Automorphic forms for GL_2 over $\mathbf{Q}$.

Kevin Buzzard

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Introduction

I just wanted to see some precise statements about automorphic representations for GL_2 over $\mathbf{Q}$, including various normalisations in the theory of modular forms, and how Maass forms fit into the picture, and so on.

1 Automorphic forms: the definitions.

Let G be the group $\mathrm{GL}_2/\mathbf{Q}$. Let K_∞ be the subgroup $\mathrm{O}_2(\mathbf{R})$ of $G(\mathbf{R})$. Say that $f : G(\mathbf{A}) \to \mathbf{C}$ is *smooth* if it's continuous and, when viewed as a function in two variables $(x, y) \in G(\mathbf{A}_f) \times G(\mathbf{R})$, it's C^∞ in y for fixed x, and locally constant in x for fixed y. Recall (for example from Borel-Jacquet in Corvallis) that an automorphic form for (G, K_∞) is a smooth $f : G(\mathbf{A}) \to \mathbf{C}$ such that

(a) $f(\gamma x) = f(x)$ for all $\gamma \in G(\mathbf{Q})$

(b1) There exists some compact open $K \in G(\mathbf{A}_f)$ such that $f(xk) = f(x)$ for all $x \in G(\mathbf{A})$ and $k \in G(\mathbf{A}_f)$

(b2) There is a finite-dimensional (semisimple) representation ρ of K_∞ with the property that the sub-$\mathbf{C}$-vector space of $\mathrm{Hom}(G(\mathbf{A}), \mathbf{C})$ generated by the maps $x \mapsto f(xk)$ as $k \in K_\infty$ varies, is, as a representation of K, isomorphic to a finite direct sum of Jordan-Hoelder factors of ρ.

(c) There is an ideal of finite codimension of the centre of the universal enveloping algebra of the complexification of the Lie algebra of $G(\mathbf{R})$, which annihilates f.

(d) For each $x \in G(\mathbf{A}_f)$, the function on $G(\mathbf{R})$ defined by $y \mapsto f(xy)$ is slowly increasing (definition below).

Definition of slowly increasing. Define a norm on $M_2(\mathbf{R})$ by $|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)| = \max\{|a|, |b|, |c|, |d|\}$. Then define a norm on $\mathrm{GL}_2(\mathbf{R})$ by $||\gamma|| = \max\{|\gamma|, |\gamma^{-1}|\}$. Note that if $\gamma^{-1} = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ then this norm is equivalent to the norm given by $||\gamma|| = \sqrt{(}a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)$: the L^2 norm and the L^∞ norm on a finite-dimensional vector space. See the remark on p190 of Borel-Jacquet to deduce that this is hence a norm on $\mathrm{GL}_2(\mathbf{R})$ in the right sense.

A function $\alpha : G(\mathbf{R}) \to \mathbf{C}$ is *slowly increasing* if there exists C and n such that for all $y \in G(\mathbf{R})$ we have $|\alpha(y)| \leq C||y||^n$.

We say f is a *cusp form* if furthermore

(e) $\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} f(nx)dn = 0$ for all $x \in G(\mathbf{A})$ and for all N in GL_2 conjugate (over $\mathrm{GL}_2(\mathbf{Q})$) to the upper triangular unipotent matrices.

Now let's construct some examples of these things.

2 The holomorphic case.

2.1 Review of classical modular forms.

Say f is a modular form in $M_k(\Gamma_1(N))$ in the classical sense. Let's recall the normalations of the classical theory. For $k \in \mathbf{Z}$ and $\gamma \in \mathrm{GL}_2^+(\mathbf{R})$ (the plus means positive determinant), and f a function on the upper half plane, let's define $f|_k\gamma$ by

$$f|_k\gamma(\tau) = (\det(\gamma))^{k-1} j(\gamma,\tau)^{-k} f(\gamma\tau).$$

Here $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = c\tau + d$.

The classical definition of Hecke operators involves using $f(\frac{\tau+i}{p})$ and hence the matrices $\left(\begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix}\right)$. To make sure this happens we need to use the double coset associated to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. The general machine will be that if f is a modular form for $\Gamma_1(N)$ and p is a prime then we want T_pf to be $\sum f|_k\gamma_i$ where $\Gamma_1(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma_1(N) = \coprod_i \Gamma_1(N)\gamma_i$. If $p|N$ then we can choose the γ_i to be $\left(\begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix}\right)$ for $0 \leq i < p$ (easy proof). This is great because it means that if $f = \sum a_n q^n$ then $T_pf = \sum a_{np}q^n$ (easy check).

For $p \nmid N$ we can take the γ_i to be as above, but then we need one more: if $\gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ and $p|a$ then $p \nmid b$ and it's not hard to check that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\gamma$ is not in any of the cosets we've written so far. If we were working with $\Gamma_0(N)$ then we'd be able to take $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for the other coset, but if we're working with Γ_1 then we need to use something a bit more subtle because, unfortunately, $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not in the double coset space! (look mod N for a trivial proof of this). Here's a modification of it that is. Choose a matrix $M \in \mathrm{SL}_2(\mathbf{Z})$ with $M \equiv 1 \bmod N$ and $M \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \bmod p^2$ (this is possible because the map $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/p^2N)$ is surjective). Then $M \in \Gamma_1(N)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)M = W\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ will do for the other coset. Here $W \in \mathrm{SL}_2(\mathbf{Z})$ and one checks that $W \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \bmod p$ and $W \equiv \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & p \end{smallmatrix}\right) \bmod N$. Note in particular that $W \in \Gamma_0(N)$ but typically $W \notin \Gamma_1(N)$ and this is what's causing the problems. Now, doing the calculations, we get that if $f = \sum a_n q^n$ then $T_pf = \sum a_{np}q^n + g$ where $g = f|_kW\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Classically we know that we want g to be $p^{k-1}\chi(p)f(q^p)$, so we want $f|_kW$ to be $\chi(p)f$ and this tells us how to normalise characters now.

Conclusion: the isomorphism $\Gamma_0(N)/\Gamma_1(N) \to (\mathbf{Z}/N\mathbf{Z})^\times$ should be defined by sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to d. Then a Dirichlet character χ of level N gives a map $\Gamma_0(N) \to \mathbf{C}^\times$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi(d)$, and we define a modular form of character χ to be one that transforms according to this character: $f|_k\gamma = \chi(\gamma)f$.

2.2 "The" automorphic form associated to a classical modular form.

Let s be an *arbitrary* complex number (this explains the quotes around "The" above). Let $f \in M_k(\Gamma_1(N);\chi)$ be a modular form of weight k, level N and character χ, with conventions as above. Let's associate an automorphic form to f. Let K be the subgroup $K_0(N)$ of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$, that is, the matrices congruent to $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ mod N, and let χ denote the map $K \to \mathbf{C}$ sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to d mod N. Let $K_1(N)$ denote the kernel of χ. It's standard stuff that $\mathrm{GL}_2(\mathbf{A}) = \mathrm{GL}_2(\mathbf{Q})K\,\mathrm{GL}_2^+(\mathbf{R})$, the reason being that $\det(K) = (\widehat{\mathbf{Z}})^\times$. Given f as above, define $\phi : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ by $\phi(\gamma\kappa u) = \chi(\kappa)^{-1}(f|_ku)(i)(\det u)^s$ with $\gamma \in \mathrm{GL}_2(\mathbf{Q})$ and $\kappa \in K$. This is well-defined as if $\gamma_1\kappa_1u_1 = \gamma_2\kappa_2u_2$ then $t := \gamma_2^{-1}\gamma_1 = \kappa_2\kappa_1^{-1}u_2u_1^{-1}$ is in $K \cap \mathrm{GL}_2(\mathbf{Q}) \subseteq \mathrm{GL}_2(\mathbf{Z})$ and it's also in $\mathrm{GL}_2^+(\mathbf{R})$ so it's in $\mathrm{SL}_2(\mathbf{Z})$ so it's in $\Gamma_0(N)$. Hence $\det(u_1) = \det(u_2)$, $u_2 = tu_1$ so $f|_ku_2 = (f|_kt)|_ku_1 = \chi(t)f|_ku_1$ and finally $\chi(\kappa_2) = \chi(t)\chi(\kappa_1)$ so the $\chi(t)$s cancel.

