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Rational points on pencils of conics and quadrics with many degenerate fibres

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Abstract

For any pencil of conics or higher-dimensional quadrics over $\mathbb{Q}$, with all degenerate fibres defined over $\mathbb{Q}$, we show that the Brauer–Manin obstruction controls weak approximation. The proof is based on the Hasse principle and weak approximation for some special intersections of quadrics over $\mathbb{Q}$, which is a consequence of recent advances in additive combinatorics.

1. Introduction

The arithmetic of conic bundle surfaces X over number fields k has long been the object of intensive study. Such varieties are defined to be projective nonsingular surfaces X over k, which are equipped with a dominant k-morphism $\pi : X \to \mathbb{P}^1_k$, all of whose fibres are conics. Colliot-Thélène and Sansuc [4] conjectured in 1979 that the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation for conic bundle surfaces. They also showed in [5] how one may use Schinzel's hypothesis (also known as the hypothesis of Bouniakowsky, Dickson and Schinzel) to study the Hasse principle, weak approximation and the Brauer–Manin obstruction on conic bundles over $\mathbb{Q}$ given by an equation $x^2 - ay^2 = P(t)$, with $a \in \mathbb{Q}^*$ and $P(t)$ a polynomial of arbitrary degree. Later the same method was taken up again and generalised by Serre [20, Ch. II, Annexe] and by Swinnerton-Dyer [24], general results for pencils of Severi–Brauer varieties or 2-dimensional quadrics being given by Colliot-Thélène and Swinnerton-Dyer in [10].

To discuss unconditional resolutions of the conjecture it is convenient to assume without loss of generality that the conic bundle $\pi : X \to \mathbb{P}^1_k$ is relatively minimal, which means that no irreducible component of a degenerate fibre is defined over the field of definition of that fibre. The work to date has been restricted to the case in which the number r of degenerate geometric fibres of π is small. When $0 \leqslant r \leqslant 5$ everything is known about the qualitative arithmetic unconditionally. Thus for $0 \leqslant r \leqslant 3$, the Hasse principle holds and, furthermore, X is k-rational as soon as $X(k) \neq \emptyset$. When $r = 4$, one finds that X is either a Châtelet surface or else a quartic del Pezzo surface with a conic bundle structure. The former is handled by Colliot-Thélène, Sansuc and Swinnerton-Dyer [8] and the latter by Colliot-Thélène [2] and Salberger [18], using independent approaches. When $r = 5$, X is k-isomorphic to a smooth cubic surface containing a line defined over k. In particular $X(k)$ is nonempty. On contracting the line to obtain a del Pezzo surface of degree 4 with a k-point, work of Salberger and Skorobogatov [19] ensures that the Brauer–Manin obstruction is the only obstruction to weak approximation in this case. Finally, when $r = 6$, some special cases have been dealt with by Swinnerton-Dyer [25] (cf. [23, §7.4]).

In this article we demonstrate how recent advances in additive combinatorics can help deal unconditionally with conic bundle surfaces X defined over $\mathbb{Q}$, in which the degenerate fibres are all defined over $\mathbb{Q}$. The point where two components of a degenerate fibre meet is then a $\mathbb{Q}$-rational point, so that $X(\mathbb{Q}) \neq \emptyset$ for the surfaces under consideration. Thus the main arithmetic questions of interest concern whether or not $X(\mathbb{Q})$ is dense in X under the Zariski topology, or in $X(\mathbf{A})$ under the product topology, where $\mathbf{A}$ denotes the ring of adèles for $\mathbb{Q}$. Our first result resolves these basic questions.

THEOREM 1.1. *Let $X/\mathbb{P}^1_{\mathbb{Q}}$ be a conic bundle surface over $\mathbb{Q}$, in which degenerate fibres exist and are all defined over $\mathbb{Q}$. Then the set $X(\mathbb{Q})$ is Zariski dense in X. Furthermore, the Brauer–Manin obstruction is the only obstruction to weak approximation for X.*

We assume without loss of generality that $X \to \mathbb{P}^1_{\mathbb{Q}}$ is relatively minimal and the fibre at infinity is smooth. Let $P \in \mathbb{Q}[t]$ be the separable monic polynomial of degree r that vanishes at the points of $\mathbb{A}^1_{\mathbb{Q}} = \mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty\}$ that produce degenerate fibres. Our hypotheses are therefore equivalent to a factorisation $P(t) = (t - e_1) \cdots (t - e_r)$, with $e_1, \ldots, e_r \in \mathbb{Q}$ pairwise distinct, and $a_1, \ldots, a_r \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$ such that each irreducible component of the fibre above e_i is defined over $\mathbb{Q}(\sqrt{a_i})$ for $1 \leqslant i \leqslant r$.

The elements of the Brauer group $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$ have the following explicit description. Since $X(\mathbb{Q}) \neq \emptyset$, the natural map $\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(X)$ is injective. Let

$$\delta : (\mathbb{Z}/2\mathbb{Z})^r \to \mathbb{Q}^*/\mathbb{Q}^{*2}$$

be the map that sends $(n_1, \ldots, n_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ to the class of $\prod_{i=1}^r a_i^{n_i}$ in $\mathbb{Q}^*/\mathbb{Q}^{*2}$. By the Faddeev reciprocity law we have $a_1 \ldots a_r \in \mathbb{Q}^{*2}$, so that $(1, \ldots, 1) \in \mathrm{Ker}(\delta)$. For $1 \leqslant i \leqslant r$, the quaternion algebras $(a_i, t - e_i)$ form classes in $\mathrm{Br}(\mathbb{Q}(t))$. An integral linear combination $\sum_{i=1}^r n_i(a_i, t - e_i)$ gives rise to an element of $\mathrm{Br}(X)$ if and only if $(n_1, \ldots, n_r) \in \mathrm{Ker}(\delta)$. This defines a homomorphism $\mathrm{Ker}(\delta) \to \mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$. It is well known that it is surjective with the kernel generated by $(1, \ldots, 1)$ (cf. [23, Prop. 7.1.2]). The second part of Theorem 1.1 states that $X(\mathbb{Q})$ is dense in $X(\mathbf{A})^{\mathrm{Br}}$, under the product of v-adic topologies, where $X(\mathbf{A})^{\mathrm{Br}}$ denotes the left kernel in the Brauer–Manin pairing $X(\mathbf{A}) \times \mathrm{Br}(X) \to \mathbb{Z}/2\mathbb{Z}$. We have $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$ if and only if $\mathrm{Ker}(\delta)$ is generated by $(1, \ldots, 1)$, in which case X satisfies weak approximation.

