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THE GEOMETRIZATION OF LOCAL LANGLANDS
[after L. Fargues and P. Scholze]

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Introduction

The Langlands program is an intricate network of conjectures that touch on number theory, representation theory, harmonic analysis, and even parts of theoretical physics. At its heart lies the principle of reciprocity, or the *Langlands correspondence*, which can be thought of as a bridge that connects different mathematical worlds.

There are several flavors of the Langlands program: global and local, arithmetic and geometric. The arithmetic Langlands program takes place over (global) number fields, such as the field of rational numbers $\mathbb{Q}$, and over (local) p-adic fields, such as $\mathbb{Q}_p$, and can be traced back to the work of Euler, Legendre and Gauss on the law of quadratic reciprocity. A famous instance of the Langlands correspondence in the number field setting, pioneered in Wiles (1995) and Taylor and Wiles (1995), is the modularity of elliptic curves over $\mathbb{Q}$. This was the cornerstone to Wiles's celebrated proof of Fermat's last theorem.

There is a deep and fruitful analogy between the arithmetic of the integers, with the special role played by prime numbers, and the geometry of algebraic curves, where prime numbers are replaced by the points of the curve[(1)]. The latter is the setting of function fields, such as the field of meromorphic functions on a compact Riemann surface or the field of rational functions $\mathbb{F}_p(t)$ on the projective line over $\mathbb{F}_p$. A parallel set of conjectures and results about the Langlands correspondence has developed in the function field setting.

Note that, in the case of an algebraic curve, all the residue fields have the same characteristic and we can form the product of the curve with itself to obtain a surface. It is not clear how to do this with $\mathbb{Z}$ or even $\mathbb{Z}_p$, or what to take the product over. These properties lead to additional flexibility in the function field setting. This additional flexibility was exploited over several decades in breakthrough results on the global Langlands

[(1)]For example, this analogy inspired Weil to conjecture that the zeta functions of smooth projective curves (and, more generally, of smooth projective varieties) defined over finite fields behave, in many ways, like the Riemann zeta function. These became the celebrated Weil conjectures.

correspondence for function fields of smooth projective curves defined over finite fields, due to Drinfeld, L. Lafforgue and V. Lafforgue. In the setting of Riemann surfaces, the Langlands correspondence was upgraded to a much richer categorical equivalence, inspired in part by ideas coming from theoretical physics. A recent breakthrough in this setting is the proof of the categorical unramified geometric Langlands conjecture in Gaitsgory and Raskin (2024a), Arinkin, Beraldo, Campbell, et al. (2024), Campbell, L. Chen, Faergeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024), Arinkin, Beraldo, L. Chen, et al. (2024), and Gaitsgory and Raskin (2024b).

Traditionally, the arithmetic Langlands program has not been able to benefit from the flexibility available in other, more geometric settings. With the discovery of the Fargues–Fontaine curve and the development of p-adic geometry due to Scholze, this additional flexibility became a tantalizing possibility, at least in the local setting of p-adic fields, as foreseen in Scholze and Weinstein (2020) and Fargues (2025). In Fargues and Scholze (2024), this promise was realized, with a transformative effect on the local Langlands correspondence over p-adic fields and, more generally, on the Langlands program in the arithmetic setting.

Fargues–Scholze laid the foundations of the geometric Langlands program over p-adic fields, using the Fargues–Fontaine curve as a substitute for the algebraic curve featured in the function field setting. More precisely, they defined, studied, and ultimately connected geometric objects on each of the two sides of the local Langlands correspondence. On the often more mysterious automorphic side, the key geometric object is Bun_G, the moduli space of G-bundles on the Fargues–Fontaine curve. By introducing powerful new techniques and structures from the function field setting, Fargues–Scholze gave a completely general construction of semi-simple local Langlands parameters attached to irreducible smooth representations of p-adic groups. They also formulated a geometric version of the categorical local Langlands conjecture.

There are a number of excellent resources the reader can use to learn about the work of Fargues–Scholze and subsequent developments. For example, the extensive introduction to Fargues and Scholze (2024) gives an overview, as do the IHES lecture notes of Fargues and Scholze (2022) and the survey article of Imai (2024). The Eilenberg/Hausdorff lectures of Fargues (2024) give a lot of examples and historical motivation and, in particular, discuss the Jacobian criterion for smoothness in great detail. The Beijing lecture notes of Hansen (2025) study the categorical local Langlands conjecture further and discuss subsequent developments.

Because there are already so many resources available, this article is brief and relatively less technical than these references. The goal is merely to introduce the reader to the groundbreaking ideas of Fargues–Scholze, and to demonstrate how their ideas connected different strands of research within the Langlands program and led to the solution of long-standing problems. For simplicity, we focus below on the case of the geometrization of the local Langlands correspondence over p-adic fields, even though Fargues–Scholze also treat the case of local fields of equal characteristic.

Organization

This article is structured as follows. In Section 1, we formulate the more classical statement of the local Langlands correspondence. We discuss smooth representations of p-adic groups, L-parameters, and the refined local Langlands conjecture that parameterizes the members of individual L-packets. This culminates in Conjecture 1.15, which shows that the internal structure of L-packets already contains glimpses of a richer, more geometric picture. Note that this section does not assume any knowledge of p-adic geometry.

In Section 2, we discuss the Fargues–Fontaine curve in its many incarnations and the moduli stack Bun_G of G-bundles on it. This culminates in Theorem 2.11, which exhibits a certain category of sheaves $D(\mathrm{Bun}_G, \Lambda)$ as a geometrization of the derived category $D(G(E), \Lambda)$ of smooth representations of the p-adic group $G(E)$. The constructions in this section rely on p-adic geometry and on the formalism of condensed mathematics, but we try to keep the discussion light on technical details and instead convey intuition.

In Section 3, we explain the main ideas in the work of Fargues–Scholze on the geometrization of local Langlands correspondence and sketch their construction of semi-simple local Langlands parameters. In particular, we emphasize the role of the geometric Satake equivalence and the connection between excursion operators and the moduli stack of Langlands parameters.

In Section 4, we aim to demonstrate the tremendous impact of the ideas introduced by Fargues–Scholze by discussing in detail two applications of their work. The first of these applications concerns the representation theory of p-adic groups, namely the results of Dat, Helm, Kurinczuk, and Moss (2024a) on the finiteness of integral Hecke algebras and Bernstein's second adjointness. The second of these applications concerns the cohomology of local and global Shimura varieties with both characteristic 0 and torsion coefficients.

Notation

We let p and ℓ be distinct prime numbers. We let $E/\mathbb{Q}_p$ be a finite extension, with ring of integers $\mathcal{O}_E$, uniformiser $\varpi_E \in \mathcal{O}_E$, and residue field $\mathbb{F}_q$ of cardinality q.

We let G/E be a connected reductive group. The p-adic group $G(E)$ is locally profinite. For a closed subgroup $H \subset G(E)$ and a smooth representation V of H, we denote by $\mathrm{Ind}_H^{G(E)} V$ the smooth induction and by $\text{c-Ind}_H^{G(E)} V$ the smooth compact induction of V.

We denote by $\mathbb{1}$ the trivial representation of an algebraic or p-adic group (the group and the coefficients should be clear from context).

Fixing an algebraic closure $\overline{E}$ of E with residue field $\overline{\mathbb{F}}_q$, denote by $\Gamma_E \coloneqq \mathrm{Gal}(\overline{E}/E)$ the absolute Galois group of E, equipped with its usual profinite topology. This group sits in a short exact sequence

$$1 \to I_E \to \Gamma_E \to \Gamma_{\mathbb{F}_q} \to 1,$$

where I_E is the inertia subgroup and $\Gamma_{\mathbb{F}_q} := \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the absolute Galois group of the residue field. The group $\Gamma_{\mathbb{F}_q}$ is topologically generated by the arithmetic Frobenius element $\sigma\colon x \mapsto x^q$ and this choice of topological generator makes it abstractly isomorphic to $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$.

The Weil group W_E of E is the dense subgroup of Γ_E defined as the preimage of $\mathbb{Z} \subset \widehat{\mathbb{Z}}$. It is endowed with the topology that makes $I_E \subset W_E$, with its induced topology from Γ_E, into an open subgroup. We have the Artin reciprocity map of local class field theory

$$\mathrm{Art}_E\colon E^\times \xrightarrow{\sim} W_E^{\mathrm{ab}},$$

that we normalise such that uniformizers are sent to lifts of the geometric Frobenius element. For $w \in W_E$, we denote by $|w|$ the absolute value of $\mathrm{Art}_E^{-1}(w)$.

We let $\breve{E}$ denote the p-adic completion of the maximal unramified extension of E. The action of the arithmetic Frobenius σ on $\overline{\mathbb{F}}_q$ lifts canonically to $\breve{E}$.

We let $\mathrm{Perf}_{\mathbb{F}_q}$ denote the slice category of perfectoid spaces in characteristic p over $* := \mathrm{Spd}\ \mathbb{F}_q$. We make this into a site by equipping it with the v-topology. For an $\mathbb{F}_q$-algebra R, we let $W_{\mathcal{O}_E}(R)$ denote its ramified Witt vectors, given explicitly by the formula

$$W_{\mathcal{O}_E}(R) := W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E,$$

where $W(\ )$ denotes the usual Witt vectors functor. Note that we could alternatively define $\breve{E}$ as $W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)\left[\frac{1}{p}\right]$. There is a unique multiplicative lift $[\ ]\colon R \to W_{\mathcal{O}_E}(R)$ of the identity morphism on R.

If S is a perfectoid space, a diamond, or, most generally, a small v-stack on $\mathrm{Perf}_{\mathbb{F}_q}$, we denote by $|S|$ its underlying topological space.

If X is a topological space, we write $\underline{X}$ for the v-sheaf that sends $S \in \mathrm{Perf}_{\mathbb{F}_q}$ to the continuous functions $C^0(|S|, X)$.

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1. The local Langlands correspondence

In this section, we discuss the local Langlands conjectures, gradually building up to more and more general versions of these conjectures. The goal is not to write down the most general form of the conjecture, but rather to exhibit some glimpses of "geometrization" already in the statement of local Langlands. The following references provide a more in-depth treatment of this topic: Borel (1979), Kaletha (2016b), and Taïbi (2022).

The goal of the local Langlands correspondence is to describe the local constituents of automorphic representations in terms of local Galois-theoretic or spectral data. We start by describing the objects of interest on the automorphic side and then on the spectral side. We then discuss the local Langlands conjectures for quasi-split groups and, more generally, for extended pure inner forms of quasi-split groups. We highlight how the general form of the local Langlands conjectures already contains some glimpses of a richer, more geometric picture.

1.1. Smooth representations of p-adic groups

On the automorphic side, the objects of interest for the local Langlands correspondence (at least a priori) are the irreducible smooth admissible representations of the p-adic group $G(E)$ on $\mathbb{C}$-vector spaces. The $\mathbb{C}$-vector spaces for these representations are typically infinite-dimensional vector spaces. We recall what smoothness and admissibility mean, working, more generally, with coefficients in a $\mathbb{Z}[1/p]$-algebra Λ.

Definition 1.1. — *A representation π of $G(E)$ on a Λ-module V is* smooth *if, for every $v \in V$, the stabilizer of v,*

$$\mathrm{Stab}(v) := \{g \in G(E) \mid \pi(g)v = v\},$$

is an open subgroup of $G(E)$.

For any compact open subgroup $K \subset G(E)$, we define the space of invariants

$$\pi^K := \{v \in V \mid \forall k \in K,\ \pi(k)v = v\}.$$

The compact open subgroups form a basis of neighbourhoods of identity in $G(E)$. The smoothness condition can be phrased equivalently by saying that we have

$$\pi = \mathrm{colim}_K \pi^K, \tag{1}$$

where the colimit runs over compact open subgroups $K \subset G(E)$. Yet another equivalent formulation of smoothness is obtained by requiring the action map $G(E) \times V \to V$ to be continuous when $G(E)$ is endowed with its natural p-adic topology and V is endowed with the discrete topology. For any abstract isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, we obtain a bijection from smooth representations $G(E)$ on $\mathbb{C}$-vector spaces to smooth representations of $G(E)$ on $\overline{\mathbb{Q}}_\ell$-vector spaces, for any prime ℓ.

A priori, the objects of interest for the local Langlands correspondence are *irreducible* smooth representations of $G(E)$. Because we equipped the underlying vector space with the discrete topology, the notions of sub-representation, quotient, irreducible representation and so on are defined in the same way as for abstract group representations. One can construct a large class of irreducible smooth representations of $G(E)$ inductively, by taking subquotients of smooth parabolic inductions of representations of Levi subgroups of G. The building blocks are those representations that never arise as subquotients of parabolic inductions, which are called supercuspidal representations and which are more mysterious. For an introduction to the representation theory of p-adic groups, see Renard (2010) and for recent advances in our understanding of supercuspidal representations, see Fintzen (2023).

Later on, when we reach the geometrization of local Langlands, we will aim to understand all smooth representations of $G(E)$ and the homomorphisms and (higher) extensions between them. It is not hard to see that the category of smooth representations of $G(E)$ is abelian and has enough injectives and projectives. We will work with its much larger derived category, which we denote by $D(G(E), \Lambda)$.

We note that $D(G(E), \Lambda)$ is compactly generated, with a set of compact generators given by the compact inductions $\text{c-Ind}_K^{G(E)} \mathbb{1}$, with K running over compact open pro-p subgroups of $G(E)$. The fact that this is a set of generators can be checked from the equivalent definition of smoothness in (1) and from Frobenius reciprocity for compact induction, which tells us that there are canonical isomorphisms

$$\mathrm{Hom}_{G(E)}\left(\text{c-Ind}_K^{G(E)} \mathbb{1}, \pi\right) \simeq \mathrm{Hom}_K(\mathbb{1}, \pi\mid_K) \simeq \pi^K.$$

The following finiteness condition plays an important role in understanding the structure of irreducible objects.

Definition 1.2. — *A smooth representation (π, V) of $G(E)$ is* admissible *if, for every compact open subgroup $K \subset G(E)$, the space of invariants π^K is finitely generated over Λ.*

Admissibility allows us to recover certain properties of finite-dimensional representations in this infinite-dimensional setting, such as Schur's lemma. Assume, for example that Λ is an algebraically closed field of characteristic 0. Then, if V is irreducible admissible, any endomorphism $T \in \mathrm{End}_{G(E)}(V)$ acts by a scalar.

A deeper result, maintaining the assumption on Λ, is that every irreducible smooth representation is automatically admissible. This was originally proved by Jacquet, see also Renard (2010, Theorem VI.2.2). The idea is to prove first that supercuspidal representations are admissible. With characteristic 0 coefficients, any other irreducible smooth representation can be realized as a sub-representation of a representation that is parabolically induced from supercuspidal. One then verifies that admissibility is preserved under parabolic induction using the Iwasawa decomposition.

1.2. L-parameters

On the spectral side of the local Langlands correspondence, the objects we are interested in are called *Langlands parameters* or *L-parameters*. In order to define them precisely, we first need to introduce the *L-group* of G, which in turn relies on the notion of the *Langlands dual group*.

The Langlands dual group of a connected reductive group G/E is a split reductive group $\widehat{G}$ over $\mathbb{C}$ (or over $\overline{\mathbb{Q}}_\ell$), which is obtained as follows. We consider a root datum attached to G over an algebraic closure $\overline{E}$ of E and take the dual root datum, obtained by interchanging the roles of characters and cocharacters, and the roles of roots and coroots. This dual root datum determines the reductive group $\widehat{G}$ we wanted. Several examples of split groups and their Langlands dual groups are given in Figure 1.

G	GL_n	SL_n	PGL_n	Sp_{2n}	SO_{2n+1}	SO_{2n}
$\widehat{G}$	GL_n	PGL_n	SL_n	SO_{2n+1}	Sp_{2n}	SO_{2n}

FIGURE 1. Examples of connected reductive groups and their Langlands dual groups.

The L-group of G is a variant of the Langlands dual group that remembers the structure of G as a connected reductive group over E. More precisely, this structure determines an action of the absolute Galois group $\Gamma_E := \mathrm{Gal}(\overline{E}/E)$ on $\widehat{G}$ by outer automorphisms. This action factors through the Galois group of a finite extension of E. For details on how this action is constructed and the choices involved, see Borel (1979, § 1). Recall that $W_E \subset \Gamma_E$ denotes the Weil group of E. In these notes, we work with the *Weil form* of the L-group:

$$^{L}G := \widehat{G} \rtimes W_E.$$

When G is split over E, the semi-direct product above is simply a direct product and it is harmless to ignore the W_E-factor. We also note that two connected reductive groups over E that are inner forms of each other give rise to isomorphic L-groups.

Remark 1.3. — While this may seem an ad hoc construction, it turns out that the L-group ^{L}G has a canonical description in terms of algebraic geometry, in the sense that it arises naturally from the geometric Satake equivalence via Tannaka duality. The geometric Satake equivalence is discussed more in § 3.1.

We let Λ be an algebraically closed field of characteristic 0. We say that an element of $^{L}G(\Lambda) := \widehat{G}(\Lambda) \rtimes W_E$ is *semi-simple* if it becomes semi-simple under projection to $\widehat{G}(\Lambda) \rtimes \mathrm{Gal}(E'/E)$, where E'/E is some (equivalently, any) finite Galois extension that splits G.

DEFINITION 1.4. — *A* (local) Langlands parameter *or* L-parameter *is a homomorphism*

$$\varphi\colon W_E \times \mathrm{SL}_2(\Lambda) \to {}^{L}G(\Lambda)$$

which satisfies the following properties:

1. *The restriction $\varphi|_{W_E}$ is an L-homomorphism, i.e. it is a group homomorphism that is a section of the projection $^{L}G(\Lambda) \to W_E$.*
2. *As a consequence of the first condition, we can write $(\varphi|_{W_E})(w) = (\phi(w), w)$ for a 1-cocycle $\phi\colon W_E \to \widehat{G}(\Lambda)$. We ask that ϕ has open kernel — this is a continuity condition for the discrete topology on the target.*
3. *The restriction $\varphi|_{W_E}$ sends all elements of W_E to semi-simple elements of $^{L}G(\Lambda)$.*
4. *The restriction $\varphi|_{\mathrm{SL}_2(\Lambda)}$ is algebraic.*

We say that two Langlands parameters are equivalent if they are conjugate under $\widehat{G}(\Lambda)$.

Remark 1.5. — Some authors impose an additional *relevance* condition in the definition of a Langlands parameter for a connected reductive group G/E, which is only a non-trivial assumption when G is not quasi-split over E. The relevance condition restricts the image of φ in ${}^LG(\Lambda)$. More precisely, φ should only factor through parabolic subgroups of ${}^LG$ that are relevant for G, i.e. that correspond to parabolic subgroups of G defined over E. See Taïbi (2022, § 4.3) for more details on how to relate parabolic subgroups of G and parabolic subgroups of ${}^LG$.

