Recent progress on Langlands reciprocity for GL_n: Shimura varieties and beyond

Ana Caraiani and Sug Woo Shin

ABSTRACT. The goal of these lecture notes is to survey progress on the global Langlands reciprocity conjecture for GL_n over number fields from the last decade and a half. We highlight results and conjectures on Shimura varieties and more general locally symmetric spaces, with a view towards the Calegari–Geraghty method to prove modularity lifting theorems beyond the classical setting of Taylor–Wiles.

CONTENTS

1. Introduction

The Langlands reciprocity conjecture for GL_n predicts a precise correspondence between automorphic representations of GL_n and n-dimensional ℓ-adic Galois representations over each global field. Over global function fields, this is a celebrated theorem by L. Lafforgue. While the conjecture is still open over number fields, Shimura varieties have been at the center of our approach to the problem ever since Eichler, Shimura, Deligne, Ihara, and Langlands analyzed the special case of modular curves. This led to the birth of the Langlands–Kottwitz–Rapoport (LKR) method, consisting of a web of conjectures concerning the geometry of Shimura varieties and the endoscopic classification of automorphic representations, to study the ℓ-adic cohomology of Shimura varieties. The LKR method almost reached the current form at the conjectural level in the 1990s, though there still remains much to be investigated.

In the early 2010s, new perspectives on Shimura varieties and their cohomology emerged in connection with the Langlands reciprocity conjecture. Calegari–Geraghty proposed that many new cases of modularity would be accessible if we better understood the torsion cohomology of locally symmetric spaces, of which (the underlying manifolds of) Shimura varieties are examples; in particular if we had "torsion" Galois representations attached to the torsion cohomology classes. The existence of such Galois representations appeared to be a mere dream apart from some computational evidence in low rank, but Scholze's revolutionary use of perfectoid Shimura varieties realized the torsion Galois representations. These developments stimulated further progress on the cohomology of Shimura varieties and locally symmetric spaces with torsion coefficients, as well as the modularity lifting theorems themselves, with a view towards implementing the Calegari–Geraghty method in some key cases, for example for elliptic curves over CM fields or for abelian surfaces over totally real fields.

Our goal in this survey is to give a gentle introduction to the aforementioned conjectures and results with an emphasis on the role of Shimura varieties. In § 2.1 we review the Langlands reciprocity conjecture for GL_n with both characteristic 0 and torsion coefficients. The rest of § 2 is devoted to one direction of the conjecture, from the automorphic side to the Galois side. After a brief review of the LKR method and its consequence on the so-called conjugate self-dual case (§ 2.3), we proceed to the perfectoid geometry of Shimura varieties and its application to the construction of torsion Galois representations (§ 2.4).

In § 3 we give a brief introduction to the Calegari–Geraghty method for proving modularity lifting theorems in the setting of the Betti cohomology of locally symmetric spaces for GL_n over a general number field F. This is a generalization of the famous Taylor–Wiles method. Firstly, in § 3.1, we discuss the prerequisites to the Calegari–Geraghty method - what we need to understand about torsion in the cohomology of these locally symmetric spaces in order to implement the method. We state several conjectures and discuss the state-of-the-art results towards them. Secondly, in § 3.2, we aim to give the reader a flavour of the method by contrasting it with the original Taylor–Wiles method. We sketch both methods in special cases in order to highlight the role of the various conjectures on locally symmetric spaces.

In § 4, we revisit Shimura varieties and the emerging web of conjectures concerning their cohomology with torsion coefficients. In § 4.1, we discuss the geometry of the Hodge–Tate period morphism, which has been instrumental in computing

a suitably generic part of the cohomology of these Shimura varieties and which is likely to play a fundamental role in obtaining a full understanding. In § 4.2, we discuss several different approaches towards computing the generic part of the cohomology of Shimura varieties PEL type A and point towards generalizations. In § 4.3, we come back full circle and discuss applications of these results to modularity via the boundary of the Borel–Serre compactification of these Shimura varieties.

Finally, we confess that we have barely touched on exciting new directions in the Langlands program such as the derived and categorical aspects as well as further new geometry over local nonarchimedean fields. The reader is urged to refer to [**55**], [**48**], and [**52**] in these proceedings to learn about them.

1.1. Notation and conventions.

1.1. Notation and conventions. Given a mathematical object V over a ring R (or Spec R) such as a module or a scheme, let $V_{R'}$ denote the base change of V along a ring homomorphism $R \to R'$.

When k is a field, let $\overline{k}$ denote its separable closure. For a finite extension k'/k write $\mathrm{Res}_{k'/k}$ for the functor carrying algebraic groups over k' to those over k. When k'/k is a Galois extension, write $\Gamma_{k'/k}$ for the Galois group, and set $\Gamma_k := \Gamma_{\overline{k}/k}$. If k is a global field, $\mathbb{A}_k$ will mean the ring of adèles. Given S a finite set of places of k, denote by $\mathbb{A}_k^S$ the ring of adèles away from S (with the S-components removed from $\mathbb{A}_k$). When S is the set of infinite places of k, we also write $\mathbb{A}_k^\infty$. If k is a global field and v is a finite place of k, we let Frob_v denote the geometric Frobenius element at v or its lift in Γ_k.

Let G be a connected reductive group over a local or global field F. We fix a quasi-split inner form G^* of G over F together with an inner twist $G_{\overline{F}} \simeq G^*_{\overline{F}}$ and a pinning for G^* defined over F. Accordingly, as in [**87**, §1.3], we fix the dual group $\hat{G} = \hat{G}^*$ with an L-action and Γ_F-equivariant bijection between the dual root datum of G and the root datum of $\hat{G}$. The pinning determines a hyperspecial subgroup $K_v \subset G(F_v)$ at almost all finite places v where G is unramified (e.g., see [**141**, §4.1]).

When ρ is a finite dimensional representation of a group, write ρ^{ss} for its semisimplification. If ρ is a continuous representation, so is ρ^{ss}.

By convention, the algebraic groups Sp_{2g} and GSp_{2g} are defined over $\mathbb{Z}$ by the symplectic form on $\mathbb{Z}^g \times \mathbb{Z}^g$ given by $((a_i), (b_j)) \mapsto \sum_{i=1}^g (a_i b_{g+i} - a_{g+i} b_i)$.

Write $\mathbb{C}_p$ for the completion of $\overline{\mathbb{Q}}_p$, and $\mathcal{O}_{\mathbb{C}_p}$ for the ring of integers in $\mathbb{C}_p$.

2. Construction of Galois representations with torsion coefficients

2.1. Conjecture and Main Theorems. We recall Buzzard–Gee's formulation of the global Langlands correspondence for reductive groups in the automorphic-to-Galois direction. Then we specialize to the case of general linear groups to state the main theorem of this section, with both characteristic zero and torsion coefficients.

Throughout this section, let G be a connected reductive group over a number field F. We adopt the convention of § 1.1 for the pinning and L-group ${}^L G$ of G.

2.1.1. *On L-normalization vs C-normalization.* Buzzard–Gee [**23**] introduce the notion of L-*algebraicity*, resp. C-*algebraicity*, for automorphic representations π of $G(\mathbb{A}_F)$, according as the infinitesimal characters of the archimedean components of π are integral, resp. integral after shifting by the half sum of positive roots of G. (These conditions will be made explicit for $G = \mathrm{GL}_n$ below.)

They go on to formulate precise conjectures on the corresponding Galois representations, building upon Clozel's work [**39**]. In the C-algebraic case, the Galois representations are valued in the so-called C-group ${}^C G$ of G, which can be thought of either as the L-group of a natural $\mathbb{G}_m$-extension of G [**23**, §5.3], or as a certain semi-direct product of ${}^L G$ with $\mathbb{G}_m$, cf. [**144**, §1.1].

In this article, we are mainly interested in automorphic representations contributing to the cohomology of locally symmetric spaces, which are always C-algebraic [**23**, §7.2] (but not necessarily L-algebraic). Hence the associated Galois representations are to be valued in ${}^C G(\overline{\mathbb{Q}}_\ell)$; compare the conjectures in § 3.2 and § 5.3 of [**23**]. As such, Conjecture 2.1 below should be stated with ${}^C G$ in place of ${}^L G$ in general. However we made the deliberate choice to use ${}^L G$ everywhere in favor of simplicity, as this is harmless for $G = \mathrm{GL}_n$, our main case of interest. In fact, ${}^C G$ is isomorphic to the *direct* product of ${}^L G$ and $\mathbb{G}_m$ for GL_n and many other reductive groups (for instance if ρ_{ad} lifts to a cocharacter of $\hat{G}$; see [**144**, §1.1, Example 2 (3)]). For such groups, we can project from ${}^C G$ to ${}^L G$ and forget about the $\mathbb{G}_m$-factor in ${}^C G$ (since the Galois representation into $\mathbb{G}_m(\overline{\mathbb{Q}}_\ell)$ should be given by the ℓ-adic cyclotomic character, no information is lost). Thereby ${}^L G$ can be used as the target group of Galois representations.

Here is an alternative perspective. Whenever G admits a twisting element in the sense of [**23**, §5.2], which is the case for $G = \mathrm{GL}_n$, one can always twist a C-algebraic representation π by a character to make it L-algebraic. Since L-algebraic automorphic representations are expected to correspond to ${}^L G$-valued Galois representations, Conjecture 2.1 should be fine as written as long as the twisting element is incorporated into the statement.

2.1.2. *The Buzzard–Gee conjecture.* Let ℓ be a prime. Fix a field isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ (disregarding topology). Let S be a finite set of places of F including all places above ℓ or ∞ as well as the places where G is ramified. Write $S^\infty \subset S$ for the subset of finite places. Let $K_v \subset G(F_v)$ be a family of hyperspecial subgroups at finite places $v \notin S$, chosen at almost all places following § 1.1. Set $F_\infty := F \otimes_\mathbb{Q} \mathbb{R}$. Let $K_\infty \subset G(F_\infty)$ be a maximal compact subgroup, and K_∞^0 its neutral component in the archimedean topology. Set $K_\infty' := K_\infty^0 A_G(\mathbb{R})^0$, where A_G is the maximal $\mathbb{Q}$-split subtorus in $\mathrm{Res}_{F/\mathbb{Q}} Z_G$.

Define $\mathcal{A}_{\mathrm{ac}}(G, F)$ to be the set of C-algebraic cuspidal automorphic representation of $G(\mathbb{A}_F)$. Write $\mathcal{A}_{\mathrm{rac}}(G, F)$ for the subset consisting of $\pi \in \mathcal{A}_{\mathrm{ac}}(G, F)$ that is regular, i.e., π_v has regular infinitesimal character at all infinite places v

of F. Define $\mathcal{A}_{\mathrm{coh}}(G, F)$ to be the set of L^2-discrete automorphic representations π of $G(\mathbb{A}_F)$ which are cohomological, i.e., there exists an irreducible algebraic representation ξ of $(\mathrm{Res}_{F/\mathbb{Q}}G) \times_{\mathbb{Q}} \mathbb{C}$ such that the relative Lie algebra cohomology $H^*(\mathrm{Lie}\, G(F_\infty), K'_\infty, \pi_\infty \otimes \xi)$ is non-vanishing in some degree. Write $\mathcal{A}_{\mathrm{coh,c}}(G, F)$ for the subset of cuspidal members in $\mathcal{A}_{\mathrm{coh}}(G, F)$. We have the inclusions

$$(2.1) \qquad \mathcal{A}_{\mathrm{coh,c}}(G, F) \subset \mathcal{A}_{\mathrm{rac}}(G, F) \subset \mathcal{A}_{\mathrm{ac}}(G, F).$$

The second inclusion is strict unless G is a torus. The first inclusion turns out to be an equality for $G = \mathrm{GL}_n$ by [**39**, §3]; for general G, the equality should follow from [**111**, Thm. 1.8]. A superscript S over each set will mean the subset consisting of π such that π_v is unramified for every $v \notin S$.

Let $k \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell, \mathbb{C}\}$ with the discrete topology on $\overline{\mathbb{F}}_\ell$ and the usual topology on $\overline{\mathbb{Q}}_\ell, \mathbb{C}$. Here is a variant of $\mathcal{A}_{\mathrm{coh}}(G, F)$ with k-coefficients. Let $K_{S^\infty} \subset \prod_{v \in S^\infty} G(F_v)$ be a sufficiently small open compact subgroup, which is allowed to vary. Take $K = K_{S^\infty} \times \prod_{v \notin S} K_v \subset G(\mathbb{A}_F^\infty)$. Consider the locally symmetric space

$$(2.2) \qquad Y_K := G(F) \backslash G(\mathbb{A}_F) / K K'_\infty.$$

The double coset algebra $\mathbb{T}^S := \mathbb{Z}[K^S \backslash G(\mathbb{A}_F^\infty) / K^S]$, referred to as the (abstract) Hecke algebra (away from S), acts on the Betti cohomology:

$$(2.3) \qquad \mathbb{T}^S \;\circlearrowright\; \bigoplus_{i \geq 0} \left(\varinjlim_{K_{S^\infty}} H^i(Y_K, k) \right).$$

Let $\mathcal{HE}^S_{\mathrm{coh}}(G, F)_k$ denote the set of eigencharacters $\mathbb{T}^S \to k$ appearing in (2.3), which are also known as *Hecke eigensystems* or *Hecke eigencharacters*. We often denote an eigencharacter by its kernel $\mathfrak{m} \subset \mathbb{T}^S$, which is a maximal ideal, understanding that the eigencharacter also comes with an embedding $\mathbb{T}^S/\mathfrak{m} \hookrightarrow k$. For $S \subset S'$ there is a natural inclusion $\mathcal{HE}^S_{\mathrm{coh}}(G, F)_k \subset \mathcal{HE}^{S'}_{\mathrm{coh}}(G, F)_k$ compatibly with the obvious map $\mathbb{T}^{S'} \to \mathbb{T}^S$.

We can define a more general set $\mathcal{HE}^S_{\mathrm{coh}}(G, F)'_k$ by replacing the coefficient field k in (2.3) with local systems arising from irreducible algebraic representations of $\mathrm{Res}_{F/\mathbb{Q}}G$ (or a suitable integral model thereof, if $\mathrm{char}(k) > 0$) on k-vector spaces. When $k = \mathbb{C}$, we have a natural map

$$(2.4) \qquad \mathcal{A}^S_{\mathrm{coh}}(G, \mathbb{C}) \to \mathcal{HE}^S_{\mathrm{coh}}(G, F)'_{\mathbb{C}}.$$

The map assigns to each π the Hecke eigensystem arising from π^S. This map is an injection for $G = \mathrm{GL}_n$ by the strong multiplicity one theorem but not in general. When $k = \overline{\mathbb{F}}_\ell$, it turns out that $\mathcal{HE}^S_{\mathrm{coh}}(G, F)'_k = \mathcal{HE}^S_{\mathrm{coh}}(G, F)_k$, namely no new Hecke eigensystems are obtained from non-constant coefficients. This ultimately comes from the fact that a continuous representation of an ℓ-adic reductive group on a finite-dimensional $\overline{\mathbb{F}}_\ell$-space is trivial on an open subgroup.

Define $\mathcal{G}(^L G, F)_k$ to be the set of weak equivalence classes of continuous semisimple representations $\rho : \Gamma_F \to {}^L G(k)$ that are unramified at almost all places of F, where ρ_1 and ρ_2 are considered weakly equivalent if the semisimple parts $\rho_1(\mathrm{Frob}_v)_{\mathrm{ss}}$ and $\rho_2(\mathrm{Frob}_v)_{\mathrm{ss}}$ are $\hat{G}(\overline{\mathbb{Q}}_\ell)$-conjugate at every finite place $v \nmid \ell$ where both ρ_1 and ρ_2 are unramified. When $k = \overline{\mathbb{Q}}_\ell$, we introduce the subset $\mathcal{G}_{\mathrm{dR}}(^L G, F)_{\overline{\mathbb{Q}}_\ell}$ consisting of $\rho \in \mathcal{G}(^L G, F)_{\overline{\mathbb{Q}}_\ell}$ such that $\rho|_{\Gamma_{F_v}}$ is a de Rham representation at every $v | \ell$. (An L-group valued representation is said to be de Rham if its composition with some, thus every, faithful representation of $^L G$ is de Rham in the usual sense.) As in

the automorphic case, if we write $\mathcal{G}^S$ in place of $\mathcal{G}$, it denotes the subset of (weak equivalence classes of) ρ which are unramified outside S.

We are ready to state a coarse form of the Buzzard–Gee conjecture (cf. the conjectures in [**23**, §3.2, §5.3], and their refinements [**73**, §4] in the cohomological case); this is statement (1) below. An analogous conjecture with $\overline{\mathbb{F}}_\ell$-coefficients was suggested by Ash [**8, 7**], at least for $G = \mathrm{GL}_n$; see statement (2) below. (Recall from § 2.1.1 that C-groups should be used to state the conjecture in general, but that we are glossing over this point.)

CONJECTURE 2.1. The following are true.

(1) There exists a map (which depends on the choice of ι)
$$\mathrm{GLC}_{G,\overline{\mathbb{Q}}_\ell} : \mathcal{A}^S_{\mathrm{ac}}(G, F) \to \mathcal{G}^S_{\mathrm{dR}}(^L G, F)_{\overline{\mathbb{Q}}_\ell}, \qquad \pi = \otimes'_v \pi_v \mapsto \rho_{\pi,\iota},$$
such that $\iota \pi_v$ corresponds to $(\rho_{\pi,\iota}|_{\Gamma_{F_v}})^{\mathrm{ss}}$ via the unramified local Langlands correspondence at all $v \notin S$.

(2) There exists a map
$$\mathrm{GLC}_{G,\overline{\mathbb{F}}_\ell} : \mathcal{HE}^S_{\mathrm{coh}}(G, F)_{\overline{\mathbb{F}}_\ell} \to \mathcal{G}^S(^L G, F)_{\overline{\mathbb{F}}_\ell}, \qquad \mathfrak{m} \mapsto \overline{\rho}_{\mathfrak{m}},$$
such that $\overline{\rho}_{\mathfrak{m}}$ is unramified at $v \notin S$ and such that $\mathrm{char}(\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))$ is explicitly described in terms of the Hecke eigenvalues at v encoded in $\mathfrak{m}$ for every $v \notin S$.

In fact the conjecture by Buzzard–Gee is more precise than part (1) above in that the Hodge–Tate cocharacter and the image of complex conjugations are predicted for $\rho_{\pi,\iota}$. On the other hand, it is nontrivial to formulate the correct local-global compatibility at ramified places in both (1) and (2). Already in (1) (the problem is worse in (2)), one reason is that the local L-packets and A-packets are only conjectural in general; another problem is the so-called CAP (cuspidal but associated with parabolic subgroups) automorphic representations. However the situation is somewhat favorable for $G = \mathrm{GL}_n$ as we will see below.

The maps $\mathrm{GLC}_{G,\overline{\mathbb{Q}}_\ell}$ and $\mathrm{GLC}_{G,\overline{\mathbb{F}}_\ell}$ do not uniquely characterize the isomorphism classes of $\rho_{\pi,\iota}$ and ρ_τ even if the isomorphism classes of $\rho_{\pi,\iota}|_{\Gamma_{F_v}}$ are prescribed at all places v of F. (See [**95, 94**] for related discussions.) This is why we phrase the conjecture in terms of weak equivalence classes of Galois representations. Moreover, the maps $\mathrm{GLC}_{G,\overline{\mathbb{Q}}_\ell}$ and $\mathrm{GLC}_{G,\overline{\mathbb{F}}_\ell}$ are not expected to be either injective or surjective. When the coefficient is $\overline{\mathbb{Q}}_\ell$, the failure of injectivity is related to the presence of local packets; the problem with surjectivity amounts to describing the image of $\mathrm{GLC}_{G,\overline{\mathbb{Q}}_\ell}$, which should incorporate the phenomenon of global A-packets among other things.

REMARK 2.2. The preceding discussion suggests that the characteristic 0 analogue of $\mathcal{HE}^S_{\mathrm{coh}}(G, F)_{\overline{\mathbb{F}}_\ell}$ should be $\mathcal{A}^S_{\mathrm{coh,c}}(G, F)$ or $\mathcal{HE}^S_{\mathrm{coh}}(G, F)'_k$ with $k \in \{\mathbb{C}, \overline{\mathbb{Q}}_\ell\}$, which is less general than $\mathcal{A}^S_{\mathrm{ac}}(G, F)$. Then it is natural to ask whether $\mathcal{A}^S_{\mathrm{ac}}(G, F)$ admits mod ℓ automorphic analogues, such that the latter accommodates "mod ℓ" of non-regular members of $\mathcal{A}^S_{\mathrm{ac}}(G, F)$ in particular. The authors have no idea.

2.1.3. *The* GL_n-*case.* Now we specialize to the case of $G = \mathrm{GL}_n$, and give more precise statements and results. Let $k \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell, \mathbb{C}\}$ as before. In this case, we write $\mathcal{A}(n, F)$, $\mathcal{HE}^S_{\mathrm{coh}}(n, F)_k$, $\mathcal{G}(n, F)_k$ for $\mathcal{A}(\mathrm{GL}_n, F)$, $\mathcal{HE}^S_{\mathrm{coh}}(G, F)_k$, $\mathcal{G}(\mathrm{GL}_n, F)_k$, and likewise with more superscripts and subscripts. Each member of $\mathcal{G}(n, F)$ is

represented by an n-dimensional representation $\Gamma_F \to \mathrm{GL}_n(k)$, and each weak equivalence is the same as an isomorphism class in the usual sense.

For each $\pi \in \mathcal{A}(n, F)$ and each embedding $\tau : F \hookrightarrow \mathbb{C}$, which determines an archimedean place v of F together with $F_v \hookrightarrow \mathbb{C}$, the infinitesimal character of π_v may be viewed as a multi-set of n complex numbers, to be denoted by $\mathrm{HC}_\tau(\pi)$. Then π is C-algebraic if and only if $\mathrm{HC}_\tau(\pi) - \frac{n-1}{2}$ is integral (i.e., $a \in \mathbb{Z} + \frac{n-1}{2}$ for all $a \in \mathrm{HC}_\tau(\pi)$) for every τ.

We start by specifying the normalization of local Langlands in part (1) of Conjecture 2.1. Let LL denote the unramified local Langlands correspondence with $\mathbb{C}$-coefficients carrying Satake parameters to the eigenvalues of geometric Frobenius elements. Then our requirement on $\pi \mapsto \rho_{\pi,\iota}$ is that the semisimplification of $\iota^{-1}\rho_{\pi,\iota}$, restricted to the Weil group over F_v, is isomorphic to $\mathrm{LL}(\pi \otimes |\det|^{(1-n)/2})$. In this case, the Hodge–Tate weights of $\rho_{\pi,\iota}$ relative to $\tau : F \hookrightarrow \overline{\mathbb{Q}}_\ell$ should be given by the multi-set $\mathrm{HC}_{\iota^{-1}\tau}(\pi) + \frac{n-1}{2}$.

Next we make part (2) of Conjecture 2.1 concrete. At each finite place v of F not above ℓ, write q_v for the residue field cardinality at v. The square root $q_v^{1/2} \in \mathbb{R}_{>0}^\times$ is viewed as an element of $\overline{\mathbb{Q}}_\ell^\times$ via ι, or as an element of $\overline{\mathbb{F}}_\ell^\times$ via the reduction map (since $q_v^{1/2}$ is an ℓ-unit). There is a "universal" Hecke polynomial

$$H_v(x) := 1 - q_v^{(n+1)/2}T_{1,v}x + q_v^{(n+1)}T_{2,v}x + \cdots + (-1)^n q_v^{n(n+1)/2}T_{n,v}x^n \in \mathbb{T}^S[q_v^{1/2}][x].$$

The Hecke polynomial for each Hecke eigencharacter $\mathfrak{m} \in \mathcal{HE}_{\mathrm{coh}}^S(n, F)_k$ at v is defined by applying it to the coefficients of H_v:

$$H_v[x]_{/\mathfrak{m}} \in (\mathbb{T}^S/\mathfrak{m})[x] \subset k[x].$$

When $k = \overline{\mathbb{F}}_\ell$, the condition at each v in Conjecture 2.1 (2) can now be made concrete: it is the equality

$$(2.5) \qquad\qquad \mathrm{char}(\rho_\tau(\mathrm{Frob}_v)) = H_v[x]_{/\mathfrak{m}}.$$

When $k = \overline{\mathbb{Q}}_\ell$, if $\mathfrak{m}_\pi$ denotes the image of $\pi \in \mathcal{A}_{\mathrm{coh}}^S(n, F)$ under (2.4) via $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, then the equality (2.5) is equivalent to the condition in Conjecture 2.1 (1) that $\iota \pi_v$ corresponds to $(\rho_{\pi,\iota}|_{\Gamma_{F_v}})^{\mathrm{ss}}$. Thus the two parts of Conjecture 2.1 are consistent through the operation of taking mod ℓ on the Galois side.

We make further observations on Conjecture 2.1. When $G = \mathrm{GL}_n$, the representations $\rho_{\pi,\iota}$ and $\overline{\rho}_{\mathfrak{m}}$ are uniquely characterized by the Brauer–Nesbitt theorem and the Chebotarev density theorem. The map $\mathrm{GLC}_{n,\overline{\mathbb{Q}}_\ell}$, if exists, must be injective by the strong multiplicity one theorem for GL_n. In this case, the image of $\mathrm{GLC}_{n,\overline{\mathbb{Q}}_\ell}$ is believed to be exactly the irreducible representations in $\mathcal{G}_{\mathrm{dR}}^S(n, F)_{\overline{\mathbb{Q}}_\ell}$. (The dictionary in the GL_n-case is that cuspidality corresponds to irreducibility through $\mathrm{GLC}_{n,\overline{\mathbb{Q}}_\ell}$.) At least when F is totally real, it is reasonable to expect that an irreducible $\overline{\rho} \in \mathcal{G}^S(n, F)_{\overline{\mathbb{F}}_\ell}$ belongs to the image of $\mathrm{GLC}_{n,\overline{\mathbb{F}}_\ell}$ exactly when $\overline{\rho}$ is odd (e.g., see [51, Def. 1.2] for a definition), based on the following heuristic. The result [51, Thm. A] shows that $\overline{\rho}$ lifts to $\rho \in \mathcal{G}_{\mathrm{dR}}^S(n, F)_{\overline{\mathbb{Q}}_\ell}$ (at the expense of "raising the level"); then ρ should correspond via $\mathrm{GLC}_{n,\overline{\mathbb{Q}}_\ell}$ to some $\pi_\rho \in \mathcal{A}_{\mathrm{coh,c}}^S(n, \mathbb{C})$. Then the reduction mod ℓ of the Hecke eigensystem for π_ρ should correspond to $\overline{\rho}$.[1]

---

[1] This may appear to be at odds with [13, Thm. 1.4], which roughly says that $\mathcal{HE}^S(3, \mathbb{Q})_{\overline{\mathbb{F}}_\ell}$ grows exponentially faster than $\mathcal{A}_{\mathrm{coh,c}}^S(3, \mathbb{Q})$ as level goes to infinity. (More precisely, the analogue is shown for "strongly cyclic" coefficient modules.) However there is no contradiction since the

To state results towards the above conjecture, we introduce some terminology. Let F be a totally real field, or a CM field (a totally imaginary quadratic extension of a totally real field). Such an F admits a complex conjugation c that is independent of how F is embedded into $\mathbb{C}$. Write F^+ for the fixed subfield of F by c, and N_{F/F^+} for the norm map (which is the identity map if F is totally real). Consider $\pi \in \mathcal{A}_{\mathrm{ac}}(n, F)$ *essentially conjugate self-dual* if $\pi \circ c \simeq \pi^\vee \otimes (\chi \circ N_{F/F^+})$ for a Hecke character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$. Write $\tilde{\mathcal{A}}_\star(n, F)$ for the subset of essentially conjugate self-dual representations in $\mathcal{A}_\star(n, F)$ for $\star \in \{\mathrm{ac}, \mathrm{rac}\}$. Let $\mathcal{A}_\star^1(n, F)$ denote the subset of conjugate self-dual members, namely π such that $\pi \circ c \simeq \pi^\vee$. If F is totally real, the word "conjugate" is usually omitted everywhere.

