Lie algebras

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Introduction

For this course you need a very good understanding of linear algebra; a good knowledge of group theory and the representation theory of finite groups will also help. The main sources for these notes are the books [6] and [8].

We give complete proofs of all statements with the exception of the conjugacy of Cartan subgroups, the uniqueness theorem for semisimple Lie algebras, and the existence theorem for exceptional semisimple Lie algebras. These results can be found in [4].

Contents

1 Basic definitions and examples

Let k be a field of characteristic zero. Most of the time k will be $\mathbb{R}$ or $\mathbb{C}$.

A not necessarily associative algebra is a vector space over k with a bilinear product, i.e. a product satisfying left and right distributivity laws. Two classes of such algebras have particularly good properties and so are very useful: *associative algebras*, where multiplication satisfies the associativity axiom

$$(ab)c = a(bc),$$

and *Lie algebras*, where the product is traditionally written as the bracket $[ab]$, and satisfies *skew-symmetry* and the *Jacobi identity*:

$$[aa] = 0, \qquad [a[bc]] + [b[ca]] + [c[ab]] = 0.$$

We have $0 = [a+b, a+b] = [aa] + [bb] + [ab] + [ba] = [ab] + [ba]$, so that the bracket is indeed skew-symmetric in the usual sense.

Examples of Lie algebras 1. *Abelian Lie algebras.* Any vector space with the zero product $[ab] = 0$.

2. $\mathbb{R}^3$ with vector product $a \times b$. It can be defined by bilinearity and skew-symmetry once we postulate

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2.$$

The Jacobi identity is a standard exercise in vector algebra.

3. From any associative algebra A we construct a Lie algebra on the same vector space by setting $[ab] = ab - ba$. The Lie algebra of $n \times n$-matrices is called $\mathfrak{gl}(n)$.

4. Let $\mathfrak{sl}(n)$ be the subspace of $\mathfrak{gl}(n)$ consisting of matrices with zero trace. Since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, the set $\mathfrak{sl}(n)$ is closed under $[ab] = ab - ba$, and hence is a Lie algebra.

5. Let $\mathfrak{o}(n)$ be the subspace of $\mathfrak{gl}(n)$ consisting of skew-symmetric matrices, that is, $A^T = -A$. Then

$$(AB - BA)^T = B^T A^T - A^T B^T = (-B)(-A) - (-A)(-B) = -(AB - BA), \quad (1)$$

so that $\mathfrak{o}(n)$ is closed under $[ab] = ab - ba$, and hence is a Lie algebra.

If we want to emphasize the dependence of the ground field k we write $\mathfrak{gl}(n, k)$, $\mathfrak{sl}(n, k)$, $\mathfrak{o}(n, k)$.

To define a Lie bracket on a vector space with basis $e_1, \ldots, e_n$ we need to specify the *structure constants* c^r_{lm}, that is, elements of k such that

$$[e_l, e_m] = \sum_{r=1}^{n} c^r_{lm} e_r.$$

For example,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

is a basis of the vector space $\mathfrak{sl}(2)$. One easily checks that

$$[H, X_+] = 2X_+, \quad [H, X_-] = -2X_-, \quad [X_+, X_-] = H.$$

Similarly,

$$R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a basis of the vector space $\mathfrak{o}(3)$. We have

$$[R_x, R_y] = R_z, \quad [R_y, R_z] = R_x, \quad [R_z, R_x] = R_y.$$

Definition 1.1 *A homomorphism of Lie algebras is a linear transformation which preserves the bracket. An isomorphism is a bijective homomorphism. A Lie subalgebra is a vector subspace closed under the bracket. An ideal of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ such that $[\mathfrak{a}\mathfrak{g}] \subset \mathfrak{a}$. By skew-symmetry of the bracket any ideal is two-sided. The quotient algebra $\mathfrak{g}/\mathfrak{a}$ is then defined in the obvious way, as a quotient vector space with the inherited bracket operation.*

More examples of Lie algebras 6. Upper triangular $n \times n$-matrices $A = (a_{ij})$, $a_{ij} = 0$ if $i > j$, form a subalgebra of the full associative algebra of $n \times n$-matrices. The attached Lie algebra will be denoted by $\mathfrak{t}(n)$.

7. Upper triangular $n \times n$-matrices $A = (a_{ij})$ such that $a_{ii} = 0$ for all i also form a subalgebra of the algebra of $n \times n$-matrices. The attached Lie algebra will be denoted by $\mathfrak{n}(n)$.

Exercises. 1. Prove that $\mathfrak{o}(2)$ and $\mathfrak{n}(2)$ are abelian 1-dimensional Lie algebras, hence they are isomorphic to k with zero bracket.

2. Prove that the Lie algebra from Example 2 is isomorphic to $\mathfrak{o}(3)$ by comparing the structure constants.

3. Let $k = \mathbb{R}$ or $\mathbb{C}$. The Lie algebras $\mathfrak{sl}(2)$, $\mathfrak{o}(3)$, $\mathfrak{t}(2)$, $\mathfrak{n}(3)$ all have dimension 3. Are any of these isomorphic? (The answer will depend on k.)

Definition 1.2 *The* **derived algebra** *$\mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$ is the ideal $[\mathfrak{g}\mathfrak{g}]$ generated by $[ab]$, for all a, $b \in \mathfrak{g}$.*

It is clear that $\mathfrak{g}/[\mathfrak{g}\mathfrak{g}]$ is the maximal abelian quotient Lie algebra of $\mathfrak{g}$. This construction can be iterated as follows to define the *derived series* of $\mathfrak{g}$:

$$\mathfrak{g}^{(1)} = [\mathfrak{g}\mathfrak{g}], \quad \mathfrak{g}^{(r+1)} = [\mathfrak{g}^{(r)}\mathfrak{g}^{(r)}].$$

By induction we show that the $\mathfrak{g}^{(r)}$ are ideals of $\mathfrak{g}$ (the first inclusion is due to the Jacobi identity):

$$[\mathfrak{g}^{(r+1)}\mathfrak{g}] = [[\mathfrak{g}^{(r)}\mathfrak{g}^{(r)}]\mathfrak{g}] \subset [[\mathfrak{g}^{(r)}\mathfrak{g}]\mathfrak{g}^{(r)}] \subset [\mathfrak{g}^{(r)}\mathfrak{g}^{(r)}] = \mathfrak{g}^{(r+1)}.$$

If $\mathfrak{g}^{(n)} = 0$ for some n, then $\mathfrak{g}$ is called *solvable*. We note that any automorphism of $\mathfrak{g}$ preserves $\mathfrak{g}^{(r)}$. The last non-zero ideal $\mathfrak{g}^{(r)}$ is visibly abelian.

An example of a solvable Lie algebra is $\mathfrak{t}(n)$, or any abelian Lie algebra.

We can also iterate the construction of the derived algebra in another way: $\mathfrak{g}^1 = \mathfrak{g}^{(1)} = [\mathfrak{g}\mathfrak{g}]$, $\mathfrak{g}^{r+1} = [\mathfrak{g}^r\mathfrak{g}]$. This means that $\mathfrak{g}^r$ is generated by iterated brackets of $r+1$ elements $[a_1[a_2[a_3 \ldots [a_r a_{r+1}] \ldots]]]$. This series of ideals is called the *lower central series*, and if it goes down to zero, then $\mathfrak{g}$ is called *nilpotent*. The ideals $\mathfrak{g}^r$ are also preserved by the automorphisms of $\mathfrak{g}$. An example of a nilpotent Lie algebra is $\mathfrak{n}(n)$. Any abelian Lie algebra is also nilpotent.

More exercises. 4. Compute the derived series of $\mathfrak{t}(n)$, $\mathfrak{n}(n)$, $\mathfrak{sl}(n)$. Hence determine which of these Lie algebras are solvable.

5. Compute the lower central series of $\mathfrak{t}(n)$, $\mathfrak{n}(n)$, $\mathfrak{sl}(n)$. Hence determine which of these Lie algebras are nilpotent.

6. Prove that $\mathfrak{g}^{(r)} \subset \mathfrak{g}^r$ for any $\mathfrak{g}$ and r. In particular, this implies that every nilpotent algebra is solvable. Show by an example that the converse is false. (Hint: consider the Lie algebra attached to the algebra of affine transformations of the line $x \mapsto ax + b$, a, $b \in k$. This algebra can be realized as the algebra of 2×2-matrices with the zero bottom row.)

2 Theorems of Engel and Lie

A (matrix) *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism $\mathfrak{g} \to \mathfrak{gl}(n)$; the number n is called the dimension of the representation. A representation is called *faithful* if its kernel is 0. A representation $\mathfrak{g} \to \mathfrak{gl}(V)$, where V is a vector space, is called *irreducible* if the only $\mathfrak{g}$-invariant subspace $W \subset V$, $W \neq V$, is $W = 0$. (Recall that a subspace $W \subset V$ is $\mathfrak{g}$-invariant if $\mathfrak{g}W \subset W$.)

Any subalgebra of $\mathfrak{gl}(n)$ comes equipped with a natural representation of dimension n. To an arbitrary Lie algebra $\mathfrak{g}$ we can attach a representation in the following way.

Definition 2.1 *Elements $a \in \mathfrak{g}$ act as linear transformations of the vector space $\mathfrak{g}$ by the rule $x \mapsto [ax]$. The linear transformation of $\mathfrak{g}$ attached to $a \in \mathfrak{g}$ is denoted by* $\mathrm{ad}(a)$.

Lemma 2.2 ad *is a homomorphism* $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, *called the* **adjoint representation**.

Proof To check that ad is a homomorphism we need to prove that

$$\mathrm{ad}([ab]) = [\mathrm{ad}(a)\mathrm{ad}(b)] = \mathrm{ad}(a)\mathrm{ad}(b) - \mathrm{ad}(b)\mathrm{ad}(a).$$

The left hand side sends x to $[[ab]x]$, whereas the right hand side sends x to $[a[bx]] - [b[ax]]$. By the Jacobi identity this is the same as $[a[bx]] + [a[xb]] + [x[ba]]$, and by skew-symmetry this equals $[x[ba]] = -[[ba]x] = [[ab]x]$. QED

Recall that a linear transformation is nilpotent if its n-th power is zero for some n. It easily follows from the definition of a nilpotent Lie algebra that if $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}(x)$ is nilpotent for any $x \in \mathfrak{g}$. The converse is also true. Engel's theorem says that if $\mathfrak{g}$ has a faithful representation $\rho : \mathfrak{g} \to \mathfrak{gl}(n)$ such that $\rho(x)$ is nilpotent for every $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent. A key result in the direction of Engel's theorem is the following proposition.

Proposition 2.3 *If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation on a (non-zero) vector space V such that $\rho(a)$ is a nilpotent linear transformation for any $a \in \mathfrak{g}$, then V has a non-zero vector x such that $\mathfrak{g}x = 0$.*

Proof We prove this for all finite-dimensional Lie algebras inductively by $\dim \mathfrak{g}$, the case of $\dim \mathfrak{g} = 1$ being clear. If ρ is not faithful, then $\dim(\mathfrak{g}/\mathrm{Ker}\,\rho) < \dim \mathfrak{g}$, so that the statement for $\mathfrak{g}$ follows from that for $\dim(\mathfrak{g}/\mathrm{Ker}\,\rho)$. Hence we assume $\mathfrak{g} \subset \mathfrak{gl}(V)$, so that the elements of $\mathfrak{g}$ are nilpotent linear transformations of V given by nilpotent matrices. Consider the restriction of the adjoint representation of $\mathfrak{gl}(V)$ to $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{gl}(V)$ sending the elements of $\mathfrak{g}$ to $\mathfrak{g}$. Thus $\mathrm{ad}(a)$, $a \in \mathfrak{g}$, is a linear transformation of $\mathfrak{g}$ given by $x \mapsto ax - xa$. We have

$$\mathrm{ad}(a)x = ax - xa,\ \mathrm{ad}(a)^2x = a^2x - 2axa + xa^2,\ \mathrm{ad}(a)^3x = a^3x - 3a^2xa + 3axa^2 - xa^3, \ldots \tag{2}$$

so that $\mathrm{ad}(a)^r$ consists of the terms like $a^m x a^{r-m}$. Thus $a^n = 0$ implies $\mathrm{ad}(a)^{2n-1} = 0$.

Let $\mathfrak{m} \subset \mathfrak{g}$, $\mathfrak{m} \neq \mathfrak{g}$, be a maximal subalgebra (possibly zero). The restriction of ad to $\mathfrak{m}$ leaves $\mathfrak{m}$ invariant, hence we have the quotient representation of $\mathfrak{m}$ on $\mathfrak{g}/\mathfrak{m}$. By the inductive assumption it has a non-zero vector on which $\mathfrak{m}$ acts by 0. Choose its representative $v \in \mathfrak{g}$, then $[\mathfrak{m}, v] \in \mathfrak{m}$, $v \notin \mathfrak{m}$. The vector space spanned by $\mathfrak{m}$ and v is a subalgebra of $\mathfrak{g}$; by the maximality of $\mathfrak{m}$ it equals $\mathfrak{g}$.

Let $U \neq 0$ be the subspace of V consisting of all the vectors killed by $\mathfrak{m}$; note that $U \neq 0$ by the inductive assumption. Let us show that U is $\mathfrak{g}$-invariant. Since $\mathfrak{m}U = 0$ we need to show that $vU \subset U$, that is, $mvu = 0$ for any $m \in \mathfrak{m}$, $u \in U$. Write $mv = vm + [mv]$, thus $mvu = vmu + [mv]u = 0$ since $[mv] \in \mathfrak{m}$, and every element of $\mathfrak{m}$ kills u. Now consider the nilpotent linear transformation of U defined by v. Clearly, U has a non-zero vector killed by v, but this vector is killed by the whole $\mathfrak{g}$. QED

Let V^* be the space of linear functions $V \to k$. A linear transformation $\phi : V \to V$ induces a linear transformation $\phi^* : V^* \to V^*$ by the rule $\phi^* f(x) = f(\phi(x))$.

If $e_1, \ldots, e_n$ is a basis of V, then V^* has the dual basis consisting of linear functions $f_1, \ldots, f_n$ defined by $f_i(e_j) = 0$ for all i and j such that $i \neq j$, and $f_i(e_i) = 1$ for all i. If a linear transformation ϕ is given by a $n \times n$-matrix A, $\phi(x) = Ax$, then the matrix of ϕ^* in the dual basis is A^T.

A representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ defines the *dual* (or *contragredient*) representation $\rho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ as follows: $a \in \mathfrak{g}$ sends a linear function $f(x)$ to $-f(\rho(a)x)$ (or $a \mapsto -\rho(a)^T$ in the matrix form). Let us check that ρ^* is a representation, that is, $\rho^*([ab]) = \rho^*(a)\rho^*(b) - \rho^*(b)\rho^*(a)$. Choose a basis in V, and let A, B be the matrices of $\rho(a)$, $\rho(b)$, respectively. The left hand side sends $f(x)$ to $-(f(ABx) - f(BAx)) = -f(ABx) + f(BAx)$, whereas the right hand side sends $f(x)$ to $f(BAx) - f(ABx)$, so these are the same. (Note that the minus sign was crucial for the success of this calculation!)

Theorem 2.4 (Engel) *If every element of $\mathfrak{g} \subset \mathfrak{gl}(n)$ is a nilpotent linear transformation, then there exists a decreasing sequence of subspaces*

$$V \supset V_1 \supset V_2 \supset \ldots \supset V_{n-1} \supset V_n = 0, \tag{3}$$

such that every $a \in \mathfrak{g}$ sends V_i to V_{i+1}. In particular, $\mathfrak{g}$ is nilpotent.

Proof Consider the dual representation $\mathfrak{g} \to \mathfrak{gl}(V^*)$. It is clear that every element in the image of this representation is a nilpotent linear transformation, and so by Proposition 2.3 there exists a non-zero linear function $f \in V^*$ such that $f(ax)$ is the zero element of V^* for every $a \in \mathfrak{g}$. Hence $V_1 = \mathfrak{g}V$ is a proper subspace of V. Iterating this construction we get a decreasing sequence of subspaces (3) such that every $a \in \mathfrak{g}$ sends V_i to V_{i+1}. Hence the product of any n elements of $\mathfrak{g}$ is zero, which implies $\mathfrak{g}^n = 0$. QED

Corollary 2.5 *If* $\mathrm{ad}(a)$ *is a nilpotent linear transformation of $\mathfrak{g}$ for every $a \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.*

Proof Apply Engel's theorem to the quotient Lie algebra $\mathfrak{g}/\mathrm{Ker}\,(\mathrm{ad})$. Then for some n we have $\mathrm{ad}(a_1) \ldots \mathrm{ad}(a_n) = 0$ for any $a_i \in \mathfrak{g}$. This means that $[a_1[a_2 \ldots [a_n, a]]] = 0$ for any $a_1, a_2, \ldots, a_n, a \in \mathfrak{g}$, so that $\mathfrak{g}^{n+1} = 0$. QED

Exercises. 1. Prove that every subalgebra of a solvable (resp. nilpotent) Lie algebra is solvable (resp. nilpotent). The same statement for quotient algebras.

2. Exercise 1 shows that every subalgebra of $\mathfrak{n}(n)$ is nilpotent. Clearly, every element of $\mathfrak{n}(n)$ is nilpotent. Find a nilpotent subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n)$ such that no non-zero element of $\mathfrak{g}$ is nilpotent. (Hint: think of diagonal matrices.)

Lie's theorem proves that for every solvable subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ there exists a basis of V such that $\mathfrak{g} \subset \mathfrak{t}(n)$. This works only for $k = \mathbb{C}$.

Lemma 2.6 (Dynkin) *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra, $\mathfrak{a} \subset \mathfrak{g}$ an ideal, and $\lambda(x) \in \mathfrak{a}^*$ a linear function $\mathfrak{a} \to k$. Define*

$$W = \{v \in V | av = \lambda(a)v \textit{ for any } a \in \mathfrak{a}\}.$$

Then $\mathfrak{g}W \subset W$.

Comment To make this statement a bit clearer let us note the following. Suppose V is a representation of a Lie algebra $\mathfrak{a}$, and $v \in V$ is a common eigenvector of all the elements of $\mathfrak{a}$. For $a \in \mathfrak{a}$ we denote by $f(a) \in k$ the eigenvalue of a, that is, $av = f(a)v$. Then $f : \mathfrak{a} \to k$ is a linear function. Indeed, for x_1, $x_2 \in k$, a_1, $a_2 \in \mathfrak{a}$ we obviously have

$$(x_1a_1 + x_2a_2)v = x_1(a_1v) + x_2(a_2v) = x_1f(a_1)v + x_2f(a_2)v = (x_1f(a_1) + x_2f(a_2))v,$$

so that f is linear. Dynkin's lemma says that *given a representation V of $\mathfrak{g}$, and a linear function λ on an ideal $\mathfrak{a} \subset \mathfrak{g}$, the λ-eigenspace of $\mathfrak{a}$ in V is $\mathfrak{g}$-invariant.*

Proof Let $x \in \mathfrak{g}$, $a \in \mathfrak{a}$. Observe that $ax = xa + [ax]$. Since $\mathfrak{a}$ is an ideal we have $[ax] \in \mathfrak{a}$. If $v \in W$, then we obtain

$$axv = xav + [ax]v = \lambda(a)xv + \lambda([ax])v. \tag{4}$$

We want to show that $xv \in W$, that is, $axv = \lambda(a)xv$, and for this it is enough to show that $[ax] \in \mathrm{Ker}\,(\lambda(x))$.

We define an increasing family of subspaces $U_0 \subset U_1 \subset U_2 \ldots$ so that U_0 is spanned by v, and U_r is spanned by $v, xv, \ldots, x^rv$. Let n be such that $v, xv, \ldots, x^nv$ are linearly independent, but $x^{n+1}v \in U_n$. Visibly, $xU_n \subset U_n$. By induction we deduce from (4) that $\mathfrak{a}U_i \subset U_i$. Indeed, $\mathfrak{a}U_0 \subset U_0$ by the choice of v, formula (4) shows that $axv \in \lambda(a)xv + U_1$,

$$ax^2v = xaxv + [ax]xv = x^2av + x[ax]v + [ax]xv \in \lambda(a)x^2 + U_1,$$

and then by induction it follows that $ax^mv \in \lambda(a)x^m + U_{m-1}$. This implies that in the basis $v, xv, \ldots, x^nv$ every $a \in \mathfrak{a}$ is given by an upper triangular matrix with all the diagonal entries equal to $\lambda(a)$.

Hence for any $a \in \mathfrak{a}$ the trace of a on U_n is $(n+1)\lambda(a)$. In particular, this is true for $[ax] \in \mathfrak{a}$, but $\mathrm{Tr}[ax] = \mathrm{Tr}(ax) - \mathrm{Tr}(xa) = 0$, thus $\lambda([ax]) = 0$. (It is crucial here that the characteristic of k is zero: if k has characteristic p and $p|n+1$, the argument will collapse.) QED

Theorem 2.7 (Lie) *Let $k = \mathbb{C}$, and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable Lie subalgebra. There exists a non-zero vector which is an eigenvector of every element of $\mathfrak{g}$. In other words, there exist a vector $w \in V$, $w \neq 0$, and a linear function $\lambda(x) \in \mathfrak{g}^*$ such that $xw = \lambda(x)w$ for any $x \in \mathfrak{g}$.*

Proof We prove this by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$ the statement follows from linear algebra since we assumed that $k = \mathbb{C}$. Now suppose the statement is proved in all dimensions less than $\dim \mathfrak{g}$. Since $\mathfrak{g}' = [\mathfrak{g}\mathfrak{g}]$ is a proper subspace of $\mathfrak{g}$ we can choose a vector subspace $\mathfrak{a} \subset \mathfrak{g}$ of codimension 1 such that $\mathfrak{g}' \subset \mathfrak{a}$. Then $[\mathfrak{a}\mathfrak{g}] \subset \mathfrak{g}' \subset \mathfrak{a}$, so that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Clearly, $\mathfrak{a}$ is a solvable Lie algebra. By induction hypothesis, for some linear function $\lambda \in \mathfrak{a}^*$ the $\lambda(x)$-eigenspace $W \subset V$ defined as in Dynkin's lemma, is non-zero. Choose $x \in \mathfrak{g} \setminus \mathfrak{a}$. By Dynkin's lemma $xW \subset W$. By linear algebra x has an eigenvector $w \in W$, $w \neq 0$, with eigenvalue $\lambda_0 \in \mathbb{C}$. Any element of $\mathfrak{g}$ can be written as $a + sx$, $a \in \mathfrak{a}$, $s \in k$, and $(a + sx)w = (\lambda(a) + s\lambda_0)w$. QED

Corollary 2.8 *For any representation of a solvable Lie algebra $\mathfrak{g}$ in a complex vector space V there exist a basis of V such that all the matrices of the elements of $\mathfrak{g}$ are upper triangular.*

Proof This follows from Lie's theorem by induction on $\dim V$ (consider the quotient space $V/\mathbb{C}w$). QED

Corollary 2.9 *A Lie algebra $\mathfrak{g}$ is solvable if and only if its derived algebra $\mathfrak{g}' = [\mathfrak{g}\mathfrak{g}]$ is nilpotent.*

Proof If $\mathfrak{g}'$ is nilpotent, then $\mathfrak{g}$ is clearly solvable. Conversely, by Corollary 2.8 $\mathfrak{g}$ has a basis in which $\mathrm{ad}(x)$ is upper triangular for any $x \in \mathfrak{g}$. Thus $\mathrm{ad}(y)$ is strictly upper triangular for any $y \in \mathfrak{g}'$. Therefore $\mathfrak{g}'$ is a nilpotent Lie algebra, by Corollary 2.5. QED

More exercises. 3. Prove that any complex (resp. real) irreducible representation of a solvable Lie algebra has dimension 1 (resp. 2).

3 The Killing form and Cartan's criteria

Recall that the adjoint representation associates to an element $a \in \mathfrak{g}$ a linear transformation $\mathrm{ad}(a) : \mathfrak{g} \to \mathfrak{g}$. Let us define a bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to k$ by the formula $K(a, b) = \mathrm{Tr}(\mathrm{ad}(a)\mathrm{ad}(b))$ (the trace of the composition of linear transformations $\mathrm{ad}(a)$ and $\mathrm{ad}(b)$, sending $x \in \mathfrak{g}$ to $[a[bx]]$). It is called the *Killing form* of $\mathfrak{g}$. Since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ the Killing form is symmetric. We shall see that the properties of the Killing form of a Lie algebra $\mathfrak{g}$ say a lot about $\mathfrak{g}$. The following lemma exhibits various functoriality properties of the Killing form.

Lemma 3.1 (i) *If $\phi : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$, then $K(\phi a, \phi b) = K(a, b)$.*

(ii) *The Killing form is invariant in the following sense:*

$$K([ab], c) + K(b, [ac]) = 0.$$

(iii) *If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{a}$ is the Killing form of $\mathfrak{a}$.*

Proof (i) Recall that an automorphism is a linear transformation $\mathfrak{g} \to \mathfrak{g}$ which preserves the Lie bracket. We have $\mathrm{ad}(a)(x) = [ax]$ hence

$$\mathrm{ad}(\phi a)(x) = [\phi a, x] = \phi[a, \phi^{-1}x] = (\phi \circ \mathrm{ad}(a) \circ \phi^{-1})(x).$$

Therefore, $\mathrm{ad}(\phi a) \circ \mathrm{ad}(\phi b) = \phi \circ \mathrm{ad}(a) \circ \phi^{-1} \circ \phi \circ \mathrm{ad}(b) \circ \phi^{-1} = \phi \circ \mathrm{ad}(a) \circ \mathrm{ad}(b) \circ \phi^{-1}$. The traces of equivalent matrices are equal, thus

$$\mathrm{Tr}(\mathrm{ad}(\phi a) \circ \mathrm{ad}(\phi b)) = \mathrm{Tr}(\phi \circ \mathrm{ad}(a) \circ \mathrm{ad}(b) \circ \phi^{-1}) = \mathrm{Tr}(\mathrm{ad}(a) \circ \mathrm{ad}(b)).$$

(ii) We need to prove that the trace of the linear transformation $\mathrm{ad}([ab])\mathrm{ad}(c) + \mathrm{ad}(b)\mathrm{ad}([ac])$ is zero. Since ad is a representation, this transformation can also be written as

$$\begin{aligned}&(\mathrm{ad}(a)\mathrm{ad}(b) - \mathrm{ad}(b)\mathrm{ad}(a))\mathrm{ad}(c) + \mathrm{ad}(b)(\mathrm{ad}(a)\mathrm{ad}(c) - \mathrm{ad}(c)\mathrm{ad}(a)) =\\ &\mathrm{ad}(a)(\mathrm{ad}(b)\mathrm{ad}(c)) - (\mathrm{ad}(b)\mathrm{ad}(c))\mathrm{ad}(a).\end{aligned}$$

Since $\mathrm{Tr}(AB - BA) = 0$, this linear transformation has trace 0.
(iii) We note that $\mathfrak{a}$ is an $\mathrm{ad}(\mathfrak{a})$-invariant subspace of $\mathfrak{g}$. Since $[\mathfrak{a}\mathfrak{g}] \subset \mathfrak{a}$ the quotient adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}/\mathfrak{a}$ is trivial. Therefore, for a, $b \in \mathfrak{a}$ we have

$$K_{\mathfrak{g}}(a, b) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}(a)\mathrm{ad}(b)) = \mathrm{Tr}_{\mathfrak{a}}(\mathrm{ad}(a)\mathrm{ad}(b)) + \mathrm{Tr}_{\mathfrak{g}/\mathfrak{a}}(\mathrm{ad}(a)\mathrm{ad}(b)) = K_{\mathfrak{a}}(a, b). \quad \text{QED}$$

Next we explore what happens if the Killing form is (almost) identically zero.