Remark 1.6. — Because of the second condition, the third condition is equivalent to asking that $\varphi(\sigma, 1)$ be semi-simple, for some lift $\sigma \in W_E$ of the arithmetic Frobenius in $\Gamma_{\mathbb{F}_q}$. This condition is called *Frobenius semi-simplicity*. When we discuss moduli spaces of Langlands parameters in § 3.2, we will not impose the Frobenius semi-simplicity condition because it does not behave well in families.

Definition 1.4 uses homomorphisms from the group $W_E \times \mathrm{SL}_2$, which is a form of the so-called *Weil–Deligne group*. We can consider another form, namely $\mathrm{WD}_E := \mathbb{G}_a \rtimes W_E$, which is the one used in Borel (1979). In this case, one can define the notion of a *Weil–Deligne Langlands parameter*. This is a pair (ρ, N), where

1. $\rho\colon W_E \to {}^LG(\Lambda)$ is an L-homomorphism with open kernel,
2. $N \in \mathrm{Lie}\, \widehat{G}$ satisfies $\rho(w)N\rho(w)^{-1} = |w|N$ for all $w \in W_E$, forcing N to be nilpotent, and
3. $\rho(w)$ is semi-simple for all $w \in W_E$.

Once again, two Weil–Deligne Langlands parameters are said to be equivalent if they are conjugate under $\widehat{G}(\Lambda)$. If we choose a square root of q in Λ, there is a natural map from the set of Langlands parameters to the set of Weil–Deligne Langlands parameters, given by

$$\varphi \mapsto (\rho, N) := \left(\varphi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right), d(\varphi|_{\mathrm{SL}_2(\Lambda)})\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right).$$

This map induces a bijection on equivalence classes by a refinement of the Jacobson–Morozov theorem (Gross and Reeder, 2010, Proposition 2.2).

When $\Lambda = \overline{\mathbb{Q}}_\ell$ with $\ell \neq p$, the notion of a Weil–Deligne Langlands parameter is useful because Grothendieck's ℓ-adic monodromy theorem relates them to L-homomorphisms

$$W_E \to {}^LG(\overline{\mathbb{Q}}_\ell) \tag{2}$$

that are continuous for the natural ℓ-adic topology on $\widehat{G}(\overline{\mathbb{Q}}_\ell)$. These are closely related to continuous ℓ-adic representations of Γ_E, which occur in algebraic geometry, for example from the étale cohomology of algebraic varieties defined over E. For this reason, N is usually referred to as "the monodromy operator" — it encodes the action of the pro-ℓ-part of tame inertia.

The notion of Weil–Deligne Langlands parameter is furthermore useful for our purposes because it motivates the definition of the semi-simplification of a Langlands parameter. Given a Langlands parameter φ, we define the associated *semi-simple Langlands parameter* to be $\varphi^{\mathrm{ss}}\colon W_E \to {}^LG(\Lambda)$, where

$$\varphi^{\mathrm{ss}}(w) := \varphi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right). \tag{3}$$

From the point of view of the associated Weil–Deligne Langlands parameter, this amounts to forgetting the monodromy operator N.

Remark 1.7. — We shall see in § 3 that Fargues–Scholze construct semi-simple Langlands parameters $\pi \mapsto \varphi_{\mathrm{FS},\pi}$, which are expected to be compatible with the L-parameters predicted by the classical local Langlands correspondence $\pi \mapsto \varphi_\pi$ via $\varphi_{\mathrm{FS},\pi} \simeq \varphi_\pi^{\mathrm{ss}}$. This compatibility is known in many cases when a classical local Langlands map has been constructed.

1.3. The refined local Langlands conjectures

The local Langlands correspondence, in its most naive form, is meant to associate Langlands parameters to irreducible smooth representations of $G(E)$.

Let Λ be an algebraically closed field of characteristic 0 and fix a square root of q in Λ. We denote the set of equivalence classes of irreducible smooth representations of $G(E)$ with Λ-coefficients by $\Pi(G(E))$. We denote the set of equivalence classes of L-parameters for G/E with Λ-coefficients by $\Phi(G)$.

Conjecture 1.8. — *There exists a canonical local Langlands map*

$$\mathrm{LL}_G\colon \Pi(G(E)) \to \Phi(G)$$

with finite fibers $\Pi_\varphi(G) := \mathrm{LL}_G^{-1}(\varphi)$*, which are called* L-packets*.*

The word "canonical" in the statement of Conjecture 1.8 does not have a precise mathematical meaning, as there is no completely general characterization of the local Langlands correspondence. Nevertheless, it signifies that the map $\mathrm{LL_G}$ should have a number of nice properties: we should understand its image in terms of the relevance condition of Remark 1.5, it should be compatible with the Satake isomorphism in the unramified case, and with parabolic induction in general etc. For a precise statement of the desired compatibilities, see Taïbi (2022, Conjecture 6.1) and for a discussion of the possible characterizations of the local Langlands correspondence, see Harris (2022).

Example 1.9. — When $G = \mathrm{GL}_n$, the local Langlands map is known and gives a bijection

$$\mathrm{LL}_{\mathrm{GL}_n} : \Pi(\mathrm{GL}_n) \xrightarrow{\sim} \Phi(\mathrm{GL}_n).$$

Note that ${}^LG = \mathrm{GL}_n \times W_E$, so that we can ignore the W_E-factor in the target in Definition 1.4. The local Langlands correspondence for GL_n over p-adic fields was originally established by Harris and Taylor (2001) and Henniart (2000) with different methods; it was later reproved by Scholze (2013).

For more general groups G, including for example in the quasi-split case, the local Langlands map LL_G is far from a bijection. In general, an important question for the local Langlands correspondence is how to describe L-packets systematically in terms of the spectral side. Such a description comes up, for example, in the global automorphic multiplicity formulae conjectured by Arthur and Kottwitz. The question of refining Conjecture 1.8 has now been resolved, at least conjecturally, cf. Kaletha (2016a). Some of the work involved in formulating a precise answer to this question has informed the geometrization program of Fargues and Scholze (Fargues, 2025). At the same time, the work of Fargues–Scholze shows that the answer fits into a much richer geometric and categorical framework.

In order to give a flavor of how to describe L-packets systematically in the simplest non-trivial case, assume that G/E is a quasi-split group. This means that G contains a Borel subgroup defined over E. We choose a *Whittaker datum* $\mathfrak{w} := (B, \psi)$ for G. This consists of a Borel subgroup $B \subset G$ with Levi decomposition $B = T \ltimes U$ (where T is a maximal E-torus of G and U is the unipotent radical) and of a generic additive character

$$\psi\colon U(E) \to \Lambda.$$

The condition for the character ψ to be generic is that its stabilizer in $T(E)$ under the usual adjoint action is $Z(G)(E)$. For example, we could obtain a generic character ψ by composing the character corresponding to the sum of positive roots with respect to B with any non-trivial additive character $E \to \Lambda$. Once we have chosen a Whittaker datum, we define a notion of genericity for irreducible smooth representations of $G(E)$.

Definition 1.10. — *An irreducible smooth representation π of $G(E)$ with Λ-coefficients is* $\mathfrak{w}$-generic *if the space*

$$\mathrm{Hom}_{U(E)}(\pi, \psi) = \mathrm{Hom}_{G(E)}\left(\pi, \mathrm{Ind}_{U(E)}^{G(E)}\psi\right)$$

is non-zero[(2)].

Let $\varphi\colon W_E \times \mathrm{SL}_2(\Lambda) \to {}^LG(\Lambda)$ be a Langlands parameter. We consider its centraliser in $\widehat{G}(\Lambda)$, defined as

$$S_\varphi := \{g \in \widehat{G}(\Lambda) \mid g\varphi g^{-1} = \varphi\}.$$

This is a (possibly disconnected) reductive group that contains $Z(\widehat{G})^{\Gamma_E}$. We set $\bar{S}_\varphi := S_\varphi/Z(\widehat{G})^{\Gamma_E}$. The group of connected components $\pi_0(\bar{S}_\varphi)$ is a finite group and we denote by $\mathrm{Irr}(\pi_0(\bar{S}_\varphi))$ the set of equivalence classes of its irreducible representations. We call the parameter φ *discrete* if $\bar{S}_\varphi$ is itself finite.

We have a refinement of Conjecture 1.8 given by the following parameterization of the L-packet $\Pi_\varphi(G)$.

(2) By a result known as the uniqueness of Whittaker models, the space is at most one-dimensional.

Conjecture 1.11. — 1. *There is a canonical bijection*

$$\iota_{\mathfrak{w}} : \Pi_\varphi(G(E)) \xrightarrow{\sim} \mathrm{Irr}(\pi_0(\bar{S}_\varphi))$$

that depends on the choice of Whittaker datum $\mathfrak{w}$.

2. *If* φ *is discrete, there exists a unique generic constituent of* $\Pi_\varphi(G(E))$. *This constituent gets mapped to the trivial representation of* $\pi_0(\bar{S}_\varphi)$ *under* $\iota_{\mathfrak{w}}$.

When $\Lambda = \mathbb{C}$, the bijection $\iota_{\mathfrak{w}}$ can be interpreted as a perfect pairing

$$\langle\ ,\ \rangle \colon \Pi_\varphi(G(E)) \times \pi_0(\bar{S}_\varphi) \to \mathbb{C},$$

which should satisfy certain endoscopic character identities. Furthermore, in this case, the second part of Conjecture 1.11 is expected to hold under the weaker condition that φ is *tempered*, which means that the image of φ projects onto a bounded subset of $\widehat{G}(\mathbb{C})$. The refined conjecture therefore motivates the notion of a generic representation, which can be thought of as a base point of a tempered L-packet. A further motivation for the notion of genericity comes from the perspective of global automorphic representations, where it is related, under Arthur's conjectures, to the generalised Ramanujan conjecture; see Shahidi (2011). We shall see one final motivation when we formulate the categorical local Langlands conjecture precisely in § 3.4.

1.4. The Kottwitz set and local Langlands for extended pure inner forms

When the group G is not quasi-split, the refined version of local Langlands breaks down. An influential idea to remedy this on the automorphic side has been to treat *all inner forms together*. This idea originates from Vogan and was first exploited in the work of Adams, Barbasch, and Vogan (1992) in the archimedean case. In the non-archimedean case, the idea has gradually evolved from the use of so-called pure inner forms of Vogan (1993), through extended pure inner forms (Kaletha, 2014), to the completely general case that covers all inner forms (Kaletha, 2016a).

Each connected reductive group G/E has a unique inner form G^*/E that is quasi-split. We choose this quasi-split inner form as our base point for stating the more general version of the refined local Langlands conjectures, i.e. we set $G \coloneqq G^*$. We will not describe the most general version of the local Langlands conjecture, but rather the version that parameterises unions of L-packets $\Pi_\varphi(G_b)$, for all the *extended pure inner forms* G_b of G. This is already very general, as we shall see below, and relates most closely to the work of Fargues–Scholze.

In order to define the notion of an extended pure inner form, we start with some recollections on the Kottwitz set $B(G)$, first studied in Kottwitz (1985a,b). The lift of the qth power Frobenius σ acts on $\breve{E}$ and, therefore, on $G(\breve{E})$. We say that two elements $b, b' \in G(\breve{E})$ are σ-conjugate if $gb\sigma(g)^{-1} = b'$ for some $g \in G(\breve{E})$; this is an equivalence relation.

Definition 1.12. — *The* Kottwitz set $B(G)$ *is the set of* σ*-conjugacy classes in* $G(\breve{E})$.

Alternatively, the set $B(G)$ can be defined as the set of isomorphism classes of isocrystals with G-structure. We define isocrystals with G-structure as follows. An *isocrystal* over E is a pair (V,ϕ), where V is a finite-dimensional $\breve{E}$-vector space and $\phi : V \to V$ is a σ-semi-linear automorphism. A morphism of isocrystals $(V,\phi) \to (V',\phi')$ is a morphism of the underlying $\breve{E}$-vector spaces that intertwines ϕ and ϕ'. We let Isoc_E denote the category of isocrystals over E. We let $\mathrm{Rep}_E(G)$ denote the exact symmetric monoidal category of algebraic representations of G over E. An *isocrystal with G-structure* is an exact $\otimes$-functor

$$\mathrm{Rep}_E(G) \to \mathrm{Isoc}_E,$$

which is then automatically faithful. We denote by $G-\mathrm{Isoc}_E$ the category of isocrystals with G-structure over E.

Given $\tilde{b} \in G(\breve{E})$ we define an associated isocrystal with G-structure as follows: to each $(V,\rho) \in \mathrm{Rep}_E G$, we associate an isocrystal via $\big(V \otimes_E \breve{E}, \rho(\tilde{b})(\mathrm{id}_V \otimes \sigma)\big)$. One can check that this defines an exact, faithful $\otimes$-functor $\mathrm{Rep}_E(G) \to \mathrm{Isoc}_E$ and that the isomorphism class of the resulting isocrystal with G-structure depends only on the σ-conjugacy class of $\tilde{b}$. The fact that any isocrystal with G-structure is isomorphic to one arising in this way from some $\tilde{b} \in G(\breve{E})$ follows from a theorem of Steinberg (1965) on Galois cohomology, which implies that all G-torsors on Spec $\breve{E}$ are trivial.

The isocrystal perspective highlights, for each element $\tilde{b} \in G(\breve{E})$ lifting $b \in B(G)$, the algebraic group over E given by

$$G_{\tilde{b}}(R) := \{g \in G(R \otimes_E \breve{E}) \mid g\tilde{b}\sigma(g)^{-1} = \tilde{b}\},$$

which is the group of automorphisms of the isocrystal with G-structure parameterized by $\tilde{b}$. For two different elements $\tilde{b}, \tilde{b}' \in G(\breve{E})$ in the same σ-conjugacy class $b \in B(G)$, there exists $g \in G(\breve{E})$ such that $g\tilde{b}\sigma(g)^{-1} = \tilde{b}'$. Usual conjugation by g induces an isomorphism $G_{\tilde{b}} \xrightarrow{\sim} G_{\tilde{b}'}$ and changing g changes this isomorphism by an inner automorphism. We therefore write G_b for the corresponding algebraic group. The group G_b turns out to be a connected reductive group which is an inner form of a Levi subgroup of G. (Note that different elements $b, b' \in B(G)$ could determine twisted Levi subgroups G_b and $G_{b'}$ of G that are abstractly isomorphic — we will see this phenomenon in Example 1.16.)

Definition 1.13. — *When G_b is an inner form of G itself, we call the element $b \in B(G)$* basic *and we call G_b an* extended pure inner form. *We denote the subset of basic elements by $B(G)_{\mathrm{bas}} \subset B(G)$.*

The Kottwitz set can be understood using two invariants: the Newton map and the Kottwitz map. While we will only use the restriction of the Kottwitz map to $B(G)_{\mathrm{bas}}$ in the formulation of Conjecture 1.15 below, the entire Kottwitz set plays a role in the work of Fargues–Scholze. Therefore, we review these notions; see (Rapoport and Richartz, 1996) for more details.

Consider a maximal torus $T \subset B_{\overline{E}} \subset G_{\overline{E}}$ and let $X_*(T)$ denote its cocharacter lattice; it has an action of the Weyl group W and of the absolute Galois group Γ_E. The Newton map

$$\nu\colon B(G) \to (X_*(T)_{\mathbb{Q}}/W)^{\Gamma_E} \simeq (X_*(T)^+_{\mathbb{Q}})^{\Gamma_E} \tag{4}$$

generalizes the Newton polygon attached to an isocrystal over E. Denote by $\pi_1(G)$ the algebraic fundamental group of G, defined as the quotient of $X_*(T)$ by the coroot lattice. This is also equipped with an action of Γ_E and we denote the coinvariants by $\pi_1(G)_{\Gamma_E}$. The Kottwitz map has the form

$$\kappa\colon B(G) \to \pi_1(G)_{\Gamma_E}. \tag{5}$$

Once ν and κ have been defined, one can prove that there is an injection

$$(\nu, \kappa) : B(G) \hookrightarrow (X_*(T)_{\mathbb{Q}}/W)^{\Gamma_E} \times \pi_1(G)_{\Gamma_E}.$$

Therefore, the Kottwitz set is determined by these two invariants. After restricting to the subset of basic elements, one can furthermore prove that there is a bijection $\kappa|_{B(G)_{\mathrm{bas}}}\colon B(G)_{\mathrm{bas}} \xrightarrow{\sim} \pi_1(G)_{\Gamma_E}$. There is a canonical isomorphism $\gamma\colon \pi_1(G)_{\Gamma_E} \simeq X^*(Z(\widehat{G})^{\Gamma_E})$ and we will use the latter group in the formulation of Conjecture 1.15 below.

We can use the Newton and Kottwitz maps to define a partial order relation on $B(G)$: we set $b \leq b'$ if $\kappa(b) = \kappa(b')$ and if $\nu(b) \leq \nu(b')$ with respect to the dominance order on cocharacters (after choosing dominant representatives). The basic elements are minimal elements for this partial order relation. We equip $B(G)$ with the order topology, which is defined by the condition

$$\{b\} \in \overline{\{b'\}} \text{ iff } b \geq b'.$$

This identifies the target of κ with the set of connected components of $B(G)$ and realizes each basic element as the unique open element in its connected component.

Remark 1.14. — There are a natural partial order and a natural topology that can be defined on the set $B(G)$ coming from considering specializations of families of isocrystals with G-structure. These are studied in Rapoport and Richartz (1996) and show up, for example, when describing the closure relations among Newton strata in special fibers of Shimura varieties. They also show up in the recent work of Zhu (2025) on the tame categorical local Langlands conjecture. The partial order relation and topology we define above are *opposite* to the ones coming from specializations of isocrystals; we shall see in § 2.2 that they model the specialization behaviour of G-bundles on the Fargues–Fontaine curve instead. This phenomenon is explored further by Gleason, Ivanov, and Zillinger (2025), who introduce certain objects, meromorphic G-bundles on the Fargues–Fontaine curve, that mediate between the settings of Zhu and of Fargues–Scholze.