Remark: one could restrict to $k \in K_1(N)$ in the definition of ϕ; this makes the definition a bit simpler (no χ) but on the other hand one then has to unravel how $K_0(N)$ acts explicitly, and this information is useful when computing central characters. We will usually restrict to $K_1(N)$ to simplify notation, unless we need $K_0(N)$.

We need to check ϕ is an automorphic form. Axioms (a) and (b1) are clear. We need to think a bit about the others. For (b2) consider first $v = \left(\begin{smallmatrix} c & -s \\ s & c \end{smallmatrix}\right) \in \mathrm{SO}_2(\mathbf{R})$, with $c = \cos(\theta)$ and $s = \sin(\theta)$.

One checks easily that if $x = \gamma\kappa u$ with $\kappa \in K_1(N)$ then

$$\begin{aligned}\phi(xv) &= (f|_k uv)(i)\det(u)^s \\ &= j(v,i)^{-k}(f|_k u)(i)\det(u)^s \\ &= e^{-ik\theta}\phi(x)\end{aligned}$$

so $\mathrm{SO}_2(\mathbf{R})$ is acting via a character. Because $\mathrm{O}_2(\mathbf{R}) = \mathrm{SO}_2(\mathbf{R})\coprod \mathrm{SO}_2(\mathbf{R})w$ for $w = \left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ it suffices to understand how w acts. Note that the reason we've chosen it this way around is because of our choice of K; if we'd put the 1 in the top left of $K_1(N)$ we would have used $-w$. Now $\phi(\gamma\kappa uw) = \phi((\gamma w)(w^{-1}\kappa)(w^{-1}uw))$ with terrible abuse of notation: the first w is diagonal, the second is at the finite places and the third and fourth are at the infinite places. If one defines g by $g(\tau) = \overline{f(-\overline{\tau})}$ then one checks that for $u = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ we have $w^{-1}uw = \left(\begin{smallmatrix}a&-b\\-c&d\end{smallmatrix}\right)$ and

$$\begin{aligned}\phi(\gamma\kappa uw) &= (f|_k(w^{-1}uw))(i)\det(u)^s \\ &= f((ai-b)/(-ci+d))(-ci+d)^{-k}\det(u)^{k-1+s} \\ &= \overline{g(ui)(ci+d)^{-k}}\det(u)^{k-1+s} \\ &= \overline{(g|_k u)(i)}\det(u)^s\end{aligned}$$

and by abstract group theory $\mathrm{SO}_2(\mathbf{R})$ acts on this via the character $e^{+ik\theta}$, because conjugating by w sends θ to $-\theta$. So if $k \neq 0$ and $f \neq 0$ then we get a genuinely 2-dimensional space spanned by the K_∞-translates of ϕ and we can let ρ be this 2-dimensional representation. A basis for it is ϕ and the map $x \mapsto \phi(xw)$, where here w is thought of as at infinity.

Next we have to work out how the universal enveloping algebra acts. The way this works, I think, is that if $X \in \mathfrak{g} = \mathfrak{gl}_2(\mathbf{R})$, then $(X\phi)(g) = (d/dt)(t \mapsto \phi(ge^{Xt}))|_{t=0}$, and more generally

$$(X_1X_2\ldots X_n\phi)(g) = (d^n/dt_1dt_2\ldots dt_n)(\phi(ge^{X_1t_1}e^{X_2t_2}\ldots e^{X_nt_n}))|_{t_i=0}.$$

We now have to work all this out explicitly.

Now say $g = \gamma\kappa u$, with $\kappa \in K_1(N)$, and let α be the function on the upper half plane defined by $\alpha = \det(u)^s(f|_k u)$, so $\alpha(i) = \phi(g)$. Then we deduce

$$\begin{aligned}X_1X_2\ldots X_n\phi(g) &= (d^n/dt_1dt_2\ldots dt_n)((\det ue^{t_1X_1}e^{t_2X_2}\ldots)^s f|_k(ue^{t_1X_1}e^{t_2X_2}\ldots)(i))|_{t_i=0} \\ &= (d^n/dt_i)((\det e^{t_1X_1}e^{t_2X_2}\ldots)^s(\alpha|_k e^{t_1X_1}e^{t_2X_2}\ldots)(i))|_{t_i=0}.\end{aligned}$$

For $E = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $F = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ we have $e^{tE} = \left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)$ and $e^{tF} = \left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)$, and we get

$$\begin{aligned}E\phi(g) &= (d/dt)(t \mapsto \alpha|_k\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)(i))|_{t=0} \\ &= (d/dt)(t \mapsto \alpha(i+t))|_{t=0} \\ &= \alpha'(i)\end{aligned}$$

and

$$\begin{aligned}F\phi(g) &= (d/dt)(t \mapsto \alpha|_k\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)(i))|_{t=0} \\ &= (d/dt)((it+1)^{-k}\alpha(i+t+\cdots))|_{t=0} \\ &= -ik\alpha(i) + \alpha'(i).\end{aligned}$$

Note that this is not enough to compute $EF\phi$; we need to do it all again. We find

$$\begin{aligned}EF\phi(g) &= (d^2/dt_1dt_2)[(t_2i+1)^{-k}\alpha(i/(t_2i+1)+t_1)] \\ &= (d/dt_2)(\alpha'(i+t_2)(t_2i+1)^{-k}) \\ &= \alpha''(i) - ki\alpha'(i)\end{aligned}$$

and

$$
\begin{aligned}
FE\phi(g) &= (d^2/dt_1dt_2)((\alpha|_k\left(\begin{smallmatrix}1 & t_2\\ t_1 & 1+t_1t_2\end{smallmatrix}\right))(i))|_{t_i=0}\\
&= (d^2/dt_1dt_2)((it_1+1+t_1t_2)^{-k}\alpha(i+t_1+t_2-2it_1t_2+\ldots))|_{t_i=0}\\
&= (d/dt_1)[-kt_1(it_1+1)^{-k-1}\alpha(i+t_1)+(it_1+1)^{-k}(1-2it_1)\alpha'(i+t_1)]|_{t_1=0}\\
&= -k\alpha(i)-(k+2)i\alpha'(i)+\alpha''(i).
\end{aligned}
$$