Proposition 3.2 *Let $k = \mathbb{C}$. If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a Lie subalgebra such that $\mathrm{Tr}(xy) = 0$ for any x, $y \in \mathfrak{g}$, then $\mathfrak{g}'$ is nilpotent.*

Proof We need the full strength of the Jordan normal form theorem: any linear transformation x is uniquely written as $x = s + n$, where n is nilpotent, $sn = ns$, and, in a certain basis, s is diagonal, $s = \mathrm{diag}\,(s_1, \ldots, s_m)$. Moreover, s and n are polynomials in x (with complex coefficients). Let us consider the complex conjugate matrix $\overline{s} = (\overline{s}_1, \ldots, \overline{s}_m)$. If $s^{(1)}, \ldots, s^{(r)}$ are the distinct eigenvalues of s, then

$$\sigma^{-1}(s - s^{(2)}I) \ldots (s - s^{(m)}I) = \mathrm{diag}\,(1, \ldots, 1, 0, \ldots, 0),$$

where $\sigma = (s^{(1)} - s^{(2)}) \dots (s^{(1)} - s^{(m)}) \neq 0$. An appropriate linear combinations of such matrices is $\overline{s}$, in particular, $\overline{s}$ is a polynomial in s. It follows that $\overline{s}$ commutes with n, and so $\overline{s}\,n$ is nilpotent. The trace of a nilpotent transformation is zero, thus

$$\mathrm{Tr}(\overline{s}\,x) = \mathrm{Tr}(\overline{s}\,s) = \sum_i s_i \overline{s}_i. \tag{5}$$

The Lie algebra $\mathfrak{g}$ acts on the algebra of matrices $\mathfrak{gl}(V)$ via the restriction of the adjoint representation of $\mathfrak{gl}(V)$ to $\mathfrak{g}$. Clearly, $\mathrm{ad}(x) = \mathrm{ad}(s) + \mathrm{ad}(n)$. The linear transformation $\mathrm{ad}(n)$ is nilpotent (the powers of n pile up on one or the other side of x, cf. (2)). Since ad is a representation, $[sn] = 0$ implies $\mathrm{ad}(s)\mathrm{ad}(n) - \mathrm{ad}(n)\mathrm{ad}(s) = 0$. The matrix of the linear transformation $\mathrm{ad}(s)$ in the standard basis of $\mathfrak{gl}(V)$ is $\mathrm{diag}\,(s_i - s_j)$. By the uniqueness part of the Jordan normal form theorem we conclude that $\mathrm{ad}(x) = \mathrm{ad}(s) + \mathrm{ad}(n)$ is the Jordan decomposition of $\mathrm{ad}(x)$. Thus $\mathrm{ad}(s)$ is a polynomial in $\mathrm{ad}(x)$. Since the matrix of $\mathrm{ad}(\overline{s})$ in the standard basis of $\mathfrak{gl}(V)$ is $\mathrm{diag}\,(\overline{s}_i - \overline{s}_j)$, we conclude in the same way as above that $\mathrm{ad}(\overline{s}) = P(\mathrm{ad}(s))$ for some $P(t) \in \mathbb{C}[t]$, hence also $\mathrm{ad}(\overline{s}) = Q(\mathrm{ad}(x))$ for some $Q(t) \in \mathbb{C}[t]$. The subspace $\mathfrak{g} \subset \mathfrak{gl}(V)$ is $\mathrm{ad}(x)$-invariant, and so is also $\mathrm{ad}(\overline{s})$-invariant. In particular, $[\overline{s}a] \in \mathfrak{g}$ for any $a \in \mathfrak{g}$.

If $x \in \mathfrak{g}'$, then x is the sum of brackets like $[ab] = ab - ba$, for a, $b \in \mathfrak{g}$. We have

$$\mathrm{Tr}(\overline{s}\,[ab]) = \mathrm{Tr}(\overline{s}ab - \overline{s}ba) = \mathrm{Tr}(\overline{s}ab - a\overline{s}b) = \mathrm{Tr}([\overline{s}a]b).$$

Since $[\overline{s}a] \in \mathfrak{g}$, our assumption implies that this trace is zero. By (5) we see that $\sum_i s_i \overline{s}_i = 0$, hence $s = 0$ and $x = n$ is nilpotent. By Engel's theorem $\mathfrak{g}'$ is nilpotent. QED

Theorem 3.3 (Cartan's first criterion) *A Lie algebra $\mathfrak{g}$ is solvable if and only if $K(\mathfrak{g}, \mathfrak{g}') = 0$.*

Proof Consider $\mathrm{ad}(\mathfrak{g}) = \mathfrak{g}/Z(\mathfrak{g})$, where $Z(\mathfrak{g}) = \{a \in \mathfrak{g} | [ab] = 0 \text{ for any } b \in \mathfrak{g}\}$ is the centre of $\mathfrak{g}$. Obviously $K(\mathfrak{g}, \mathfrak{g}') = 0$ implies $K(\mathfrak{g}', \mathfrak{g}') = 0$, that is, $\mathrm{Tr}(xy) = 0$ for any x, $y \in \mathrm{ad}(\mathfrak{g})' = \mathrm{ad}(\mathfrak{g}')$. By Proposition 3.2 $\mathrm{ad}(\mathfrak{g})''$ is nilpotent, and this implies that $\mathrm{ad}(\mathfrak{g})$ is solvable. Thus $\mathrm{ad}(\mathfrak{g})^{(r)} = 0$ if r is large enough, that is, $\mathfrak{g}^{(r)} \subset Z(\mathfrak{g})$. But then $\mathfrak{g}^{(r+1)} = 0$ so that $\mathfrak{g}$ is solvable.

Let us prove the converse. In an appropriate basis all the elements of $\mathrm{ad}(\mathfrak{g})$ are given by upper triangular matrices, by Lie's theorem. Therefore, all the elements of $\mathrm{ad}(\mathfrak{g}')$, which are sums of expressions like $\mathrm{ad}(a)\mathrm{ad}(b) - \mathrm{ad}(b)\mathrm{ad}(a)$, a, $b \in \mathfrak{g}$, are given by strictly upper triangular matrices. This clearly implies that $K(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)) = 0$ for any $x \in \mathfrak{g}$, $y \in \mathfrak{g}'$. QED

Exercise. 1. Let $\mathfrak{g}$ be the Lie algebra of affine transformations of the line. Compute the Gram matrix of the Killing form of $\mathfrak{g}$. Check that the kernel of the Killing form of $\mathfrak{g}$ is $\mathfrak{g}'$, so that the Killing form is not identically zero.

Cartan's first criterion characterizes solvable Lie algebras as those for which the derived Lie algebra $\mathfrak{g}'$ is contained in the kernel of the Killing form of $\mathfrak{g}$. The opposite case are the algebras for which the Killing form is non-degenerate. To discuss the properties of such algebras we need a lemma and a definition.

Lemma 3.4 *If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals of a Lie algebra $\mathfrak{g}$, then the ideal $\mathfrak{a}+\mathfrak{b}$ is solvable.*

Proof This follows from a more general fact that any extension of a solvable Lie algebra by a solvable Lie algebra is solvable. (If $\mathfrak{a}$ is a solvable ideal of $\mathfrak{g}$, and $\mathfrak{g}/\mathfrak{a}$ is solvable, then so is $\mathfrak{g}$. Indeed, $(\mathfrak{g}/\mathfrak{a})^{(r)} = 0$ if r is large enough, so that $\mathfrak{g}^{(r)} \subset \mathfrak{a}$. If $\mathfrak{a}^{(s)} = 0$, then $\mathfrak{g}^{(r+s)} = 0$.) To conclude we note that the ideal $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$ is solvable, and $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \simeq \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ is a quotient of a solvable Lie algebra $\mathfrak{b}$, and hence is solvable. QED

Definition 3.5 *The union of all solvable ideals of $\mathfrak{g}$ is called the* **radical** *of $\mathfrak{g}$. This is the maximal solvable ideal of $\mathfrak{g}$.*

The existence of the radical follows from Lemma 3.4. The last non-zero term of the derived series of the radical is an abelian ideal of $\mathfrak{g}$. Therefore, the radical of $\mathfrak{g}$ is zero if and only if $\mathfrak{g}$ has no non-zero abelian ideals. The Lie algebras possessing these equivalent properties are called *semisimple*; these algebras and their representation will be the main focus of this course.

Exercise. 2. Prove that the quotient of a Lie algebra $\mathfrak{g}$ by its radical is semisimple.

3. A Lie algebra $\mathfrak{g}$ is called *simple* if it has no ideals different from 0 and $\mathfrak{g}$, and $\dim \mathfrak{g} > 1$. Prove that any simple Lie algebra is semisimple.

4. Prove that $\mathfrak{sl}(n)$ is a simple Lie algebra. (Hint: Let E_{ij}, $i \neq j$, be the matrix with the ij-entry equal to 1, and all the other entries equal to 0. If $\mathfrak{a} \subset \mathfrak{sl}(n)$ is a non-zero ideal, and $a \in \mathfrak{a}$, $a \neq 0$, then $[E_{ij}, a] \in \mathfrak{a}$. Use this to prove that $E_{mn} \in \mathfrak{a}$ for some m and n. Deduce that $\mathfrak{a} = \mathfrak{sl}(n)$.)

Theorem 3.6 (Cartan's second criterion) *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate.*

Proof If $\mathfrak{g}$ is not semisimple, it contains a non-zero abelian ideal $\mathfrak{a} \neq 0$. Choose $a \in \mathfrak{a}$, $a \neq 0$. We claim that $\mathrm{ad}(a)$ is in the kernel of the Killing form. Indeed, let $x \in \mathfrak{g}$ be an arbitrary element. Since $\mathfrak{a}$ is an ideal, $\mathrm{ad}(x)\mathrm{ad}(a)$ sends $\mathfrak{g}$ to $\mathfrak{a}$, and thus $\mathrm{ad}(a)\mathrm{ad}(x)\mathrm{ad}(a)$ sends $\mathfrak{g}$ to 0. We see that $\mathrm{ad}(x)\mathrm{ad}(a)\mathrm{ad}(x)\mathrm{ad}(a) = 0$, or $(\mathrm{ad}(x)\mathrm{ad}(a))^2 = 0$, so that $\mathrm{ad}(x)\mathrm{ad}(a)$ is nilpotent. The trace of a nilpotent linear transformation is 0, by the Jordan normal form theorem. Therefore, $K(x,a) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(a)) = 0$ for any $x \in \mathfrak{g}$.

Conversely, if the Killing form of $\mathfrak{g}$ is degenerate, its kernel is a non-zero ideal of $\mathfrak{g}$. Indeed, if $x \in \mathfrak{g}$ is such that $K(x, y) = 0$ for any $y \in \mathfrak{g}$, then $K([zx], y) = -K(x, [zy]) = 0$ by Lemma 3.1 (ii). Call this ideal $\mathfrak{a}$. By Lemma 3.1 (iii) the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{a}$ is the Killing form of $\mathfrak{a}$, so that the Killing form of $\mathfrak{a}$ is identically zero. By Cartan's first criterion $\mathfrak{a}$ is solvable, and so $\mathfrak{g}$ is not semisimple. QED

Corollary 3.7 *A Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} = \oplus \mathfrak{g}_i$, where the $\mathfrak{g}_i$ are simple Lie algebras (a direct sum of vector spaces with component-wise Lie bracket).*

Proof The orthogonal complement $\mathfrak{a}^\perp$ to an ideal $\mathfrak{a} \subset \mathfrak{g}$ is also an ideal, as follows from Lemma 3.1 (ii). The restriction of the Killing form of $\mathfrak{g}$ to the ideal $\mathfrak{a} \cap \mathfrak{a}^\perp$, which is the Killing form of this ideal by Lemma 3.1 (iii), is zero. Hence $\mathfrak{a} \cap \mathfrak{a}^\perp$ is a solvable ideal by Cartan's first criterion, and thus $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$ because $\mathfrak{g}$ is semisimple. Thus $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ is the direct sum of vector spaces. On the other hand, $[\mathfrak{a}, \mathfrak{a}^\perp] \subset \mathfrak{a} \cap \mathfrak{a}^\perp = 0$ so that $[ab] = 0$ for any $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. This means that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ is the direct sum of Lie algebras. The Killing form is non-degenerate on $\mathfrak{g}$, and $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are orthogonal, thus the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{a}$ is non-degenerate. This restriction is the Killing form of $\mathfrak{a}$. Cartan's second criterion now says that $\mathfrak{a}$ is semisimple, and similarly for $\mathfrak{a}^\perp$. Applying this argument to $\mathfrak{a}$ and $\mathfrak{a}^\perp$ we will have to stop eventually since $\mathfrak{g}$ is finite dimensional. The resulting components $\mathfrak{g}_i$ will be simple.

Conversely, any simple Lie algebra is semisimple, and the direct sum of semisimple algebras is semisimple. (If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semisimple Lie algebras, and $\mathfrak{a} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is an abelian ideal, then the projection of $\mathfrak{a}$ to $\mathfrak{g}_i$ is an abelian ideal of $\mathfrak{g}_i$, hence is zero.). QED

Corollary 3.8 *If a Lie algebra $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}\mathfrak{g}] = \mathfrak{g}$.*

This follows from the previous corollary, since the statement is true for simple Lie algebras.

Exercise 5. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{a} \subset \mathfrak{g}$ a semisimple ideal. Prove that there exists an ideal $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum of Lie algebras.

4 Cartan subalgebras

The *normalizer* of a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is the set of $x \in \mathfrak{g}$ such that $[x\mathfrak{a}] \subset \mathfrak{a}$. Equivalently, the normalizer of $\mathfrak{a} \subset \mathfrak{g}$ is the biggest subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}$ is an ideal of it.

Definition 4.1 *A subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is called a* **Cartan subalgebra** *of* $\mathfrak{g}$ *if* $\mathfrak{h}$ *is nilpotent and equal to its own normalizer.*

Cartan subalgebras will be our main tool in uncovering the structure and the classification of semisimple Lie algebras. Our next goal is to prove that any Lie algebra has a Cartan subalgebra.

Let $n = \dim \mathfrak{g}$. Let $P_x(t) = \det(tI - \mathrm{ad}(x))$ be the characteristic polynomial of $\mathrm{ad}(x)$, $x \in \mathfrak{g}$; we write $P_x(t) = a_0(x) + a_1(x)t + \ldots + a_n(x)t^n$, $a_n(x) = 1$.

Definition 4.2 *The* **rank** *of* $\mathfrak{g}$ *is the minimal value of* m *for which* $a_m(x)$ *is not identically equal to zero. Visibly, the rank is at most* n. *An element* $x \in \mathfrak{g}$ *is* **regular** *if* $a_m(x) \neq 0$.

Exercises 1. A linear transformation is nilpotent if and only if its characteristic polynomial is t^n. In particular, $\mathrm{ad}(x)$ for $x \in \mathfrak{g}$ is nilpotent if and only if $a_n(x)$ is the only non-zero coefficient of $P_x(t)$.

2. If $n \neq 0$, then $a_0(x)$ is identically zero. Hence the rank is at least 1. (Hint: $[x, x] = 0$ so that x is in the kernel of $\mathrm{ad}(x)$, thus $\mathrm{ad}(x)$ has determinant 0.)

Fix $x \in \mathfrak{g}$. Let

$$\mathfrak{g}_x^\lambda = \{y \in \mathfrak{g} | (\mathrm{ad}(x) - \lambda I)^r y = 0 \text{ for some } r\}.$$

For almost all $\lambda \in k$ we have $\mathfrak{g}_x^\lambda = 0$. By the Jordan normal form theorem we have $\mathfrak{g} = \oplus \mathfrak{g}_x^\lambda$. Note that λ is the only eigenvalue of $\mathrm{ad}(x)$ on $\mathfrak{g}_x^\lambda$.

