norm if and only if its image in $D_M(\mathbb{Q}_v)$ is a norm. It remains to check that an element of $D_M(\mathbb{Q}_v)$ is a norm if and only if its image under the canonical map $D_M(\mathbb{Q}_v) \rightarrow D_{\bar{n}}(\mathbb{Q}_v)$ is a norm.

The Levi M is $G_{\bar{n}}(\mathbb{Q}_v)$ conjugate to a standard Levi subgroup of $G_{\bar{n}}$ defined over \mathbb{Q} . We may assume that M is a standard Levi subgroup of $G_{\bar{n}}$ defined over \mathbb{Q} , in which case we have explicit descriptions of M, D_M and $D_{\bar{n}}$. There exist $s, m_1, m_2 \in \mathbb{N}$ such that $n_i - m_i$ is non-negative and even for $i = 1, 2$ and M is the product of G_{m_1, m_2} and of s groups obtained by restriction of scalars of general linear groups over F . We have isomorphisms

$$D_{\bar{n}} \xrightarrow{\sim} \{(\lambda, z_1, z_2) \in \mathbb{G}_m \times (\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m)^2 \mid z_i \bar{z}_i = \lambda^{n_i}\}.$$

and

$$D_M \xrightarrow{\sim} D_{M,l} \times D_{M,h},$$

where $D_{M,l} = (\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m)^s$ and

$$D_{M,h} \xrightarrow{\sim} \{(\lambda, z_1, z_2) \in \mathbb{G}_m \times (\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m)^2 \mid z_i \bar{z}_i = \lambda^{m_i} \text{ if } m_i > 0 \text{ and } z_i = 1 \text{ if } m_i = 0.\}$$

The map $D_M \rightarrow D_{\bar{n}}$ is 1 on $D_{M,l}$ and given by

$$(\lambda, z_1, z_2) \mapsto (\lambda, \lambda^{(n_1 - m_1)/2} z_1, \lambda^{(n_2 - m_2)/2} z_2)$$

on $D_{M,h}$. Every element of $D_{M,l}(\mathbb{Q}_p)$ is the norm of some element in $\mathbb{D}_{M,l}(\mathbb{Q}_p)$, where $\mathbb{D}_{M,l} := \text{Res}_{F_0/\mathbb{Q}_p}(D_{M,l} \times_{\mathbb{Q}_p} F_0)$. The lemma follows, since an element of $D_{\bar{n}}$ or $D_{M,h}$ is a norm if and only if the corresponding multiplier λ is a norm. \square

By combining Proposition 5.4.2, Lemma 5.4.3 and the observation that the group of norms in a torus contains an open neighborhood of the identity, we deduce that every function in $C_c^\infty(G_{\bar{n}}(\mathbb{Q}_v))$ with support a small enough neighborhood of the identity is the BC -transfer of some function in $C_c^\infty(\mathbb{G}_{\bar{n}}(\mathbb{Q}_v))$.

Lemma 5.4.4. *Let $v \in S_f^p$. Every function $\phi_v \in C_c^\infty(G_{2n}(\mathbb{Q}_v))$ with support in a small enough neighborhood of the identity has the property that its transfer $\phi^{\bar{n}} \in C_c^\infty(G_{\bar{n}}(\mathbb{Q}_v))$ is the BC -transfer of some $f^{\bar{n}} \in C_c^\infty(\mathbb{G}_{\bar{n}}(\mathbb{Q}_v))$.*

Remark 5.4.5. We can now explain the choice of N_0 : It is such that the principal congruence subgroup of level N_0 is small enough in the sense of this lemma, for all $v \in S_f^p$. (Recall that $G \cong G_{2n}$.)

Proof. This is essentially the same as the proof of the first part of Lemma 8.4.1 of [Mor10]. The only difference is that we use Lemma 5.4.3 instead of Lemma 8.3.6 of [Mor10]. We sketch the proof. For every $G_{\bar{n}} \in \mathcal{E}^{\text{ell}}(G_{2n})$, choose an embedding $G_{\bar{n}} \hookrightarrow G_{2n}$. By Lemma 5.4.3 and the observation that the group of norms in a torus contains an open neighborhood of the identity, there exists an open neighborhood of the identity $U_{\bar{n}} \subset G_{\bar{n}}$ such that every semisimple element of $U_{\bar{n}}$ is a norm. By Lemma 8.4.2 of [Mor10] (with $\mathbb{G}_{\bar{n}}$ the Levi subgroup of \mathbb{G}_{2n}), there exists a neighborhood $V_{\bar{n}}$ of the identity in $G_{2n}(F_0 \otimes_{\mathbb{Q}} \mathbb{Q}_v)$ such that every semisimple element of $G_{\bar{n}}(\mathbb{Q}_v)$ that is $G_{2n}(F_0 \otimes_{\mathbb{Q}} \mathbb{Q}_v)$ -conjugate to an element of $V_{\bar{n}}$ is $G_{\bar{n}}(\mathbb{Q}_v)$ -conjugate to an element of $U_{\bar{n}}$.

Let $V := \cap_{G_{\bar{n}} \in \mathcal{E}^{\text{ell}}(G)} V_{\bar{n}}$, then $U := V \cap G_{2n}(\mathbb{Q}_v)$ is an open neighborhood of the identity in $G_{2n}(\mathbb{Q}_v)$. Any function $\phi \in C_c^\infty(G_{2n}(\mathbb{Q}_v))$ with support contained in U will satisfy the desired property. Choose a transfer $\phi^{\bar{n}}$ of ϕ to $G_{\bar{n}}$. To show that

$\phi^{\bar{n}}$ is the base change transfer of some $f^{\bar{n}} \in C_c^\infty(\mathbb{G}_{\bar{n}}(\mathbb{Q}_v))$, it is enough, by Proposition 5.4.2 to show that, if for some semisimple $\gamma \in G_{\bar{n}}(\mathbb{Q}_v)$ we have $SO_\gamma(\phi^{\bar{n}}) \neq 0$, then γ is a norm. By the definition of transfer, if $SO_\gamma(\phi^{\bar{n}}) \neq 0$, then there exists $\delta \in G_{2n}(\mathbb{Q}_v)$ which is associated to γ such that the orbital integral $O_\delta(\phi) \neq 0$. The fact that γ and δ are associated means that they are $G_{2n}(F_0 \otimes_{\mathbb{Q}} \mathbb{Q}_v)$ -conjugate and the fact that $O_\delta(\phi) \neq 0$ means that $\delta \in U \subseteq V_{\bar{n}}$. By the construction of $V_{\bar{n}}$, we see that γ is $G_{\bar{n}}(\mathbb{Q}_v)$ -conjugate to an element of $U_{\bar{n}}$, which is a norm, so γ is a norm itself. \square

5.5. The twisted trace formula. The first part of this section is identical to the corresponding section of [CS17]. The essential difference is that we give more detail in the proof of Proposition 5.5.1, since we want to emphasize the role that *cuspidal subgroups* play in it.

Define the group

$$\mathbb{G}_{\bar{n}}^+ := (\text{Res}_{F_0/\mathbb{Q}} \mathbb{G}_m \times \text{Res}_{F/\mathbb{Q}} \text{GL}_{\bar{n}}) \rtimes \{1, \theta\},$$

where $\theta(\lambda, g)\theta^{-1} = (\lambda^c, \lambda^c g^\sharp)$ and $g^\sharp = \Phi_{\bar{n}}^t g^c \Phi_{\bar{n}}^{-1}$. There is a natural \mathbb{Q} -isomorphism $\mathbb{G}_{\bar{n}} \xrightarrow{\sim} \mathbb{G}_{\bar{n}}^\circ$, which extends to an isomorphism

$$\mathbb{G}_{\bar{n}} \rtimes \text{Gal}(F_0/\mathbb{Q}) \xrightarrow{\sim} \mathbb{G}_{\bar{n}}^+$$

so that $c \in \text{Gal}(F_0/\mathbb{Q})$ maps to θ . Using this isomorphism, we write $\mathbb{G}_{\bar{n}}$ and $\mathbb{G}_{\bar{n}}\theta$ for the two cosets.

If $f \in C_c^\infty(\mathbb{G}_{\bar{n}}(\mathbb{A}))$ (with trivial character on $A_{\mathbb{G}_{\bar{n}}, \infty}^\circ$), then we define $f\theta$ to be the function on $\mathbb{G}_{\bar{n}}\theta(\mathbb{A})$ obtained via translation by θ . The (invariant) twisted trace formula (see [Art88a, Art88b]) gives an equality

$$(5.5.1) \quad I_{\text{geom}}^{\mathbb{G}_{\bar{n}}\theta}(f\theta) = I_{\text{spec}}^{\mathbb{G}_{\bar{n}}\theta}(f\theta).$$

The left hand side of the equation is defined in Section 3 of [Art88b], while the right hand side is defined in Section 4 of loc. cit.

Let $f_{\mathbb{G}_{\bar{n}}, \Xi}$ and $\phi_{G_{\bar{n}}, \xi}$ be as defined above. The following is analogous to Corollary 4.7 of [Shi11].