An important feature of Theorem 1.1 is that it covers, unconditionally, conic bundle surfaces with arbitrarily many degenerate fibres. It can be applied, for example, to the surfaces given by the equation

$$f(t)x^2 + g(t)y^2 + h(t)z^2 = 0,$$

where t is a coordinate function on $\mathbb{A}^1_{\mathbb{Q}}$, $(x : y : z)$ are homogeneous coordinates in $\mathbb{P}^2_{\mathbb{Q}}$, and f, g, h are products of linear polynomials with rational coefficients. In Section 5 we use Theorem 1.1 to construct explicit families of minimal del Pezzo surfaces X of degree 1 and 2 over $\mathbb{Q}$, for which the set $X(\mathbb{Q})$ is nonempty and dense in $X(\mathbf{A})^{\mathrm{Br}}$.

The proof of Theorem 1.1 rests upon a confirmation of the Hasse principle and weak approximation for an auxiliary class of varieties. For the moment, let $a_1, \ldots, a_r \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$ and let $f_1, \ldots, f_r \in \mathbb{Q}[u_1, \ldots, u_s]$ be a system of pairwise nonproportional homogeneous linear polynomials, with $s \geqslant 2$. We consider smooth varieties $\mathscr{V} \subset \mathbb{A}^{2r+s}_{\mathbb{Q}}$, defined by

$$(1.1) \qquad 0 \neq x_i^2 - a_i y_i^2 = f_i(u_1, \ldots, u_s), \quad i = 1, \ldots, r.$$

We will establish the following result in Section 2.

Theorem 1.2. *$\mathscr{V}(\mathbb{Q})$ is Zariski dense in $\mathscr{V}$ as soon as it is nonempty. Furthermore, $\mathscr{V}$ satisfies the Hasse principle and weak approximation.*

The proof of Theorem 1.2 relies on recent work of Matthiesen [16], [17], which itself builds on fundamental work of Green and Tao [11] and Green–Tao–Ziegler [12]. Our approach is based on counting suitably constrained integer points on $\mathscr{V}$ and the shape of our asymptotic formula allows us to deduce the existence of a global solution close to any finite set of local solutions, provided that the system of equations is everywhere locally soluble. This follows the pattern of the Hardy–Littlewood circle method, although it must be stressed that Theorem 1.2 is beyond the reach of the circle method.

The fact that Theorem 1.1 follows from Theorem 1.2 has been known for a long time (see [6], [7], [18]). We recall this argument here, although Theorem 1.1 is also a special case of Theorem 1.3 below. Let $X/\mathbb{P}^1_{\mathbb{Q}}$ be a conic bundle surface as in Theorem 1.1. According to work of Colliot-Thélène and Sansuc [7, Thm. 2.6.4(iii)], any universal torsor $\mathscr{T}$ over X is $\mathbb{Q}$-birationally equivalent to $W_{\boldsymbol{\lambda}} \times C \times \mathbb{A}^1_{\mathbb{Q}}$, where C is a conic over $\mathbb{Q}$ and $W_{\boldsymbol{\lambda}} \subset \mathbb{A}^{2r+2}_{\mathbb{Q}}$ is the variety defined by

$$u - e_i v = \lambda_i(x_i^2 - a_i y_i^2), \quad i = 1, \ldots, r,$$

for suitable $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in (\mathbb{Q}^*)^r$. An application of Theorem 1.2 in the special case $s = 2$ shows that all universal torsors $\mathscr{T}$ over X satisfy the Hasse principle and weak approximation. Since $X(\mathbb{Q}) \neq \emptyset$, it therefore follows from the descent theory of Colliot-Thélène and Sansuc [7, Thm. 3.5.1, Prop. 3.8.7] that $X(\mathbb{Q})$ is dense in $X(\mathbf{A})^{\mathrm{Br}}$ under the product topology, as required for the second part of Theorem 1.1. This implies that *weak-weak approximation* holds; namely, there is a finite set S of places of $\mathbb{Q}$ such that weak approximation holds away from S. In particular, for almost all primes p, the set $X(\mathbb{Q})$ is dense in $X(\mathbb{Q}_p)$ under the p-adic topology. This shows that the first part of Theorem 1.1 follows from its second part.

Recall that if $X \to B$ and $Y \to B$ are two varieties with morphisms to the same base variety B, then the fibred product $X \times_B Y$ can be defined as the restriction of $X \times Y \to B \times B$ to the diagonal $B \subset B \times B$. In Section 3 we shall employ "open descent," as in [9], to prove the following result.

THEOREM 1.3. *Let $X_i/\mathbb{P}^1_{\mathbb{Q}}$, $i = 1, \ldots, n$, be conic bundle surfaces over $\mathbb{Q}$, in which the degenerate fibres are all defined over $\mathbb{Q}$. Let*

$$X = X_1 \times_{\mathbb{P}^1_{\mathbb{Q}}} X_2 \times_{\mathbb{P}^1_{\mathbb{Q}}} \cdots \times_{\mathbb{P}^1_{\mathbb{Q}}} X_n$$

be the fibred product. Assume that whenever two or more of these conic bundles have degenerate fibres over the same point of $\mathbb{P}^1_{\mathbb{Q}}$, the components of their fibres at this point are defined over the same quadratic field. Then the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on any smooth and proper model of X.