Before generalizing Conjecture 1.11, we need one more definition, namely we need to rigidify the notion of an extended pure inner form. This gives a canonical identification between the L-group of the extended pure inner form and the L-group of the quasi-split form G. An *extended pure inner twist* is a pair (b,ξ), where $b \in B(G)_{\mathrm{bas}}$ determines the extended pure inner form G_b and $\xi\colon G_b \times_E \breve{E} \xrightarrow{\sim} G \times_E \breve{E}$ is an inner twist determined by a lift $\tilde{b} \in G(\breve{E})$ of the σ-conjugacy class b. Any two extended pure inner twists (b,ξ) and (b,ξ') are isomorphic, and the set of automorphisms of a fixed extended pure inner twist (b,ξ) is isomorphic to $G_b(E)$, acting on itself by inner automorphisms.

Given two isomorphic extended pure inner twists (b,ξ) and (b,ξ') determined by $\tilde{b}, \tilde{b}' \in G(\breve{E})$, respectively, any isomorphism between them induces an isomorphism $G_{\tilde{b}} \simeq G_{\tilde{b}'}$ defined over E and, therefore, a bijection between the sets of isomorphism classes of irreducible smooth representations $\Pi(G_{\tilde{b}}(E)) \simeq \Pi(G_{\tilde{b}'}(E))$. This bijection is independent of the choice of isomorphism (as inner automorphisms of $G_b(E)$ fix isomorphism classes of representations), and therefore we can view the set of isomorphism classes of irreducible smooth representations of $G_b(E)$ more canonically as $\Pi_b := \varprojlim_{\tilde{b}} \Pi(G_{\tilde{b}}(E))$.

Maintain the choice of our Whittaker datum $\mathfrak{w}$ for the quasi-split group G. Let $\widehat{G}_{\mathrm{der}}$ denote the derived subgroup of $\widehat{G}$. For an L-parameter $\varphi\colon W_E \times \mathrm{SL}_2(\Lambda) \to {}^LG(\Lambda)$, we define the (possibly disconnected) reductive group $S_\varphi^\natural := S_\varphi/(S_\varphi \cap \widehat{G}_{\mathrm{der}})^\circ$. We denote by $\mathrm{Irr}(S_\varphi^\natural)$ the set of isomorphism classes of irreducible algebraic representations of $S_\varphi^\natural$. Conjecture 1.11 can be extended as follows.

Conjecture 1.15. — *For any $b \in B(G)_{\mathrm{bas}}$, there exists a canonical local Langlands map*

$$\mathrm{LL}_{G,b}\colon \Pi_b \to \Phi(G)$$

with finite fibers $\Pi_{\varphi,b} := \mathrm{LL}_{G,b}^{-1}(\varphi)$ and such that the following two properties hold.

1. *There exists a commutative diagram*

$$\begin{array}{ccc} \bigsqcup_b \Pi_{\varphi,b} & \xrightarrow{\ \iota_{\mathfrak{w}}\ } & \mathrm{Irr}(S_\varphi^\natural) \\ \downarrow & & \downarrow \\ B(G)_{\mathrm{bas}} & \xrightarrow{\ \gamma\circ\kappa\ } & X^*(Z(\widehat{G})^{\Gamma_E}), \end{array} \tag{6}$$

   where the horizontal maps are both bijections. The left vertical map is the natural projection and the right vertical map is the restriction of representations along the natural map $Z(\widehat{G})^{\Gamma_E} \to S_\varphi^\natural$.
2. *The map $\iota_{\mathfrak{w}}$ restricted to the neutral component, i.e. taking $b = 1$*[3] *in* (6) *recovers the map denoted $\iota_{\mathfrak{w}}$ in Conjecture 1.11. In particular, if φ is discrete, the unique $\mathfrak{w}$-generic representation in the L-packet $\Pi_\varphi(G)$ gets mapped to the trivial representation of $S_\varphi^\natural$.*

[3] Strictly speaking, if we set $b = 1$, we obtain the restriction of $\iota_{\mathfrak{w}}$ to the subset $\Pi_{\varphi,1}$ of Π_1, which was defined as a limit over all extended pure inner twists above $1 \in B(G)$. In order to identify $\Pi_{\varphi,1}$ with $\Pi_\varphi(G(E)) \subset \Pi(G(E))$ as in Conjecture 1.11, we should further rigidify the situation by taking the extended pure inner twist $\tilde{b} = 1 \in G(\breve{E})$.

Example 1.16. — We make Conjecture 1.15 explicit in the case when $G = \mathrm{GL}_n$. First, we note that the Dieudonné–Manin classification completely describes Isoc_E: the category of isocrystals is semi-simple and the simple objects V_λ are determined by rational numbers $\lambda \in \mathbb{Q}$, called slopes. Indeed, writing $\lambda = \frac{s}{r}$ for rational numbers s, r with $(s, r) = 1$ and $r > 0$, we can define the isocrystal

$$(V_\lambda, \phi_\lambda) = \left(\breve{E}^r, \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \varpi_E^s & & & \end{pmatrix} \sigma\right),$$

where all the non-zero entries are 1 except for the bottom left one, which is equal to ϖ_E^s. A general isocrystal is isomorphic to

$$V = \bigoplus_{\lambda \in \mathbb{Q}} V_\lambda^{\oplus n_\lambda}$$

and the basic elements are the same as the isoclinic elements, i.e. the ones with a single slope λ. The extended pure inner forms of GL_n have the form $\mathrm{GL}_{n_\lambda}(D_{\overline{\lambda}})$ for some $n_\lambda \in \mathbb{Z}_{\geq 1}$, where $D_{\overline{\lambda}}$ is the division algebra with invariant $\overline{\lambda} \in \frac{1}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. One can see that these cover all inner forms of GL_n.

On the RHS of diagram (6), when φ is discrete, we have $S_\varphi = S_\varphi^\natural = Z(\widehat{G})^{\Gamma_E} = \mathbb{G}_m$ and the vertical map is the identity on $\mathbb{Z}$. The Newton map is injective (this is a consequence of Hilbert's theorem 90), so the subset of basic elements can be identified with the corresponding subset of slopes. We have $B(\mathrm{GL}_n)_{\mathrm{bas}} = \frac{1}{n}\mathbb{Z}$ and the Kottwitz map $B(G)_{\mathrm{bas}} \xrightarrow{\sim} X^*(Z(\widehat{G})^{\Gamma_E}) = \mathbb{Z}$ is given by $\lambda \mapsto \lambda \cdot n$. Each extended pure inner form shows up infinitely many times on the LHS of diagram (6), as the isomorphism class of the extended pure inner form only depends on the image $\overline{\lambda} \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}$. Each individual L-packet has size 1.

Remark 1.17. — Assume that $\Lambda = \overline{\mathbb{Q}}_\ell$. We explain, heuristically, some glimpses of geometrization that already show up in the statement of Conjecture 1.15. On the spectral side, we have the following phenomenon. Assume that we can make sense of a moduli stack of Langlands parameters. Assume also that G is semi-simple and that φ is discrete, so that $S_\varphi = S_\varphi^\natural$. The substack corresponding to φ in this moduli stack is open and looks like $[*/S_\varphi]$. The elements of $\mathrm{Irr}(S_\varphi)$, occurring in the top right corner of diagram (6) give rise to coherent sheaves on this substack.

On the automorphic side, we shall see in § 2 that each element $b \in B(G)$ gives rise to a point on the stack Bun_G of G-bundles on the Fargues–Fontaine curve, with corresponding group of automorphisms closely related to the p-adic group $G_b(E)$. As a result, Fargues–Scholze embed fully faithfully the entire derived category $D(G_b(E), \Lambda)$ of smooth representations of $G_b(E)$ into an appropriate derived category of ℓ-adic sheaves on Bun_G. In particular, each element in the top left corner of diagram 6 gives rise to an ℓ-adic sheaf on Bun_G.

The moduli stack of Langlands parameters can be constructed rigorously, with several different approaches due to Dat–Helm–Kurinczuk–Moss, Zhu and Fargues–Scholze. This is discussed more in § 3.2. The work of Fargues–Scholze upgrades the top arrow in (6) to a much richer conjectural equivalence of derived categories between ℓ-adic sheaves on Bun_G and so-called ind-coherent sheaves on the moduli stack of Langlands parameters. This much richer conjecture is stated precisely in § 3.4.

Remark 1.18. — In their geometrization of the local Langlands correspondence, Fargues–Scholze work with the entire set $B(G)$ rather than just with the subset of basic elements $B(G)_{\mathrm{bas}}$. This raised the question of whether there is an extension of Conjecture 1.15 to the entire set $B(G)$. Such an extension was formulated in Bertoloni Meli and Oi (2023), after the work of Fargues–Scholze was completed. In the parameterization of Bertoloni Meli and Oi, the object in the top right corner of their version of diagram (6) is $\mathrm{Irr}(S_\varphi)$, the set of equivalence classes of irreducible algebraic representations of S_φ itself, leading to a cleaner formulation that is also compatible with the heuristic in Remark 1.17.[4]

Remark 1.19. — Conjecture 1.15 captures all inner forms of G if $Z(G)$ is connected. If this condition does not hold, one can still formulate a more general version of Conjecture 1.15 using Kaletha's notion of *rigid inner twists.* In (Kaletha, 2018), it is shown that knowing Conjecture 1.15 for all reductive groups over E is equivalent to knowing this more general version of the local Langlands conjecture for all reductive groups over E.

2. The Fargues–Fontaine curve and the stack Bun_G

The work of Fargues–Scholze lays the foundations of the geometric Langlands program over the Fargues–Fontaine curve, as foreseen in Scholze and Weinstein (2020) and Fargues (2025). The key geometric object on the automorphic side is Bun_G, the stack of G-bundles on the Fargues–Fontaine curve over E. The goal of this section is to introduce the Fargues–Fontaine curve in its different incarnations, to discuss G-bundles on it, and to state some preliminary results that connect the geometry of Bun_G to the representation theory of $G(E)$ and its inner forms, as described in Section 1.4 and in particular in Remark 1.17.

In the discussion below, we will assume knowledge of p-adic geometry as developed by Scholze and his collaborators, particularly of the theory of perfectoid spaces, diamonds and v-stacks. The original references for these topics are Scholze (2012), Scholze and Weinstein (2020), and Scholze (2022). In addition, the reader can find an overview in the Bourbaki talk of Fontaine (2013) and in the plenary ICM lecture of Scholze (2018), and a number of surveys on different aspects of p-adic geometry in the book by Bhatt, Caraiani, Kedlaya, and Weinstein (2019).

(4) More precisely, in this level of generality, one would see on the spectral side the classifying stack for the centralizer of the Weil–Deligne form of the L-parameter, whose reductive quotient is S_φ. This gives rise to the same set of equivalence classes of irreducible algebraic representations.

2.1. The three incarnations of the Fargues–Fontaine curve

The Fargues–Fontaine curve is known as "the fundamental curve of p-adic Hodge theory" because, in some sense, it geometrizes many of the period rings of p-adic Hodge theory. It was introduced in Fargues and Fontaine (2018) as a geometric object defined in terms of a local field, in our case the p-adic field E, and in terms of a perfectoid field of characteristic p. With the introduction of the theories of perfectoid spaces and diamonds, it became possible to consider this object in families, i.e. to construct relative versions of the Fargues–Fontaine curve.

There are a number of excellent references on "the curve", particularly in the case over a perfectoid field, such as Fargues and Fontaine (2014) and Morrow (2019). Here, we discuss briefly the more general version, in families. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space in $\mathrm{Perf}_{\mathbb{F}_q}$. We let $\varpi \in R^+$ be a pseudo-uniformiser (meaning, a topologically nilpotent unit of R). We will construct three incarnations of the relative Fargues–Fontaine curve parameterized by S: as an adic space, as a diamond, and as a scheme. The constructions will glue to give versions of the relative Fargues–Fontaine curve parameterized by general perfectoid spaces in $\mathrm{Perf}_{\mathbb{F}_q}$.

1. We first explain the construction of the Fargues–Fontaine curve as an *adic space*. We first consider the adic space

$$Y_S \coloneqq \mathrm{Spa}(W_{\mathcal{O}_E}(R^+)) \setminus V(\varpi_E \cdot [\varpi]),$$

where we have removed the vanishing locus $V(\varpi_E \cdot [\varpi])$. This is an analytic adic space over $\mathrm{Spa}(E, \mathcal{O}_E)$. One can check that the lift φ to $W_{\mathcal{O}_E}(R)$ of the qth power Frobenius on R^+ acts freely and properly discontinuously on Y_S and we can form the quotient

$$X_S \coloneqq Y_S/\varphi^{\mathbb{Z}},$$

which is itself an analytic adic space over $\mathrm{Spa}(E, \mathcal{O}_E)$. The space X_S is the *adic relative Fargues–Fontaine curve* over S. This is a particularly nice kind of adic space, namely a *sous-perfectoid* space, in the sense that it becomes perfectoid after base change from E to $\widehat{\overline{E}}$, the p-adic completion of our fixed algebraic closure of E.

This construction is functorial in S. In particular, if $x = \mathrm{Spa}(K(x), K(x)^+) \to S$ is a point, corresponding to a perfectoid field $K(x)$, we have a natural morphism $X_{\mathrm{Spa}(K(x),K(x)^+)} \to X_S$. Therefore, one can heuristically think of X_S as a family of curves $\left(X_{\mathrm{Spa}(K(x),K(x)^+)}\right)_{x\to S}$ indexed by the points of S. Note, however, that there is no morphism of adic spaces $X_S \to S$, as X_S lives over $\mathrm{Spa}(E, \mathcal{O}_E)$ and S is a characteristic p perfectoid space.

2. Setting $\mathrm{Spd}\ E \coloneqq \mathrm{Spa}(E, \mathcal{O}_E)^\diamond$, the *diamond* incarnation of the relative Fargues–Fontaine curve is given by

$$X_S^\diamond \simeq (S^\diamond \times \mathrm{Spd}\ E)\,/(\varphi^{\mathbb{Z}} \times \mathrm{id}).$$

More precisely, there is a natural isomorphism between the space $X_S^\diamond$ obtained by applying the diamond functor to the adic incarnation X_S and the quotient

$(S^\diamond \times \mathrm{Spd}\ E)/(\varphi \times \mathrm{id})$. This follows from the natural isomorphism

$$Y_S^\diamond \simeq S^\diamond \times \mathrm{Spd}\ E,$$

which is established as a consequence of Fargues and Scholze (2024, Proposition II.1.2), using the functor-of-points perspective.

Indeed, set $\mathcal{Y}_S := \mathrm{Spa}(W_{\mathcal{O}_E}(R^+)) \setminus V([\varpi])$, an adic space over Spa $\mathcal{O}_E$ with adic generic fiber Y_S. We claim that the functor-of-points of $\mathcal{Y}_S^\diamond$ can be described as follows: for $T \in \mathrm{Perf}_{\mathbb{F}_q}$, we have

$$\mathcal{Y}_S^\diamond(T) := \left\{T^\#, T^{\#\flat} \xrightarrow{\sim} T, T^\# \to \mathrm{Spa}\ \mathcal{O}_E, T \to S\right\}/\sim,$$

i.e. an untilt $T^\#$, equipped with an isomorphism $T^{\#\flat} \xrightarrow{\sim} T$, and with maps $T^\# \to \mathrm{Spa}\ \mathcal{O}_E$ and $T \to S$. By the definition of the diamondification functor, for any $T \in \mathrm{Perf}_{\mathbb{F}_q}$, we have

$$(\mathrm{Spd}\ \mathcal{O}_E)(T) = \left\{T^\#, T^{\#\flat} \xrightarrow{\sim} T, T^\# \to \mathrm{Spa}\ \mathcal{O}_E\right\}/\sim .$$

One then concludes by using the universal property of the Witt vectors functor to observe that a map $T^\# \to \mathcal{Y}_S$, for any perfectoid space $T^\#$ over $\mathcal{O}_E$, is equivalent, when R^+ is perfect, to a map $T^{\#\flat} \to S$. As a consequence, we obtain a natural isomorphism

$$\mathcal{Y}_S^\diamond \simeq S^\diamond \times \mathrm{Spd}\ \mathcal{O}_E,$$

from which the desired isomorphisms follow by taking the adic generic fiber and the quotient by $\varphi^{\mathbb{Z}}$.

3. We can use descent along the morphism $Y_S \to X_S$ to construct vector bundles on X_S. For example, consider the structure sheaf $\mathcal{O}_{Y_S}$ equipped with the descent datum given by

$$\frac{1}{\varpi_E}\varphi\colon \mathcal{O}_{Y_S} \xrightarrow{\sim} \mathcal{O}_{Y_S}.$$

This gives rise to an ample line bundle $\mathcal{O}_{X_S}(1)$ on X_S. We define $\mathcal{O}_{X_S}(n) := \mathcal{O}_{X_S}(1)^{\otimes n}$ and set

$$P := \bigoplus_{n\geq 0} H^0(X_S, \mathcal{O}_{X_S}(n)).$$

We set $X_S^{\mathrm{alg}} := \mathrm{Proj}\ P$. This is the *algebraic incarnation* of the relative Fargues–Fontaine curve. It is a scheme over Spec E. If $S = \mathrm{Spa}(C, C^+)$, for an algebraically closed perfectoid field C of characteristic p, then X_C^{alg} is a regular, Noetherian scheme of Krull dimension 1, locally the spectrum of a principal ideal domain. These properties are the reason that the Fargues–Fontaine curve is referred to as a “curve”.

The scheme X_S^{alg} can alternatively be constructed without reference to the adic or diamond versions, using instead a relative version of Fontaine’s period rings. For example, if we denote by $B^+_{\mathrm{crys},S}$ the crystalline period ring $A_{\mathrm{crys}}(R^+)[\frac{1}{\varpi_E}]$, we have an isomorphism

$$P \simeq \bigoplus_{n\geq 0} (B^+_{\mathrm{crys},S})^{\varphi=\varpi_E^n}.$$

This isomorphism is one instance of the general heuristic that the Fargues–Fontaine curve geometrizes the period rings introduced by Fontaine in p-adic Hodge theory.

Remark 2.1. — The formula $Y_S^\diamond \simeq S^\diamond \times \mathrm{Spd}\ E$ is reminiscent of the formula $S \times_{\mathrm{Spa}\ \mathbb{F}_q} \mathrm{Spa}\ \mathbb{F}_q((t))$ for the punctured open unit disc over S, which is the analogous object in the local Langlands program in the function field setting. However, note that the product $S \times_{\mathbb{F}_q} \mathbb{F}_q((t))$ already makes sense when working with usual algebraic geometry. See the introduction of Scholze and Weinstein (2020) for more details on this analogy, which inspired the development of diamonds in order to make sense of constructions such as this one over p-adic fields.