Now if $H=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ then $e^{tH}=\left(\begin{smallmatrix}e^t & 0\\ 0 & e^{-t}\end{smallmatrix}\right)$ so

$$
\begin{aligned}
H\phi(g) &= (d/dt)(\alpha|_k\left(\begin{smallmatrix}e^t & 0\\ 0 & e^{-t}\end{smallmatrix}\right)(i))|_{t=0}\\
&= (d/dt)(e^{kt}\alpha(e^{2t}i))|_{t=0}\\
&= (d/dt)(e^{kt}\alpha(i+2it+\cdots))|_{t=0}\\
&= k\alpha(i)+2i\alpha'(i)
\end{aligned}
$$

which is a relief, because $EF-FE=H$! Furthermore,

$$
\begin{aligned}
HH\phi(g) &= (d^2/dt_1dt_2)(\alpha|_k\left(\begin{smallmatrix}e^{t_1+t_2} & 0\\ 0 & e^{-t_1-t_2}\end{smallmatrix}\right))|_{t_i=0}\\
&= (d^2/dt_1dt_2)(e^{k(t_1+t_2)}\alpha(e^{2(t_1+t_2)}i))|_{t_i=0}\\
&= (d^2/dt_1dt_2)(e^{kt_1}e^{kt_2}\alpha(i+2it_1+2it_2(1+2t_1)+\ldots))|_{t_i=0}\\
&= (d/dt_1)(ke^{kt_1}\alpha(i+2it_1)+e^{kt_1}(2i)(1+2t_1)\alpha'(i+2it_1))\\
&= k^2\alpha(i)+(2ik+2ik+4i)\alpha'(i)-4\alpha''(i).
\end{aligned}
$$

Finally, if $Z=\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ then

$$
\begin{aligned}
Z\phi(g) &= (d/dt)(e^{2st}\alpha|_k\left(\begin{smallmatrix}e^t & 0\\ 0 & e^t\end{smallmatrix}\right)))|_{t=0}\\
&= (d/dt)(e^{2t(s+k-1)}e^{-kt}\alpha(i))|_{t=0} \qquad = (2s+k-2)\alpha(i)
\end{aligned}
$$

So $((H^2+2EF+2FE)\phi)(g)=(k^2-2k)\alpha(i)=(k^2-2k)\phi(g)$ and $Z\phi=(2s+k-2)\phi$ and axiom (c) is verified.

Next let's try axiom (d). If $x\in\mathrm{GL}_2(\mathbf{A}_f)$ then write $x=\gamma\kappa u$ with $\kappa\in K_1(N)$, $\gamma\in\mathrm{GL}_2(\mathbf{Q})$ and $u\in\mathrm{GL}_2^+(\mathbf{R})$. If $y\in\mathrm{GL}_2(\mathbf{R})$ then there are two cases:
(i) $\det(y)>0$: then $xy=\gamma\kappa(uy)$, and
(ii) $\det(y)<0$. Then with the usual $w=\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and the usual abuse of notation, we have $xy=(\gamma w)(w^{-1}\kappa)(w^{-1}uy)$ and the third bracket has positive determinant (and the second is still in $K_1(N)$).

So we must check that the following function on $\mathrm{GL}_2(\mathbf{R})$ is slowly increasing: $y\in\mathrm{GL}_2^+(\mathbf{R})\mapsto(f|_k(uy))(i)(\det uy)^s$ and $y\in\mathrm{GL}_2^-(\mathbf{R})\mapsto(f|_kw^{-1}uy)(i)(\det(w^{-1}uy))^s$. Now $x\in\mathrm{GL}_2(\mathbf{A}_f)$ and hence $u\in\mathrm{GL}_2^+(\mathbf{Q})$, as is $w^{-1}uw$, and so $f_|ku$ and $f_k(w^{-1}uw)$ will be modular forms of some level. We are left to check that if F and G are modular forms for some congruence subgroup, then the function on $\mathrm{GL}_2(\mathbf{R})$ sending $y\in\mathrm{GL}_2^+(\mathbf{R})$ to $(F|_ky)(i)(\det y)^s$ and sending $y\in\mathrm{GL}_2^-(\mathbf{R})$ to $(G|_k(w^{-1}y))(i)(\det(w^{-1}y))^s$ is slowly increasing. Now the determinant is just noise, because if $ad-bc>0$ then $ad-bc\leq|ad|+|bc|\leq a^2+b^2+c^2+d^2$ so this is slowly increasing, and the two components are just noise, and we're left with showing that if F is a modular form for a congruence subgroup then the function on $\mathrm{GL}_2^+(\mathbf{R})$ sending y to $(F|_ky)(i)$ is slowly increasing, or equivalently again that the function sending $y=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ to $(ci+d)^{-k}F(yi)$ is slowly increasing. Now if F were a *cusp* form we'd be away, because for a cusp form it's well-known that $|F(z)|(\Im(z))^{k/2}$ is bounded on all the upper half plane, and hence $|F(yi)|(\det(y)/|ci+d|^2)^{k/2}$ is bounded, and we're home. On the other hand if F is an Eisenstein series, well, it also looks fine doesn't it, really.

I am pretty sure that the usual cuspidality condition translates exactly into the cuspidality condition defined above, and I am too lazy to check this carefully.

2.3 Central characters and normalisations for ϕ.

The automorphic representation associated to ϕ (as above) is just the space spanned by the images of ϕ under the Hecke action. Borel and Jacquet seem to want the action on the right, but I don't really understand this, because the compact open naturally acts on the left by $(kf)(g) = f(gk)$.

Regardless of whether things are left or right, one can compute the central character of this represesentation by computing it on ϕ. If $z \in \mathrm{GL}_1(\mathbf{A})$ then z can be written as $\gamma\kappa u$ with $\gamma \in \mathrm{GL}_1(\mathbf{Q})$, $\kappa \in \widehat{\mathbf{Z}}^\times$ and $u \in \mathbf{R}_{>0}$. All of these things can be thought of as lying in the centres of the appropriate GL_2s, and $\kappa \in K_0(N)$. Hence $\phi(xz) = \phi(x)\chi(\kappa)^{-1}u^{k-2+2s}$ so there's the central character. Note in particular that one can now guess which normalisations people usually use for s:

Gelbart in his orange book: firstly he uses a not d. He defines what it means to have character χ by putting in $\chi(a)^{-1}$ instead of $\chi(d)$, and then uses a in his adelic definition. This is tantamount to twisting by χ and means his level $K_1(N)$ is not the same as mine. Note that his definition of $j(\gamma, z)$ is also not the same as mine; he inserts a factor of $\det(\gamma)^{-1/2}$. His choice of s is hence such that $s + k - 1 = k/2$, the left hand side being the power of det I use at the end of the day, and the right hand side being what he uses. So his s is $1 - k/2$ and note that this makes his central character unitary (in fact his central character is the inverse of the Dirichlet character associated to f).

Taylor in his CalTech notes: he goes for $s + k - 1 = 1$, that is $s = 2 - k$, and we'll see the motivation for this later: it makes the Hecke operator T_p work compatibly on the adelic side of things.

Carayol: he wants the central character on $\mathbf{R}_{>0}$ to be $t \mapsto t^{-w}$ for some arbitrary integer w congruent to k mod 2, so he uses $k - 2 + 2s = -w$, that is, $s = (2 - w - k)/2$.

Nyssen: she wants $w = k$ so she sets $s = 1 - k$.

Deligne: didn't look yet.

Hooray! Everyone uses a different choice!

2.4 Hecke operators.

At the finite places, the natural thing to use is I think the matrix t_p which is $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ at p and the identity elsewhere. Because f is invariant under the compact open, we have to decompose $K_1(N)t_pK_1(N)$ into $\coprod_i \gamma_i K_1(N)$. If $p|N$ then we can set $\gamma_i = \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)$ (at p) for $0 \leq i < p$. If $p \nmid N$ then we also have to use $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Note that there is currently no issues with characters, not like in the messy $\mathrm{SL}_2(\mathbf{Z})$ case.