Here is the main theorem of this section. (See the end of § 2.2, resp. § 2.4, for known results on local-global compatibility at $v \in S$ and other comments.) Part (1) is due to Harris–Lan–Taylor–Thorne [69]. Shortly thereafter, Scholze [116] proved both parts by a different method based on perfectoid Shimura varieties and Hodge–Tate period morphisms. Boxer's thesis [21] suggests an alternative congruence argument for both parts.

THEOREM 2.3. *Let F be a totally real or CM field, and ι as above.*

(1) *There exists an injection*

$$\mathrm{GLC}_{n,\overline{\mathbb{Q}}_\ell} : \mathcal{A}_{\mathrm{rac}}^S(n, F) \to \mathcal{G}_{\mathrm{dR}}^S(n, F)_{\overline{\mathbb{Q}}_\ell}, \qquad \pi \mapsto \rho_{\pi,\iota},$$

such that $\iota\pi_v$ corresponds to $(\rho_{\pi,\iota}|_{\Gamma_{F_v}})^{\mathrm{ss}}$ via the unramified LLC at all $v \notin S$.

(2) *There exists a map*

$$\mathrm{GLC}_{n,\overline{\mathbb{F}}_\ell} : \mathcal{HE}_{\mathrm{coh}}^S(n, F)_{\overline{\mathbb{F}}_\ell} \to \mathcal{G}^S(n, F)_{\overline{\mathbb{F}}_\ell}, \qquad \mathfrak{m} \mapsto \overline{\rho}_\mathfrak{m},$$

such that $\mathrm{char}(\rho_\tau(\mathrm{Frob}_v)) = H_v[x]_{/\mathfrak{m}}$ for every $v \notin S$.

From now, restrict F to be a totally real or CM field. We follow Scholze's proof as it has further applications as presented in § 4. The proof breaks up into two main parts, carried out in the following order:

(A) Prove Theorem 2.3 (1) for $\pi \in \tilde{\mathcal{A}}_{\mathrm{rac}}(n, F)$.
(B) Prove Theorem 2.3 (2) and its analogue for $\overline{\mathbb{Z}}_p/(p^m)$-coefficients, $m \geq 1$.

Part (A) was completed by a combination of works by Clozel, Kottwitz, et al., spanning over nearly two decades, before Theorem 2.3 was proved in general; see § 2.2.5 for a guide to references. The argument will be reviewed in § 2.2. Explanation of Part (B) will take up § 2.3–§ 2.4 below. Once this is done, Theorem 2.3 (1) in full follows from (2) by realizing π in the cohomology and gluing Galois representations modulo powers of p. See the proof of [116, Cor. 5.4.2] for details.

REMARK 2.4. Before we restrict ourselves to the regular or cohomological case from now on, let us mention several results on Conjecture 2.1 in the *non-regular* case. The basic principle is that we can often prove the conjecture when π contributes to the so-called coherent cohomology of Shimura varieties (even if π is not cohomological, i.e., not appearing in the Betti cohomology). When $G = \mathrm{GL}_2$ over $\mathbb{Q}$, such π come from weight 1 cuspforms. Deligne–Serre [46] proved the conjecture

---

lifting from $\overline{\rho}$ to ρ increases level. We thank Naomi Sweeting for her insightful question, which led to this comment.

in this case. For GL$_2$ over a totally real field, the problem is about Hilbert cusp-forms of partial weight 1. Here the result is due to Rogawski–Tunnell, Ohta, and Jarvis. (See [**72**] and the references therein.) In semisimple rank greater than 1, there has been much progress since Taylor's work [**134**] on GSp$_4$ over $\mathbb{Q}$; see the last paragraph of § 2.4 for recent results. All of these results rest on variants and generalizations of Hasse invariants to be discussed in § 2.4.

However we stress that Conjecture 2.1 is still unsettled even when $G = $ GL$_2$ over $\mathbb{Q}$. The remaining π (which does not show up in either coherent or Betti cohomology of modular curves) correspond to Maass cuspforms. Except when π arise from Hecke characters over real quadratic fields via automorphic induction (the conjecture is reduced to class field theory for such π), very little is known.

2.2. The conjugate self-dual case. This subsection sketches the proof of the following special case for essentially conjugate self-dual representations.

THEOREM 2.5. *Theorem 2.3 (1) is true if $\pi \in \tilde{\mathcal{A}}_{\mathrm{rac}}(n, F)$.*

In fact there is a standard argument, cf. the proofs of [**11**, Thm. 1.1, 1.2] and [**44**, Prop. 4.3.1], to reduce the proof of Theorem 2.5 to the following case, so we will assume it throughout § 2.2:

- F is a CM field, and $\pi \in \tilde{\mathcal{A}}^1_{\mathrm{rac}}(n, F)$.

2.2.1. *Setup.* Before getting to the proof of Theorem 2.5, we explain the basic strategy to make progress towards constructing the map GLC$_{G, \overline{\mathbb{Q}}_\ell}$ in an idealized setup, based on the Langlands–Kottwitz–Rapoport method for the cohomology of Shimura varieties.

Let (G, X) be a Shimura datum [**45**, §2.1.1]; see § 2.3.1 of Caraiani's article in [**14**] for further explanation. Thus G is a connected reductive group over $\mathbb{Q}$, and X is a Hermitian symmetric domain with a transitive $G(\mathbb{R})$-action. From X we obtain a conjugacy class $[\mu_X]$ of cocharacters $\mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$. Then $[\mu_X]$ is defined over a number field $E := E(G, X) \subset \mathbb{C}$ called the reflex field. We fix a $\Gamma_{\mathbb{Q}}$-pinning $(\hat{B}, \hat{T}, \{\hat{X}_\alpha\})$. From the inverse of $[\mu_X]$ we obtain a $\hat{B}$-dominant character $-\mu \in X^*(\hat{T})$, which is then fixed by the Γ_E-action. So there exists a representation

$$r_{-\mu} : {}^L G_E = \hat{G} \rtimes \Gamma_E \to \mathrm{GL}(V_{-\mu}),$$

characterized uniquely up to isomorphism by the two conditions: $r_{-\mu}|_{\hat{G}}$ is an irreducible representation of highest weight $-\mu$, and Γ_E acts trivially on the highest weight space.

We have a projective system of canonical models of Shimura varieties Sh $=$ (Sh$_K$), which are quasi-projective smooth varieties over E and labeled by neat open compact subgroups $K \subset G(\mathbb{A}^\infty)$. The system Sh is equipped with the "Hecke" action defined over E by $G(\mathbb{A}^\infty)$. Each Sh$_K(\mathbb{C})$ is a complex manifold, identified with Y_K in (2.2) by construction (except the technical point that K'_∞ may be replaced with a larger subgroup of $G(\mathbb{R})$ in which K'_∞ has finite index). Each $\xi \in \mathrm{Irr}((\mathrm{Res}_{F/\mathbb{Q}} G)_{\mathbb{C}})$ gives rise to lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_{\xi,K}$ over Sh$_K$ compatibly with respect to the transition maps between (Sh$_K$). The étale cohomology with compact support

$$(2.6) \qquad H^i_{\xi,c} := \varinjlim_K H^i(\mathrm{Sh}_K \times_E \overline{E}, \mathcal{L}_{\xi,K}), \qquad i \in \mathbb{Z}_{\geq 0},$$

is a $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{E}/E)$-module, and

$$[H_{\xi,c}] := \sum_{i \geq 0} (-1)^i [H^i_{\xi,c}]$$

is a virtual $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{E}/E)$-module (viewed in the Grothendieck group; see [**70**, pp.23–25]).

2.2.2. *The Langlands–Kottwitz–Rapoport (LKR) method.* This is a systematic way to compute $[H_{\xi,c}]$ at each rational prime $p \neq \ell$ of good reduction. More precisely, let $\mathfrak{p}$ be a prime of E above p; assume that $G_{\mathbb{Q}_p}$ is an unramified group and fix a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$. As (sufficiently small) open compact subgroups $K^p \subset G(\mathbb{A}^{\infty,p})$ vary, we have a projective system $\mathrm{Sh}_{K_p} := \varprojlim_{K^p} \mathrm{Sh}_{K_p K^p}$ over E equipped with a $G(\mathbb{A}^{\infty,p})$-action. By taking the limit in (2.6) over K^p, we can define H^i_{ξ,c,K_p}, which is the K_p-fixed subspace of $H^i_{\xi,c}$, as well as a virtual $G(\mathbb{A}^{\infty,p}) \times \mathrm{Gal}(\overline{E}/E)$-module $[H_{\xi,c,K_p}]$. Let $\mathfrak{p}$ be a prime of E above p. The LKR method aims to show that the local Galois action at $\mathfrak{p}$ is unramified and to describe

the virtual $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_{\mathfrak{p}}$-module $[H_{\xi,c,K_p}]$

via automorphic representations. The method consists of the following steps.

(1) Construct an integral model $\mathscr{S}_{K_p}$ of $\mathrm{Sh}_{K_p} := \varprojlim_{K^p} \mathrm{Sh}_{K_p K^p}$ over $\mathcal{O}_{E,\mathfrak{p}}$, which is canonical in the sense of [**81**].

(2) Establish a trace formula for the action of $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_{\mathfrak{p}}^{\mathbb{Z}}$ on $[H_{\xi,c,K_p}]$ by studying the structure of points on $\mathscr{S}_{K_p}$ modulo $\mathfrak{p}$.

(3) Stabilize the trace formula.

(4) Compare the outcome with the stabilization of the Arthur-Selberg trace formula to obtain a representation-theoretic description of $[H_{\xi,c,K_p}]$.

Part (1) was done for PEL-type Shimura varieties by Kottwitz [**85**] and for abelian-type Shimura varieties by Kisin [**81**] ($p > 2$) and Kim–Madapusi Pera [**77**] ($p = 2$). Until recently, the best result on (2) and (3) has been due to Kottwitz [**84, 85**] in the case of PEL-type Shimura varieties. This has been used to carry out (4) in some important special cases, which were enough to prove Theorem 2.5.

Now the first two steps are complete more generally for Shimura varieties of abelian type in [**79**] following [**82**]; the key is the proof of a version of the Langlands–Rapoport (LR) conjecture regarding the structure of points in (2). (The work of Kottwitz [**85**], and its recent extension to the case of Hodge type by Lee [**96**] bypasses the LR conjecture. However the LR conjecture seems essential to deal with abelian-type Shimura varieties.) The outcome has been employed to prove new cases of Conjecture 2.1 (2) for the groups GSp_{2n} and certain quasi-split forms of GSO_{2n} over totally real fields. Further applications are being worked out in [**78**].

2.2.3. *Idealized situation.* Kottwitz gave a conjectural description of the intersection cohomology version of $[H_{\xi,c}]$ (thus also $[H_{\xi,c,K_p}]$) in [**84**, §10] but it is quite complicated. To convey the basic principle, we make the following simplifying hypotheses:

(a) G^{ad} is anisotropic over $\mathbb{Q}$,

(b) $\pi \in \mathcal{A}_{\mathrm{coh}}(G, \mathbb{Q})$ is essentially tempered at every place,

(c) we pretend that endoscopy disappears (e.g., conjugacy classes are stable conjugacy classes, and L-packets are singletons).

Hypothesis (c) is satisfied if G is an inner form of (restriction of scalars of) GL_2 but typically false unless we restrict attention to only a certain subset of π. Hypothesis (a) implies that Shimura varieties and their canonical integral models are proper

over the base, so that there is no need for compactification; thus we freely write H_ξ instead of $H_{\xi,c}$. The same hypothesis also implies that each $H^i_{\xi,c}$ is a semisimple $G(\mathbb{A}^\infty)$-module via Matsushima's formula, since the L^2-automorphic spectrum is semisimple. Hypothesis (b) is supposed to ensure that π contributes to cohomology only in the middle degree, namely H^d_ξ where $d := \dim \mathrm{Sh}$. Consider the assignment

$$(2.7) \qquad \pi \quad \mapsto \quad R_{\pi,\mu,\iota} := \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi^\infty, H^d_\xi).$$

If we had $\rho_{\pi,\iota}$ (whose construction is our goal), then our expectation is essentially that

$$(2.8) \qquad R_{\pi,\mu,\iota} = r_{-\mu} \circ \rho_{\pi,\iota}$$

up to twists having to do with normalizations. This leads to:

An approach to Conjecture 2.1. Suppose $F = \mathbb{Q}$. Given G, ι and π, choose (possibly several) Shimura data and study the resulting representations $R_{\pi,\mu,\iota}$. From this, construct $\rho_{\pi,\iota}$ and show the desired properties of $\rho_{\pi,\iota}$.

REMARK 2.6. If $F \neq \mathbb{Q}$ in Conjecture 2.1, then the group G over F in the conjecture will differ from the reductive groups used in Shimura data, because the latter are always over $\mathbb{Q}$. Often the latter is chosen to be a form of $\mathrm{Res}_{F/\mathbb{Q}}G$.

Let us illustrate the mechanism for unitary groups in relation to Theorem 2.5. Let F be a CM field. We fix an algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$. Put $F^+ := F^{c=1}$ for the totally real subfield fixed by c. Choose a CM type Φ, namely a representative for the coset space $\mathrm{Gal}(F/F^+)\backslash\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}})$, so that we have the identification

$$F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\tau \in \Phi} \mathbb{C}, \qquad s \otimes t \mapsto (\tau(s)t).$$

Consider a unitary group U over F^+ for the n-dimensional Hermitian space W over F (with respect to F/F^+) whose signature at each $\tau \in \Phi$ is (a_τ, b_τ) with $a_\tau + b_\tau = n$. We identify $U \times_{F^+} \mathbb{R} = U(a_\tau, b_\tau)$, where the latter is the real unitary group for the form

$$((z_i), (w_j)) \in \mathbb{C}^n \times \mathbb{C}^n \quad \mapsto \quad \sum_{i=1}^{a_\tau} z_i \overline{w}_j - \sum_{i=a_\tau+1}^{n} z_i \overline{w}_i.$$

Now define $G := \mathrm{Res}_{F^+/\mathbb{Q}}U$. Consider the $\mathbb{R}$-morphism $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}} = \prod_{\tau \in \Phi} U(a_\tau, b_\tau)$, which on $\mathbb{R}$-points sends $z \in \mathbb{C}^\times$ to the block diagonal matrix $\mathrm{diag}((z/\overline{z})I_{a_\tau}, I_{b_\tau})$. Setting X to be the $G(\mathbb{R})$-conjugacy class of h, we obtain a Shimura datum (G, X). In this example, we have

$$\hat{G} = \prod_{\tau \in \Phi} \mathrm{GL}_n(\mathbb{C}), \qquad \mu(z) = (\mathrm{diag}(zI_{a_\tau}, I_{b_\tau})),$$

if we take the upper triangular Borel subgroup in the pinning of $\hat{G}$. So

$$(2.9) \qquad V_{-\mu}|_{\hat{G}} = \otimes_{\tau \in \Phi} \wedge^{a_\tau} (\mathrm{Std}^\vee),$$

where $\mathrm{Std}^\vee$ denotes the dual of the standard representation of GL_n. We leave it as an exercise for the reader to describe the $\Gamma_{\mathbb{Q}}$-action on $\hat{G}$ and compute the reflex field E. In light of (2.9), an important case is when U has signatures

$$(a_{\tau_0}, b_{\tau_0}) = (1, n-1) \text{ for some } \tau_0 \in \Phi \quad \text{and} \quad (a_\tau, b_\tau) = (0, n) \text{ for } \tau \in \Phi\backslash\{\tau_0\}.$$

In this case, view F as a subfield of $\overline{\mathbb{Q}}$ (thus also $\mathbb{C}$) via τ_0. Then $E = F$ (unless $E = \mathbb{Q}$ and $n = 2$, but this case is excluded by (a)). The representation $V_{-\mu}|_{\hat{G}}$ is $\mathrm{Std}^\vee$ on the τ_0-component and trivial on the other components. This extends to

a representation of ${}^L G_F = \hat{G} \rtimes \Gamma_F$ by making Γ_F act trivially. So the expected output from the cohomology of Shimura varieties is that, under hypotheses (a), (b), and (c),

$$\pi \in \mathcal{A}_{\mathrm{coh}}(G, \mathbb{Q}) = \mathcal{A}_{\mathrm{coh}}(U, F^+) \quad \mapsto \quad R_{\pi,\mu,\iota} = \rho_{\pi,\iota}^{\vee} \in \mathcal{G}_{\mathrm{dR}}(n, F)$$

up to twists.

REMARK 2.7. When Theorem 2.5 was proved, it was based on Kottwitz's result for PEL-type Shimura data [**85**] in the case of certain unitary *similitude* groups. We have chosen to avoid similitude groups to keep the notation and exposition simpler. The price to pay is that the Shimura datum above is not of PEL or Hodge type but only of abelian type. So the recent work [**79**] is needed to apply the LKR method; the overall argument mostly stays the same other than that.

Going back to Theorem 2.5, recall that our main interest lies in $\Pi \in \tilde{\mathcal{A}}_{\mathrm{rac}}^1(n, F)$. The connection between the two sets $\mathcal{A}_{\mathrm{coh}}(U, F^+)$ and $\tilde{\mathcal{A}}_{\mathrm{rac}}^1(n, F)$ is provided by an automorphic base change due to Clozel and Labesse: [2]

$$\mathrm{BC} : \mathcal{A}_{\mathrm{coh}}(U, F^+) \to \tilde{\mathcal{A}}_{\mathrm{ra,iso}}^1(n, F),$$

where the target is by definition the set of isobaric sums $\boxplus_{i \in I} \Pi_i$ such that $\Pi_i \in \tilde{\mathcal{A}}_{\mathrm{ra}}^1(n_i, F)$ with $\sum_i n_i = n$. Conversely, they also prove that every $\Pi \in \tilde{\mathcal{A}}_{\mathrm{rac}}^1(n, F)$ is the image of some $\pi \in \mathcal{A}_{\mathrm{coh}}(U, F^+)$ under BC, namely Π admits a "descent" π (such a π is typically far from unique), provided that U is quasi-split at all finite places. In fact there is a parity obstruction (e.g., see [**40**, §2]) for finding such a U with signatures $(1, n-1), (0, n), ..., (0, n)$ at ∞ if n is even (and $[F^+ : \mathbb{Q}]$ is even at the same time), leading to complication in § 2.2.4 (b) and (c) below, but let us ignore it until we revisit the issue.

Thus the construction of $\rho_{\Pi,\iota}$ for Theorem 2.5 would go as follows:

$$(2.10) \quad \Pi \in \tilde{\mathcal{A}}_{\mathrm{rac}}^1(n, F) \overset{\mathrm{descent}}{\rightsquigarrow} \pi \in \mathcal{A}_{\mathrm{coh}}(U, F^+) \overset{[H_\xi]}{\rightsquigarrow} \rho_{\pi,\iota}^{\vee} =: R_{\Pi,\iota} \in \mathcal{G}_{\mathrm{dR}}(n, F).$$

A key point is to compute the restriction of $R_{\Pi,\iota}$ to local Galois groups, at the places v of F where Π is unramified. The unramified components of π can be described from those of Π as this is part of the automorphic base change and descent. From this information, the LKR method discussed above gets us to understand $R_{\Pi,\iota}$ at each v to complete the argument.

2.2.4. *Reality.* The reader is cautioned that the argument sketched in the preceding paragraph is not rigorous but idealistic. In fact (a) is somewhat harmless since it is satisfied if $F^+ \neq \mathbb{Q}$; even if $F^+ = \mathbb{Q}$ there is a standard argument [**132**] to reduce to the case $F^+ \neq \mathbb{Q}$. However we do not know (b) a priori, which amounts to a generalized Ramanujan conjecture. In other words, (b) cannot be used as an ingredient. (Rather, it turns out to be a consequence, cf. [**29**].) More seriously, (c) cannot be implemented since endoscopy does exhibit itself for unitary groups.

These difficulties can be overcome with more effort, at the expense of modifying the ideal version (2.10). At first, Theorem 2.5 was proven by [**86, 40, 70**] under a technical assumption that Π is essentially square-integrable at a finite place. The

---

[2]For instance, see [**90**, Cor. 5.3, Thm. 5.4] for both base change and descent results, which are proven under mild technical hypotheses. In practice, one can overcome the hypothesis to obtain Theorem 2.5. A general base change and descent result follows from the endoscopic classification for unitary groups in [**101, 74**], but these works are conditional on some papers to be written. (See the introduction of [**74**] for details.)

assumption is made to avoid endoscopy (i.e., to implement (c)) via the Jacquet–Langlands transfer to the unit group $D^\times$ of a division algebra and by using a unitary group that is a form of $D^\times$. When Ngô proved the fundamental lemma [**104**] (also thanks to earlier contributions by Cluckers–Loeser, Hales, Waldspurger and others), it became possible to embrace endoscopy and prove Theorem 2.5 in full generality. This is done in [**43, 130, 38**]. Now we explain more precise ideas on how to work without hypotheses (a), (b), and (c).

(a) *Reducing to the anisotropic case.* As mentioned above, the only nontrivial case is when $F^+ = \mathbb{Q}$ (so F is imaginary quadratic over $\mathbb{Q}$); then $G = U$ is a unitary group of $\mathbb{Q}$-rank one so condition (a) fails.

Suppose that the theorem was proved in all cases for $F^+ \neq \mathbb{Q}$. We want to construct $R_{\Pi,\iota} \in \mathcal{G}_{\mathrm{dR}}(n, F)$ starting from $\Pi \in \tilde{\mathcal{A}}^1_{\mathrm{rac}}(n, F)$. The reduction step is based on the fact that there is a "huge" supply of real quadratic fields F_0^+. For each F_0^+, set $F_0 := F_0^+ F$. Then we have a diagram

$$
\begin{array}{ccc}
\Pi \in \tilde{\mathcal{A}}^1_{\mathrm{rac}}(n, F) & \dashrightarrow^{\ ?\ } & R_{\Pi,\iota} \in \mathcal{G}_{\mathrm{dR}}(n, F) \\
{\scriptstyle \text{base change}} \downarrow & & \downarrow {\scriptstyle \text{Res}} \\
\Pi_0 \in \tilde{\mathcal{A}}^1_{\mathrm{rac}}(n, F_0) & \xrightarrow[\text{assumed}]{} & R_{\Pi_0,\iota} \in \mathcal{G}_{\mathrm{dR}}(n, F_0),
\end{array}
$$

where the vertical arrows are the Arthur–Clozel base change and the restriction of Galois representations along F_0/F, respectively. If $R_{\Pi,\iota}$ exists with the correct restriction to local Galois groups, then it should make the diagram commutative. Thus the problem is whether we can construct $R_{\Pi,\iota}$ from the data of $R_{\Pi,\iota}|_{\Gamma_{F_0}}$ for the (infinite) family of real quadratic fields F_0^+. This algebraic problem was solved by Blasius–Ramakrishnan [**15**, §4.3]; see [**132**] for a generalization.

The above argument is flexible. For instance, one can analogously reduce to the case when $[F^+ : \mathbb{Q}]$ is even by making similar quadratic base changes.

(b) *Concentration in the middle degree and temperedness.* We begin with a slogan: tempered automorphic representations should contribute to the ℓ-adic cohomology of Shimura varieties only in the middle degree. Although we cannot use the principle since the relevant case of the Ramanujan conjecture is a priori unknown, the idea is that an approximation towards Ramanujan can still do the job thanks to the Weil conjectures. Moreover the Ramanujan conjecture for relevant cuspidal automorphic representations can be shown as a by-product.

To be more precise, let Π and π be as in (2.10). We want to show that the π^∞-isotypic part of the cohomology is concentrated in degree $d = \dim \mathrm{Sh}$, so that $R_{\pi,\mu,\iota}$ in (2.7) is equal to the alternating sum of the π^∞-isotypic part of the cohomology over all degrees (up to the sign $(-1)^d$). The alternating sum is amenable to computation by the fixed-point formula in the context of the LKR method, so we would then have a grip on the local Galois action of $R_{\pi,\mu,\iota}$.

As we are in the anisotropic case thanks to (a), mod $\mathfrak{p}$ Shimura varieties are proper (and smooth, since the level subgroup K_p at p is hyperspecial), so that the Weil conjectures apply to the Frobenius eigenvalues on the cohomology. In particular there is no cancellation between different degrees in the alternating sum. Now the LKR method relates the Frob$_{\mathfrak{p}}$-eigenvalues in the π^∞-isotypic part to the Satake parameters of π at $\mathfrak{p}$. Now the the Jacquet–Shalika bound on the Satake parameters of Π show that they are not far from essentially tempered. On the

other hand, the Frob$_{\mathfrak{p}}$-eigenvalues in degree $i \neq d$ are "far" from those in degree d by the Weil conjectures. From this one can conclude that the Frob$_{\mathfrak{p}}$-eigenvalues in the π^∞-isotypic part must appear in degree d, and further that $\pi_{\mathfrak{p}}$ is essentially tempered (a fortiori its Satake parameters are Weil numbers of appropriate weight).

If we start from $\Pi \in \tilde{\mathcal{A}}^1_{\mathrm{rac}}(n, F)$ (rather than π) then we would prove the Ramanujan conjecture for Π by descending to π as in the first arrow of (2.10) and run the preceding argument. This can be made to work if n is odd, but there is a parity obstruction if n is even; see the paragraph above (2.10). For even n (when $[F^+ : \mathbb{Q}]$ is also even), the Ramanujan conjecture for $\tilde{\mathcal{A}}^1_{\mathrm{rac}}(n, F)$ was completed in [**42, 29**] by switching to the signature $(2, n-2), (0, n), ..., (0, n)$ or $(1, n-1), (1, n-1), (0, n), ..., (0, n)$ at ∞ to avoid the obstruction and by making a more elaborate argument.

(c) Working in the endoscopic setting. This is the most serious problem. A starting point is the stabilization of the fixed point formula for Sh. This can be done thanks to the fundamental lemma (which also implies that the Langlands–Shelstad transfer exists by [**140**]).