Each of the three perspectives on the Fargues–Fontaine curve has its own advantages. The diamond perspective allows us to glue the relative versions for affinoid perfectoid spaces and construct adic space and diamond versions of the Fargues–Fontaine curve X_S and $X_S^\diamond$ over general perfectoid spaces $S \in \mathrm{Perf}_{\mathbb{F}_q}$. The gluing relies on the fact that the construction is functorial in S and on the fact that there is a morphism on the level of topological spaces

$$|Y_S| \simeq |Y_S^\diamond| \simeq |S^\diamond \times \mathrm{Spd}\ E| \to |S|,$$

because the underlying topological spaces of an analytic adic space and of its associated diamond are naturally homeomorphic. In fact, this morphism factors over a morphism

$$|X_S| \simeq |(S^\diamond \times \mathrm{Spd}\ E)/(\varphi^{\mathbb{Z}} \times \mathrm{id})| \simeq |(S^\diamond \times \mathrm{Spd}\ E)/(\mathrm{id} \times \varphi^{\mathbb{Z}})| \to |S|, \tag{7}$$

because the absolute Frobenius $\varphi_S \times \varphi_E$ acts trivially on the topological space $|S^\diamond \times \mathrm{Spd}\ E|$. We warn the reader once more that the morphism (7) cannot be upgraded to a morphism of adic spaces, or even to one of diamonds.

For S an affinoid perfectoid space in $\mathrm{Perf}_{\mathbb{F}_q}$, recall that we have

$$(\mathrm{Spd}\ E)(S) = \left\{S^\#, \iota\colon S^{\#\flat} \xrightarrow{\sim} S, S^\# \to \mathrm{Spa}\ E\right\} / \sim,$$

and the action of φ on it is given by $\iota \mapsto \varphi_S \circ \iota$ (post-composition with the qth power Frobenius on S). Fargues–Scholze consider the diamond

$$\mathrm{Div}^1_{\mathbb{F}_q} := \mathrm{Spd}\ E/\varphi^{\mathbb{Z}} \tag{8}$$

over $* := \mathrm{Spd}\ \mathbb{F}_q$; they show that it has nice geometric properties, e.g. that it is proper and ℓ-cohomologically smooth[(5)]. Given a map $S \to \mathrm{Div}^1_{\mathbb{F}_q}$, one can define an associated closed Cartier divisor $D_S \subset X_S$ that is locally given by an untilt $D_S := S^\# \hookrightarrow X_S$, well-defined up to Frobenius equivalence (i.e. up to post-composing the map ι by φ_S).

[(5)] This is a notion of smoothness for diamonds and (small) v-stacks introduced in Scholze (2022, Definition 23.8) for any prime $\ell \neq p$. See also Fargues and Scholze (2024, Proposition IV.2.33) for an equivalent characterisation. As v-stacks are defined in terms of the category $\mathrm{Perf}_{\mathbb{F}_q}$ of characteristic p perfectoid spaces, the usual definition of a smooth morphism from algebraic geometry, using Kähler differentials, does not make sense. Instead, one uses a definition with a more cohomological flavor, where the key condition needed for a morphism $f : X \to Y$ to be ℓ-cohomologically smooth is for $Rf^!\mathbb{F}_\ell$ to define an invertible sheaf whose formation commutes with arbitrary base change.

They define the notion of a *closed Cartier divisor of degree* 1 on X_S as consisting of a closed Cartier divisor that arises from such a map $S \to \mathrm{Div}^1_{\mathbb{F}_q}$, identifying $\mathrm{Div}^1_{\mathbb{F}_q}$ with the moduli space of such divisors.

Example 2.2. — Let C be an algebraically closed perfectoid field of characteristic p. In this case, one can define the classical points of the Fargues–Fontaine curve via $|X_C|^{\mathrm{cl}} := |Y_C|^{\mathrm{cl}}/\varphi^{\mathbb{Z}}$. By Fargues and Scholze (2024, Definition / Proposition II.1.22), these classical points are in bijection with the Frobenius equivalence classes of untilts $C^\#$ of C and, equivalently, with the closed Cartier divisors of degree 1 on X_C, i.e. with the C-valued points of $\mathrm{Div}^1_{\mathbb{F}_q}$. Moreover, the bijection has the property that, if a classical point corresponds to an untilt $C^\#$, this untilt is the residue field at that point. This makes precise the idea that the Fargues–Fontaine curve should be a moduli space of untilts, recovering an earlier result of Fargues–Fontaine.

We set $\mathrm{Div}^1 := \mathrm{Div}^1_{\mathbb{F}_q} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. For any finite set I, local systems on $(\mathrm{Div}^1)^I$ turn out to be naturally equivalent to continuous representations of I copies of the Weil group W_E (Fargues and Scholze, 2024, Proposition VI.9.2), so the diamonds $(\mathrm{Div}^1)^I$ will play an important role in the construction of L-parameters and in that of the spectral action. This is discussed more in § 3.

Remark 2.3. — A smooth algebraic curve X over a perfect field S can be identified with the moduli space of effective degree 1 Cartier divisors on X, by taking sections of the structure morphism $X \to S$. This identification is implicit in the local Langlands programs in the function field setting. In the case of the Fargues–Fontaine curve, there is no structure morphism, so we have to work with Div^1, often referred to as the *mirror curve*. The mirror curve already played an important role in Fargues (2020b), where Fargues studies the geometrization conjecture for GL_1, i.e. the case of local class field theory, using the moduli stack of line bundles on the Fargues–Fontaine curve.

The algebraic version of the Fargues–Fontaine curve is equipped, by construction, with a natural morphism of locally ringed spaces

$$X_S \to X_S^{\mathrm{alg}}.$$

This leads to a GAGA-style comparison result for vector bundles in the two settings, originally established in Kedlaya and Liu (2015). They showed that pullback along the morphism $X_S \to X_S^{\mathrm{alg}}$ induces an equivalence of categories on the level of vector bundles. The algebraic perspective is helpful in classifying line bundles on the Fargues–Fontaine curve, and, ultimately, vector bundles, via a Beauville–Laszlo-style gluing result that is established on the level of the algebraic curve.

The algebraic perspective is also helpful in constructing Hecke correspondences over the moduli stack Bun_G of G-bundles on the Fargues–Fontaine curve, as in the diagram (15), particularly the local Hecke stack denoted by $\mathcal{H}\mathrm{ck}$. This diagram plays an important role in the geometrization of local Langlands and the local Hecke stack makes the link with the representation theory of ${}^LG$ via the geometric Satake equivalence.

2.2. G-bundles on the Fargues–Fontaine curve

Recall that, for a connected reductive group G/E, we have defined the category $\mathrm{Rep}_E G$ of finite-dimensional algebraic representations of G over E. For a scheme X over Spec E or for a sous-perfectoid adic space X over $\mathrm{Spa}(E, \mathcal{O}_E)$, let $\mathrm{Bun}(X)$ denote the category of vector bundles on X.

Definition 2.4. —

1. *A (Tannakian) G-bundle on X is an exact tensor functor*

$$\mathrm{Rep}_E G \to \mathrm{Bun}(X),$$

   which is then automatically faithful.
2. *A (cohomological) G-bundle on X is an étale sheaf $\mathcal{Q}$ on X equipped with a G-action such that, étale locally on X, we have a G-equivariant isomorphism $\mathcal{Q} \simeq G$.*
3. *A (geometric) G-bundle on X is a scheme, resp. adic space, $T \to X$, equipped with a G-action, such that, étale locally on X, there is a G-equivariant isomorphism*

$$T \simeq X \times G.$$

By Scholze and Weinstein (2020, § 9.5), all three of these notions give rise to equivalent categories $\mathrm{Bun}_G(X)$ of G-bundles on X, both in the setting of schemes and in the setting of sous-perfectoid adic spaces. The condition for X to be sous-perfectoid is imposed in order to guarantee that the product $X \times G$ is also an adic space.

Example 2.5. — For $G = \mathrm{GL}_n$, the isomorphism classes of GL_n-bundles on X are in bijection with the isomorphism classes of vector bundles of rank n on X, as follows. Given a rank n vector bundle $\mathcal{V}$ on X, we can define the associated GL_n-bundle using the cohomological definition as the étale sheaf $U \mapsto \mathcal{E}(U) := \mathrm{Isom}_U(\mathcal{V}|_U, \mathcal{O}_U^n)$. Given a GL_n-bundle $\mathcal{E}$ on X, we take $\mathcal{V}$ to be the vector bundle corresponding to the standard representation, using the Tannakian definition.

Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$. We apply Definition 2.4 to either $X := X_S^{\mathrm{alg}}$ or to $X := X_S$. The GAGA result of Kedlaya–Liu mentioned above implies that the notions are equivalent, independently of whether we work with the schematic or with the adic incarnation of the Fargues–Fontaine curve.

Recall that, in the definition of the Fargues–Fontaine curve as an adic space, we set $X_S := Y_S/\varphi^{\mathbb{Z}}$, where φ is the canonical lift of Frobenius to Y_S. We have a natural, exact $\otimes$-functor $\mathrm{Isoc}_E \to \mathrm{Bun}(X_S)$ that sends an isocrystal (V, ϕ) over E to the vector bundle $\mathcal{E}(V, \phi)$ on X_S obtained by descent from the vector bundle $V \otimes_{\breve{E}} \mathcal{O}_{Y_S}$ by taking the quotient under $\phi \otimes \varphi$. This functor can be upgraded to a functor from the category of isocrystals with G-structure over E to the category of G-bundles on X_S. In particular, if $b \in B(G)$, we denote the associated constant G-bundle on X_S by $\mathcal{E}_b$.

Definition 2.6. — *We define* Bun_G *to be the pre-stack that sends* $S \in \mathrm{Perf}_{\mathbb{F}_q}$ *to the groupoid of G-bundles on X_S.*

Fargues–Scholze prove that this pre-stack is in fact a stack for the v-topology on $\mathrm{Perf}_{\mathbb{F}_q}$ and, moreover, that it is a so-called small v-stack[(6)]. Therefore, we have a well-defined underlying topological space $|\mathrm{Bun}_G|$. The following result, known as the classification of G-bundles on the Fargues–Fontaine curve, is a first step towards understanding the geometry of Bun_G, as it describes the points of the topological space $|\mathrm{Bun}_G|$.

Theorem 2.7. — *Let (C, C^+) be a Huber pair where C is an algebraically closed perfectoid field of characteristic p. The functor*

$$G-\mathrm{Isoc}_E \to \mathrm{Bun}_G\left(X_{\mathrm{Spa}(C,C^+)}\right)$$

given by $b \mapsto \mathcal{E}_b$ induces a bijection on the level of isomorphism classes of objects.

The classification result for vector bundles, i.e. the GL_n case of Theorem 2.7, was proved first and has an intricate history. In the case of local fields of equal characteristic, the vector bundle case of Theorem 2.7 was established in Hartl and Pink (2004). In the case of p-adic fields, such as our chosen field $E/\mathbb{Q}_p$, this was originally proved in Kedlaya (2004) and then given a more elegant proof in Fargues and Fontaine (2018). A proof of the result for p-adic fields is also implicit in Colmez (2002).

The starting point for the proof of Fargues–Fontaine is to establish a version over the curve $X_{\mathrm{Spa}(C,C^+)}$ of the formalism developed in Harder and Narasimhan (1974) to study vector bundles on a smooth projective curve over an algebraically closed field, such as $\mathbb{P}^1_{\mathbb{C}}$. This leads to the notion of a semi-stable bundle of some slope $\lambda \in \mathbb{Q}$ and to a proof that a general vector bundle admits a decreasing $\mathbb{Q}$-filtration with semi-stable graded pieces, called the Harder–Narasimhan filtration.

The category Isoc_E satisfies a Harder–Narasimhan formalism as well, as can be seen from the Dieudonné–Manin classification. However, as the category Isoc_E is semi-simple, Theorem 2.7 provides a strong constraint on vector bundles on the Fargues–Fontaine curve. For example, it implies that the Harder–Narasimhan filtration on a general vector bundle (still working over an algebraically closed perfectoid field) is split. To prove their deep classification result, Fargues–Fontaine study modifications of vector bundles associated to p-divisible groups using two period morphisms coming from p-adic Hodge theory, namely the Hodge–Tate period morphism and the Hodge–de Rham period morphism. A sketch of this original proof is given in Fargues and Fontaine (2014, § 6.3).

Fargues and Scholze (2024) give a new, more streamlined proof of the classification result for vector bundles over the Fargues–Fontaine curve, which uses ideas introduced in Colmez (2002) as well as the geometry of diamonds.

Granted the result for vector bundles, the extension of Theorem 2.7 to general G was established in Fargues (2020a) over p-adic fields and subsequently in Anschütz (2019)

[(6)]This roughly means a v-stack that admits a reasonable surjection from a perfectoid space, which allows us to endow its set of points with a topology inherited from that of the perfectoid space. See Definition 12.4 of Scholze (2022) for the precise definition and Proposition 12.7 of *loc. cit.* for the construction of the underlying topological space.

over both p-adic and equal characteristic local fields. The starting point in both these references is to develop the Harder–Narasimhan formalism in the setting of G-bundles on $X_{\mathrm{Spa}(C,C^+)}$.

Assume, from now on, that the connected reductive group G is quasi-split[(7)]. Theorem 2.7 determines a bijection

$$|\mathrm{Bun}_G| \xrightarrow{\sim} B(G). \tag{9}$$

Recall that, in § 1.4, we equipped the Kottwitz set $B(G)$ with a topology using the Newton and Kottwitz invariants. Fargues–Scholze prove that the bijection (9) is continuous. This amounts to showing that the (pullback to Bun_G of) the Newton map is upper semi-continuous and that the (pullback to Bun_G of) the Kottwitz map is locally constant. Furthemore, a result of Viehmann (2024) implies that the continuous map (9) is open, hence a homeomorphism. This implies that the topology we defined on the Kottwitz set $B(G)$ reflects the specialization behaviour of G-bundles on the relative Fargues–Fontaine curve, as promised.

The functor of Theorem 2.7 is far from an equivalence of categories[(8)]. Indeed, for $b \in B(G)$, the group of automorphisms of the corresponding isocrystal with G-structure is isomorphic to $G_b(E)$. We define the v-sheaf of groups $\widetilde{G}_b$ on $\mathrm{Perf}_{\mathbb{F}_q}$ by

$$\widetilde{G}_b(S) \coloneqq \mathrm{Aut}_{X_S}(\mathcal{E}_b). \tag{10}$$

Let ρ denote half the sum of simple roots in G. Fargues–Scholze prove that $\widetilde{G}_b$ admits a semi-direct product decomposition

$$\widetilde{G}_b = \widetilde{G}_b^\circ \rtimes \underline{G_b(E)}, \tag{11}$$

where $G_b(E)$ can be identified with the group of connected components $\pi_0(\widetilde{G}_b)$ and where the identity component $\widetilde{G}_b^\circ$ is ℓ-cohomologically smooth of dimension $\langle 2\rho, \nu(b)\rangle$. In the special case when b is basic, we have an identification $\widetilde{G}_b \simeq \underline{G_b(E)}$ (note that, in this case, $\nu(b)$ is central and $\langle 2\rho, \nu(b)\rangle = 0$.) In Example 2.8 below, we discuss what $\widetilde{G}_b$ can look like for a non-basic b.

2.3. The geometry of Bun_G and the category $D(\mathrm{Bun}_G, \Lambda)$

Granted the classification result of Theorem 2.7, we define for each $b \in B(G)$, the Newton stratum in Bun_G corresponding to it as the fiber product

$$\mathrm{Bun}_G^b \coloneqq \mathrm{Bun}_G \times_{|\mathrm{Bun}_G|} \{b\}.$$

[(7)] This is not such a restrictive assumption. If G is an extended pure inner form of its unique quasi-split inner form G^* (which is always the case if $Z(G^*)$ is connected), we can still understand the geometry of Bun_G as follows. Assume that G is the extended pure inner form determined by some $\tilde{b} \in G^*(\breve{E})$. Fargues–Scholze show that there is a natural isomorphism $\mathrm{Bun}_G \xrightarrow{\sim} \mathrm{Bun}_{G^*}$ that takes the neutral point corresponding to geometrically fiberwise trivial G-bundles on Bun_G to the point on Bun_{G^*} corresponding to bundles that are geometrically fiberwise isomorphic to $\mathcal{E}_{\tilde{b}}$.

[(8)] However, it does become an equivalence of categories when working with the "absolute Fargues–Fontaine curve", see Anschütz (2023).

This is a locally closed sub-stack of Bun_G and, in the special case when b is basic, it is an open substack. Fargues–Scholze furthermore prove that we have a natural isomorphism

$$\mathrm{Bun}_G^b \simeq [*/\widetilde{G}_b], \tag{12}$$

which, in the special case when b is basic, identifies Bun_G^b with the classifying stack $[*/\underline{G_b(E)}]$ of the p-adic group $G_b(E)$. The description in (12) will be key to relating the representation theory of G and its extended pure inner forms to objects of a more geometric nature. Even when b is not basic, the connected component of identity $\widetilde{G}_b^\circ$ is, in some sense, ℓ-adically contractible, which will allow us to realize the derived category $D(G_b(E), \Lambda)$ as a certain category of ℓ-adic sheaves on the classifying stack $[*/\widetilde{G}_b]$.

Example 2.8. — We give a sense for what Bun_G looks like in the case when $G = \mathrm{GL}_2/E$. The Kottwitz invariant gives a decomposition into connected components

$$\mathrm{Bun}_G = \bigsqcup_{\alpha \in \mathbb{Z}} \mathrm{Bun}_G^\alpha.$$

Specializing the discussion from Example 1.16 to $n = 2$, we have $B(G)_{\mathrm{bas}} = \frac{1}{2}\mathbb{Z}$ and the extended pure inner form G_b is isomorphic to GL_2 if $b \in \mathbb{Z}$ and to $D^\times$ otherwise, where D is the unique quaternion algebra over E. Since the Kottwitz map can be identified with multiplication by 2, the semi-stable locus in each connected component Bun_G^α is an open subset of the form $[*/\underline{GL_2(E)}]$ if $\alpha \in 2\mathbb{Z}$, or of the form $[*/\underline{D^\times}]$ if $\alpha \in 2\mathbb{Z} + 1$.

Beyond the semi-stable locus, we see extensions of line bundles that are geometrically fiberwise isomorphic to $\mathcal{O}_{X_S}(i) \oplus \mathcal{O}_{X_S}(j)$, with $i < j \in \mathbb{Z}$ satisfying $i + j = \alpha$. For example, if b corresponds to $\mathcal{O}_{X_S} \oplus \mathcal{O}_{X_S}(1)$, we have

$$\widetilde{G}_b = \begin{pmatrix} \underline{E^\times} & \mathcal{BC}(\mathcal{O}_{X_S}(1)) \\ 0 & \underline{E^\times} \end{pmatrix},$$

where $\mathcal{BC}(\mathcal{O}_{X_S}(1))$ is a so-called relative *Banach–Colmez space.* The space $\mathcal{BC}(\mathcal{O}_{X_S}(1))$ is closely related to the universal cover in the sense of Scholze and Weinstein (2013) of the Lubin–Tate formal group for E, cf. Fargues and Scholze (2024, § II.2.1), and, therefore, it is representable by the perfectoid open unit disc $\mathrm{Spd}\ \mathbb{F}_q[\![x^{1/p^\infty}]\!]$. In particular, $\mathcal{BC}(\mathcal{O}_{X_S}(1))$ is ℓ-cohomologically smooth of dimension 1.

