On the image of complex conjugation in certain Galois representations

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Abstract

We compute the image of any choice of complex conjugation on the Galois representations associated to regular algebraic cuspidal automorphic representations and to torsion classes in the cohomology of locally symmetric spaces for GL_n over a totally real field F.

1. Introduction

The goal of this note is to describe the image of any choice of complex conjugation on the Galois representations associated to regular algebraic cuspidal automorphic representations and to (mod p) torsion classes in the cohomology of locally symmetric spaces for GL_n over a totally real field F. Since any choice of complex conjugation has eigenvalues ± 1, the key computation is to determine how many $+1$'s and how many -1's occur; we do this by showing that their numbers differ by at most 1. Our results are conditional on Arthur's work [Art13].

In the case of regular algebraic cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ which are essentially self-dual this is known in almost all cases, due to Taylor [Tay12] (when n is odd and under the assumption that the corresponding Galois representation is irreducible) and Taibi [Tai12] (all cases when n is odd and most cases when n is even). We note that in the essentially self-dual case when n is odd, the corresponding Galois representation occurs in the étale cohomology of a certain Shimura variety. Taylor makes use of a geometric realization of complex conjugation and studies its action on the Hodge filtration of the Betti cohomology of this Shimura variety. Taibi uses p-adic interpolation techniques (eigenvarieties) to extend Taylor's result to almost all essentially self-dual cases.

Recently, Harris, Lan, Taylor and Thorne used more geometric p-adic interpolation techniques in [HLTT13] to construct Galois representations associated to regular algebraic cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ which do not need to be essentially self-dual. Later, Scholze gave a different construction in [Sch], still via p-adic interpolation, which also applies to torsion classes in the cohomology of the corresponding locally symmetric space.

In this paper, we extend the result concerning the image of complex conjugation beyond the essentially self-dual case using the very techniques which led to the construction of the Galois representations we are interested in. We follow Scholze's approach rather than that of [HLTT13]. Just as the construction of Galois representations for torsion classes in the case when F is totally real, our result makes use of the transfer of a cusp form on Sp_{2n} to GL_{2n+1} and is therefore dependent on [Art13], which is still conditional on the stabilization of the twisted trace formula.

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In addition, we rely on Taibi's main result [Tai12], which is conditional on the analogue of Arthur's results for inner forms of quasi split classical groups.

Let F be a totally real field and let π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ such that π_∞ is regular L-algebraic. Let S be a finite set of places of F, which contains all the places where π is ramified, and let $G_{F,S}$ denote the Galois group of the maximal extension of F unramified outside S. Then there exists a Galois representation

$$\sigma_\pi : G_{F,S} \to GL_n(\bar{\mathbb{Q}}_p)$$

which satisfies local-global compatibility at all finite places $v \notin S$. More precisely, for every finite place $v \notin S$, the Satake parameters of π_v are the same as the eigenvalues of $\sigma_\pi(\mathrm{Frob}_v)$ (see, for example, Corollary V.4.2 of [Sch]). We prove the following:

Theorem 1.1. *Let π be a regular L-algebraic, cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$, *with associated (p-adic) Galois representation σ_π. Let $c \in \mathrm{Gal}(\bar{F}/F)$ be a choice of complex conjugation. Then $\mathrm{tr}(\sigma_\pi)(c) = 0$ if n is even and $\mathrm{tr}(\sigma_\pi)(c) = \pm 1$ if n is odd.*

Let

$$\mathbb{T}_{F,S} := \bigotimes_{v \notin S} \mathbb{T}_v, \mathbb{T}_v = \mathbb{Z}_p[GL_n(F_v)//GL_n(\mathcal{O}_{F_v})]$$

be the abstract Hecke algebra. For a sufficiently small level $K \subset GL_n(\mathbb{A}_{F,f})$, define the locally symmetric space

$$X_K := GL_n(F)\backslash \left(GL_n(\mathbb{A}_{F,f})/K \times GL_n(F \otimes_{\mathbb{Q}} \mathbb{R})/\mathbb{R}_{>0}K_\infty\right),$$

where $K_\infty \subset GL_n(F \otimes_{\mathbb{Q}} \mathbb{R})$ is a maximal compact subgroup. The representation $\mathfrak{p}'$ determines a homomorphism $\psi : \mathbb{T}_{F,S} \to \bar{\mathbb{Z}}_p$ (a *system of Hecke eigenvalues*), which factors through some

$$\mathrm{Im}(\mathbb{T}_{F,S} \to \mathrm{End}_{\bar{\mathbb{Z}}_p}(H^i(X_K, \mathcal{M}_{\xi,K}))).$$

(To be precise, we may need to twist π by some quadratic character, as in the proof of Corollary V.4.2 of [Sch], to get a cuspidal automorphic representation π' which contributes to $H^i(X_K, \mathcal{M}_{\xi,K})$. This is because we are working with the *connected* locally symmetric space, but twisting by a quadratic character is harmless for our purposes. Moreover, π' a priori determines a homomorphism of the Hecke algebra into $\bar{\mathbb{Q}}_p$, but local-global compatibility at places $v \notin S$ guarantees that this actually lands inside $\bar{\mathbb{Z}}_p$).

Let $\bar{\psi} : \mathbb{T}_{F,S} \to \bar{\mathbb{F}}_p$ be obtained from ψ by composing with the natural map $\bar{\mathbb{Z}}_p \to \bar{\mathbb{F}}_p$. Since π occurs in $H^i(X_K, \mathcal{M}_{\xi,K}) \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p$, we see by Proposition 1.2.3 of [AS86] that

$$H^i(X_K, \mathcal{M}_{\xi,K} \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p)[\bar{\psi}] \neq 0.$$

An argument using the Hochschild-Serre spectral sequence (see, for example, the proof of Theorem V.4.1 of [Sch]) tell us that

$$H^i(X_K, \bar{\mathbb{F}}_p)[\bar{\psi}] \neq 0,$$

where we have possibly replaced K by a smaller compact open subgroup. Thus, the reduction mod p of the system of Hecke eigenvalues corresponding to π occurs in the mod p cohomology of X_K. There is also a mod p^n version of the above picture.

Theorem 1.2. *Let p be an odd prime. Let ψ be a system of Hecke eigenvalues occurring in $H^i(X_K, \bar{\mathbb{F}}_p)$ and let σ_ψ be the corresponding Galois representation. Let $c \in \mathrm{Gal}(\bar{F}/F)$ be a choice of complex conjugation. Then $\sigma_\pi(c)$ has $+1$ as an eigenvalue with multiplicity $\lceil \frac{n-1}{2} \rceil$ and -1 as an eigenvalue with multiplicity $\lfloor \frac{n+1}{2} \rfloor$.*

The Galois representation σ_ψ is obtained by specializing an n-dimensional continuous determinant, which is extracted from a $2n+1$-dimensional determinant which in turn interpolates the Galois representations associated to regular algebraic automorphic representations of Sp_{2n}/F. This essentially shows that $1 \oplus \sigma_\psi \oplus \check{\sigma}_\psi$ is congruent to a Galois representation associated to (the transfer to GL_{2n+1} of) a cuspidal automorphic representation of Sp_{2n}. We then compute the characteristic polynomial of any choice of complex conjugation on the latter Galois representation (using Taibi's main theorem) and therefore determine the characteristic polynomial of any complex conjugation on the former. This gives Theorem 1.2. Adapting this for mod p^n systems of Hecke eigenvalues then gives us Theorem 1.1.