Let $a_i \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$ and $c_i \in \mathbb{Q}^*$ for $i = 1, \ldots, n$. Given pairwise distinct rational numbers $e_1, \ldots, e_{2n}$, Theorem 1.3 can be applied to the intersection of quadrics

$$(u - e_{2i-1}v)(u - e_{2i}v) = c_i(x_i^2 - a_i y_i^2), \quad i = 1, \ldots, n,$$

in $\mathbb{P}^{2n+1}_{\mathbb{Q}}$. Indeed no two of the conic bundles in the fibred product have degenerate fibres over the same point of $\mathbb{P}^1_{\mathbb{Q}}$. The interest here is that such varieties do not necessarily have $\mathbb{Q}$-points. In fact, counterexamples to the Hasse principle and weak approximation are known (see [3, §7]). Theorem 1.3 tells us that all such counter-examples are explained by the Brauer–Manin obstruction. This was previously known only when $n = 2$, by using a descent argument to reduce the problem to an intersection of two quadrics in $\mathbb{P}^6_{\mathbb{Q}}$ covered by [8, Thm. 6.7].

It possible to generalise Theorem 1.1 to families of higher-dimensional quadrics. By [8, Prop. 3.9] any variety with a surjective morphism to an open subset of affine space, such that the fibres are smooth projective quadrics of dimension at least 3, satisfies the Hasse principle and weak approximation. Thus we focus on the case of a variety with a surjective map to $\mathbb{P}^1_{\mathbb{Q}}$ such that the fibres are 2-dimensional quadrics. Progress so far has been restricted to the case in which there are at most three geometric fibres that are quadrics of rank 2 or less, as in [9] and [21]. The following result will be proved in Section 3.

THEOREM 1.4. *Let X be a smooth, proper and geometrically integral threefold over $\mathbb{Q}$ equipped with a surjective morphism $X \to \mathbb{P}^1_{\mathbb{Q}}$ such that the generic fibre is a 2-dimensional quadric. If all the fibres that are not geometrically integral are defined over $\mathbb{Q}$, then the set $X(\mathbb{Q})$ is dense in $X(\mathbf{A})^{\mathrm{Br}}$.*

One can also deduce analogous statements for suitable higher-dimensional varieties. Let $m \geqslant 1$ and $n \geqslant 3$. The equation

$$\sum_{i=1}^{n} f_i(\mathbf{t}) X_i^2 = 0 \tag{1.2}$$

defines a variety in $\mathbb{P}^{n-1}_{\mathbb{Q}} \times \mathbb{A}^m_{\mathbb{Q}}$, where $\mathbf{t} = (t_1, \dots, t_m)$ and $f_1, \dots, f_n \in \mathbb{Q}[\mathbf{t}]$. The following result will be established in Section 4 using the fibration method.

THEOREM 1.5. *The Brauer–Manin obstruction is the only obstruction to weak approximation on smooth and proper models of the varieties* (1.2)*, provided that $f_1, \dots, f_n$ are products of nonzero linear polynomials defined over $\mathbb{Q}$.*

As mentioned above, in the case $n \geqslant 5$ the smooth Hasse principle and weak approximation are known to hold for (1.2) without any assumption on $f_1, \dots, f_n$ (see [8, Prop. 3.9]).

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2. Proof of Theorem 1.2

Before we begin with the proof of Theorem 1.2, we observe that a change of variables allows us to assume without loss of generality that $a_1, \dots, a_r$ are integers in the definition (1.1) of $\mathscr{V}$ and that $f_1, \dots, f_r$ are all defined over $\mathbb{Z}$.

In order to establish Theorem 1.2 it suffices to assume that the varieties $\mathscr{V}$ in (1.1) are everywhere locally soluble and to show, under this hypothesis, that $\mathscr{V}(\mathbb{Q})$ is nonempty and that $\mathscr{V}$ satisfies weak approximation. Since individual conics with $\mathbb{Q}$-points satisfy weak approximation, it suffices to place weak approximation conditions on the variables $\mathbf{u} = (u_1, \dots, u_s)$ in $\mathscr{V}$ alone.

Let Ω denote the set of places of $\mathbb{Q}$. For any $v \in \Omega$, let $|\cdot|_v$ denote the v-adic norm. When $v = \infty$, we will simply write $|\cdot|_\infty = |\cdot|$. Let $S \subset \Omega$ be any finite set, let $\varepsilon > 0$ and suppose we are given a point $(\mathbf{x}^{(v)}, \mathbf{y}^{(v)}, \mathbf{u}^{(v)}) \in \mathscr{V}(\mathbb{Q}_v)$ for every $v \in \Omega$. Our task is show that there exists a point $(\mathbf{x}, \mathbf{y}, \mathbf{u}) \in \mathscr{V}(\mathbb{Q})$ such that

$$|\mathbf{u} - \mathbf{u}^{(v)}|_v < \varepsilon \tag{2.1}$$

for each $v \in S$. We will prove this assertion by counting points in $\mathscr{V}(\mathbb{Q})$ satisfying a stronger set of conditions than (2.1). First of all, it is permissible to place additional constraints on the point $(\mathbf{x}, \mathbf{y}, \mathbf{u})$. Since we assumed that there are points $(\mathbf{x}^{(v)}, \mathbf{y}^{(v)}, \mathbf{u}^{(v)}) \in \mathscr{V}(\mathbb{Q}_v)$ for every place $v \in \Omega$, we may suppose without loss of generality that S contains the infinite place and all finite places v bounded by L for some parameter L to be specified in due course.

On rescaling appropriately it suffices to assume that the given solutions $(\mathbf{x}^{(v)}, \mathbf{y}^{(v)}, \mathbf{u}^{(v)})$ belong to $\mathbb{Z}_v^{2r+s}$ for every finite $v \in S$. Applying the Chinese remainder theorem for $\mathbb{Z}^{2r+s}$ we can then produce an integer vector $(\mathbf{x}^{(M)}, \mathbf{y}^{(M)}, \mathbf{u}^{(M)})$ such that

$$|\mathbf{x}^{(M)} - \mathbf{x}^{(v)}|_v < \varepsilon, \quad |\mathbf{y}^{(M)} - \mathbf{y}^{(v)}|_v < \varepsilon, \quad |\mathbf{u}^{(M)} - \mathbf{u}^{(v)}|_v < \varepsilon$$

for all finite $v \in S$. For each finite place in S, we replace (2.1) by the sufficient condition that $\mathbf{u} \in \mathbb{Z}^s$ and

$$u_j \equiv u_j^{(M)} \pmod{M}, \quad j = 1, \dots, s \tag{2.2}$$

for an appropriate modulus $M \in \mathbb{Z}_{>0}$. For technical reasons we require that M has the following property. If $p \mid M$ is a prime divisor and if we write $m = \mathrm{val}_p(M)$, then

$$m \geqslant \max_{1 \leqslant i \leqslant r} \{\mathrm{val}_p(4a_i)\} \tag{2.3}$$

and

$$f_i(\mathbf{u}^{(M)}) \not\equiv 0 \pmod{p^m}, \quad i = 1, \dots, r. \tag{2.4}$$

Since $f_i(\mathbf{u}^{(v)}) \neq 0$ in $\mathbb{Q}_v$, we can arrange for this property to hold by possibly decreasing the value of ε in (2.1).