Remark 2.9. — Banach–Colmez spaces were first introduced in Colmez (2002). The category of Banach–Colmez spaces turns out to be closely related to the category of coherent sheaves on the Fargues–Fontaine curve. For this more modern perspective, see Le Bras (2018). In general, if $b \in B(G)$, the Harder–Narasimhan formalism implies that $\widetilde{G}_b^\circ$ is a successive extension of positive Banach–Colmez spaces, and this leads to the ℓ-cohomological smoothness result and to the dimension computation mentioned above.

Fargues and Scholze (2024, § V) study the geometry of Bun_G further and prove that it is an ℓ-cohomologically smooth Artin v-stack of dimension 0. To establish this, they construct, for each $b \in B(G)$, a chart

$$\begin{array}{ccc} \mathcal{M}_b & \xrightarrow{\pi_b} & \mathrm{Bun}_G \\ \downarrow{\scriptstyle q_b} & & \\ [*/\underline{G_b(E)}] & & \end{array} \tag{13}$$

and prove that both maps q_b and π_b are representable in locally spatial diamonds, ℓ-cohomologically smooth and partially proper. The v-stack $\mathcal{M}_b$ is defined moduli-theoretically: it parameterises G-bundles $\mathcal{E}$ equipped with an increasing $\mathbb{Q}$-filtration[9] whose associated graded is geometrically fiberwise isomorphic to the associated graded of the constant G-bundle $\mathcal{E}_b$, equipped with its Harder–Narasimhan filtration. The map q_b is going to the associated graded and the map π_b forgets the filtration. The image of $\mathcal{M}_b$ in Bun_G is open and is equal to the set of generalizations of b.

The map q_b can be described explicitly in terms of negative Banach–Colmez spaces, so the desired geometric properties are established relatively painlessly, see Example 2.10 for the flavor of the argument. On the other hand, proving the ℓ-cohomological smoothness of the map π_b in (13) relies on one of the major technical innovations of the paper, as ℓ-cohomological smoothness is a condition that is often difficult to check in practice. To achieve this, Fargues–Scholze develop a general *Jacobian criterion of smoothness* (Fargues and Scholze, 2024, § IV). This new criterion is discussed extensively in Fargues (2024).

Example 2.10. — We continue Example 2.8 here, to give some more intuition for the geometry of Bun_G. Let $b_s \in B(G)$ correspond to rank 2 vector bundles that are geometrically fiberwise isomorphic to $\mathcal{O}_{X_S} \oplus \mathcal{O}_{X_S}(1)$. This point has one generalization b_η in Bun_G, corresponding to the semi-stable bundle $\mathcal{O}_{X_S}(\frac{1}{2})$ of rank 2. (In other words, $\mathcal{O}_{X_S}(\frac{1}{2})$ can degenerate to $\mathcal{O}_{X_S} \oplus \mathcal{O}_{X_S}(1)$.) The v-stack $\mathcal{M}_{b_s}$ parameterizes short exact sequences

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0,$$

where $\mathcal{L}$ is geometrically fiberwise isomorphic to $\mathcal{O}_{X_S}$ and $\mathcal{L}'$ is geometrically fiberwise isomorphic to $\mathcal{O}_{X_S}(1)$. Fixing such isomorphisms leads to a Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{b_s} & \longrightarrow & \mathcal{M}_{b_s} \\ \downarrow & & \downarrow{\scriptstyle q_b} \\ * & \longrightarrow & [*/\underline{E^\times} \times \underline{E^\times}] \end{array} \tag{14}$$

where $\widetilde{\mathcal{M}}_{b_s}$ parameterizes extensions of $\mathcal{O}_{X_S}(1)$ by $\mathcal{O}_{X_S}$ and is thus isomorphic to the negative Banach–Colmez space $\mathcal{BC}(\mathcal{O}_{X_S}(-1)[1])$. In this example, one can also understand the individual fibers over b_s and b_η, see Fargues and Scholze (2024, Example V.3.1).

[9] The Harder–Narasimhan filtration decreases with the slope, so this is the opposite condition.

We now discuss the category of ℓ-adic sheaves on Bun_G considered by Fargues–Scholze. Let Λ be a $\mathbb{Z}_\ell$-algebra. The desired category should have the property that its restriction to Bun_G^b recovers the derived category of smooth representations $D(G_b(E), \Lambda)$ under the the isomorphism (12) and the decomposition (11). If Λ is ℓ-power torsion, one can consider the category $D(\mathrm{Bun}_G, \Lambda)$ defined in Scholze (2022). If $\Lambda = \overline{\mathbb{Q}}_\ell$, the usual procedure for passing from torsion coefficients to $\overline{\mathbb{Q}}_\ell$-coefficients via completion would give rise to the wrong category of representations of $G_b(E)$. Instead, Fargues–Scholze use the theory of solid modules developed in Clausen and Scholze (2025+) to define $D(\mathrm{Bun}_G, \Lambda)$ in general and to prove the following result.

Theorem 2.11. —

1. *The Newton stratification on* Bun_G *induces, via excision triangles, an infinite semi-orthogonal decomposition of* $D(\mathrm{Bun}_G, \Lambda)$ *in terms of the categories* $D(\mathrm{Bun}_G^b, \Lambda)$ *for* $b \in B(G)$.
2. *For each* $b \in B(G)$*, the isomorphism* (12) *and the decomposition* (11) *induce a morphism*

$$\mathrm{Bun}_G^b \simeq [*/\widetilde{G}_b] \to [*/\underline{G_b(E)}].$$

Pullback along this morphism induces an equivalence

$$D([*/\underline{G_b(E)}], \Lambda) \simeq D(\mathrm{Bun}_G^b, \Lambda),$$

and we have a further equivalence $D([*/\underline{G_b(E)}], \Lambda) \simeq D(G_b(E), \Lambda)$ *to the derived category of smooth representations of* $G_b(E)$ *on* Λ*-modules.*

In light of Theorem 2.11, $D(\mathrm{Bun}_G, \Lambda)$ can be thought of as a geometrization of the derived category of smooth representations $D(G(E), \Lambda)$. Indeed, we have a fully faithful embedding $D(G(E), \Lambda) \hookrightarrow D(\mathrm{Bun}_G, \Lambda)$ coming from the open inclusion of the neutral point corresponding to $b = 1$ into Bun_G. Fargues–Scholze geometrize further notions from representation theory, such as admissible representations, smooth duality, Bernstein–Zelevinsky duality etc. The notion of parabolic induction is geometrized via the functor of geometric Eisenstein series, which is studied further in Hamann (2025b) and Hamann, Hansen, and Scholze (2024).

Furthermore, Fargues–Scholze prove that the category $D(\mathrm{Bun}_G, \Lambda)$ is compactly generated. The charts (13) (and the Jacobian criterion of smoothness, which was used in understanding them) play, once again, a crucial role. These charts are used to construct an explicit set of compact generators, which generalize the set of compact generators $\left\{\text{c-Ind}_K^{G(E)} \Lambda\right\}$ of $D(G(E), \Lambda)$, as K runs over open pro-p-subgroups of $G(E)$. Later on, the existence of a set of compact generators is used in a critical way to prove the main results of Fargues–Scholze, particularly in their construction of the spectral action.

3. Connecting the automorphic side to the spectral side

The category $D(\mathrm{Bun}_G, \Lambda)$ lives on the automorphic side of the local Langlands conjecture. In order to connect this to the Galois, or spectral side, Fargues–Scholze were inspired by developments from the global geometric Langlands program over $\mathbb{C}$ and from the global Langlands program over function fields. Working with the Fargues–Fontaine curve and with its mirror $\mathrm{Div}^1 = \mathrm{Spd}\ \breve{E}/\varphi^{\mathbb{Z}}$, Fargues–Scholze implement in the setting of p-adic fields ideas and constructions due to Beilinson, Drinfeld, Gaitsgory, L. Lafforgue, V. Lafforgue and others, and developed over several decades.

In particular, Fargues–Scholze attach semi-simple local Langlands parameters to irreducible smooth representations of $G(E)$ (and its extended pure inner forms $G_b(E)$) using a method introduced by V. Lafforgue to construct semi-simple global Langlands parameters in the function field setting (Lafforgue, 2018). See Stroh (2017) and Heinloth (2018) for surveys of V. Lafforgue's work.

We give a brief overview of the construction of semi-simple local Langlands parameters due to Fargues–Scholze because it is a more concrete aspect of their work. The starting point is to consider moduli stacks over Bun_G that parameterise modifications of G-bundles on the relative Fargues–Fontaine curve and whose relative étale cohomology also produces representations of the Weil group W_E. These are mixed-characteristic analogues of the moduli spaces of local G-shtukas that were used to study the Langlands conjectures in the function field setting. To make this more precise, let Λ be a $\mathbb{Z}_\ell[\sqrt{q}]$-algebra and let I be a finite set. Fargues–Scholze construct a diagram of v-stacks of the form

$$\begin{array}{ccccc} & & \mathrm{Hck}_G^I & \xrightarrow{\ q\ } & \mathcal{H}\mathrm{ck}_G^I \\ & \swarrow^{p_1} & & \searrow^{p_2} & & \searrow \\ \mathrm{Bun}_G & & & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \longrightarrow (\mathrm{Div}^1)^I. \end{array} \tag{15}$$

This diagram has the following moduli-theoretic interpretation:

- For $S \in \mathrm{Perf}_{\mathbb{F}_q}$, a section $S \to (\mathrm{Div}^1)^I$ parameterizes a set of closed Cartier divisors of X_S of degree 1 labelled by I. We denote by $D_S \subset X_S$ the union of these divisors.
- The *global Hecke stack* $\mathrm{Hck}_G^I \to (\mathrm{Div}^1)^I$ parameterizes in addition two G-bundles $(\mathcal{E}_1, \mathcal{E}_2)$ on X_S together with an isomorphism

$$f\colon \mathcal{E}_1 \mid_{X_S \setminus D_S} \xrightarrow{\sim} \mathcal{E}_2 \mid_{X_S \setminus D_S}$$

that is meromorphic along D_S[10]. This latter piece of data is denoted by

$$(f\colon \mathcal{E}_1 \dashrightarrow \mathcal{E}_2)$$

[10] The meromorphy condition is the following: for each representation in $\mathrm{Rep}_E G$, the Tannakian perspective induces an isomorphism of the corresponding vector bundles $\mathcal{F}_1 \mid_{X_S \setminus D_S} \xrightarrow{\sim} \mathcal{F}_2 \mid_{X_S \setminus D_S}$. This should extend to a morphism $\mathcal{F}_1 \to \mathcal{F}_2(kD_S)$ for some $k \gg 0$, where we have the natural inclusion of vector bundles $\mathcal{F}_2 \hookrightarrow \mathcal{F}_2(kD_S)$ by allowing poles along D_S.

and is called a meromorphic modification of G-bundles on X_S. The maps p_1 and p_2 form a Hecke correspondence: they send a tuple $(S \to (\mathrm{Div}^1)^I, \mathcal{E}_1, \mathcal{E}_2, f)$ to $\mathcal{E}_1$, respectively to $(\mathcal{E}_2, S \to (\mathrm{Div}^1)^I)$.

– The *local Hecke stack* $\mathcal{H}\mathrm{ck}_G^I \to (\mathrm{Div}^1)^I$ also parameterizes a meromorphic modification of G-bundles $(f : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2)$, except that these G-bundles are only defined on the completion of X_S along D_S. (We work locally on S to make sense of this completion, via Frobenius equivalence classes of untilts $S^\# \hookrightarrow X_S$ of S.) There is a natural restriction map $q\colon \mathrm{Hck}_G^I \to \mathcal{H}\mathrm{ck}_G^I$.

The L-group ${}^L G = \widehat{G} \rtimes W_E$ enters the picture (15) via the *geometric Satake equivalence* and via the local Hecke stack. We briefly discuss the geometric Satake equivalence in the setting of Fargues–Scholze in § 3.1, but for now the upshot is the following. Let Q be the finite quotient of W_E through which the action of W_E on $\widehat{G}$ factors. We will use the semi-direct product $\widehat{G} \rtimes Q$ instead of the L-group because it has a model as an algebraic group over Λ. The geometric Satake equivalence tells us that each algebraic representation $V \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$ determines a certain kind of perverse sheaf $\mathcal{S}_V$, the so-called Satake sheaf, on the local Hecke stack $\mathcal{H}\mathrm{ck}_G^I$. The diagram (15) allows us then to define a Hecke operator $T_{V,I}$ via the formula

$$T_{V,I}\colon A \mapsto p_{2,*}(p_1^* A \otimes_\Lambda^{\mathbb{L}} q^* \mathcal{S}_V), \tag{16}$$

where $\otimes^{\mathbb{L}}$ denotes a derived tensor product.

The $T_{V,I}$ are a priori functors from $D(\mathrm{Bun}_G, \Lambda)$ to $D(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda)$. However, using the formalism of condensed mathematics, the target can ultimately be identified with the category $D(\mathrm{Bun}_G, \Lambda)^{BW_E^I}$ of $(W_E)^I$-equivariant objects in $D(\mathrm{Bun}_G, \Lambda)$[(11)]. This also relies on the identification between local systems of Λ-modules on $(\mathrm{Div}^1)^I$ and continuous representations of $(W_E)^I$ on finite projective Λ-modules, a version of Drinfeld's lemma (Drinfeld, 1980, Theorem 1.2) in the setting of the mirror curve Div^1.

The monoidal structure of the geometric Satake equivalence implies that the Hecke operators $T_{V,I}$ commute and that, for two elements $V, W \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$, the composition $T_{V,I} \circ T_{W,I}$, when restricted to the diagonal copy of W_E^I, is naturally isomorphic to $T_{V \otimes W, I}$. Furthermore, the Hecke operators preserve the subcategory $D(\mathrm{Bun}_G, \Lambda)^\omega \subset D(\mathrm{Bun}_G, \Lambda)$ of compact objects (Fargues and Scholze, 2024, Theorem IX.2.2). The upshot is that we can package the system of Hecke operators $(T_{V,I})_V$ into an exact $\mathrm{Rep}_\Lambda Q^I$-linear monoidal functor

$$T_I\colon \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda\left(D(\mathrm{Bun}_G, \Lambda)^\omega\right)^{BW_E^I}, V \mapsto T_{V,I}. \tag{17}$$

The Hecke functors T_I are also functorial with respect to the finite sets I. Using V. Lafforgue's idea of excursion operators, this turns out to be the categorical structure needed in order to construct L-parameters up to semi-simplification. More precisely, we use the notion of an *excursion datum*: a tuple $(I, V, \alpha, \beta, (\gamma_i)_{i \in I})$, consisting of a

(11) To define the notion of W_E^I-equivariant objects, one needs to upgrade $D(\mathrm{Bun}_G, \Lambda)$ to a condensed ∞-category, denoted by $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ in Fargues and Scholze (2024, § 9). The notation BW_E^I refers to the classifying stack of W_E^I.

finite set I, a representation $V \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$, $\widehat{G}$-equivariant morphisms $\alpha : \mathbb{1} \to V|_{\widehat{G}}$, $\beta\colon V|_{\widehat{G}} \to \mathbb{1}$, and elements $\gamma_i \in W_E$ for $i \in I$. Here, the restriction $V\ |_{\widehat{G}}$ is to the diagonal copy $\widehat{G} \subset \widehat{G}^I \subset (\widehat{G} \rtimes Q)^I$. For any $A \in D(\mathrm{Bun}_G, \Lambda)$, the functoriality of the T_I with respect to I induces an endomorphism of A via the composition

$$A = T_{\mathbb{1}}(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_{\mathbb{1}}(A) = A \tag{18}$$

This endomorphism is known as an *excursion operator* and its construction is explained in detail in § 3.3.

Set $\Lambda = \overline{\mathbb{Q}}_\ell$ and consider *Schur-irreducible* objects $A \in D(\mathrm{Bun}_G, \Lambda)$, i.e. objects with $\mathrm{End}(A) = \Lambda$. For example, irreducible smooth representations of $G(E)$ on Λ-modules are admissible and, therefore, they give rise to Schur-irreducible objects in $D(\mathrm{Bun}_G, \Lambda)$. Explicitly, by Theorem 2.11, such a representation π determines a sheaf on the neutral point $\mathcal{F}_\pi \in D(\mathrm{Bun}^1_G, \Lambda)$ and we consider its extension by zero to all of Bun_G. For any Schur-irreducible object A, the endomorphism defined in (18) acts by a scalar. Fargues–Scholze conclude by the following proposition, which is a version of Lafforgue (2018, Proposition 11.7).

Proposition 3.1. — *For any Schur-irreducible object $A \in D(\mathrm{Bun}_G, \Lambda)$, there exists a unique (up to $\widehat{G}(\Lambda)$-conjugacy) semi-simple L-parameter $\varphi_{\mathrm{FS},A}\colon W_E \to \widehat{G}(\Lambda) \rtimes W_E$ such that, for any excursion datum $(I, V, \alpha, \beta, (\gamma_i)_{i\in I})$, the corresponding excursion operator*

$$A = T_{\mathbb{1}}(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_{\mathbb{1}}(A) = A$$

acts via the scalar

$$\Lambda \xrightarrow{\alpha} V \xrightarrow{(\varphi_{\mathrm{FS},A}(\gamma_i))_{i\in I}} V \xrightarrow{\beta} \Lambda.$$

Excursion operators turn out to be closely related to regular functions on the quotient, in the sense of geometric invariant theory, of the moduli stack of L-parameters up to $\widehat{G}(\Lambda)$-conjugacy. This fact underlies the proof of Proposition 3.1 and, therefore, the construction of semi-simple L-parameters. We discuss the moduli stack of L-parameters in § 3.2 and we explain the connection to excursion operators in § 3.3. Furthermore, we explain how the construction of semi-simple L-parameters can be deduced from the existence of a morphism on the level of Bernstein centers, which can even be defined integrally (under a technical assumption on ℓ).

Remark 3.2. — Fargues–Scholze prove that their construction $\pi \mapsto \varphi_{\mathrm{FS},\pi}$ of semi-simple L-parameters satisfies some of the desiderata of local Langlands conjectures. For example, they establish compatibility with local class field theory in the case when G is a torus and compatibility with parabolic induction in general.