Remark 1.3. Theorem 1.1 applies in particular to the essentially self-dual representations not covered by Taibi's theorem. However, our proof does not give a new proof of his results, as it was one of the inputs of our argument.

In practice, some technical complications arise. Theorem V.4.1 of [Sch] guarantees that there is a determinant valued in the quotient of $\mathbb{T}_{F,S}$ which acts faithfully on $H^i(X_K, \bar{\mathbb{F}}_p)$, glued out of determinants valued in similar quotients acting on the interior cohomology $H^i_!(X_K, \bar{\mathbb{F}}_p)$ and on the cohomology of the boundary of the Borel-Serre compactification of X_K, $H^{i+1}(X_K^{\mathrm{BS}}, \bar{\mathbb{F}}_p)$.

The determinant valued in the Hecke algebra acting on $H^i_!(X_K, \bar{\mathbb{F}}_p)$ is constructed by showing that the interior cohomology above contributes to the cohomology of the boundary of the Borel-Serre compactification of the locally symmetric space for $G_0 := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}_{2n}$. The torsion cohomology of the locally symmetric space for G_0 is then related to classical cusp forms. Only this part is directly related to *cuspidal* automorphic forms on G_0. We review the construction of this determinant in Section 2.

On the other hand, the determinant valued in the Hecke algebra acting on $H^{i+1}(X_K^{\mathrm{BS}}, \bar{\mathbb{F}}_p)$ is glued together out of the determinant for interior cohomology and determinants for locally symmetric spaces for GL'_n with $n' < n$, whose cohomology contributes to the boundary cohomology of the Borel-Serre compactification of X_K. This allows an induction argument. We review the geometry of the boundary of the Borel-Serre compactification and explain the construction of the determinant in Section 3. In section 4, we put all of this together to compute the characteristic polynomial of complex conjugation.

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2. Determinants and completed cohomology

In this section, we recall the construction due to Scholze of determinants (and, therefore, Galois representations) associated to systems of Hecke eigenvalues occurring in the completed cohomology of Shimura varieties for symplectic groups.

Recall that $G_0 := \mathrm{Res}_{F/\mathbb{Q}} Sp_{2n}$. In this section, all Shimura varieties and Hecke algebras will be with respect to G_0. For $K^p \subset G_0(\mathbb{A}^{\infty,p})$ a sufficiently small compact open subgroup, let

$$\mathbb{T}_{K^p} := \mathbb{Z}_p[G(\mathbb{A}_f^p//K^p)]$$

be the abstract Hecke algebra over $\mathbb{Z}_p$ of K^p-biinvariant compactly-supported functions on $G(\mathbb{A}_\mho)$. For $K_p \subset G_0(\mathbb{Q}_p)$ compact open, let $X_{K^pK_p}$ be the Shimura variety for G_0 of level K^pK_p and let $X^*_{K^pK_p}$ be its minimal compactification. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$, which we will use as our coefficients, and let π be a uniformizer.

Let d be the dimension of $X_{K^pK_p}$ and let $\tilde{\mathbb{T}}_{K^p}(m)$ be the inverse limit of the images $\mathbb{T}_{K_pK^p}(m)$ of $\mathbb{T}_{K^p}$ acting on $\oplus_{i=0}^{2d} H^i_c(X_{K^pK_p}, \mathcal{O}/\pi^m)$ as K_p becomes arbitrarily small. It is an inverse limit of finite discrete rings, and hence the inverse limit topology makes it a compact topological ring. Let $\tilde{H}^i_{c,K^p}(\mathcal{O}/\pi^m) = \varinjlim_{K_p} H^i_c(X_{K^pK_p}, \mathcal{O}/\pi^m)$ denote the completed compactly supported cohomology with $\mathcal{O}/\pi^m$-coefficients. Then $\tilde{\mathbb{T}}_{K^p}(m)$ is the image of $\mathbb{T}_{K^p}$ in

$$\mathrm{End}_{\mathcal{O}/\pi^m}\left(\oplus_{i=0}^{2d}(\tilde{H}^i_{c,K^p}(\mathcal{O}/\pi^m))\right),$$

where the ring of endomorphisms is endowed with the weakest topology which makes the action continuous for the discrete topology on $\tilde{H}^i_{c,K^p}(\mathcal{O}/\pi^m)$.

Each $\mathbb{T}_{K_pK^p}(m)$ is a finite ring, and hence is the product of finitely many local rings, which are in bijection with its maximal ideals. A maximal ideal is the same data as a homomorphism $\mathbb{T}_{K_pK^p}(m) \to \overline{\mathbb{F}}_p$ (i.e. an $\bar{\mathbb{F}}_p$-system of Hecke eigenvalues), up to an automorphism of $\overline{\mathbb{F}}_p$. Each such system of eigenvalues is valued in a finite field. If $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}_{K_pK^p}(m)$, then $\oplus_{i=0}^{2d} H^i_c(X_{K^pK_p}, \bar{\mathbb{F}}_p)_{\mathfrak{m}} \neq 0$, and (equivalently) $\oplus_{i=0}^{2d} H^i(X_{K^pK_p}, \bar{\mathbb{F}}_p)[\mathfrak{m}] \neq 0$. If $H^i(X_{K^pK_p}, \bar{\mathbb{F}}_p)[\mathfrak{m}] \neq 0$, we say that the system of Hecke eigenvalues corresponding to $\mathfrak{m}$ *occurs* in $H^i_c(X_{K^pK_p}, \bar{\mathbb{F}}_p)$. A non-zero cohomology class in this space is an eigenvector for $\mathbb{T}_{K^p}$, with the given system of Hecke eigenvalues, justifying the terminology. Observe that the maximal ideals of $\mathbb{T}_{K_pK^p}(m)$ and $\mathbb{T}_{K_pK^p}(1)$ are naturally in bijection with each other.

The following result is well-known to experts and essentially due to [AS86], but we could not find a reference for it in this form. It is not strictly necessary for our purposes, but it clarifies the structure of the Hecke algebras $\tilde{\mathbb{T}}_{K^p}(m)$.