It is clear that $\dim \mathfrak{g}_x^0$ is the smallest m for which $a_m(x) \neq 0$. Hence $x \in \mathfrak{g}$ is regular if and only if the function $\mathfrak{g} \to \mathbf{Z}$ given by $x \mapsto \dim \mathfrak{g}_x^0$ takes its minimal value on x.

Lemma 4.3 $[\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] \subset \mathfrak{g}_x^{\lambda+\mu}$

Proof The Jacobi identity is equivalent to

$$\mathrm{ad}(x)[yz] = [\mathrm{ad}(x)y, z] + [y, \mathrm{ad}(x)z].$$

Hence we have

$$(\mathrm{ad}(x) - (\lambda + \mu))[yz] = [(\mathrm{ad}(x) - \lambda)y, z] + [y, (\mathrm{ad}(x) - \mu)z].$$

By induction on n this implies

$$(\mathrm{ad}(x) - (\lambda + \mu))^n[yz] = \sum_{r=1}^{n} \binom{n}{r} [(\mathrm{ad}(x) - \lambda)^r y, (\mathrm{ad}(x) - \mu)^{n-r} z].$$

If n is large, then one of r and $n - r$ is large. Thus for any $y \in \mathfrak{g}_x^\lambda$ and $z \in \mathfrak{g}_x^\mu$ all the terms in the right hand side vanish. Thus $[yz] \in \mathfrak{g}_x^{\lambda+\mu}$. QED

In particular, $\mathfrak{g}_x^0$ is a subalgebra of $\mathfrak{g}$. Since $[xx] = 0$ we have $x \in \mathfrak{g}_x^0$.

Proposition 4.4 *If $x \in \mathfrak{g}$ is regular, then $\mathfrak{g}_x^0$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof Let $x_1 \in \mathfrak{g}_x^0$. We note that $x_t = tx_1 + (1-t)x \in \mathfrak{g}_x^0$ for any $t \in \mathbb{C}$, hence $\mathrm{ad}(x_t)$ preserves $\mathfrak{g}_x^\lambda$ for every λ, by Lemma 4.3. If t is close to 0, then x_t is close to x, and so $\mathrm{ad}(x_t)$ is close to $\mathrm{ad}(x)$. But then the eigenvalues of $\mathrm{ad}(x_t)$ on $\mathfrak{g}_x^\lambda$, $\lambda \neq 0$, are close to a non-zero number, in particular, they can be assumed to be non-zero. Thus $\mathfrak{g}_{x_t}^0 \subset \mathfrak{g}_x^0$. If $\mathfrak{g}_{x_t}^0$ is strictly smaller than $\mathfrak{g}_x^0$, then $\dim \mathfrak{g}_{x_t}^0 < \dim \mathfrak{g}_x^0$ contradicts the regularity of x. Therefore $\mathfrak{g}_{x_t}^0 = \mathfrak{g}_x^0$, in particular $\mathrm{ad}(x_t)$ is nilpotent on the vector space $\mathfrak{g}_x^0$. This property is a polynomial condition on t. This polynomial vanishes for all t in a small neighbourhood of 0 in $\mathbb{C}$, hence it vanishes identically. Thus $\mathrm{ad}(x_1)$ is a nilpotent linear transformation of $\mathfrak{g}_x^0$ for every $x_1 \in \mathfrak{g}_x^0$. By Engel's theorem $\mathfrak{g}_x^0$ is a nilpotent Lie algebra.

It remains to show that $\mathfrak{g}_x^0$ coincides with its normalizer. Let $y \in \mathfrak{g}$ be such that $[y\mathfrak{g}_x^0] \subset \mathfrak{g}_x^0$. In particular, $[xy] \in \mathfrak{g}_x^0$. But then x is nilpotent on the linear span of y and $\mathfrak{g}_x^0$, so that $y \in \mathfrak{g}_x^0$ by the definition of $\mathfrak{g}_x^0$. QED

Remark. It can be proved that for any two Cartan subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$ of a Lie algebra $\mathfrak{g}$ there exists an automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$. This implies that any Cartan subalgebra of $\mathfrak{g}$ is of the form $\mathfrak{g}_x^0$ for a regular element $x \in \mathfrak{g}$.

Theorem 4.5 *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of a complex Lie algebra $\mathfrak{g}$. There exists a finite subset $\Phi \subset \mathfrak{h}^* \setminus \{0\}$ such that $\mathfrak{g}$ is a direct sum of $\mathfrak{h}$-invariant vector spaces*

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi \cup \{0\}} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \mathfrak{h}, \tag{6}$$

satisfying the property that for each $\alpha \in \Phi$ the space $\mathfrak{g}_\alpha$ contains a common eigenvector of $\mathfrak{h}$ with eigenvalue $\alpha : \mathfrak{h} \to k$, and this is the only eigenvalue of $\mathfrak{h}$ on $\mathfrak{g}_\alpha$. Moreover, if we set $\mathfrak{g}_\alpha = 0$ for $\alpha \notin \Phi \cup \{0\}$, then for any α, $\beta \in \mathfrak{h}$ we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}. \tag{7}$$

Proof For any $x \in \mathfrak{h}$ the linear transformation $\mathrm{ad}(x)$ of $\mathfrak{h}$ is nilpotent, hence $\mathfrak{h} \subset \mathfrak{g}_x^0$. It follows that for any x, $y \in \mathfrak{h}$ each space $\mathfrak{g}_x^\lambda$ is $\mathrm{ad}(y)$-invariant. Choose a basis $x_1, \ldots, x_n$ of $\mathfrak{h}$. A linear function $\alpha : \mathfrak{h} \to k$ is uniquely determined by its values $\alpha_i = \alpha(x_i)$. By induction on i one establishes the decomposition

$$\mathfrak{g} = \bigoplus_{\alpha_1,\ldots,\alpha_i \in \mathbb{C}} \mathfrak{g}_{x_1}^{\alpha_1} \cap \ldots \cap \mathfrak{g}_{x_i}^{\alpha_i}.$$

Define $\mathfrak{g}_\alpha = \mathfrak{g}_{x_1}^{\alpha_1} \cap \ldots \cap \mathfrak{g}_{x_n}^{\alpha_n}$. Then we obtain

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha.$$

Each $\mathfrak{g}_\alpha$ is $\mathfrak{h}$-invariant since $\mathfrak{h} \subset \mathfrak{g}_h^0$ for any $h \in \mathfrak{h}$. Let us denote by Φ the set of non-zero linear functions $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$. The set Φ is finite since $\mathfrak{g}$ is finite-dimensional. Thus we obtain the decomposition in (6). By Lie's theorem $\mathfrak{g}_\alpha$ contains a common eigenvector of $\mathfrak{h}$. For any common eigenvector $v \in \mathfrak{g}_a lpha$ we have $v \in \mathfrak{g}_{x_i}^{\alpha_i}$ so that $\mathrm{ad}(x_i)v = \alpha_i v$, which says that $\mathrm{ad}(h)v = \alpha(h)v$.

It remains to show that $\mathfrak{g}_0 = \mathfrak{h}$. If this is not true, Proposition 2.3 shows that $\mathfrak{h}$ kills a non-zero vector in $\mathfrak{g}_0/\mathfrak{h}$. Then $[\mathfrak{h}, x] \subset \mathfrak{h}$ for some $x \notin \mathfrak{h}$, but this contradicts the condition that $\mathfrak{h}$ is its own normalizer.

The last property follows from Lemma 4.3. QED

Corollary 4.6 *For each $\alpha \in \Phi$ and any $x \in \mathfrak{g}_\alpha$ the linear transformation* $\mathrm{ad}(x)$ *of* $\mathfrak{g}$ *is nilpotent.*

Proof Recall that Φ does not contain 0, so that $\alpha \neq 0$. By (7) $\mathrm{ad}(x)^r$ sends each space $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\beta + r\alpha}$. Since the direct sum (6) is finite, for some large r we shall have $\mathrm{ad}(x)^r = 0$. QED

The elements of Φ are called the *roots* of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and the $\mathfrak{g}_\alpha$ are called the *root spaces.*

The root decomposition (6) behaves nicely regarding the Killing form.

Lemma 4.7 (i) *If α, $\beta \in \Phi \cup \{0\}$ are such that $\alpha + \beta \neq 0$, then* $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

(ii) *For x, $y \in \mathfrak{h}$ we have*

$$K(x, y) = \sum_{\alpha \in \Phi} \alpha(x)\alpha(y) \dim \mathfrak{g}_\alpha.$$

Proof (i) Applying (7) twice we see that $[\mathfrak{g}_\alpha[\mathfrak{g}_\beta \mathfrak{g}_\gamma]] \subset \mathfrak{g}_{\alpha+\beta+\gamma}$ so that if $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, then $\mathrm{ad}(x)\mathrm{ad}(y)$ sends the root space $\mathfrak{g}_\gamma$ into $\mathfrak{g}_{\alpha+\beta+\gamma}$. Since there are only finitely many terms in the direct sum (6), the linear transformation $\mathrm{ad}(x)\mathrm{ad}(y)$ is nilpotent whenever $\alpha + \beta \neq 0$, and hence $K(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)) = 0$.

(ii) By Lie's theorem every subspace $\mathfrak{g}_\alpha$ has a basis in which all the elements of $\mathfrak{h}$ act by upper-triangular matrices. In particular, $\mathrm{ad}(x)$ acts by an upper-triangular matrix with $\alpha(x)$ on the main diagonal, and similarly for y. Thus $\mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ is given by the formula in (ii). QED

5 Semisimple Lie algebras

The root decomposition (6) has particularly nice properties when $\mathfrak{g}$ is a semisimple Lie algebra.

Theorem 5.1 *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the following properties hold:*

(i) Φ *spans* $\mathfrak{h}^*$*;*

(ii) *the restriction of* $K(\cdot,\cdot)$ *to* $\mathfrak{h}$ *is non-degenerate;*

(iii) $\mathfrak{h}$ *is abelian;*

(iv) $\Phi = -\Phi$.

Proof (i) If Φ is contained in a proper subspace of $\mathfrak{h}^*$, then there exists $x \in \mathfrak{h}$, $x \neq 0$, such that $\alpha(x) = 0$ for all $\alpha \in \Phi$ (for any subspace of $\mathfrak{h}^*$ of codimension 1 there exists $x \in \mathfrak{h}$, $x \neq 0$, such that this subspace is the set of linear functions f vanishing at x, $f(x) = 0$). This x is orthogonal to the $\mathfrak{g}_\alpha$ for $\alpha \neq 0$ by Lemma 4.7 (i), and it is orthogonal to $\mathfrak{h}$ by Lemma 4.7 (ii). However, the Killing form on $\mathfrak{g}$ has no kernel by Cartan's second criterion. This contradiction proves that Φ spans $\mathfrak{h}^*$.

(ii) By Lemma 4.7 (i) $\mathfrak{h}$ is orthogonal to $\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$, and since the Killing form is non-degenerate, its restriction to $\mathfrak{h}$ is also non-degenerate.

(iii) Let $x \in [\mathfrak{h}\mathfrak{h}]$. Then the eigenvalue of $\mathrm{ad}(x)$ on $\mathfrak{g}_\alpha$ is 0, that is, $\alpha(x) = 0$ for any $\alpha \in \Phi$. By (ii) we see that $x = 0$.

(iv) The only space to which $\mathfrak{g}_\alpha$ can be non-orthogonal is $\mathfrak{g}_{-\alpha}$, and this must be non-zero since the Killing form is non-degenerate. QED

Comment A part of this result can be restated by saying that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

is an orthogonal direct sum, and the restriction of the Killing form to $\mathfrak{h}$ and to each $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is non-degenerate. Since the form is zero on each $\mathfrak{g}_\alpha$, the Killing form induces a non-degenerate pairing (bilinear form) $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$. Let $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.

Lemma 5.2 *For each* $\alpha \in \Phi$ *we have*

(i) $\mathfrak{h}_\alpha \cap \mathrm{Ker}\,(\alpha) = 0 \ \subset \mathfrak{h}$*;*

(ii) $\dim \mathfrak{h}_\alpha = 1$*;*

(iii) $\dim \mathfrak{g}_\alpha = 1$*, and* $\mathfrak{g}_{n\alpha} = 0$ *for any* $n \geq 2$*;*

(iv) *every* $h \in \mathfrak{h}$ *acts on* $\mathfrak{g}_\alpha$ *as multiplication by* $\alpha(h)$*.*

Proof (i) Choose any $\beta \in \Phi$. For any $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ we consider the action of $\mathrm{ad}([xy])$ on the "α-string of β", that is, the space $\oplus_{n\in\mathbb{Z}}\, \mathfrak{g}_{\beta+n\alpha}$. This space is both $\mathrm{ad}(x)$- and $\mathrm{ad}(y)$-invariant, hence the trace of $\mathrm{ad}([xy]) = \mathrm{ad}(x)\mathrm{ad}(y) - \mathrm{ad}(y)\mathrm{ad}(x)$ is zero. On the other hand, the trace of any $h \in \mathfrak{h}$ on this space is $(\dim \mathfrak{g}_\beta)\beta(h) + r\alpha(h)$ for some $r \in \mathbb{Z}$. Applying this to $h = [xy]$ we see that if $\alpha([xy]) = 0$, then $\beta([xy]) = 0$ as well. Since this is true for any $\beta \in \Phi$, and Φ spans $\mathfrak{h}^*$ by Theorem 5.1 (i), we must have $[xy] = 0$.

(ii) By (i) it is enough to show that $\mathfrak{h}_\alpha \neq 0$. By the non-degeneracy of the Killing form we can find $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) \neq 0$. This implies $[xy] \neq 0$. Indeed, otherwise $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$, which are nilpotent by Corollary 4.6, commute. But since the composition of two commuting nilpotent transformations is nilpotent, $\mathrm{ad}(x)\mathrm{ad}(y)$ is a nilpotent linear transformation of $\mathfrak{g}$, which implies $K(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)) = 0$.

(iii) By Lie's theorem $\mathfrak{h}$ has a eigenvector $x \in \mathfrak{g}_{-\alpha}$ (that is, a common eigenvector of all the elements of $\mathfrak{h}$ with eigenvalue $-\alpha \in \mathfrak{h}^*$). Choose y as in (ii) so that $[xy] \neq 0$; this implies $\alpha([xy]) \neq 0$ by (i). Define $V \subset \mathfrak{g}$ to be the linear span of x, $\mathfrak{h}$ and the $\mathfrak{g}_{n\alpha}$ for $n > 0$. By (7) the vector space V is invariant under $\mathrm{ad}(y)$; V is also invariant under the action of $\mathrm{ad}(x)$: this follows from (7), the facts that $[xx] = 0$ and $[xh] = -[hx] = \alpha(h)x$. Hence V is invariant under $\mathrm{ad}([xy]) = \mathrm{ad}(x)\mathrm{ad}(y) - \mathrm{ad}(y)\mathrm{ad}(x)$, by this formula we also conclude that the trace of $\mathrm{ad}([xy])$ in V is zero. On the other hand, this trace equals $(-1 + \sum_{n>0} n\dim \mathfrak{g}_{n\alpha})\alpha([xy])$, since for any $n \in \mathbb{Z}$, $\mathrm{ad}(h)$ is upper-triangular on $\mathfrak{g}_{n\alpha}$ with $n\alpha(h)$ on the main diagonal. But $\alpha([xy]) \neq 0$, thus (iii) follows.