They also show that, when $G = \mathrm{GL}_n$, their construction recovers the local Langlands correspondence of Harris–Taylor and Henniart, up to semi-simplification, i.e. in the sense of the formula (3) and Remark 1.7. This compatibility was extended to all inner forms of GL_n by Hansen, Kaletha, and Weinstein (2022). The compatibility with more classical approaches to Conjecture 1.8 is also known for GSp_4 and its inner forms (Hamann, 2025a), for odd unitary groups (Bertoloni Meli, Hamann, and Nguyen, 2024) and in a number of other special cases. These compatibility results are typically established using global Shimura varieties and the trace formula.

Remark 3.3. — A version of V. Lafforgue's method was also used in the work of Genestier–Lafforgue on the local Langlands correspondence over function fields (Genestier and Lafforgue, 2018) to construct semi-simple local Langlands parameters in that setting. On the other hand, the method of Fargues–Scholze also works over local fields of equal characteristic. The compatibility between these two constructions was proved in Li-Huerta (2023).

Remark 3.4. — The result that the Hecke operators $T_{V,I}$ preserve the subcategory of compact objects $D(\mathrm{Bun}_G, \Lambda)^\omega \subset D(\mathrm{Bun}_G, \Lambda)$ implies non-trivial finiteness results for the cohomology of Rapoport–Zink spaces and, more generally, for the cohomology of local Shimura varieties (Fargues and Scholze, 2024, § IX.3). This gives unconditional proofs and refinements of results of Rapoport and Viehmann (2014, § 6).

The system of Hecke functors $(T_I)_I$ from (17) contains much more information than just the construction of semi-simple L-parameters. This categorical structure allowed Fargues–Scholze to construct the spectral action and to formulate the local Langlands conjecture as an equivalence of categories, inspired by analogous developments in the setting of the geometric Langlands program (Arinkin and Gaitsgory, 2015; Nadler and Yun, 2019; Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, 2022). We discuss the spectral action and the categorical local Langlands conjecture over p-adic fields in § 3.4.

3.1. The geometric Satake equivalence

In this subsection, we discuss the geometric Satake equivalence, with an emphasis on what is new in the work of Fargues–Scholze compared to earlier approaches. An excellent survey on the geometric Satake equivalence that pre-dates this work is Zhu (2017b).

Classically, the geometric Satake equivalence relates the representation theory of the L-group ${}^LG$ to the geometry of the affine Grassmannian for the original group G. For example, if G is a connected reductive group over $\mathbb{C}$, Mirković and Vilonen (2007), building on Lusztig (1983), Ginzburg (1990), and Beilinson and Drinfeld (unpublished), established a monoidal equivalence of categories between the category of algebraic representations of the Langlands dual group $\widehat{G}/\overline{\mathbb{Q}}_\ell$ and the so-called Satake category, i.e. the category of ℓ-adic perverse sheaves on the affine Grassmannian Gr_G, which are equivariant with respect to the action of the loop group L^+G.

The monoidal structure on the Satake category is given by the convolution product of sheaves. To establish the geometric Satake equivalence, one needs to know that the Satake category is a Tannakian category and, in particular, that this convolution product is commutative. This is achieved by comparing it to the another operation, the fusion product, which is defined via the so-called Beilinson–Drinfeld Grassmannian, a generalization of the affine Grassmannian Gr_G.

In our mixed characteristic setting, i.e. for a connected reductive group G over a p-adic field E, the geometric Satake equivalence was established in Zhu (2017a) with $\overline{\mathbb{Q}}_\ell$-coefficients, using a version of the affine Grassmannian defined via Witt vectors. However, the analogue of the Beilinson–Drinfeld Grassmannian does not make sense in Zhu's setting. Instead, Zhu used the equal characteristic version of the geometric Satake equivalence at a crucial point in his proof, to establish the commutativity constraint via a combinatorial identity and without directly appealing to the fusion product. Zhu's result was later refined to work with $\mathbb{Z}_\ell$- and $\mathbb{F}_\ell$-coefficients by Yu (2022).

We consider the form of the L-group of G given by $\widehat{G} \rtimes Q$. To construct the system of functors $(T_I)_I$ from (17), indexed by finite sets I, Fargues–Scholze need to relate the representation theory of $(\widehat{G} \rtimes Q)^I$ to the geometry of the local Hecke stack $\mathcal{H}\mathrm{ck}_G^I$, a small v-stack on $\mathrm{Perf}_{\mathbb{F}_q}$ and, thus, an object of p-adic geometry. When $I = \{*\}$ consists of one element, the local Hecke stack $\mathcal{H}\mathrm{ck}_G^{\{*\}}$ is the analogue in p-adic geometry of the stack quotient $[\mathrm{L}^+G\backslash\mathrm{Gr}_G]$ and thus closely related to the Witt vector affine Grassmannian considered by Zhu and Yu. When I has more than one element, the local Hecke stack $\mathcal{H}\mathrm{ck}_G^I$ is closely related to the analogue of the Beilinson–Drinfeld Grassmannian, which now makes sense as an object of p-adic geometry, via the constructions of Scholze and Weinstein (2020). This additional flexibility of p-adic geometry allows Fargues–Scholze to reprove the results of Zhu and Yu, but also to establish the additional functoriality properties needed for the construction for the construction of semi-simple L-parameters and of the spectral action.

However, working in the setting of p-adic geometry makes the definition of the Satake category of sheaves on $\mathcal{H}\mathrm{ck}_G^I$ more difficult. For example, it is more subtle to define the perverse t-structure because there is not, in general, a good notion of perversity in p-adic geometry. The problem is that there is not a well behaved notion for the dimension of a point on an adic space. Instead, Fargues–Scholze define a notion of relative perversity[(12)] with respect to the morphism $\mathcal{H}\mathrm{ck}_G^I \to (\mathrm{Div}^1)^I$. This amounts to perversity in all the geometric fibers, where the dimension can be defined "by hand". To show that this notion of relative perversity is well-defined and well-behaved under convolution, Fargues–Scholze prove a version of the hyperbolic localization theorem of Braden (2003) in the setting of p-adic geometry.

Finally, one can define the Satake category $\mathrm{Sat}_G^I(\Lambda)$ as a certain category of perverse sheaves on $\mathcal{H}\mathrm{ck}_G^I$ that are, in addition, flat over Λ and universally locally acyclic with respect to $(\mathrm{Div}^1)^I$ (Fargues and Scholze, 2024, Definition VI.0.1). There is a monoidal equivalence of categories between $\mathrm{Sat}_G^I(\Lambda)$ and the category of representations $\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$ (Fargues and Scholze, 2024, Theorem VI.0.2). The proof of this latter result goes

[(12)]This construction led to the observation that there is a good notion of relative perversity in usual algebraic geometry (Hansen and Scholze, 2023). This is shown to be equivalent to perversity in all the geometric fibers, using the fact that nearby cycles preserve perversity.

along similar general lines as the proof of Mirković and Vilonen (2007), employing the convolution and the fusion products, hyperbolic localization, and a Tannakian reconstruction of the group scheme $\widehat{G}$ with its W_E-action[13].

3.2. The moduli stack of local Langlands parameters

The content of Theorem 2.11 is that the category $D(\mathrm{Bun}_G, \Lambda)$ geometrizes the category $D(G(E), \Lambda)$ of smooth representations of $G(E)$. This takes place on the automorphic side. In order to upgrade the local Langlands conjectures, for example Conjecture 1.15, to an equivalence of categories, we need to define the counterpart of $D(\mathrm{Bun}_G, \Lambda)$ on the spectral side. In this subsection, we briefly discuss the underlying geometric object, namely the moduli stack of L-parameters.

There are several approaches to constructing such a moduli stack over $\mathbb{Z}_\ell$, due to Dat, Helm, Kurinczuk, and Moss (2025), Zhu (2021), and Fargues and Scholze (2024). An excellent recent survey on this subject is Dat (2022), which discusses the history of the problem, motivates the definition of L-parameter used for the general moduli problem, and compares the different approaches mentioned above.

In this discussion, we will follow the approach of Fargues–Scholze. In § 1.2, we have seen three possible definitions for the notion of an L-parameter valued in $\overline{\mathbb{Q}}_\ell$, namely:

- an L-parameter $\varphi \colon W_E \times \mathrm{SL}_2(\overline{\mathbb{Q}}_\ell) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ as in Definition 1.4;
- a Frobenius semi-simple Weil–Deligne Langlands parameter (ρ, N), where $\rho \colon W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$ is an L-homomorphism with open kernel and N is the monodromy operator;
- an ℓ-adically continuous L-homomorphism $\varphi \colon W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$, such that the associated Weil–Deligne parameter is Frobenius semi-simple.

These notions give the same equivalence classes over $\overline{\mathbb{Q}}_\ell$ (the first and the second by the Jacobson–Morozov theorem, the second and the third by Grothendieck's ℓ-adic monodromy theorem), but they do not give rise to the same moduli spaces. As explained in Dat (2022), it is the *third definition*, without the Frobenius semi-simplicity requirement, that gives the "correct" moduli space over $\mathbb{Z}_\ell$ from the perspective of the categorical local Langlands conjecture.[14] However, when working with the third definition, the problem is how to define the notion of continuity for an L-parameter valued in a general $\mathbb{Z}_\ell$-algebra Λ.

To solve this problem, Fargues–Scholze use once again the formalism of condensed mathematics. More precisely, let Λ be any $\mathbb{Z}_\ell$-algebra. This can be viewed as a condensed

[13]More precisely, the W_E-action on $\widehat{G}$ that arises from Tannaka duality only agrees with the one in the definition of the L-group ${}^L G$ up to an explicit cyclotomic twist, which can be trivialized if $\sqrt{q} \in \Lambda$.
[14]In fact, the third definition gives the only correct moduli stack over $\mathbb{Z}_\ell$ already from the perspective of the local Langlands correspondence in families of Emerton and Helm (2014). The second definition, without the Frobenius semi-simplicity condition, does give the correct moduli stack over $\mathbb{Q}_\ell$. More recently, Scholze (2025, §4) constructed a canonical moduli stack over $\mathbb{Z}\left[\frac{1}{p}\right]$, using something closer in spirit to the second definition.

algebra that is relatively discrete over $\mathbb{Z}_\ell$ via $\Lambda_{\mathrm{disc}} \otimes_{\mathbb{Z}_{\ell,\mathrm{disc}}} \mathbb{Z}_\ell$. Fargues–Scholze define the notion of an L-parameter for G with coefficients in Λ as a section

$$\varphi \colon W_E \to \widehat{G}(\Lambda) \rtimes W_E$$

of the natural map $\widehat{G}(\Lambda) \rtimes W_E \to W_E$ of condensed groups. This is equivalent to the datum of a condensed 1-cocycle

$$\phi \colon W_E \to \widehat{G}(\Lambda)$$

for the usual W_E-action on $\widehat{G}$. With this definition, we have the following result.

THEOREM 3.5. — *There exists a scheme $Z^1(W_E, \widehat{G})$ over $\mathbb{Z}_\ell$ whose Λ-valued points are the L-parameters for G with coefficients in Λ.*

The scheme $Z^1(W_E, \widehat{G})$ is a union of open and closed subschemes $Z^1(W_E/P, \widehat{G})$, with P running over open subgroups of the wild inertia subgroup in W_E that are normal in W_E. Each $Z^1(W_E/P, \widehat{G})$ is a flat local complete intersection over $\mathbb{Z}_\ell$ of dimension $\dim G = \dim \widehat{G}$.

The representability of the moduli problem would be straightforward to prove if W_E was a discrete and finitely generated group, by considering a system of generators of W_E. The theorem is proved by reducing to this case, in two steps. Firstly, because the wild inertia subgroup is pro-p, each L-parameter for G with coefficients in a $\mathbb{Z}_\ell$-algebra Λ factors through some open subgroup P of the wild inertia subgroup that is a normal subgroup of W_E. The moduli space of L-parameters for G breaks up as a disjoint union of open and closed subspaces of L-parameters according to the kernel P of the L-parameter restricted to the wild inertia. This reduces the problem to groups of the form W_E/P.

The second step is to "discretize" the tame inertia, i.e. to replace W_E/P by the discrete dense subgroup $W \subset W_E/P$ given by the preimage of the subgroup $\tau^{\mathbb{Z}[\frac{1}{p}]} \rtimes \sigma^{\mathbb{Z}}$ of the tame quotient of W_E, where σ is a lift of the arithmetic Frobenius and τ is a topological generator of tame inertia. One checks that the discretization step gives an equivalent moduli problem, by checking that any 1-cocycle $W \to \widehat{G}(\Lambda)$ extends uniquely to a condensed 1-cocycle $W_E/P \to \widehat{G}(\Lambda)$. The nice geometric properties of each $Z^1(W_E/P, \widehat{G}) \coloneqq Z^1(W, \widehat{G})$ are deduced from the relatively simple structure of the tame quotient of W, which is only subject to the relation $\sigma\tau\sigma^{-1} = \tau^q$.

Remark 3.6. — In the $\ell = p$ case, the action of the wild inertia subgroup on a p-adic Galois representation can be highly non-trivial — there is no simple analogue of Grothendieck's ℓ-adic monodromy theorem. Nevertheless, the analogous moduli stack has been constructed in this case by Emerton and Gee (2023). In that setting, the correct notion that interpolates well in families turns out to be a (φ, Γ)-module, a notion coming from p-adic Hodge theory.

The scheme $Z^1(W_E, \widehat{G})$ is equipped with a natural action of $\widehat{G}$ coming from the moduli problem. We define the *moduli stack of Langlands parameters* to be the stack quotient $Z^1(W_E, \widehat{G})/\widehat{G}$. Using geometric invariant theory, Fargues–Scholze construct a concrete bijection between the $\overline{\mathbb{Q}}_\ell$-valued points of the associated coarse moduli space $Z^1(W_E, \widehat{G})/\!/\widehat{G}$ (i.e. the quotient taken in the category of schemes) and semi-simple L-parameters $\varphi\colon W_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes W_E$. Furthermore, they prove the following result about the $\widehat{G}$-action on each $Z^1(W_E/P, \widehat{G})$.

Theorem 3.7. — *Assume that ℓ does not divide the order of the torsion subgroup of $\pi_1(\widehat{G})$. Then $H^i\left(\widehat{G}, \mathcal{O}(Z^1(W_E/P, \widehat{G}))\right) = 0$ for $i > 0$ and the formation of the ring of invariants $\mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}$ commutes with any base change.*

We set $\operatorname{Exc}(W, \widehat{G}) := \operatorname{colim}_{n, I_n \to W} \mathcal{O}(Z^1(I_n, \widehat{G}))^{\widehat{G}}$*, where the colimit runs over all $n \in \mathbb{Z}_{\geq 1}$ and over all maps from a free group I_n on n generators to $W \subset W_E/P$.*[15] *We then have an isomorphism*

$$\operatorname{Exc}(W, \widehat{G}) \xrightarrow{\sim} \mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}}. \tag{19}$$

Remark 3.8. — Even if ℓ does divide the order of the torsion subgroup of $\pi_1(\widehat{G})$, there exists a morphism

$$\operatorname{Exc}(W, \widehat{G}) \to \mathcal{O}(Z^1(W_E/P, \widehat{G}))^{\widehat{G}},$$

which is a universal homeomorphism of finite type $\mathbb{Z}_\ell$-algebras and which becomes an isomorphism after inverting ℓ (Fargues and Scholze, 2024, § VIII.3.2).

We define the *spectral Bernstein center* to be the algebra of regular functions on the coarse moduli space of L-parameters for G:

$$\mathcal{Z}^{\mathrm{spec}}(G, \Lambda) := \mathcal{O}(Z^1(W_E, \widehat{G})_\Lambda)^{\widehat{G}}. \tag{20}$$

The second part of Theorem 3.7 can be thought of as a presentation of $\mathcal{Z}^{\mathrm{spec}}(G, \Lambda)$ in terms of an algebra generated by excursion operators. The reason for this interpretation will be clarified in § 3.3.

3.3. Excursion operators

In this subsection, we discuss excursion operators in more detail and we connect them to the moduli stack of L-parameters discussed in the previous subsection.

We assume for simplicity that the group G is split over E, so that we only need to consider the Langlands dual group $\widehat{G}$. We also assume, for now, that W is a discrete group. We let $\mathcal{C}$ be a $\mathbb{Z}_\ell$-linear category and denote by $\operatorname{End}(\mathrm{id}_{\mathcal{C}})$ the Bernstein center of $\mathcal{C}$. Assume that we have, for each finite set I, a monoidal functor

$$T_I\colon \operatorname{Rep}_{\mathbb{Z}_\ell} \widehat{G}^I \to \operatorname{End}(\mathcal{C})^{BW^I}, V \mapsto T_{I,V}.$$

[15] Here, the action of I_n on $\widehat{G}$ comes via the map $I_n \to W$ and the action of W on $\widehat{G}$. We can identify $Z^1(I_n, \widehat{G})$ with $\widehat{G}^n$, with $\widehat{G}$-action given by simultaneous twisted conjugation.

Assume also that the functors $(T_I)_I$ satisfy an additional functoriality in I. This abstract situation models the properties of the Hecke functors constructed by Fargues–Scholze, including the additional functoriality in I. Using excursion operators, we will show that this data induces a morphism of $\mathbb{Z}_\ell$-algebras

$$\mathrm{Exc}(W,\widehat{G}) \to \mathrm{End}(\mathrm{id}_{\mathcal{C}}).$$

We make the construction of excursion operators more explicit, using the original perspective introduced in Lafforgue (2018). For any map $\zeta\colon I \to J$ of finite sets, we have a morphism of groups $\widehat{G}^J \to \widehat{G}^I$ induced by restriction along ζ; explicitly, this is given by $(g_j)_{j\in J} \mapsto (g_{\zeta(i)})_{i\in I}$. This induces a restriction morphism on the level of representations $\mathrm{Rep}_{\mathbb{Z}_\ell}\widehat{G}^I \to \mathrm{Rep}_{\mathbb{Z}_\ell}\widehat{G}^J$ that we denote by $V \mapsto V^\zeta$. The functoriality condition in I implies[16] that we have a system of isomorphisms

$$\chi_{\zeta,V}\colon T_{I,V} \xrightarrow{\sim} T_{J,V^\zeta},$$

indexed by maps $\zeta : I \to J$ of finite sets and by $V \in \mathrm{Rep}_{\mathbb{Z}_\ell}\widehat{G}^I$, and satisfying the following additional compatibilities:

- they are functorial in V;
- they are W^J-equivariant, where the action of W^J on the LHS is induced by restriction along the diagonal morphism $W^J \to W^I$;
- they are compatible with the composition of maps of finite sets.