The adelic definition of a Hecke operator is now $(T_p\phi)(x) = \sum \phi(x\gamma_i)$. If one writes $x = \gamma\kappa u$ with $\kappa \in K_1(N)$ (warning: γ is embedded diagonally, but the γ_i are concentrated at p) then $x\gamma_i = \gamma\kappa\gamma_i u = \gamma\gamma_j\kappa' u$ where $\kappa' \in K_1(N)$ still. Now $\gamma_j \in \mathrm{GL}_2(\mathbf{Q})$ so we can incorporate it into γ and the result is that u changes to $\gamma_j^{-1}u$ and furthermore if $p \nmid N$ then κ' changes to $\gamma_j^{-1}\kappa'$ (with γ_j^{-1} now meaning something which is trivial at p and γ_j^{-1} at all the other finite places!), so

$$\begin{aligned}
(T_p\phi)(x) &= \sum_j \chi(\gamma_j^{-1})^{-1} \det(\gamma_j^{-1}u)^s (f|_k\gamma_j^{-1}u)(i) \\
&= \det(u)^s p^{-s} \sum \chi(\gamma_j)(f|_k\gamma_j^{-1}u)(i) \\
&= \det(u)^s p^{-s} \sum \chi(\gamma_j)(f|_k p^{-1}(p\gamma_j^{-1})(u))(i) \\
&= \det(u)^s p^{-s} p^{2-k} \sum \chi(\gamma_j)(f|_k(p\gamma_j^{-1})(u))(i)
\end{aligned}$$

Now if $p \mid N$ then this becomes

$$\begin{aligned}
&\det(u)^s p^{2-k-s} \sum_j (f|_k \left(\begin{smallmatrix} 1 & -j \\ 0 & p \end{smallmatrix}\right)(u))(i) \\
&= \det u^s p^{2-k-s} (T_p f)|_k u(i) \\
&= \det u^s p^{2-k-s} a_p f|_k u(i) \\
&= p^{2-k-s} a_p \phi(x)
\end{aligned}$$

and if $p \nmid N$ then it becomes

$$\begin{aligned}
&\det(u)^s p^{2-k-s} \left(\sum_j (f|_k \left(\begin{smallmatrix} 1 & -j \\ 0 & p \end{smallmatrix}\right)(u))(i) + \chi(p)(f|_k \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)(u))(i) \right) \\
&= \det u^s p^{2-k-s} ((T_p f)|_k u)(i) \\
&= \det u^s p^{2-k-s} a_p f|_k u(i) \\
&= p^{2-k-s} a_p \phi(x)
\end{aligned}$$

where a_p is the T_p-eigenvalue for f. Note that Taylor's choice of s gives $p^{2-k-s} = 1$, which was his motivation.

Now S_p, $p \nmid N$, is much easier: $S_p\phi(x) = \phi(x\varpi)$ where ϖ is p at p and trivial elsewhere. Writing this as $\gamma\kappa u$ as usual with $\kappa \in K_0(N)$ central gives $\kappa = p^{-1}$ at all primes dividing N, and $u = p^{-1}$ too, so one ends up multiplying ϕ by $\chi(p)p^{-2s+2-k}$.

Now Langlands' normalisations of the Satake parameters must be as follows: if π has trivial central character then it descends to an automorphic form on PGL_2 and hence the Satake parameters should land in $\mathrm{SL}_2(\mathbf{C})$. Hence if $p \nmid N$ then the Satake parameters (for Langlands) are γ and δ at p, with $(\gamma+\delta)\sqrt{p} = \lambda(T_p)$ and $\gamma\delta = \lambda(S_p)$, where $\lambda(X)$ denotes the eigenvalue of X, so we get $(X-\gamma)(X-\delta) = X^2 - a_p p^{3/2-k-s} X + p^{-2s+2-k}\chi(p)$. If $(X-\alpha)(X-\beta) = X^2 - a_p X + p^{k-1}\chi(p)$ then one sees instantly that $\gamma = p^{3/2-k-s}\alpha$ and $\delta = p^{3/2-k-s}\beta$. If you follow Gelbart and put $s = 1-k/2$ then $\gamma = p^{1/2-k/2}\alpha$ and because $|\alpha| = p^{(k-1)/2}$ we see that $|\gamma| = 1$ which is presumably what he was gunning for.

2.5 The L-function.

If f is a cusp form with trivial character (and hence even weight) then $L(f,s) = \sum a_n/n^s$, and if N is the level of f and $\Lambda(f,s) = N^{s/2}\Gamma(s)(2\pi)^{-s}L(f,s)$ then

$$\Lambda(f, k-s) = \epsilon.(-i)^k.N^{k/2-1}.\Lambda(f,s)$$

with ϵ the eigenvalue of the Atkin-Lehner involution w_N.

More on this later.

2.6 Carayol's normalisation and Galois representations.

The construction above gives a $\pi_{f,s}$ associated to an eigenform f and a complex number s. Now which ones would Carayol attach Galois representations to? Well he demands $k \geq 2$, and he sets $s = (2-w-k)/2$ for some integer w congruent mod 2 to k. He then gets an automorphic
representation π_f which looks the way he wants to look at infinity[1] , and hence he gets a Galois 1
representation. For p an unramified prime, his representation sends a geometric Frobenius to something with eigenvalues $p^{1/2}/\gamma$ and $p^{1/2}/\delta$ so the eigenvalues look like $p^{k+s-1}\alpha$ and so on. Because $s = 1-w/2-k/2$ we get $p^{k/2-w/2}\alpha$ etc for our eigenvalues of geometric Frobenius, and lo and behold this looks exactly right. Finally, Nyssen sets $w = k$ so we really do deduce that for the automorphic representation she associates to a modular form coming from an elliptic curve E, the associated Galois representation is $H^1(E, \mathbf{Z}_p)$.

[1] I should check this. I've only checked the central char is right. I bet it's easy.

2.7 Various notions of algebraicity.

At the minute we have four definitions for GL_2 and two near-definitions. For an automorphic representation $\pi = \pi_f \otimes \pi_\infty$ of GL_2, we say π is C-arithmetic if π_f is defined over a number field, L-arithmetic if the Satake parameters at the unramified primes are defined over a number field, L-algebraic if the Weil rep attached to π_∞ (restricted to $\mathbf{C}^\times$) looks integral, C-algebraic if it looks like Clozel wants it to look (that is, a shift of 1/2 from integrality), and then two conditions on the infinitesimal character involving the induced linear form on the complexification of the Cartan landing in some lattice perhaps shifted by ρ, and at the time of writing I still haven't figured these out.

But we can collect up what we know. It follows easily from what we have said earlier that, for a modular form of weight 1 or more, and with notation as above, the corresponding automorphic rep is C-arithmetic if $s \in \mathbf{Z}$, L-arithmetic if $s - \frac{1}{2} \in \mathbf{Z}$. In his paper Clozel asserts without proof that the representation of $\mathbf{C}^\times$ attached to π_∞ via local Langlands and restriction sends z to $(z^p\overline{z}^q, z^q\overline{z}^p)$ with (WLOG) $p - q = k - 1$ (this fixes the order of p and q). I am guessing that $p + q$ matches up with the central character so I think it should be (see below) $p + q = 2s + k - 2$. If I've for all his right then $p = s + k - \frac{3}{2}$ and $q = s - \frac{1}{2}$. Clozel says it's C-algebraic iff $s \in \mathbf{Z}$, which sounds OK, and if he's right we can deduce that it's L-algebraic iff $s - \frac{1}{2} \in \mathbf{Z}$. I should be able to check this rigorously but have no time at the time of writing.