To take care of the infinite place we will seek points in $\mathscr{V}(\mathbb{Z})$ satisfying

$$|\mathbf{u} - B\mathbf{u}^{(\infty)}| < \varepsilon B, \tag{2.5}$$

with $B = C^2$ for a positive integer $C \equiv 1 \pmod{M}$ tending to infinity. It is clear that any solution $(\mathbf{x}, \mathbf{y}, \mathbf{u}) \in \mathscr{V}(\mathbb{Z})$ satisfying (2.2) and (2.5) will give rise to a solution $(C^{-1}\mathbf{x}, C^{-1}\mathbf{y}, C^{-2}\mathbf{u}) \in \mathscr{V}(\mathbb{Q})$ satisfying our original condition (2.1).

Let us decompose the set of indices $\{1, \ldots, r\}$ as $I_- \cup I_+$, where $i \in I_\pm$ if and only if $\operatorname{sign}(a_i) = \pm$. We claim that after possibly decreasing ε, any $\mathbf{u} \in \mathbb{R}^s$ satisfying (2.5) will produce positive values of $f_i(\mathbf{u})$ for $i \in I_-$. This claim follows since $(\mathbf{x}^{(\infty)}, \mathbf{y}^{(\infty)}, \mathbf{u}^{(\infty)})$ belongs to $\mathscr{V}(\mathbb{R})$ and hence $f_i(\mathbf{u}^{(\infty)}) > 0$ for $i \in I_-$.

Let $q_i(x, y) = x^2 - a_i y^2$ for $i = 1, \ldots, r$. This is a primitive binary quadratic form of discriminant $4a_i$, which is positive definite for $i \in I_-$ and indefinite for $i \in I_+$. For $d \leqslant -4$, let

$$w(d) = \begin{cases} 4, & \text{if } d = -4, \\ 2, & \text{if } d < -4, \end{cases}$$

and for $d > 0$ let $\eta(d)$ denote the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Let us call a solution $(\mathbf{x}, \mathbf{y}, \mathbf{u}) \in \mathbb{Z}^{2r+s}$ of (1.1) *primary* if the pair (x_i, y_i) lies in a fixed fundamental domain for the action of the group of automorphisms $\mathscr{E}_i$ of q_i for $i = 1, \ldots, r$. Our strategy will be to estimate asymptotically, as $B \to \infty$, the total number $N(B)$ of primary solutions $(\mathbf{x}, \mathbf{y}, \mathbf{u}) \in \mathbb{Z}^{2r+s}$ of (1.1), which satisfy (2.2) and (2.5), and to show that this quantity is positive for large enough B.

We will henceforth view ε, M, together with the coefficients of $\mathscr{V}$ and $\mathbf{u}^{(M)}, \mathbf{u}^{(\infty)}$ as being fixed once and for all. Any implied constants in this section will therefore be allowed to depend on these quantities. Given $n \in \mathbb{Z}$, we define the representation functions

$$R_i(n) = \#\{(x, y) \in \mathbb{Z}^2/\mathscr{E}_i : q_i(x, y) = n\}$$

for $i = 1, \ldots, r$, with $R_i(n) = 0$ if $n \leqslant 0$ and $i \in I_-$. We may clearly write

$$N(B) = \sum_{\substack{\mathbf{u} \in \mathbb{Z}^s \\ (2.2) \text{ and } (2.5) \text{ hold}}} \prod_{i=1}^{r} R_i(f_i(\mathbf{u})).$$

The success of our investigation rests upon work of the second author [16], [17], who has shown how recent innovations of Green and Tao [11] and Green–Tao–Ziegler [12] in additive combinatorics can be brought to bear on sums like $N(B)$ for arbitrary r.

We eliminate the constraint (2.2) in $N(B)$ by writing $u_j = u_j^{(M)} + Mt_j$ for $j = 1, \ldots, s$. This leads to the expression

$$N(B) = \sum_{\mathbf{t} \in \mathbb{Z}^s \cap K} \prod_{i=1}^{r} R_i(g_i(\mathbf{t})),$$

where $K = \left\{ \mathbf{t} \in \mathbb{R}^s : \ |M\mathbf{t} + \mathbf{u}^{(M)} - B\mathbf{u}^{(\infty)}| < \varepsilon B \right\}$ and

$$g_i(\mathbf{t}) = f_i(\mathbf{u}^{(M)} + M\mathbf{t}), \quad i = 1, \ldots, r.$$

The region K is convex and contained in $[-cB, cB]^s$ for an appropriate absolute positive constant c. It has measure $(2\varepsilon M^{-1}B)^s \gg B^s$. Our choice of ε ensures that $g_i(K)$ is positive for every $i \in I_-$. Moreover, $(g_1, \dots, g_r) : \mathbb{Z}^s \to \mathbb{Z}^r$ defines a system of linear polynomials of "finite complexity," in the language of Green and Tao [11]. Indeed, the linear parts of any two g_i, g_j, with $i \neq j$, are nonproportional. Given $A \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$, let

$$\varrho_i(q; A) = \#\{(x, y) \in (\mathbb{Z}/q\mathbb{Z})^2 : x^2 - a_i y^2 \equiv A \pmod{q}\}.$$

It then follows from [17, Thm. 1.1] that

$$N(B) = \beta_\infty \prod_p \beta_p + o(B^s) \tag{2.6}$$

as $B \to \infty$. Here the main term is a product of local densities, given by

$$\beta_\infty = \mathrm{meas}(K) \prod_{i \in I_-} \frac{\pi}{w(4a_i)\sqrt{|a_i|}} \prod_{j \in I_+} \frac{\log \eta(a_j)}{\sqrt{a_j}}$$

and

$$\beta_p = \lim_{k \to \infty} p^{-(s+r)k} \sum_{\mathbf{t} \in (\mathbb{Z}/p^k\mathbb{Z})^s} \prod_{i=1}^r \varrho_i(p^k; g_i(\mathbf{t}))$$

for each prime p. Since $\beta_\infty \gg \mathrm{meas}(K) \gg B^s$, we see that in order to complete the proof of Theorem 1.2 it remains to show that $\prod_p \beta_p \gg 1$ in (2.6).