The situation is somewhat favorable when n is odd. Then the construction goes as in (2.10) as there is no obstruction for finding U and descending Π to π. In this case the main problem is to run the trace formula portion of the LKR method to compute the local Galois action of $R_{\Pi,\iota} = \rho^\vee_{\pi,\iota}$. (At primes where π is ramified, one uses a variant method following [**70, 130**].) This involves the stabilized formula for Sh, a comparison with the twisted trace formula for GL_n, and the strong multiplicity one theorem to single out the Π-isotypic part in the twisted trace formula.

The case of even n is complicated due to the obstruction mentioned in part (b). Here switching to a different signature at ∞ is not a good idea in view of (2.8) and (2.9). Instead, we go from $\Pi \in \tilde{\mathcal{A}}^1_{\mathrm{rac}}(n, F)$ to an isobaric sum $\Pi \boxplus \chi$ on $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ by making an auxiliary choice of conjugate self-dual Hecke character χ of $\mathrm{GL}_1(\mathbb{A}_F)$. Then we consider a unitary group U in $n + 1$ variables which is quasi-split at all finite places and signature $(1, n), (0, n+1), ..., (0, n+1)$; there is no obstruction since $n+1$ is odd. This unitary group gives rise to Shimura varieties as before. Then the recipe (2.10) is modified as

$$\Pi \boxplus \chi \overset{\mathrm{descent}}{\rightsquigarrow} \pi \in \mathcal{A}_{\mathrm{coh}}(U, F^+) \overset{[H_\xi]}{\rightsquigarrow} \rho^\vee_{\pi,\iota} =: R_{\Pi,\iota}.$$

One can think of the first arrow as taking *advantage* of endoscopy. The local Galois action at $\mathfrak{p}$ of $\rho^\vee_{\pi,\iota}$ can be computed using a similar method as for the case of odd n. However the computation shows that the Galois representation $\rho^\vee_{\pi,\iota}$ sometimes corresponds to Π and sometimes to χ; we definitely want the former to obtain the correct Galois representation $R_{\Pi,\iota}$. It turns out that the archimedean components of Π and χ control which case should occur. As long as Π satisfies a mild regularity condition at ∞, it is shown in [**43, 130**] that χ can be chosen such that $R_{\Pi,\iota}$ corresponds to Π rather than χ.

In summary, if n is even then we are done as long as Π satisfies a mild regularity hypothesis. The remaining case is covered by a congruence argument due to Chenevier–Harris [**38**] utilizing the eigenvarieties for definite unitary groups.

2.2.5. *Complements and further references.* There are several expository articles on the Langlands–Kottwitz–Rapoport method with different emphases, including [**16, 41, 100, 60, 145**]. In the case of bad reduction (when the level subgroup is parahoric but not hyperspecial), one can start from the surveys [**108, 63**]. See

also [**126**] for a discussion of the LKR method and a variety of related approaches, results, and further references.

The proof of Theorem 2.5 along the lines of [**70, 130**] is surveyed in [**131**], where some aspects of § 2.2.1–§ 2.2.4 are more detailed and more references are given. The argument to prove Theorem 2.5 may be partly simplified by using [**120**] (which circumvents Mantovan's formula and the works [**43, 130**]) or [**56**] (which further avoids endoscopy and reduces to the case covered by [**40**]).

For a stronger version of Theorem 2.5, including the local-global compatibility at *all* finite places, refer to [**10**, Thm. 2.1.1] and the references therein due to many mathematicians. This theorem is complemented by [**105**] (removing a parity condition) and [**135, 133, 32**] (describing the image of complex conjugation).

From Theorem 2.5, one can deduce many cases of part (1) of Conjecture 2.1 for classical groups G over totally real fields F^+. The additional work consists in establishing a Langlands functoriality for the standard embedding of $^L G$ into a general linear group, and showing that the Galois representations in Theorem 2.5 satisfy a certain sign condition so as to yield Galois representations valued in the L-group (or C-group) of G. The sign has been pinned down by [**12**]. As for the functoriality, given $\pi \in \mathcal{A}_{\mathrm{ac}}(G, F^+)$ we want to find a (not necessarily cuspidal) automorphic representations Π of $\mathrm{GL}_N(\mathbb{A}_F)$ for a suitable N such that the map $\pi \mapsto \Pi$ is compatible with the embedding of $^L G$ via the unramified local Langlands at almost all places. Such a map is obtained by [**6, 101**] in a very precise form when G is quasi-split, conditionally on some works in preparation. There is a simpler argument to prove a weaker functoriality that is enough for the application to Conjecture 2.1. Either way, the result for G requires Π to be regular and algebraic (up to a character twist) in order to apply Theorem 2.5. This is the case if π is regular in $\mathcal{A}_{\mathrm{ac}}(G, F^+)$, and if π satisfies a mild extra regularity in the even orthogonal case. See [**127**] for further details regarding this paragraph.

As in [**89, 88**], the LKR method can be applied to obtain Conjecture 2.1 (1) when G is GSp_{2n} or GSO_{2n} (general symplectic or special orthogonal group) over a totally real field under technical hypotheses. The argument also makes use of the known case of Conjecture 2.1 (1) for Sp_{2n} and SO_{2n} in the last paragraph.

2.3. Perfectoid Shimura varieties. As a preparation for Theorem 2.3 (2), we need some fundamental results on perfectoid Shimura varieties. We assume familiarity with basics of adic spaces and perfectoid spaces; see [**14, 123**] for expositions and more references therein. From here until the end of § 2 we are in the setting $p = \ell$. This makes our notation close to that of [**116**].

Let $\mathcal{X}$ be an adic space and $(\mathcal{X}_i)$ be a filtered projective system of adic spaces with qcqs transition maps. We write

$$(2.11) \qquad\qquad \mathcal{X} \sim \varprojlim_i \mathcal{X}_i$$

to mean that $\mathcal{X}$ is equipped with maps $f_i : \mathcal{X} \to \mathcal{X}_i$ compatible with the system $(\mathcal{X}_i)$ such that (i) (f_i) induce a homeomorphism $|\mathcal{X}| \simeq \varprojlim_i |\mathcal{X}_i|$ and (ii) over an affinoid cover of $\mathcal{X}$, the map on the level of Huber rings has dense image. (See [**121**, Def. 2.4.1] for the precise definition.) In this article (2.11) will be considered only when $\mathcal{X}$ is perfectoid over a perfectoid field. Then $\mathcal{X}$ is *unique* up to a unique isomorphism [**121**, Prop. 2.4.5]. (Given $(\mathcal{X}_i)$ as above, the limit always exists in the category of diamonds. The point of the definition is that the diamond is actually

representable by a perfectoid space.) When a group Δ acts on each of $\mathcal{X}$ and the system $(\mathcal{X}_i)$, we say that (2.11) is Δ-equivariant if (f_i) are Δ-equivariant.

EXAMPLE 2.8. Define $R_r^+ := \mathbb{Z}_p^{\mathrm{cyc}}[t]$ and $R_r := R_r^+[1/p]$ for $r \geq 1$ so that $\mathrm{Spa}\,(R_r, R_r^+)$ is a closed unit disc. Let $R^+ := \varinjlim_r R_r^+$ and $R := \varinjlim_r R_r$ via the "relative Frobenius" transition maps $R_r^+ \to R_{r+1}^+$ given by $f(t) \mapsto f(t^p)$. (They become relative Frobenius maps modulo p.) Then $R^+ = \mathbb{Z}_p^{\mathrm{cyc}}[T^{1/p^\infty}]$ and $R = R^+[1/p]$, where $t \in R_r^+$ is identified with $T^{1/p^r} \in R^+$. Then $\mathrm{Spa}\,(R, R^+)$ is affinoid perfectoid, known as the "perfectoid closed unit disc", and

$$\mathrm{Spa}\,(R, R^+) \sim \varprojlim_{r \geq 1} \mathrm{Spa}\,(R_r, R_r^+).$$

Let (G, X) be a Shimura datum of Hodge type. This means that (G, X) embeds in the Siegel Shimura datum associated with the symplectic similitude group GSp_{2g} for some $g \in \mathbb{Z}_{\geq 1}$.[3] In this case the Shimura varieties Sh_K over $\mathbb{C}$ for neat open compact subgroups $K \subset G(\mathbb{A}^\infty)$ are equipped with closed embeddings into Siegel modular varieties of genus g at suitable levels. (We could work over the reflex field but it is enough for our purpose to stay over a bigger field like $\mathbb{C}$ or $\mathbb{C}_p$.) The Satake–Baily–Borel (a.k.a. minimal) compactification [9] of Sh_K is denoted by Sh_K^*. The latter is a projective normal variety over $\mathbb{C}$ which is typically singular. Write $\mathcal{S}_K^*$ for the adic space over $\mathbb{C}_p$ associated with the base change of Sh_K^* to $\mathbb{C}_p$ via a fixed isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$ and the embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$. Denote by $\mathcal{S}_{K^p}^\circ$ the good reduction locus in $\mathcal{S}_K^*$. (This is the adic space associated with the p-adic completion of Sh_K, which is an open subspace of the adic space associated with Sh_K, in analogy with an open unit disc sitting inside an affine line.)

The datum (G, X) determines the so-called Hodge cocharacter $\mu : \mathbb{G}_m \to G_\mathbb{C}$, canonical up to $G(\mathbb{C})$-conjugacy. The cocharacter μ gives rise to a parabolic subgroup

$$P_\mu := \{g \in G : \lim_{t \to 0} \mathrm{Ad}(\mu(t))g \text{ exists}\} \subset G_\mathbb{C},$$

well-defined up to conjugacy, and to a flag variety G/P_μ [35, §2.1]. (The parabolic subgroup P_μ is opposite to the parabolic subgroup used to construct the flag variety that receives the Borel embedding of the complex Shimura variety.) This flag variety is a priori defined over $\mathbb{C}$, but in fact it has a model over the reflex field of the Shimura datum. Write $\mathscr{F}\ell_G$ for the adic space over $\mathbb{C}_p$ associated with this flag variety. The dependence of $\mathscr{F}\ell_G$ on μ is clear but we omit μ from the notation for simplicity. When (G, X) is a Siegel Shimura datum, $\mathscr{F}\ell_G$ parametrizes maximal isotropic subspaces of the underlying symplectic space.

THEOREM 2.9. *Let (G, X) be a Siegel Shimura datum. Then there exists a perfectoid space $\mathcal{S}_{K^p}^*$ over $\mathbb{C}_p$ with a $G(\mathbb{Q}_p)$-action, as well as a $G(\mathbb{A}^{\infty,P})$-action as K^p varies, such that we have $G(\mathbb{Q}_p)$-equivariantly*

$$(2.12) \qquad\qquad \mathcal{S}_{K^p}^* \sim \varprojlim_{K_p \subset G(\mathbb{Q}_p)} \mathcal{S}_{K_p K^p}^*.$$

Moreover $\mathcal{S}_{K^p}^*$ is equipped with the Hodge–Tate period morphism

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p}^* \to \mathscr{F}\ell_G$$

---

[3]In fact [116] uses a variant of Hodge-type Shimura data in the context of *connected* Shimura data; see [116, §4.1]. We ignore this point in our exposition.

enjoying the following properties.

    (1) *π_{HT} is $G(\mathbb{Q}_p)$-equivariant. As K^p varies, the morphism is also $G(\mathbb{A}^{\infty,p})$-equivariant with respect to the trivial action on $\mathscr{F}\ell_G$.*

    (2) *There exists a finite cover $(\mathcal{U}_i)$ of $\mathscr{F}\ell_G$ by open affinoid subsets such that $\mathcal{V}_i := \pi_{HT}^{-1}(\mathcal{U}_i)$ are affinoid perfectoid satisfying: (i) each $\mathcal{V}_i$ is the preimage of an affinoid subset of $\mathcal{S}_{K_pK^p}$ for some open compact $K_p \subset G(\mathbb{Q}_p)$, (ii) a suitable analogue of (2.12) with $\mathcal{V}_i$ in place of $\mathcal{S}_{K^p}^*$ holds true.*

    (3) *There is an ample line bundle $\omega_{\mathscr{F}\ell}$ over $\mathscr{F}\ell_G$ whose pullback $\omega_{K^p}^* := \pi_{HT}^* \omega_{\mathscr{F}\ell}$ is $G(\mathbb{Q}_p)$-equivariantly isomorphic to an automorphic ample line bundle.*

REMARK 2.10. Assume now that (G, X) is a general Shimura datum of Hodge type. The analogous statements are true except that (2.12) has to be slightly modified by introducing certain *ad hoc* minimal compactifications, which are good enough for applications. The relation (2.12) still holds if it is stated for Shimura varieties without compactification, and it holds for the usual minimal compactifications if $\sim$ in (2.12) is weakened to an isomorphism of the associated diamonds. For more details and references, see the discussion between Theorem 3.1 and Example 3.2 in [**28**].

REMARK 2.11. It is fruitful to study π_{HT} with respect to the Newton stratification of $\mathscr{F}\ell_G$ to gather further information. Informally speaking, the geometric fibers of π_{HT} are perfectoid Igusa varieties, and are constant over each individual Newton stratum. This fact is shown and exploited to great effect in [**35, 34**]. We discuss this more in § 4.1.

SKETCH OF PROOF. The rest of § 2.3 is devoted to sketching the proof of Theorem 2.9 in the *Siegel case*. So $G = GSp_{2g}$ until the end of § 2.3. The case of Hodge type can be deduced from this case with further work; refer to [**116**, §4] and [**35**, §2]. See [**125**] for an analogous result in the case of abelian type.

Recall the convention for the groups Sp_{2g} and GSp_{2g} over $\mathbb{Z}$ from § 1.1. Write Id_g for the $g \times g$ identity matrix. For $r \in \mathbb{Z}_{\geq 0}$, we introduce the three congruence subgroups of $Sp_{2g}(\mathbb{Z}_p)$:

$$\Gamma(p^r) \subset \Gamma_1(p^r) \subset \Gamma_0(p^r),$$

given by the conditions that $\gamma \bmod p^m$ is congruent to the forms[4]

$$\begin{pmatrix} Id_g & 0 \\ 0 & Id_g \end{pmatrix}, \quad \begin{pmatrix} Id_g & * \\ 0 & Id_g \end{pmatrix}, \quad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad \text{respectively.}$$

Step 0. Hasse invariants. It is helpful to recall some key facts about the Hasse invariant, denoted by Ha, in the setting of modular forms and modular curves, following [**75**, §2.1].

    (H1) Ha is a Katz modular form of level 1 and weight $p-1$ over $\mathbb{F}_p$. In particular it is a section of an ample line bundle.

    (H2) Ha lifts to a classical modular form $\widetilde{Ha}$ if $p \geq 5$. (The lift can be taken to be the Eisenstein series E_{p-1} of level 1 and weight $p - 1$.)

    (H3) Ha is invertible on the ordinary locus and zero on the supersingular locus.

    (H4) The q-expansion of Ha is 1.

---

[4]The definition of $\Gamma_0(p^r)$ in [**116**] also requires that $\det \gamma \equiv 1 \bmod p^m$, but we ignore this point.

In general, given an abelian scheme $f : A \to S$ of dimension g such that $p = 0$ in S, consider the line bundle $\omega_{A/S} := \wedge^g(\pi_*\Omega_{A/S})$ over S. Put $A^{(p)} := A \times_{S,\mathrm{Frob}_S} S$, where $\mathrm{Frob}_S : S \to S$ is the absolute Frobenius. Then the Verschiebung morphism $A^{(p)} \to A$ induces a map $\omega_{A/S} \to \omega_{A^{(p)}/S} \simeq \omega_{A/S}^{\otimes p}$, hence a section $\mathrm{Ha}(A/S) \in \Gamma(S, \omega_{A/S}^{\otimes(p-1)})$. The Hasse invariant Ha over $\mathrm{Sh}_{K_pK^p,\mathbb{F}_p}$ is obtained by applying this construction to the universal abelian scheme; when $g = 1$, we recover the classical Hasse invariant above. Property (H3) generalizes to the fact that Ha is invertible exactly on the ordinary locus. (If $g > 1$, there are several intermediate strata between the ordinary and supersingular loci.)

Even though Ha is defined only in characteristic p, it can be used to study the p-adic formal scheme $\mathfrak{S}_{K_pK^p}$ and its adic generic fiber $\mathcal{S}^\circ_{K_pK^p}$. For this, one lifts Ha to $\widetilde{\mathrm{Ha}}$ over $\mathfrak{S}_{K_pK^p}$, which is always possible locally on the base. This weaker version of (H2) is enough to define $\mathfrak{S}_{K_pK^p}(\epsilon)$ and $\mathcal{S}_{K_pK^p}(\epsilon)$ for $0 \le \epsilon < 1/2$, which are the loci where $\widetilde{\mathrm{Ha}}$ divides p^ϵ. The locus $\mathcal{S}_{K_pK^p}(\epsilon)$ of $\mathcal{S}^\circ_{K_pK^p}$ should be thought of as the ϵ-neighborhood of the ordinary locus $\mathcal{S}_{K_pK^p}(0)$. Loosely speaking, the points of $\mathcal{S}_{K_pK^p}(\epsilon)$ correspond to abelian varieties that are "not too supersingluar" in the sense quantified by ϵ.

The above construction extends to minimal compactifications. It is classical that Ha is defined over $\mathrm{Sh}^*_{K_pK^p,\mathbb{F}_p}$ if $g = 1$. For $g \ge 2$, Ha is extended to $\mathrm{Sh}^*_{K_pK^p,\mathbb{F}_p}$ by Hartog's extension principle (cf. [116, Lem. 3.2.10]; see Remark 3.2.11 therein for an analogy between $\mathbb{C}$-analytic and p-adic settings), using the normality of $\mathrm{Sh}^*_{K_pK^p,\mathbb{F}_p}$ and the fact that the boundary has codimension g. By means of lifts $\widetilde{\mathrm{Ha}}$ locally over $\mathfrak{S}^*_{K_pK^p}$, we can define[5]

$$\mathfrak{S}^*_{K_pK^p}(\epsilon) \to \mathfrak{S}^*_{K_pK^p} \text{ over } \mathcal{O}_{\mathbb{C}_p} \quad \text{and} \quad \mathcal{S}^*_{K_pK^p}(\epsilon) \to \mathcal{S}^*_{K_pK^p} \text{ over } \mathbb{C}_p.$$

The former is an admissible blow-up in Raynaud's sense, and the latter is an open immersion.

Step 1. Anti-canonical tower at $\Gamma_0(p^\infty)$-level. We want to show that the projective limit $\varprojlim_{K_p} \mathcal{S}^*_{K_pK^p}$ is perfectoid. In light of Example 2.8, we would hope that the transition maps induce relative Frobenius modulo p. While this is overly optimistic, the situation is better if we restrict ourselves to the *anti-canonical* tower over $\mathcal{S}^*_{K_pK^p}(\epsilon)$ as K_p goes through Γ_0-level structures. In such a restricted setup, (anti-)canonical subgroups are available to help us show that the mod p picture of such a tower is given by relative Frobenius maps.

Let $0 < \epsilon < 1/2$ and $r \in \mathbb{Z}_{\ge 1}$. Let A be an abelian scheme of dimension g over a p-adically complete $\mathcal{O}_{\mathbb{C}_p}$-algebra R. If $A \times_R R/pR$ is not too supersingular in a way depending on r (measured by the Hasse invariant), then A possesses a unique subgroup $C_r \subset A[p^r]$ of order p^{rg} which becomes the kernel of of the r-th power Frobenius on A modulo $p^{1-\epsilon}$. (See [116, Cor. III.2.6].) Such a C_r is called the weak canonical subgroup of level p^r. If A contains C_1, i.e., a canonical subgroup of level p, a level p^r *anti-canonical subgroup* of A is defined to be a totally isotropic subgroup $D \subset A[p^r]$ of order p^{rg} such that $D[p] \cap C_1 = \{0\}$.

---

[5]Actually they are defined over $\mathbb{Z}_p^{\mathrm{cyc}}$ and $\mathbb{Q}_p^{\mathrm{cyc}}$ respectively, but we do not keep track of optimal base rings in these notes.

Let $\mathcal{S}_{\Gamma_0(p^r)K^p}(\epsilon)_{\mathrm{anti}}$ denote the open subspace of $\mathcal{S}_{\Gamma_0(p^r)K^p}(\epsilon)$ on which the universal subgroup of $\mathcal{A}[p^r]$ coming from the $\Gamma_0(p^r)$-level structure is the anti-canonical subgroup. The following commutative diagram is a key:

$$
\begin{array}{ccc}
\mathcal{S}_{\Gamma_0(p^{r+1})K^p}(\epsilon)_{\mathrm{anti}} \xrightarrow{\;\sim\;} \mathcal{S}_{K^p}(p^{-r-1}\epsilon) & \qquad & (A, D_{r+1}) \longmapsto A/D_{r+1} \\
\downarrow \qquad\qquad\qquad \downarrow & & \downarrow \qquad\qquad\qquad \downarrow \\
\mathcal{S}_{\Gamma_0(p^r)K^p}(\epsilon)_{\mathrm{anti}} \xrightarrow{\;\sim\;} \mathcal{S}_{K^p}(p^{-r}\epsilon) & & (A, D_r) = (A, D_{r+1}[p^r]) \longmapsto A/D_r,
\end{array}
$$

where the diagram on the right gives the moduli-theoretic description of the maps in terms of principally polarized abelian varieties, together with anti-canonical subgroups in the left column. The two vertical maps are natural projections. The horizontal maps are isomorphisms as the inverse maps can be given as the quotient maps by suitable canonical subgroups, cf. [**116**, Thm. 3.2.15.(ii)]. The projective system in the left column as $r \to \infty$ is the anti-canonical tower.

The use of anti-canonical subgroups is significant in two ways. First, ϵ does not shrink as $r \to \infty$ in the anti-canonical tower. (In a similar construction via canonical subgroups, ϵ tends to 0.) Second, the right vertical map $A/D_{r+1} \mapsto A/D_r$ is a lift of relative Frobenius. Analogously with Example 2.8, this essentially tells us that a perfectoid limit $\mathcal{S}_{\Gamma_0(p^\infty)K^p}(\epsilon)_{\mathrm{anti}}$ of the anti-canonical tower exists, satisfying $\mathcal{S}_{\Gamma_0(p^\infty)K^p}(\epsilon)_{\mathrm{anti}} \sim \varprojlim_r \mathcal{S}_{\Gamma_0(p^r)K^p}(\epsilon)_{\mathrm{anti}}$.

The anti-canonical tower extends to the tower of $\mathcal{S}^*_{\Gamma_0(p^r)K^p}(\epsilon)_{\mathrm{anti}}$ over the compactifications by Hartog's extension principle (cf. [**116**, Lem. 3.2.10]). Essentially the same argument as above shows the existence of a perfectoid limit

$$
\mathcal{S}^*_{\Gamma_0(p^\infty)K^p}(\epsilon)_{\mathrm{anti}} \sim \varprojlim_r \mathcal{S}^*_{\Gamma_0(p^r)K^p}(\epsilon)_{\mathrm{anti}}.
$$

Moreover it is affinoid perfectoid. The intuitive reason is that each $\mathcal{S}^*_{\Gamma_0(p^r)K^p}(\epsilon)_{\mathrm{anti}}$ is isomorphic to $\mathcal{S}^*_{K^p}(p^{-r}\epsilon)$, and the latter is affinoid as it is the $p^{-r}\epsilon$-neighborhood given in terms of a section of an ample line bundle $\omega^{\otimes p^r(p-1)}$, namely a lift of the p^r-th power of Ha, cf. (H1) above. (Such a lift exists globally for $r \gg 1$ by ampleness of ω.) A precise argument is made from computing the tilts in characteristic p [**116**, Cor. 3.2.20].

Step 2. Going up to $\Gamma(p^\infty)$-level. We have shown that $\varprojlim_r \mathcal{S}^*_{\Gamma_0(p^r)K^p}(\epsilon)_{\mathrm{anti}}$ is perfectoid. Since $\Gamma_0(p^r)$ do not form a neighborhood basis at 1 in $G(\mathbb{Q}_p)$ but $\Gamma(p^r)$ do, we would like to go from the $\Gamma_0(p^\infty)$-level all the way up to the $\Gamma(p^\infty)$-level. This is done in two steps via $\Gamma_1(p^\infty)$.

Going from $\Gamma_1(p^\infty)$ to $\Gamma(p^\infty)$ is relatively easy. Since

$$
\mathcal{S}^*_{\Gamma(p^r)K^p}(\epsilon)_{\mathrm{anti}} \to \mathcal{S}^*_{\Gamma_1(p^r)K^p}(\epsilon)_{\mathrm{anti}}
$$

are finite étale for all $r \geq 1$, one can appeal to almost purity, which means that an adic space that is finite étale over an affinoid perfectoid space is also affinoid perfectoid. The hard part is to pass from $\Gamma_0(p^\infty)$ to $\Gamma_1(p^\infty)$. Since

$$
\mathcal{S}_{\Gamma_1(p^r)K^p}(\epsilon)_{\mathrm{anti}} \to \mathcal{S}_{\Gamma_0(p^r)K^p}(\epsilon)_{\mathrm{anti}}
$$

is finite étale, there is no problem with obtaining a perfectoid limit over the interior. The main difficulty is how to extend to the boundary since the map between compactifications is ramified at the boundary. The idea is to tilt the interior to characteristic p, extend it to an affinoid perfectoid space including the boundary,

and prove that this extension is the correct tilt. The proof involves Hartog's extension principle in characteristic 0 and a p-adic version of Riemann's Hebbarkeitssatz ("bounded functions have removable singularities") in characteristic p.

The output is an affinoid perfectoid space

$$\mathcal{S}^*_{K^p}(\epsilon)_{\mathrm{anti}} \sim \varprojlim_r \mathcal{S}^*_{\Gamma(p^r)K^p}(\epsilon)_{\mathrm{anti}}.$$

Step 3. Topological Hodge–Tate morphism. Motivated by the definition of (2.11), we define the topological space

$$|\mathcal{S}^*_{K^p}| := \varprojlim_{K_p \subset G(\mathbb{Q}_p)} |\mathcal{S}^*_{K_p K^p}|,$$

which is naturally equipped with a continuous $G(\mathbb{Q}_p)$-action. One of our goals is to upgrade $|\mathcal{S}^*_{K^p}|$ to a perfectoid space. As a preparation and also as a step towards the Hodge–Tate period map, cf. Step 5 below, we construct a continuous $G(\mathbb{Q}_p)$-equivariant map of topological spaces

$$|\pi_{\mathrm{HT}}| : |\mathcal{S}^\circ_{K^p}| \to |\mathscr{F}\ell_G|,$$

described pointwise as follows. Each $x \in |\mathcal{S}^\circ_{K^p}|$ can be represented by a $\mathrm{Spa}\,(\kappa, \kappa^+)$-point for a complete nonarchimedean extension $\kappa \supset \mathbb{C}_p$, which corresponds to a principally polarized abelian variety A_x over κ with a symplectic isomorphism $T_p A_x \simeq \mathbb{Z}_p^{2g}$. Then x is sent to the Hodge–Tate filtration

$$0 \to \mathrm{Lie}\, A_x(1) \to T_p A_x \otimes_{\mathbb{Z}_p} C_x \simeq C_x^{2g} \to (\mathrm{Lie}\, A_x^\vee)^\vee \to 0,$$

which determines a point of $|\mathscr{F}\ell_G|$. Continuity of $|\pi_{\mathrm{HT}}|$ essentially follows from the fact that the Hodge–Tate filtration works well in families. In fact, the same construction defines $|\pi_{\mathrm{HT}}|$ on a larger domain than the good reduction locus $\mathcal{S}^\circ_{K^p}$, namely $|\mathcal{S}^*_{K^p}|$ minus the boundary, the point being that the Hodge–Tate filtration belongs in characteristic zero.