Finally, we have done enough to read off the infinitesimal character of π, the automorphic representation associated to the pair (f, s). We know by the Harish-Chandra homomorphism that the centre of the universal enveloping algebra of the complexified Lie algebra of $\mathrm{GL}_2(\mathbf{R})$ is a polynomial ring generated by Z and $H^2 + 2EF + 2FE$, and we have computed the eigenvalues of these operators on ϕ earlier. Now the Harish-Chandra homomorphism sends Z to Z, and $H^2 + 2EF + 2FE = H^2 + 2H + 4FE$ to, well, the un-normalised one sends it to $H^2 + 2H$, and now we replace H by $H - 1$ to see that the normalised one (which is the one we want because it's the normalised one that is canonical, that is, independent of the choice of positivity) sends $H^2 + 2EF + 2FE$ to $(H-1)^2 + 2(H-1) = H^2 - 1$.

We conclude that ϕ has infinitesimal character sending Z to $2s + k - 2$ and $H^2 - 1$ to $k^2 - 2k$, and hence the linear forms on the Cartan it induces send H to $\pm(k-1)$ and Z to $2s + k - 2$. The choice of sign is because really we can only see an orbit of linear maps under the Weyl group. Finally, the "canonical lattice" that Toby and I are looking for is presumably just the lattice of characters of the torus? If this is so then for a pair $\lambda(H)$, $\lambda(Z)$ of eigenvalues to lie in this lattice we must have $\lambda(H) + \lambda(Z)$ and $\lambda(H) - \lambda(Z)$ both even integers, which is true iff $s - \frac{1}{2} \in \mathbf{Z}$. Finally, half the sum of the positive roots is the map sending H to 1 and Z to zero, so the shift of the lattice by this has an eigenvalue λ in it iff $\lambda(H) \pm \lambda(Z)$ are odd integers, which is true iff $s \in \mathbf{Z}$.

3 Maass forms.

3.1 Review of the basics of Maass forms.

We're going to play exactly the same game now, in the non-holomorphic case. I have never worked out for myself exactly how Maass forms with non-trivial and non-quadratic character work out, so let's just take the same normalisations as in the holomorphic case and define $K_1(N)$, for example, as in the holomorphic case, and assume that it's OK. With this assumption (which I'm sure is fine and would be tedious to check) we see that we should make the following definition:

Let ϵ be a sign (i.e., $\epsilon \in \{+1, -1\}$), let N be a positive integer, let χ be a Dirichlet character of level N. and let λ be a complex number (although other things will force λ to be a positive real). A *cuspidal Maass form* of level N, character χ, sign ϵ and eigenvalue λ is a smooth complex-valued function F on the upper half plane such that $F((az + b)/(cz + d)) = \chi(d)F(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, and such that $\Delta F = \lambda F$ (Δ a certain second order differential operator, the Laplace-Beltrami operator $-y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$) and F satisfies a boundedness condition (F tends to zero at all cusps, which means that for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ we have $F(\gamma(x + iy)) \to 0$ as

$y \to \infty$). Oh, lastly, $F(-\overline{z}) = \epsilon F(z)$. I know examples of forms with the same level, character and eigenvalue, and different signs, for what it's worth.

For such a form let me keep track of not one but two "q-expansion coefficients". Firstly some standard arguments imply that F has a Q-expansion

$$F(z) = \sum_{n \geq 1} b_n Q_n^{\epsilon}(z)$$

. satisfying the usual bounds. Furthermore for a Hecke eigenform normalised with $b_1 = 1$, the b_n will satisfy the usual bounds and recurrence relations for b_{p^r} ($r = 0, 1, 2, \ldots$) as modular forms *of weight 0*. So we should have $|b_p| \leq 2p^{-1/2}$ for example. The Q_n^{ϵ} are functions I define in my Maass form notes, and $Q_n^{+}(x+iy) = \sqrt{ny}K_{\nu}(2\pi ny)\cos(2\pi nx)$ (and Q_n^{-} has a sin instead of a cos). The thing is that sometimes people drop the $\sqrt{n}$ from Q_n, which motivates the definition $a_n = b_n\sqrt{n}$. For the a_n we have $|a_p| \leq 2$ conjecturally, and it's the a_n that are used in the definition of the L-function $\sum_n a_n/n^s$ associated to the Maass form.

I have no idea what the classical normalisation of a Hecke operator is, if any, so let's just make one up inspired by the holomorphic case: write $\Gamma_1(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma_1(N) = \coprod_j \Gamma_1(N)\gamma_j$ and define $(T_p f)(z) = p^{-1}\sum_j f(\gamma_j z)$; this is well-defined and so on. If $p \mid N$ then we can take the γ_j to be $\left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right)$ for $0 \leq j \leq p-1$, and an easy calculation shows that T_p has eigenvalue b_p. If $p \nmid N$ then we need one extra γ, namely $W\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as in the holomorphic case, and we get an extra factor of $p^{-1}f(W\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)z)$ which gives the usual $p^{-1}\chi(p)\sum b_n Q_{np}^{\epsilon}(z)$ and so again we see that T_p has eigenvalue b_p.

3.2 "The" automorphic form associated to a Maass form.

Let F be a level N Maass form on the upper half plane with Δ-eigenvalue λ (Δ the Laplace-Beltrami operator) and sign ϵ. The sign is all to do with $\rho(c)$ and is a phenomenon that one doesn't see in the holomorphic case. Say F has character χ (a Dirichlet character of level N), by which I will mean the following: extend χ to a function $\Gamma_0(N) \to \mathbf{C}^{\times}$ by $\chi(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) := \chi(d)$, and then we ask that for $\gamma \in \Gamma_0(N)$ we have $F(\gamma\tau) = \chi(\gamma)F(\tau)$. Also extend χ to a function on $K_0(N)$ by $\chi(\kappa) = \chi(d)$ where d is the bottom right hand entry of the image of κ in $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$.

Let s be an arbitrary complex number, as in the holomorphic case, and define[2] a function ϕ on $\mathrm{GL}_2(\mathbf{A}) = \mathrm{GL}_2(\mathbf{Q})K_0(N)\,\mathrm{GL}_2^{+}(\mathbf{R})$ by $\phi(\gamma\kappa u) = \chi(\kappa)^{-1}(\det u)^s F(ui)$. As ever, we must check that this is well-defined, and it is because if $\gamma_1\kappa_1 u_1 = \gamma_2\kappa_2 u_2$ then $t := \gamma_2^{-1}\gamma_1 \in \mathrm{GL}_2(\mathbf{Q})$ and $t = \kappa_2\kappa_1^{-1}$ and $t = u_2 u_1^{-1}$, so $t \in \mathrm{GL}_2(\mathbf{Q}) \cap K_0(N) \cap \mathrm{GL}_2^{+}(\mathbf{R})$ and hence $t \in \Gamma_0(N)$, and $tu_1 = u_2$ and $t\kappa_1 = \kappa_2$ so

$$\begin{aligned} &\chi(\kappa_2)^{-1}(\det u_2)^s F(u_2 i) \\ &= \chi(t\kappa_1)^{-1}(\det u_1)^s F(tu_1 i) \\ &= \chi(t)^{-1}\chi(\kappa_1)^{-1}(\det u_1)^s \chi(t) F(u_1 i) \end{aligned}$$

which is what we wanted. Note also that because $1 \in \mathrm{GL}_2(\mathbf{A})$ can actually be written as $\gamma\kappa u$ with $\gamma = \kappa = u = -1$ we see that if $F \neq 0$ then $\chi(-1) = 1$.

