

THE ARITHMETIC OF THE LANGLANDS PROGRAM

SYLLABUS

The goal of this course is to give an introduction to the Langlands program, with a focus on Langlands reciprocity for the group GL_2 over \mathbb{Q} . This will prepare students for the HIM trimester program with the same name, that is scheduled for summer 2023. Please see the webpage at

<https://www.him.uni-bonn.de/programs/future-programs/future-trimester-programs/langlands-program/langlands-programdescription/>

for details on the program activities.

Some topics that will be covered in the course are: modular forms and their relationship to automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, the geometry of modular curves and their integral models, and the construction of Galois representations associated to modular forms of weight $k \in \mathbb{Z}_{\geq 1}$. In the last 4 weeks of the course, we will discuss various generalisations of these topics in self-contained, weeklong mini-courses on Shimura varieties, p -adic modular forms, modularity lifting via the Taylor–Wiles method, and the local Langlands correspondence.

COURSE INFORMATION

Instructor: Prof. Dr. Ana Caraiani, caraiani@math.uni-bonn.de

Assessment: Oral exam after the end of the course.

Prerequisites:

- An introductory course on algebraic number theory, including Galois theory, splitting of primes in extensions of \mathbb{Q} , and at least some familiarity with the statements of local and global class field theory.
- An introductory course on algebraic geometry, preferably including scheme theory.
- Some familiarity with étale cohomology and some basic representation theory are desirable. The course by Dr. Johannes Anschütz on étale cohomology can be followed in parallel.
- Some familiarity with the classical theory of modular forms, Hecke operators, and modular curves as Riemann surfaces is desirable.

References (to be updated): The first part of the course will follow roughly the same outline as the course taught in Summer 2020 by Dr. Anschütz. Further recommended references (some classical, some very recent) are as follows:

- Modular forms and modular curves: [3], [4], [5], [11].
- Automorphic representations: [8]
- Shimura varieties: [12], [2, §2].
- p -adic modular forms: [10], [1].
- Galois representations, modularity lifting: [7].

- The local Langlands correspondence: [13], [9]. For a categorical perspective: [6].

REFERENCES

- [1] George Boxer and Vincent Pilloni, *Higher Hida and Coleman theories on the modular curve*, to appear in EPIGA.
- [2] Ana Caraiani, *Perfectoid Shimura varieties*, in Perfectoid spaces: Lectures from the 2017 Arizona Winter School, edited by Bryden Cais, Mathematical Surveys and Monographs 242, American Mathematical Society, 2019, 245–297.
- [3] Pierre Deligne, *Formes modulaires et représentations ℓ -adiques*, in Séminaire Bourbaki vol. 1968/69 Exposés 347-33, Springer, 1971, 139–172.
- [4] Pierre Deligne and Michael Rapoport, *Les schémas de modules de courbes elliptiques*, in Modular functions of one variable II, Lecture Notes in Math 349, Springer, 1973, 143–316.
- [5] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids 1*, Annales ENS 7, 1974, 1 507–530.
- [6] Matthew Emerton, Toby Gee and Eugen Hellmann, *An introduction to the categorical p -adic Langlands program*, available online at <https://arxiv.org/abs/2210.01404>.
- [7] Toby Gee, *Modularity lifting theorems*, to appear in Essential Number Theory.
- [8] Jayce Getz and Heekyoung Hahn, *An introduction to automorphic representations*, available online at https://sites.duke.edu/jgetz/files/2022/04/Graduate_Text.pdf
- [9] Tasho Kaletha, *The local Langlands conjectures for non-quasi-split groups*, in Families of Automorphic Forms and the Trace Formula, Simons Symposia, Springer 2016, 217–257.
- [10] Nicholas Katz, *p -adic properties of modular schemes and modular forms*, in Modular functions of one variable III, Lecture notes in Math 350, Springer, 1973.
- [11] Nicholas Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals Math Studies 108, Princeton University Press, Princeton NJ, 1985.
- [12] Kai-Wen Lan, *An example-based introduction to Shimura varieties*, to appear in the proceedings of the ETHZ Summer School on Motives and Complex Multiplication.
- [13] Torsten Wedhorn, *The local Langlands correspondence for GL_n over p -adic fields*, in School on automorphic forms on GL_n , ICTP Lecture Notes 21, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008, 237–320.