Algebraic number theory

Solutions to test 2

March 12, 2012

You can use any results from lectures without proof.

1. **6 marks**

Let d be a square free integer. It is known from lectures (and is easily computed) that the discriminant of $\mathbb{Q}(\sqrt{d})$ is $D = 4d$ if d is 2 or 3 modulo 4, and $D = d$ if d is 1 modulo 4. The first case gives $d = 2, 3, -1, -2$ with discriminants $D = 8, 12, -4, -8$, respectively. The second case gives $d = D = 5, -3, -7, -11$.

2. **8 marks**

(a) From lectures we know that 2 is ramified and $P = (2, 1 + \sqrt{-13})$. (1 mark)

(b) We need to show that P is not principal. (Then its class in $\mathrm{Cl}(K)$ is non-trivial. Since $P^2 = 2\mathcal{O}_K$ is principal, the order of the class of P in $\mathrm{Cl}(K)$ is 2.) Now P has norm 2, so if P has one generator, then the norm of this generator is 2. Since $x^2 + 13y^2 = 2$ has no integral solutions, we conclude that P is not principal. (6 marks for a complete proof)

(c) Any Euclidean domain is a PID, so $\mathcal{O}_K$ is not one. (1 mark)

3. **6 marks**

The norm $(1 + \sqrt{-13})$ is 14, so $(1 + \sqrt{-13}) = PQ$, where P is the unique prime ideal over 2 (described in Q2(a)), and Q is a prime ideal over 7. Now -13 is $1 = (\pm 1)^2$ modulo 7, so 7 is split and Q is either $(7, 1 + \sqrt{-13})$ or its conjugate. Since $1 + \sqrt{-13} \in (7, 1 + \sqrt{-13})$ we conclude that $Q = (7, 1 + \sqrt{-13})$.