As in the holomorphic case we should strictly speaking check that this guy is an automorphic form. I'll run through this noting any differences between this case and the holomorphic case. Axioms (a) and (b1) are again clear. Axiom (b2) is a bit different this time. We have $\mathrm{SO}_2(\mathbf{R})$ acting trivially but to see the action of $w = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we float it through as in the holomorphic case and see that $\phi(\gamma\kappa uw) = \phi((\gamma w)(w^{-1}\kappa)(w^{-1}uw))$ and so the only change from $\phi(\gamma\kappa u)$ is that

[2] To be completely careful I should note the following: I have just made an arbitrary choice of "orientation" for χ here; I have never checked that when one unravels everything on Q-expansions one gets the "usual" formulae for T_p, or whether there's perhaps a $\chi^{-1}(p)$ in there instead of a $\chi(p)$. I am not sure that I care at this point because I'm more concerned about algebraic automorphic forms here, but perhaps one day I'll check to see if I've got it right.

$F(ui)$ becomes $F(w^{-1}uwi)$, or, in other words, $F(z)$ gets changed to $F(-\overline{z}) = \epsilon F(z)$ if ϵ is the sign of the Maass form. So we can take ρ 1-dimensional in this case, depending on the sign of F.

The computation of the action of the universal enveloping algebra really is different this time, not least because the analogue of the function α in the holomorphic case is something typically non-holomorphic and one has to keep track of whether one is differentiating with respect to x or y.

I will use notation as in the holomorphic case. Fix $g = \gamma\kappa u$ with $\kappa \in K_1(N)$, and let α be the function on the upper half plane defined by $\alpha(\tau) = \det(u)^s F(u\tau)$, so $\alpha(i) = \phi(g)$ (note α depends on g). Now F is a Maass form so $\Delta F = \lambda F$ for some λ, with Δ the Laplace-Beltrami operator. This implies that α is also an eigenfunction of Δ, because $\Delta\alpha = \det(u)^s\Delta(F \circ u) = \det(u)^s(\Delta F) \circ u = \det(u)^s\lambda F \circ u = \lambda\alpha$.

First let's recall the definition of the Lie algebra action, and what it boils down to in this case. We see that for X_i in the Lie algebra of $\mathrm{GL}_2(\mathbf{R})$ we have $e^{tX_i} \in \mathrm{GL}_2^+(\mathbf{R})$ and hence for $g = \gamma\kappa u$ with $\kappa \in K_1(N)$ we have

$$\begin{aligned}
(X_1X_2\ldots X_n\phi)(g) &= (d^n/dt_1dt_2\ldots dt_n)(\phi(ge^{X_1t_1}e^{X_2t_2}\ldots e^{X_nt_n}))|_{t_i=0} \\
&= (d^n/dt_i)(\det(ue^{X_1t_1}e^{X_2t_2}\ldots e^{X_nt_n})^s F(ue^{X_1t_1}e^{X_2t_2}\ldots e^{X_nt_n}i))|_{t_i=0} \\
&= (d^n/dt_i)(\det(e^{X_1t_1}e^{X_2t_2}\ldots e^{X_nt_n})^s \alpha(e^{X_1t_1}e^{X_2t_2}\ldots e^{X_nt_n}i))|_{t_i=0}
\end{aligned}$$

OK so now let's go. Let ϕ be attached to F as above. Then we have

$$\begin{aligned}
(E\phi)(g) &= (d/dt)\left(t \mapsto \alpha\left(\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)i\right)\right)|_{t=0} \\
&= (d/dt)\alpha(i+t)|_{t=0} \\
&= \alpha_x(i)
\end{aligned}$$

and

$$\begin{aligned}
(F\phi)(g) &= (d/dt)(t \mapsto \alpha((\begin{smallmatrix}1 & 0\\ t & 1\end{smallmatrix})i))|_{t=0} \\
&= (d/dt)\alpha(i+t+\ldots)|_{t=0} \\
&= \alpha_x(i)
\end{aligned}$$

and

$$\begin{aligned}
(EF\phi)(g) &= (d^2/dt_1dt_2)(\alpha((\begin{smallmatrix}1 & t_1\\ 0 & 1\end{smallmatrix})(\begin{smallmatrix}1 & 0\\ t_2 & 1\end{smallmatrix})i))|_{t_i=0} \\
&= (d^2/dt_1dt_2)(\alpha(i+t_1+t_2+\cdots))|_{t_i=0} \\
&= (d/dt_1)\alpha_x(i+t_1)|_{t_1=0} \\
&= \alpha_{xx}(i)
\end{aligned}$$

and

$$\begin{aligned}
(FE\phi)(g) &= (d^2/dt_1dt_2)(\alpha((\begin{smallmatrix}1 & 0\\ t_1 & 1\end{smallmatrix})(\begin{smallmatrix}1 & t_2\\ 0 & 1\end{smallmatrix})i))|_{t_i=0} \\
&= (d^2/dt_1dt_2)(\alpha(i+t_1+t_2-2it_1t_2+\cdots))|_{t_i=0} \\
&= (d/dt_1)(\alpha_x(i+t_1+\cdots) - 2t_1\alpha_y(i+t_1+\cdots))|_{t_1=0} \\
&= \alpha_{xx}(i) - 2\alpha_y(i).
\end{aligned}$$

Note for that last calculation that when expanding as a power series in t_1 and t_2 one only needs to consider the constant term and the coefficients of t_1, t_2 and t_1t_2. Note also that the FE calculation is proof that you can't just naively "do E then F"; it's somehow more complicated than that.

Next

$$\begin{aligned}
(H\phi)(g) &= (d/dt)\alpha(e^{2t}i)|_{t=0} \\
&= (d/dt)\alpha(i+2it+\cdots)|_{t=0} \\
&= 2\alpha_y(i)
\end{aligned}$$

and our sanity check, that $EF - FE = H$, works again. Next

$$\begin{aligned}(HH\phi)(g) &= (d^2/dt_1dt_2)\alpha(e^{2t_1+2t_2}i)|_{t_i=0}\\ &= (d^2/dt_1dt_2)\alpha(i + 2it_1 + 2it_2 + 4it_1t_2 + \cdots)|_{t_i=0}\\ &= (d/dt_1)((2 + 4t_1)\alpha_y(i + 2it_1 + \cdots))|_{t_1=0}\\ &= 4\alpha_y(i) + 4\alpha_{yy}(i)\end{aligned}$$

and finally

$$\begin{aligned}(Z\phi)(g) &= (d/dt)(e^{2st}\alpha(i))|_{t=0}\\ &= 2s\alpha(i) = 2s\phi(g)\end{aligned}$$

and so the central character sends Z to $2s$ and $H^2 + 2EF + 2FE$ to the map $g \mapsto 4\alpha_y(i) + 4\alpha_{yy}(i) + 2\alpha_{xx}(i) + 2\alpha_{xx}(i) - 4\alpha_y(i) = 4\alpha_{yy}(i) + 4\alpha_{xx}(i)$. Now $(\Delta\alpha)(i) = -(\alpha_{xx}(i) + \alpha_{yy}(i))$, so $(H^2 + 2EF + 2FE)\phi$ sends g to $-4(\Delta\alpha)(i) = -4\lambda\alpha(i) = -4\lambda\phi(g)$.

(d) and (e) I will currently pass on.