Proposition 2.1. *There are finitely many systems of Hecke eigenvalues for $\mathbb{T}_{K_pK^p}(1)$ as K_p varies.*

Proof. Define $K(m) := \{g \in G_0(\mathbb{Z}_p) | g \equiv 1 \pmod{p^m}\}$. It suffices to see that every system of Hecke eigenvalues occurring in $H^i(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p)$ also occurs in some $H^{i'}(X_{K^pK(m)}, \bar{\mathbb{F}}_p)$ with $i' \leqslant i$, whenever $m \geqslant 1$. For this, we use the Hochschild-Serre spectral sequence:

$$E_2^{ij} = H^i(K(m)/K(m+1), H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p)) \Rightarrow H^{i+j}(X_{K^pK(m)}, \bar{\mathbb{F}}_p),$$

which is $\mathbb{T}_{K^p}$-equivariant. First, note that $K(m+1)/K(m)$ is an abelian p-group, so any element of $K(m+1)/K(m)$ has 1 as its only eigenvalue. Second, note that $\mathbb{T}_{K^p}[K(m+1)/K(m)]$ is commutative. Therefore, every system of $\mathbb{T}_{K^p}$-eigenvalues that occurs in $H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p)$ also occurs in $H^0(K(m+1)/K(m), H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p))$.

If the eigenvector survives in the E_∞ page of the Hochschild-Serre spectral sequence, we are done. Otherwise, a diagram chase and Proposition 1.2.2 of [AS86] tell us that the system of Hecke eigenvalues has to occur in some $H^{i'}(K(m+1)/K(m), H^{j'}(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p))$ with $j' < j$. But then Lemma 2.2 and the argument above tells us it must also occur in $H^0(K(m+1)/K(m), H^{j'}(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p))$. We then proceed as above until we possibly reach $j' = 0$, in which case the system of Hecke eigenvalues occurs in $H^0(X_{K^pK(m)}, \bar{\mathbb{F}}_p)$. ∎

Lemma 2.2. *Every system of $\mathbb{T}_{K^p}$-eigenvalues occurring in*

$$H^i(K(m+1)/K(m), H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p))$$

also occurs in $H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p)$.

Proof. Define

$$\mathbb{T}_1 := \mathrm{Im}\left(\mathbb{T}_{K^p} \to \mathrm{End}_{\bar{\mathbb{F}}_p}\left(H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p)\right)\right)$$

and

$$\mathbb{T}_2 := \mathrm{Im}\left(\mathbb{T}_{K^p} \to \mathrm{End}_{\bar{\mathbb{F}}_p}\left(H^i(K(m+1)/K(m), H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p))\right)\right).$$

Then $\mathbb{T}_2$ is a quotient of $\mathbb{T}_1$, so any system of Hecke eigenvalues occurring in $H^i(K(m+1)/K(m), H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p))$ determines a maximal ideal of $\mathbb{T}_1$. Now, it suffices to notice that, since $H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p)$ is a finite-dimensional $\bar{\mathbb{F}}_p$-vector space, every maximal ideal of $\mathbb{T}_1$ is in the support of $H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p)$. Using Nakayama's lemma and Proposition 1.2.2 of [AS86], we see that every maximal ideal of $\mathbb{T}_1$ determines a system of Hecke eigenvalues occurring in $H^j(X_{K^pK(m+1)}, \bar{\mathbb{F}}_p)$. □

Corollary 2.3. *The ring $\tilde{\mathbb{T}}_{K^p}(m)$ is a product of finitely many complete profinite local rings, each with finite residue field.*

Proof. $\tilde{\mathbb{T}}_{K^p}(m) = \varprojlim_{K_p} \mathbb{T}_{K_pK^p}(m)$, and by the previous proposition, the transition maps are eventually surjective maps between products of local Artinian rings which the same number of factors. This gives the factorization of $\tilde{\mathbb{T}}_{K^p}(m)$ into a product of projective limits of local finite rings. □

Fix once and for all an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_p$ and let C be the completion of $\bar{\mathbb{Q}}_p$. Let $\mathbb{T}_{K^p,\mathrm{cl}} = \mathbb{T}_{K^p,\mathrm{cl},m}$ denote $\mathbb{T}_{K^p}$ endowed with the weakest topology for which all the maps

$$\mathbb{T}_{K^p} \to \mathrm{End}_C(H^0(X^*_{K^pK_p}, \omega^{\otimes m_0 k}_{K^pK_p} \otimes \mathcal{I}) \otimes_{\mathbb{C},\iota} C)$$

are continuous, for varying $k \geqslant 1$ and $K_p \subset G_0(\mathbb{Q}_p)$. Here, $\omega_{K^pK_p}$ is the ample line bundle on $X^*_{K^pK_p}$ defined in Section 3 of [Sch] and $\mathcal{I}$ is the ideal sheaf of the boundary. The space $H^0(X^*_{K^pK_p}, \omega^{\otimes m_0 k}_{K^pK_p} \otimes \mathcal{I})$ is the space of classical cups forms of level K^pK_p and parallel weight m_0k. We want our Hecke algebras to have p-adic coefficients, so we base change from $\mathbb{C}$ to C. The right hand side is then a finite-dimensional C-vector space, endowed with the p-adic topology.

The following follows from Theorem IV.3.1 of [Sch], which implies that the map $\mathbb{T}_{K^p,\mathrm{cl}} \to \tilde{\mathbb{T}}_{K^p}(m)$ is continuous, from Corollary V.1.11 of loc. cit. for large enough m_0 (where we can take $\mathbb{T}_{K_pK^p}(m)$ for varying K_p to be the discrete quotients) and from the construction of $2n+1$-dimensional determinants valued in

$$\mathbb{T}_{K^pK_p,k} := \mathrm{Im}\left(\mathbb{T}_{K^p} \to \mathrm{End}_C(H^0(X^*_{K^pK_p}, \omega^{\otimes m_0 k} \otimes \mathcal{I}) \otimes_{\mathbb{C},\iota} C)\right).$$

The latter step interpolates the determinants coming from Galois representations associated to classical cusp forms for G_0 of high enough parallel weight.

Theorem 2.4. *There is a continuous $(2n+1)$-dimensional determinant $\tilde{D}$ of $G_{F,S}$ with values in $\tilde{\mathbb{T}}_{K^p}(m)$, such that*

$$\tilde{D}(1 - X \cdot \mathrm{Frob}_v) = 1 - T_{1,v}X + T_{2,v}X^2 - \cdots + (-1)^{2n+1}T_{n,v}X^{2n+1}.$$

As promised, we can now also describe the structure of the Hecke algebras $\tilde{\mathbb{T}}_{K^p}(m)$.