(iv) By (iii) $\mathfrak{g}_\alpha$ is spanned by an eigenvector x of $\mathfrak{h}$ (cf. the beginning of the proof of (iii)). QED

Comment Statement (iv) implies that all elements of $\mathfrak{h}$ are diagonal in the basis of $\mathfrak{g}$ consisting of a non-zero vector in each $\mathfrak{g}_\alpha$, $\alpha \in \Phi$, and some basis of $\mathfrak{h}$.

To each $\alpha \in \Phi$ we associate its *root vector* $h_\alpha \in \mathfrak{h}$ defined as an element of $\mathfrak{h}$ such that $\alpha(h) = K(h_\alpha, h)$ for any $h \in \mathfrak{h}$. The root vector h_α exists and is unique by the non-degeneracy of the restriction of the Killing form to $\mathfrak{h}$ (the form K defines an isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$, under which $h_\alpha \in \mathfrak{h}$ corresponds to $\alpha \in \mathfrak{h}^*$).

Lemma 5.3 *The 3-dimensional vector subspace $\mathfrak{s}_\alpha = \mathfrak{h}_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a Lie subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}(2)$.*

Proof For any $h \in \mathfrak{h}$, and any $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) = 1$, we have

$$K([xy], h) = -K(y, [xh]) = K(y, [hx]) = \alpha(h)K(x, y) = \alpha(h),$$

hence $h_\alpha = [xy]$. Thus x, y, h_α is a basis of $\mathfrak{s}_\alpha$. By Lemma 5.2 (i) we have $\alpha(h_\alpha) = K(h_\alpha, h_\alpha) \neq 0$. Define

$$H_\alpha = \frac{2}{K(h_\alpha, h_\alpha)} h_\alpha.$$

Then $\alpha(H_\alpha) = 2$. We now choose $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha X_{-\alpha}] = H_\alpha$. We obtain

$$[X_\alpha X_{-\alpha}] = H_\alpha, \quad [H_\alpha X_\alpha] = 2X_\alpha, \quad [H_\alpha X_{-\alpha}] = -2X_{-\alpha},$$

where the last two equalities follow from $\alpha(H_\alpha) = 2$. We identify H_α, X_α, $X_{-\alpha}$ with the natural basis H, X_+, X_- of $\mathfrak{sl}(2)$, thus proving the lemma. QED

Proposition 5.4 (i) *If* α, $\beta \in \Phi$, *then* $\beta(H_\alpha) \in \mathbb{Z}$, *and* $\beta - \beta(H_\alpha)\alpha \in \Phi$.

(ii) *If* $\alpha + \beta \neq 0$, *then* $[\mathfrak{g}_\alpha \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof (i) If $\beta = \pm\alpha$ the statement is already proved, so assume that α and β are not collinear. We have a representation of $\mathfrak{s}_\alpha$ on the α-string of β, that is, on the space $\oplus_{n\in\mathbb{Z}}\ \mathfrak{g}_{\beta+n\alpha}$. Statement (i) is a consequence of the following general fact about the representations of $\mathfrak{sl}(2)$:

Claim *Let* $\mathfrak{sl}(2) \to \mathfrak{gl}(V)$ *be a finite-dimensional representation. Then all the eigenvalues of* H *in* V *are integers, and if* $n \geq 0$ *is an eigenvalue of* v, *then* $n-2$, $n-4, \ldots, -n$ *are the eigenvalues of the following* H*-eigenvectors:* X_-v, $X_-^2v, \ldots, X_-^n v$, *respectively. If* $n \leq 0$, *then* $n+2$, $n+4, \ldots, -n$ *are the eigenvalues of the following* H*-eigenvectors:* X_+v, $X_+^2v, \ldots, X_+^{-n}v$, *respectively.*

Indeed, a non-zero vector $v \in \mathfrak{g}_\beta$ is an eigenvector of H_α with eigenvalue $\beta(H_\alpha)$, so this must be an integer. If $n = \beta(H_\alpha) \geq 0$, then $X_{-\alpha}^n v$ is an eigenvector of H_α, and so is non-zero; therefore, $\mathfrak{g}_{\beta-n\alpha} \neq 0$. If $n = \beta(H_\alpha) \leq 0$, then $X_\alpha^n v$ is an eigenvector of H_α, and so is non-zero; therefore, $\mathfrak{g}_{\beta+(-n)\alpha} \neq 0$. This proves (i).

Proof of Claim Let $v \in V$, $v \neq 0$, be a non-zero vector such that $Hv = \lambda v$. We have $HX_+v = [HX_+]v + X_+Hv = (\lambda+2)X_+v$, and, similarly, $HX_-v = (\lambda-2)X_-v$. This implies that $X_+^i v$, if non-zero, is an eigenvector of H with eigenvalue $\lambda + 2i$, and, similarly, $X_-^j v$, if non-zero, is an eigenvector of H with eigenvalue $\lambda - 2j$. Since V is finite-dimensional, we let e (respectively, f) be the last non-zero term in the sequence v, X_+v, $X_+^2v, \ldots$ (respectively, v, X_-v, $X_-^2v, \ldots$). Thus e is an eigenvector of H with eigenvalue $\lambda + 2m$, for some $m \in \mathbb{Z}$, $m \geq 0$. Similarly, f is an eigenvector of H with eigenvalue $\lambda - 2l$, for some $l \in \mathbb{Z}$, $l \geq 0$. For $r \geq 0$ define $e_r = X_-^r e/r!$, and $e_{-1} = 0$ (respectively, $f_r = X_+^r f/r!$, and $f_{-1} = 0$). The following identities are true for all $r \geq 0$:

$$He_r = (\lambda + 2m - 2r)e_r, \quad X_-e_r = (r+1)e_{r+1}, \quad X_+e_r = (\lambda + 2m - r + 1)e_{r-1}. \quad (8)$$

The first and second properties are clear, so let us prove the last one. We argue by induction on r. For $r = 0$ we have $X_+e = 0 = e_{-1}$, so that the formula is true. Let $r \geq 1$. Then we have

$$\begin{aligned} &rX_+e_r = X_+X_-e_{r-1} = [X_+X_-]e_{r-1} + X_-X_+e_{r-1} = \\ &He_{r-1} + (\lambda + 2m - r + 2)X_-e_{r-2} = \\ &((\lambda + 2m - 2r + 2) + (r-1)(\lambda + 2m - r + 2))e_{r-1} = \\ &r(\lambda + 2m - r + 1)e_{r-1}, \end{aligned}$$

hence the last formula of (8) is true. In a similar way we obtain

$$Hf_r = (\lambda - 2l + 2r)f_r, \quad X_+f_r = (r+1)f_{r+1}, \quad X_-f_r = (-\lambda + 2l - r + 1)f_{r-1}. \quad (9)$$

Since V is finite-dimensional, there exists an $n \geq 0$ such that e_n is the last non-zero term in the sequence $e, e_1, e_2, \ldots$ Then $0 = X_+ e_{n+1} = (\lambda + 2m - n)e_n$ implies that $\lambda = n - 2m \in \mathbb{Z}$. This shows that all eigenvalues of H in V are integers.

Consider the case $\lambda \geq 0$. Since $n = \lambda + 2m, n-2, \ldots, -n = \lambda + 2m - 2n$ are eigenvalues of H (corresponding to eigenvectors $e, e_1, \ldots, e_n$), and $m \geq 0$ the integers $\lambda, \lambda - 2, \ldots, -\lambda$ are eigenvalues of H.

In the case $\lambda \leq 0$, define $n \geq 0$ such that f_n is the last non-zero term in the sequence $f, f_1, f_2, \ldots$ The integers $-n = \lambda - 2l, -n+2, \ldots, n$ are eigenvalues of H (corresponding to eigenvectors $f, f_1, f_2, \ldots, f_n$). Since $l \geq 0$, the integers $\lambda, \lambda - 2, \ldots, -\lambda$ are eigenvalues of H. The claim is proved.

Proof of Proposition 5.4 (ii). The proof of the claim shows that if l is a non-negative integer such that either l or $-l$ is an eigenvalue of H in V, then V contains a linearly independent set of vectors $v_{-l}, v_{-l+2}, \ldots, v_{l-2}, v_l$ such that $Hv_n = nv_n$, $X_+ v_n$ is a non-zero multiple of v_{n+2} for $n \neq l$, $X_- v_n$ is a non-zero multiple of v_{n-2} for $n \neq -l$, and $X_+ v_l = 0$, $X_- v_{-l} = 0$, see (8) and (9).

We can assume that $\alpha \neq \pm\beta$ and $\alpha + \beta \in \Phi$. Let $q \geq 1$ be the maximal number such that $\beta + q\alpha \in \Phi$, and let $p \geq 0$ be the maximal number such that $\beta - p\alpha \in \Phi$. The corresponding eigenvalues of H_α are $\beta(H_\alpha) - 2p$ and $\beta(H_\alpha) + 2q$. Since these are respectively the smallest and the greatest eigenvalues of H_α in $\oplus_n \mathfrak{g}_{\beta+n\alpha}$, by what has been said in the previous paragraph we see that $\beta(H_\alpha) - 2p = -(\beta(H_\alpha) + 2q)$, and the integers of the same parity in the range

$$[\beta(H_\alpha) - 2p, \beta(H_\alpha) + 2q] = [-(p+q), p+q]$$

are eigenvalues of H_α. The corresponding eigenvectors can be chosen as $X_\alpha^i u$, $i = 0, 1, \ldots, p+q$, where u is non-zero vector in $\mathfrak{g}_{\beta - p\alpha}$. Thus $X_\alpha^i u$, where $0 \leq i \leq p+q$, is a non-zero vector in $\mathfrak{g}_{\beta+(i-p)\alpha}$. In particular, $X_\alpha^p u$ generates $\mathfrak{g}_\beta$, and $X_\alpha^{p+1} u$ generates $\mathfrak{g}_{\beta+\alpha}$. This proves (ii). QED

Remark. For future reference we point out that in the notation of this proof $\beta(H_\alpha) = p - q$.

6 Root systems

We now summarize some of the previous constructions. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$, that is, the eigenvalues of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. The non-degeneracy of the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ allows us to define $h_\alpha \in \mathfrak{h}$ such that $\alpha(h) = K(h_\alpha, h)$ for any $h \in \mathfrak{h}$. Lemma 5.2 (i) implies that $K(h_\alpha, h_\alpha) \neq 0$, which permits us to define

$$H_\alpha = \frac{2}{K(h_\alpha, h_\alpha)} h_\alpha.$$

For any α, $\beta \in \Phi$ we define the Cartan numbers as

$$n_{\beta\alpha} = \beta(H_\alpha) = 2\frac{K(h_\alpha, h_\beta)}{K(h_\alpha, h_\alpha)}.$$

By Lemma 5.4 the Cartan numbers are integers.

Define $\mathfrak{h}_\mathbb{R}$ to be the vector space over $\mathbb{R}$ spanned by the H_α, $\alpha \in \Phi$. Since $n_{\beta\alpha} \in \mathbb{Z}$, any root defines a linear function $\mathfrak{h}_\mathbb{R} \to \mathbb{R}$, that is to say $\Phi \subset \mathfrak{h}_\mathbb{R}^*$.

Lemma 6.1 *The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}_\mathbb{R}$ is a positive definite symmetric bilinear form.*

Proof By Lemma 4.7 (ii) for any complex semisimple Lie algebra $\mathfrak{g}$, and any x, $y \in \mathfrak{h}$ we have

$$K(x, y) = \sum_{\alpha \in \Phi} \alpha(x)\alpha(y).$$

Thus for $x \in \mathfrak{h}_\mathbb{R}$ we have $K(x, x) = \sum_{\alpha\in\Phi} \alpha(x)^2$. If this equals 0, then $\alpha(x) = 0$ for all $\alpha \in \Phi$, but since Φ spans $\mathfrak{h}^*$, by Theorem 5.1 (i), we must have $x = 0$. QED

One consequence of this lemma is that the obvious sum $\mathfrak{h} = \mathfrak{h}_\mathbb{R} + i\mathfrak{h}_\mathbb{R}$ of real vector spaces is a direct sum. Indeed, if $v \in \mathfrak{h}_\mathbb{R} \cap i\mathfrak{h}_\mathbb{R}$, then $K(v, v) \geq 0$ and $K(v, v) \leq 0$, so that $v = 0$. This can be rephrased by saying that $\mathfrak{h}_\mathbb{R}$ is a *real form* of $\mathfrak{h}$, or that $\mathfrak{h}$ is the *complexification* of $\mathfrak{h}_\mathbb{R}$.

We have $K(H_\alpha, H_\alpha) = 4/K(h_\alpha, h_\alpha)$, so that h_α differs from H_α by a non-zero positive multiple. In particular, $h_\alpha \in \mathfrak{h}_\mathbb{R}$.

By Lemma 6.1 the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}_\mathbb{R}$ is non-degenerate. Hence it defines an isomorphism $\mathfrak{h}_\mathbb{R} \simeq \mathfrak{h}_\mathbb{R}^*$ sending x to the linear form $K(x, \cdot)$. Under this isomorphism $h_\alpha \in \mathfrak{h}_\mathbb{R}$ corresponds to $\alpha \in \mathfrak{h}_\mathbb{R}^*$. Let us write $(\cdot, \cdot)$ for the positive definite symmetric bilinear from on $\mathfrak{h}_\mathbb{R}^*$ which corresponds to the Killing form under this isomorphism. Then we have

$$(\alpha, \beta) = K(h_\alpha, h_\beta) = \alpha(h_\beta) = \beta(h_\alpha),$$

hence

$$2\frac{(\alpha, \beta)}{(\alpha, \alpha)} = \beta(H_\alpha) = n_{\beta,\alpha} \in \mathbb{Z}.$$

One immediately checks that the linear transformation

$$s_\alpha(x) = x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha = x - x(H_\alpha)\alpha$$

preserves the form $(\cdot, \cdot)$, and is an involution. Moreover, $s_\alpha(\alpha) = -\alpha$, and $s_\alpha(x) = x$ for all vectors $x \in \mathfrak{h}_\mathbb{R}$ orthogonal to α. Such an orthogonal transformation is called the *reflection* in α. Lemma 5.4 says that $s_\alpha(\Phi) = \Phi$ for any $\alpha \in \Phi$.