Proposition 5.5.1. *We have the following equality:*

$$(5.5.2) \quad I_{\text{geom}}^{\mathbb{G}_{\bar{n}}\theta}(f^{\bar{n}}\theta) = \sum_{\gamma} SO_{\gamma}^{G_{\bar{n}}(\mathbb{A})}(\phi^{\bar{n}}) = \tau(G_{\bar{n}})^{-1} \cdot ST_e^{G_{\bar{n}}}(\phi^{\bar{n}}),$$

when $\phi^{\bar{n}}$ and $f^{\bar{n}}$ satisfy

$$\phi^{\bar{n}} = (\phi^{\bar{n}})^S \cdot \phi_{S_f}^{\bar{n}} \cdot \phi_{G_{\bar{n}}, \xi}^{\bar{n}} \text{ and } f^{\bar{n}} = (f^{\bar{n}})^S \cdot f_{S_f}^{\bar{n}} \cdot f_{\mathbb{G}_{\bar{n}}, \Xi}$$

with $(\phi^{\bar{n}})^S$ a BC-transfer of $(f^{\bar{n}})^S$, $\phi_{S_f}^{\bar{n}}$ a BC-transfer of $f_{S_f}^{\bar{n}}$. The sum in the center runs over a set of representatives of \mathbb{Q} -elliptic semisimple stable conjugacy classes in $G_{\bar{n}}(\mathbb{Q})$ and $\tau(G_{\bar{n}})$ is the Tamagawa number of $G_{\bar{n}}$.

Proof. The second equality follows exactly as in the proof of Corollary 4.7 of [Shi11]. Moreover, Theorem 4.3.4 of [Lab99] rewrites the sum of stable orbital integrals in the center as the *elliptic* part of the twisted trace formula for $\mathbb{G}_{\bar{n}}\theta$. Comparing with the definition of the elliptic part in Section 4.1 of [Lab99], we get

$$\sum_{\gamma} SO_{\gamma}^{G_{\bar{n}}(\mathbb{A})}(\phi^{\bar{n}}) = \sum_{\delta} \text{vol}(I_{\delta\theta}(\mathbb{Q})A_{\mathbb{G}_{\bar{n}}\theta} \setminus I_{\delta\theta}(\mathbb{A})) \cdot O_{\delta\theta}^{\mathbb{G}_{\bar{n}}(\mathbb{A})}(f^{\bar{n}}),$$

where γ is as above and δ runs over a set of representatives for θ -elliptic θ -conjugacy classes in $\mathbb{G}_{\bar{n}}(\mathbb{Q})$.

The key point in proving the first equality is to simplify the geometric side of the twisted trace formula for $\mathbb{G}_{\bar{n}}\theta$, using similar techniques to those in Chapter 7 of [Art88b] to show that only this part contributes. This is done in Proposition 4.5 of [Shi11], but we repeat the argument here in order to emphasize the role that cuspidal subgroups play. Theorem 7.1.(b) and Corollary 7.4 of [Art88b] give a way to simplify the geometric side of the trace formula if the test function $f^{\bar{n}}$ is cuspidal at two places. The test function $f^{\bar{n}}$ that we constructed is only necessarily cuspidal at ∞ but we will use the assumption that $[F^+ : \mathbb{Q}] \geq 2$ instead and we will explain why twisted analogues of these two results hold.

The geometric side of the twisted trace formula is a linear combination of invariant distributions $I_{\mathbb{M}}^{\mathbb{G}_{\bar{n}}\theta}(\delta\theta, f^{\bar{n}}\theta)$, where \mathbb{M} runs over Levi subsets of $\mathbb{G}_{\bar{n}}$. (The twisted analogue of) Theorem 7.1.(b) of [Art88b] shows that the geometric side can be simplified to a linear combination of distributions on $\mathbb{G}_{\bar{n}}\theta$ itself. The proof of Theorem 7.1.(b) of [Art88b] goes through as long as one can show that the distributions $I_{\mathbb{M}}^{\mathbb{G}_{\bar{n}}\theta}(\delta\theta, f^{\bar{n}}\theta)$ vanish for every proper subset $\mathbb{M} \subset \mathbb{G}_{\bar{n}}$ and semisimple element $\delta \in \mathbb{M}(\mathbb{Q})$. Since $f_{\mathbb{G}_{\bar{n}}, \Xi}^{\bar{n}}$ is cuspidal, Proposition 9.1 (the splitting formula) and Corollary 8.3 of [Art88a] imply that $I_{\mathbb{M}}^{\mathbb{G}_{\bar{n}}\theta}(\delta\theta, f^{\bar{n}}\theta)$ is a multiple of $I_{\mathbb{M}}^{\mathbb{G}_{\bar{n}}\theta}(\delta\theta, f_{\mathbb{G}_{\bar{n}}, \Xi}^{\bar{n}})$ so it is enough to show that the distribution at ∞ vanishes. Proposition 8.2.3 of [Mor10] shows that the term $I_{\mathbb{M}}^{\mathbb{G}_{\bar{n}}\theta}(\delta\theta, f_{\mathbb{G}_{\bar{n}}, \Xi}^{\bar{n}})$ vanishes if \mathbb{M} is not a *cuspidal* Levi subset of $\mathbb{G}_{\bar{n}}\theta$. We recall what cuspidal subgroups and subsets are in Definition 5.5.2 below and show in Lemma 5.5.3 that, when $[F^+ : \mathbb{Q}] \geq 2$, $\mathbb{G}_{\bar{n}}\theta$ does not admit any cuspidal subsets.

The argument above implies that the conclusion of Theorem 7.1.(b) of [Art88b] holds: the geometric side of the twisted trace formula in this case is a linear combination of distributions on $\mathbb{G}_{\bar{n}}\theta$ itself. Recall that $f^{\bar{n}}$ is cuspidal at ∞ . Theorem A.1.1 of [Lab99] implies that $O_{\delta\theta}^{\mathbb{G}_{\bar{n}}(\mathbb{A})}(f^{\bar{n}}) = 0$ unless δ is θ -elliptic in $\mathbb{G}_{\bar{n}}(\mathbb{R})$. We deduce the twisted analogue of the first condition of Corollary 7.4 of [Art88b]. The second condition (that $f^{\bar{n}}$ is cuspidal at an additional place) is not needed for the proof to go through: it was only imposed in order to invoke Theorem 7.1 of loc. cit.

By the twisted analogue of Corollary 7.4 of [Art88b], we deduce that

$$I_{\text{geom}}^{\mathbb{G}_{\bar{n}}\theta}(f^{\bar{n}}\theta) = \sum_{\delta} \text{vol}(I_{\delta\theta}(\mathbb{Q})A_{\mathbb{G}_{\bar{n}}\theta} \setminus I_{\delta\theta}(\mathbb{A})) \cdot O_{\delta\theta}^{\mathbb{G}_{\bar{n}}(\mathbb{A})}(f^{\bar{n}}),$$

where δ runs over a set of representatives for θ -elliptic θ -conjugacy classes in $\mathbb{G}_{\bar{n}}(\mathbb{Q})$. The proposition follows. \square

Definition 5.5.2.

- (1) A *cuspidal group* is a reductive group M over \mathbb{Q} such that $(M/A_M)_{\mathbb{R}}$ contains a maximal \mathbb{R} -torus that is \mathbb{R} -anisotropic. Here A_M denotes the maximal \mathbb{Q} -split torus in the center of M .
- (2) A *cuspidal Levi subgroup* M of $\mathbb{G}_{\bar{n}}$ is a subgroup which is a cuspidal group.
- (3) For a Levi subset \mathbb{M} of $\mathbb{G}_{\bar{n}}\theta$, we let \mathbb{M}° be the intersection of the subgroup of $\mathbb{G}_{\bar{n}}^{\pm}$ generated by M with $\mathbb{G}_{\bar{n}}$. We say that a subset \mathbb{M} of $\mathbb{G}_{\bar{n}}\theta$ is *cuspidal* if $\mathbb{M}^{\circ} \cap \mathbb{G}_{\bar{n}}$ is a cuspidal Levi subgroup of $\mathbb{G}_{\bar{n}}$.

Lemma 5.5.3. $\mathbb{G}_{\bar{n}}\theta$ admits no proper cuspidal subsets.

Proof. It is enough to show that $\mathbb{G}_{\bar{n}}$ admits no proper cuspidal Levi subgroups. Suppose that $M \subsetneq \mathbb{G}_{\bar{n}}$ is a cuspidal Levi subgroup. Then M contains a direct factor of the form $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_h$ for some $0 < h \leq n$, which would also have to be cuspidal group. However, the center of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_h \times_{\mathbb{Q}} \mathbb{R}$ contains a split torus of rank $[F^+ : \mathbb{Q}] > 1$, whereas the maximal split torus in the center of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_h$ has rank 1. \square

Remark 5.5.4. When F is an imaginary quadratic field, so $F^+ = \mathbb{Q}$, the unitary similitude groups $G_{\bar{n}}$ do admit proper cuspidal subgroups. For example, for G_{2n} , these are the Levi subgroups of the form $(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m)^s \times G_{2(n-s)}$ for some $s \in \mathbb{Z}_{>0}$. In this case, one cannot directly identify the stable trace formula for Igusa varieties with the geometric side of the (twisted) trace formula. The latter contains more terms coming from the cuspidal subgroups.

5.6. Construction of test functions. In this section, we explain how to construct our test functions. This differs slightly from the construction of test functions in Section 6 of [Shi11] and Section 5 of [CS17], but only at primes in S_f^p . We fix the isomorphism $G \cong G_{2n}$. We let $(f^{2n})^S$ be any function in $\mathcal{H}^{\mathrm{ur}}(\mathbb{G}_{2n}(\mathbb{A}^S))$ and take $\phi^S \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}^S))$ be its BC -transfer. We let $\phi_{S_f^p} \in C_c^\infty(G(\mathbb{A}_{S_f^p}))$ be the characteristic function of a principal congruence subgroup for some sufficiently large level N , as guaranteed by Lemma 5.4.4. We let $\phi_p \in C_c^\infty(J_b(\mathbb{Q}_p))$ and set

$$\phi := \phi_p \cdot \phi_{S_f^p} \cdot \phi^S,$$

and assume that it is acceptable.