For each prime p, let

$$\beta_p' = \lim_{k \to \infty} p^{-(s+r)k} \sum_{\mathbf{u} \in (\mathbb{Z}/p^k\mathbb{Z})^s} \prod_{i=1}^r \varrho_i(p^k; f_i(\mathbf{u}))$$

be the local factor associated to the original system of equations. By [16, Lemma 8.3] these factors satisfy $\beta_p' = 1 + O(p^{-2})$. Since the change of variables from $f_i(\mathbf{t})$ to $g_i(\mathbf{t}) = f_i(\mathbf{u}^{(M)} + M\mathbf{t})$ is nonsingular modulo p whenever $p \nmid M$, we conclude that $\beta_p = \beta_p'$ for $p \nmid M$. Recall that primes $p \nmid M$ satisfy $p > L$. We may now specify the parameter $L = O(1)$ to be such that $\beta_p' > 0$ for all $p > L$. Hence, for this choice of L, we have

$$\prod_{p \nmid M} \beta_p = \prod_{p \nmid M} \beta_p' \gg 1.$$

Our final task is to show that $\beta_p > 0$ for primes $p \mid M$. It will be convenient to write

$$\begin{aligned} G(p^k) &= \sum_{\mathbf{t} \in (\mathbb{Z}/p^k\mathbb{Z})^s} \prod_{i=1}^r \varrho_i(p^k; g_i(\mathbf{t})) \\ &= \#\left\{ (\mathbf{x}, \mathbf{y}, \mathbf{t}) \in (\mathbb{Z}/p^k\mathbb{Z})^{2r+s} : \begin{array}{l} x_i^2 - a_i y_i^2 \equiv g_i(\mathbf{t}) \pmod{p^k} \\ \text{for } i = 1, \dots, r \end{array} \right\}, \end{aligned}$$

so that $\beta_p = \lim_{k \to \infty} p^{-(s+r)k} G(p^k)$. Suppose $\mathrm{val}_p(M) = m > 0$. To start with, observe that the integer vector $(\mathbf{x}^{(M)}, \mathbf{y}^{(M)}, \mathbf{u}^{(M)})$ satisfies (1.1) modulo M.

This implies $G(p^m) \geqslant p^{sm}$ since $g_i(\mathbf{t}) = f_i(\mathbf{u}^{(M)} + M\mathbf{t})$. To analyse $G(p^k)$ for $k > m$, we shall employ [16, Cor. 6.4]. This yields

$$\varrho_i(p^k, A) = \frac{1}{p}\varrho_i(p^{k+1}, A + \ell p^k)$$

for any $\ell \in \mathbb{Z}/p\mathbb{Z}$, providing that $k \geqslant \mathrm{val}_p(4a_i)$ and $A \not\equiv 0 \,(\mathrm{mod}\, p^k)$. We have arranged things so that M satisfies (2.3) and (2.4). Thus the conditions hold for $k > m$ when $A = g_i(\mathbf{t})$ and $\mathbf{t} \in \mathbb{Z}^s$, and we deduce that $G(p^{k+1}) = p^{s+r}G(p^k)$. Hence

$$\beta_p = p^{-(s+r)m}G(p^m) \geqslant p^{-rm} > 0$$

for $p \mid M$. This establishes the desired lower bound for the product of local densities and so concludes the proof of Theorem 1.2.

3. **Proof of Theorems 1.3 and 1.4**

Let Y be a variety over a number field k, and let $f : Z \to Y$ be a torsor of a k-torus T. We write $\mathbf{A}_k$ for the ring of adèles of k. Specialising the torsor at an adelic point defines the evaluation map $Y(\mathbf{A}_k) \to \prod_v H^1(k_v, T)$, where the product is taken over all completions k_v of k. Let $Y(\mathbf{A}_k)^f$ be the set of adelic points for which the image of the evaluation map is contained in the image of the natural map $H^1(k, T) \to \prod_v H^1(k_v, T)$. It is clear that the diagonal image of $Y(k)$ in $Y(\mathbf{A}_k)$ is in $Y(\mathbf{A}_k)^f$.

There is an equivalent way to define $Y(\mathbf{A}_k)^f$. Up to isomorphism, the k-torsors R of T are classified by their classes $[R] \in H^1(k, T)$. The twist of $f : Z \to Y$ by R is defined as the quotient of $Z \times R$ by the diagonal action of T, with the morphism to Y induced by the first projection. We denote the twisted torsor by $f^R : Z^R \to Y$. Then $Y(\mathbf{A}_k)^f$ is the union of the images of projections $f^R : Z^R(\mathbf{A}_k) \to Y(\mathbf{A}_k)$ for all $[R] \in H^1(k, T)$ (see [23, §5.3]).

The following slight variation on [9, Prop. 1.1] was found after our discussions with J.-L. Colliot-Thélène.