The pair $(\mathfrak{h}^*_{\mathbb{R}}, \Phi)$ is thus an example of a pair (V, R) consisting of a finite set R of non-zero vectors spanning a real vector space V equipped with a positive definite symmetric bilinear form $(\cdot, \cdot)$, such that

(1) any two distinct proportional vectors of R are negatives of each other;

(2) the reflections with respect to the elements of R preserve R; and

$$2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \text{for any} \quad \alpha,\, \beta \in \Phi.$$

Definition 6.2 *A finite set of vectors R in a real vector space V with a positive definite symmetric bilinear form satisfying* (1) *and* (2) *is called a* **root system**. *The elements of R are called* **roots**, *and the dimension of V is called the* **rank** *of R. The group of linear transformations of V generated by the reflections in the roots of R is called the* **Weyl group** *$W = W(R)$.*

Since R spans V, and W permutes the roots, W is a subgroup of the symmetric group on $|R|$ elements. In particular, W is finite.

Note that multiplying the scalar product by a positive multiple does not change the property of R to be a root system. More generally, we have the following

Definition 6.3 *The roots systems (V_1, R_1) and (V_2, R_2) are equivalent if the exists an isomorphism of vector spaces $\phi : V_1 \to V_2$ such that $\phi(R_1) = R_2$ and for some constant $c \in \mathbb{R}^*$ we have $(\phi(x), \phi(y)) = c(x, y)$ for any $x,\, y \in V_1$.*

The importance of the concept of a root system is due to the fact that *the isomorphism classes of complex semisimple Lie algebras bijectively correspond to equivalence classes of root systems.* This is a central result of this course. We mentioned earlier (without proof) that for any two Cartan subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$ of a Lie algebra $\mathfrak{g}$ there exists an automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$. Also, the Killing form is preserved by automorphisms, by Lemma 3.1 (i). Thus for a given complex semisimple Lie algebra the various choices of a Cartan subalgebra give rise to equivalent root systems.

We now turn to a study of abstract root systems.

Lemma 6.4 *Let (V, R) be a root system. The set of vectors $\alpha^* = \frac{2}{(\alpha,\alpha)}\alpha$ is a root system in V, called the* **dual** *root system and denoted by R^*. The Cartan numbers of R^* are $n^*_{\beta\alpha} = n_{\alpha\beta}$. The Weyl group of R^* is canonically isomorphic to $W(R)$.*

Proof Indeed, R^* is a finite set of non-zero vectors which span V. Property (1) holds because it holds for R. The reflection in α^* is the same orthogonal transformation as the reflection in α, and hence it preserves R^*. It follows that $W(R^*) = W(R)$. Finally,

$$n^*_{\beta\alpha} = 2\frac{(\beta^*,\alpha^*)}{(\alpha^*,\alpha^*)} = 2\frac{(\beta,\alpha)}{(\beta,\beta)} = n_{\alpha\beta}. \qquad \text{QED}$$

In the above context the dual root system Φ^* consists of the vectors H_α, $\alpha \in \Phi$ (we identify $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$ using the Killing form on $\mathfrak{h}_\mathbb{R}$, as usual).

Let us explore which metric properties of roots follow from the definition of a root system. We write $|x| = (x,x)^{1/2}$ for the length of a vector x. Let α, $\beta \in R$ be roots, and ϕ be the angle between α and β. Then $(\alpha, \beta) = |\alpha| \cdot |\beta| \cdot \cos(\phi)$. It follows that

$$n_{\beta\alpha} = 2\frac{|\beta|}{|\alpha|}\cos(\phi), \quad \text{whence} \quad 4\cos^2(\phi) = n_{\beta\alpha} n_{\alpha\beta} \in \mathbb{Z}. \tag{10}$$

Thus $\cos(\phi)$ can be 0, $\pm 1/2$, $\pm\sqrt{2}/2$, $\pm\sqrt{3}/2$ or ± 1. The last case corresponds to collinear roots. If $\phi = \pi/3$ or $-\pi/3$ we must have $n_{\beta\alpha} = \pm 1$, and then α and β have the same length. If $\phi = \pi/4$ or $3\pi/4$, then the square of the length of one of the roots is twice that of the other. Finally, if $\phi = \pi/6$ or $5\pi/6$, then the square of the length of one of the roots is three times that of the other. We note the following curious property.

Lemma 6.5 *If $0 < \phi < \pi/2$, then $\alpha - \beta \in R$.*

Proof If 1, 2 or 3 is written as a product of two positive integers, then one of the factors is 1. Up to swapping α and β we can assume that $n_{\beta\alpha} = 1$. The reflection in α sends β to $\beta - n_{\beta\alpha}\alpha$, thus $\beta - \alpha \in R$. Since $R = -R$, we are done. QED

Exercises: classical root systems In the following cases prove that R is a root system in V, determine the number of roots in R, find its Weyl group and its dual root system.

A_n Consider the vector space $\mathbb{R}^{n+1}$ with basis $e_1, \ldots, e_{n+1}$ and the standard scalar form, and the subspace V consisting of the vectors with the zero sum of coordinates. Let R be the set of vectors of the form $e_i - e_j$, $i \neq j$.

B_n Let $V = \mathbb{R}^n$ with basis $e_1, \ldots, e_n$ and the standard scalar form. Let R be the set of vectors of the form $\pm e_i$ or $\pm e_i \pm e_j$, $i \neq j$.

C_n The same V, and the set of vectors of the form $\pm 2e_i$ or $\pm e_i \pm e_j$, $i \neq j$.

D_n The same V, and the set of vectors $\pm e_i \pm e_j$, $i \neq j$.

Definition 6.6 *A root system $R \subset V$ is* **irreducible** *if V cannot be written as an orthogonal direct sum $V = V_1 \oplus V_2$ such that $R = R_1 \cup R_2$, where $R_i \subset V_i$, $i = 1, 2$, is a root system.*

In the opposite case R is called reducible, and we write $R = R_1 \times R_2$. All irreducible root systems other than A_n, B_n, C_n, D_n are called *exceptional.*

Exercises: exceptional isomorphisms Prove that the following root systems are equivalent: $A_1 \simeq B_1 \simeq C_1$, $C_2 \simeq B_2$, $D_2 \simeq A_1 \times A_1$, $D_3 \simeq A_3$.

Definition 6.7 *A subset S of a root system $R \subset V$ is called a* **basis** *of R if S is a basis of V, and every root of R is an integral linear combination of the elements of S all of whose coefficients have the same sign. The elements of S are called* **simple roots**, *and the elements of R that can be written as linear combinations of simple roots with positive coefficients, are called* **positive roots**. *The set of positive roots is denoted by R^+.*

Proposition 6.8 *Any root system has a basis. Moreover, any basis of $R \subset V$ can be obtained by the following construction. Let $\ell : V \to \mathbb{R}$ be a linear function such that $\ell(\alpha) \neq 0$ for any $\alpha \in R$. Then the roots $\alpha \in R$ such that $\ell(\alpha) > 0$, which cannot be written as $\alpha = \beta + \gamma$, where $\beta, \gamma \in R$, and $\ell(\beta) > 0$, $\ell(\gamma) > 0$, form a basis of R.*

Proof Choose such a linear function ℓ. Let us prove that the corresponding set S is a basis of R. It is clear that any root $\alpha \in R$ such that $\ell(\alpha) > 0$ is a linear combination of the elements of S with positive integral coefficients. Next, we show that the angles between the elements of S are obtuse or right. Otherwise, by Lemma 6.5, $\gamma = \alpha - \beta$ is a root, and so is $-\gamma$. If $\ell(\gamma) > 0$, then α cannot be in S. If $\ell(-\gamma) > 0$, then β cannot be in S. It remains to show that *a set of vectors $\alpha \in V$ such that $\ell(\alpha) > 0$, and all the angles between these vectors are right or obtuse, is linearly independent.* If not, there exist non-empty subsets $S' \subset S$, $S'' \subset S$, $S' \cap S'' = \emptyset$, and numbers $y_\beta > 0$, $z_\gamma > 0$ such that

$$\sum_{\beta \in S'} y_\beta \beta = \sum_{\gamma \in S''} z_\gamma \gamma.$$

Call this vector v. Then

$$0 \leq (v, v) = \sum y_\beta z_\gamma (\beta, \gamma) \leq 0,$$

so that $v = 0$, a contradiction. We proved that S is a basis of R.

Remark Let us denote by S_ℓ the basis of $R \subset V$ defined by a linear function $\ell : V \to \mathbb{R}$. It is clear that the positive roots are precisely the roots α such that $\ell(\alpha) > 0$; we denote this set by R^+_ℓ.

End of proof of Proposition. Conversely, suppose that a linear function ℓ takes positive values on the elements of S. We need to show that $S = S_\ell$. Recall that R^+ is the set of positive roots with respect to the basis S. We have $R^+ \subset R^+_\ell$, but since $R = R^+ \cup -R^+ = R^+_\ell \cup -R^+_\ell$ we have $R^+ = R^+_\ell$. Obviously, $S \subset S_\ell$ but each set is a basis of V, hence $S = S_\ell$. QED

What are the linear functions $\ell : V \to \mathbb{R}$ giving rise to the same basis $S = S_\ell$? The set of such ℓ's is called the *Weyl chamber* defined by S. By the previous proof this set is given by the inequalities $\ell(\alpha) > 0$ for any $\alpha \in S$. We will show that the Weyl chambers are precisely the connected components of the complement in the real vector space V^* to the finite union of hyperplanes $\ell(\beta) = 0$, $\beta \in R$. For this it is enough to show that for any $\ell \in V^*$ there exists $w \in W$ such that $\ell(w\alpha) \geq 0$ for all $\alpha \in S$. In this case ℓ is in the Weyl chamber corresponding to the basis wS. But how to choose such a w?

Lemma 6.9 *Define*

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

Then $s_\alpha(\rho) = \rho - \alpha$ for any $\alpha \in S$.

Proof Any positive root $\beta \neq \alpha$ is sent by s_α to a positive root. (The coefficients of β and $s_\alpha(\beta) = \beta - n_{\beta\alpha}\alpha$ corresponding to the simple roots other than α are the same, in particular, they are positive. Thus $s_\alpha(\beta)$ cannot be negative, and hence is positive.) Since $s_\alpha(\alpha) = -\alpha$ the statement follows. QED

Definition 6.10 *Let S be a basis of a root system R. The matrix of size $r \times r$, where $r = |S|$ is the rank of R, whose entries are the Cartan numbers $n_{\alpha\beta}$, is called the* **Cartan matrix** *of R.*

Since $n_{\alpha\alpha} = \alpha(H_\alpha) = 2$, the diagonal entries of the Cartan matrix are equal to 2. The other entries can be 0, -1, -2 or -3 (since the angles between the simple roots are right or obtuse).

Our next goal is to show that the Cartan matrix of R is well defined, that is, it does not depend on the choice of a basis $S \subset R$, and defines the root system R up to isomorphism.

Exercises *Root systems of rank* 1 *and* 2. 1. What are the root systems of rank 1? Give an example of a semisimple Lie algebra which defines such a root system.

2. Let $R \subset V$, $\dim V = 2$, be a root system of rank 2. Then R has a basis $\{\alpha, \beta\}$. Write all possible Cartan matrices of R, draw their respective root systems in $\mathbb{R}^2$, compute their Weyl groups. (Hint: make a list of possible angles ϕ between α and β, taking into account that ϕ is right or obtuse.)

3. List all pairs of dual root systems of rank 2.

Exercises: exceptional root systems In the examples below show that R is a root system, make a list of elements of R, find its Weyl group and its dual root system.

1. The unique exceptional root system of rank 2 is called G_2. Show that it can be identified with the set of integers of norm 1 or 3 in $\mathbb{Q}(\sqrt{-3})$. (Describe the remaining root systems of rank 2 in a similar way.)

2. Consider the lattice $L \subset \mathbb{R}^4$ generated by the basis vectors e_i and the vector $(e_1 + e_2 + e_3 + e_4)/2$. Let R be the set of vectors $v \in L$ such that $(v, v) = 1$ or $(v, v) = 2$. This root system is called F_4.

3. Consider the lattice $L \subset \mathbb{R}^8$ generated by the basis vectors e_i and the vector $(e_1 + \ldots + e_8)/2$, and let $L_0 \subset L$ be the sublattice consisting of the vectors with even sum of coordinates. Let R be the set of vectors $v \in L'$ such that $(v, v) = 2$. This root system is called E_8.

4. The intersection of the root system of type E_8 with the linear span of $e_1, \ldots, e_6$ (resp. $e_1, \ldots, e_7$) defines a root system in $\mathbb{R}^6$ (resp. $\mathbb{R}^7$). This root system is called E_6 (resp. E_7). (Describe the root system obtained as the intersection of E_8 with the linear span of $e_1, \ldots, e_n$ for $n = 2, 3, 4, 5$.)

Theorem 6.11 *Let S be a basis of the root system R. Then*

(i) *any other basis of R has the form wS for some $w \in W$;*

(ii) *$R = WS$, that is, any root can be obtained from a simple root by applying an element of W;*

(iii) *the Weyl group W is generated by the reflections in the simple roots s_α, $\alpha \in S$;*

(iv) *if the root systems $R_1 \subset V_1$ and $R_2 \subset V_2$ have bases S_1 and S_2, respectively, with equal Cartan matrices, then R_1 and R_2 are equivalent.*

Proof (i) By Proposition 6.8 any basis has the form S_ℓ for some $\ell \in V^*$, $\ell(\beta) \neq 0$ for all $\beta \in R$. Let W_S be the subgroup of W generated by the reflections in the simple roots. Choose $w \in W_S$ such that $\ell(w\rho)$ is maximal, where ρ is defined in Lemma 6.9. Then

$$\ell(w\rho) \geq \ell(ws_\alpha\rho) = \ell(w\rho) - \ell(w\alpha),$$

where the last equality comes from Lemma 6.9. Thus $\ell(w\alpha) > 0$ for all $\alpha \in S$, and (i) follows from the discussion before Lemma 6.9.

(ii) Let $\beta \in R$. Choose a linear form $\ell_0 \in V^*$ such that $\ell_0(\beta) = 0$ but $\ell_0(\gamma) \neq 0$ for all roots $\gamma \in R$, $\gamma \neq \pm\beta$. There exists a small deformation ℓ of ℓ_0 such that $|\ell(\gamma)| > \ell(\beta) > 0$. Then $\beta \in S_\ell$ (see Proposition 6.8), so that β is in the W-orbit of some simple root, by (i).