From these test functions, we construct all the other test functions we will need. First, for each elliptic endoscopic group $G_{\bar{n}}$, we let $\phi^{\bar{n}}$ be the function constructed from ϕ as in Definition 5.3.1. Let $(f^{n_1, n_2})^S$ be obtained from $(f^{2n})^S$ by transfer along the L -morphism $\tilde{\zeta}_{n_1, n_2}$. We choose $f_{S_f^p}^{\bar{n}}$ as in Lemma 5.4.4 to be any function such that $\phi_{S_f^p}^{\bar{n}}$ is a BC -transfer of $f_{S_f^p}^{\bar{n}}$. We choose $f_p^{\bar{n}}$ so that $BC_{\bar{n}}^*(f_p^{\bar{n}}) = \phi_p^{\bar{n}}$ (recall that $BC_{\bar{n}}^*$ is surjective at p). We define $f_\infty^{\bar{n}}$ explicitly, as a linear combination of Lefschetz functions for representations $\Xi(\varphi_{\bar{n}})$ of $\mathbb{G}_{\bar{n}}$ for which $\tilde{\zeta}_{\bar{n}} \circ \varphi_{\bar{n}}$ corresponds to the trivial representation of \mathbb{G}_{2n} (see (6.7) of [Shi11] for the precise formula). Finally, we set

$$f^{\bar{n}} := f_p^{\bar{n}} \cdot (f^{\bar{n}})_{S_f^p} \cdot (f^{\bar{n}})^S \cdot f_\infty^{\bar{n}}.$$

By the commutative diagram (5.4.2), we see that $(\phi^{\bar{n}})^S$ is the transfer of $(f^{\bar{n}})^S$ along $BC_{\bar{n}}^*$. We can therefore apply Proposition 5.5.1 to $f^{\bar{n}}$ and $\phi^{\bar{n}}$. We obtain the following result, analogous to Theorem 5.3.2 of [CS17].

Theorem 5.6.1. *We have an equality*

$$\mathrm{tr}(\phi | [H_c(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)]) = \tau(G) \sum_{G_{\bar{n}} \in \mathcal{E}^{\mathrm{ell}}(G)} \epsilon_{\bar{n}} \cdot I_{\mathrm{spec}}^{\mathbb{G}_{\bar{n}} \theta}(f^{\bar{n}} \theta),$$

where $\epsilon_{\bar{n}} = \frac{1}{2}$ if $\bar{n} = (n, n)$ and $\epsilon_{\bar{n}} = 1$ otherwise.

Recall the group morphism

$$\mathrm{Red}_{\bar{n}}^b : \mathrm{Groth}(G_{\bar{n}}(\mathbb{Q}_p)) \rightarrow \mathrm{Groth}(J_b(\mathbb{Q}_p))$$

from Section 5.4 of [CS17]. This is the representation-theoretic counterpart of constructing the transfer $\phi_p^{\bar{n}}$ of ϕ_p . The following is an analogue of Lemma 5.5.1 of [CS17].

Lemma 5.6.2. *The trace $\mathrm{tr}(\phi|[H_c(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)])$ can be written as a linear combination of terms of the form*

$$\mathrm{tr}\left(\mathrm{Red}_n^b(\pi_p^{\bar{n}})(\phi_p)\right) \mathrm{tr}\left((\Pi^{\bar{n}})_{S_f^p}((f^{\bar{n}})_{S_f^p}) \circ A_{S_f^p}\right) \mathrm{tr}\left((\tilde{\zeta}_{\bar{n},*} \Pi^{\bar{n}})^S((f^{2n})^S)\right),$$

where $\pi_p^{\bar{n}} \in \mathrm{Rep}(G_{\bar{n}}(\mathbb{Q}_p))$ base changes to $\Pi_p^{\bar{n}} \in \mathrm{Rep}(G_{\bar{n}}(\mathbb{Q}_p))$, the component at p of a θ -stable isobaric irreducible automorphic representation $\Pi^{\bar{n}}$ of $G_{\bar{n}}$. Moreover, $\Pi_\infty^{\bar{n}}$ is cohomological (with respect to the trivial algebraic representation).

Proof. We sketch the proof. Since we are only interested in obtaining a linear combination, we do not keep track of endoscopic signs or constants. First, assume that ϕ was chosen to be an acceptable function. Then we can apply Theorem 5.6.1 to $\mathrm{tr}(\phi|[H_c(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)])$.

For each $G_{\bar{n}} \in \mathcal{E}^{\mathrm{ell}}(G)$, we simplify the spectral side $I_{\mathrm{spec}}^{G_{\bar{n}}\theta}(f^{\bar{n}}\theta)$ as in Section of [Shi11] and Section 5.3 of [CS17]. Fix a minimal Levi subgroup M_0 of $G_{\bar{n}}$. For each Levi subgroup M containing M_0 , choose a parabolic subgroup Q with Levi M . The spectral side $I_{\mathrm{spec}}^{G_{\bar{n}}\theta}(f^{\bar{n}}\theta)$ can be written as a linear combination of terms of the form $\mathrm{tr}\left(\mathfrak{n} - \mathrm{Ind}_Q^{G_{\bar{n}}}(\Pi_M)_\xi(f^{\bar{n}}) \circ A\right)$, where Π_M runs over irreducible, $\Phi_{\bar{n}}^{-1}\theta$ -stable subrepresentations of the (relatively) discrete spectrum $R_{M,\mathrm{disc}}$. The subscript ξ indicates a possible twist by a character of $A_{G_{\bar{n}},\infty}^\circ$ corresponding to an irreducible algebraic representation ξ of $G_{\bar{n}}$ and A is a normalized intertwiner on $\mathfrak{n} - \mathrm{Ind}_Q^{G_{\bar{n}}}(\Pi_M)_\xi$. This follows by combining Proposition 4.8 and Corollary 4.14 of [Shi11].

We take $\Pi^{\bar{n}}$ to be $\mathfrak{n} - \mathrm{Ind}_Q^{G_{\bar{n}}}(\Pi_M)_\xi$. Choose a decomposition

$$A = A_p \cdot A_{S_f^p} \cdot A^S \cdot A_\infty$$

as a product of normalized intertwining operators. Then we can rewrite the desired trace as a linear combination of terms of the form

$$\mathrm{tr}\left(\Pi_p^{\bar{n}}(f_p^{\bar{n}}) \circ A_p\right) \mathrm{tr}\left((\Pi^{\bar{n}})_{S_f^p}((f^{\bar{n}})_{S_f^p}) \circ A_{S_f^p}\right) \mathrm{tr}\left((\Pi^{\bar{n}})^S((f^{\bar{n}})^S) \circ A^S\right).$$

In order to rewrite the traces at p , use the fact that the base change transfer of representations at p is injective, since p splits in F_0 , to construct $\pi_p^{\bar{n}}$; then the term at p is equal to $\mathrm{tr}(\pi_p^{\bar{n}}(\phi_p^{\bar{n}}))$. Then appeal to Lemma 5.4.2 of [CS17] which identifies the latter trace with $\mathrm{tr}\left(\mathrm{Red}_n^b(\pi_p^{\bar{n}})(\phi_p)\right)$. On the other hand, if we ignore the sign that comes from the choice of the normalized intertwiner $A^{\infty,p}$, then recalling that $(f^{\bar{n}})^S$ is the transfer of $(f^{2n})^S$ along $\tilde{\zeta}_{\bar{n},*}$, we can rewrite

$$\mathrm{tr}\left((\Pi^{\bar{n}})^S((f^{\bar{n}})^S)\right) = \mathrm{tr}\left(\tilde{\zeta}_{\bar{n},*}(\Pi^{\bar{n}})^S((f^{2n})^S)\right).$$

We now explain why $\Pi^{\bar{n}}$ has the desired properties. The fact that $\Pi^{\bar{n}}$ is θ -stable follows from the fact that Π_M is $\Phi_{\bar{n}}^{-1}\theta$ -stable. The fact that $\Pi^{\bar{n}}$ is irreducible follows from the fact that Π_M is irreducible and unitary, and Π_M is isobaric because it contributes to the (relatively) discrete automorphic spectrum $R_{M,\mathrm{disc}}$ and M is a product of general linear groups. Then $\Pi^{\bar{n}}$ is also isobaric because it is irreducible.

Finally, we remove the assumption that ϕ be acceptable using Lemma 6.4 of [Shi09]. (The idea is that the twist $\phi_p^{(N')}$ of any ϕ_p by a power of Frobenius makes ϕ an acceptable function for any large enough N' and that, as long as we keep ϕ^p fixed, we have expressed the desired trace as a finite linear combination of traces of $\phi_p^{(N')}$

against irreducible representations of $J_b(\mathbb{Q}_p)$. The argument in the proof of Lemma 6.4 of [Shi09] proves that the desired equality holds for every integer N' and, in particular, for $N' = 0$. \square

Remark 5.6.3. A fixed \mathbb{T}^S -character of $[H_c(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)]$ could be obtained from several $\Pi^{\vec{n}}$ for several different $G_{\vec{n}} \in \mathcal{E}^{\mathrm{ell}}(G)$. For example, in the Case *ST* which is discussed in Section 6 of [Shi11], the contribution is from an endoscopic group \mathbb{G}_{n_1, n_2} but also from a Levi subgroup M of \mathbb{G} .