The central character for ϕ is easy to work out: if $z \in \mathrm{GL}_1(\mathbf{A})$ then $z = \gamma\kappa u$ with $\gamma \in \mathrm{GL}_1(\mathbf{Q})$ and so on (all embedded diagonally), and $\phi(xz) = \phi(x)\chi(\kappa)^{-1}u^{2s}$ so the central character sends $\gamma\kappa u$ to $\chi(\kappa)^{-1}u^{2s}$.

4 Hecke operators.

We just need to unravel, as in the holomorphic case. Exactly the same calculation (with the γ_j representing the matrices in the adelic, not the classical, calculation) gives us that

$$T_p\phi(x) = \sum_j \chi(\gamma_j^{-1})^{-1}\det(\gamma_j^{-1}u)^s F(\gamma_j^{-1}ui)$$

and again we split up into the cases $p \mid N$ and $p \nmid N$. If $p \mid N$ then the sum becomes

$$\begin{aligned}&\sum_{j=0}^{p-1} p^{-s}\det(u)^s F(\left(\begin{smallmatrix}1 & -j\\ 0 & p\end{smallmatrix}\right)ui)\\ &= p^{-s}\det(u)^s p(T_pF)(ui)\\ &= p^{1-s}\det(u)^s b_pF(ui)\\ &= p^{1-s}b_p\phi(x)\end{aligned}$$

and if $p \nmid N$ then we get

$$\begin{aligned}&\sum_{j=0}^{p-1} p^{-s}\det(u)^s F(\left(\begin{smallmatrix}1 & -j\\ 0 & p\end{smallmatrix}\right)ui) + \chi(p)p^{-s}\det(u)^{-s}F(pui)\\ &= p^{-s}\det(u)^s p(T_pF)(ui)\\ &= p^{1-s}\det(u)^s b_pF(ui)\\ &= p^{1-s}b_p\phi(x)\end{aligned}$$

so in either case the eigenvalue is $p^{1-s}b_p$.

As usual S_p is much easier, we just need to compute the central character evaluated at $\gamma\kappa u$ with $\kappa = u = p^{-1}$ so it's $\chi(p)p^{-2s}$.

To get the Satake parameters we divide the T_p eigenvalue by $\sqrt{p}$; the parameters are the roots of $X^2 - p^{1/2-s}b_pX + \chi(p)p^{-2s}$, which can also be written $X^2 - p^{-s}a_pX + \chi(p)p^{-2s}$. Recall that in the Maass forms induced from finite order Grossencharacters on real quadratic fields we have $\lambda = 1/4$ and a_p the sum of two roots of unity.

5 Various notions of algebraicity.

For π to be defined over a number field you certainly need the T_p and S_p to be, for all good p, and this should happen when $\lambda = 1/4$ and $p^{1/2-s}a_p$ is in a number field (independent of p), so $s - 1/2 \in \mathbf{Z}$ conjecturally, so C-arithmetic iff $s - 1/2 \in \mathbf{Z}$ and $\lambda = 1/4$ conjecturally. For the Satake parameters to be defined over a number field we again conjecture that this happens iff $\lambda = 1/4$ and $s \in \mathbf{Z}$, so L-arithmetic iff $s \in \mathbf{Z}$.

Now for the infinitesimal character computation: The infinitesimal character sends $H^2 - 1$ goes to -4λ and Z to $2s$, so it sends H to $\pm\sqrt{1-4\lambda}$ and because λ is going to be a positive real for a cusp form, this is an integer iff $\lambda = 1/4$, and in this case it's zero. So we're in the lattice (L-algebraic) iff $s \in \mathbf{Z}$ and in the shift by δ (C-algebraic) iff $s - 1/2 \in \mathbf{Z}$.

6 L-functions a la Tate.

Here's something that really bewilders me. If f is a cusp form then there's an "elementary" proof that the L-function of f has a meromorphic continuation and functional equation. But there's also a "general machine" proof, and the "general machine" proof seems to use the Fourier Inversion theorem. What is going on here?

6.1 The elementary proof.

Let $f : \mathbf{H} \to \mathbf{C}$ be a cuspidal newform of level $\Gamma_1(N)$ and weight k. Write $f = \sum_{n\geq 1} a_n q^n$. Now define

$$\Lambda(f,s) := \int_{y=0}^{\infty} f(iy) y^{s-1} \mathrm{d}y.$$

I claim that this converges for $\mathrm{Re}(s)$ sufficiently large, and the proof is that if we substitute in the power series for f we see

$$\begin{aligned}\Lambda(f,s) &= \sum_{n\geq 1} \int_{y=0}^{\infty} a_n e^{-2\pi n y} y^{s-1} \mathrm{d}y \\ &= \sum_{n\geq 1} a_n \int_{y=0}^{\infty} e^{-2\pi n y} y^{s-1} \mathrm{d}y\end{aligned}$$

and making the substitution $x = 2\pi n y$ we get

$$\begin{aligned}\Lambda(f,s) &= \sum_{n\geq 1} a_n \int x = 0^{\infty} e^{-x} (x/2\pi n)^s \mathrm{d}x/x \\ &= \sum_{n\geq 1} a_n (2\pi n)^{-s} \int_{x=0}^{\infty} e^{-s} x^{s-1} \mathrm{d}x \\ &= (2\pi)^{-s} \Gamma(s) \sum_{n\geq 1} a_n n^{-s}\end{aligned}$$

so by standard bounds we have convergence for $\mathrm{Re}(s)$ sufficiently large.

But now if we write $w = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ then $f|_k w$ is a cusp form for $w^{-1}\Gamma_1(N)w$ which is $\Gamma_1(N)$ again, and hence $g(z) := (Nz)^{-k} f(-1/(Nz))$ is also a cusp form for $\Gamma_1(N)$ (it's some constant times the conjugate of f, in fact), and we see that for $0 < t < \infty$ we have

$$\Lambda(f,s) = \int_{y=0}^{t} f(iy) y^{s-1} \mathrm{d}y + \int_{y=t}^{\infty} f(iy) y^{s-1} \mathrm{d}y,$$

with the second integral converging for all s, because $f(iy)$ will be about $e^{-2\pi y}$ which beats y^{s-1} for all s. Moreover, setting $z = i/(Ny)$ in the definition of g we see that $g(i/(Ny)) = (i/y)^{-k} f(iy)$

and hence

$$\int_{y=0}^{t} f(iy)y^{s-1}\mathrm{d}y = \int_{y=0}^{t} (i/y)^k g(i/(Ny))y^s \mathrm{d}y/y,$$

so writing $u = 1/(Ny)$ we get that this integral is

$$\begin{aligned}&\int_{u=1/(Nt)}^{\infty} (iuN)^k g(iu)(1/(Nu))^s \mathrm{d}u/u \\ &= (iN)^k N^{-s} \int_{u=1/Nt}^{\infty} u^{k-s-1} g(iu)\mathrm{d}u\end{aligned}$$

and this latter integral again converges for all s, giving the holomorphic continuation of $\Lambda(f,s)$. Even better, if we define $R(f,s) := \Lambda(f,s).N^{s/2}$ then we see $R(f,s) = N^{s/2}\int_{y=t}^{\infty} f(iy)y^s \mathrm{d}y/y + i^k N^{k/2} N^{(k-s)/2} \int_{u=t}^{\infty} u^{k-s}(i^k N^{k/2} g)(iu)\mathrm{d}u/u$ and one can go on to check that something like $R(f,s) = R(h,k-s)$ will hold, if $h = i^k N^{k/2} g$. I am too lazy to do this properly. I need to compute $h|w$ etc; there's very little left. Maybe this will come back to haunt me.

6.2 Tate's way.

I'm now following Godement-Jacquet, the thrust of which is actually rather easy to see once you've read Tate's thesis.