Corollary 2.5. *$\tilde{\mathbb{T}}_{K^p}(m)$ is a product of finitely many complete local Noetherian rings.*

Proof. After localizing at each of the finitely many maximal ideals $\mathfrak{m}_i$ of $\tilde{\mathbb{T}}_{K^p}(m)$, Theorem 2.4 implies that $\tilde{\mathbb{T}}_{K^p,\mathfrak{m}_i}(m)$ receives a surjection from a Galois pseudo-deformation ring, and hence it is complete local Noetherian. □

3. The boundary of the Borel-Serre compactification

In this section, we go back to studying the locally symmetric spaces for general linear groups. Assume that the level $K \subset GL_n(\mathbb{A}_{F,f})$ is neat, which can be achieved by increasing the level at p, as in the introduction. Let X_K^{BS} be the Borel-Serre compactification of the locally symmetric space X_K defined in the introduction. This is a real manifold with corners, which is a compactification of X_K such that the inclusion $X_K \hookrightarrow X_K^{\mathrm{BS}}$ is a homotopy equivalence. We have the excision long exact sequence for compactly supported cohomology:

$$\cdots \to H^i_c(X_K, \bar{\mathbb{F}}_p) \to H^i(X_K, \bar{\mathbb{F}}_p) \to H^i(X_K^{\mathrm{BS}} \setminus X_K, \bar{\mathbb{F}}_p) \to \dots$$

Theorem 2.4 allows us to understand Galois representations associated to systems of Hecke eigenvalues which contribute to the interior cohomology $H^i_!(X_K, \bar{\mathbb{F}}_p)$. To extend this to all systems of Hecke eigenvalues in $H^i(X_K, \bar{\mathbb{F}}_p)$, we also need to account for those systems of Hecke eigenvalues occurring in $H^i(X_K^{\mathrm{BS}} \setminus X_K, \bar{\mathbb{F}}_p)$. In this subsection, we recall the geometry of the Borel-Serre and reductive Borel-Serre compactifications and use induction to construct a determinant valued in (a quotient of) the Hecke algebra acting on $H^i(X_K^{\mathrm{BS}} \setminus X_K, \bar{\mathbb{F}}_p)$. We follow the original construction in [BS73] as well as the exposition in [GHM94].

Note that the group $\mathrm{Res}_{F/\mathbb{Q}} GL_n$ has a non-trivial $\mathbb{Q}$-split torus in its center. Let H be the group

$$({}^0(\mathrm{Res}_{F/\mathbb{Q}} GL_n))^0 := \bigcap_{\chi} \ker \chi,$$

where χ runs over all rationally defined characters of $\mathrm{Res}_{F/\mathbb{Q}} GL_n$. The locally symmetric spaces X_K can be identified with finitely many disjoint copies of generalized locally symmetric spaces for the algebraic group H in the sense of [GHM94]. This identification follows from the definition in [BS73] and Paragraph 3.4 of [GHM94]. The boundary of the Borel-Serre compactification of each connected component of X_K has a stratification which runs over finitely many conjugacy classes of rational parabolic subgroups $P_H \subseteq H$.

We make the indexing set for the strata more precise (see Section 6 of [GHM94] for more details). Borel and Serre construct a partial compactification $\tilde{D}$ of the symmetric domain $D := H(\mathbb{R})/K^H_\infty$ corresponding to H by attaching strata indexed by rationally-defined parabolic subgroups of H. The group $H(\mathbb{Q})$ acts as a group of automorphisms on this partial compactification. For a discrete subgroup $\Gamma \subset H(\mathbb{Q})$ (obtained, in our situation, by intersecting some conjugate of the compact open subgroup $K \cap H(\mathbb{A}_f)$ with $H(\mathbb{Q})$), the quotient $\Gamma\backslash\tilde{D}$ is a compact manifold with corners. The corners, which are the Borel-Serre boundary strata, are now indexed by Γ-conjugacy classes of rational parabolic subgroups $P_H \subset H$. (Taking a quotient by Γ identifies the Γ-conjugate components of the boundary of $\tilde{D}$ and, in addition, makes certain identifications within each component.) An analogous stratification is induced from the components of the partial compactification $\tilde{D}$ on adelic double quotients for H. The conjugacy classes of rational parabolics will now depend on the level $K \cap H(\mathbb{A}_f)$.

Fix a representative P_H for each conjugacy class; we now describe the stratum corresponding to P_H. The parabolic subgroup P_H has a decomposition $P_H = L_H U_H$, where L_H is the Levi quotient and U_H is the unipotent radical. Let

$$M_H =^0 (L_H) := \bigcap_\chi \ker \chi^2,$$

where χ runs over all rationally defined characters of L_H. The group of real points of every rational parabolic subgroup $P_H \subseteq H$ has a Langlands decomposition $P_H(\mathbb{R}) = M_H(\mathbb{R})A_H(\mathbb{R})U_H(\mathbb{R})$. The stratum corresponding to a parabolic subgroup P_H can be identified with the locally symmetric space:

$$X^{P_H} := P_H(\mathbb{Q})\backslash \left(P_H(\mathbb{A}_f)/K_f^{P_H} \times P_H(\mathbb{R})/A_H(\mathbb{R})K_\infty^{P_H}\right)$$

where $K_f^{P_H} := K \cap P_H(\mathbb{A}_f) \subset P_H(\mathbb{A}_f)$ is a compact open subgroup and $K_\infty^{P_H} \subset M_H(\mathbb{R})$ is a maximal compact subgroup. (See Section 7 of [GHM94] for the formula in the case of quotients of $\tilde{D}$ by discrete subgroups of $H(\mathbb{Q})$. To compute the contribution of P_H in our situation, compare the adelic picture with the definition via quotients of $\tilde{D}$ by discrete subgroups of $H(\mathbb{Q})$, and use weak approximation and the Iwasawa decomposition for H at non-archimedean places.)

This space is a nilmanifold bundle over the locally symmetric space associated to the Levi quotient L_H of P_H

$$X^{L_H} := L_H(\mathbb{Q})\backslash \left(L_H(\mathbb{A}_f)/K_f^L \times L_H(\mathbb{R})/A_H(\mathbb{R})K_\infty^{P_H}\right)$$

where $K_f^L \subset L_H(\mathbb{A}_f)$ is the image of $K_f^{P_H}$ under the projection $P_H(\mathbb{A}_f) \to L_H(\mathbb{A}_f)$ and is a compact open subgroup. The fibers of this nilmanifold bundle are isomorphic to $(U_H(\mathbb{Q}) \cap K_f^{P_H})\backslash U_H(\mathbb{R})$, as in the proof of Lemma V.2.2 of [Sch]. We note that $X^L_{K_f^L}$ can be identified with a locally symmetric space for the connected component of the identity in M_H, in the sense of (7.2.2) of [GHM94] (we can go to the connected component of the identity since $K_\infty^{P_H} \subset M_H(\mathbb{R})$ is a maximal compact subgroup).

We now reinterpret X^{L_H} as a product of generalized locally symmetric spaces. Each parabolic subgroup $P_H \subset H$ is the intersection with H of a parabolic subgroup P of $\mathrm{Res}_{F/\mathbb{Q}}GL_n$. Assume that P has Levi quotient isomorphic to $L := \prod_i \mathrm{Res}_{F/\mathbb{Q}}GL_{n_i}$, with $\sum_i n_i = n$. From the definition of the locally symmetric space X^{L_H}, we see that

$$X^{L_H} \simeq L(\mathbb{Q})\backslash \left(L(\mathbb{A}_f)/K_f^L \times L(\mathbb{R})/A_L(\mathbb{R})K_L\right),$$

where $K_L \subset L(\mathbb{R})$ is a maximal compact subgroup.