5.7. Galois representations. Lemma 5.6.2 essentially finishes the proof of Theorem 5.1.2; it only remains to apply the known results on existence of Galois representations for regular L -algebraic, essentially self-dual, cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_F)$. (Here we use the notions of L -algebraic and C -algebraic representations due to Buzzard-Gee [BG14] and note that in the case of general linear groups these notions only differ by a character twist.)

Our goal is now to construct a Galois representation

$$\rho_\Pi : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$$

attached to the automorphic representation $\Pi := \tilde{\zeta}_{\vec{n}, *}(\Pi^{\vec{n}})$ (or rather the automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ obtained from Π by forgetting the similitude factor) as in Lemma 5.6.2.

By Lemma 5.6.2, $\Pi_\infty^{\vec{n}}$ is cohomological, which implies that $\Pi^{\vec{n}}$ is C -algebraic. Write $\Pi^{\vec{n}} = \psi \otimes \Pi_1 \otimes \Pi_2$ according to the decomposition $\mathbb{G}_{n_1, n_2}(\mathbb{A}) = \mathbb{A}_{F_0}^\times \times \mathrm{GL}_{n_1}(\mathbb{A}_F) \times \mathrm{GL}_{n_2}(\mathbb{A}_F)$. Each Π_i is a regular C -algebraic, θ -stable isobaric automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$. The automorphic representation $\Pi_i | \det |^{(1-n_i)/2}$ is regular L -algebraic.

Recall that we have chosen an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$.

Theorem 5.7.1. *There exists a Galois representation*

$$\rho_i : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_{n_i}(\overline{\mathbb{Q}}_\ell)$$

such that for any place \mathfrak{q} of F ,

$$WD \left(\rho_i |_{\mathrm{Gal}(\overline{F}_\mathfrak{q}/F_\mathfrak{q})} \right)^{\mathrm{F-ss}} \simeq \iota_\ell^{-1} \mathrm{rec} \left(\Pi_{i, \mathfrak{q}} | \det |^{(1-n_i)/2} \right),$$

where rec denotes the local Langlands correspondence normalized as in [HT01].

Proof. This is proved just as Theorem 5.5.3 of [CS17]. Recall that the representation $\Pi^{\vec{n}}$ was constructed as $\mathfrak{n} - \mathrm{Ind}_{Q_i}^{\mathbb{G}_{\vec{n}}}(\Pi_M)$ (recall that for us ξ is trivial), for some automorphic representation Π_M which is $\Phi_{\vec{n}}^{-1}\theta$ -stable and which occurs in the discrete automorphic spectrum of some Levi subgroup M of $\mathbb{G}_{\vec{n}}$. This means we can write

$$\Pi_i = \mathfrak{n} - \mathrm{Ind}_{Q_i}^{\mathrm{GL}_{n_i}}(\Pi_{M_i}),$$

where M_i is the Levi subgroup of parabolic subgroup Q_i of GL_{n_i} and Π_{M_i} is $\Phi_{n_i}^{-1}\theta$ -stable and occurs in the discrete automorphic spectrum of M_i . The classification of the discrete automorphic spectrum for general linear groups due to Harris–Taylor and Mœglin–Waldspurger [MW89] together with the fact that Π_{M_i} is $\Phi_{n_i}^{-1}\theta$ -stable tells us that Π_{M_i} can be expressed in terms of regular L -algebraic, conjugate self-dual cuspidal automorphic representations of (possibly a product of) general linear groups. The existence of the Galois representation ρ_i and the compatibility with

the local Langlands correspondence now follows from the main theorems of [Shi11, CH13, Car12]. \square

Now we can prove Theorem 5.1.2.

Proof of Theorem 5.1.2. We use the description from Lemma 5.6.2, where we fix ϕ_f^p to be the characteristic function of $K(N)$, and note that we can realize all elements of \mathbb{T}^S as the base change of some $(f^{2n})^S$ as we only used places split in F_0 in the definition of \mathbb{T}^S . Lemma 5.6.2 shows that for each $j \in J$, there is some cuspidal automorphic representation $\Pi^{\vec{n}} = \psi \otimes \Pi_1 \otimes \Pi_2$ of $\mathbb{G}_{\vec{n}}$ such that $(\Pi^{\vec{n}})^S$ gives the Hecke character ψ_j , and $\pi_j = \text{Red}_{\vec{n}}^b(\pi_p^{\vec{n}})$ where $\pi_p^{\vec{n}}$ is an irreducible representation of $G_{\vec{n}}(\mathbb{Q}_p)$ base changing to $\Pi_p^{\vec{n}}$. By Theorem 5.7.1, there exist Galois representations ρ_i associated to the L -algebraic representations $\Pi_i | \det |^{(1-n_i)/2}$.

The character ϖ satisfies $\varpi_{\infty}(z) = (z/\bar{z})^{\delta/2}$ for some odd integer δ , since $\varpi_{\infty} : \mathbb{C}^{\times} \rightarrow \mathbb{C}^{\times}$ extends the sign character on \mathbb{R}^{\times} . The character of $\text{GL}_{n_i}(\mathbb{A}_F)$ defined by $|\det |^{n_i/2-n} \varpi(N_{F/F_0} \circ \det)^{\epsilon(n_i)}$ is L -algebraic, since $(n_i + \epsilon(n_i)\delta)/2 \in \mathbb{Z}$, so it corresponds to a character $\epsilon_i : \text{Gal}(\bar{F}/F) \rightarrow \bar{\mathbb{Q}}_{\ell}^{\times}$.

Let $\Pi := \tilde{\zeta}_{n_1, n_2, *}(\Pi^{\vec{n}})$. Write $\Pi = \psi' \otimes \Pi^0$, according to the decomposition $\mathbb{G}_{2n}(\mathbb{A}) = \mathbb{A}_{F_0}^{\times} \times \text{GL}_{2n}(\mathbb{A}_F)$. Set

$$\Pi_i^0 := \Pi_i | \det |^{(1-n_i)/2} | \det |^{n_i/2-n} \varpi(N_{F/F_0} \circ \det)^{\epsilon(n_i)}$$

By the definition of $\tilde{\zeta}_{n_1, n_2, *}$, we get the identity

$$\Pi^0 | \det |^{1/2-n} = \mathfrak{n} - \text{Ind}_{\text{GL}_{n_1} \times \text{GL}_{n_2}}^{\text{GL}_n} (\Pi_1^0 \otimes \Pi_2^0).$$

The representation on the RHS is L -algebraic and normalized parabolic induction is compatible with this notion and with the local Langlands correspondence rec, so the term on the LHS, $\Pi^0 | \det |^{(1-n)/2}$, is also L -algebraic, with corresponding Galois representation $\rho_{\Pi^S, \ell} := \bigoplus_{i=1}^2 \rho_i \otimes \epsilon_i$ (matching via rec).

In particular, one can compare the Hecke eigenvalues at good places, as stated in Theorem 5.1.2. On the other hand, regarding the representation π_j of $J_b(\mathbb{Q}_p)$, we recall that it is given by $\text{Red}_{\vec{n}}^b(\pi_p^{\vec{n}})$, and then follow the semisimple Langlands parameter through all normalized parabolic inductions and Langlands correspondences. Note that $\text{Red}_{\vec{n}}^b$ is also by construction compatible with semisimple Langlands parameters. \square

6. BOUNDARY COHOMOLOGY OF IGUSA VARIETIES

In this section, we compute the cohomology of the (partial) boundary of Igusa varieties.

6.1. Statements. More precisely, fix as usual a prime p that is unramified in F . Fix any p -divisible group with G -structure \mathbb{X} over an algebraically closed field k of characteristic p . By abuse of notation, we write $\mathfrak{I}\mathfrak{g}_{K(N)}^b = \mathfrak{I}\mathfrak{g}_{K^{\mathbb{X}}(N)}^{\mathbb{X}}$ for the associated Igusa variety, where $b = b(\mathbb{X}) \in B(G_{\mathbb{Q}_p}, \mu^{-1})$ is as usual; here $N \geq 3$ is any integer prime to p . Let $\partial\mathfrak{I}\mathfrak{g}^{b,*} \subset \mathfrak{I}\mathfrak{g}^{b,*}$ be the boundary of the Igusa variety. It admits a natural stratification in terms of conjugacy classes $[P]$ of maximal rational parabolic subgroups $P \subsetneq G_{\mathbb{Q}}$: Note that a set of representatives for these are given by the stabilizer P_r of the chain

$$0 \subset F^r \subset F^{2n-r} \subset F^{2n}$$

for $r = 1, \dots, n$. Then the stratum $\mathfrak{I}\mathfrak{g}_{[P]}^{b,*} \subset \partial\mathfrak{I}\mathfrak{g}^{b,*}$ can be defined as the preimage of all strata $\mathcal{S}_Z \subset \mathcal{S}^*$ where the cusp label $Z = (Z_N, X)$ has an \mathcal{O}_F -lattice X of rank r . The strata are naturally Hecke-equivariant.

We can pass to the inverse limit over all $N \geq 3$ prime to p : Let

$$\mathfrak{I}\mathfrak{g}_{\infty}^b = \varprojlim_N \mathfrak{I}\mathfrak{g}_{K(N)}^b$$

and define similarly $\mathfrak{I}\mathfrak{g}_{\infty}^{b,*}$, $\mathfrak{I}\mathfrak{g}_{\infty}^{b,\text{tor}}$, $\partial\mathfrak{I}\mathfrak{g}_{\infty}^{b,*}$ and $\mathfrak{I}\mathfrak{g}_{\infty,[P]}^{b,*}$.