Proposition 3.1. *Let X be a smooth geometrically integral variety over a number field k. Let $Y \subset X$ be a dense open set, and let $f : Z \to Y$ be a torsor of a k-torus T. Then $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $Y(\mathbf{A}_k)^f \neq \emptyset$. If X is proper, then $X(\mathbf{A}_k)^{\mathrm{Br}}$ is contained in the closure of $Y(\mathbf{A}_k)^f$ in $X(\mathbf{A}_k) = \prod_v X(k_v)$. In this case, if all the twists of Z by k-torsors of T satisfy the Hasse principle and weak approximation, then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}}$.*

Proof. Let $\hat{T}$ be the group of homomorphisms $T \times_k \overline{k} \to \mathbb{G}_{m,\overline{k}}$ of algebraic groups. Equipped with the discrete topology, $\hat{T}$ is a continuous $\mathrm{Gal}(\overline{k}/k)$-module. The natural pairing of discrete $\mathrm{Gal}(\overline{k}/k)$-modules $T(\overline{k}) \times \hat{T} \to \overline{k}^*$ gives rise to the $\cup$-product pairing

$$\cup : H^1_{\text{ét}}(Y, T) \times H^1(k, \hat{T}) \to H^1_{\text{ét}}(Y, T) \times H^1_{\text{ét}}(Y, \hat{T}) \to H^2_{\text{ét}}(Y, \mathbb{G}_m) = \mathrm{Br}(Y);$$

cf. [23, pp. 63-64]. Let $[Z] \in H^1_{\text{ét}}(Y,T)$ be the class of the torsor Z/Y, and let $B \subset \mathrm{Br}(Y)$ be the subgroup $[Z] \cup H^1(k,\hat{T})$. Since $H^1(k,\hat{T})$ is finite, B is also finite. Let $Y(\mathbf{A}_k)^B$ be the set of adelic points of Y that are orthogonal to B with respect to the Brauer–Manin pairing. By [9, Prop. 1.1] (based on Harari's "formal lemma" [13, Cor. 2.6.1]) we have $X(\mathbf{A}_k)^{B\cap \mathrm{Br}(X)} \neq \emptyset$ if and only if $Y(\mathbf{A}_k)^B \neq \emptyset$, and the latter set is dense in the former when X is proper. Since $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{B\cap \mathrm{Br}(X)}$, it remains to prove that $Y(\mathbf{A}_k)^B = Y(\mathbf{A}_k)^f$. This is a well-known consequence of the Poitou–Tate duality for tori; see, e.g., the proof of statement (2) in [23, pp. 115, 119–121]. □

Proof of Theorem 1.3. Without loss of generality we assume that $X_j \to \mathbb{P}^1_{\mathbb{Q}}$ is relatively minimal and the fibre at infinity of $X_j \to \mathbb{P}^1_{\mathbb{Q}}$ is smooth for each $j = 1, \dots, n$. Then there are $e_1, \dots, e_r$ in $\mathbb{Q} = \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$ such that the restriction of $X_j \to \mathbb{P}^1_{\mathbb{Q}}$ to $\mathbb{P}^1_{\mathbb{Q}} \setminus \{e_1, \dots, e_r\}$ is a smooth morphism, for $j = 1, \dots, n$. By assumption, for $i = 1, \dots, r$ there exists $a_i \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$, defined up to a square, such that the fibre of each $X_j \to \mathbb{P}^1_{\mathbb{Q}}$ at e_i is either a smooth conic or a union of two conjugate lines defined over $\mathbb{Q}(\sqrt{a_i})$.

Let $U = \mathbb{A}^1_{\mathbb{Q}} \setminus \{e_1, \dots, e_r\}$. For $j = 1, \dots, n$, define $Y_j \subset X_j$ as the inverse image of $U \subset \mathbb{P}^1_{\mathbb{Q}}$, and let Y be the fibred product of $Y_1, \dots, Y_n$ over U. To apply Proposition 3.1 we now introduce a certain torsor over Y. Let $\mathscr{W}_{\boldsymbol{\lambda}} \subset \mathbb{A}^{2r+2}_{\mathbb{Q}}$, for $\boldsymbol{\lambda} \in (\mathbb{Q}^*)^r$, be the variety given by

$$(3.1) \qquad u - e_i v = \lambda_i(x_i^2 - a_i y_i^2), \quad i = 1, \dots, r, \qquad v \prod_{i=1}^r (u - e_i v) \neq 0.$$

The morphism $\mathscr{W}_{\boldsymbol{\lambda}} \to U$ that sends the point (u, v, x_i, y_i) to the point with the coordinate $t = u/v$ is a torsor of the following $\mathbb{Q}$-torus T:

$$v = x_1^2 - a_1 y_1^2 = \dots = x_r^2 - a_r y_r^2 \neq 0.$$

The fibred product $Y \times_U \mathscr{W}_{\boldsymbol{\lambda}}$ is a Y-torsor of T for any $\boldsymbol{\lambda}$.

The $\mathbb{Q}$-torsors of T are the affine varieties $R_{\mathbf{c}}$ given by

$$v = c_1(x_1^2 - a_1 y_1^2) = \dots = c_r(x_r^2 - a_r y_r^2) \neq 0,$$

where $\mathbf{c} = (c_1, \dots, c_r) \in (\mathbb{Q}^*)^r$. The isomorphism classes of $\mathbb{Q}$-torsors of T bijectively correspond to $\mathbf{c} \in (\mathbb{Q}^*)^r$ up to a common nonzero rational multiple and multiplication of each c_i by the norm of a nonzero element of $\mathbb{Q}(\sqrt{a_i})$. The twist $\mathscr{W}_{\boldsymbol{\lambda}}^{R_{\mathbf{c}}}$ is the torsor $\mathscr{W}_{\mathbf{c}\boldsymbol{\lambda}}$, where $\mathbf{c}\boldsymbol{\lambda} = (c_1\lambda_1, \dots, c_r\lambda_r)$. Thus the set of torsors $Y \times_U \mathscr{W}_{\boldsymbol{\lambda}} \to Y$ for all $\boldsymbol{\lambda} \in (\mathbb{Q}^*)^r$ is closed under all twists by $\mathbb{Q}$-torsors of T.