Choose an excursion datum $(I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ with I non-empty and let $\zeta : I \to \{*\}$ denote the unique map. The corresponding map on the level of groups is the diagonal morphism $\Delta\colon \widehat{G} \to \widehat{G}^I$. The corresponding excursion operator is the composition

$$\begin{array}{ccccccc} \mathrm{id}_{\mathcal{C}} & \xrightarrow{\sim} & T_{\mathbb{1},\{*\}} & \xrightarrow{T_{\{*\}}(\alpha)} & T_{V^\zeta,\{*\}} & \xrightarrow{\sim} & T_{V,I} \\ &&&&&& \downarrow{\scriptstyle (\gamma_i)_{i\in I}} \\ \mathrm{id}_{\mathcal{C}} & \xleftarrow[\sim]{} & T_{\mathbb{1},\{*\}} & \xleftarrow[T_{\{*\}}(\beta)]{} & T_{V^\zeta,\{*\}} & \xleftarrow[\sim]{} & T_{V,I} \end{array} \tag{21}$$

This consists of three steps:

- a creation step induced by α, which creates I "legs", i.e. closed Cartier divisors of degree 1 on the Fargues–Fontaine curve, where the modifications of G-bundles parameterized by I copies of the Hecke stack should be supported;
- the action of $(\gamma_i)_{i\in I}$, which moves the I legs independently;
- an annihilation step induced by β, which annihilates the I legs.

[16] The precise condition in Fargues and Scholze (2024) is that we view both $(\mathrm{Rep}_{\mathbb{Z}_\ell}\widehat{G}^I)_I$ and $\left(\mathrm{End}(\mathcal{C})^{BW^I}\right)_I$ as coCartesian fibrations over the category of finite sets and the functors $(T_I)_I$ are required to lift to the total space of these coCartesian fibrations. This condition is morally equivalent to the explicit conditions on the isomorphisms $(\chi_{\zeta,V})_{\zeta,V}$, but only if these explicit conditions are understood to include all higher coherences.

Using the properties of the system of functors $(T_I)_I$, one can prove that excursion operators satisfy certain important compatibilities, which make the link to the algebra of regular functions $\mathcal{O}(Z^1(W,\widehat{G}))^{\widehat{G}}$.

Indeed, we can view α as an element in $V^{\Delta(\widehat{G})}$ and β as an element in $(V^\vee)^{\Delta(\widehat{G})}$, where $V^\vee$ denotes the $\mathbb{Z}_\ell$-dual of V. A first observation is that the excursion operator only depends on (V,α,β) via the function $f\colon \widehat{G}^I \to \mathbb{Z}_\ell$ given by $f((g_i)) = \langle \beta, (g_i)\alpha\rangle$. Furthermore, the function f is invariant under the diagonal action of $\widehat{G}$ on both the left and the right. If we rewrite our non-empty finite set I as $I \cup \{*\}$, we can equivalently define excursion operators in terms of elements of $\mathcal{O}[\widehat{G}^I]^{\widehat{G}}$, where the $\widehat{G}$-action is given by simultaneous conjugation. In fact, we define a morphism of $\mathbb{Z}_\ell$-modules

$$\Theta^I\colon \mathcal{O}[\widehat{G}^I]^{\widehat{G}} \to \mathrm{Map}\left(W^I, \mathrm{End}(\mathrm{id}_{\mathcal{C}})\right).$$

These morphisms are functorial with respect to I and they can also be shown to respect the algebra structures on both sides.

We now take the colimit over all finite sets I and over all group morphisms $F(I) \to W$, where $F(I)$ is the free group on I generators, to obtain an algebra morphism

$$\mathrm{Exc}(W,\widehat{G}) = \mathrm{colim}_{I,F(I)\to W}\mathcal{O}[\widehat{G}^I]^{\widehat{G}} \to \mathrm{End}(\mathrm{id}_{\mathcal{C}}). \tag{22}$$

In order to go from the system of the algebra morphisms $(\Theta^I)_I$ to the algebra morphism in (22), we need to verify that the $(\Theta^I)_I$ satisfy an additional compatibility with respect to group morphisms $F(I) \to F(J)$ that do not arise from maps of the underlying sets. The key point is to check compatibility with group morphisms that multiply subsets of generators.

The colimit in (22) looks formally similar to the RHS of (19). In order to make the simplified situation described above apply to the Weil group W_E, we want to let P run over open subgroups of the wild inertia subgroup of W_E and let W be a discretization of W_E/P as in the (sketched) proof of Theorem 3.5. We can think of the group morphisms $F(I) \to W$ as a way to "probe" the discrete group W by a free group on finitely many generators; the colimit of $F(I)$ over all such morphisms recovers W (Zhu, 2021, § 2.1). Taking the colimit of $\mathcal{O}[\widehat{G}^I]^{\widehat{G}}$ over all group morphisms $F(I) \to W$ from a free, finitely generated group should heuristically recover the algebra of regular functions on the coarse moduli space of $\widehat{G}$-valued representations of W. Theorem 3.7 makes this heuristic precise.

We upgrade $D(\mathrm{Bun}_G,\Lambda)$ to a condensed ∞-category $\mathcal{D}(\mathrm{Bun}_G,\Lambda)$, and we let $\mathcal{D}(\mathrm{Bun}_G,\Lambda)^\omega \subset \mathcal{D}(\mathrm{Bun}_G,\Lambda)$ denote the stable condensed ∞-subcategory of compact objects. The category $\mathcal{C}$ is taken to be the full stable condensed ∞-subcategory $\mathcal{C}_P \subset D(\mathrm{Bun}_G,\Lambda)^\omega$ where all copies of P act trivially after applying any Hecke operator. Fargues–Scholze show that every object of $\mathcal{D}(\mathrm{Bun}_G,\Lambda)^\omega$ lies in some $\mathcal{C}_P$. The functors T_I are taken to be the restrictions of the Hecke functors (17) to $\mathcal{C}_P$.

We define the *geometric Bernstein center* as

$$\mathcal{Z}^{\mathrm{geom}}(G,\Lambda) := \pi_0\mathrm{End}(\mathrm{id}_{\mathcal{D}(\mathrm{Bun}_G,\Lambda)}). \tag{23}$$

Assume that the order of the torsion subgroup of $\pi_1(\widehat{G})$ (or, equivalently, the order of $\pi_0(Z(G))$) is invertible in Λ. The discussion of excursion operators from above, together with the presentation of Theorem 3.7, leads to a morphism

$$\mathcal{Z}^{\mathrm{spec}}(G,\Lambda) \to \mathcal{Z}^{\mathrm{geom}}(G,\Lambda) \tag{24}$$

on the level of Bernstein centers (Fargues and Scholze, 2024, Theorem IX.5.2). Note that the passage to the subcategory $\mathcal{D}(\mathrm{Bun}_G,\Lambda)^{\omega} \subset \mathcal{D}(\mathrm{Bun}_G,\Lambda)$ of compact objects is essential for constructing this morphism, as is the statement that the category $\mathcal{D}(\mathrm{Bun}_G,\Lambda)$ is compactly generated.

Denote by $\mathcal{Z}(G(E),\Lambda)$ the usual Bernstein center of the category of smooth representations of $G(E)$ (Bernstein, 1984). The fully faithful embedding $D(G(E),\Lambda) \hookrightarrow D(\mathrm{Bun}_G,\Lambda)$, which is a consequence of Theorem 2.11, induces a restriction morphism $\mathcal{Z}^{\mathrm{geom}}(G,\Lambda) \to \mathcal{Z}(G(E),\Lambda)$. The composition of the morphism (24) with this restriction morphism induces a morphism

$$\mathcal{Z}^{\mathrm{spec}}(G,\Lambda) \to \mathcal{Z}(G(E),\Lambda). \tag{25}$$

The construction of semi-simple L-parameters $\pi \mapsto \varphi_{\mathrm{FS},\pi}$ is deduced from the morphism (25) by specializing to the point of $\mathcal{Z}(G(E),\overline{\mathbb{Q}}_\ell)$ corresponding to the irreducible smooth representation π.

When $G = \mathrm{GL}_n$, the morphism (25) recovers a result of Helm and Moss (2018).

Remark 3.9. — For each prime $\ell \neq p$, one obtains a morphism on the level of Bernstein centers

$$\mathcal{Z}^{\mathrm{spec}}\left(G,\mathbb{Q}_\ell(\sqrt{q})\right) \to \mathcal{Z}\left(G(E),\mathbb{Q}_\ell(\sqrt{q})\right). \tag{26}$$

Both sides can already be defined over $\mathbb{Q}$: indeed, one can define $\mathcal{Z}^{\mathrm{spec}}(G,\mathbb{Q})$ using the moduli stack of Langlands parameters of Dat, Helm, Kurinczuk, and Moss (2025), which is already constructed over $\mathbb{Z}\left[\frac{1}{p}\right]$. Fargues–Scholze conjectured that each morphism (26) arises from a unique morphism

$$\mathcal{Z}^{\mathrm{spec}}\left(G,\mathbb{Q}(\sqrt{q})\right) \to \mathcal{Z}\left(G(E),\mathbb{Q}(\sqrt{q})\right).$$

This conjecture, known as independence of ℓ, was recently proved in Scholze (2025), by redoing the constructions of Fargues–Scholze using motivic sheaves instead of ℓ-adic sheaves.

3.4. The spectral action and the categorical conjecture

In this subsection, we briefly discuss the construction of the spectral action and state the geometric version of the categorical local Langlands conjecture. For an introduction to the categorical Langlands program, including global and p-adic aspects, the reader should consult Zhu (2021) and Emerton, Gee, and Hellmann (2022). For the state of the art on the categorical equivalence conjecture stated below, the reader should consult Hansen (2025).

We continue to assume that G/E is a quasi-split connected reductive group. We also assume that Λ is either the ring of integers in a finite extension of $\mathbb{Q}_\ell(\sqrt{q})$, such that ℓ does not divide the order of the torsion subgroup of $\pi_1(\widehat{G})$, or a finite extension of $\mathbb{Q}_\ell(\sqrt{q})$.

Maintaining the notation from the previous section, recall that, for each finite set I, we have an exact, $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor

$$T_I\colon \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{End}_\Lambda\left(\mathcal{D}(\mathrm{Bun}_G, \Lambda)^\omega\right)^{BW_E^I}.$$

These are, in addition, functorial in I. With higher categorical techniques, one can repackage this system of functors into the so-called *spectral action*. We denote by $\mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})$ the stable ∞-category of perfect complexes on the moduli stack $Z^1(W_E, \widehat{G})_\Lambda/\widehat{G}$ of L-parameters. Fargues and Scholze (2024, Theorem X.0.1) prove that giving the system of functors $(T_I)_I$ is equivalent to giving a compactly supported Λ-linear action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})$ on $\mathcal{D}(\mathrm{Bun}_G, \Lambda)^\omega$. The condition that the action be compactly supported is that, for any compact object, the action factors over some $\mathrm{Perf}\left(Z^1(W_E/P, \widehat{G})_\Lambda/\widehat{G}\right)$.

This result works for an abstract small, idempotent-complete, Λ-linear stable ∞-category $\mathcal{C}$ in place of $\mathcal{D}(\mathrm{Bun}_G, \Lambda)^\omega$ and uses as an input the presentation of Theorem 3.7 and the higher category theory developed by Lurie (2017). A version of this result with characteristic 0 coefficients appears in Nadler and Yun (2019) and in Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky (2022). However, obtaining the result with integral coefficients is more subtle.

We explain how to recover the system of functors $(T_I)_I$ from the spectral action. The universal object parameterized by the moduli stack $Z^1(W_E, \widehat{G})/\widehat{G}$, with its universal W_E-equivariance, defines an exact $\mathrm{Rep}_\Lambda Q$-linear, symmetric monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q) \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})^{BW_E}.$$

By taking tensor products, we obtain for each finite set I, an exact, $\mathrm{Rep}_\Lambda(Q^I)$-linear symmetric monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \to \mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})^{BW_E^I}.$$

This is now composed with the spectral action of $\mathrm{Perf}(Z^1(W_E, \widehat{G})_\Lambda/\widehat{G})$ on $\mathcal{D}(\mathrm{Bun}_G, \Lambda)^\omega$ to obtain T_I.

Example 3.10. — Assume that G is split over E and that $I = \{*\}$ is a singleton. An element $V \in \mathrm{Rep}_\Lambda \widehat{G}$ can be regarded as a vector bundle on the classifying stack of $\widehat{G}$ and this can be pulled back to a W_E-equivariant vector bundle $\mathcal{V}$ on $Z^1(W_E, \widehat{G})_\Lambda/\widehat{G}$ under the natural map from $Z^1(W_E, \widehat{G})_\Lambda/\widehat{G}$ to the classifying stack. (The W_E-action on $\mathcal{V}$ is induced from the W_E-action on the universal representation over $Z^1(W_E, \widehat{G})$.) The spectral action of $\mathcal{V}$ on $\mathcal{D}(\mathrm{Bun}_G, \Lambda)^\omega$ is given by applying the Hecke operator $T_{V,\{*\}}$ that corresponds to V under the geometric Satake equivalence.

The spectral action connects our two main geometric objects: the moduli stack $Z^1(W_E,\widehat{G})/\widehat{G}$ on the spectral side and the stack Bun_G on the automorphic side. The existence of the spectral action can be deduced if one assumes a geometric version of the categorical local Langlands conjecture. However, the spectral action can also be used to formulate the categorical local Langlands conjecture precisely.

To state the conjecture, we choose a Whittaker datum $\mathfrak{w} = (B,\psi)$ for G (up to $G(E)$-conjugacy). Writing $B = T \ltimes U$, we have the (co-)Whittaker representation $\text{c-Ind}_{U(E)}^{G(E)}\psi$. We denote by $\mathcal{W}_\psi$ its extension by zero from the neutral point to all of Bun_G, the *Whittaker sheaf*. Via the spectral action discussed above, the Whittaker sheaf will rigidify the (conjectural) equivalence of categories.

Recall that $Z^1(W_E,\widehat{G})/\widehat{G}$ is not quasi-compact, as it is an increasing union of substacks $Z^1(W_E/P,\widehat{G})/\widehat{G}$, where P runs over open subgroups of the wild inertia in W_E. We denote by $\mathrm{Perf}^{\mathrm{qc}}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G})$ the full stable ∞-subcategory of $\mathrm{Perf}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G})$ consisting of objects supported on finitely many connected components of $Z^1(W_E,\widehat{G})_\Lambda/\widehat{G}$. There is an extension of the spectral action to the ind-completion $\mathrm{IndPerf}^{\mathrm{qc}}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G})$ that preserves colimits. We denote the action of $M \in \mathrm{IndPerf}^{\mathrm{qc}}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G})$ on $\mathcal{W}_\psi$ by $M * \mathcal{W}_\psi$. This induces a functor

$$\mathrm{IndPerf}^{\mathrm{qc}}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G}) \to \mathcal{D}(\mathrm{Bun}_G,\Lambda), M \mapsto M * \mathcal{W}_\psi \tag{27}$$

that preserves colimits.

Conjecture 3.11. — *The right adjoint of the functor* (27) *restricts to a fully faithful functor on* $\mathcal{D}(\mathrm{Bun}_G,\Lambda)^\omega$ *and this induces a* $\mathrm{Perf}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G})$*-linear equivalence of stable* ∞*-categories*

$$\mathcal{D}(\mathrm{Bun}_G,\Lambda)^\omega \simeq \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh},\mathrm{Nilp}}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G}).$$

With coefficients of characteristic 0 or banal characteristic[17], $\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G})$ denotes the stable ∞-category of bounded complexes of quasi-coherent sheaves on $Z^1(W_E,\widehat{G})_\Lambda/\widehat{G}$, whose cohomology sheaves are coherent and have quasi-compact support. The condition Nilp is automatic in this case and can therefore be ignored. If the coefficients are integral and the characteristic is not banal, the condition Nilp denotes nilpotent singular support. This is a condition first introduced in the context of the geometric Langlands program by Arinkin and Gaitsgory (2015).

Remark 3.12. — After taking ind-completions of the categories in Conjecture 3.11, the categorical local Langlands equivalence should take the Whittaker sheaf $\mathcal{W}_\psi$ to the

[17] The banal characteristic assumption holds if $\ell \gg 0$. On the automorphic side, a prime ℓ is *banal* for $G(E)$ if ℓ does not divide the pro-order of any compact open subgroup of $G(E)$. This ensures that there are no interesting congruences modulo ℓ between irreducible smooth representations of $G(E)$. On the spectral side, one needs to impose a related, but in general stronger condition to guarantee that the singularities of the moduli stack of Langlands parameters are not too bad. See Fargues and Scholze (2024, Theorem VIII.2.11) and the discussion in Dat, Helm, Kurinczuk, and Moss (2025, § 1.5) on primes that are ${}^LG$-banal.

structure sheaf $\mathcal{O}_{Z^1(W_E,\widehat{G})_\Lambda/\widehat{G}}$ of the moduli stack of L-parameters. This is related to the requirement in Conjecture 1.15 that generic members of discrete (or, more generally, tempered) L-packets $\Pi_{\varphi,1}$ correspond to the trivial representation of S_φ.

Remark 3.13. — A version of this conjecture, restricted to the neutral point of Bun_G, was previously formulated by Hellmann (2023) and Zhu (2021) and Ben-Zvi, H. Chen, Helm, and Nadler (2024). More precisely, consider the subcategory $D^b_{\mathrm{f.g.}}(G(E),\Lambda) \subset D(G(E),\Lambda)$ consisting of bounded complexes whose cohomology groups are finitely generated smooth representations of $G(E)$. The conjecture for the neutral point is that there is a fully faithful embedding

$$D^b_{\mathrm{f.g.}}(G(E),\Lambda) \hookrightarrow \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh},\mathrm{Nilp}}(Z^1(W_E,\widehat{G})_\Lambda/\widehat{G}).$$

This conjecture was proved with characteristic 0 coefficients in the case of the Iwahori block (and deduced for all blocks for $G = \mathrm{GL}_n$) by Ben-Zvi, H. Chen, Helm, and Nadler (2024).

Remark 3.14. — Zhu also formulated a variant of Conjecture 3.11 using Isoc_G on the automorphic side, a moduli stack of isocrystals with G-structure, as a replacement for Bun_G. This stays within the realm of usual algebraic geometry and has a more direct relationship with Deligne–Lusztig theory. In the very recent preprint Zhu (2025), the tame case of the Isoc_G variant of the conjecture is established under certain technical assumptions.