(iii) It is enough to prove that $s_\beta \in W_S$. By (ii) we have $\beta = w\alpha$ for some $w \in W_S$, but

$$s_\beta(x) = s_{w\alpha}(x) = w \cdot s_\alpha \cdot w^{-1}(x),$$

whence $s_\beta \in W_S$.

(iv) The bijection between S_1 and S_2 uniquely extends to an isomorphism $V_1 \to V_2$. The Cartan matrix allows us to identify all reflections in the simple roots. By (iii), the respective Weyl groups are canonically isomorphic. Now (ii) implies that the isomorphism $V_1 \to V_2$ identifies R_1 with R_2. QED

Remark. It can be proved that the Weyl group $W = W(R)$ acts simply transitively on the set of Weyl chambers (equivalently, on the set of bases) of R.

Definition 6.12 *Let S be a basis of a root system R of rank n with the Cartan matrix $(n_{\alpha\beta})$, α, $\beta \in S$. The Dynkin diagram of R is the graph with n vertices defined by the simple roots $\alpha \in S$, and to each vertex we attach the weight (α, α). The vertices $\alpha \neq \beta$ are joined by $n_{\alpha\beta} n_{\beta\alpha}$ lines.*

We agree not to distinguish between the diagrams that can be obtained from each other by multiplying the weights of the vertices by a common positive multiple.

Since $n_{\alpha\beta} n_{\beta\alpha} = 4\cos^2(\phi)$, the distinct vertices can be connected by 0, 1, 2 or 3 lines. α and β are not connected if and only if α and β are perpendicular. Thus the Dynkin diagram of R is connected if and only if R is irreducible.

Proposition 6.13 *A root system is uniquely determined by its Dynkin diagram.*

Proof We have seen in Theorem 6.11 (iv) that two root systems with identical Cartan matrices are equivalent. It remains to show that the Cartan matrix can be recovered from the Dynkin diagram. Indeed, $n_{\alpha\alpha} = 2$ for any $\alpha \in S$. Let us use (10), taking into account the fact that all the angles between simple roots are right or obtuse. If α and β are not connected, then $n_{\alpha\beta} = n_{\beta\alpha} = 0$. If α and β are connected by one line, then $n_{\alpha\beta} = n_{\beta\alpha} = -1$. If α and β are connected by two lines, then $\cos(\phi) = -\sqrt{2}/2$, and hence $n_{\alpha\beta} = -2$, $n_{\beta\alpha} = -1$, if the weight of α is greater than the weight of β. If α and β are connected by three lines, then $\cos(\phi) = -\sqrt{3}/2$, and hence $n_{\alpha\beta} = -3$, $n_{\beta\alpha} = -1$, if the weight of α is greater than the weight of β. QED

Exercise Check that the following sets are bases of the classical root systems. Compute their Cartan matrices and Dynkin diagrams. (See [1] or [8] for the explicit description of bases in exceptional root systems.)

A_n $e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}$

B_n $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n$

C_n $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n$

D_n $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n$

Theorem 6.14 *Any irreducible root system is one of the classical root systems A_n, $n \geq 1$, B_n, $n \geq 2$, C_n, $n \geq 3$, D_n, $n \geq 4$, or one of the exceptional root systems G_2, F_4, E_6, E_7, E_8.*

Proof We temporarily forget the weights of vertices. By multiplying every simple root by an appropriate non-zero multiple we ensure that all the resulting vectors have length 1. The possible angles between them are $\pi/2$, $2\pi/3$, $3\pi/4$ or $5\pi/6$. Call such a system of vectors an *allowable configuration*. The allowable configurations can be classified using elementary geometry, see [4], pages 130–135. Then it is not hard to prove that the only Dynkin diagrams that an irreducible root system can have are those listed above. QED

7 Classification and examples of semisimple Lie algebras

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, and $\Phi \subset \mathfrak{h}^*_{\mathbb{R}}$ the corresponding root system. The choice of a basis S of Φ allows us to define a system of generators of $\mathfrak{g}$ as follows. To each *simple* root $\alpha \in S$ we associate $H_\alpha \in \mathfrak{h}_{\mathbb{R}}$ as before, and $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$.

Theorem 7.1 (Uniqueness and existence) (i) *The semisimple Lie algebra* $\mathfrak{g}$ *is generated by* $3n$ *elements* H_α, X_α, $X_{-\alpha}$ *for* $\alpha \in S$.
(ii) *These generators satisfy the relations (for all* $\alpha, \beta \in S$*)*

$$\begin{gathered}[H_\alpha, H_\beta] = 0,\ [X_\alpha, X_{-\alpha}] = H_\alpha,\ [H_\alpha, X_\beta] = n_{\beta\alpha} X_\beta,\ [H_\alpha, X_{-\beta}] = -n_{\beta\alpha} X_{-\beta}, \\ [X_\alpha, X_{-\beta}] = 0, \quad \textit{if} \quad \alpha \neq \beta, \quad \textit{and}\end{gathered} \tag{11}$$

$$\mathrm{ad}(X_\alpha)^{-n_{\beta\alpha}+1}(X_\beta) = \mathrm{ad}(X_{-\alpha})^{-n_{\beta\alpha}+1}(X_{-\beta}) = 0 \quad \textit{if} \quad \alpha \neq \beta. \tag{12}$$

(iii) *Any two Lie algebras with such generators and relations are isomorphic.*
(iv) *For any root system* R *there exists a complex semisimple Lie algebra whose root system is equivalent to* R.

Proof (i) It is clear that H_α, $\alpha \in S$, span the Cartan subalgebra $\mathfrak{h}$. Because of the decomposition (6) it is enough to show how to generate $\mathfrak{g}_\gamma$ for every positive (not necessarily simple) root $\gamma \in \Phi$, starting from X_α, $\alpha \in S$. Write $\gamma = \sum_{\alpha \in S} m_\alpha(\gamma)\alpha$, and set $m(\gamma) = \sum_{\alpha \in S} m_\alpha(\gamma)$. Let us show by induction in $m(\gamma)$ that for every $\gamma \in \Phi^+ \setminus S$ there exists a simple root $\alpha \in S$ such that $\gamma - \alpha \in \Phi^+$. Indeed, we cannot have $(\gamma, \alpha) \leq 0$ for all $\alpha \in S$ since otherwise the vectors in $S \cup \{\alpha\}$ are linearly independent (by the argument in the proof of Proposition 6.8). If $(\gamma, \alpha) > 0$, then $\gamma - \alpha \in \Phi$, by Lemma 6.5. Since $m_\beta(\gamma) > 0$ for some simple root $\beta \neq \alpha$, and every root is a linear combination of simple roots with coefficients of the same sign, we see that $\gamma - \alpha \in \Phi^+$. By induction in $m(\gamma)$ we conclude that $\gamma = \alpha_1 + \ldots + \alpha_m$, where $\alpha_i \in S$, and the partial sums $\alpha_1 + \ldots + \alpha_i$ are roots for all $1 \leq i \leq m$. Applying Proposition 5.4 (ii) m times we see that $\mathfrak{g}_\gamma$ is spanned by

$$[X_{\alpha_1}[X_{\alpha_2} \ldots [X_{\alpha_{m-1}} X_{\alpha_m}]]] \neq 0.$$

This finishes the proof of (i).

(ii) All the relations in (11) are clear except $[X_\alpha, X_{-\beta}] = 0$. But if this element is not zero, then $\mathfrak{g}_{\alpha-\beta} \neq 0$. However, $\alpha - \beta \notin \Phi$ since the coefficient of α is positive, whereas that of β is negative.

Similarly, $\mathrm{ad}(X_\alpha)^{-n_{\beta\alpha}+1}(X_\beta)$ has weight $\beta - n_{\beta\alpha}\alpha + \alpha = s_\alpha(\beta - \alpha)$, which is not in Φ because $\beta - \alpha \notin \Phi$, as seen above. Thus this element is zero.

(iii) and (iv) This (complicated) proof is omitted, see [4]. However, the existence theorem for classical Lie algebras will follow from their explicit descriptions (see below). QED

This shows that a semisimple Lie algebra is determined by its root system up to isomorphism, and that every root system is obtained from some semisimple Lie algebra. Irreducible root systems correspond to simple Lie algebras.

Definition 7.2 *For a semisimple Lie algebra* $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ *define*

$$\mathfrak{n}_+ = \oplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha\,, \quad \mathfrak{n}_- = \oplus_{\alpha\in\Phi^+}\mathfrak{g}_{-\alpha}\,, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+.$$

Proposition 7.3 *We have* $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$*, where* $\mathfrak{n}_+$ *and* $\mathfrak{n}_-$ *are nilpotent subalgebras of* $\mathfrak{g}$*, and* $\mathfrak{b} \subset \mathfrak{g}$ *is a solvable subalgebra. Moreover,* $[\mathfrak{b}\mathfrak{b}] = \mathfrak{n}_+$*.*

Proof By Engel's theorem it is enough to prove that $\mathrm{ad}(x)$ is nilpotent for every $x \in \mathfrak{n}_+$. Consider the following filtration in $\mathfrak{n}_+ = \mathfrak{n}_1 \supset \mathfrak{n}_2 \supset \mathfrak{n}_3 \supset \ldots$ given by

$$\mathfrak{n}_m = \bigoplus_{\gamma\in\Phi^+,\, m(\gamma)\geq m} \mathfrak{g}_\gamma\,,$$

with the notation $m(\gamma)$ from the proof of Theorem 7.1 (i). Then $\mathrm{ad}(x)$ sends $\mathfrak{n}_m$ to $\mathfrak{n}_{m+1}$, and so this linear transformation is nilpotent.

From the definition of $\mathfrak{n}_+$ it follows that $[\mathfrak{h}\mathfrak{n}_+] = \mathfrak{n}_+$, and so $[\mathfrak{b}\mathfrak{b}] = [\mathfrak{h}\mathfrak{h}] + [\mathfrak{h}\mathfrak{n}_+] + [\mathfrak{n}_+\mathfrak{n}_+] = \mathfrak{n}_+$. QED

Definition 7.4 $\mathfrak{b}$ *is called the Borel subalgebra of* $\mathfrak{g}$ *defined by the Cartan subalgebra* $\mathfrak{h}$ *and the basis* S *of the root system* Φ*.*

$\mathfrak{sl}(n+1)$, $n \geq 1$, is a semisimple Lie algebra of type A_n The semisimplicity of $\mathfrak{sl}(n+1)$ follows from Exercise 4 in Section 3; alternatively, this follows from Theorem 7.5 below, since checking that $\mathfrak{sl}(n+1)$ has trivial centre is straightforward.

Let E_{ij} be the matrix whose only non-zero entry is 1 in the i-th row and the j-th column. The vector space $\mathfrak{sl}(n+1)$ is spanned by the E_{ij}, $i \neq j$, and the diagonal matrices $E_{ii} - E_{jj}$. Let $\mathfrak{h}$ be the span of the $E_{ii} - E_{jj}$; we shall see in a while that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n+1)$ justifying this choice of notation. Obviously $[\mathfrak{h}\mathfrak{h}] = 0$, so $\mathfrak{h}$ is abelian. It is immediate to check that if $x = diag(x_1, \ldots, x_{n+1})$ is a diagonal matrix, then

$$\mathrm{ad}(x)E_{ij} = (x_i - x_j)E_{ij}, \tag{13}$$

so that the E_{ij} are eigenvectors of $\mathfrak{h}$. This implies that $\mathfrak{h}$ is equal to its own normalizer: any element y of the normalizer of $\mathfrak{h}$, $y \notin \mathfrak{h}$, must contain some E_{ij}, $i \neq j$, in its decomposition with respect to our basis of $\mathfrak{sl}(n+1)$, but then $[y\mathfrak{h}] \not\subset \mathfrak{h}$. Thus $\mathfrak{h} \subset \mathfrak{sl}(n+1)$ is indeed a Cartan subalgebra.

The Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(n+1)$ is none other than the algebra of upper triangular matrices with trace zero, $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the algebra of strictly upper (resp. lower) triangular matrices.

Let $\alpha_i \in \mathfrak{h}^*$, $i = 1, \ldots, n$, be the linear forms defined by $\alpha_i(x) = x_i - x_{i+1}$. The $n \times n$-matrix whose rows are the coefficients of the linear forms $\alpha_1, \ldots, \alpha_n$ in the basis $E_{11} - E_{22}, \ldots, E_{nn} - E_{n+1,n+1}$ of $\mathfrak{h}$, is the Cartan matrix of A_n. By induction one shows that its determinant is $n+1 \neq 0$. Thus $\alpha_1, \ldots, \alpha_n$ are linearly independent, and so form a basis of $\mathfrak{h}^*$. The root defined by the linear form $x \mapsto x_i - x_j$, $i < j$, equals $\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$. This shows that every root is an integral linear combination of the roots α_i with coefficients of the same sign, thus $S = \{\alpha_1, \ldots, \alpha_n\}$ is a basis of the root system Φ of $\mathfrak{sl}(n+1)$. To identify Φ we compute the Cartan matrix using the remark after the proof of Proposition 5.4: we have $n_{\alpha_i \alpha_{i+1}} = n_{\alpha_{i+1} \alpha_i} = -1$ and $n_{\alpha_i \alpha_j} = 0$ if $|i - j| > 1$. Thus the root system of $\mathfrak{sl}(n+1)$ is of type A_n.

To compute the Killing form we use the formula

$$K(x,x) = \sum_{\alpha \in \Phi} \alpha(x)^2 = 2 \sum_{1 \leq i < j \leq n+1} (x_i - x_j)^2 = 2n \sum_{i=1}^{n+1} x_i^2 - 4 \sum_{1 \leq i < j \leq n+1} x_i x_j,$$

where $x = (x_1, \ldots, x_{n+1})$.

To construct more examples of semisimple Lie algebras we use the following criterion.

Theorem 7.5 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra such that V is an irreducible representation of $\mathfrak{g}$. Then $\mathfrak{g}$ is the direct sum of Lie algebras $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$, where $Z(\mathfrak{g})$ is the centre, and the derived algebra $\mathfrak{g}' \subset \mathfrak{g}$ is semisimple. In particular, if $Z(\mathfrak{g}) = 0$, then $\mathfrak{g}$ is semisimple.*

If $\mathfrak{g}$ is the direct sum of a semisimple Lie algebra and an abelian Lie algebra, then $\mathfrak{g}$ is called a *reductive* Lie algebra. Such algebras can be also characterized by the property that $\mathfrak{g}'$ is semisimple, or by an equivalent property that the radical of $\mathfrak{g}$ coincides with the centre of $\mathfrak{g}$, see [1], I.6.4.