The local story is this. Recall that for GL_1, Tate's local zeta integrals in the unramified case worked like this. If $\chi : \mathbf{Q}_p \to \mathbf{C}^\times$ is unramified then let Φ be the characteristic function of $\mathbf{Z}_p$ and consider

$$\int_{\mathbf{Q}_p^\times} \Phi(x)\chi(x)|x|^s \mathrm{d}^\times x$$

with $\mathrm{d}^\times x$ the multiplicative Haar measure on $\mathbf{Q}_p^\times$ normalised in such a way that the integral of $\mathbf{Z}_p^\times$ is 1. The integrand is constant on $p^n\mathbf{Z}_p^\times$, the measure of this latter set is also 1, and the integrand is zero if $n < 0$. So the integral is

$$\sum_{n\geq 0} \chi(p)^n p^{-ns} = (1-\chi(p)p^{-s})^{-1}$$

which is precisely the local L-factor. In Godement-Jacquet the correct analogue of this for $\mathrm{GL}_n(\mathbf{Q}_p)$ is given. The idea is that Φ is a Bruhat-Schwarz function on $M_n(\mathbf{Q}_p)$ and that χ is a matrix coefficient for the representation χ of $\mathrm{GL}_n(\mathbf{Q}_p)$. If we now go to infinite-dimensional representations, they still have matrix coefficients, and if π is unramified principal series corresponding (via Satake normalised in Langlands' way) to the pair of non-zero complex numbers (z_1, z_2) and we choose s_1, s_2 with $p^{-s_i} = z_i$ then the matrix coefficient corresponding to the $K := \mathrm{GL}_2(\mathbf{Z}_p)$-fixed vectors is (up to scalar) equal to $\omega : g \mapsto \int_K \phi(kg)\mathrm{d}k$, with ϕ defined by

$$\phi\left(\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)k\right) = |a|^{1/2+s_1}|d|^{-1/2+s_2}.$$

Now for Φ the characteristic function of $M_2(\mathbf{Z}_p)$, and Haar measure $\mathrm{d}^\times g$ on $\mathrm{GL}_2(\mathbf{Q}_p)$ normalised such that the integral of K is 1, we can attempt to compute

$$\int_{\mathrm{GL}_2(\mathbf{Q}_p)} \Phi(g)\omega(g)|\det(g)|^{s+1/2}\mathrm{d}^\times g$$

This is the analogue of the local L-factor for π. Note the $|.|^{s+1/2}$; the $+1/2$ is some kind of appropriate normalising factor to make things unitary, I *think*. Writing d for $\mathrm{d}^\times$ out of laziness, we get that this local factor is

$$\int_{g\in\mathrm{GL}_2(\mathbf{Q}_p)} \Phi(g) \int_{k\in\mathrm{GL}_2(\mathbf{Z}_p)} \phi(kg)|\det(g)|^{s+1/2}\mathrm{d}g\mathrm{d}k$$

and if we switch the integrals and then write $g' = kg$ we see that, because $\Phi(g') = \Phi(g)$ and so on, the k vanishes completely and we just get (writing g for g')

$$\int_{g\in \mathrm{GL}_2(\mathbf{Q}_p)} \Phi(g)\phi(g)|\det(g)|^{s+1/2}\mathrm{d}g.$$

Now the integrand is zero off the semigroup S, the matrices in $M_2(\mathbf{Z}_p)$ with non-zero determinant. We can write S as a disjoint union $K\left(\begin{smallmatrix} p^{m+n} & 0 \\ 0 & p^n \end{smallmatrix}\right)K$ with $0 \leq m, n$, by some standard result, and because ϕ is right K invariant we may as well break the double coset above up into single cosets: up to a factor of p^n in the centre, it's $K\left(\begin{smallmatrix} p^m & 0 \\ 0 & 1 \end{smallmatrix}\right)K = \cup\left(\begin{smallmatrix} p^e & i \\ 0 & p^f \end{smallmatrix}\right)K$ with $0 \leq e, f$, $e + f = m$, $0 \leq i < p^e$, and $(p^e, p^f, i) = 1$. Hence we can write $S = \cup\left(\begin{smallmatrix} p^e & i \\ 0 & p^f \end{smallmatrix}\right)K$ with $0 \leq e, f$ and $0 \leq i < p^e$. Now we can do the integral; it's

$$\begin{aligned}
&\sum_{e,f\geq 0}\sum_{i=0}^{p^e-1} p^{-e(1/2+s_1)}p^{-f(-1/2+s_2)}p^{(-e-f)(s+1/2)} \\
&= \sum_{e,f\geq 0} z_1^e z_2^f p^{-es}p^{-fs} \\
&= \sum_{e,f\geq 0} (z_1p^{-s})^e(z_2p^{-s})^f \\
&= (1-p^{-s}z_1)^{-1}(1-p^{-s}z_2)^{-1}
\end{aligned}$$

which, surprise surprise, is the usual local L-factor.

I am guessing that at the ramified places the story is different but I'm not sure I know enough about matrix coefficients to be able to work it out myself.

There is a general result in Godement-Jacquet that says that if you know the integrals for π (a rep of the Levi) then you know them for $\mathrm{Ind}(\pi)$ and hence you know a lot about them for the subquotients of this induction. In the arch case it's Theorem 8.8 and Corollary 8.9.

But let's try to do the infinite places by myself. I think that the discrete series representations look like this: let D_k ($k \geq 2$) denote the following representation of $\mathrm{SL}_2^{\pm}(\mathbf{R})$ (the matrices with determinant ± 1): the underlying space is the holomorphic functions $\mathbf{C}\backslash\mathbf{R} \to \mathbf{C}$ with $||f||^2 := \int_{x\in\mathbf{R}}\int_{y\in\mathbf{R}^\times} |f(x+iy)|^2|y|^{n-2}\mathrm{d}x\mathrm{d}y$ finite, and let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ send f to the function $z \mapsto (bz+d)^{-n}f((az+c)/(bz+d))$. This is irreducible and unitary. Extend to a representation of $\mathrm{GL}_2(\mathbf{R})$ by "pasting on" a character of $\mathbf{R}_{>0}$. Note that our action of $\mathrm{GL}_2(\mathbf{R})$ is on the left; the usual action on modular forms is on the right, so we have applied that funny involution ι.

When you unravel things, you have to choose a matrix coefficient, and hence a function in D_n. I think $(z+i)^{-n}$ (on the upper half plane) is the natural choice, but now I can't do the integrals. Aah well.

Globally, let Φ be a Bruhat-Schwarz function on $\mathrm{GL}_2(\mathbf{A})$, let π be a unitary cuspidal irreducible automorphic representation of $\mathrm{GL}_2(\mathbf{A})$, let ω be a matrix coefficient, and set

$$Z(\Phi, \omega, s) = \int_{\mathrm{GL}_2(\mathbf{A})} \Phi(g)\omega(g)|\det(g)|^{s+1/2}\mathrm{d}g.$$

Note I'm assuming π is unitary so the functional equation should relate s to $1 - s$. The global integral breaks up as a product of local integrals, so for some sensible choices of Φ and ω (for example, ω can be thought of as $g \mapsto \int_{\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A})^1} f_1(hg)\overline{f}_2(h)\mathrm{d}h$ for f_1 and f_2 in π) we should get the L-function attached to the modular form giving rise to π. Just as in the GL_1 case one can analytically continue the integral to all $s \in \mathbf{C}$ but this seems to use Poisson summation—and that is what baffles me.