Let

$$j : \mathfrak{I}\mathfrak{g}_{\infty}^b \hookrightarrow \mathfrak{I}\mathfrak{g}_{\infty}^{b,*}$$

be the open immersion and

$$i_{[P]} : \mathfrak{I}\mathfrak{g}_{\infty,[P]}^{b,*} \hookrightarrow \mathfrak{I}\mathfrak{g}_{\infty}^{b,*}$$

the locally closed immersion. We consider

$$R\Gamma_c(\mathfrak{I}\mathfrak{g}_{\infty,[P]}^{b,*}, i_{[P]}^* Rj_* \mathbb{F}_{\ell}).$$

This is naturally a complex of smooth representations of $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ (where we note that both groups act naturally on all objects in the definition).

The main theorem of this section is the following. For the statement, we fix the standard rational parabolic $P \in [P]$ given by the stabilizer of $0 \subset F^r \subset F^{2n-r} \subset F^{2n}$. Its Levi group M is given by $\text{Res}_{F/\mathbb{Q}} \text{GL}_r \times G_{2(n-r)}$ where $G_{2(n-r)}$ is the variant of G with n replaced by $n-r$. Let

$$X_r = \left(\prod_{\tau: F^+ \hookrightarrow \mathbb{R}} M_r^{\text{herm}, >0}(\mathbb{C}) \right) / \mathbb{R}_{>0}$$

where $M_r^{\text{herm}, >0}(\mathbb{C})$ denotes the space of positive definite hermitian matrices, which is the symmetric space for $\text{GL}_r(F \otimes_{\mathbb{Q}} \mathbb{R})$. Finally, fix a symplectic \mathcal{O}_F -stable filtration

$$Z_b : 0 \subset Z_{b,-2} \subset Z_{b,-1} \subset \mathbb{X}_b$$

with an isomorphism $Z_{b,-2} \cong \text{Hom}(\mathcal{O}_F^r, \mu_{p^\infty})$. This induces a parabolic subgroup $P_b(\mathbb{Q}_p) \subset J_b(\mathbb{Q}_p)$ of the self-quasi-isogenies preserving this filtration, and a p -divisible group $\mathbb{X}_P = Z_{b,-2}/Z_{b,-1}$ with $G_{2(n-r)}$ -structure; we denote by b_P its associated isocrystal with $G_{2(n-r)}$ -structure.

Theorem 6.1.1. *There is a natural $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ -equivariant isomorphism*

$$R\Gamma_c(\mathfrak{Jg}_{\infty,[P]}^{b,*}, i_{[P]}^* Rj_* \mathbb{F}_\ell) \cong \text{Ind}_{P_b(\mathbb{Q}_p) \times P(\mathbb{A}_f^p)}^{J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)} R\Gamma(\text{GL}_r(F) \backslash (X_r \times \text{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) \otimes R\Gamma_c(\mathfrak{Jg}_{\infty}^{bP}, \mathbb{F}_\ell).$$

The action of $P_b(\mathbb{Q}_p) \times P(\mathbb{A}_f^p)$ on

$$R\Gamma(\text{GL}_r(F) \backslash (X_r \times \text{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) \otimes R\Gamma_c(\mathfrak{Jg}_{\infty}^{bP}, \mathbb{F}_\ell)$$

is through its Levi quotient.

We will prove this result in several steps. First, we realize the structure of parabolic induction geometrically and reduce to a statement about $P_b(\mathbb{Q}_p) \times P(\mathbb{A}_f^p)$ -representations. Then we construct an equivariant map, as the cup product of two different maps, realizing the two factors. The factor $R\Gamma_c(\mathfrak{Jg}_{\infty}^{bP}, \mathbb{F}_\ell)$ is easy, while the other factor $R\Gamma(\text{GL}_r(F) \backslash (X_r \times \text{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell)$ will arise by realizing this real manifold inside the perfectoid space that is the punctured formal neighborhood of $\mathfrak{Jg}_{\infty,P}^{b,*}$. The latter is probably the most novel part of the proof. Finally, we show that the map is an isomorphism. For this, we use the explicit toroidal boundary charts. (We could have done this computation directly, but this would have obscured the Hecke-equivariance).

In the last subsection, we deduce Theorem 2.8.7 on possible systems of Hecke eigenvalues in the boundary. In particular, we show, under the same technical assumptions as in the last section, that all systems of Hecke eigenvalues appearing in the cohomology of Igusa varieties admit associated Galois representations.

6.2. Construction of the map. The goal of this section is to construct a natural map

$$\text{Ind}_{P_b(\mathbb{Q}_p) \times P(\mathbb{A}_f^p)}^{J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)} R\Gamma(\text{GL}_r(F) \backslash (X_r \times \text{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) \otimes R\Gamma_c(\mathfrak{Jg}_{\infty}^{bP}, \mathbb{F}_\ell) \rightarrow R\Gamma_c(\mathfrak{Jg}_{\infty,[P]}^{b,*}, i_{[P]}^* Rj_* \mathbb{F}_\ell).$$

6.2.1. Parabolic induction. First, we prove that

$$R\Gamma_c(\mathfrak{Jg}_{\infty,[P]}^{b,*}, i_{[P]}^* Rj_* \mathbb{F}_\ell)$$

is necessarily parabolically induced, by showing that the space $\mathfrak{Jg}_{\infty,[P]}^{b,*}$ is parabolically induced. More precisely, we note that there is a natural $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ -equivariant map

$$\mathfrak{Jg}_{\infty,[P]}^{b,*} \rightarrow J_b(\mathbb{Q}_p)/P_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)/P(\mathbb{A}_f^p)$$

induced by cusp labels, as $J_b(\mathbb{Q}_p)/P_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)/P(\mathbb{A}_f^p)$ parametrizes pairs (Z_b, Z^p) “of rank r ”. Let

$$\mathfrak{Jg}_{\infty,P}^{b,*} \subset \mathfrak{Jg}_{\infty,[P]}^{b,*}$$

be the fibre over the identity. It is then formal that

$$R\Gamma_c(\mathfrak{Jg}_{\infty,[P]}^{b,*}, i_{[P]}^* Rj_* \mathbb{F}_\ell) = \text{Ind}_{P_b(\mathbb{Q}_p) \times P(\mathbb{A}_f^p)}^{J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)} R\Gamma_c(\mathfrak{Jg}_{\infty,P}^{b,*}, i_P^* Rj_* \mathbb{F}_\ell)$$

as $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ -representations.

6.2.2. *Cup products.* It remains to construct a map

$$R\Gamma(\mathrm{GL}_r(F)\backslash(X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) \otimes R\Gamma_c(\mathfrak{Jg}_{\infty}^{b_P}, \mathbb{F}_\ell) \rightarrow R\Gamma_c(\mathfrak{Jg}_{\infty,P}^{b,*}, i_P^* Rj_* \mathbb{F}_\ell)$$

as a representation of $P_b(\mathbb{Q}_p) \times P(\mathbb{A}_f^p)$. To do this, we will construct two maps:

(1) A map

$$R\Gamma_c(\mathfrak{Jg}_{\infty}^{b_P}, \mathbb{F}_\ell) \rightarrow R\Gamma_c(\mathfrak{Jg}_{\infty,P}^{b,*}, \mathbb{F}_\ell).$$

(2) A map

$$R\Gamma(\mathrm{GL}_r(F)\backslash(X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) \rightarrow R\Gamma(\mathfrak{Jg}_{\infty,P}^{b,*}, i_P^* Rj_* \mathbb{F}_\ell).$$

Both maps will by construction be $P_b(\mathbb{A}_f^p) \times G(\mathbb{A}_f^p)$ -equivariant, where the action on the left-hand side will factor over $J_{b_P}(\mathbb{Q}_p) \times G_{2(n-r)}(\mathbb{A}_f^p)$ respectively $\mathrm{GL}_r(\mathbb{A}_{F,f})$, both of which are quotients of the Levi. The desired map will then arise as the cup product of the two maps, via the natural map

$$R\Gamma_c(\mathfrak{Jg}_{\infty,P}^{b,*}, \mathbb{F}_\ell) \otimes R\Gamma(\mathfrak{Jg}_{\infty,P}^{b,*}, i_P^* Rj_* \mathbb{F}_\ell) \rightarrow R\Gamma_c(\mathfrak{Jg}_{\infty,P}^{b,*}, i_P^* Rj_* \mathbb{F}_\ell).$$

The first map comes directly from the profinite map

$$\mathfrak{Jg}_{\infty,P}^{b,*} \rightarrow \mathfrak{Jg}_{\infty}^{b_P}$$

obtained by passing to the limit in Theorem 3.3.15.

6.2.3. *Perfectoid magic.* Thus, it remains to construct a natural equivariant map

$$R\Gamma(\mathrm{GL}_r(F)\backslash(X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) \rightarrow R\Gamma(\mathfrak{Jg}_{\infty,P}^{b,*}, i_P^* Rj_* \mathbb{F}_\ell).$$

Note that this seems a priori tricky, as the space on the left is naturally only a real manifold, while on the right we have a perfect scheme.

Let us indicate how one can bridge the gap by using analytic geometry; in the present situation, we will naturally get perfectoid spaces. Let us first give an idealized version of the argument and then add in the details.