Step 4. Spreading perfectoid property. Fix $0 \le \epsilon < 1/2$. Recall from Step 2 that the open subset $|\mathcal{S}^*_{K^p}(\epsilon)_{\mathrm{anti}}|$ of $|\mathcal{S}^*_{K^p}|$ is affinoid perfectoid. In fact

$$|\mathcal{S}^*_{K^p}(\epsilon)| = G(\mathbb{Z}_p) \cdot |\mathcal{S}^*_{K^p}(\epsilon)_{\mathrm{anti}}|$$

so there is a perfectoid space $\mathcal{S}^*_{K^p}(\epsilon)$ whose underlying space is $|\mathcal{S}^*_{K^p}(\epsilon)|$. It is covered by finitely many translates of $\mathcal{S}^*_{K^p}(\epsilon)_{\mathrm{anti}}$. (Finiteness comes from continuity of the $G(\mathbb{Z}_p)$-action and the fact that an open subgroup of $G(\mathbb{Z}_p)$ has finite index.) Denote by $\mathcal{S}_{K^p}(\epsilon)$ the open subspace of $\mathcal{S}^*_{K^p}(\epsilon)$ away from the boundary.

To promote $|\mathcal{S}^*_{K^p}|$ to the desired perfectoid space in Theorem 2.9, it is thus enough to show that finitely many translates of $\mathcal{S}^*_{K^p}(\epsilon)$ under the $G(\mathbb{Q}_p)$-action cover $|\mathcal{S}^*_{K^p}|$. The starting point is the observation that the ordinary locus $\mathcal{S}_{K^p}(0)$ maps to $\mathscr{F}\ell_G(\mathbb{Q}_p)$ under $|\pi_{\mathrm{HT}}|$. Now it is important to take $\epsilon > 0$ to work with a strict neighborhood of the ordinary locus. Then one can show the existence of an open subset $U \subset \mathscr{F}\ell_G$ containing $\mathscr{F}\ell_G(\mathbb{Q}_p)$ such that

$$|\pi_{\mathrm{HT}}|^{-1}(U) \subset |\mathcal{S}^*_{K^p}(\epsilon)|$$

(and $|\pi_{\mathrm{HT}}|^{-1}(U)$ does not meet the boundary). On the other hand, $G(\mathbb{Q}_p)U = \mathscr{F}\ell_G$ as this is true for every open subset containing $\mathscr{F}\ell_G(\mathbb{Q}_p)$; the proof uses an explicit open affinoid covering $\mathscr{F}\ell_G = \cup_J \mathscr{F}\ell_{G,J}$ coming from the Plücker embedding, indexed by cardinality g subsets $J \subset \{1, 2, ..., 2g\}$. Since $\mathscr{F}\ell_G$ is quasi-compact, a

finite subset $\{g_i\}_{i \in I} \subset G(\mathbb{Q}_p)$ can be chosen such that

$$\mathscr{F}\ell_G = \cup_{i \in I} g_i U.$$

The two displayed facts together with the $G(\mathbb{Q}_p)$-equivariance of $|\pi_{\mathrm{HT}}|$ imply that $|\mathcal{S}^*_{K^p}| = \cup_{i \in I} g_i |\pi_{\mathrm{HT}}|^{-1}(U)$. (The boundary is taken care of by a topological argument.) Hence

$$|\mathcal{S}^*_{K^p}| = \cup_{i \in I} g_i |\mathcal{S}^*_{K^p}(\epsilon)|.$$

Step 5. Finishing the construction of π_{HT}. Now that we have a perfectoid space $\mathcal{S}_{K^p}$, the topological map $|\pi_{\mathrm{HT}}|$ can be upgraded to a map of adic spaces

$$\pi_{\mathrm{HT}} : \mathcal{S}^*_{K^p} \to \mathscr{F}\ell_G$$

by largely repeating the same idea for constructing $|\pi_{\mathrm{HT}}|$ and extending the map to the boundary by a p-adic Hebbarkeitssatz.

Step 6. Further properties of π_{HT}. We briefly comment on verifying properties (2) and (3) of π_{HT} in Theorem 2.9. For (2), it turns out that the covering $(\mathscr{F}\ell_{G,J})$ of $\mathscr{F}\ell_G$ from Step 4 works. The argument again utilizes the outcome of Step 2 and the $G(\mathbb{Q}_p)$-equivariance of π_{HT}. For (3), if $W_{\mathscr{F}\ell_G} \subset \mathcal{O}^{2g}_{\mathscr{F}\ell_G}$ denotes the universal totally isotropic subspace, then it follows from the construction of π_{HT} that there is a $G(\mathbb{Q}_p)$-equivariant isomorphism of locally free rank g modules

$$\mathrm{Lie}\,\mathcal{A}_{K^p} \simeq \pi^*_{\mathrm{HT}} W_{\mathscr{F}\ell_G}$$

away from the boundary. Property (3) is obtained by taking the dual of the top exterior power and extending to the boundary. □

2.4. Construction of torsion Galois representations.

THEOREM 2.12. *Theorem 2.3 (2) holds true. We also have a suitable analogue for mod p^m coefficients for $m \geq 1$, in terms of pseudo-determinants.*

A more precise statement is as follows. Denote by $\mathbb{T}^S_{\mathrm{GL}_n}$ the abstract Hecke algebra for GL_n over F away from S as in § 2.1.2. Then $H^*(Y_{\mathrm{GL}_n,K'}, \mathbb{Z}/p^m\mathbb{Z})$ is naturally a $\mathbb{T}^S_{\mathrm{GL}_n}$-module. Write

$$\mathbb{T}^S_{\mathrm{GL}_n}(K', m) := \mathrm{image}(\mathbb{T}^S_{\mathrm{GL}_n} \to \mathrm{End}_{\mathbb{Z}/p^m\mathbb{Z}}(H^*(Y_{\mathrm{GL}_n,K'}, \mathbb{Z}/p^m\mathbb{Z}))).$$

Then the theorem asserts that there exists an n-dimensional pseudo-determinant D of Γ_F which is unramified away from S and valued in $\mathbb{T}^S_{\mathrm{GL}_n}(K', m)/J$ for a nilpotent ideal $J \subset \mathbb{T}^S_{\mathrm{GL}_n}(K', m)$ of bounded nilpotence exponent, such that the characteristic polynomial of D at each Frob_v, $v \notin S$, is given by $H_v(x)$. (See [**37**] for the foundation of pseudo-determinants.) A technical remark is that it is advantageous to deal with Hecke algebras and Hecke modules mod p^m rather than Hecke eigensystems mod p^m.

The existence of such a D subsumes Theorem 2.3 (2). Indeed, $\mathfrak{m} \in \mathcal{HE}^S_{\mathrm{coh}}(n, F)_{\overline{\mathbb{F}}_\ell}$ corresponds to a morphism $f : \mathbb{T}^S_{\mathrm{GL}_n}(K', 1)/J \to \overline{\mathbb{F}}_p$, so $f \circ D$ is an n-dimensional pseudo-determinant of Γ_F valued in $\overline{\mathbb{F}}_p$ with the correct Frobenius characteristic polynomials away from S. Since a pseudo-determinant valued in an algebraically closed field always comes from an actual n-dimensional representation, we obtain the desired $\overline{\rho}_{\mathfrak{m}}$ as in Theorem 2.3 (2).

SKETCH OF PROOF. We concentrate on the case where F is a CM field and employ unitary Shimura varieties. If F is totally real, then the argument below can

be adapted to the setting of Siegel modular varieties for the group $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}_{2n}$, cf. [**116**, §5.1].

Recall that F^+ denotes the maximal totally real subfield of F. Let U_{2n} denote the quasi-split unitary group in $2n$ variables relative to the quadratic extension F/F^+. For $G := \mathrm{Res}_{F^+/\mathbb{Q}}U_{2n}$, there is a standard choice for X to make (G, X) a Shimura datum. One has the tower of connected Shimura varieties Sh_K for G over $\mathbb{Q}(\mu_\infty)$ indexed by neat open compact subgroups $K = \prod_{v \neq \infty} K_v \subset G(\mathbb{A}^\infty)$. (This differs from the convention in the proof part of § 2.3, where G was GSp_{2g} and Sh_K denoted the Siegel modular varieties.) Let S be a finite set of places as in § 2.1.2, which in particular contains $\{p, \infty\}$, and keep K_v hyperspecial at $v \notin S$. Write $\mathcal{I}$ for the ideal sheaf of the boundary of $\mathcal{S}^*_{K_p K^p}$, and $\mathcal{O}^+$ for the +-part of the structure sheaf of $\mathcal{S}^*_{K_p K^p}$. Set $\mathcal{I}^+ := \mathcal{I} \cap \mathcal{O}^+$ over $\mathcal{S}^*_{K_p K^p}$. We still denote by $\mathcal{I}^+$ the pullback of $\mathcal{I}^+$ to $\mathcal{S}^*_{K^p}$. This sheaf may be thought of as the sheaf of cuspforms. Define

$$\tilde{H}^i(\mathrm{Sh}_{K^p}, \mathbb{Z}/p^m\mathbb{Z}) := \varinjlim_{K_p} H^i(\mathrm{Sh}_{K_p K^p}, \mathbb{Z}/p^m\mathbb{Z}), \quad i \geq 0.$$

The tilde indicates that it is the mod p^m version of the *completed* cohomology.

(0) Overview. The basic strategy is to reduce to the case of § 2.2 where Galois representations are already constructed, via p-adic congruences. The proof progresses by studying various cohomology spaces in order:

$$H^*(Y_{\mathrm{GL}_n, K'}, \mathbb{Z}/p^m\mathbb{Z}) \overset{(1)}{\rightsquigarrow} \tilde{H}^*(\mathrm{Sh}_{K^p}, \mathbb{Z}/p^m\mathbb{Z}) \overset{(2)}{\rightsquigarrow} H^*(\mathcal{S}^*_{K^p}, \mathcal{I}^+/p^m)$$

$$\overset{(3)}{\rightsquigarrow} H^0(\mathcal{V}, \mathcal{I}^+/p^m) \overset{(4)}{\rightsquigarrow} H^*(\mathcal{S}^*_{K_p K^p}, \omega^{\otimes mk}_{K_p K^p} \otimes \mathcal{I}),$$

where $\mathcal{V}$ is an open affinoid perfectoid subspace of $\mathcal{S}^*_{K^p}$ (see (3) below), and $\omega_{K_p K^p}$ is the automorphic ample line bundle at a finite level $K_p \subset G(\mathbb{Q}_p)$ whose pullback to the infinite level is as in property (3) of Theorem 2.9. The five cohomology groups are computed on their respective topological spaces (the first two of which are real; the next three are p-adic). It is important to compute the second and third groups also in the étale topology to make the transitions (2) and (3).

Step (1) is a reduction of the problem about locally symmetric spaces for GL_n over F (which are not Shimura varieties if $n > 1$) to one about Shimura varieties associated with $G = \mathrm{Res}_{F^+/\mathbb{Q}}U_{2n}$. The next steps (2)–(4) establish p-adic congruences, namely that the mod p^m (and p-adically completed) cohomology of the Shimura varieties can be p-adically approximated by classical cuspforms.

In summary, if we keep a careful track of Hecke algebras along the way, then the above steps reduce the construction of Galois representations (or rather pseudo-determinants) for $H^*(Y_{\mathrm{GL}_n, K'}, \mathbb{Z}/p^m\mathbb{Z})$ to that for $H^0(\mathcal{S}^*_{K_p K^p}, \omega^{\otimes mk}_{K_p K^p} \otimes \mathcal{I})$. The latter consists of classical cuspidal automorphic forms on U_{2n} over F^+, so we do have associated Galois representations from the conjugate self-dual case treated in § 2.2. This completes the proof. Now we give more details on each step below.

(1) Borel–Serre compactification. The Borel–Serre compactification $\mathrm{Sh}^{\mathrm{BS}}_K$ is a compact real manifold with corners such that $\mathrm{Sh}_K \hookrightarrow \mathrm{Sh}^{\mathrm{BS}}_K$ is a homotopy equivalence. (See [**116**, §5.2], cf. the original work [**17**] and a survey [**62**, §4].) Write $\partial\,\mathrm{Sh}^{\mathrm{BS}}_K$ for the complement of Sh_K. There is a long exact sequence

$$\cdots \to H^i_c(\mathrm{Sh}_K, \mathbb{Z}/p^m\mathbb{Z}) \to H^i(\mathrm{Sh}^{\mathrm{BS}}_K, \mathbb{Z}/p^m\mathbb{Z}) \to H^i(\partial\,\mathrm{Sh}^{\mathrm{BS}}_K, \mathbb{Z}/p^m\mathbb{Z}) \to \cdots,$$

where the middle term is identified with $H^i(\mathrm{Sh}_K, \mathbb{Z}/p^m\mathbb{Z})$.

The boundary strata in $\partial \mathrm{Sh}_K^{\mathrm{BS}}$ are indexed by conjugacy classes of $\mathbb{Q}$-rational proper parabolic subgroups of U_{2n}. From the fundamental fact that $\mathrm{Res}_{F/F^+} \mathrm{GL}_n$ is a Levi subgroup of such a parabolic subgroup, it follows that the "parabolic induction" (for modules over Hecke algebras, as we are working at fixed levels) of $H^*(Y_{\mathrm{GL}_n, K'}, \mathbb{Z}/p^m\mathbb{Z})$ contributes to $H^*(\partial \mathrm{Sh}_K^{\mathrm{BS}}, \mathbb{Z}/p^m\mathbb{Z})$. According to the long exact sequence, such a contribution to each $H^i(\partial \mathrm{Sh}_K^{\mathrm{BS}}, \mathbb{Z}/p^m\mathbb{Z})$ is captured by $H^i(\mathrm{Sh}_K, \mathbb{Z}/p^m\mathbb{Z})$ and $H_c^{i+1}(\mathrm{Sh}_K, \mathbb{Z}/p^m)$. So we are essentially reduced to attaching mod p^m Galois representations to $H^*(\mathrm{Sh}_K, \mathbb{Z}/p^m\mathbb{Z})$ and $H_c^*(\mathrm{Sh}_K, \mathbb{Z}/p^m\mathbb{Z})$.[6] Thanks to Poincaré duality, it is enough to deal with $H^*(\mathrm{Sh}_K, \mathbb{Z}/p^m\mathbb{Z})$. By the Hochschild–Serre spectral sequence with respect to level subgroups at p, it is enough to consider the limit over K_p, namely $\tilde{H}^i(\mathrm{Sh}_{K^p}, \mathbb{Z}/p^m\mathbb{Z})$.

(2) A comparison isomorphism. Write $j : \mathrm{Sh}_{K_p K^p} \hookrightarrow \mathrm{Sh}_{K_p K^p}^*$ for the open immersion. The main point of (2) is the following isomorphisms which are equivariant for the Hecke action away from p, where the tensor products are over $\mathbb{Z}/p^m\mathbb{Z}$:

$$\tilde{H}^*(\mathrm{Sh}_{K^p}, \mathbb{Z}/p^m\mathbb{Z}) \otimes \mathcal{O}_{\mathbb{C}_p}/p^m \;\simeq\; \tilde{H}^*(\mathrm{Sh}_{K^p}^*, j_! \mathbb{Z}/p^m\mathbb{Z}) \otimes \mathcal{O}_{\mathbb{C}_p}/p^m$$

$$\stackrel{a}{\simeq}\; H_{\text{ét}}^*(\mathcal{S}_{K^p}^*, j_! \mathcal{O}^+/p^m) \;\simeq\; H_{\text{ét}}^*(\mathcal{S}_{K^p}^*, \mathcal{I}^+/p^m).$$

The first isomorphism is formal. The second map is a comparison [**114**, Thm. 3.13] for étale cohomology between a proper (not necessarily smooth) scheme over $\mathbb{C}_p$ and its adification with constructible sheaf coefficients. (The theorem for $\mathbb{F}_p$-sheaves therein is easily extended to the case of $\mathbb{Z}/p^m\mathbb{Z}$-sheaves.) Actually it is only an *almost* isomorphism, denoted by the symbol $\stackrel{a}{\simeq}$, but we ignore the technical point as it does not affect our goal.

(3) Hecke-stable affinoid perfectoid Čech covering. Now it truly matters to pass to infinite-level at p. We employ the affinoid perfectoid covering $(\mathcal{V}_i)$ of $\mathcal{S}_{K^p}^*$ from Theorem 2.9 to compute $H_{\text{ét}}^*(\mathcal{S}_{K^p}^*, \mathcal{I}^+/p^m)$ as the Čech cohomology of $\mathcal{S}_{K^p}^*$ as a topological space. Here we appeal to cohomological vanishing of affinoid perfectoid spaces as in [**113**, Prop. 6.14, 7.13], and also the fact that the boundary of the perfectoid space $\mathcal{S}_{K^p}^*$ is strongly Zariski closed in each $\mathcal{V}_i$.

The outcome is that $H_{\text{ét}}^*(\mathcal{S}_{K^p}^*, \mathcal{I}^+/p^m)$ can be computed by a Čech complex whose terms consist of

$$H^0(\mathcal{V}, \mathcal{I}^+/p^m),$$

where $\mathcal{V}$ stands for finite intersections made from the covering $(\mathcal{V}_i)$. Since each $\mathcal{V}_i$ is stable under the Hecke action away from S, so is $\mathcal{V}$. Thus the Hecke action away from S is encoded by each $H^0(\mathcal{V}, \mathcal{I}^+/p^m)$.

(4) Fake Hasse invariants. The point of this step is to extend the sections of $\mathcal{I}^+/p^m$ to the whole Shimura variety and also lift to characteristic 0, at the expense of multiplying the coefficient sheaf by a power of an automorphic line bundle. This is achieved by introducing fake Hasse invariants, mimicking the classical Hasse invariants. The name "fake" suggests that, unlike the classical Hasse invariants, they do not arise as sections of vector bundles over typical integral models of Shimura varieties at finite level.

---

[6]More precisely, we construct a pseudo-determinant valued in the Hecke algebra corresponding to each cohomology group. A priori we get a $2n$-dimensional pseudo-determinant as Sh_K are associated with U_{2n}. It takes an extra twisting argument to obtain the desired n-dimensional pseudo-determinant from this. See [**116**, §5.3], cf. [**69**, §7].

In the case of modular forms, when a mod p^m modular form is multiplied by a sufficiently divisible power of the Hasse invariant (or its lift to characteristic 0), the poles outside the ordinary locus are removed while the q-expansion mod p^m is unchanged. (This argument appears in the proof of [**75**, Thm. 4.5.1], for instance.) Moreover the resulting modular form lifts to characteristic 0 by a cohomological vanishing theorem for ample line bundles.

Fake Hasse invariants are designed to play similar roles. They are sections of ω_{K^p} pulled back via π_{HT} from explicit sections of $\omega_{\mathscr{Fl}}$, via Theorem 2.9 (3). One can choose "exotic" formal integral models of $\mathcal{S}^*_{K^p}$ and ω_{K^p} at finite level $K_p \subset G(\mathbb{Q}_p)$ where these sections mod p^m are defined. (See footnote 19 of [**116**, p.1030] for "exotic".) The key properties of fake Hasse invariants are as follows, cf. (H1)–(H4) in Step 0 of the proof of Theorem 2.9, besides abundance of fake Hasse invariants thanks to Theorem 2.9 (2) (i.e., the Hodge–Tate period map is "affinoid").

(i) They are sections of ample line bundles.
(ii) Multiplying them commutes with the prime-to-p Hecke action.
(iii) The formal integral model of $\mathcal{V}$ at level K_p is the invertible locus of a suitable fake Hasse invariant.

With these inputs, one can descend to finite levels K_p at p and mimic the classical arguments, by multiplying a sufficiently divisible power of fake Hasse invariants, to extend sections from $\mathcal{V}_{K_p}$ to $\mathcal{S}^*_{K_p K^p}$ and lift the sections from mod p^m to characteristic 0 coefficients, compatibly with the Hecke actions away from p. This completes Step (4). □

Complements and further references. Regarding §§ 2.3–2.4, we recommend the survey articles [**115, 118, 117, 102, 142**] as well as Caraiani's chapter in [**14**].

Varma [**137**] has strengthened Theorem 2.3 (1) by showing the local-global compatibility with characteristic 0 coefficients (up to semisimplification) at all primes away from ℓ, not just at unramified places. See [**2**, §§3–5] for some local-global compatibility results at all primes including those above ℓ (the "$\ell = p$ case", which is most subtle), where Galois representations are valued in suitable Hecke algebras; this requires understanding of the torsion setting.

Scholze's method was adapted by Pilloni–Stroh [**107**, §3] to prove the analogue of Theorem 2.3 (1) for automorphic representations contributing to the *coherent cohomology* of certain Shimura varieties of Hodge type via analogues of Hasse invariants. Since such representations can be *non-regular* (e.g., the archimedean components can be non-degenerate limits of discrete series), their theorem applies to new cases that are not covered by Theorem 2.3 (1). Getting around perfectoid Shimura varieties, Boxer and Goldring–Koskivirta [**21, 61**] (independently of each other) develop different analogues of Hasse invariants and their applications; see therein for further references. The results of [**107**] are also obtained by [**61**].

3. The Calegari–Geraghty method for Betti cohomology

The goal of this section is to give a rough outline of the Calegari–Geraghty method in the Betti setting for the group GL_n and to discuss its prerequisites. The Calegari–Geraghty method is a vast extension of the Taylor–Wiles method for proving modularity lifting, first proposed in [**25**]. There are two versions of the

Calegari–Geraghty method in the literature: one that applies to the Betti cohomology of locally symmetric spaces and one that applies to the coherent cohomology of Shimura varieties. In both cases, it is essential to apply the Taylor–Wiles patching method to complexes rather than to individual cohomology groups.

The remarkable feature of the Betti version is that it applies to locally symmetric spaces that are not Shimura varieties - this is the case on which we will focus. The coherent version applies to coherent cohomology groups of Shimura varieties, particularly in non-regular weight. This version has been spectacularly applied in the work of Boxer–Calegari–Gee–Pilloni to prove the potential modularity of abelian surfaces over totally real fields [19]. Another essential ingredient for this result comes from higher Hida theory, which was introduced by Pilloni [106] and is now being further developed by Boxer and Pilloni (see [20] for the modular curve case of higher Hida theory). There are many beautiful ideas here - see, for example, [24] for a survey focused on this topic.

3.1. Prerequisites: conjectures on the cohomology of locally symmetric spaces. For this section, we let our connected reductive group G be GL_n and we work over an arbitrary number field F. We let ℓ be a prime and S be a finite set of places of F containing all the ramified places and all the places above ℓ and ∞. For a sufficiently small compact open subgroup $K \subset G(\mathbb{A}_F^\infty)$ as in § 2.1.2, we have the locally symmetric space

$$Y_K = G(F) \backslash G(\mathbb{A}_F) / K K'_\infty,$$

which is a smooth Riemannian manifold. This locally symmetric space does not arise from a Shimura variety except in very special cases.

EXAMPLE 3.1.
   (1) Let $n = 2$ and F be a totally real field. If $F = \mathbb{Q}$, the locally symmetric space Y_K arises from a modular curve defined over $\mathbb{Q}$. If $[F : \mathbb{Q}] > 1$, the locally symmetric space Y_K is not a Shimura variety itself. However, it is closely related to one, since it is essentially a torus bundle over a Hilbert modular variety. Hilbert modular varieties are Shimura varieties of abelian type.
   (2) Let $n = 2$ and F be an imaginary quadratic field. Then $G(F_\infty) = GL_2(\mathbb{C})$ and $K'_\infty = SU_2(\mathbb{R}) \cdot \mathbb{R}_{>0}$. The quotient $G(F_\infty)/K'_\infty$ can be identified with the 3-dimensional hyperbolic space $\mathbb{H}^3$. Therefore, the locally symmetric space Y_K is an arithmetic hyperbolic 3-manifold as it can be obtained from a disjoint union of finitely many quotients of $\mathbb{H}^3$ by congruence subgroups of $GL_2(\mathcal{O}_F)$. In particular, Y_K does not admit a complex structure. The manifolds Y_K are also called *Bianchi manifolds*.
   (3) More generally, if $n \geq 2$ and F is not totally real, or if $n \geq 3$ and F is an arbitrary number field, the locally symmetric spaces Y_K cannot be directly related to Shimura varieties. This follows from a classification of groups that admit a Shimura datum.

3.1.1. *Existence of Galois representations.* The Calegari–Geraghty method relies on a deep understanding of the Betti cohomology of the locally symmetric spaces Y_K, including a formulation of the Buzzard–Gee conjecture for torsion classes in the cohomology. To make this more precise, we introduce certain local systems on Y_K.

We will use the following notation throughout this subsection. Let $E/\mathbb{Q}_\ell$ be a finite extension which will be our field of coefficients, with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k. We always assume that E is large enough, for example so that it contains the image of every embedding of F into $\overline{\mathbb{Q}}_\ell$.

We let λ be a highest weight vector for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$. In other words, we take $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}_\mathbb{Q}(F,E)}$ such that $\lambda_{\tau,1} \geq \lambda_{\tau,2} \geq \cdots \geq \lambda_{\tau,n}$ for every embedding $\tau \in \mathrm{Hom}_\mathbb{Q}(F,E)$. We let σ_λ denote the irreducible algebraic representation of G over $\overline{\mathbb{Q}}_\ell$ of highest weight λ. Since E is assumed large enough, we may assume that σ_λ is defined over E. We let σ_λ° denote a $\prod_{v|\ell} \mathrm{GL}_n(\mathcal{O}_{F,v})$-stable lattice in σ_λ.

We assume that the compact open subgroup K is of the form $\prod_v K_v$, where v runs over finite places of F and $K_v \subset \mathrm{GL}_n(\mathcal{O}_{F,v})$ is a compact open subgroup. We let K act on σ_λ° via the projection to $\prod_{v|\ell} K_v$, pull back σ_λ° to a K-equivariant sheaf on $G(F)\backslash G(\mathbb{A}_F)/K'_\infty$ and the descend to a local system of $\mathcal{O}$-modules $\mathcal{V}_\lambda$ on Y_K. We denote $\mathcal{V}_\lambda[1/p]$ by V_λ.