We start with a lemma.

Lemma 7.6 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra such that V is an irreducible representation of $\mathfrak{g}$. Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal all of whose elements are nilpotent linear transformations of V. Then $\mathfrak{a} = 0$.*

Proof By Engel's theorem there exists a basis of V such that all the elements of $\mathfrak{a}$ are given by strictly upper triangular matrices. This implies that $\mathfrak{a}V \neq V$. Note that $W = \mathfrak{a}V$ is $\mathfrak{g}$-invariant, that is, $\mathfrak{g}W \subset W$. Indeed, for any $x \in \mathfrak{g}$, $a \in \mathfrak{a}$, $v \in V$

we have $xav = [xa]v + axv \in W$ since $[xa] \in \mathfrak{a}$. By the irreducibility of V we must have $\mathfrak{a}V = 0$ which implies that $\mathfrak{a} = 0$. QED

Proof of Theorem 7.5. Let $\mathfrak{r} \subset \mathfrak{g}$ be the radical. Since $\mathfrak{r}$ is solvable, V has a basis in which the elements of $\mathfrak{r}$ are given by upper triangular matrices. Thus the elements of $\mathfrak{r}' = [\mathfrak{r}\mathfrak{r}]$ are given by strictly upper triangular matrices. The Jacobi identity implies that $\mathfrak{r}' \subset \mathfrak{g}$ is an ideal. By Lemma 7.6 we have $\mathfrak{r}' = 0$.

Lemma 7.7 *We have* $\mathfrak{r} \cap \mathfrak{g}' = 0$.

Proof Consider the ideal $[\mathfrak{r}\mathfrak{g}] \subset \mathfrak{g}$. If $x \in \mathfrak{g}$, $a \in \mathfrak{r}$, and s is any linear transformation of V which commutes with all transformations defined by the elements of $\mathfrak{r}$, then

$$\mathrm{Tr}\,[xa]s = \mathrm{Tr}\,(xas - axs) = \mathrm{Tr}\,(xas - xsa) = \mathrm{Tr}\,x(as - sa) = 0.$$

Since $\mathfrak{r}$ is abelian, we can take s to be any power of $[xa] \in \mathfrak{r}$. Then the trace of any power of $[xa]$ is zero, and this implies that $[xa]$ is a nilpotent linear transformation. (The coefficients of the characteristic polynomial are up to sign the symmetric functions in the eigenvalues, and these can be expressed in terms of sums of powers. Thus if $\mathrm{Tr}\,A^n = 0$ for all $n > 0$, then the characteristic polynomial of A is t^m, $m = \dim V$.) In particular, every element of the ideal $[\mathfrak{r}\mathfrak{g}] \subset \mathfrak{g}$ is nilpotent. By Lemma 7.6 we conclude that $[\mathfrak{r}\mathfrak{g}] = 0$.

Now, for any $x, y \in \mathfrak{g}$ we have

$$\mathrm{Tr}\,[xy]s = \mathrm{Tr}\,(xys - yxs) = \mathrm{Tr}\,(xys - xsy) = \mathrm{Tr}\,x(ys - sy) = 0,$$

if s is a power of an element of $\mathfrak{r}$, because $[\mathfrak{r}\mathfrak{g}] = 0$. Then $\mathrm{Tr}\,bs = 0$ for any $b \in \mathfrak{g}'$. In particular, $\mathrm{Tr}\,a^n = 0$ for any $a \in \mathfrak{r} \cap \mathfrak{g}'$ and any $n > 0$. As above, this implies that every element of the ideal $\mathfrak{r} \cap \mathfrak{g}'$ is nilpotent, hence $\mathfrak{r} \cap \mathfrak{g}' = 0$ by Lemma 7.6. QED

End of proof of Theorem 7.5. From $[\mathfrak{r}\mathfrak{g}] \subset \mathfrak{r} \cap \mathfrak{g}' = 0$ we conclude that $\mathfrak{r} \subset Z(\mathfrak{g})$. Since the centre is a solvable ideal, we have $Z(\mathfrak{g}) \subset \mathfrak{r}$, so that $\mathfrak{r} = Z(\mathfrak{g})$. Thus the subalgebra $\mathfrak{r} + \mathfrak{g}' \subset \mathfrak{g}$ is the direct sum of Lie algebras $\mathfrak{r} \oplus \mathfrak{g}'$.

Let $\mathfrak{a} = (\mathfrak{g}')^\perp$ be the orthogonal complement to $\mathfrak{g}'$ with respect to the Killing form. This is an ideal of $\mathfrak{g}$, and the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{a}$ is the Killing form of $\mathfrak{a}$. We have $K(\mathfrak{a}, \mathfrak{a}') = 0$ since $\mathfrak{a}' \subset \mathfrak{g}'$, thus $\mathfrak{a}$ is solvable by Cartan's first criterion. Hence $\mathfrak{a} \subset \mathfrak{r}$. Thus

$$\dim \mathfrak{r} \geq \dim\,(\mathfrak{g}')^\perp \geq \dim \mathfrak{g} - \dim \mathfrak{g}',$$

so that $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{g}'$ is a direct sum of Lie algebras. Finally, $\mathfrak{g}'$ is isomorphic to $\mathfrak{g}/\mathfrak{r}$ and so is semisimple. QED

We shall apply Theorem 7.5 in conjunction with the following

Proposition 7.8 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be the subalgebra consisting of skew-symmetric linear transformations with respect to a non-degenerate symmetric or skew-symmetric bilinear form, that is, satisfying* (14) *below. If* $\dim V > 2$, *then* V *is an irreducible representation of* $\mathfrak{g}$.

Proof A calculation similar to (1) shows that $\mathfrak{g}$ is indeed a Lie algebra. For any $u, v \in V$ the linear transformation $Ax = (x, u)v - (v, x)u$ satisfies (14), and so is in $\mathfrak{g}$.

Now let $W \subset V$ be a non-zero $\mathfrak{g}$-invariant subspace, $W \neq V$. Let $n = \dim V$. Take $z \in W$, $z \neq 0$, and let $u \neq 0$ be any vector in the orthogonal complement $z^\perp$, that is, such that $(z, u) = 0$. Finally, choose $v \in V$ such that $(z, v) \neq 0$. Then $Az = -(v, z)u \neq 0$ is an element of W. Thus $z^\perp \subset W$, hence $\dim W = n - 1$ and $W = z^\perp$ for any non-zero vector $z \in W$. This means that the restriction of the non-degenerate form $(,)$ to W is identically zero. By linear algebra $2\dim W \leq n$, thus $2(n-1) \leq n$ implying $n \leq 2$. QED

Note that the natural 2-dimensional complex representation of $\mathfrak{o}(2)$ is a direct sum of two 1-dimensional representations.

$\mathfrak{o}(2l+1)$, $l \geq 2$, is a semisimple Lie algebra of type B_l Consider a complex vector space V of dimension $2l+1$ with a non-degenerate symmetric bilinear form, and define $\mathfrak{o}(2l+1)$ as the set of linear transformations A such that

$$(Ax, y) = -(x, Ay). \tag{14}$$

If the form is the 'standard' diagonal form, then these are precisely the skew-symmetric matrices, see Example 5 in Section 1. However, to explicitly exhibit a Cartan subalgebra it is more practical to choose the scalar product with the $(2l+1) \times (2l+1)$ Gram matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix},$$

where I_l is the identity matrix of size $l \times l$. Our ground field is $\mathbb{C}$, so that all non-degenerate symmetric forms are equivalent, that is, can be obtained from each other by an automorphism of the vector space V. Let us write the $(2l+1)\times(2l+1)$-matrix A as follows

$$A = \begin{pmatrix} a & v_1 & v_2 \\ u_1 & X & Y \\ u_2 & Z & W \end{pmatrix},$$

where $a \in \mathbb{C}$, X, Y, Z, W are matrices of size $l \times l$, and v_1, v_2 (resp. u_1, u_2) are row (resp. column) vectors with l coordinates. Then (14) says that

$$A = \begin{pmatrix} 0 & v_1 & v_2 \\ -v_2^T & X & Y \\ -v_1^T & Z & -X^T \end{pmatrix},$$

where $Y^T = -Y$ and $Z^T = -Z$. This suggest the following choice of a basis of V:

$$H_i = E_{i+1,i+1} - E_{i+l+1,i+l+1},$$
$$X_{e_i-e_j} = E_{j+1,i+1} - E_{i+l+1,j+l+1},\ i \neq j,$$
$$X_{e_i+e_j} = E_{i+l+1,j+1} - E_{j+l+1,i+1},\ X_{-e_i-e_j} = E_{j+1,i+l+1} - E_{i+1,j+l+1},\ i < j,$$
$$X_{e_i} = E_{1,i+1} - E_{i+l+1,1},\ X_{-e_i} = E_{i+1,1} - E_{1,i+l+1},$$

where i and j range from 1 to l. Let $\mathfrak{h}$ be the linear span of H_i, $i = 1, \ldots, l$. We have $[\mathfrak{h}\mathfrak{h}] = 0$. The convention is that $e_1, \ldots, e_l$ is the basis of $\mathfrak{h}^*$ dual to the basis $H_1, \ldots, H_l$ of $\mathfrak{h}$. Thus the basis vectors of $\mathfrak{o}(2l+1)$, other than the H_i, are numbered by certain linear forms $\alpha : \mathfrak{h} \to \mathbb{C}$. The subscripts here have been so chosen that

$$[hX_\alpha] = \alpha(h)X_\alpha \quad \text{for any} \quad h \in \mathfrak{h}. \tag{15}$$

We representation of $\mathfrak{o}(2l+1)$ on V is irreducible, by Proposition 7.8. Hence Theorem 7.5 will imply that $\mathfrak{o}(2l+1)$ is semisimple, once we prove that the centre of $\mathfrak{o}(2l+1)$ is trivial.

Lemma 7.9 *The centre of* $\mathfrak{o}(2l+1)$ *is* 0.

Proof Let $z = \xi + \sum t_\alpha X_\alpha$ be an element of the centre, $\xi \in \mathfrak{h}$. We have $[zh] = \sum t_\alpha \alpha(h) X_\alpha = 0$ for any $h \in \mathfrak{h}$, which implies that $t_\alpha \alpha(h) = 0$ since the vectors X_α are linearly independent. This holds identically on $\mathfrak{h}$, thus $t_\alpha = 0$. Now $[\xi, X_\alpha] = \alpha(\xi)X_\alpha = 0$ for all α. Since the linear forms $\alpha : \mathfrak{h} \to \mathbb{C}$ span $\mathfrak{h}^*$ we conclude that $\xi = 0$. QED

An argument similar to that we used for $\mathfrak{sl}(n+1)$ shows that $\mathfrak{h} \subset \mathfrak{o}(2l+1)$ is a Cartan subalgebra. Finally, the roots $\alpha_1 = e_1 - e_2$, ..., $\alpha_{l-1} = e_{l-1} - e_l$, $\alpha_l = e_l$ form a basis of the root system of $\mathfrak{o}(2l+1)$. The corresponding Dynkin diagram is B_l.

$\mathfrak{o}(2l)$, $l \geq 4$, **is a semisimple Lie algebra of type** D_l This case is similar to the previous one, so we only explain the key steps in this analysis. Consider a complex vector space V of dimension $2l$ with a non-degenerate symmetric bilinear form, and define $\mathfrak{o}(2l)$ as the set of linear transformations A satisfying (14). We choose the scalar product with the $2l \times 2l$ Gram matrix

$$\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}.$$

Let us write the $2l \times 2l$-matrix A as

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where X, Y, Z, W are matrices of size $l \times l$. Then (14) says that

$$A = \begin{pmatrix} X & Y \\ Z & -X^T \end{pmatrix},$$

where $Y^T = -Y$ and $Z^T = -Z$. This suggest the following choice of a basis of V:

$$H_i = E_{i,i} - E_{i+l,i+l},$$
$$X_{e_i - e_j} = E_{j,i} - E_{i+l,j+l}, \ i \neq j,$$
$$X_{e_i + e_j} = E_{i+l,j} - E_{j+l,i}, \ X_{-e_i - e_j} = E_{j,i+l} - E_{i,j+l}, \ i < j.$$

Let $\mathfrak{h}$ be the linear span of H_i, $i = 1, \ldots, l$. Then we obtain (15) and the analogue of Lemma 7.9, and conclude that $\mathfrak{o}(2l)$ is semisimple, and $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{o}(2l)$. The roots $\alpha_1 = e_1 - e_2$, ..., $\alpha_{l-1} = e_{l-1} - e_l$, $\alpha_l = e_{l-1} + e_l$ form a basis of the root system of $\mathfrak{o}(2l)$. The corresponding Dynkin diagram is D_l.

$\mathfrak{sp}(2l)$, $l \geq 3$, **is a semisimple Lie algebra of type** C_l Now we equip a complex vector space V of dimension $2l$ with a non-degenerate skew-symmetric bilinear form, and define $\mathfrak{sp}(2l)$ as the set of linear transformations A satisfying (14). We choose the form given by the matrix

$$\begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

Then (14) says that

$$A = \begin{pmatrix} X & Y \\ Z & -X^T \end{pmatrix},$$

where $Y^T = Y$ and $Z^T = Z$, so a natural basis is

$$H_i = E_{i,i} - E_{i+l,i+l},$$
$$X_{e_i - e_j} = E_{j,i} - E_{i+l,j+l}, \ i \neq j,$$
$$X_{e_i + e_j} = E_{i+l,j} - E_{j+l,i}, \ X_{-e_i - e_j} = E_{j,i+l} - E_{i,j+l}, \ i < j,$$
$$X_{2e_i} = E_{i+l,i}, \ X_{-2e_i} = E_{i,i+l}.$$

The same statements as in the previous cases hold, and the roots $\alpha_1 = e_1 - e_2$, ..., $\alpha_{l-1} = e_{l-1} - e_l$, $\alpha_l = 2e_l$ form a basis of the root system of $\mathfrak{sp}(2l)$. The corresponding Dynkin diagram is C_l.

We constructed semisimple Lie algebras of types A_n, $n \geq 1$, B_n, $n \geq 2$, C_n, $n \geq 3$, D_n, $n \geq 4$. This is the full list of classical root systems. Explicit constructions of the exceptional semisimple Lie algebras are much more involved, see [4] and references therein for details.

Exercise Determine the positive roots and hence describe $\mathfrak{b}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ for $\mathfrak{o}(2l+1)$, $\mathfrak{sp}(2l)$, and $\mathfrak{o}(2l)$.

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