Let $\mathcal{I}g_{\infty,P}^b$ be the perfectoid space obtained from $\mathfrak{Jg}_{\infty}^{b,*}$ by taking the punctured formal neighborhood of $\mathfrak{Jg}_{\infty,P}^{b,*}$. This agrees with the perfectoid space obtained from $\mathfrak{Jg}_{\infty}^{b,\mathrm{tor}}$ by taking the punctured formal neighborhood of the preimage of $\mathfrak{Jg}_{\infty,P}^{b,*}$. Using the corresponding description of the toroidal boundary, and restricting for the moment to a specific cusp label \tilde{Z} , we see that over $\mathcal{I}g_{\infty,\tilde{Z}}^b$ we have the universal abelian variety \mathcal{A} as well as a Raynaud extension

$$0 \rightarrow \mathcal{T} \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0$$

and a lift of the corresponding map $f_0 : X \rightarrow \mathcal{B}$ to a symmetric map $f : X \rightarrow \mathcal{G}$; equivalently, a section of the Poincaré bundle $\mathcal{P} \rightarrow \mathcal{B} \times \mathcal{B}$ over $X \times X$; such that $\mathcal{A} = \mathcal{G}/X$. Note that the principally polarized abelian variety \mathcal{B} has locally good reduction, and in particular the Poincaré bundle \mathcal{P} over $\mathcal{B} \times \mathcal{B}$ has a canonical integral structure. If one takes a point $x \in \mathcal{I}g_{\infty,\tilde{Z}}^b$ and fixes a norm $|\cdot| : K(x) \rightarrow \mathbb{R}_{\geq 0}$ on the residue field, then the map taking $x, y \in X$ to the logarithm of the norm of the section of \mathcal{P} gives a positive definite symmetric hermitian form

$$X \times X \rightarrow \mathbb{R};$$

changing the norm $|\cdot|$ changes this only by a scalar. We see that there is a canonical map

$$|\mathcal{I}g_{\infty,\tilde{Z}}^b| \rightarrow \mathrm{GL}_{\mathcal{O}_F}(X)\backslash X_r.$$

In fact, working globally and Hecke-equivariantly, we note that $X_{\mathbb{Q}}$ is a well-defined F -vector space of rank r over all of $\mathcal{I}g_{\infty, P}^b$. Fixing an isomorphism $X_{\mathbb{Q}} \cong F^r$, we get (for each $x \in \mathcal{I}g_{\infty, P}^b$) a point of X_r . Moreover, we have level structures: In the present situation, this is an \mathcal{O}_F -linear symplectic isomorphism $V_f^p(\mathcal{A}) \cong V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ matching the subspace $V_f^p(\mathcal{T})$ with $F^r \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. This gives another isomorphism $X_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong F^r \otimes_{\mathbb{Q}} \mathbb{A}_f^p$, i.e. an element of $\mathrm{GL}_r(F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)$. Similarly, the Igusa level structure induces an isomorphism $X_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong F^r \otimes_{\mathbb{Q}} \mathbb{Q}_p$, i.e. an element of $\mathrm{GL}_r(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. In total, we get a natural continuous map

$$f : |\mathcal{I}g_{\infty, P}^b| \rightarrow \mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F, f}))$$

and it follows from the construction that it is $J_b(\mathbb{Q}_p) \times P(\mathbb{A}_f^p)$ -equivariant.

There is a natural map

$$R\Gamma(\mathfrak{J}g_{\infty, P}^{b,*}, i_P^* Rj_* \mathbb{F}_{\ell}) \rightarrow R\Gamma(\mathcal{I}g_{\infty, P}^b, \mathbb{F}_{\ell})$$

(as the left-hand side can be computed as the cohomology of the perfect scheme obtained by taking the henselization of $\mathfrak{J}g_{\infty, P}^{b,*}$ along $\mathfrak{J}g_{\infty, P}^{b,*}$ and deleting the boundary, and there is a natural map from $\mathcal{I}g_{\infty, P}^b$ to that scheme) that is probably an isomorphism. However, as $\mathcal{I}g_{\infty, P}^b$ is in general highly non-quasicompact – in fact, at least as noncompact as the locally symmetric space $\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F, f}))$ – it is nontrivial to justify this. If the map were an isomorphism, we would now get a natural map

$$\begin{aligned} R\Gamma(\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F, f})), \mathbb{F}_{\ell}) &\xrightarrow{f^*} R\Gamma(\mathcal{I}g_{\infty, P}^b, \mathbb{F}_{\ell}) \\ &\cong R\Gamma(\mathfrak{J}g_{\infty, P}^{b,*}, i_P^* Rj_* \mathbb{F}_{\ell}). \end{aligned}$$

For this reason, we make the following small circumlocutions. Recall that the natural map

$$\varinjlim_{K \subset \mathrm{GL}_r(\mathbb{A}_{F, f})} R\Gamma(\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F, f})/K), \mathbb{F}_{\ell}) \rightarrow R\Gamma(\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F, f})), \mathbb{F}_{\ell})$$

is an isomorphism, where K runs over compact open subgroups of $\mathrm{GL}_r(\mathbb{A}_{F, f})$; this follows for example from the Borel–Serre compactification.¹⁷

¹⁷Indeed, let X_r^{BS} be the Borel–Serre compactification of X_r . Then both X_r and X_r^{BS} are contractible. It follows that for any paracompact Hausdorff space S , one has

$$R\Gamma(X_r \times S, A) = R\Gamma(S, A) = R\Gamma(X_r^{\mathrm{BS}} \times S, A)$$

for any coefficient module A ; here cohomology means Čech (or equivalently sheaf) cohomology. By descent along the $\mathrm{GL}_r(F)$ -quotient, it follows that the natural map

$$R\Gamma(\mathrm{GL}_r(F) \backslash (X_r^{\mathrm{BS}} \times \mathrm{GL}_r(\mathbb{A}_{F, f})/H), A) \rightarrow R\Gamma(\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F, f})/H), A)$$

is an isomorphism, for any closed subgroup $H \subset \mathrm{GL}_r(\mathbb{A}_{F, f})$. On the other hand, the quotients $\mathrm{GL}_r(F) \backslash (X_r^{\mathrm{BS}} \times \mathrm{GL}_r(\mathbb{A}_{F, f})/H)$ are compact Hausdorff, and the natural maps

$$\mathrm{GL}_r(F) \backslash (X_r^{\mathrm{BS}} \times \mathrm{GL}_r(\mathbb{A}_{F, f})/H) \rightarrow \varinjlim_{K \supset H} \mathrm{GL}_r(F) \backslash (X_r^{\mathrm{BS}} \times \mathrm{GL}_r(\mathbb{A}_{F, f})/K)$$

are homeomorphisms (as continuous bijections between compact Hausdorff spaces), where K runs through open compact subgroups containing H . This implies that on Čech cohomology,

$$\varinjlim_{K \supset H} R\Gamma(\mathrm{GL}_r(F) \backslash (X_r^{\mathrm{BS}} \times \mathrm{GL}_r(\mathbb{A}_{F, f})/K), A) \rightarrow R\Gamma(\mathrm{GL}_r(F) \backslash (X_r^{\mathrm{BS}} \times \mathrm{GL}_r(\mathbb{A}_{F, f})/H), A)$$

is an isomorphism.

Similarly, for each finite m and $N \geq 3$ prime to p , let

$$\mathrm{Ig}_{m,K(N),P}^{b,*} \subset \mathrm{Ig}_{m,K(N)}^{b,*}$$

be the image of $\mathfrak{I}\mathfrak{g}_{\infty,P}^{b,*}$, and let $\mathcal{I}g_{m,K(N),P}^b$ be formed similarly as the punctured formal completion, which is an analytic adic space. Then it follows from [Hub96, Corollary 3.5.14] applied at each of these finite levels, plus passage to the limit on the left-hand side, that

$$R\Gamma(\mathfrak{I}\mathfrak{g}_{\infty,P}^{b,*}, i_P^* Rj_* \mathbb{F}_\ell) \cong \varinjlim_{m,N} R\Gamma(\mathcal{I}g_{m,K(N),P}^b, \mathbb{F}_\ell).$$

Moreover, the map

$$f : |\mathcal{I}g_{\infty,P}^b| \rightarrow \mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f}))$$

is the limit of a map of pro-systems

$$\{\mathcal{I}g_{m,K(N),P}^b\}_{m,N} \rightarrow \{\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f})/K)\}_K.$$

Thus, passing to the colimit, we get the desired map

$$\begin{aligned} R\Gamma(\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) &\cong \varinjlim_{K \subset \mathrm{GL}_r(\mathbb{A}_{F,f})} R\Gamma(\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f})/K), \mathbb{F}_\ell) \\ &\rightarrow \varinjlim_{m,N} R\Gamma(\mathcal{I}g_{m,K(N),P}^b, \mathbb{F}_\ell) \\ &\cong R\Gamma(\mathfrak{I}\mathfrak{g}_{\infty,P}^{b,*}, i_P^* Rj_* \mathbb{F}_\ell). \end{aligned}$$

On the level of pro-systems (and passing to perfections where appropriate), everything carries natural $P_b(\mathbb{Q}_p) \times P(\mathbb{A}_f^p)$ -actions for which the maps are equivariant, implying the desired equivariance.

6.3. Local computation. In this section, we do a local computation with toroidal boundary charts to finish the proof of Theorem 6.1.1.