For $j = 1, \dots, n$, we denote by I_j the subset of $\{1, \dots, r\}$ such that the fibre of $X_j \to \mathbb{P}^1_{\mathbb{Q}}$ at e_i is singular if and only if $i \in I_j$. Let $r_j = |I_j|$. We define $\mathscr{W}_{\boldsymbol{\lambda}}^{(j)} \subset \mathbb{A}^{2r_j+2}_{\mathbb{Q}}$ to be the variety given by

$$u - e_i v = \lambda_i(x_i^2 - a_i y_i^2), \quad i \in I_j, \quad v \prod_{i=1}^r (u - e_i v) \neq 0$$

for $\boldsymbol{\lambda} \in (\mathbb{Q}^*)^{r_j}$. As proved in [7] (Thm. 2.6.4(ii)(a) and Remarque 2.6.8), there exists a conic C_j over $\mathbb{Q}$ such that $Y_j \times_U \mathscr{W}_{\boldsymbol{\lambda}}^{(j)}$ is birationally equivalent to $C_j \times \mathscr{W}_{\boldsymbol{\lambda}}^{(j)}$. There is a natural morphism $\mathscr{W}_{\boldsymbol{\lambda}} \to \mathscr{W}_{\boldsymbol{\lambda}}^{(j)}$ that forgets the coordinates x_i, y_i for $i \notin I_j$. This morphism is obviously compatible with the projection to U, hence $Y_j \times_U \mathscr{W}_{\boldsymbol{\lambda}}$ is birationally equivalent to $C_j \times \mathscr{W}_{\boldsymbol{\lambda}}$. Therefore $Y \times_U \mathscr{W}_{\boldsymbol{\lambda}}$ is birationally equivalent to $C_1 \times \cdots \times C_n \times \mathscr{W}_{\boldsymbol{\lambda}}$. By Theorem 1.2 and the Hasse–Minkowski theorem this variety satisfies the Hasse principle and weak approximation. It now follows from Proposition 3.1 that $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$. □

Let us now turn to the arithmetic of pencils of 2-dimensional quadrics. We start with recalling the relevant definitions from [21]. Let k be an arbitrary field of characteristic different from 2.

Definition 3.2. (1) A geometrically integral variety X over k with a morphism $p : X \to \mathbb{P}^1_k$ is a *quadric bundle* if every closed point $P \in \mathbb{P}^1_k$ has a Zariski open neighbourhood $U_P \subset \mathbb{P}^1_k$ such that $p^{-1}(U_P)$ is the closed subset of $U_P \times \mathbb{P}^3_k$ given by the vanishing of a quadratic form $Q_P(x_1, x_2, x_3, x_4) = 0$ with coefficients in the k-algebra of regular functions on U_P such that $\det(Q_P)$ is not identically zero.

(2) A quadric bundle $X/\mathbb{P}^1_k$ is *admissible* if for every closed point $P \in \mathbb{P}^1_k$ for which the fibre X_P is singular, U_P and Q_P in (1) can be chosen so that $Q_P(x_1, x_2, x_3, x_4) = \sum_{i=1}^4 f_i x_i^2$, where each f_i is invertible outside P with at most a simple zero at P, and $f_1(P) f_2(P) \neq 0$.

(3) Let us call an admissible quadric bundle *relatively minimal* if, in the notation of (2), for each closed point $P \in \mathbb{P}^1_k$ such that $f_3(P) = f_4(P) = 0$, the (well-defined) values of the functions $-f_1/f_2$ and $-f_3/f_4$ at P are both nonsquares in the residue field $k(P)$.

If $X/\mathbb{P}^1_k$ is a relatively minimal admissible quadric bundle, then the closed fibre X_P is not geometrically integral if and only if X_P is the zero set of a quadratic form of rank 2. In our notation, X_P is $f_1(P)x_1^2 + f_2(P)x_2^2 = 0$. Thus X_P is the union of two conjugate projective planes defined over the quadratic extension $k(P)(\sqrt{a_P})$ of the residue field $k(P)$, where $a_P = -f_1(P)/f_2(P)$. In particular, the (nontrivial) class of a_P in $k(P)^*/k(P)^{*2}$ is uniquely determined by $X/\mathbb{P}^1_k$.

The singular locus $(X_P)_{\mathrm{sing}}$ of X_P is the projective line given by $x_1 = x_2 = 0$. An easy calculation (see [21, Cor. 2.1]) shows that the singular locus X_{sing} is contained in the union of singular loci of the closed fibres of $X/\mathbb{P}^1_k$ that are not geometrically integral. Let $b_P \in k(P)^*$ be the value of $-f_3/f_4$ at P. By [21, Prop. 2.2], $X_{\mathrm{sing}} \cap X_P$ is the subscheme of $(X_P)_{\mathrm{sing}}$ given by $x_4^2 = b_P x_3^2$. In particular, the (nontrivial) class of b_P in $k(P)^*/k(P)^{*2}$ is uniquely determined by $X/\mathbb{P}^1_k$.

Recall that a scheme over k is called *split* if it contains a nonempty geometrically integral open subscheme [22, Def. 0.1, p. 906]. Let us denote by $\tilde{X}$ the blow-up of X_{sing} in X. In [21, Prop. 2.4] it is shown that $\tilde{X}$ is a smooth projective threefold. Since $X/\mathbb{P}^1_k$ is relatively minimal, each fibre of $\tilde{X}/\mathbb{P}^1_k$ that is not geometrically integral consists of two irreducible components, none of them geometrically integral (since a_P and b_P are both nonsquares in $k(P)^*$); cf. [21, Rem. 2.2]. Hence a fibre of $\tilde{X}/\mathbb{P}^1_k$ is split if and only if it is geometrically integral.

Proof of Theorem 1.4. By [21, Prop. 2.1 and its proof, Prop. 2.3] there exists a relatively minimal admissible quadric bundle $X'/\mathbb{P}^1_{\mathbb{Q}}$ such that the generic fibres of $X/\mathbb{P}^1_{\mathbb{Q}}$ and $X'/\mathbb{P}^1_{\mathbb{Q}}$ are isomorphic. (In particular, X and X' are birationally equivalent.) If a fibre X_P is geometrically integral, hence split, then $\tilde{X}'_P$ is split too [22, Cor. 1.2]. By the previous paragraph $\tilde{X}'_P$ is then a geometrically integral quadric, hence so is X'_P. It follows that X'_P is geometrically integral whenever X_P is geometrically integral.

If all the fibres of $X'/\mathbb{P}^1_{\mathbb{Q}}$ are geometrically integral, the variety X satisfies the Hasse principle and weak approximation (see [8, Thm. 3.10] or [22, Thm. 2.1]). Thus we may assume that at least one $\mathbb{Q}$-fibre X'_P of $X'/\mathbb{P}^1_{\mathbb{Q}}$ is given by a quadratic form of rank 2. (Then almost all $\mathbb{Q}$-points on the common line of the two planes of X'_P are smooth in X', hence $X(\mathbb{Q}) \neq \emptyset$.)