From now on, we will be interested in understanding the cohomology groups

$$(3.1) \qquad\qquad\qquad H^i(Y_K, \mathcal{V}_\lambda),$$

which can be equipped with an action of the abstract Hecke algebra $\mathbb{T}^S$. Perhaps the most conceptual way of constructing the Hecke action in this setting is by observing that the pullback of σ_λ° to $G(F)\backslash G(\mathbb{A}_F)/K'_\infty$ is actually a $G(\mathbb{A}_F^S) \times K_S$-equivariant sheaf. The shadow of the $G(\mathbb{A}_F^S)$-action on this equivariant sheaf at infinite level induces an action of the Hecke algebra $\mathbb{T}^S$ on the finite level complex $R\Gamma(Y_K, \mathcal{V}_\lambda)$, viewed as an object in an appropriate derived category. Taking cohomology, one obtains an action of $\mathbb{T}^\mathbb{S}$ on the $H^i(Y_K, \mathcal{V}_\lambda)$. See [**103**, §2] for more details.

The cohomology groups in (3.1) are finitely generated $\mathcal{O}$-modules: this follows from the existence of the Borel–Serre compactification of Y_K, which is a compact manifold with corners that is homotopy equivalent to Y_K. However, these cohomology groups can and do contain torsion. For example, in the Bianchi case the torsion is conjectured to grow exponentially with the index of K in $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ - see [**13**] for more details. In what follows, the case of torsion classes will be the most subtle. We define

$$(3.2) \qquad \mathbb{T}^S(K,\lambda) := \mathrm{Im}\left(\mathbb{T}^S \to \mathrm{End}_\mathcal{O}\left(\bigoplus_{i=0}^{\dim_\mathbb{R} Y_K} H^i(Y_K, \mathcal{V}_\lambda)\right)\right)$$

and formulate several conjectures about $\mathbb{T}^S(K,\lambda)$ and about the cohomology groups it acts on.

To make precise statements, we introduce explicit generators for the spherical Hecke algebra at a place $v \notin S$ of F. We assume that, for such a place v, we have $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$. We let ϖ_v denote a uniformizer of F_v and q_v denote the cardinality of the residue field $k_v := \mathcal{O}_{F,v}/\varpi_v$. For each $i = 1, \ldots, n$, we let $T_{v,i}$ denote the double coset operator in the spherical Hecke algebra of $\mathrm{GL}_n(F_v)$ given by

$$T_{v,i} := \mathrm{GL}_n(\mathcal{O}_{F_v})\mathrm{diag}[\varpi_v, \ldots, \varpi_v, 1, \ldots 1]\mathrm{GL}_n(\mathcal{O}_{F_v}),$$

where ϖ_v occurs i times on the diagonal.

CONJECTURE 3.2. Let $\mathfrak{m} \subset \mathbb{T}^S(K,\lambda)$ be a maximal ideal; its residue field is a finite extension k' of the residue field k of E. Then there exists a unique continuous,

semi-simple Galois representation

$$\bar{\rho}_{\mathfrak{m}} : \Gamma_F \to \mathrm{GL}_n(k')$$

characterized by the fact that, for any $v \notin S$, $\bar{\rho}_{\mathfrak{m}} \mid_{\Gamma_{F_v}}$ is unramified and the characteristic polynomial of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of the polynomial

$$X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_{v,n}.$$

modulo $\mathfrak{m}$.

REMARK 3.3. This is an explicit version of the second part of Conjecture 2.1 in the case when $G = \mathrm{GL}_n$. This can be seen from the Hochschild–Serre spectral sequence and from the fact that one can trivialize the local system $\mathcal{V}_\lambda/\varpi$ by raising the level $\prod_{v|\ell} K_v$. The normalization is compatible with the C-algebraic version of the Buzzard–Gee conjecture.

We say that a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S(K,\lambda)$ is *non-Eisenstein* if the Galois representation $\bar{\rho}_{\mathfrak{m}}$ exists and is absolutely irreducible. (This is a stronger condition than the notion of non-Eisenstein ideal going back to the work of Mazur [99]. See [27, §3.8] for a discussion of various related notions for $n = 2$.)

CONJECTURE 3.4. Let $\mathfrak{m} \subset \mathbb{T}^S(K,\lambda)$ be a non-Eisenstein maximal ideal. Then there exists a unique continuous Galois representation

$$\rho_{\mathfrak{m}} : \Gamma_F \to \mathrm{GL}_n \left(\mathbb{T}^S(K,\lambda)\widehat{_{\mathfrak{m}}} \right)$$

characterized by the fact that, for any finite place $v \notin S$ of F, $\rho_{\mathfrak{m}} \mid_{\Gamma_{F_v}}$ is unramified and the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of the polynomial

$$X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_{v,n}.$$

in $\mathbb{T}^S(K,\lambda)\widehat{_{\mathfrak{m}}}$.

We now discuss the status of Conjectures 3.2 and 3.4. For a general number field, they are wide open. From now on, we restrict to the case when F is a totally real or imaginary CM field. In this case, Conjectures 3.2 and 3.4 are essentially known and their proof, due to Scholze, has been discussed in § 2.4. The Galois representation $\rho_{\mathfrak{m}}$ ℓ-adically interpolates both the Galois representations attached to torsion classes occurring in $H^*(Y_K, \mathcal{V}_\lambda)_{\mathfrak{m}}$ as well as the characteristic 0 automorphic Galois representations first constructed in [69].

We emphasise that Conjecture 3.4 is only proved up to a nilpotent ideal. More precisely, by work of Newton–Thorne [103], there exists a nilpotent ideal $J \subset \mathbb{T}^S(K,\lambda)\widehat{_{\mathfrak{m}}}$ with $J^4 = 1$ such that there exists a Galois representation

$$\rho_{\mathfrak{m}} : \Gamma_F \to \mathrm{GL}_n \left(\mathbb{T}^S(K,\lambda)\widehat{_{\mathfrak{m}}}/J \right)$$

with the right characteristic polynomials at places $v \notin S$ of F. This does not usually cause trouble for applications to modularity lifting theorems, since the key point there is to understand the support of a certain patched module inside a Galois deformation ring. Nilpotent ideals do not affect the support. However, nilpotent ideals can cause trouble for more subtle questions, such as those concerning the Bloch–Kato conjecture, as in the appendix to [26]. If the prime ℓ splits completely in the totally real or imaginary CM field F, then Conjecture 3.4 is known as stated by the refined construction in [31].

3.1.2. *Local-global compatibility.* Continue to assume that F is a totally real or imaginary CM field and assume further that $\mathfrak{m} \subset \mathbb{T}^S(K, \lambda)$ is a non-Eisenstein maximal ideal. Let $R_{\bar{\rho}_{\mathfrak{m}}}$ denote the global deformation ring of $\bar{\rho}_{\mathfrak{m}}$, which parametrizes deformations of $\bar{\rho}_{\mathfrak{m}}$ that are unramified outside S^∞. The Galois representation $\rho_{\mathfrak{m}}$ is itself such a deformation, so by the universal property of $R_{\bar{\rho}_{\mathfrak{m}}}$ we obtain a map

$$(3.3) \qquad\qquad R_{\bar{\rho}_{\mathfrak{m}}} \to \mathbb{T}^S(K, \lambda)\widehat{}_{\mathfrak{m}}/J.$$

This is an incarnation of the usual map from a Galois deformation ring to a Hecke ring that shows up in both the classical Taylor–Wiles method and in the extension due to Calegari–Geraghty. The ultimate goal is to prove that this map is an isomorphism, or at least that it is an isomorphism on the underlying reduced rings. However, in order for the map (3.3) to have a chance of being an isomorphism, we first need to refine it by imposing appropriate local conditions at the primes in S. The next conjecture concerns local-global compatibility for the Galois representation $\rho_{\mathfrak{m}}$, which will help us refine this map by showing that it factors through a quotient of $R_{\bar{\rho}_{\mathfrak{m}}}$ with the appropriate local conditions.

Let p be a prime and let $v \mid p$ be a place of F, possibly contained in S^∞. If $\ell = p$, we set $\lambda_v = (\lambda_\tau)_\tau$, where $\tau \in \mathrm{Hom}_{\mathbb{Q}}(F, E)$ runs over embeddings that induce the place v. Local-global compatibility can roughly be formulated as the following question: how does the restriction $\rho_{\mathfrak{m}}|_{\Gamma_{F_v}}$ depend on the level K_v at which $\mathfrak{m}$ occurs, on the Hecke eigenvalues at v, and, if $\ell = p$, on λ_v? [7]

For the characteristic 0 automorphic Galois representations constructed in [69], this can be formulated more precisely by requiring compatibility with a suitable normalization of the classical local Langlands correspondence for GL_n/F_v. In the case $\ell \neq p$, this compatibility up to semi-simplification is a theorem of Varma [137]. In the case $\ell = p$, one needs to first show that the automorphic Galois representations in question are de Rham at v (or, equivalently, potentially semi-stable at v), then identify their Hodge–Tate numbers in terms of the sets λ_v, and finally establish the compatibility with classical local Langlands. In [1], A'Campo recently proved that the automorphic Galois representations in question are indeed de Rham at primes $v \mid \ell$ under certain technical assumptions and identified their Hodge–Tate weights.

The formulation of local-global compatibility for the integral Galois representations $\rho_{\mathfrak{m}}$ is more subtle, particularly when $\ell = p$. This is because, outside a restricted family of cases such as the Fontaine–Laffaille and ordinary ones, it is not (a priori) clear how to formulate integral p-adic Hodge-theoretic conditions that should be satisfied by $\rho_{\mathfrak{m}}|_{\Gamma_{F_v}}$. However, it turns out that we can formulate the local-global compatibility conjecture at $v \mid \ell$ using the potentially semi-stable Galois deformation rings constructed by Kisin in [80]. This version of the conjecture first appears in the case $n = 2$ in [59] and it is particularly well-suited for applications to modularity lifting theorems.

For simplicity, we restrict to the crystalline case. Let $\bar{\rho}_{\mathfrak{m},v}$ denote the local representation $\bar{\rho}_{\mathfrak{m}}|_{\Gamma_{F_v}}$. We let $R^{\square}_{\bar{\rho}_{\mathfrak{m},v}}$ denote its framed unrestricted local deformation ring. We let $R^{\square,\mathrm{crys}}_{\bar{\rho}_{\mathfrak{m},v}}(\lambda_v)$ denote its quotient whose characteristic 0 points parametrize crystalline lifts of $\bar{\rho}_{\mathfrak{m},v}$ with Hodge–Tate weights equal to $\{\lambda_{\tau,n}, \lambda_{\tau,n-1} + 2, \ldots, \lambda_{\tau,1} + n - 1\}$ at each embedding $\tau : F \hookrightarrow E$ inducing the place v. This

---

[7] If F is a totally real field, the description of $\rho_{\mathfrak{m}}(c)$ for a choice of complex conjugation $c \in \mathrm{Gal}(\overline{F}/F)$ can also be interpreted as a form of local-global compatibility – at the infinite places.

quotient was constructed by Kisin in [**80**], first after inverting ℓ and then integrally by taking Zariski closure.

CONJECTURE 3.5. Assume that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$, that $\mathfrak{m} \subset \mathbb{T}^S(K, \lambda)$ is a non-Eisenstein maximal ideal, and that Conjecture 3.4 holds for $\mathfrak{m}$. Consider the composition

(3.4)
$$R^{\square}_{\bar{\rho}_{\mathfrak{m}, v}} \longrightarrow R^{\square}_{\bar{\rho}_{\mathfrak{m}}} \longrightarrow \mathbb{T}^S(K, \lambda)\widehat{_{\mathfrak{m}}}$$

$$R^{\square, \mathrm{crys}}_{\bar{\rho}_{\mathfrak{m}, v}}(\lambda_v)$$

where the first horizontal map is induced by restriction and the second horizontal map is induced from the existence of the Galois representation $\rho_{\mathfrak{m}}$. This composition factors through $R^{\square, \mathrm{crys}}_{\bar{\rho}_{\mathfrak{m}, v}}(\lambda_v)$.

REMARK 3.6. We give an equivalent formulation of the local-global compatibility conjecture at a place $v \mid \ell$ when we are in the Fontaine–Laffaille setting. Concretely, we assume that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ as above, but also that ℓ is unramified in F and that the vectors λ_v are all in the Fontaine–Laffaille range, in other words

$$0 \leq \lambda_{\tau, n} \leq \cdots \leq \lambda_{\tau, 1} \leq \ell - n - 1$$

for every $\tau : F \hookrightarrow E$ that induces v. Then it is equivalent to conjecture that $\rho_{\mathfrak{m}}|_{\Gamma_{F_v}}$ is in the image of the Fontaine–Laffaille functor and has Hodge–Tate weights equal to $\{\lambda_{\tau, n}, \lambda_{\tau, n-1} + 2, \ldots, \lambda_{\tau, 1} + n - 1\}_\tau$ with $\tau : F \hookrightarrow E$ running over embeddings that induce v. One can state a similar explicit local-global compatibility conjecture in the ordinary case.

REMARK 3.7. It should be possible to formulate a version of Conjecture 3.5 in the potentially crystalline case. For this, one could replace the algebraic representation σ_λ with a locally algebraic representation $\sigma_{\tau, \lambda} := \sigma_\tau \otimes \sigma_\lambda$ of $\mathrm{GL}_n(\mathcal{O}_{F_v})$. Here, σ_τ should be taken to be a smooth representation of $\mathrm{GL}_n(\mathcal{O}_{F_v})$ over a finite-dimensional E'-vector space, which should correspond to an inertial type $\tau : I_{F_v} \to \mathrm{GL}_n(E')$ under the inertial local Langlands correspondence, for some finite extension E'/E. For more details, see [**30**, §§3–4], which describes the inertial local Langlands correspondence for GL_n in the potentially crystalline case. One can further generalize to the potentially semi-stable case by handling the monodromy operator carefully.

To make progress on Conjecture 3.5, we need a new way to access the Galois representations $\rho_{\mathfrak{m}}$. The reason is that the construction of $\rho_{\mathfrak{m}}$, which is described in § 2.4, involves increasing the level $\prod_{v \mid \ell} K_v$ as well as losing track of the weight λ by trivializing the local systems $\mathcal{V}_\lambda / \varpi^m$.

At the moment, all progress on Conjecture 3.5 has relied crucially on a series of increasingly more powerful vanishing theorems for the cohomology of unitary Shimura varieties with integral coefficients [**35, 34, 83**]. We discuss the general strategy in detail in § 4. Based on this strategy, one can obtain partial results towards Conjecture 3.5 in the Fontaine–Laffaille case, in the formulation of Remark 3.6. This is done in [**2**, §4]. One can also obtain partial results in the ordinary case - see [**2**, §5].

The recent paper [**33**] proves Conjecture 3.5 in the case when F is an imaginary CM field under some additional technical assumptions. Below, we give a flavour of the main assumptions; see Theorem 4.2.15 of *loc. cit.* for the precise statement.

(1) There is an assumption on the image of the residual representation $\bar{\rho}_{\mathfrak{m}}$, which is needed in order to appeal to known vanishing results for the cohomology of unitary Shimura varieties with torsion coefficients. With a better understanding of these cohomology groups of Shimura varieties, this assumption can be progressively weakened.

(2) There is also an assumption that there are "enough" places above ℓ in the totally real subfield F^+ of F. This is needed in order to play these places against each other: one focuses on one place $\bar{v} \mid \ell$ of F^+ where one wishes to prove local-global compatibility, using the remaining places $\bar{v}' \mid \ell, \bar{v}' \neq \bar{v}$ as auxiliary ones. At auxiliary places, one employs congruences to vary the weight. This assumption excludes, for example, the case when F is an imaginary quadratic field and the Y_K are Bianchi manifolds.

It should be possible to adapt the method of [**33**] to the potentially semi-stable case, under similar technical assumptions as described above. This is the subject of the upcoming PhD thesis of Bence Hevesi.

3.1.3. *Vanishing conjectures for locally symmetric spaces.* Conjectures 3.4 and 3.5 are natural generalizations of results that hold true in the classical Taylor–Wiles setting. For example, they hold true in the conjugate self-dual case, when one can implement the classical Taylor–Wiles method starting from algebraic automorphic forms on a definite unitary group, as is done for example in [**44**].

We now discuss a third conjecture, which roughly says that the non-degenerate part of the cohomology $H^*(Y_K, \mathbb{F}_\ell)$ is concentrated in a restricted range of degrees. This is the most novel prerequisite to the Calegari–Geraghty method and it illustrates some of the differences between general locally symmetric spaces and Shimura varieties.

To state the conjecture, we define the following numerical invariants:

$$l_0 := \operatorname{rk} \operatorname{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R}) - \operatorname{rk} K'_\infty \text{ and } q_0 := \frac{1}{2}(\dim_{\mathbb{R}} Y_K - l_0),$$

which turn out to be non-negative integers. Concretely, if the number field F has r_1 real places and r_2 complex places, one can compute that

$$l_0 = \begin{cases} r_1\left(\frac{n-2}{2}\right) + r_2(n-1) \text{ for } n \text{ even} \\ r_1\left(\frac{n-1}{2}\right) + r_2(n-1) \text{ for } n \text{ odd.} \end{cases}$$

These invariants were first introduced by Borel–Wallach in [**18**] for a general connected reductive group G over $\mathbb{Q}$. There, they show up naturally in the computation of $(\mathfrak{g}, K'_\infty)$-cohomology of tempered representations of G($\mathbb{R}$).

We consider the range of cohomological degrees $[q_0, q_0 + l_0]$, which is symmetric about the middle $\frac{1}{2}\dim_{\mathbb{R}} Y_K$ of the total range of cohomology. The following conjecture is formulated in [**49**] – see the discussion around Conjecture 3.3 in *loc. cit.* – and in [**25**, Conj. B].

CONJECTURE 3.8. *Let $\mathfrak{m} \subset \mathbb{T}^S$ be a non-Eisenstein maximal ideal in the support of $H^*(Y_K, \mathbb{F}_\ell)$. Then $H^i(Y_K, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$ only if $i \in [q_0, q_0 + l_0]$.*

This conjecture is motivated by heuristics to do with Langlands reciprocity, since the invariant l_0 can be recovered naturally from a computation on the Galois side. We explain this more in § 3.2. We also discuss the analogue of Conjecture 3.8 for Shimura varieties in § 4.

EXAMPLE 3.9.

(1) If $n = 2$ and $F = \mathbb{Q}$, we have $l_0 = 0$ and $q_0 = 1$. There is only one interesting degree for the Betti cohomology of modular curves, which is H^1. This is the setting in which the Taylor–Wiles method was first implemented. Conjecture 3.8 can be verified by hand in the case of modular curves. In fact, the maximal ideals $\mathfrak{m} \subset \mathbb{T}^S$ in the support of $H^0(Y_K, \mathbb{F}_\ell)$ can be shown to satisfy

$$\bar{\rho}_{\mathfrak{m}} \simeq \chi \oplus \chi_{\mathrm{cyclo}} \cdot \chi,$$

where $\chi : \Gamma_{\mathbb{Q}} \to \overline{\mathbb{F}}_\ell^{\times}$ is some continuous mod ℓ character and χ_{cyclo} is the mod ℓ cyclotomic character.

The Taylor–Wiles method was later generalized to other settings in which $l_0 = 0$. For example, in the conjugate self-dual case, one can arrange to work with algebraic automorphic forms on a definite unitary group, so that $q_0 = l_0 = 0$.

(2) If $n = 2$ and F is imaginary quadratic, we have $l_0 = 1$ and $q_0 = 1$. The interesting degrees of Betti cohomology for Bianchi manifolds are H^1 and H^2. One can again prove Conjecture 3.8 by hand in this case. By Poincaré duality, which applies after localization at a non-Eisenstein ideal $\mathfrak{m}$, it is enough to control H^0, which can be computed explicitly.

(3) It should be possible to control another handful of low-dimensional cases by employing the congruence subgroup property to control H^1. If $n = 2$ and F is totally real, the locally symmetric spaces are closely related to Hilbert modular varieties and one should be able to deduce (most of) Conjecture 3.8 from [**36**, Thm. A]. For arbitrary n and an arbitrary number field F, Conjecture 3.8 is still open, even in the case when F is an imaginary CM field.

One can formulate an analogous conjecture with $\mathbb{Q}_\ell$ rather than $\mathbb{F}_\ell$ coefficients. This is a strictly weaker conjecture, but it can be proved for arbitrary n when F is a CM or totally real field and this turns out to be extremely important for applications.

CONJECTURE 3.10. Let $\mathfrak{m} \subset \mathbb{T}^S$ be a non-Eisenstein maximal ideal in the support of $H^*(Y_K, \mathbb{Z}_\ell)$. Then $H^i(Y_K, \mathbb{Q}_\ell)_{\mathfrak{m}} \neq 0$ only if $i \in [q_0, q_0 + l_0]$.

To prove Conjecture 3.10, one needs to compute the Betti cohomology groups $H^*(Y_K, \mathbb{Q}_\ell)$ in terms of automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. This goes back to a theorem of Franke [**57**] that uses harmonic analysis. If $\mathfrak{m}$ is non-Eisenstein and in the support of $H^*(Y_K, \mathbb{Q}_\ell)$, then it is the reduction modulo ℓ of a system of Hecke eigenvalues occurring in a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. In that case, one can eliminate all the terms in Franke's computation coming from proper parabolic subgroups of GL_n/F, and one controls what is left by appealing to the computation of tempered cohomology in [**18**]. See [**2**, Thm. 2.4.10] for the

detailed argument. We emphasise, however, that one needs Conjecture 3.2 in order to formulate the notion of non-Eisenstein maximal ideal, so even for rational cohomology one needs to work over a CM or totally real field.

REMARK 3.11. When $l_0 = 0$ and $\mathfrak{m} \subset \mathbb{T}^S$ is a maximal ideal in the support of $H^*(Y_K, \mathbb{Z}_\ell)$, Conjecture 3.8 implies that $H^*(Y_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ is concentrated in degree q_0 and ℓ-torsion free. This can be shown by playing around with the long exact cohomology sequences associated to the short exact sequences of sheaves

$$0 \to \mathbb{Z}/\ell^{m-1}\mathbb{Z} \xrightarrow{\cdot \ell} \mathbb{Z}/\ell^m\mathbb{Z} \to \mathbb{F}_\ell \to 0$$

on Y_K for $m \in \mathbb{Z}_{\geq 2}$.

REMARK 3.12. All the conjectures discussed in this section can be formulated for the faithful quotient of $\mathbb{T}^S$ that acts on the cohomology of Y_K with compact support. A useful observation is that, after localizing at a non-Eisenstein maximal ideal, the natural map from cohomology with compact support to usual cohomology becomes an isomorphism. This follows from a now standard argument that involves a detailed study of strata in the boundary of the Borel–Serre compactification of Y_K. See [103, §4] for more details. Moreover, one could formulate Conjecture 3.8 instead by asking that $H^*(Y_K, \mathbb{F}_\ell)_{\mathfrak{m}}$ is only non-zero in some range of degrees of length l_0. By the observation just above, Poincaré duality shows that this apparently weaker conjecture is equivalent to the original one.

3.2. A sketch of the Calegari–Geraghty method. In this subsection, we briefly sketch the Taylor–Wiles patching method for proving modularity lifting theorems and then we discuss its improvement due to Calegari–Geraghty. The goal is to motivate and illustrate the role of Conjectures 3.4 and 3.5 and especially of Conjecture 3.8 in proving modularity lifting theorems. We suppress many technical details in our account. For a more thorough expository account of the classical Taylor–Wiles method, see [58], and for the same on the Calegari–Geraghty enhancement of this method, see [136].

3.2.1. *The case $l_0 = 0$.* We let $G = \mathrm{GL}_2/\mathbb{Q}$, in which case the Y_K are (open) modular curves, considered as Riemann surfaces. We let $\mathfrak{m} \subset \mathbb{T}^S$ be a non-Eisenstein maximal ideal in the support of $H^1(Y_K, \mathcal{V}_\lambda)$. This cohomology group is torsion-free, as discussed above, and so $\bar{\rho}_{\mathfrak{m}}$ is (up to a character twist) the reduction modulo ℓ of the Galois representation associated to some cuspidal eigenform (of weight determined by λ).

In this setting, Conjectures 3.4, 3.5 and 3.8 are known. For simplicity, we assume that $\ell > 2$, that the level K_ℓ is the hyperspecial maximal compact subgroup $\mathrm{GL}_2(\mathbb{Z}_\ell)$ and that the weight λ is trivial. This means that $\bar{\rho}_{\mathfrak{m}}$ is the reduction modulo ℓ of the Galois representation attached to a cuspidal eigenform f of weight 2 and level K (prime to ℓ). The lift

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2\left(\mathbb{T}(K,\lambda)_{\widehat{\mathfrak{m}}}\right).$$

can be constructed by interpolating the 2-dimensional ℓ-adic Galois representations attached to *all* the cuspidal eigenforms of weight 2 and level K and which are congruent to f modulo $\mathfrak{m}$. Local-global compatibility for each of these Galois representations implies that $\rho_{\mathfrak{m}}|_{\Gamma_{\mathbb{Q}_\ell}}$ is Barsotti–Tate, which after inverting ℓ means crystalline with Hodge–Tate weights equal to $\{0, 1\}$.

We let $R^{\mathrm{BT}}_{\bar{\rho}_{\mathfrak{m}}}$ denote the global deformation ring for $\bar{\rho}_{\mathfrak{m}}$ which parametrizes deformations that are unramified outside a fixed finite set of primes containing ℓ, and that are Barsotti–Tate at ℓ. (In practice, we might impose some additional local conditions at ramified primes other than ℓ, but we suppress this in this sketch.) The existence and local-global compatibility for the automorphic Galois representation $\rho_{\mathfrak{m}}$ give rise to a diagram

$$(3.5) \qquad\qquad\qquad R^{\mathrm{BT}}_{\bar{\rho}_{\mathfrak{m}}} \to \mathbb{T}(K,\lambda)\widehat{_{\mathfrak{m}}} \circlearrowleft H^1(Y_K, \mathcal{V}_\lambda)_{\mathfrak{m}}.$$

To prove a modularity lifting theorem, we wish to show that the cohomology group $H^1(Y_K, \mathcal{V}_\lambda)_{\mathfrak{m}}$ has full support over the global deformation ring $R^{\mathrm{BT}}_{\bar{\rho}_{\mathfrak{m}}}$. (Since $\mathbb{T}(K,\lambda)\widehat{_{\mathfrak{m}}}$ acts faithfully on $H^1(Y_K, \mathcal{V}_\lambda)_{\mathfrak{m}}$, this is equivalent to the statement that the surjective map $R^{\mathrm{BT}}_{\bar{\rho}_{\mathfrak{m}}} \to \mathbb{T}(K,\lambda)\widehat{_{\mathfrak{m}}}$ induces an isomorphism on the underlying reduced rings.)