Note that

$$\mathfrak{I}\mathfrak{g}_{\infty,P}^{b,*}$$

admits a further decomposition according to Igusa cusp labels above P . Concretely, this amounts to finite projective \mathcal{O}_F -modules X of rank r with isomorphisms $X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \cong \mathcal{O}_F^r \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$ and $X \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}_F^r \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (as we have fixed an isomorphism $\mathbb{Z}_{b,-2} \cong \mathrm{Hom}(\mathcal{O}_F^r, \mu_{p^\infty})$). Noting that we are really only mapping to the limit over all levels K of K -equivalence classes of cups labels, we get a decomposition according to

$$\bigsqcup_{X/\cong} \overline{\mathrm{GL}_{\mathcal{O}_F}(X)} \backslash \mathrm{GL}_{\mathcal{O}_F}(X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}).$$

On the other hand, in the last section, we constructed a map from the punctured formal neighborhoods to

$$\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f})),$$

and in particular to $\overline{\mathrm{GL}_r(F)} \backslash \mathrm{GL}_r(\mathbb{A}_{F,f})$. Note that the natural map

$$\bigsqcup_{X/\cong} \overline{\mathrm{GL}_{\mathcal{O}_F}(X)} \backslash \mathrm{GL}_{\mathcal{O}_F}(X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) \rightarrow \overline{\mathrm{GL}_r(F)} \backslash \mathrm{GL}_r(\mathbb{A}_{F,f})$$

is a bijection. It is readily seen that these two maps are compatible.

Thus, from now on we fix a finite projective \mathcal{O}_F -module X of rank r with an isomorphism $X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \cong \mathcal{O}_F^r \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, corresponding to an Igusa cusp label \tilde{Z} , and let

$$\mathfrak{Jg}_{\infty, \tilde{Z}}^{b,*} \subset \mathfrak{Jg}_{\infty, P}^{b,*}$$

be the corresponding closed subset, defined as the inverse limit of the corresponding strata at finite level. Note that the corresponding part of the cohomology of $\mathrm{GL}_r(F) \backslash (X_r \times \mathrm{GL}_r(\mathbb{A}_{F,f}))$ is given by

$$\varinjlim_{\Gamma \subset \mathrm{GL}_{\mathcal{O}_F}(X)} R\Gamma(\Gamma, \mathbb{F}_\ell),$$

where Γ runs through congruence subgroups of $\mathrm{GL}_{\mathcal{O}_F}(X)$ (recall that X_r is contractible). Thus, we have to prove the following statement.

Proposition 6.3.1. *The map*

$$R\Gamma_c(\mathfrak{Jg}_{\infty}^{b_P}, \mathbb{F}_\ell) \otimes \varinjlim_{\Gamma \subset \mathrm{GL}_{\mathcal{O}_F}(X)} R\Gamma(\Gamma, \mathbb{F}_\ell) \rightarrow R\Gamma_c(\mathfrak{Jg}_{\infty, \tilde{Z}}^{b,*}, i_{\tilde{Z}}^* Rj_* \mathbb{F}_\ell)$$

constructed in the previous section is an isomorphism.

Proof. Using the toroidal compactification to do the computation and the explicit description of the toroidal boundary charts from Theorem 3.3.12, this is a by now standard computation due to Pink, [Pin92, Theorem 4.2.1], see also [LS18a, Theorem 4.3.10]. The key point is to notice that the inverse system $(\mathrm{Ig}_{K^p}^{b, \mathrm{tor}})_{K^p}$ of partial toroidal compactifications of Igusa varieties has the axiomatic properties described in [LS18a, Lemma 4.3.2]. \square

This finishes the proof of Theorem 6.1.1.

Remark 6.3.2. Theorem 6.1.1 can be interpreted as a version of Pink's formula [Pin92] for Igusa varieties. A similar argument as in § 6.2.3 in the setting of Shimura varieties can be used to prove that the original version of Pink's formula is Hecke-equivariant. This provides an alternative to the argument in [Pin92, §4.8].

6.4. Applications. We now use Theorem 6.1.1 to construct Galois representations associated to maximal ideals $\mathfrak{m} \subset \mathbb{T}$ in the support of $R\Gamma_{(c-\partial)}(\mathrm{Ig}^b, \mathbb{F}_\ell)$.

From now on, assume that F contains (properly) an imaginary quadratic field $F_0 \subset F$ in which p splits and fix the finite set S of places of \mathbb{Q} and a level $N \geq 3$ prime to p as in the last section. We let \mathbb{T}^S be the same unramified Hecke algebra, and fix a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ containing ℓ .

We will consider the Igusa varieties with implicit tame level $K^p(N)$.

Theorem 6.4.1. *Assume that for some $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$ one of the cohomology groups*

$$H_{c-\partial}^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}}, H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}}$$

is nonzero. Then there exists a continuous semisimple Galois representation

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_{2n}(\bar{\mathbb{F}}_\ell)$$

such that for all primes v dividing a rational prime $q \notin S$ that splits in F_0 , the characteristic polynomial of $\bar{\rho}_{\mathfrak{m}}$ is given by

$$X^{2n} - T_{1,v} X^{2n-1} + \dots + (-1)^i q_v^{i(i-1)/2} T_{i,v} X^{2n-i} + \dots + q_v^{n(2n-1)} T_{2n,v}$$

with notation as in the introduction.

Moreover, if the map

$$H_{c-\partial}^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m} \rightarrow H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m}$$

is not an isomorphism and b is not ordinary, then $\bar{\rho}_\mathfrak{m}$ is of length at least 3.

Before proving the theorem, we record some preliminary notation and results. We denote by

$$R\Gamma_{\mathrm{cont}}(K^P(N),) : D_{\mathrm{sm}}^+(G(\mathbb{A}_f^P), \mathbb{F}_\ell) \rightarrow D^+(\mathbb{T}^S)$$

the derived functor of $K^P(N)$ -invariants in the bounded below derived category of smooth $G(\mathbb{A}_f^P)$ -representations with \mathbb{F}_ℓ -coefficients. We also consider the forgetful functor $D^+(\mathbb{T}^S) \rightarrow D^+(\mathbb{Z})$. After applying this forgetful functor to the left hand sides, we obtain \mathbb{T}^S -equivariant isomorphisms

$$(6.4.1) \quad R\Gamma_{\mathrm{cont}}(K^P(N), R\Gamma_c(\mathfrak{J}\mathfrak{g}_\infty^b, \mathbb{F}_\ell)) \xrightarrow{\sim} R\Gamma_c(\mathrm{Ig}^b, \mathbb{F}_\ell)$$

and

$$(6.4.2) \quad R\Gamma_{\mathrm{cont}}(K^P(N), R\Gamma_c(\mathfrak{J}\mathfrak{g}_\infty^{b,*}, Rj_*\mathbb{F}_\ell)) \xrightarrow{\sim} R\Gamma_c(\mathrm{Ig}^{b,*}, Rj_*\mathbb{F}_\ell).$$

in $D^+(\mathbb{Z})$.

We set $K_P^S := K^S \cap P(\mathbb{A}^S)$ and consider the corresponding abstract Hecke algebra \mathbb{T}_P^S for P . Recall from [NT16, §2.2.3] the ring homomorphism $r_P : \mathbb{T}^S \rightarrow \mathbb{T}_P^S$ given by “restriction of functions”. The following is a version of [NT16, Corollary 2.6] for smooth representations.

Lemma 6.4.2. *There is a natural isomorphism of functors*

$$R\Gamma_{\mathrm{cont}}\left(K^S, \mathrm{Ind}_{P(\mathbb{A}^S)}^{G(\mathbb{A}^S)}\right) \xrightarrow{\sim} r_P^* R\Gamma_{\mathrm{cont}}(K_P^S,)$$

from $D_{\mathrm{sm}}^+(P(\mathbb{A}^S), \mathbb{F}_\ell)$ to $D^+(\mathbb{T}_P^S)$.

Proof. Since K^S is hyperspecial for every place $v \notin S$ of F , we have the Iwasawa decomposition $G(\mathbb{A}^S) = K^S \cdot P(\mathbb{A}^S)$. For non-derived functors, we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{sm}}(G(\mathbb{A}^S), \mathbb{F}_\ell) & \xrightarrow{\Gamma(K^S,)} & \mathrm{Mod}(\mathbb{T}^S) \\ \mathrm{Ind}_{P(\mathbb{A}^S)}^{G(\mathbb{A}^S)} \uparrow & & \uparrow r_P^* \\ \mathrm{Mod}_{\mathrm{sm}}(P(\mathbb{A}^S), \mathbb{F}_\ell) & \xrightarrow{\Gamma(K_P^S,)} & \mathrm{Mod}(\mathbb{T}_P^S), \end{array}$$

where the smooth induction $\mathrm{Ind}_{P(\mathbb{A}^S)}^{G(\mathbb{A}^S)}$ is exact and preserves injectives, r_P^* is exact, and $\Gamma(K^S,)$ and $\Gamma(K_P^S,)$ are left exact. The proof that the diagram commutes is identical to that of [NT16, Lemma 2.4 (iii)]. Taking derived functors, we obtain the lemma. \square

Let $M = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_r \times G_{n-r, \mathbb{Q}}$ be the Levi and N be the unipotent radical of P ; we have a Levi decomposition $P = M \rtimes N$. The compact open subgroup $K_P^S \subset P(\mathbb{A}^S)$ is decomposed in the sense of [NT16, §2.2.4] with respect to this Levi decomposition. We set $K_M^S := K_P^S \cap M(\mathbb{A}^S)$ and $K_N^S := K_P^S \cap N(\mathbb{A}^S)$. Recall also from [NT16, §2.2.4] the ring homomorphism $r_M : \mathbb{T}_P^S \rightarrow \mathbb{T}_M^S$ given by “integration along unipotent fibers”. The following is an analogue of [NT16, Corollary 2.8] for smooth representations.