Moreover, the latter obviously decomposes as a product of locally symmetric spaces associated to the $\mathrm{Res}_{F/\mathbb{Q}}GL_{n_i}$. Therefore, we can compute the compactly supported cohomology of a stratum corresponding to P using the Leray-Serre spectral sequence of a fibration and the Kunneth formula for compactly supported cohomology.

For $\mathrm{Res}_{F/\mathbb{Q}}P$ running over our chosen rational maximal parabolic subgroups of $\mathrm{Res}_{F/\mathbb{Q}}GL_n$ (corresponding to the representatives P_H), denote the corresponding stratum by $X^P_{K_f^P}$. The level is defined as $K_f^P := K \cap P(\mathbb{A}_f)$. We have an open immersion $X^P_{K_f^P} \hookrightarrow X_K^{\mathrm{BS}} \setminus X_K$, which leads to the long exact sequence for cohomology with compact support

$$\cdots \to H_c^i(X^P_{K_f^P}, \bar{\mathbb{F}}_p) \to H^i(X_K^{\mathrm{BS}} \setminus X_K, \bar{\mathbb{F}}_p) \to H^i(X_K^{\mathrm{BS}} \setminus (X_K \cap X^P_{K_f^P}), \bar{\mathbb{F}}_p) \to \dots.$$

Thus $H^i(X_K^{\mathrm{BS}} \backslash X_K, \bar{\mathbb{F}}_p)$ is an extension between subquotients of $H_c^i(X^P_{K_f^P}, \bar{\mathbb{F}}_p)$ and $H^i(X_K^{\mathrm{BS}} \backslash (X_K \cap X^P_{K_f^P}), \bar{\mathbb{F}}_p)$. After removing the strata corresponding to all maximal parabolic subgroups this way, we can continue this process for smaller parabolic subgroups. The corresponding stratum will be open in what is left of the boundary and we can iterate the process above. We see that $H^i(X_K^{\mathrm{BS}} \setminus X_K, \bar{\mathbb{F}}_p)$ has a filtration whose graded pieces are subquotients of $H^i_c(X^P_{K^P_f}, \bar{\mathbb{F}}_p)$ where P runs through the chosen rational parabolic subgroups $\mathrm{Res}_{F/\mathbb{Q}}P$ of $\mathrm{Res}_{F/\mathbb{Q}}GL_n$. Furthermore, the length of the filtration depends only on G and not on K.

Let $\pi : X^P_{K^P_f} \to X^L_{K^L_f}$ be the projection map, with $K^L_f \subset L(\mathbb{A}_f)$ the image of K^P_f. The projection map is proper. The Leray spectral sequence tells us that

$$E_2^{i,j} := H^i_c(X^L_{K^L_f}, R^j\pi_*\bar{\mathbb{F}}_p) \implies H^{i+j}_c(X^P_{K^P_f}, \bar{\mathbb{F}}_p).$$

Therefore, $H^{i+j}_c(X^P_{K^P_f}, \bar{\mathbb{F}}_p)$ has a filtration whose graded pieces are subquotients of $H^i_c(X^L_{K^L_f}, R^j\pi_*\bar{\mathbb{F}}_p)$, and whose length is bounded in terms of G.

Let S be a finite set of rational places, containing p, ∞ and all the primes l, where the compact open subgroup $K_l \subset (\mathrm{Res}_{F/\mathbb{Q}}GL_n)(\mathbb{Q}_l)$ is not a hyperspecial maximal compact subgroup. Since $K^P_f = K \cap P(\mathbb{A}_f)$, and $K^L_f \subset L(\mathbb{A}_f)$ is the image of K^P under the projection $P \twoheadrightarrow L$, $K^{L,S} \subset (\mathrm{Res}_{F/\mathbb{Q}}L)(\mathbb{A}^S)$ is a product of hyperspecial maximal compact subgroups. We define the auxiliary Hecke algebras $\mathbb{T}^P_{F,S} := \mathbb{Z}_p[P(\mathbb{A}^S_F)//K^{P,S}]$ and $\mathbb{T}^L_{F,S} := \mathbb{Z}_p[L(\mathbb{A}^S_F)//K^{L,S}]$.

Remark 3.1. The sheaves $R^j\pi_*\bar{\mathbb{F}}_p$ encode the cohomology of the nilmanifold

$$(U_H(\mathbb{Q}) \cap K^P_f)\backslash U_H(\mathbb{R}).$$

As in Lemma 1.9 of [HLTT13], we see that $R^j\pi_*\bar{\mathbb{F}}_p$ can be identified with the local system corresponding to the algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}L$ given by

$$\rho^j_L := \wedge^j \left(\bigoplus_{\tau: F \hookrightarrow \mathbb{R}} \left(\bigoplus_{k<l} \mathrm{Std}_{n_k} \otimes \mathrm{Std}^{-1}_{n_l} \right) \right),$$

where n_k, n_l correspond to blocks in the Levi subgroup L and Std_n is the standard representation of $GL_n(\mathcal{O}_F)$ on $(\bar{\mathbb{F}}_p)^n$. (The formula above comes from the fact that the action of $\mathrm{Res}_{F/\mathbb{Q}}L$ on $U_H(\mathbb{R})$ is the adjoint action.) Since the local system $R^j\pi_*\bar{\mathbb{F}}_p$ on $X^L_{K^L_f}$ corresponds to the algebraic representation ρ^j_L, it will have an action of $\mathbb{T}^L_{F,S}$ compatible with the one on $X^L_{K^L_f}$, which will induce an action of $\mathbb{T}^L_{F,S}$ on the cohomology groups $H^i_c(X^L_{K^L_f}, R^j\pi_*\bar{\mathbb{F}}_p)$.

We have maps $\mathbb{T}_{F,S} \to \mathbb{T}^P_{F,S}$ by restriction and $\mathbb{T}^P_{F,S} \to \mathbb{T}^L_{F,S}$ by integration along unipotent fibres. Their composite

$$\eta : \mathbb{T}_{F,S} \to \mathbb{T}^P_{F,S}$$

is the unnormalized Satake transform. The following is an analogue of Lemma V.2.3 of [Sch].

Lemma 3.2. *The Leray spectral sequence*

$$E_2^{i,j} = H^i_c(X^L_{K^L_f}, R^j\pi_*\bar{\mathbb{F}}_p) \implies H^{i+j}_c(X^P_{K^P_f}, \bar{\mathbb{F}}_p)$$

is equivariant for the action of $\mathbb{T}_{F,S}$ given by η composed with the action of $\mathbb{T}^L_{F,S}$ on the E_2-page, as described in Remark 3.1, and for the natural action (via restriction to $\mathbb{T}^P_{F,S}$) on $H^{i+j}_c(X^P_{K^P_f}, \bar{\mathbb{F}}_p)$.