Rather than study the diagram (3.5) directly, we first put everything in ℓ-adic families, in order to smooth out the underlying geometry. For this, we wish to "thicken" the global deformation ring $R^{\mathrm{BT}}_{\bar{\rho}_{\mathfrak{m}}}$ into a power series ring $R_\infty = \mathcal{O}[\![x_1,\ldots,x_g]\!]$. The dimension of R_∞ is determined via a Galois cohomology computation that identifies g with the dimension over $\mathbb{F}$ of a certain Selmer group $H^1_{\mathcal{L}}(\Gamma_{\mathbb{Q}}, \mathrm{ad}\,\bar{\rho}_{\mathfrak{m}})$.

In order to thicken the diagram (3.5) in a controlled fashion, we "patch" (co)homology groups[8] $H^1(Y_{K_{Q_N}}, \mathcal{V}_\lambda/\ell^N)_{\mathfrak{m}_N}$ with additional tame level at sets Q_N of so-called Taylor–Wiles primes. Patching is a highly non-canonical process that involves varying the sets Q_N of Taylor–Wiles primes as N goes to infinity. However, the cardinality of these sets is a fixed integer r, which can be computed as the dimension over $\mathbb{F}$ of a certain dual Selmer group $H^1_{\mathcal{L}^\perp}(\Gamma_{\mathbb{Q}}, \mathrm{ad}\,\bar{\rho}_{\mathfrak{m}}(1))$. This fixed cardinality ensures that the tangent spaces of the deformation rings with additional tame ramification at the Taylor–Wiles primes stay bounded.

The output of Taylor–Wiles patching is a diagram

$$(3.6) \qquad\qquad\qquad\qquad S_\infty \to R_\infty \circlearrowleft M_\infty$$

where $S_\infty = \mathcal{O}[\![z_1,\ldots,z_{2r}]\!]$ and R_∞ is a priori a complete local $\mathcal{O}$-algebra which only admits a surjection from $\mathcal{O}[\![x_1,\ldots,x_g]\!]$. The module M_∞ is patched from (co)homology groups of the modular curve with additional tame level. The ring R_∞ is patched from global deformation rings with additional tame ramification at Taylor–Wiles primes. The ring S_∞ keeps track of the additional tame ramification coming from Taylor–Wiles primes.

Because the growth of the (co)cohomology groups as we add tame level is strictly controlled, M_∞ will be finite free over S_∞; it is also finite over R_∞. We have a sequence of (in)equalities

$$(3.7) \quad 2r+1 = \dim S_\infty = \mathrm{depth}_{S_\infty}(M_\infty) \le \mathrm{depth}_{R_\infty}(M_\infty) \le \dim R_\infty \le g+1.$$

The equality $\dim S_\infty = \mathrm{depth}_{S_\infty}(M_\infty)$ follows from the freeness of M_∞ over S_∞. The first inequality comes from the fact that the action of S_∞ on M_∞ factors through R_∞. The second inequality comes from the fact that the depth of a finitely generated module is less than or equal to the dimension of its support.

At this point, Wiles makes use of an Euler characteristic formula in Galois cohomology that gives $2r - g = l_0$, which is 0 in this setting. This implies that

---

[8]Technically, patching amounts to taking a projective limit as the level varies. For this reason, it is better to switch from cohomology to homology at this point.

all the above inequalities are in fact equalities! In particular, this means that the given surjection $\mathcal{O}[\![x_1, \ldots, x_g]\!] \twoheadrightarrow R_\infty$ is in fact an isomorphism. We now apply the Auslander–Buchsbaum formula for the finitely generated module M_∞ over the Noetherian local ring R_∞:

$$(3.8) \qquad \mathrm{depth}_{R_\infty}(M_\infty) + \mathrm{proj.\ dim.}_{R_\infty}(M_\infty) = \dim R_\infty = g + 1.$$

Since we have also found that $\mathrm{depth}_{R_\infty}(M_\infty) = g + 1$, we deduce M_∞ is finite free over R_∞.

To recover the original diagram (3.5), we take a tensor product of (3.6) with $\otimes_{S_\infty} \mathcal{O}$. We have $R_\infty \otimes_{S_\infty} \mathcal{O} = R_{\bar{\rho}_\mathfrak{m}}^{\mathrm{BT}}$ and $M_\infty \otimes_{S_\infty} \mathcal{O}$ is (the $\mathcal{O}$-linear dual of) $H^1(Y_K, \mathcal{V}_\lambda)_\mathfrak{m}$. This shows that the cohomology group $H^1(Y_K, \mathcal{V}_\lambda)_\mathfrak{m}$ is finite free over the global deformation ring $R_{\bar{\rho}_\mathfrak{m}}^{\mathrm{BT}}$ and, in particular, has full support.

REMARK 3.13. In the above sketch, we can weaken the assumptions that the level K_ℓ is hyperspecial and that the weight λ is trivial. The right perspective here is due to Kisin [80] and involves constructing R_∞ as a power series ring over a product of local (possibly framed) deformation rings. We can then show that M_∞ is maximal Cohen–Macaulay over R_∞ and thus supported on a union of irreducible components of $\mathrm{Spec}\, R_\infty$. Therefore, the geometry of the local deformation rings determines the kind of modularity lifting result we can prove. A key problem is how to extend the known support of M_∞ from one irreducible component to another, e.g. as in Taylor's Ihara avoidance. See [58] for more details. The same phenomenon arises in the case when $l_0 > 0$ and, in this case, it is even more subtle to move between different irreducible components.

3.2.2. *The case $l_0 > 0$.* For simplicity and to parallel the case $l_0 = 0$ discussed above, we restrict to $G = \mathrm{GL}_2/F$, where F is now an imaginary CM field. For a neat compact open subgroup $K \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$, the spaces Y_K are smooth real manifolds. We assume that the prime $\ell > 2$ is unramified in F, that the level K_v at each $v \mid \ell$ is hyperspecial, and that the weight λ is trivial. We let $\mathfrak{m} \subset \mathbb{T}^S$ be a non-Eisenstein maximal ideal in the support of $H^*(Y_K, \mathcal{V}_\lambda)$.

We assume Conjectures 3.4, 3.5 and 3.8. As explained above, Conjecture 3.4 is essentially known by [116], and there are many settings where Conjecture 3.5 is known as well in the case at hand, see [33] for the state of the art result. However, Conjecture 3.8 is still a significant assumption. The main exception to this is the case when F is imaginary quadratic, where Conjecture 3.8 is known, but Conjecture 3.5 is surprisingly subtle and not known even in the case at hand.

We let $R_{\bar{\rho}_\mathfrak{m}}^{\mathrm{BT}}$ denote the universal deformation ring of $\bar{\rho}_\mathfrak{m}$ with appropriate local conditions: in particular, we require these deformations to be unramified outside a fixed finite set of finite primes of F and Barsotti–Tate at each prime $v \mid \ell$ of F. As before, we obtain a diagram

$$(3.9) \qquad R_{\bar{\rho}_\mathfrak{m}}^{\mathrm{BT}} \to \mathbb{T}(K, \lambda)_{\widehat{\mathfrak{m}}} \circlearrowleft \bigoplus_{i=q_0}^{q_0+l_0} H^i(Y_K, \mathcal{V}_\lambda)_\mathfrak{m}..$$

To prove a modularity lifting theorem in this setting, we want to show that the direct sum has full support in $R_{\bar{\rho}_\mathfrak{m}}^{\mathrm{BT}}$. It is, of course, enough to establish this for any individual cohomological degree.

We want to proceed as in the case $l_0 = 0$ discussed above. To thicken the diagram (3.9), we again need to consider a patched deformation ring R_∞, with

$\dim R_\infty = g + 1$, where g is the dimension over $\mathbb{F}$ of a Selmer group $H^1_{\mathcal{L}}(\Gamma_F, \mathrm{ad}\,\bar{\rho}_{\mathfrak{m}})$. At the same time, the cardinality of the sets of Taylor–Wiles primes must equal r, the dimension over $\mathbb{F}$ of the dual Selmer group $H^1_{\mathcal{L}^\perp}(\Gamma_F, \mathrm{ad}\,\bar{\rho}_{\mathfrak{m}}(1))$. We therefore set $S_\infty = \mathcal{O}[\![z_1, \ldots, z_{2r}]\!]$ once again. However, if we try to implement the classical Taylor–Wiles method as outlined above, we get stuck at the Euler characteristic formula:

$$(3.10) \qquad\qquad \dim S_\infty - \dim R_\infty = l_0 > 0.$$

This means that there are not enough Galois representations to fill out the entire space $\mathrm{Spec}\, S_\infty$. On the other hand, because the cohomology groups of Y_K are likely supported in multiple degrees, if we add tame level at sets of Taylor–Wiles primes and patch (co)homology groups (in a fixed degree or even taking a direct sum of cohomological degrees), we will likely not obtain a finite free module M_∞ over S_∞. Thus, it seems like there are also not enough automorphic forms to fill out $\mathrm{Spec}\, S_\infty$. Fortunately, as Calegari and Geraghty have observed, these two problems precisely cancel each other out, when considered from the right point of view!

The idea is to work with a patched complex C_∞ instead of a single patched module M_∞. The complex C_∞ is obtained from patching chain complexes that compute the homology of $Y_{K_{Q_N}}$, with additional tame level at a cardinality r set of Taylor–Wiles primes Q_N, and after localizing at a non-Eisenstein maximal ideal $\mathfrak{m}_N$ obtained from $\mathfrak{m}$. Because we are working with the homology of the $Y_{K_{Q_N}}$ as chain complexes, rather than with individual homology groups, we obtain after patching a complex C_∞ of finite projective S_∞-modules. Moreover, Conjecture 3.8 ensures that, up to replacing C_∞ by a quasi-isomorphic complex, we can assume that it is a complex of finite projective S_∞-modules concentrated in a range of degrees of length l_0. The following lemma in commutative algebra is now crucial, see [**25**, Lem. 6.2] for its proof.

LEMMA 3.14. *Let S be a regular Noetherian local ring of dimension d. Let C be a chain complex[9] of finite projective S-modules concentrated in the range of degrees $[0, l]$ for some non-negative integer $l \le d$. Then $\dim_S(H_*(C)) \le d - l$. Moreover, if equality holds, then $H_i(C)$ is non-zero only for $i = 0$,*

$$\mathrm{proj.\,dim.}_S(H_0(C)) = l, \text{ and } \mathrm{depth}_S(H_0(C)) = d - l.$$

(In particular, C is a projective resolution of $H_0(C)$ of minimal length.)

In the Calegari–Geraghty extension of the Taylor–Wiles method, we apply this lemma to $S = S_\infty$ and $C = C_\infty[q_0]$ (we shift our patched complex q_0 degrees to the left in order to get the concentration in the range $[0, l_0]$). We have a diagram

$$S_\infty \to R_\infty \circlearrowleft C_\infty,$$

giving us the sequence of inequalities

$$(3.11) \qquad 2r + 1 - l_0 \le \dim_{S_\infty}(H_*(C_\infty)) \le \dim_{R_\infty}(H_*(C_\infty)) \le g + 1.$$

The Euler characteristic formula (3.10) implies that all the inequalities in (3.11) are equalities. Lemma 3.14 implies that the homology of C_∞ is non-zero only in degree q_0 and that

$$\mathrm{depth}_{S_\infty}(H_{q_0}(C_\infty)) = 2r + 1 - l_0 = g + 1.$$

---

[9]In [**25**, Lem. 6.2] the result is stated and proved for a cochain complex.

At the same time, because the action of S_∞ on C_∞ factors through R_∞, we have the (in)equalities

$$g + 1 = \mathrm{depth}_{S_\infty}(H_{q_0}(C_\infty)) \leq \mathrm{depth}_{R_\infty}(H_{q_0}(C_\infty)) \leq \dim R_\infty = g + 1.$$

These are all equalities and the Auslander–Buchsbaum formula applied to $H_{q_0}(C_\infty)$ viewed as an R_∞-module implies that $H_{q_0}(C_\infty)$ is free over R_∞. To recover (part of) the original diagram (3.9), we take a tensor product with $\otimes_{S_\infty} \mathcal{O}$[10] and an $\mathcal{O}$-linear dual to go from homology to cohomology. We ultimately deduce that $H^{q_0}(Y_K, \mathcal{V}_\lambda)_{\mathfrak{m}}$ has full support over $R_{\bar\rho_{\mathfrak{m}}}^{\mathrm{BT}}$, as desired.

4. Shimura varieties, p-adic geometry, and torsion classes

In this section, we discuss the steps that led to an unconditional implementation of the Calegari–Geraghty method for GL_n over CM fields. A key input is an analogue of the vanishing Conjecture 3.8 for Shimura varieties. This analogue predicts that the part of the cohomology of Shimura varieties outside the middle degree is somewhat degenerate. In § 4.1, we discuss the geometry of the Hodge–Tate period morphism, which has allowed us to make significant progress towards this conjecture, and which is a beautiful and important topic in its own right. In § 4.2, we discuss several different approaches towards the conjecture in the case of Shimura varieties PEL type A and point towards generalizations. Finally, in § 4.3, we briefly describe how to use these results to make progress on the local-global compatibility Conjecture 3.5 and mention further applications.

4.1. The geometry of the Hodge–Tate period morphism. The goal of this subsection is to discuss more in depth the geometry of the Hodge–Tate period morphism that has already played a key role in Section 2.3. We will briefly explain how this geometry is illuminated by an infinite-level version of Mantovan's product formula [98]. This product formula describes Newton strata in Shimura varieties in terms of local analogues of Shimura varieties, known as Rapoport–Zink spaces or local Shimura varieties, and objects of a semi-global nature, known as Igusa varieties. We end by mentioning an even more general version of the product formula, which allows us to vary the Newton stratum, conjectured by Scholze and recently established in many cases by Mingjia Zhang in her Bonn PhD thesis [143].

For simplicity, we work with a Shimura datum (G, X) of PEL type. We assume that p is an unramified prime for this Shimura datum and ℓ is a rational prime different from p. We fix a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}^{p,\infty})$. Consider the minimally compactified perfectoid Shimura variety $\mathcal{S}_{K^p}^*$ over $\mathbb{C}_p$ introduced in Theorem 2.9 and the associated Hodge–Tate period morphism

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p}^* \to \mathscr{F}\ell_G.$$

For technical reasons, we restrict the source of this morphism from now on to the good reduction locus $\mathcal{S}_{K^p}^\circ \subset \mathcal{S}_{K^p}^*$.

---

[10]Here, one could ask what would happen if we took a derived tensor product $\otimes_{S_\infty}^{\mathrm{L}} \mathcal{O}$ instead. This will recover the entire cohomology as an object in a derived category. See the article [55] of Feng–Harris in these proceedings for more details on derived aspects.

4.1.1. *The Newton stratification.* Recall the Kottwitz set $B(G_{\mathbb{Q}_p})$ classifying isocrystals with $G_{\mathbb{Q}_p}$-structure, cf. [**109**] for example. This set is equipped with a partial order, known as the Bruhat order. The Hodge cocharacter μ defines a subset $B(G_{\mathbb{Q}_p}, \mu^{-1}) \subset B(G_{\mathbb{Q}_p})$ of μ^{-1}-admissible elements. The special fiber of the Shimura variety with hyperspecial level at p admits a Newton stratification

$$(4.1) \qquad \overline{\mathrm{Sh}}_{K^p K_p} = \bigsqcup_{b \in B(G_{\mathbb{Q}_p}, \mu^{-1})} \overline{\mathrm{Sh}}^b_{K^p K_p}$$

into locally closed strata indexed by this subset. This stratification is defined moduli-theoretically and detects at geometric points the isogeny class of the universal p-divisible group $\mathscr{G} = \mathcal{A}[p^\infty]$ with $G_{\mathbb{Q}_p}$-structure living over $\overline{\mathrm{Sh}}_{K^p K_p}$. The closure relations between strata are determined by the degenerations of isocrystals with $G_{\mathbb{Q}_p}$-structure: for example, the μ-ordinary locus is the unique open and dense stratum and the basic locus is the unique Zariski closed stratum. The Newton stratification can be extended in a natural way to the minimal compactification $\overline{\mathrm{Sh}}^*_{K^p K_p}$ using the notion of a well-positioned subset of the special fiber, cf. [**91**].

There is an analogue of the Newton stratification on $\mathscr{F}\ell_G$ that is defined using purely local ingredients. More precisely, we have a decomposition

$$(4.2) \qquad \mathscr{F}\ell_G = \bigsqcup_{b \in B(G_{\mathbb{Q}_p}, \mu^{-1})} \mathscr{F}\ell^b_G$$

into locally closed strata, cf. [**35**, §3]. Morally, the argument is as follows (although this language did not exist at the time when [**35**] was written): one first constructs a map of v-stacks $\mathscr{F}\ell_G \to \mathrm{Bun}_G$, where the latter is the v-stack of G-bundles on the Fargues–Fontaine curve. To construct this map of v-stacks, it is convenient to notice that one can identify the diamond associated to $\mathscr{F}\ell_G$ with the minuscule Schubert cell defined by μ inside the B_{dR}^+-Grassmannian for G. Once we have obtained the map $\mathscr{F}\ell_G \to \mathrm{Bun}_G$, we can use Fargues's result that the points of Bun_G are in bijection with the Kottwitz set $B(G_{\mathbb{Q}_p})$, cf. [**53**] (see also [**5**] for an alternative proof that also works in equal characteristic). Moreover, the Newton decomposition (4.2) is a *stratification*, in the sense that, for $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$, we have

$$\overline{\mathscr{F}\ell^b_G} = \bigsqcup_{b' \geq b} \mathscr{F}\ell^{b'}_G,$$

where $\geq$ denotes the Bruhat order. The latter fact follows from the analogous result of Viehmann that applies to Bun_G, see [**138**, Thm. 1.1].

REMARK 4.1. The closure relations between Newton strata on the special fiber $\overline{\mathrm{Sh}}^*_{K^p K_p}$ are *opposite* to the closure relations between Newton strata on $\mathscr{F}\ell_G$. This is not an inconsistency, because the Hodge–Tate period map respects the Newton stratification only on points of rank one. The behaviour of π_{HT} on higher rank points is more subtle and this accounts for the difference in closure relations.

4.1.2. *Igusa varieties.* For each $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$, we can choose a (completely slope divisible) p-divisible group with $G_{\mathbb{Q}_p}$-structure $\mathbb{X}_b/\overline{\mathbb{F}}_p$, which is in the isogeny class determined by b and also compatible with μ, and define the corresponding *Oort central leaf.* This is a smooth closed subscheme $\mathscr{C}^{\mathbb{X}_b}$ of the Newton stratum $\overline{\mathrm{Sh}}^b_{K^p K_p}$, such that the isomorphism class of the p-divisible group $\mathscr{G}$ with $G_{\mathbb{Q}_p}$-structure over each geometric point of the leaf is constant and equal to that of

$\mathbb{X}_b$:

$$\mathscr{C}^{\mathbb{X}_b} = \left\{ x \in \overline{\mathrm{Sh}}^b_{K^p K_p} \mid \mathscr{G} \times \kappa(\bar{x}) \simeq \mathbb{X}_b \times \kappa(\bar{x}) \right\}.$$

In general, there can be infinitely many non-empty leaves inside a given Newton stratum. Over each central leaf, one has the *perfect Igusa variety* $\mathrm{Ig}^b/\overline{\mathbb{F}}_p$, a pro-finite cover of $\mathscr{C}^{\mathbb{X}^b}$ which parametrizes trivializations of $\mathscr{G}$ together with its $G_{\mathbb{Q}_p}$-structure.

Variants of Igusa varieties were introduced in [70] in the special case of Shimura varieties of Harris–Taylor type. They were defined more generally for Shimura varieties of PEL type by Mantovan [97] and their ℓ-adic cohomology was computed in many cases using a counting point formula; see [70, Ch.V] as well as [128, 129, 130]. The overall strategy is a variation of the LKR method (§ 2.2.2). Steps (1)–(4) of § 2.2.2 have analogues for Igusa varieties. See [126, §7] for more details and the generalization to the case of Hodge or abelian type.

In these references, Igusa varieties are defined as pro-finite étale covers of central leaves, which trivialize the graded pieces of the slope filtration, which $\mathscr{G}$ admits after restriction to $\mathscr{C}^{\mathbb{X}^b}$. Providing a splitting of the slope filtration amounts to taking perfection on the level of the Igusa variety, recovering the object Ig^b introduced above. From now on, we only consider the perfect Igusa varieties, since they have a more elegant moduli-theoretic interpretation, but still have the same ℓ-adic cohomology as the original objects.

REMARK 4.2. While the central leaf $\mathscr{C}^{\mathbb{X}_b}$ depends on the choice of $\mathbb{X}_b$ in its isogeny class, one can show that the perfect Igusa variety Ig^b only depends on the isogeny class: this follows from the equivalent moduli-theoretic description in [35, Lem. 4.3.4] (see also [36, Lem. 4.2.2], which keeps track of the extra structures more carefully). In particular, the pair (G, μ) is not intrinsically attached to the Igusa variety Ig^b – it can happen that Igusa varieties that are a priori obtained from different Shimura varieties are isomorphic. See [36, Thm. 4.2.4] for an example of this phenomenon.

Because $\mathrm{Ig}^b/\overline{\mathbb{F}}_p$ is perfect, the base change $\mathrm{Ig}^b \times_{\overline{\mathbb{F}}_p} \mathcal{O}_{\mathbb{C}_p}/p$ admits a canonical lift to a flat formal scheme over $\mathrm{Spf}\,\mathcal{O}_{\mathbb{C}_p}$. We let $\mathfrak{Ig}^b$ denote the adic generic fibre of this lift, which is a perfectoid space over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. The spaces Ig^b and $\mathfrak{Ig}^b$ both have an action of a locally profinite group $G_b(\mathbb{Q}_p)$, where G_b is an inner form of a Levi subgroup of G, and they have a Hecke action away from p. The ℓ-adic cohomology groups of Ig^b and $\mathfrak{Ig}^b$ are isomorphic, equivariantly for these actions.

4.1.3. *Rapoport–Zink spaces.* For each $b \in B(G_{\mathbb{Q}_p}, \mu^{-1})$, one can also consider the associated *Rapoport–Zink space*, a moduli space of p-divisible groups with $G_{\mathbb{Q}_p}$-structure that is a local analogue of a Shimura variety. Concretely in the PEL case, one considers a moduli problem of p-divisible groups equipped with $G_{\mathbb{Q}_p}$-structure, satisfying the Kottwitz determinant condition with respect to μ, and with a modulo p quasi-isogeny to the fixed p-divisible group $\mathbb{X}_b$. This moduli problem was shown by Rapoport–Zink [110] to be representable by a formal scheme. We let $\mathcal{M}^b$ denote the adic generic fibre of this formal scheme, base changed to $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$, and let $\mathcal{M}^b_\infty$ denote the corresponding infinite-level Rapoport–Zink space[11]. The latter object can be shown to be a perfectoid space using the techniques of [122]. By *loc. cit.*,

---

[11]As a consequence of the comparison with moduli spaces of local shtukas in [124], one obtains a group-theoretic characterisation of Rapoport–Zink spaces as local Shimura varieties determined by the tuple (G, b, μ). We suppress (G, μ) from the notation for simplicity.

the infinite-level Rapoport–Zink space admits a local analogue of the Hodge–Tate period morphism

$$\pi_{\mathrm{HT}}^b : \mathcal{M}_\infty^b \to \mathscr{F}\ell_G.$$

The geometry of π_{HT} is intricately tied up with the geometry of its local analogues π_{HT}^b, as we shall now explain.

4.1.4. *The product formula.* For each $b \in B(G, \mu^{-1})$, we define a locally closed subset $\mathcal{S}_{K^p}^{\circ b}$ of the good reduction locus $\mathcal{S}_{K^p}^\circ$ by intersecting the pre-image under the specialization map of the Newton stratum corresponding to b on the special fiber with the pre-image under π_{HT} of $\mathscr{F}\ell_G^b$.

THEOREM 4.3. *There exists a Cartesian diagram of diamonds over* $\mathrm{Spd}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$

(4.3)
$$
\begin{array}{ccc}
\mathcal{M}_\infty^b \times_{\mathrm{Spd}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})} \mathfrak{Ig}^b & \longrightarrow & \mathcal{M}_\infty^b \\
\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{HT}}^b} \\
\mathcal{S}_{K^p}^{\circ b} & \xrightarrow{\ \pi_{\mathrm{HT}}\ } & \mathscr{F}\ell_G^b.
\end{array}
$$

Moreover, each vertical map is a pro-étale torsor for the "unipotent group diamond" $\widetilde{G}_b$ of [54] *(identified with* $\mathrm{Aut}_G(\widetilde{\mathbb{X}}_b)$, *in the notation of* [35, §4]).

Theorem 4.3 is established in [35, §4], although it is not formulated in terms of diamonds in *loc. cit.* It has been generalized to Shimura varieties of Hodge type, in the context of Kisin–Pappas integral models (and under some additional technical assumption), by Hamacher–Kim [64]. At the time of writing, a general version of the Mantovan product formula for Shimura varieties of abelian type is not yet available.

Assume that the Shimura varieties Sh_K are compact. We have the following consequence of Theorem 4.3 for the fibres of π_{HT}: let $\bar{x} : \mathrm{Spa}(C, C^+) \to \mathscr{F}\ell_G^b$ be a geometric point. Then there is an inclusion of $\mathfrak{Ig}^b$ into $\pi_{\mathrm{HT}}^{-1}(\bar{x})$, which identifies the target with the canonical compactification of the source, in the sense of [119, Prop. 18.6].

This computation of the fibres can be extended to minimal and toroidal compactifications of (non-compact) Shimura varieties. For a model of the argument, see [34, Thm. 1.10], where the computation of the fibres is extended to Shimura varieties attached to quasi-split unitary groups. In this case, the fibres are obtained from partial minimal and toroidal compactifications of Igusa varieties. The partial minimal compactifications of Igusa varieties are affine - this is reflected in the "affinoid" nature of π_{HT} for the minimal compactification $\mathcal{S}_{K^p}^*$. This result has been generalized to Shimura varieties of PEL type in the PhD thesis of Mafalda Santos [112].

EXAMPLE 4.4. We make the geometry of π_{HT} explicit in the case of the modular curve, i.e. for $G = \mathrm{GL}_2/\mathbb{Q}$. In this case, we identify $\mathscr{F}\ell_G = \mathbb{P}^{1,\mathrm{ad}}$ and we have the decomposition into Newton strata (at least on points of rank one):

$$
\begin{array}{ccccccc}
\mathcal{S}_{K^p}^\circ & = & \mathcal{S}_{K^p}^{\circ,\mathrm{ord}} & \sqcup & \mathcal{S}_{K^p}^{\circ,\mathrm{ss}} \\
\downarrow{\scriptstyle \pi_{\mathrm{HT}}} & & \downarrow & & \downarrow \\
\mathbb{P}^{1,\mathrm{ad}} & = & \mathbb{P}^{1,\mathrm{ad}}(\mathbb{Q}_p) & \sqcup & \Omega.
\end{array}
$$

The ordinary locus inside $\mathbb{P}^{1,\mathrm{ad}}$ consists of the set of points defined over $\mathbb{Q}_p$ and the basic / supersingular locus is its complement Ω, the Drinfeld upper half-plane.