Lemma 6.4.3. *There is a natural isomorphism of functors*

$$r_M^* R\Gamma_{\text{cont}}(K_M,) \xrightarrow{\sim} R\Gamma_{\text{cont}}(K_P^S, \text{Inf}_{M(\mathbb{A}^S)}^{P(\mathbb{A}^S)})$$

from $D_{\text{sm}}^+(M(\mathbb{A}^S), \mathbb{F}_\ell)$ to $D^+(\mathbb{T}_P^S)$.

Proof. On the level of non-derived functors, we have the commutative diagram

$$\begin{array}{ccc} \text{Mod}_{\text{sm}}(P(\mathbb{A}^S), \mathbb{F}_\ell) & \xrightarrow{\Gamma(K_P^S,)} & \text{Mod}(\mathbb{T}_P^S) \\ \text{Inf}_{M(\mathbb{A}^S)}^{P(\mathbb{A}^S)} \uparrow & & \uparrow r_M^* \\ \text{Mod}_{\text{sm}}(M(\mathbb{A}^S), \mathbb{F}_\ell) & \xrightarrow{\Gamma(K_M^S,)} & \text{Mod}(\mathbb{T}_M^S), \end{array}$$

where the vertical functors are exact and the horizontal functors are left exact. We do not know that $\text{Inf}_{M(\mathbb{A}^S)}^{P(\mathbb{A}^S)}$ preserves injectives, but we get a natural transformation of functors

$$r_M^* R\Gamma_{\text{cont}}(K_M,) \rightarrow R\Gamma_{\text{cont}}(K_P^S, \text{Inf}_{M(\mathbb{A}^S)}^{P(\mathbb{A}^S)})$$

by [NT16, Lemma 2.1].

Now observe that the above natural transformation becomes an isomorphism on any object of $D_{\text{sm}}^+(M(\mathbb{A}^S), \mathbb{F}_\ell)$. It is enough to check this after applying functors that forget the Hecke action. We have $R\Gamma_{\text{cont}}(K_P^S,) = R\Gamma_{\text{cont}}(K_M^S,) \circ \Gamma(K_N^S,)$, because $\Gamma(K_N^S,)$ preserves injectives and is exact on $D_{\text{sm}}^+(P(\mathbb{A}^S), \mathbb{F}_\ell)$. This last part follows from the fact that K_N^S is profinite with order prime to ℓ . On the other hand, $\Gamma(K_N^S,) \circ \text{Inf}_{K_M^S}^{K_P^S} = \text{Id}$. \square

Proof of Theorem 6.4.1. We argue by induction on n , so we may assume that the analogous result is known for Igusa varieties on smaller unitary groups. Assume first that the map

$$H_{c-\partial}^i(\text{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}} \rightarrow H^i(\text{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}}$$

is not an isomorphism. In particular, this happens at some finite Igusa level p^m , and then by Poincaré duality the map

$$H_c^i(\text{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}^\vee} \rightarrow H_c^i(\text{Ig}^{b,*}, Rj_* \mathbb{F}_\ell)_{\mathfrak{m}^\vee}$$

is not an isomorphism for the “dual” set of Hecke eigenvalues \mathfrak{m}^\vee .

From the \mathbb{T}^S -equivariant isomorphisms (6.4.1) and (6.4.2), and by considering the stratification of the boundary of $\mathfrak{Jg}_{\infty}^{b,*}$ in terms of the conjugacy classes $[P]$ of rational parabolic subgroups of G , we deduce that for some such $[P]$ corresponding to an integer $1 \leq r \leq n$,

$$R\Gamma_{\text{cont}}(K^P(N), R\Gamma_c(\mathfrak{Jg}_{\infty, [P]}^{b,*}, i_{[P]}^* Rj_* \mathbb{F}_\ell))_{\mathfrak{m}^\vee} \neq 0.$$

In particular,

$$(6.4.3) \quad R\Gamma_{\text{cont}}(K^S, R\Gamma_c(\mathfrak{Jg}_{\infty, [P]}^{b,*}, i_{[P]}^* Rj_* \mathbb{F}_\ell))_{\mathfrak{m}^\vee} \neq 0$$

where $K^S = G(\mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^S)$.

We now wish to apply Theorem 6.1.1 in order to show that \mathfrak{m}^\vee must be pulled back under the unnormalized Satake transform $\mathbb{T}^S \rightarrow \mathbb{T}_M^S$ from a maximal ideal of the Hecke algebra \mathbb{T}_M^S . We will do this in two steps, going via the Hecke algebra \mathbb{T}_P^S

of P . By Theorem 6.1.1 and Lemma 6.4.2, we can rewrite (6.4.3) as the localization at \mathfrak{m}^\vee of

$$(6.4.4) \quad r_P^* R\Gamma_{\text{cont}} \left(K_P^S, \text{Inf}_{M(\mathbb{A}^S)}^{P(\mathbb{A}^S)} \text{Ind}_{P_b(\mathbb{Q}_p) \times P(F \otimes_{\mathbb{Q}} \mathbb{A}_{S_f^p})}^{J_b(\mathbb{Q}_p) \times G(F \otimes_{\mathbb{Q}} \mathbb{A}_{S_f^p})} R\Gamma(\text{GL}_r(F) \backslash (X_r \times \text{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) \otimes R\Gamma_c(\mathcal{I}\mathfrak{g}_\infty^{b_P}, \mathbb{F}_\ell) \right).$$

By Lemma 6.4.3, we can rewrite (6.4.4) as the localization at \mathfrak{m}^\vee of

$$(6.4.5) \quad r_M^* r_P^* R\Gamma_{\text{cont}} \left(K_M^S, \text{Ind}_{P_b(\mathbb{Q}_p) \times P(F \otimes_{\mathbb{Q}} \mathbb{A}_{S_f^p})}^{J_b(\mathbb{Q}_p) \times G(F \otimes_{\mathbb{Q}} \mathbb{A}_{S_f^p})} R\Gamma(\text{GL}_r(F) \backslash (X_r \times \text{GL}_r(\mathbb{A}_{F,f})), \mathbb{F}_\ell) \otimes R\Gamma_c(\mathcal{I}\mathfrak{g}_\infty^{b_P}, \mathbb{F}_\ell) \right).$$

After applying the forgetful functor to $D^+(\mathbb{Z})$, and using (6.4.1) for Ig^{b_P} and the analogue for the locally symmetric spaces for GL_r , we see that (6.4.5) is \mathbb{T}^S -equivariantly isomorphic to

$$r_M^* r_P^* \text{Ind}_{P_b(\mathbb{Q}_p) \times P(F \otimes_{\mathbb{Q}} \mathbb{A}_{S_f^p})}^{J_b(\mathbb{Q}_p) \times G(F \otimes_{\mathbb{Q}} \mathbb{A}_{S_f^p})} R\Gamma(\text{GL}_r(F) \backslash (X_r \times \text{GL}_r(\mathbb{A}_{F,f}) / K^{S, \text{GL}_r}), \mathbb{F}_\ell) \otimes R\Gamma_c(\text{Ig}_{K^{S, G_2(n-r)}}^{b_P}, \mathbb{F}_\ell),$$

where $K^{S, \text{GL}_r} = \text{GL}_r(\mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^S)$ and similarly $K^{S, G_2(n-r)} = G_{2(n-r)}(\mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^S)$ (satisfying $K_M^S = K^{S, \text{GL}_r} \times K^{S, G_2(n-r)}$).

Note that the composition $r_M \circ r_P : \mathbb{T}^S \rightarrow \mathbb{T}_M^S$ is the unnormalised Satake transform. Using the decomposition $M = \text{Res}_{F/\mathbb{Q}} \text{GL}_r \times G_{n-r, \mathbb{Q}}$, we find that there are maximal ideals \mathfrak{m}_1 of the Hecke algebra for $\text{Res}_{\mathcal{O}_F/\mathbb{Z}} \text{GL}_r$ and \mathfrak{m}_2 of the Hecke algebra for $G_{2(n-r)}$ such that

$$R\Gamma(\text{GL}_r(F) \backslash (X_r \times \text{GL}_r(\mathbb{A}_{F,f}) / K^{S, \text{GL}_r}), \mathbb{F}_\ell)_{\mathfrak{m}_1} \neq 0$$

and

$$R\Gamma_c(\text{Ig}_{K^{S, G_2(n-r)}}^{b_P}, \mathbb{F}_\ell)_{\mathfrak{m}_2} \neq 0,$$

and \mathfrak{m}^\vee maps into a maximal ideal containing $\mathfrak{m}_1 \otimes \mathfrak{m}_2 \subset \mathbb{T}_M$. Using [Sch15, Corollary 5.4.3], the induction hypothesis, and Poincaré duality, we get the desired Galois representation $\bar{\rho}_{\mathfrak{m}}$ associated to \mathfrak{m} .

If $r < n$, $\bar{\rho}_{\mathfrak{m}}$ will be a direct sum of three representations: Note that the representation from GL_r will contribute two summands. If b is not ordinary, then $P = P_n \subset G$ does not contribute a stratum to the Igusa variety (as there will be no Igusa cusp labels where the rank of X is n), justifying the final statement.

Thus, we are left with the case that

$$H_{c-\partial}^i(\text{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}} \rightarrow H^i(\text{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}}$$

is an isomorphism. Possibly changing b , we can then ensure, using Proposition 2.8.4 and Corollary 2.8.2, that these cohomology groups are concentrated in one degree. Now Theorem 5.1.2 finishes the proof. \square

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