Proof. We first check that the action of a Hecke correspondence $t \in \mathbb{T}_{F,S}$ on $X^P_{K^P_f}$ is compatible with the action of $\eta(t)$ on $X^L_{K^L_f}$ via the natural projection $\pi : X^P_{K^P_f} \to X^L_{K^L_f}$. This statement can be checked at the level of points. Let $l \notin S$ and t be the characteristic function of $G(\mathbb{Z}_l)hG(\mathbb{Z}_l)$. Put $P(\mathbb{Z}_l)hP(\mathbb{Z}_l) = \coprod h_i P(\mathbb{Z}_l)$, so that t acts as $[g] \mapsto \sum [gh_i]$ on $X^P_{K^P_f}$. Now if $P(\mathbb{Z}_l)hP(\mathbb{Z}_l) = \coprod u_i U(\mathbb{Q}_l)P(\mathbb{Z}_l) = \coprod u_i L(\mathbb{Z}_l)U(\mathbb{Q}_l)$, then $\pi(h_i) = \pi(h_j)$ if and only if h_i and h_j belong to the same right $L(\mathbb{Z}_l)U(\mathbb{Q}_l) = U(\mathbb{Q}_l)P(\mathbb{Z}_l)$-coset. Thus the number of coset representatives h_i whose projection to L are in the same right $L(\mathbb{Z}_l)$-coset is exactly given by integrating t along $U(\mathbb{Q}_l)$.

We now study the action of $\mathbb{T}_{F,S}$ on the sheaves $R^j\pi_*\bar{\mathbb{F}}_p$. Using the equivalence between local systems on a connected component of $X^L_{K^L_f}$ and representations of its fundamental group, we see that $R^j\pi_*\bar{\mathbb{F}}_p$ is the $\bar{\mathbb{F}}_p$-local system $\mathcal{L}_{\rho^j_L}$ corresponding to the algebraic representation ρ^j_L defined in Remark 3.1. The Hecke action of $\mathbb{T}^L_{F,S}$ is then twisted by ρ^j_L. More precisely, for an element $\lambda_p \in L(\mathbb{Z}_p)$ there is an induced map $l : l^*(\mathcal{L}_{\rho^j_L}) \to \mathcal{L}_{\rho^j_L}$, which corresponds to acting on the representation space of ρ^L_j by $\rho^j_L(\lambda_p)$. (See, for example, Section III.2 of [HT01] for all the details worked out in the case of lisse l-adic sheaves on Shimura varieties; the case of Betti local systems on locally symmetric spaces is analogous.) This means that the Hecke action of $\mathbb{T}^L_{F,S}$ on $R^j\pi_*\bar{\mathbb{F}}_p$ at some prime $l \notin S$ of $\lambda_l \in G(\mathbb{Z}_l)\backslash G(\mathbb{Q}_l)/G(\mathbb{Z}_l)$ is via $\rho^j_L(\lambda_l)^{-1}$.

We check that this is compatible with the action of $\mathbb{T}_{F,S}$ on the fibers of the nilmanifold bundle $X^P_{K^P_f}$. Again, let $t \in \mathbb{T}_{F,S}$ be the characteristic function of some double coset $G(\mathbb{Z}_l)hG(\mathbb{Z}_l)$, which acts as $[g] \mapsto \sum [gh_i]$ on $X^P_{K^P_f}$. It is enough to compute the action of each h_i on some fiber above a point of the base $X^L_{K^L_f}$. We see that h_i acts via conjugation by $\pi(h_i)^{-1} \in L(\mathbb{Q}_l)$ on $U_H(\mathbb{Q}_l)$, followed by a translation by the element $h_i\pi(h_i)^{-1} \in U_H(\mathbb{Q}_l)$. The translation has no effect on the cohomology of the nilmanifold, since it is homotopic to the identity. The induced action of each h_i on the sheaves $R^j\pi_*\bar{\mathbb{F}}_p$ is, therefore, the one coming from the adjoint representation of $L(\mathbb{Q}_l)$ on $U(\mathbb{Q}_l)$, so it is precisely $\rho^j_L(\pi(h_i))^{-1}$. It is straightforward to check that this is the same as the composition of $\eta : \mathbb{T}_{F,S} \to \mathbb{T}^L_{F,S}$ followed by the action of $\mathbb{T}^L_{F,S}$ described in the paragraph above. □

If the level K_p at p is sufficiently small (depending only on P and L), the local system $R^j\pi_*\bar{\mathbb{F}}_p$ is trivial. If this is the case, the Kunneth formula for compactly supported cohomology (which applies because the spaces we consider are manifolds) expresses the $E^{i,j}_2$ terms in terms of tensor products of $H^a_c(X^{\mathrm{GL}_{n_b}}_K, \bar{\mathbb{F}}_p)$. In general, we can always find a normal subgroup K' of K which is sufficiently small, and the Hochshild-Serre spectral sequence shows that $H^i_c(X^L_{K^L_f}, R^j\pi_*\bar{\mathbb{F}}_p)$ has a filtration whose graded pieces are subquotients of (direct sums of) $H^a(K/K', H^b_c(X^L_{K'^L_f}, \bar{\mathbb{F}}_p))$. The length of the filtration is bounded in terms of G, and the spectral sequence is equivariant with respect to $\mathbb{T}^L_{F,S}$. The following summarizes the discussion in this section:

Proposition 3.3. *$H^i(X^{\mathrm{BS}}_K \backslash X_K, \bar{\mathbb{F}}_p)$ admits a filtration by $\mathbb{T}_{F,S}$-modules whose graded pieces are modules for the quotients of $\mathbb{T}_{F,S}$ acting on $H^i_c(X^L_{K_L}, \bar{\mathbb{F}}_p)$, where $\mathbb{T}_{F,S}$ acts via the unnormalized Satake transform $\mathbb{T}_{F,S} \to \mathbb{T}^L_{F,S}$ and L is a rational Levi subgroup. The length of this filtration is bounded in terms of G only.*

Remark 3.4. The Proposition continues to hold (with exactly the same argument) if we replace cohomology with $\bar{\mathbb{F}}_p$-coefficient by cohomology with $\mathcal{O}/\pi^m$-coefficients, where $\mathcal{O}$ is a sufficiently large finite extension of $\mathbb{Z}_p$ (for example, its fraction field containing all the images of embeddings $F \hookrightarrow \bar{\mathbb{Q}}_p$ is large enough).

4. The image of complex conjugation

The group $G_0 = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp_{2n}}$ contains the group $\mathrm{Res}_{F/\mathbb{Q}}GL_n$ as a maximal Levi, therefore the interior cohomology of the locally symmetric space for $G = \mathrm{Res}_{F/\mathbb{Q}}GL_n$ contributes to the boundary cohomology of the locally symmetric space for G_0. Therefore, one can use Theorem 2.4 to obtain a determinant valued in the Hecke algebra acting on the interior cohomology of the locally symmetric space for G. This is not yet the determinant for Galois representations associated to G, but rather involves a functorial transfer from G to G_0.

