Algebraic number theory

Test 2, solutions

18 March, 2011

1. **4 marks**

d is divisible by 2011, but no other prime, otherwise some other prime will also be ramified. Thus $d = \pm 2011$. But 2011 is congruent to 3 mod 4, so $\mathbb{Q}(\sqrt{2011})$ is ramified at 2. Thus $\mathbb{Q}(\sqrt{-2011})$ is the only quadratic field ramified exactly at 2011.

2. **4 marks**

Use the multiplicativity of the norm of ideals. By inspection of split, inert and ramified cases we see that for every prime ideal P the product $P\overline{P}$ is the principal ideal generated by $||P||$. Write $I = P_1 \dots P_n$, where the P_i are prime ideals. Then $I\overline{I} = ||I||\mathcal{O}_K$, hence the result.

3. **8 marks**

$N_K(7+\sqrt{-5}) = 54 = 2 \times 3^3$, hence the prime ideals dividing $(7+\sqrt{-5})$ have norms 2 and a power of 3. There is only one prime ideal over 2 (since 2 is ramified), namely $P = (2, 1+\sqrt{-5})$, $||P|| = 2$. Next, $(3) = Q\overline{Q}$, where $Q = (3, 1+\sqrt{-5})$ and $\overline{Q} = (3, 1-\sqrt{-5})$, $||Q|| = ||\overline{Q}|| = 3$. Since 3 does not divide $7+\sqrt{-5}$, the ideals Q and $\overline{Q}$ cannot both divide $(7+\sqrt{-5})$, so we need to decide whether $(7+\sqrt{-5}) = PQ^3$ or $P\overline{Q}^3$. But $(7+\sqrt{-5}) \subset Q$ since $7+\sqrt{-5}$ is an integral linear combination of 3 and $1+\sqrt{5}$. Therefore, $(7+\sqrt{-5}) = PQ^3$.

4. **4 marks**

$\mathrm{Tr}_K((\sqrt[3]{2})^n) = 0$ unless n is a multiple of 3, hence $\det(\mathrm{Tr}_K((\sqrt[3]{2})^{i+j})) = -108$.