The fibres of π_{HT} over the ordinary locus are "perfectoid versions" of Igusa curves. The infinite-level version of the product formula reduces, in this case, to the statement that the ordinary locus is parabolically induced from $\mathfrak{Ig}^{\mathrm{ord}}$, as in [**36**, §6]. The fibres of π_{HT} over the supersingular locus are profinite sets: the corresponding Igusa varieties can be identified with double cosets $D^\times \backslash D^\times(\mathbb{A}_f)/K^p$, where $D/\mathbb{Q}$ is the quaternion algebra ramified precisely at ∞ and p. In some form, this result goes back to Deuring–Serre; see [**71**] for this precise formulation. This description of the supersingular / basic Igusa varieties together with the Mantovan product formula recover Rapoport–Zink uniformization in this special case.

REMARK 4.5. The Cartesian diagram (4.3) can be obtained from a Cartesian diagram of v-stacks

$$(4.4) \qquad \begin{array}{ccc} \mathcal{S}_{K^p}^{\circ b} & \xrightarrow{\ \pi_{\mathrm{HT}}\ } & \mathscr{F}\ell_G^b \\ \downarrow & & \downarrow \\ [\mathfrak{Ig}^b/\widetilde{G}_b] & \longrightarrow & [*/\widetilde{G}_b] \xrightarrow{\ \simeq\ } \mathrm{Bun}_G^b \end{array} \qquad .$$

Indeed, pullback of (4.4) under $* \to [*/\widetilde{G}_b]$ produces a cube with all faces Cartesian diagrams and one of the faces can be identified with (4.3). In her PhD thesis [**143**], Zhang constructs a more general version of (4.4), where the element b, and thus the Newton stratum, is allowed to vary.

More precisely, for Shimura varieties of PEL type AC and when p is a prime of good reduction, Zhang constructs a v-stack Igs living over Bun_G, called an *Igusa stack*. She then establishes a Cartesian diagram of v-stacks

$$(4.5) \qquad \begin{array}{ccc} \mathcal{S}_{K^p}^{\circ} & \xrightarrow{\ \pi_{\mathrm{HT}}\ } & \mathscr{F}\ell_G \\ \downarrow & & \downarrow \\ \mathrm{Igs} & \xrightarrow{\ \overline{\pi}_{\mathrm{HT}}\ } & \mathrm{Bun}_G \end{array}$$

which specializes to (4.4) for fixed $b \in B(G, \mu^{-1})$. Under some mild technical assumptions, Zhang also extends (4.5) to minimal compactifications. That such a construction should be possible for general Shimura varieties was conjectured by Scholze.

Morally, one should think of the diagram (4.5) as taking as an input a moduli space of global G-iso-shtukas with no legs, and creating legs via Beauville–Laszlo gluing. This intuition cannot be made precise, of course, since we do not have a good notion of global shtukas in the number field setting. However, this does suggest that Igusa varieties and, more generally, Igusa stacks are, in some ways, just as fundamental as Shimura varieties.

4.2. Vanishing of the cohomology of Shimura varieties with torsion coefficients. The goal of this subsection is to discuss local counterparts to Conjecture 3.8 in the special case of Shimura varieties, and to describe the state of the art for results towards these conjectures. We especially focus on the references [**35**],[**34**] and [**83**] which concern Shimura varieties of PEL type A and a completely split

prime p. However, we end by mentioning work in progress of Hamann and Lee [**66**] that goes significantly beyond this case.

In the case of Shimura varieties, the analogue of Conjecture 3.8 predicts that the non-Eisenstein part of the cohomology with $\mathbb{F}_\ell$-coefficients is concentrated in the middle degree $q_0 = d := \dim_{\mathbb{C}} \mathrm{Sh}_K$. The initial progress on this conjecture in the Shimura variety setting had rather strong additional assumptions: for example one needed ℓ to be an unramified prime for the Shimura datum and K_ℓ to be hyperspecial, as in the work of Dimitrov [**47**] and Lan–Suh [**92, 93**]. The geometry of the Hodge–Tate period morphism, as described in § 4.1, has been a game-changer in this area, allowing us to obtain results that are simultaneously more general and more precise.

Assume that (G, X) is a Shimura datum of PEL type A. Let ℓ be a prime number and $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal in the support of $H^*_{(c)}(\mathrm{Sh}_K(\mathbb{C}), \mathbb{F}_\ell)$. Scholze's argument with fake Hasse invariants explained in § 2.4 also allows us to associate a global modulo ℓ continuous and semi-simple Galois representation $\bar{\rho}_{\mathfrak{m}}$ to the system of Hecke eigenvalues determined by $\mathfrak{m}$. The non-Eisenstein condition on $\mathfrak{m}$ makes sense as a *global* condition imposed on the Langlands parameter $\bar{\rho}_{\mathfrak{m}}$. In what follows, we will choose an auxiliary prime $p \neq \ell$ that is unramified for the Shimura datum and impose *local* conditions on the Langlands parameter at p.

DEFINITION 4.6. Let k be a finite field of characteristic ℓ.

(1) Let $p \neq \ell$ be a prime, $L/\mathbb{Q}_p$ be a finite extension, and $\bar{\rho} : \Gamma_L \to \mathrm{GL}_n(k)$ be a continuous Galois representation. We say that $\bar{\rho}$ is *generic* if it is unramified and the eigenvalues (with multiplicity) $\alpha_1, \ldots, \alpha_n \in \overline{k}^{\times}$ of $\bar{\rho}(\mathrm{Frob}_L)$ satisfy $\alpha_i/\alpha_j \neq |\mathcal{O}_L/\mathfrak{m}_L|$ for $i \neq j$. For a product $\bar{\rho} : \Gamma_L \to \prod_{i=1}^m \mathrm{GL}_{n_i}(k)$ of continuous Galois representations, we say that $\bar{\rho}$ is *generic* if each factor in the product is generic.

(2) Let F be a number field and $\bar{\rho} : \Gamma_F \to \mathrm{GL}_n(k)$ be a continuous representation. We say that a prime $p \neq \ell$ is *decomposed generic* for $\bar{\rho}$ if p splits completely in F and, for every prime $\mathfrak{p} \mid p$ of F, $\bar{\rho}|_{\Gamma_{F_{\mathfrak{p}}}}$ is generic. We say that $\bar{\rho}$ is *decomposed generic* if there exists a prime $p \neq \ell$ which is decomposed generic for $\bar{\rho}$. (If one such prime exists, then infinitely many do, by the Chebotarev density theorem.) These notions also extend to products $\bar{\rho} : \Gamma_F \to \prod_{i=1}^m \mathrm{GL}_{n_i}(k)$ as above.

REMARK 4.7. The condition for a local representation $\bar{\rho} : \Gamma_L \to \mathrm{GL}_n(k)$ to be generic implies that any lift to characteristic 0 of $\bar{\rho}$ corresponds under the local Langlands correspondence to an irreducible and generic principal series representation of $\mathrm{GL}_n(L)$. Such a representation can never arise from a non-split inner form of GL_n/L via the Jacquet–Langlands correspondence. For this reason, a generic $\bar{\rho}$ cannot be the modulo ℓ reduction of the L-parameter of a smooth representation of a non-split inner form of GL_n/L.

The genericity condition can also be characterized in terms of the modulo ℓ semi-simple local Langlands correspondence of Vignéras [**139**]: the condition is that $\bar{\rho}$ corresponds to an irreducible (and thus generic) modulo ℓ principal series representation of $\mathrm{GL}_n(L)$. If this is the case, then the representation of $\mathrm{GL}_n(L)$ attached to $\bar{\rho}$ by the Emerton–Helm mod ℓ local Langlands in families [**50**] is the same irreducible and generic principal series representation.

REMARK 4.8. The condition for a global Galois representation $\bar{\rho} : \Gamma_F \to \mathrm{GL}_n(k)$ to be decomposed generic can be ensured in many cases when $\bar{\rho}$ has sufficiently large image, using the Chebotarev density theorem. For example, if F is totally real, $n = 2$, $\ell > 2$ and $\bar{\rho}$ is totally odd and has non-solvable image, then it is decomposed generic, cf. [**36**, Lem. 7.1.8]. A similar result holds if F is CM and Galois over $\mathbb{Q}$, cf. [**4**, Lem. 2.3].

As mentioned above, the idea is to impose some kind of genericity condition on a system of Hecke eigenvalues $\mathfrak{m}$ in order to restrict the range of degrees in which $\mathfrak{m}$ can contribute. The expectation is that, under such a condition,

(4.6) $H_c^i(\mathrm{Sh}_K(\mathbb{C}), \mathbb{F}_\ell)_\mathfrak{m} = 0$ if $i > d$ and $H^i(\mathrm{Sh}_K(\mathbb{C}), \mathbb{F}_\ell)_\mathfrak{m} = 0$ if $i < d$.

If either $\mathfrak{m}$ is non-Eisenstein or the Shimura varieties under consideration are compact, the above cohomology groups would be concentrated only in the middle degree. This would prove Conjecture 3.8 in many cases for Shimura varieties of PEL type A. For this purpose, the decomposed generic condition works extremely well, as exhibited by the following result.

THEOREM 4.9 ([**35, 34, 83, 112**]). *Let (G, X) be a Shimura datum of PEL type A. Let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal in the support of $H^*_{(c)}(\mathrm{Sh}_K(\mathbb{C}), \mathbb{F}_\ell)$ with associated Galois representation $\bar{\rho}_\mathfrak{m}$. Assume that $\bar{\rho}_\mathfrak{m}$ is decomposed generic. Then the expectation (4.6) holds true.*

REMARK 4.10. In the case when (G, X) gives rise to a compact Shimura variety of Harris–Taylor type, Theorem 4.9 was first proved by Boyer [**22**]. His argument uses the canonical integral models of these Shimura varieties. In this setting, Boyer also goes *beyond genericity*, in the following sense. Given the eigenvalues (with multiplicity) $\alpha_1, \ldots, \alpha_n$ of $\bar{\rho}_\mathfrak{m}(\mathrm{Frob}_\mathfrak{p})$, with $\mathfrak{p} \mid p$ the relevant prime of F[12], one can define a "defect" that measures how far $\bar{\rho}_\mathfrak{m}$ is from being generic at $\mathfrak{p}$. Concretely, set $\delta_\mathfrak{p}(\mathfrak{m})$ to be equal to the length of the maximal chain of eigenvalues where the successive terms have ratio equal to $|\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{m}_{F_\mathfrak{p}}|$. Boyer shows that the cohomology groups $H_{(c)}^i(\mathrm{Sh}_K(\mathbb{C}), \mathbb{F}_\ell)_\mathfrak{m}$ are non-zero at most in the range $[d - \delta_\mathfrak{p}(\mathfrak{m}), d + \delta_\mathfrak{p}(\mathfrak{m})]$. As noted by both Emerton and Koshikawa, such a result is consistent with Arthur's conjectures on the cohomology of Shimura varieties with $\mathbb{C}$-coefficients and points towards a modulo ℓ analogue of these conjectures.

REMARK 4.11. Let $p \neq \ell$ be a prime that splits completely in F, that is unramified for the Shimura datum (G, X), and such that $K_p \subset G(\mathbb{Q}_p)$ is the hyperspecial compact open subgroup. Then the genericity condition can also be formulated purely locally, using the spherical Hecke algebra at p. This is the approach taken in [**83**]. Indeed, the spherical Hecke algebra $\mathcal{H}_p$ for $G(\mathbb{Q}_p)$ is a product of spherical Hecke algebras for general linear groups. For each system of Hecke eigenvalues $\mathfrak{m}_p \subset \mathcal{H}_p$, the associated local L-parameter is a product of unramified L-parameters valued in general linear groups. The notion of generic as in the first part of Definition 4.6 makes sense and Koshikawa establishes the inequalities in (4.6) after localizing at a maximal ideal $\mathfrak{m}_p$ with generic L-parameter.

For simplicity, let us first discuss the proof of Theorem 4.9 in the case of compact Shimura varieties. The two main approaches that work in this generality are due

---

[12]In this special case, one does not have to impose the condition that p splits completely in F, and it suffices to have genericity at one prime $\mathfrak{p} \mid p$.

to [**35**] and [**83**] and they are, in some sense, complementary. They both rely on the Hodge–Tate period morphism and on the intuition that the generic part of the cohomology of a PEL type A Shimura variety at a completely split prime should come only from the ordinary locus. The geometry of the Hodge–Tate period morphism is used to control the non-ordinary Newton strata, showing that the generic part of their cohomology vanishes.

At this point, the approaches diverge. The best way to think about their difference is in terms of the infinite-level Mantovan product formula, which describes a given Newton stratum as a (twisted) product of an Igusa variety and a Rapoport–Zink space. The original approach due to [**35**] controls the cohomology of Igusa varieties using global methods, particularly the computation of their cohomology via the trace formula. The later approach due to [**83**] controls the cohomology of Rapoport–Zink spaces via Fargues–Scholze [**54**] and its comparison with the classical construction of the local Langlands correspondence for GL_n due to Harris–Taylor [**70**].

The original approach starts by analyzing the $\mathbb{T}$-equivariant diagram

(4.7)
$$
\begin{array}{ccc}
 & \mathcal{S}_{K^p} & \\
 \swarrow & & \searrow^{\pi_{\mathrm{HT}}} \\
 \mathcal{S}_{K^p K_p} & & \mathscr{F}\ell_G.
\end{array}
$$

The standard comparison theorems between various cohomology theories allow us to identify $H^*(\mathrm{Sh}_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}}$ with $H^*(\mathcal{S}_K, \mathbb{F}_\ell)_{\mathfrak{m}}$. The arrow on the left hand side of (4.7) is a K_p-torsor, so the Hochschild–Serre spectral sequence allows us to recover $H^*(\mathcal{S}_K, \mathbb{F}_\ell)_{\mathfrak{m}}$ from $H^*(\mathcal{S}_{K^p}, \mathbb{F}_\ell)_{\mathfrak{m}}$. The idea is now to compute $H^*(\mathcal{S}_{K^p}, \mathbb{F}_\ell)_{\mathfrak{m}}$ in two stages: first understand the complex of sheaves $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ on $\mathscr{F}\ell$, then compute the total cohomology using the Leray–Serre spectral sequence. In the decomposed generic case, the structure of $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ is particularly simple.

(1) Firstly, $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ behaves like a perverse sheaf on $\mathscr{F}\ell_G$ (up to shift). This is because π_{HT} is simultaneously *affinoid*, as explained in Theorem 2.9 and the subsequent discussion, and *partially proper*, because the Shimura varieties were assumed to be compact. In particular, the restriction of $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ to a highest-dimensional stratum in its support is concentrated in one degree. By the computation of the fibres of π_{HT}, this implies that the cohomology groups $R\Gamma(\mathfrak{Ig}^b, \mathbb{Z}_\ell)_{\mathfrak{m}}$ are concentrated in one degree and torsion-free.

(2) Secondly, whenever the group $G_b(\mathbb{Q}_p)$ acting on $\mathfrak{Ig}^b$ comes from a non-quasi-split inner form, the localisation $R\Gamma(\mathfrak{Ig}^b, \mathbb{Q}_\ell)_{\mathfrak{m}}$ vanishes. Recall from § 4.1 that we have an isomorphism $R\Gamma(\mathfrak{Ig}^b, \mathbb{Q}_\ell)_{\mathfrak{m}} \simeq R\Gamma(\mathrm{Ig}^b, \mathbb{Q}_\ell)_{\mathfrak{m}}$. The computation of $R\Gamma(\mathrm{Ig}^b, \mathbb{Q}_\ell)_{\mathfrak{m}}$, at least at the level of the Grothendieck group, can be done using the trace formula method pioneered in [**70**, Ch.V] and [**128**], which is particularly well-suited to Shimura varieties of PEL type A (at least when working with a unitary group defined over a totally real field $F^+ \neq \mathbb{Q}$).

(3) Finally, the condition that p splits completely in F guarantees that the only Newton stratum for which G_b is quasi-split is the ordinary one.

Therefore, the hypotheses of Theorem 4.9 guarantee that $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ is as simple as possible - it is supported in one degree on a zero-dimensional stratum! The degree in which is concentrated is in fact equal to d. The Leray–Serre spectral sequence computing $H^*(\mathcal{S}_{K^p}, \mathbb{F}_\ell)_{\mathfrak{m}}$ degenerates right away and the resulting cohomology is also concentrated in degree d. The same result can then also be deduced for $H^*(\mathcal{S}_K, \mathbb{F}_\ell)_{\mathfrak{m}}$ by analyzing the Hochschild–Serre spectral sequence.

REMARK 4.12. The intuition that $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ behaves like a perverse sheaf (up to shift) can be made precise if we instead work with the morphism of small v-stacks

$$\pi_{\mathrm{HT}/K_p} : \mathcal{S}_{K^p K_p} \to \mathscr{F}\ell_G/\underline{K_p}.$$

One can define an ad hoc perverse t-structure on $\mathscr{F}\ell_G/\underline{K_p}$ with respect to its Newton stratification and show that $(R\pi_{\mathrm{HT}/K_p*}\mathbb{F}_\ell)_{\mathfrak{m}}[d]$ is in the heart of this t-structure. The perverse t-structure on $\mathscr{F}\ell_G/\underline{K_p}$ is defined by glueing the shifted standard t-structures $(D^{\leq-\delta_b}, D^{\geq-\delta_b})$ on each Newton stratum $\mathscr{F}\ell_G^b/\underline{K_p}$, with δ_b equal to the dimension of the stratum. In fact, this perverse t-structure is pulled back from a perverse t-structure on Bun_G under the ℓ-cohomologically smooth morphism $\mathscr{F}\ell_G/\underline{K_p} \to \mathrm{Bun}_G$ (see [65] for the definition of the t-structure on Bun_G and [67, Prop. 2.10] for ℓ-cohomological smoothness). One then obtains semi-perversity of $(R\pi_{\mathrm{HT}/K_p*}\mathbb{F}_\ell)_{\mathfrak{m}}[d]$ in one direction from the affineness of Igusa varieties and Artin vanishing and in the other direction from Verdier duality, which uses properness of the Shimura varieties. We thank Matteo Tamiozzo for explaining the details of this argument to us.

We now explain the more recent approach due to [83]. In this case, one first proves a vanishing result for the generic part of the cohomology of local Shimura varieties. Let $L/\mathbb{Q}_p$ be a finite extension. Let (G, b, μ) be a local Shimura datum, with G/L a group of the form $\prod_{i \in I} \mathrm{GL}_{n_i}$ for some finite index set I. We set $K_L := \prod_{i \in I} \mathrm{GL}_{n_i}(\mathcal{O}_L) \subset G(L)$ and we consider the associated Rapoport–Zink space $\mathcal{M}_{K_L}$ at level K_L. We consider the spherical $\mathbb{F}_\ell$-Hecke algebra $\mathcal{H}_L$ acting on $H_c^*(\mathcal{M}_{K_L}, \mathbb{F}_\ell)$ with $\ell \neq p$. To a maximal ideal $\mathfrak{m}_L \subset \mathcal{H}_L$ we have an associated unramified Galois representation $\bar{\rho}_{\mathfrak{m}_L} \to \prod_{i \in I} \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_\ell)$. We also let G_b denote the twisted Levi subgroup of G determined by the element b.

THEOREM 4.13 ([83]). *Let $\mathfrak{m}_L \subset \mathcal{H}_L$ be a maximal ideal with associated Galois representation $\bar{\rho}_{\mathfrak{m}_L}$. Assume that $\bar{\rho}_{\mathfrak{m}_L}$ is generic as in Definition 4.6 and that G_b is not quasi-split. Then $H_c^*(\mathcal{M}_{K_L}, \mathbb{F}_\ell)_{\mathfrak{m}_L} = 0$*

The proof of Theorem 4.13 relies on two steps. Firstly, Koshikawa proves that, if G_b is not quasi-split and if π is an irreducible smooth $\overline{\mathbb{F}}_\ell$-representation of $G_b(L)$, then the semi-simple L-parameter φ_π associated to it by Fargues–Scholze cannot be generic. This uses the compatibility between the L-parameters constructed by Fargues–Scholze and the ones constructed via classical Jacquet–Langlands and local Langlands; this compatibility holds up to semi-simplification in the case of GL_n by [68]. Secondly, Koshikawa uses the action of the spectral Bernstein center to show that, for any irreducible subquotient π of $H_c^*(\mathcal{M}_{K_L}, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}_L}$, the Fargues–Scholze parameter satisfies $\varphi_\pi = \bar{\rho}_{\mathfrak{m}_L}$.

Let us return to the global perspective of a Shimura variety of PEL type A and assume that p is an unramified prime for the Shimura datum that splits completely in the underlying CM field F. Then the only Newton stratum for which the group

G_b is quasi-split is the ordinary one. Assume that $\mathfrak{m} \subset \mathbb{T}$ is a maximal ideal in the support of $H^*(\mathrm{Sh}_K(\mathbb{C}), \mathbb{F}_\ell)$ such that $\bar{\rho}_\mathfrak{m}$ is decomposed generic. By the cohomological version of the Mantovan product formula, cf. [83, Thm. 7.1], and by Theorem 4.13, the only contribution to $H^*(\mathcal{S}_{K^p}, \mathbb{F}_\ell)_\mathfrak{m}$ comes from the ordinary locus. At this point, Theorem 4.9 follows from the perversity of $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_\mathfrak{m}[d]$ (in the sense used originally in [35], or in the sense of Remark 4.12).

For the non-compact case of Theorem 4.9, the key conceptual difference is that one must weaken perversity to semi-perversity. For example, if one considers the restriction of the Hodge–Tate period morphism to the good reduction locus

$$\pi_{\mathrm{HT}}^\circ : \mathcal{S}_{K^p}^\circ \to \mathscr{F}\ell_G,$$

it is shown in [34, §4.6] that $(R\pi_{\mathrm{HT}*}^\circ\mathbb{F}_\ell)_\mathfrak{m}[d]$ is, in some sense, semi-perverse. This uses the computation of the fibers of the Hodge–Tate period morphism for a toroidal compactification of the Shimura variety. Such a result can again be made precise as in Remark 4.12 by showing that $(R\pi_{\mathrm{HT}/K_p*}^\circ\mathbb{F}_\ell)_\mathfrak{m}[d]$ is in ${}^pD^{\geq 0}$ for the perverse t-structure on $\mathscr{F}\ell_G/\underline{K}_p$. (There is also a dual result, which applies to $(R\pi_{\mathrm{HT}/K_p!}^\circ\mathbb{F}_\ell)_\mathfrak{m}[d]$ and shows that this is in ${}^pD^{\leq 0}$. This dual result can be obtained from the computation of the fibers of the Hodge–Tate period morphism for the minimal compactification of the Shimura variety. These fibers are related to partial minimal compactifications of Igusa varieties, which are affine and therefore satisfy Artin vanishing.)

In the original strategy of [34], once one has semi-perversity, one still needs to control the boundary of Shimura variety. This is done via an intricate induction argument; along the way, one constructs Galois representations attached to torsion classes occurring in the cohomology of Igusa varieties. In the more recent strategy of [83], semi-perversity together with Theorem 4.13 are essentially all that is needed. The proof of Theorem 4.9 in the non-compact case is completed in [112] by extending semi-perversity to more general PEL cases.

Several different generalizations of Theorem 4.9 are now emerging. Let us mention the upcoming work of Hamann–Lee [66], who work with a prime that does not necessarily split completely in F and who adapt their method to more general reductive groups under an axiomatic set-up. The key assumption in their axiomatic set-up is the compatibility between the work of Fargues–Scholze and more classical approaches to the local Langlands correspondence. This is known, for example, for GSp_4 and for odd unitary groups. The key new ingredient in their work is the theory of geometric Eisenstein series over the Fargues–Fontaine curve developed in [65] in the case of the Borel subgroup. In the cases considered by Hamann–Lee, only the so-called unramified elements in $B(G_{\mathbb{Q}_p})_{\mathrm{un}} \subset B(G_{\mathbb{Q}_p})$ contribute to the cohomology of the Shimura variety and, moreover, the contribution of each $b \in B(G_{\mathbb{Q}_p})_{\mathrm{un}}$ can be controlled with the theory of geometric Eisenstein series.

REMARK 4.14. Recall the Igusa stack $\bar{\pi}_{\mathrm{HT}} : \mathrm{Igs} \to \mathrm{Bun}_G$ constructed in [143] and discussed in Remark 4.5. One advantage of the original approach to Theorem 4.9 via Igusa varieties is that it describes the sheaves $(R\bar{\pi}_{\mathrm{HT}!}\mathbb{F}_\ell)_\mathfrak{m}$ on Bun_G, at least under the assumption that p splits completely in F. These sheaves are expected to satisfy a form of local-global compatibility for the conjectural equivalence of categories of Fargues–Scholze. In fact, with the methods introduced in [65], one can investigate this question in a slightly more general setting. The cohomology of Igusa varieties is admissible, therefore the sheaves $R\bar{\pi}_{\mathrm{HT}!}\mathbb{F}_\ell$ are ULA objects

in $D(\mathrm{Bun}_G, \mathbb{F}_\ell)$, cf. [**54**, Thm. 1.5.1]. Given a semi-simple L-parameter ϕ which satisfies an appropriate genericity condition, one can use the spectral action of Fargues–Scholze to define the "localisation" $(R\bar{\pi}_{\mathrm{HT}!}\overline{\mathbb{F}}_\ell)_\phi$. The techniques of Hamann and Hamann–Lee (specifically [**66**, Prop. 4.15]) show that

$$(4.8) \qquad (R\bar{\pi}_{\mathrm{HT}!}\overline{\mathbb{F}}_\ell)_\phi \simeq \bigoplus_{b \in B(G_{\mathbb{Q}_p})_{\mathrm{un}}} R\Gamma_{c-\partial}(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)_\phi,$$

where the terms on the RHS of the decomposition represent cohomology with compact support towards the boundary of the partial minimal compactification of the Igusa varieties and they are supported on $[*/\widetilde{G}_b]$ for $b \in B(G_{\mathbb{Q}_p})_{\mathrm{un}}$. Furthermore, in the case when the Igusa stack agrees with its minimal compactification, these terms are all supported in one degree by the perversity result discussed in Remark 4.12.

4.3. Applications to locally symmetric spaces. In this subsection, we return to the case when F is an imaginary CM field and $G = \mathrm{GL}_n/F$. For $K \subset G(\mathbb{A}_F^\infty)$ a sufficiently small compact open subgroup, recall that Y_K is the associated locally symmetric space. In this subsection, we explain how Theorem 4.9 can be used to approach Conjecture 3.5 for the cohomology groups of Y_K. We then explain how to implement the Calegari–Geraghty method unconditionally over CM fields.

As we have seen in § 2.4, G can be realized as a Levi subgroup of a maximal parabolic subgroup of a quasi-split unitary group $\widetilde{G} = U_{2n}/F^+$ and $\mathrm{Res}_{F^+/\mathbb{Q}}\widetilde{G}$ admits a Shimura datum. For a neat compact open subgroup $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+}^\infty)$, we let $\mathrm{Sh}_{\widetilde{K}}$ denote the corresponding Shimura variety for (the restriction of scalars of) $\widetilde{G}$, considered in this subsection as a complex manifold. We let $\mathrm{Sh}_{\widetilde{K}}^{\mathrm{BS}}$ denote its Borel–Serre compactification, a real manifold with corners, and $\partial\,\mathrm{Sh}_{\widetilde{K}}^{\mathrm{BS}}$ denote the boundary of the Borel–Serre compactification.