More precisely, let $\mathbb{T}_{F,S}(K,i,m) := \mathrm{Im}(\mathbb{T}^G_{F,S} \to \mathrm{End}_{\mathcal{O}/\pi^m}(H^i_!(X_K^{\mathrm{GL}_n}, \mathcal{O}/\pi^m)))$, where $\mathcal{O}$ is a finite extension of $\mathbb{Z}_p$ and π a uniformizer. Let

$$P_v(X) := 1 - q_v^{(n+1)/2}T^{GL_n}_{1,v}X + \cdots + (-1)^n q_v^{n(n+1)/2}T^{GL_n}_{n,v}X^n$$

and $P_v^\vee(X)$ be the polynomial with constant coefficient 1 which is a scalar multiple of $P_v(1/X)$. The following is one of the main results in [Sch], proved in Corollary V.2.6 and Theorem V.3.1.

Theorem 4.1. *There exists a nilpotent ideal $I \subset \mathbb{T}_{F,S}(K,i,m)$ with nilpotency index bounded only in terms of G as well as continuous $2n+1$- and n-dimensional determinants $\tilde{D}$, D of $G_{F,S}$ valued in $\mathbb{T}_{F,S}(K,i,m)/I$, such that*

$$\tilde{D}(1 - X\cdot \mathrm{Frob}_v) = (1-X)P_v(X)P_v^\vee(X)$$

$$D(1 - X\cdot \mathrm{Frob}_v) = P_v(X)$$

for all places $v \notin S$. The same statement holds for the Hecke algebras acting on $H^i_c(X_K^{\mathrm{GL}_n}, \mathcal{O}/\pi^m)$ and $H^i(X_K^{\mathrm{GL}_n}, \mathcal{O}/\pi^m)$

Remark 4.2. Let $\mathbb{T}^{G_0}_{K,k}$ be the quotient of the Hecke algebra which acts on the space $H^0(X_K^{G_0*}, \omega_K^{\otimes k} \otimes \mathcal{I})$ of classical cuspforms for G_0 (see the notation in Section 2, which match with those in [Sch] section V.1). The determinant $\tilde{D}$ for interior cohomology is obtained by gluing determinants of the type constructed in Theorem 2.4, which is glued from determinants valued in $\mathbb{T}^{G_0}_{K,k}$. Therefore, if all such determinants satisfy a certain identity, so will $\tilde{D}$. We will apply this observation to compute the coefficient of X in $\tilde{D}(1 - X\cdot c)$ and hence in $D(1 - X\cdot c)$, where $c \in G_{F,S}$ is a choice of complex conjugation.

We can compute the characteristic polynomial of any complex conjugation in the determinants on $\mathbb{T}^{G_0}_{K,k}$ by the following:

Lemma 4.3. *Let $k > n$, $x \in \mathrm{Spec}\,\mathbb{T}^{G_0}_{K,k}(\bar{\mathbb{Q}}_p)$, let $\sigma_x : G_{F,S} \to GL_{2n+1}(\bar{\mathbb{Q}}_p)$ be the Galois representation whose Frobenius eigenvalues match the Satake parameters determined by x at places not in S. Then for any complex conjugation c*

$$\mathrm{tr}(\sigma_x)(c) = \pm 1$$

Thus the characteristic polynomial of c is either $(1-X)^n(1+X)^{n+1}$ or $(1+X)^n(1-X)^{n+1}$.

Proof. By the proof of Corollary V.1.7 of [Sch], the cuspidal automorphic representation associated to x determines cuspidal automorphic representations Π_i of GL_{n_i} for $i = 1, \dots, m$ and integers $l_1, \dots, l_m$ such that

- $l_1 n_1 + \cdots + l_m n_m = 2n+1$
- each Π_i is self-dual
- each $\Pi_i|\cdot|^{2n+1-l_i}$ is regular L-algebraic

- the infinitesimal characters associated to $\Pi_i|\cdot|^{(l_i-1)/2}, \dots, \Pi_i, \dots, \Pi_i^{(1-l_i)/2}$ with $i = 1, \dots, m$ form the multiset $\{k-1, \dots, k-n, 0, n-k, \dots, 1-k\}$.
- the Galois representation associated to x satisfies

$$\sigma_x = \bigoplus_{i=1}^{m} \left(\sigma_i \oplus \sigma_i\chi_p^{-1} \oplus \cdots \oplus \sigma_i\chi_p^{\otimes(1-l_i)}\right),$$

  where σ_i is the Galois representation associated to the regular L-algebraic cuspidal automorphic representation $\Pi_i|\cdot|^{(1-l_i)/2}$ and χ_p^{-1} is the p-adic cyclotomic character (and also the Galois representation associated to the absolute value $|\cdot|$ by our normalization of class field theory).

(This is where we make use of the results of [Art13].)

We note that if l_i is even, the trace of c on $\sigma_i \oplus \cdots \oplus \sigma_i\chi_p^{\otimes(1-l_i)}$ is equal to 0. If l_i is odd, then $\Pi_i|\cdot|^{(l_i-1)/2}$ is essentially self-dual, with even multiplier character, so by [Tai12], the trace of c on σ_i is 0 if n_i is even and ± 1 if n_i is odd. Therefore, the trace of c on $\sigma_i \oplus \cdots \oplus \sigma^i\chi_p^{\otimes(1-l_i)}$ is 0 if n_i is even and ± 1 if n_i is odd.

We now show that there is at most one i for which both n_i and l_i are odd. Indeed, Π_i is self-dual, so the multiset of infinitesimal characters of $\Pi_i|\cdot|^{(l_i-1)/2}, \dots, \Pi_i, \dots, \Pi_i|\cdot|^{(1-l_i)/2}$ is symmetric about 0. If both n_i and l_i are odd, this multiset contains an odd number of elements, so it must contain 0. But from above, we see that 0 can appear only once, so n_il_i is odd for at most one Π_i. Since $2n+1$ is odd, we see that n_il_i is odd for exactly one Π_i and, therefore, $\mathrm{tr}(\sigma_x)(c) = \pm 1$.

□

Proposition 4.4. *Let p be an odd prime. Let ψ be a system of Hecke eigenvalues which occurs in the interior cohomology $H^i_!(X_K, \bar{\mathbb{F}}_p)$, where X_K is a locally symmetric space attached to G. Let σ_ψ be the Galois representation associated to the determinant D in Theorem 4.1 specialized to ψ. Then $\sigma_\psi(c)$ has $+1$ as an eigenvalue with multiplicity $\lceil \frac{n-1}{2} \rceil$ and -1 as an eigenvalue with multiplicity $\lfloor \frac{n+1}{2} \rfloor$.*

Proof. It suffices to consider the case $p > 2$, as the assertion is empty when $p = 2$. This follows from Lemma 4.3 and the construction of the determinant with the additional observation that the the determinant $\tilde{D}$ specialized at ψ gives rise to the Galois representation $1 \oplus \sigma_\psi \oplus (\sigma_\psi)^\vee$. This observation follows from the polynomial identity $\tilde{D}(1 - X \cdot \mathrm{Frob}_v) = (1-X)P_v(X)P_v^\vee(X)$ appearing in Theorem 4.1 and from the fact that σ_ψ is the Galois representation associated to the specialization at ψ of a determinant which matches $P_v(X)$ at almost all places.