4. Further directions and applications

The work of Fargues–Scholze has already had a transformative effect on the Langlands program in the arithmetic setting. It has inspired applications to long-standing open problems as well as the development of parallel geometrization programs for p-adic local Langlands and real local Langlands. In the p-adic case, several related conjectures are formulated in Emerton, Gee, and Hellmann (2022); for the real case, see Scholze (2024).

To illustrate the power of the ideas introduced by Fargues–Scholze, this section will focus in-depth on two examples of applications. The first example is the striking result of Dat, Helm, Kurinczuk, and Moss (2024a) on the finiteness of integral Hecke algebras of p-adic groups. This has important representation-theoretic consequences, most notably an integral version of Bernstein's second adjointness that shows parabolic induction is also a left adjoint functor. This topic is discussed in § 4.1.

The second example is of a global nature and concerns the cohomology of Shimura varieties. The work of Fargues–Scholze, together with the ideas introduced in Koshikawa (2021b) and Hamann and Lee (2025) and with the construction of the Igusa stack in Zhang (2023) and Daniels, Hoften, Kim, and Zhang (2024), have revolutionized the study of Shimura varieties and their cohomology. This topic is discussed in § 4.2.

4.1. Finiteness of integral Hecke algebras

For any compact open subgroup $K \subset G(E)$ and any ring Λ, the (not necessarily commutative) algebra $\Lambda[K\backslash G(E)/K] \simeq \mathrm{End}_{G(E)}(\text{c-Ind}_K^{G(E)}\mathbb{1})$ is called a *Hecke algebra.* The algebra structure is given by convolution of bi-K-invariant functions. For any smooth representation π of $G(E)$ with Λ-coefficients, the space of its K-invariants

$$\mathrm{Hom}_{G(E)}\left(\text{c-Ind}_K^{G(E)}\mathbb{1}, \pi\right) \simeq \pi^K$$

is naturally a right module over this Hecke algebra (by pre-composing with endomorphisms of $\text{c-Ind}_K^{G(E)}\mathbb{1}$). When π is finitely-generated and K is chosen small enough, the Hecke module of K-invariants completely determines the representation π. Therefore, studying Hecke algebras is highly relevant for understanding smooth representations of $G(E)$. In particular, showing that the Hecke algebras $\Lambda[K\backslash G/K]$ have good finiteness properties has important representation-theoretic consequences. Dat, Helm, Kurinczuk, and Moss (2024a,b) prove the following result.

THEOREM 4.1. — *For any noetherian $\mathbb{Z}\left[\frac{1}{p}\right]$-algebra Λ, the Hecke algebra $\Lambda[K\backslash G(E)/K]$ is a finitely generated module over its center, which is a finitely generated (commutative) Λ-algebra.*

When Λ is an algebraically closed field of characteristic 0, Theorem 4.1 was proved in Bernstein (1984). Prior to the work of Dat, Helm, Kurinczuk, and Moss (2024a), the integral version of the theorem was only known in the case $G = \mathrm{GL}_n$ due to Helm (2016). This proof relies on specific features of GL_n, such as the uniqueness of supercuspidal support.

We explain the proof of Theorem 4.1, for simplicity, in the case when Λ is a noetherian $\mathbb{Z}_\ell$-algebra for some prime $\ell \neq p$. The theorem can be equivalently formulated by requiring finitely generated smooth representations of $G(E)$ to satisfy certain finiteness / admissibility properties with respect to the Bernstein center $\mathcal{Z}(G(E), \Lambda)$. This equivalent formulation was then reduced to proving the finiteness of a map of excursion algebras

$$\mathrm{Exc}(W, \widehat{G}) \to \mathrm{Exc}(W, \widehat{M}), \tag{28}$$

for M a Levi subgroup of G and W a discretization of some W_E/P. This reduction uses in a crucial way the morphisms from the excursion algebra to the Bernstein center constructed by Fargues–Scholze, but also Bernstein's theorem in characteristic 0.

The finiteness of the morphism (28) could then be checked on the spectral side of the local Langlands correspondence by proving the finiteness of the corresponding morphism of moduli stacks of L-parameters. This works even when ℓ divides the order of the torsion subgroup of $\pi_1(\widehat{G})$. This is because the Fargues–Scholze morphism from the excursion algebra to the Bernstein center factors through the reduced quotient of the excursion algebra.

As a corollary of Theorem 4.1, Dat, Helm, Kurinczuk, and Moss establish an integral version of Bernstein's second adjointness theorem.

Corollary 4.2. — *For all* $\mathbb{Z}\left[\frac{1}{p}\right]$*-algebras* Λ *and all pairs of opposite parabolic subgroups* $(P, \overline{P})$ *in* G *with common Levi component* $M = P \cap \overline{P}$*, the parabolic induction functor*

$$\mathrm{Ind}_{P(E)}^{G(E)} \colon \mathrm{Rep}_{\Lambda}^{\mathrm{sm}}(M(E)) \to \mathrm{Rep}_{\Lambda}^{\mathrm{sm}}(G(E))$$

is left adjoint to the Jacquet functor for the opposite parabolic $\overline{P}$*, twisted by the modulus character of* P*.*

4.2. The Igusa stack and the p-adic geometry of Shimura varieties

(Global) Shimura varieties are algebraic varieties defined over number fields, which play a central role in the arithmetic global Langlands program. In this subsection, we explain how the work of Fargues–Scholze, together with the introduction of the Igusa stack, led to elegant solutions to two problems in the area of Shimura varieties: a long-standing one (Eichler–Shimura relations) and a more recent one, but with striking applications (torsion vanishing).

A Shimura variety is determined by a Shimura datum $(\mathbb{G}, \mathbb{X})$, where $\mathbb{G}/\mathbb{Q}$ is a connected reductive group and $\mathbb{X}$ is a conjugacy class of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathbb{G}_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$. Both $\mathbb{G}$ and $\mathbb{X}$ are required to satisfy certain axioms (Deligne, 1979). These, in particular, give the conjugacy class $\mathbb{X}$ a geometric structure, as a variation of polarizable Hodge structures. For a compact open subgroup $\mathbb{K} \subset \mathbb{G}(\mathbb{A}_f)$, we can form the double quotient

$$\mathbb{G}(\mathbb{Q}) \backslash \left(\mathbb{X} \times \mathbb{G}(\mathbb{A}_f)\right) / \mathbb{K}.$$

This is, a priori, a complex manifold (at least if $\mathbb{K}$ is sufficiently small). However, the axioms of a Shimura datum ensure that this complex manifold arises in a canonical way from an algebraic variety $S_{\mathbb{K}}$, called a Shimura variety, defined over a number field $\mathbb{E}$, called the reflex field of the Shimura datum.

There is a complete classification of connected reductive groups that admit a Shimura datum. For example, this holds for $\mathbb{G} = \mathrm{GSp}_{2n}/\mathbb{Q}$. The associated Shimura varieties are called Siegel modular varieties and they are moduli spaces of principally polarized abelian varieties of dimension n equipped with level structures. More generally, one can consider Shimura varieties of *PEL type*, which represent moduli problems of abelian varieties equipped with polarizations, endomorphisms and level structures. These Shimura varieties are the easiest to study with moduli-theoretic techniques. The next best understood class is that of Shimura varieties of *Hodge type*, where the Shimura datum admits a closed embedding into some Siegel Shimura datum.

The ℓ-adic étale cohomology groups of Shimura varieties are equipped with many different kinds of symmetries. There is a Hecke symmetry, coming from the action of the finite adelic group $\mathbb{G}(\mathbb{A}_f)$ on the inverse system of Shimura varieties of varying level $\mathbb{K}$. There is also a Galois symmetry, coming from the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{E}}/\mathbb{E})$. The relationship between the Hecke action and the Galois action is determined by the global Langlands correspondence. A precise conjecture was formulated in Kottwitz (1990) for the L^2 or intersection cohomology of Shimura varieties.

A weaker version of the relationship between the Galois and Hecke actions is encoded in the *Eichler–Shimura relations*, which were conjectured in Blasius and Rogawski (1994). Roughly, the expectation is that the Frobenius at an unramified prime, acting on the étale cohomology of a Shimura variety, satisfies a certain Hecke polynomial. This extends the well-known Eichler–Shimura relation for the modular curve; the proof there can be generalized in the PEL case when the group of the Shimura datum is split at the unramified prime in question (Wedhorn, 2000). There has been significant recent progress on the Eichler–Shimura relations that goes beyond this case due to Lee (2021) and Wu (2025). Wu proved the Eichler–Shimura relations for a very general class of Shimura varieties, namely those of Hodge type. His proof relies on the version of the geometric Satake equivalence established by Fargues–Scholze.

In addition to ℓ-adic cohomology, the étale cohomology of Shimura varieties with *integral* or *torsion* coefficients received a great deal of attention over the past decade. Calegari and Geraghty (2018) formulated an exciting program to extend the celebrated method of Taylor and Wiles (1995) for proving modularity to more general number fields, such as CM fields. Their program required a very precise understanding of the cohomology of Shimura varieties (and more general locally symmetric spaces) with integral coefficients. Motivated by this program, Calegari–Geraghty and Emerton independently suggested that, after localising at a sufficiently generic[(18)] system of Hecke eigenvalues, the cohomology of Shimura varieties with $\mathbb{Z}_\ell$-coefficients should be concentrated in the middle degree and torsion-free.

The first general *torsion vanishing* results of this kind were proved in Caraiani and Scholze (2017, 2024) for PEL type A Shimura varieties, treating p as an auxiliary prime and imposing a genericity condition on the Langlands parameter at p. These works use a trace formula computation at a key step in the argument. These results were crucial ingredients in new (potential) modularity results over CM fields and in the proof of new instances of the famous Sato–Tate and Ramanujan conjectures, starting with Allen et al. (2023). The Bourbaki talk of Colmez (2024) discusses these applications in more detail, together with other exciting developments in the arithmetic global Langlands program.

As observed by Koshikawa (2021b), the work of Fargues–Scholze allowed one to obtain a more elegant proof of torsion vanishing, by-passing the use of the trace formula. In fact, Koshikawa used the work of Fargues–Scholze to prove analogous vanishing results for the *local* Shimura varieties with integral coefficients. Furthermore, in Koshikawa

[(18)] A technical subtlety, which is glossed over in this exposition, is the following. In the original conjectures of Calegari–Geraghty and Emerton, the condition on the system of Hecke eigenvalues is a global one, but existing results impose local conditions and understanding what is the optimal local condition is both subtle and important. See Koshikawa and Shin (2024) for precise conjectures.

(2021a), he also proved a version of the Eichler–Shimura relations in the context of local Shimura varieties.

More recently, the study of geometric Eisenstein series by Hamann (2025b) led to even more powerful torsion-vanishing theorems and to a more flexible approach in Hamann and Lee (2025), which could be used to study a more general class of Shimura varieties.

The most elegant approach to date to both Eichler–Shimura relations and torsion vanishing results uses the so-called *Igusa stack*[19]. The Igusa stack connects Shimura varieties to the geometric Langlands program over the Fargues–Fontaine curve.

Assume, for simplicity, that $(\mathbb{G}, \mathbb{X})$ is a Shimura datum of PEL type, giving rise to a compact Shimura variety of some dimension $d \in \mathbb{Z}_{\geq 1}$. Let $v \mid p$ be a a prime of $\mathbb{E}$ and set $E := \mathbb{E}_v$. For some (sufficiently small) compact open subgroup $\mathbb{K}^p \subset \mathbb{G}(\mathbb{A}_f^p)$, denote by $\mathcal{S}_{\mathbb{K}^p}$ the corresponding perfectoid Shimura variety over E with infinite level at p and tame level $\mathbb{K}^p$. Following a conjecture of Scholze, Zhang (2023) proves that there is a v-stack $\mathrm{Igs}_{\mathbb{K}^p}$ that fits in a Cartesian diagram[20] of v-stacks in $\mathrm{Perf}_{\mathbb{F}_q}$

$$\begin{array}{ccc} \mathcal{S}_{\mathbb{K}^p} & \xrightarrow{\pi_{\mathrm{HT}}} & \mathscr{F}\ell_G \\ \downarrow & & \downarrow \\ \mathrm{Igs}_{\mathbb{K}^p} & \xrightarrow{\overline{\pi}_{\mathrm{HT}}} & \mathrm{Bun}_G \end{array} . \tag{29}$$

The top horizontal arrow is the Hodge–Tate period morphism, which measures the variation of p-adic Hodge–Tate structure on the universal abelian variety. The right hand side of the diagram is purely local: the flag variety is (essentially) part of the local Hecke stack, parameterizing meromorphic modifications of the trivial G-bundle, with poles bounded by a minuscule cocharacter. The morphism to Bun_G is given by Beauville–Laszlo gluing. The left hand side of the diagram has a global flavor; the Igusa stack is equipped with a Hecke action away from p and the morphism $\mathcal{S}_{\mathbb{K}^p} \to \mathrm{Igs}_{\mathbb{K}^p}$ is equivariant for this action.

Remark 4.3. — The Cartesian diagram (29) has its origins in the *Mantovan product formula* that was first established in Harris and Taylor (2001), Oort (2004), and Mantovan (2005). The original formula provides a uniformization of Newton strata inside Shimura varieties in terms of Igusa varieties and Rapoport–Zink spaces. In the special case of a basic Newton stratum, this goes back even earlier, to the p-adic uniformization results of Rapoport and Zink (1996). A cleaner version of the Mantovan product formula was established in Caraiani and Scholze (2017), working on the adic generic fiber with perfectoid Shimura varieties and Igusa varieties, but still restricted to individual Newton strata.

(19) The Igusa stack is not strictly necessary to prove torsion-vanishing, but it provides a conceptually clearer proof.

(20) By the uniqueness properties established in Kim (2025), the Igusa stack is essentially characterized by the fact that it fits into the Cartesian diagram (29). See Theorem A and Definition 1.1 of *loc. cit.* for the precise statement.

The relative cohomology of the Igusa stack recovers the cohomology of the Shimura variety, essentially after applying a geometric Hecke operator as in § 3. Assume, for simplicity, that Λ is a torsion $\overline{\mathbb{Z}}_\ell$-algebra and set $\mathscr{F} := R\overline{\pi}_{\mathrm{HT},\overline{\mathbb{F}}_q!}\Lambda$. The conjugacy class $\mathbb{X}$ determines a conjugacy class $\{\mu\}$ of cocharacters of $\mathbb{G}$ defined over $\mathbb{E}$ called the conjugacy class of *Hodge cocharacters*. In turn, this determines (up to isomorphism) an algebraic representation V_μ of ${}^LG$ over Λ. We denote by T_μ the Hecke operator $T_{V_\mu,\{*\}}$. We then have the formula (Daniels, Hoften, Kim, and Zhang, 2024, Theorem 8.4.10)

$$R\Gamma(S_{\mathbb{K}^p,\overline{E}},\Lambda) \simeq i_1^* T_\mu \mathscr{F}[-d]\left(\frac{d}{2}\right), \tag{30}$$

where $i_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$ denotes the inclusion of the neutral point, and where the isomorphism uses the identification of $D(\mathrm{Bun}_G^1,\Lambda)$ with $D(\mathbb{G}(\mathbb{Q}_p),\Lambda)$ from Theorem 2.11. The isomorphism (30) is equivariant for the Hecke action away from p, for the $\mathbb{G}(\mathbb{Q}_p)$-action, and for the action of the Weil group W_E.

The formula (30) decomposes down the cohomology of Shimura varieties into two parts: a global part, given by the relative cohomology of the Igusa stack, and a purely local part, given by the Hecke operator T_μ. The relative cohomology of the Igusa stack $\mathscr{F}$ is an object of $D(\mathrm{Bun}_G,\Lambda)$, so it is equipped with the spectral action of Fargues–Scholze. This brings powerful new tools from the geometric Langlands program into the study of the cohomology of Shimura varieties.

We briefly explain the proof of the Eichler–Shimura relations, following Daniels, Hoften, Kim, and Zhang (2024), who strengthened the results of Wu (2025), using a more direct argument. Their argument was inspired by the argument in Koshikawa (2021a) for local Shimura varieties. The conjugacy class $\{\mu\}$ of Hodge cocharacters determines a W_E-equivariant vector bundle $\mathcal{V}_\mu$ on the moduli stack $Z^1(W_{\mathbb{Q}_p},\widehat{G})/\widehat{G}$ of L-parameters for $G := \mathbb{G}\times_{\mathbb{Q}}\mathbb{Q}_p$. The formula (30) shows that the cohomology of the Shimura variety is obtained from the relative cohomology of the Igusa stack via the spectral action of $\mathcal{V}_\mu$, as in Example 3.10. One has a composition of morphisms

$$W_E \to \pi_0\mathrm{End}_{Z^1(W_{\mathbb{Q}_p},\widehat{G})/\widehat{G}}(\mathcal{V}_\mu) \to \mathrm{End}_\Lambda\left(R\Gamma(S_{\mathbb{K}^p,\overline{E}},\Lambda)\right),$$

which agrees with the usual (geometric) action of W_E on $R\Gamma(S_{\mathbb{K}^p,\overline{E}},\Lambda)$. One obtains similarly an action of the spectral Bernstein center $\mathcal{Z}^{\mathrm{spec}}(G,\Lambda)$ on $R\Gamma(S_{\mathbb{K}^p,\overline{E}},\Lambda)$, which factors through the action of the usual Bernstein center $\mathcal{Z}(G(E),\Lambda)$. One then proves the necessary relations on the spectral side, using a version of the Cayley–Hamilton theorem.

The proof of torsion vanishing also uses the formula (30), together with the observation that $\mathscr{F}$ is a perverse sheaf with respect to a natural perverse t-structure on Bun_G. The purely local results of Hamann (2025b) imply that the Hecke operator T_μ is perverse t-exact after localizing at a sufficiently generic local Langlands parameter at p. This is enough to deduce that the cohomology $R\Gamma(S_{\mathbb{K}^p,\overline{E}},\Lambda)$ is concentrated in the middle degree d. We note that these results use the compatibility between the semi-simple parameters constructed by Fargues–Scholze with more classical approaches to the local Langlands correspondence, which is discussed in Remark 3.3.

Remark 4.4. — Very recently, Yang and Zhu (2025) proved very general vanishing results for the generic part of the cohomology of Shimura varieties of abelian type with torsion coefficients. Their work does not directly use the work of Fargues–Scholze. Instead, it relies on the results of Zhu (2025) on the tame categorical local Langlands correspondence, but it also makes use of a version of the Igusa stack and is morally inspired by the argument described above. In Yang (2025), these results, together with the Igusa stack, are further used to prove, under certain technical assumptions, a conjecture of Clozel, Harris, and Taylor (2008) known as Ihara's lemma.

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