We let S be a finite set of finite places of F that is stable under complex conjugation. This determines a finite set of finite places $\bar{S}$ of F^+. We have the usual abstract Hecke algebra $\mathbb{T}^S$ for G. We consider an analogous Hecke algebra $\widetilde{\mathbb{T}}^{\bar{S}}$ for $\widetilde{G}$ (although we only need to take the product of local spherical Hecke algebras at places in F^+ that split in F). There is an unnormalized Satake transform map $\mathcal{S} : \widetilde{\mathbb{T}}^{\bar{S}} \to \mathbb{T}^S$; see [**103**, §2] for its definition.

Recall from § 2.4 that the excision sequence for the boundary of the Borel–Serre compactification induces the $\widetilde{\mathbb{T}}^{\bar{S}}$-equivariant long exact sequence

$$(4.9) \quad \cdots \to H_c^i(\mathrm{Sh}_{\widetilde{K}}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H^i(\mathrm{Sh}_{\widetilde{K}}, \mathbb{Z}/\ell^m\mathbb{Z}) \to$$
$$H^i(\partial\,\mathrm{Sh}_{\widetilde{K}}^{\mathrm{BS}}, \mathbb{Z}/\ell^m\mathbb{Z}) \to H_c^{i+1}(\mathrm{Sh}_{\widetilde{K}}, \mathbb{Z}/\ell^m\mathbb{Z}) \to \cdots.$$

Assume that a maximal ideal $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbb{T}}^{\bar{S}}$ satisfies an appropriate genericity condition at an auxiliary prime $p \neq \ell$. Theorem 4.9 applies to the Shimura varieties $\mathrm{Sh}_{\widetilde{K}}$[13] and can be used to simplify the long exact sequence (4.9) after localisation at $\widetilde{\mathfrak{m}}$. In degree $d = \dim_{\mathbb{C}}\mathrm{Sh}_{\widetilde{K}}$, we obtain the following diagram:

$$(4.10) \qquad H^d(\mathrm{Sh}_{\widetilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \hookleftarrow H^d(\mathrm{Sh}_{\widetilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}} \twoheadrightarrow H^d(\partial\,\mathrm{Sh}_{\widetilde{K}}^{\mathrm{BS}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}},$$

---

[13]More precisely, one considers a variant of $\widetilde{G}$ that is a unitary similitude group, giving rise to Shimura varieties of PEL type A. The difference is only on the level of geometric connected components.

which is used to approach Conjecture 3.5. Very roughly, the cohomology groups on the LHS can be described in terms of (essentially) conjugate self-dual, regular C-algebraic cuspidal automorphic representations of GL_r/F with various $r \leq 2n$[14]. Their Galois representations are known to satisfy local-global compatibility up to Frobenius semi-simplification at all places, including at the places $v \mid \ell$ of F. This ultimately gives control over the cohomology group on the RHS of (4.10).

On the other hand, if $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m})$, the cohomology groups $H^i(Y_K, \mathbb{Z}/\ell^m\mathbb{Z})_{\mathfrak{m}}$ for $i = 0, \ldots d-1$ can be shown to contribute to the cohomology group on the RHS of (4.10). This is the basic strategy employed in [2] and later refined via P-ordinary parts in [33]. The strategy requires there to be enough primes of F^+ dividing ℓ: we fix one such prime $\bar{v} \mid \ell$ where we want to prove local-global compatibility, and we use the primes $\bar{v}' \mid \ell$ with $\bar{v}' \neq \bar{v}$ to create congruences and shift to cohomological degree d on the boundary of the Borel–Serre compactification. (This is the reason why Conjecture 3.5 seems to be most subtle when F is an imaginary quadratic field – there are not enough auxiliary primes!)

REMARK 4.15. There can be genuine torsion classes in the cohomology group $H^d(\partial\mathrm{Sh}^{\mathrm{BS}}_{\widetilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}}$. The diagram (4.10) implies that these lift to characteristic 0 classes in $H^d(\mathrm{Sh}_{\widetilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. By design, the proofs of Conjecture 3.5 using the method outlined above go through these torsion classes.

At this point, we have access to Conjectures 3.4 and 3.5, at least under some technical assumptions. In order to implement the Calegari–Geraghty method unconditionally as sketched in § 3.2, we also need to access Conjecture 3.8. This seems extremely hard in this generality. However, a modification of the Calegari–Geraghty method due to Khare–Thorne [76] works in certain settings by assuming only Conjecture 3.10. This predicts the vanishing of cohomology with $\mathbb{Q}_\ell$-coefficients, a result that can be proved when F is a CM field. One of the main challenges in [2] was to make this insight of Khare–Thorne compatible with other techniques in automorphy lifting, which rely on reduction modulo ℓ. This challenge was resolved by considering reduction modulo ℓ from a derived perspective, leading to an unconditional implementation of the Calegari–Geraghty method in arbitrary dimensions.

We end by briefly mentioning some applications of this circle of ideas: a proof of the Sato–Tate conjecture for elliptic curves over CM fields, cf. [2], a proof that all elliptic curves over many imaginary fields are modular, cf. [3, 33], and proofs of many new cases of the Ramanujan conjecture for GL_2 over CM fields, cf. [2] and further work in progress of Boxer–Calegari–Gee–Newton–Thorne.

References

1. Lambert A'Campo, *Rigidity of automorphic Galois representations over CM fields*, 2022.
2. Patrick B. Allen, Frank Calegari, Ana Caraiani, Toby Gee, David Helm, Bao V. Le Hung, James Newton, Peter Scholze, Richard Taylor, and Jack A. Thorne, *Potential automorphy over CM fields*, Ann. of Math. (2) **197** (2023), no. 3, 897–1113. MR 4564261
3. Patrick B. Allen, Chandrashekhar Khare, and Jack A. Thorne, *Modularity of $\mathrm{GL}_2(\mathbb{F}_p)$-representations over CM fields*, Camb. J. Math. **11** (2023), no. 1, 1–158. MR 4600542
4. Patrick B. Allen and James Newton, *Monodromy for some rank two Galois representations over CM fields*, Doc. Math. **25** (2020), 2487–2506. MR 4213130

---

[14]In order to make this work in reality, one needs to introduce a non-trivial local system $\mathcal{V}_{\widetilde{\lambda}}$ on $\mathrm{Sh}_{\widetilde{K}}$, which can eliminate contributions from non-conjugate self-dual automorphic forms by imposing a central character constraint.

5. Johannes Anschütz, *Reductive group schemes over the Fargues-Fontaine curve*, Math. Ann. **374** (2019), no. 3-4, 1219–1260. MR 3985110

6. James Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013, Orthogonal and symplectic groups. MR 3135650

7. Avner Ash, *Galois representations and cohomology of* $\mathrm{GL}(n, \mathbf{Z})$, Séminaire de Théorie des Nombres, Paris, 1989–90, Progr. Math., vol. 102, Birkhäuser Boston, Boston, MA, 1992, pp. 9–22. MR 1476725

8. ———, *Galois representations attached to mod p cohomology of* $\mathrm{GL}(n, \mathbf{Z})$, Duke Math. J. **65** (1992), no. 2, 235–255. MR 1150586

9. W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528. MR 216035

10. Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609. MR 3152941

11. Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 1, 29–98. MR 2827723

12. Joël Bellaïche and Gaëtan Chenevier, *The sign of Galois representations attached to automorphic forms for unitary groups*, Compos. Math. **147** (2011), no. 5, 1337–1352. MR 2834723

13. Nicolas Bergeron and Akshay Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, J. Inst. Math. Jussieu **12** (2013), no. 2, 391–447. MR 3028790

14. Bhargav Bhatt, Ana Caraiani, Kiran S. Kedlaya, and Jared Weinstein, *Perfectoid spaces*, Mathematical Surveys and Monographs, vol. 242, American Mathematical Society, Providence, RI, 2019, Lectures from the 2017 Arizona Winter School, held in Tucson, AZ, March 11–17, Edited and with a preface by Bryden Cais, With an introduction by Peter Scholze. MR 3970252

15. Don Blasius and Dinakar Ramakrishnan, *Maass forms and Galois representations*, Galois groups over $\mathbf{Q}$ (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York, 1989, pp. 33–77. MR 1012167

16. Don Blasius and Jonathan D. Rogawski, *Zeta functions of Shimura varieties*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 525–571. MR 1265563

17. A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491. MR 387495

18. A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000. MR 1721403

19. George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni, *Abelian surfaces over totally real fields are potentially modular*, Publ. Math. Inst. Hautes Études Sci. **134** (2021), 153–501. MR 4349242

20. George Boxer and Vincent Pilloni, *Higher Hida and Coleman theories on the modular curve*, Épijournal Géom. Algébrique **6** (2022), Art. 16, 33. MR 4482376

21. George Andrew Boxer, *Torsion in the Coherent Cohomology of Shimura Varieties and Galois Representations*, ProQuest LLC, Ann Arbor, MI, 2015, Thesis (Ph.D.)–Harvard University. MR 3450451

22. Pascal Boyer, *Sur la torsion dans la cohomologie des variétés de Shimura de Kottwitz-Harris-Taylor*, J. Inst. Math. Jussieu **18** (2019), no. 3, 499–517. MR 3936639

23. Kevin Buzzard and Toby Gee, *The conjectural connections between automorphic representations and Galois representations*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 135–187. MR 3444225

24. Frank Calegari, *Motives and L-functions*, Current developments in mathematics 2018, Int. Press, Somerville, MA, [2020] ©2020, pp. 57–123. MR 4363376

25. Frank Calegari and David Geraghty, *Modularity lifting beyond the Taylor-Wiles method*, Invent. Math. **211** (2018), no. 1, 297–433. MR 3742760

26. _____, *Minimal modularity lifting for nonregular symplectic representations*, Duke Math. J. **169** (2020), no. 5, 801–896, With an appendix by Calegari, Geraghty and Michael Harris. MR 4079417

27. Frank Calegari and Akshay Venkatesh, *A torsion Jacquet-Langlands correspondence*, Astérisque (2019), no. 409, x+226. MR 3961523

28. Ana Caraiani, *The cohomology of Shimura varieties with torsion coefficients*, to appear in the proceedings of the ICM 2022.

29. _____, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413. MR 2972460

30. Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the p-adic local Langlands correspondence*, Camb. J. Math. **4** (2016), no. 2, 197–287. MR 3529394

31. Ana Caraiani, Daniel R. Gulotta, Chi-Yun Hsu, Christian Johansson, Lucia Mocz, Emanuel Reinecke, and Sheng-Chi Shih, *Shimura varieties at level $\Gamma_1(p^\infty)$ and Galois representations*, Compos. Math. **156** (2020), no. 6, 1152–1230. MR 4108870

32. Ana Caraiani and Bao V. Le Hung, *On the image of complex conjugation in certain Galois representations*, Compos. Math. **152** (2016), no. 7, 1476–1488. MR 3530448

33. Ana Caraiani and James Newton, *On the modularity of elliptic curves over imaginary quadratic fields*, Preprint, available online at https://arxiv.org/abs/2301.10509 (2023).

34. Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of non-compact unitary Shimura varieties*, `https://arxiv.org/abs/1909.01898`.

35. _____, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) **186** (2017), no. 3, 649–766. MR 3702677

36. Ana Caraiani and Matteo Tamiozzo, *On the étale cohomology of Hilbert modular varieties with torsion coefficients*, 2021.

37. Gaëtan Chenevier, *The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 221–285. MR 3444227

38. Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations, II*, Camb. J. Math. **1** (2013), no. 1, 53–73. MR 3272052

39. Laurent Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159. MR 1044819 (91k:11042)

40. _____, *Représentations galoisiennes associées aux représentations automorphes autoduales de* GL(n), Inst. Hautes Études Sci. Publ. Math. (1991), no. 73, 97–145. MR 1114211

41. _____, *Nombre de points des variétés de Shimura sur un corps fini (d'après R. Kottwitz)*, no. 216, 1993, Séminaire Bourbaki, Vol. 1992/93, pp. Exp. No. 766, 4, 121–149. MR 1246396

42. _____, *Purity reigns supreme*, Int. Math. Res. Not. IMRN (2013), no. 2, 328–346. MR 3010691

43. Laurent Clozel, Michael Harris, and Jean-Pierre Labesse, *Construction of automorphic Galois representations, I*, On the stabilization of the trace formula, Stab. Trace Formula Shimura Var. Arith. Appl., vol. 1, Int. Press, Somerville, MA, 2011, pp. 497–527. MR 2856383

44. Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR 2470687

45. Pierre Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289. MR 546620

46. Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975). MR 379379

47. Mladen Dimitrov, *Galois representations modulo p and cohomology of Hilbert modular varieties*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 4, 505–551. MR 2172950

48. M. Emerton, T. Gee, and E. Hellmann, *An introduction to the categorical p-adic Langlands program*, in these proceedings.

49. Matthew Emerton, *Completed cohomology and the p-adic Langlands program*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 319–342. MR 3728617

50. Matthew Emerton and David Helm, *The local Langlands correspondence for* GL_n *in families*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 655–722. MR 3250061

51. Najmuddin Fakhruddin, Chandrashekhar Khare, and Stefan Patrikis, *Relative deformation theory, relative Selmer groups, and lifting irreducible Galois representations*, Duke Math. J. **170** (2021), no. 16, 3505–3599. MR 4332672

52. L. Fargues and P. Scholze, *The Langlands program and the moduli of bundles on the curve*, in these proceedings.

53. Laurent Fargues, *G-torseurs en théorie de Hodge p-adique*, Compos. Math. **156** (2020), no. 10, 2076–2110. MR 4179595

54. Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, available online at https://arxiv.org/abs/2102.13459 (2021).

55. Tony Feng and Michael Harris, *Derived structures in the Langlands correspondence*, in these proceedings.

56. Jessica Fintzen and Sug Woo Shin, *Congruences of algebraic automorphic forms and super-cuspidal representations*, Camb. J. Math. **9** (2021), no. 2, 351–429. MR 4325284

57. Jens Franke, *Harmonic analysis in weighted* L_2*-spaces*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 2, 181–279. MR 1603257

58. Toby Gee, *Modularity lifting theorems*, Essential Number Theory **1** (2022), no. 1, 73–126.

59. Toby Gee and James Newton, *Patching and the completed homology of locally symmetric spaces*, J. Inst. Math. Jussieu **21** (2022), no. 2, 395–458. MR 4386819

60. Alain Genestier and Bao Châu Ngô, *Lectures on Shimura varieties*, Autour des motifs—École d'été Franco-Asiatique de Géométrie Algébrique et de Théorie des Nombres/Asian-French Summer School on Algebraic Geometry and Number Theory. Volume I, Panor. Synthèses, vol. 29, Soc. Math. France, Paris, 2009, pp. 187–236. MR 2730658

61. Wushi Goldring and Jean-Stefan Koskivirta, *Strata Hasse invariants, Hecke algebras and Galois representations*, Invent. Math. **217** (2019), no. 3, 887–984. MR 3989256

62. Mark Goresky, *Compactifications and cohomology of modular varieties*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 551–582. MR 2192016

63. Thomas J. Haines, *The stable Bernstein center and test functions for Shimura varieties*, Automorphic forms and Galois representations. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 415, Cambridge Univ. Press, Cambridge, 2014, pp. 118–186. MR 3444233

64. Paul Hamacher and Wansu Kim, *l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients*, Math. Ann. **375** (2019), no. 3-4, 973–1044. MR 4023369

65. Linus Hamann, *Geometric Eisenstein series, intertwining operators, and Shin's averaging formula*, 2023.

66. Linus Hamann and Si-Ying Lee, *Torsion vanishing for some Shimura varieties*, in preparation (2023).

67. David Hansen, *On the supercuspidal cohomology of basic local Shimura varieties*, available online at http://davidrenshawhansen.net/middlev2.pdf (2022).

68. David Hansen, Tasho Kaletha, and Jared Weinstein, *On the Kottwitz conjecture for local shtuka spaces*, Forum Math. Pi **10** (2022), Paper No. e13, 79. MR 4430954

69. Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne, *On the rigid cohomology of certain Shimura varieties*, Res. Math. Sci. **3** (2016), Paper No. 37, 308.

70. Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802

71. Sean Howe, *The spectral p-adic Jacquet-Langlands correspondence and a question of Serre*, Compos. Math. **158** (2022), no. 2, 245–286. MR 4406784

72. Frazer Jarvis, *On Galois representations associated to Hilbert modular forms*, J. Reine Angew. Math. **491** (1997), 199–216. MR 1476093

73. Christian Johansson and Jack Thorne, *On subquotients of the étale cohomology of Shimura varieties*, Shimura Varieties, London Mathematical Society Lecture Note Series, vol. 457, Cambridge University Press, 2020, pp. 306–334.

74. Tasho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White, *Endoscopic classification of representations: Inner forms of unitary groups*, preprint (2014), `https://arxiv.org/abs/1409.3731`.

75. Nicholas M. Katz, *p-adic properties of modular schemes and modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 69–190. Lecture Notes in Mathematics, Vol. 350. MR 0447119

76. Chandrashekhar B. Khare and Jack A. Thorne, *Potential automorphy and the Leopoldt conjecture*, Amer. J. Math. **139** (2017), no. 5, 1205–1273. MR 3702498

77. Wansu Kim and Keerthi Madapusi Pera, *2-adic integral canonical models*, Forum Math. Sigma **4** (2016), e28, 34.

78. M. Kisin, S. W. Shin, and Y. Zhu, *Cohomology of certain Shimura varieties of abelian type*, in progress.

79. _____, *The stable trace formula for Shimura varieties of abelian type*, `https://arxiv.org/abs/2110.05381`.

80. Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546. MR 2373358

81. _____, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.

82. _____, mod p *points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), no. 3, 819–914.

83. Teruhisa Koshikawa, *On the generic part of the cohomology of local and global Shimura varieties*, available online at https://arxiv.org/pdf/2106.10602.pdf (2021).

84. R. E. Kottwitz, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and *L*-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 161–209. MR 1044820 (92b:11038)

85. _____, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444. MR 1124982 (93a:11053)

86. Robert E. Kottwitz, *On the λ-adic representations associated to some simple Shimura varieties*, Invent. Math. **108** (1992), no. 3, 653–665. MR 1163241

87. Robert E. Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190. MR 1687096

88. Arno Kret and Sug Woo Shin, *Galois representations for general symplectic groups*, `https://arxiv.org/abs/2010.08408`.

89. _____, *Galois representations for general symplectic groups*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 1, 75–152. MR 4556781

90. J.-P. Labesse, *Changement de base CM et séries discrètes*, On the stabilization of the trace formula, Stab. Trace Formula Shimura Var. Arith. Appl., vol. 1, Int. Press, Somerville, MA, 2011, pp. 429–470. MR 2856380

91. Kai-Wen Lan and Benoît Stroh, *Compactifications of subschemes of integral models of Shimura varieties*, Forum Math. Sigma **6** (2018), Paper No. e18, 105. MR 3859178

92. Kai-Wen Lan and Junecue Suh, *Vanishing theorems for torsion automorphic sheaves on compact PEL-type Shimura varieties*, Duke Math. J. **161** (2012), no. 6, 1113–1170. MR 2913102

93. _____, *Vanishing theorems for torsion automorphic sheaves on general PEL-type Shimura varieties*, Adv. Math. **242** (2013), 228–286. MR 3055995

94. M. Larsen, *On the conjugacy of element-conjugate homomorphisms. II*, Quart. J. Math. Oxford Ser. (2) **47** (1996), no. 185, 73–85. MR 1380951 (97b:20070)

95. Michael Larsen, *On the conjugacy of element-conjugate homomorphisms*, Israel J. Math. **88** (1994), no. 1-3, 253–277. MR 1303498 (95k:20073)

96. Dong Uk Lee, *Galois gerbs and Lefschetz number formula for Shimura varieties of Hodge type*, `https://arxiv.org/abs/1801.03057`.

97. Elena Mantovan, *On certain unitary group Shimura varieties*, no. 291, 2004, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, pp. 201–331. MR 2074715

98. _____, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. J. **129** (2005), no. 3, 573–610. MR 2169874

99. B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 33–186 (1978), With an appendix by Mazur and M. Rapoport. MR 488287

100. James S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The zeta functions of Picard modular surfaces, Univ. Montréal, Montreal, QC, 1992, pp. 151–253. MR 1155229

101. Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248. MR 3338302

102. Sophie Morel, *Construction de représentations galoisiennes de torsion [d'après Peter Scholze]*, Astérisque (2016), no. 380, Séminaire Bourbaki. Vol. 2014/2015, Exp. No. 1102, 449–473. MR 3522182

103. James Newton and Jack A. Thorne, *Torsion Galois representations over CM fields and Hecke algebras in the derived category*, Forum Math. Sigma **4** (2016), Paper No. e21, 88. MR 3528275

104. Bao Châu Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 111, 1–169. MR 2653248

105. Stefan Patrikis, *On the sign of regular algebraic polarizable automorphic representations*, Math. Ann. **362** (2015), no. 1-2, 147–171. MR 3343873

106. V. Pilloni, *Higher coherent cohomology and p-adic modular forms of singular weights*, Duke Math. J. **169** (2020), no. 9, 1647–1807. MR 4105535

107. Vincent Pilloni and Benoît Stroh, *Cohomologie cohérente et représentations Galoisiennes*, Ann. Math. Qué. **40** (2016), no. 1, 167–202. MR 3512528

108. M. Rapoport, *On the bad reduction of Shimura varieties*, Automorphic forms, Shimura varieties, and *L*-functions, Vol. II (Ann Arbor, MI, 1988), Perspect. Math., vol. 11, Academic Press, Boston, MA, 1990, pp. 253–321. MR 1044832

109. M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181. MR 1411570

110. M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996. MR 1393439

111. Susana A. Salamanca-Riba, *On the unitary dual of real reductive Lie groups and the $A_g(\lambda)$ modules: the strongly regular case*, Duke Math. J. **96** (1999), no. 3, 521–546. MR 1671213 (2000a:22023)

112. Mafalda Santos, *On the generic part of the cohomology of non-compact PEL-type A Shimura varieties*, PhD Thesis, Imperial College London (2023).

113. Peter Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313. MR 3090258

114. _____, *Perfectoid spaces: a survey*, Current developments in mathematics 2012, Int. Press, Somerville, MA, 2013, pp. 193–227. MR 3204346

115. _____, *Perfectoid spaces and their applications*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 461–486. MR 3728623

116. _____, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) **182** (2015), no. 3, 945–1066. MR 3418533

117. _____, *Perfectoid Shimura varieties*, Jpn. J. Math. **11** (2016), no. 1, 15–32. MR 3510678

118. _____, *p-adic geometry*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 899–933. MR 3966748

119. Peter Scholze, *Etale cohomology of diamonds*, (2022).

120. Peter Scholze and Sug Woo Shin, *On the cohomology of compact unitary group Shimura varieties at ramified split places*, J. Amer. Math. Soc. **26** (2013), no. 1, 261–294.

121. Peter Scholze and Jared Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2013), no. 2, 145–237. MR 3272049

122. _____, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2013), no. 2, 145–237. MR 3272049

123. _____, *Berkeley lectures on p-adic geometry*, Annals of Mathematics Studies, vol. 207, Princeton University Press, Princeton, NJ, 2020.

124. _____, *Berkeley lectures on p-adic geometry*, Annals of Mathematics Studies, vol. 207, Princeton University Press, Princeton, NJ, 2020. MR 4446467

125. Xu Shen, *Perfectoid Shimura varieties of abelian type*, Int. Math. Res. Not. IMRN (2017), no. 21, 6599–6653. MR 3719474

126. S. W. Shin, *Points on Shimura varieties modulo primes*, to appear in the proceedings of the ICM 2022.

127. _____, *Weak transfer from classical groups to general linear groups*, `https://math.berkeley.edu/~swshin/weak_transfer.pdf`.

128. _____, *Counting points on Igusa varieties*, Duke Math. J. **146** (2009), no. 3, 509–568. MR 2484281

129. _____, *A stable trace formula for Igusa varieties*, J. Inst. Math. Jussieu **9** (2010), no. 4, 847–895. MR 2684263

130. _____, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741.

131. _____, *Construction of automorphic Galois representations: the self-dual case*, Shimura Varieties, London Mathematical Society Lecture Note Series, vol. 457, Cambridge University Press, 2020, pp. 209–250.

132. Claus Sorensen, *A patching lemma*, Shimura Varieties, London Mathematical Society Lecture Note Series, vol. 457, Cambridge University Press, 2020, pp. 297–305.

133. Olivier Taïbi, *Eigenvarieties for classical groups and complex conjugations in Galois representations*, Math. Res. Lett. **23** (2016), no. 4, 1167–1220. MR 3554506

134. Richard Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. **63** (1991), no. 2, 281–332. MR 1115109

135. _____, *The image of complex conjugation in l-adic representations associated to automorphic forms*, Algebra Number Theory **6** (2012), no. 3, 405–435. MR 2966704

136. Jack Thorne, *Beyond the Taylor–Wiles method*, Available online at https://www.dpmms.cam.ac.uk/ jat58/beyondtw.pdf.

137. Ila Kapur Varma, *On local-global compatibility for cuspidal regular algebraic automorphic representations of GLn*, ProQuest LLC, Ann Arbor, MI, 2015, Thesis (Ph.D.)–Princeton University. MR 3450147

138. Eva Viehmann, *On Newton strata in the B_{dR}^+-Grassmannian*, to appear in Duke Math. Journal (2023).

139. Marie-France Vignéras, *Correspondance de Langlands semi-simple pour* GL(n, F) *modulo* $\neq p$, Invent. Math. **144** (2001), no. 1, 177–223. MR 1821157

140. J.-L. Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Math. **105** (1997), no. 2, 153–236. MR 1440722

141. _____, *L'endoscopie tordue n'est pas si tordue*, Mem. Amer. Math. Soc. **194** (2008), no. 908, x+261. MR 2418405

142. Jared Weinstein, *Reciprocity laws and Galois representations: recent breakthroughs*, Bull. Amer. Math. Soc. (N.S.) **53** (2016), no. 1, 1–39. MR 3403079

143. Mingjia Zhang, *A PEL-type Igusa stack and the p-adic geometry of Shimura varieties*, PhD Thesis, University of Bonn (2023).

144. Xinwen Zhu, *A note on integral Satake isomorphisms*, `https://arxiv.org/abs/2005.13056`.

145. Yihang Zhu, *Introduction to the Langlands–Kottwitz method*, Shimura Varieties, London Mathematical Society Lecture Note Series, vol. 457, Cambridge University Press, 2020, pp. 115–150.

Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Department of Mathematics, Imperial College London, London SW7 2AZ, UK
Email address: `caraiani.ana@gmail.com`

Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA / Korea Institute for Advanced Study, Seoul 02455, Republic of Korea
Email address: `sug.woo.shin@berkeley.edu`