Now Remark 4.2 and Lemma 4.3 tell us that, if 1 is an eigenvalue of $\sigma_\psi(c)$ with multiplicity a and -1 is an eigenvalue of $\sigma_\psi(c)$ with multiplicity b, then

$$(1-X)^{2a+1}(1+X)^{2b} = (1 \pm X)^n(1 \mp X)^{n+1}.$$

Considering the cases n even and n odd separately gives the desired result.

□

Theorem 4.5. *Let p be an odd prime. Let ψ be a system of Hecke eigenvalues which occurs in $H^i_c(X_K, \bar{\mathbb{F}}_p)$. Let σ_ψ be the Galois representation associated to ψ. Then $\sigma_\psi(c)$ has $+1$ as an eigenvalue with multiplicity $\lceil \frac{n-1}{2} \rceil$ and -1 as an eigenvalue with multiplicity $\lfloor \frac{n+1}{2} \rfloor$.*

Proof. We prove this by induction on n. Note that the case $n = 1$ is obvious, since the locally symmetric space in this case is compact so Proposition 4.4 applies. Assume that the theorem holds for all $n' < n$. If ψ occurs in interior cohomology, we are done by Proposition 4.4. If not, then ψ occurs in the cohomology of the boundary of the Borel-Serre compactification of X_K. By Proposition 3.3, it is enough to show that for any Levi subgroup L the determinant D_L associated to $H^i_c(X^L_{K^L_f}, \bar{\mathbb{F}}_p)$ gives rise to Galois representations for which the trace of complex conjugation satisfies the conditions of the theorem.

Assume that $L = \prod_{i=1}^k GL_{n_i}(F)$. Then D_L is obtained by taking the direct product of the determinants associated to the locally symmetric spaces for $GL_{n_i}(F)$ (where we mean the connected locally symmetric spaces), each of these appropriately twisted by powers of the cyclotomic character. The reason for the twists is that the action of $\mathbb{T}_{F,S}$ is compatible with the action of $\mathbb{T}^L_{F,S}$ via the *unnormalized* Satake transform, as shown in Lemma 3.2. Explicitly, for a place v of F not in S, the unnormalized Satake transform is the map

$$\mathbb{Z}_p[T_1^{\pm 1}, \dots T_n^{\pm 1}]^{S_n} \to \prod_{i=1}^k \mathbb{Z}_p[q_v^{1/2}][(X_1^i)^{\pm 1}, \dots (X_{n_i}^i)^{\pm 1}]^{S_{n_i}},$$

where

$$T_j \mapsto q_v^{-(n_1+\dots+n_{i-1})/2+(n_{i+1}+\dots+n_k)/2} X^i_{j_i},$$

with i and j_i uniquely determined by $1 \leqslant j \leqslant n$. Since the normalized Satake transform is compatible with local Langlands up to $q_v^{(n_i+1)/2}$, we get that $D_L = \prod_{i=1}^k D_i(\chi_p^{n_{i+1}+\dots+n_k})$, where χ_p is the p-adic cyclotomic character and D_i is the determinant associated to the locally symmetric space for $GL_{n_i}(F)$. By the induction hypothesis, the characteristic polynomial of complex conjugation on each D_i satisfies the conditions of the theorem. Noting that the cyclotomic character is odd, we see that the sign of the contribution from the successive D_i switches every time an odd-dimensional determinant contributes to the sum. Therefore, D_L also satisfies the conditions of the theorem.

□

The above theorem determines the conjugacy class of complex conjugation in σ_ψ for p odd. To prove the analogous statement for characteristic 0 system of Hecke eigenvalues, we need to have a version of the above theorem for cohomology with $\mathcal{O}/\pi^m$-coefficients.

Proposition 4.6. *Let $\psi : \mathbb{T}_{F,S} \to \mathcal{O}/\pi^m\mathcal{O}$ be a system of Hecke eigenvalue factoring through the Hecke algebra quotient acting on $H^i_c(X^{\mathrm{GL}_n}_K, \mathcal{O}/\pi^m)$. There exists an integer N_0 depending only on G such that the determinant D in Theorem 4.1 specializes to $\tilde{\psi} = \psi \bmod \pi^{\lfloor m/N_0 \rfloor - \mathrm{ord}_\pi(4)}$, and $D_{\tilde{\psi}}(1 - X \cdot c) = 1 - \mathrm{tr}(c)X + \cdots$ with $\mathrm{tr} c = 0$ if n is even and $\mathrm{tr}(c) = -1$ if n is odd.*

Proof. The existence of the specialization to $\tilde{\psi} \bmod \pi^{\lfloor m/N_0 \rfloor}$ follows from the fact that the image $\psi(I)$ is nilpotent with nilpotency index bounded by that of the ideal I. Lemma 4.3 shows that the identity

$$(2\mathrm{tr}(\sigma_\psi)(c) + 1)^2 = 1$$

holds for the determinant on interior cohomology. By Proposition 3.3 and an induction on n as in the proof of Theorem 4.5, the identity holds for the determinant on the Borel-Serre boundary of X^G_K, hence it holds for the determinant on $H^i_c(X^G_K, \mathcal{O}/\pi^m)$. Specializing via $\tilde{\psi}$ gives the identity $4\mathrm{tr}(c)(\mathrm{tr}(c) + 1) = 0$ in $\mathcal{O}/\pi^{\lfloor m/N_0 \rfloor}$. Since either $\mathrm{tr}(c)$ or $\mathrm{tr}(c) + 1$ is a unit (which one depends on the parity of n), we are done. □

Corollary 4.7. *Let ψ be the system of Hecke eigenvalues of a regular algebraic cusp forms on GL_n / F, with associated (p-adic) Galois representation σ_ψ. Then $\mathrm{tr}\sigma_\psi(c) \in \{0, \pm 1\}$.*

Proof. It suffices to compute $\mathrm{tr}\sigma_\psi(c)$ in the case ψ occurs in $H^i_c(X^G_K, \mathcal{L}_\xi)$, where we've chosen an algebraic coefficient system $\mathcal{L}_\xi$ defined over (possibly a finite extension of) $\mathcal{O}$. Applying the above Proposition to ψ mod π^m gives $\mathrm{tr}\sigma_\psi(c)$ mod $\pi^{\lfloor m/N_0 \rfloor - \mathrm{ord}_\pi(4)} \in \{0, -1\}$. (Since we're looking at X^G_K, we know that the trace lies in $\{0, -1\}$ rather than $\{0, \pm 1\}$.) Letting $m \to \infty$ gives the result. □

Remark 4.8. The corollary completely determines the conjugacy class of $\sigma_\psi(c)$. In general, for systems of Hecke eigenvalues mod π^m, one can refine Proposition 4.6 to compute the entire characteristic polynomial of c modulo a smaller power of π.

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