Let us choose $\mathbb{A}^1_{\mathbb{Q}} \subset \mathbb{P}^1_{\mathbb{Q}}$ so that the fibre of $X'/\mathbb{P}^1_{\mathbb{Q}}$ at infinity is smooth, and let t be a coordinate function on $\mathbb{A}^1_{\mathbb{Q}}$. By assumption we know that there are $e_1, \dots, e_r \in \mathbb{Q}$ such that the fibres $X'_{e_1}, \dots, X'_{e_r}$ can be given by quadratic forms of rank 2, and all the other fibres of $X'/\mathbb{P}^1_{\mathbb{Q}}$ are geometrically integral. Let $a_1, \dots, a_r \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$, defined up to squares, be such that $\mathbb{Q}(\sqrt{a_i})$ is the quadratic field over which the components of X'_{e_i} are defined.

Let $U = \mathbb{A}^1_{\mathbb{Q}} \setminus \{e_1, \dots, e_r\}$, and let U_i be a Zariski open neighbourhood of e_i as in Definition 3.2(2). The restriction of $X' \to \mathbb{P}^1_{\mathbb{Q}}$ to U_i can be given by the equation

$$x_1^2 - \alpha_i x_2^2 + \gamma_i (t - e_i)(x_3^2 - \beta_i x_4^2) = 0, \tag{3.2}$$

where α_i, β_i, γ_i are invertible regular functions on U_i. We then have $a_i = \alpha_i(e_i)$.

We denote by S a finite set of places of $\mathbb{Q}$ containing 2 and the real place. Define $\mathbb{Z}_S$ as the subring of $\mathbb{Q}$ consisting of the fractions with denominators divisible only by primes in S. We choose S large enough so that for all $i = 1, \dots, r$, we have

$$e_i \in \mathbb{Z}_S, \quad a_i \in \mathbb{Z}_S^*, \quad e_i - e_j \in \mathbb{Z}_S^* \quad \text{for } i \neq j.$$

Moreover, by further increasing S we can assume that X' has an integral model $\mathscr{X}' \to \mathbb{P}^1_{\mathbb{Z}_S}$ such that for any $p \notin S$, its reduction modulo p is an admissible quadric bundle $\mathscr{X}'_{\mathbb{F}_p} \to \mathbb{P}^1_{\mathbb{F}_p}$ with exactly r fibres that are quadrics of rank 2 at the reductions of $e_1, \ldots, e_r$ modulo p. For $i = 1, \ldots, r$, we define $\mathscr{U}_i \subset \mathbb{P}^1_{\mathbb{Z}_S}$ as the complement to the Zariski closure of $\mathbb{P}^1_{\mathbb{Q}} \setminus U_i$ in $\mathbb{P}^1_{\mathbb{Z}_S}$. It is clear that $U_i = \mathscr{U}_i \times_{\mathbb{Z}_S} \mathbb{Q}$. By enlarging S we ensure that α_i, β_i, γ_i are invertible regular functions on $\mathscr{U}_i$, and (3.2) is an equation for $\mathscr{X}'$ over $\mathscr{U}_i$.

Let $a_0 = a_1 \ldots a_r$. For $\boldsymbol{\lambda} \in (\mathbb{Q}^*)^r$, we define the variety $\mathscr{W}_{\boldsymbol{\lambda}}$ as follows:

$$(3.3) \qquad u - e_i v = \lambda_i(x_i^2 - a_i y_i^2) \neq 0, \quad i = 1, \ldots, r, \quad v = x_0^2 - a_0 y_0^2 \neq 0.$$

The morphism $\mathscr{W}_{\boldsymbol{\lambda}} \to U$ that sends the point (u, v, x_i, y_i) to the point with the coordinate $t = u/v$ is a torsor of the following $\mathbb{Q}$-torus T:

$$v = x_0^2 - a_0 y_0^2 = x_1^2 - a_1 y_1^2 = \cdots = x_r^2 - a_r y_r^2 \neq 0.$$

Let $Y \subset X'$ be the inverse image of U. The fibred product $Y \times_U \mathscr{W}_{\boldsymbol{\lambda}}$ is a Y-torsor of T for any $\boldsymbol{\lambda}$. As in the proof of Theorem 1.3 we see that the family of torsors $Y \times_U \mathscr{W}_{\boldsymbol{\lambda}} \to Y$ is closed under all twists by $\mathbb{Q}$-torsors of T. By Proposition 3.1 it is enough to prove that the varieties $Y \times_U \mathscr{W}_{\boldsymbol{\lambda}}$ satisfy the Hasse principle and weak approximation.

Write $\mathscr{W} = \mathscr{W}_{\boldsymbol{\lambda}}$. Let us enlarge the set S by including into it the primes where we need to approximate, together with any primes needed to ensure that $\lambda_i \in \mathbb{Z}_S^*$ for $i = 1, \ldots, r$. We are given a family of $\mathbb{Q}_p$-points N_p, for all primes p, and a real point N_∞, in $Y \times_U \mathscr{W}$. Let M_p, M_∞ be the images of these points in $\mathscr{W}$. By Theorem 1.2 the variety $\mathscr{W}$ satisfies the Hasse principle and weak approximation. Indeed, if $a_0 \notin \mathbb{Q}^{*2}$, then Theorem 1.2 can be directly applied to $\mathscr{W}$. For $a_0 \in \mathbb{Q}^{*2}$, a change of variables in the last equation of (3.3) gives $v = x_0' y_0'$, so that $\mathscr{W}$ is birationally equivalent to the product of $\mathbb{A}^1_{\mathbb{Q}}$ and the variety (3.1), to which Theorem 1.2 can be applied.

Thus in all cases we can find a point $M \in \mathscr{W}(\mathbb{Q})$ arbitrarily close to the points M_∞ and M_p for $p \in S$, in their respective local topologies. Let $P \in U(\mathbb{Q})$ be the image of M. We can choose M so that P is contained in a given nonempty open subset of $\mathbb{P}^1_{\mathbb{Q}}$, for example in the open set $U_0 \subset U \cap U_1 \cap \cdots \cap U_r$ defined by the property that $Y_P = X'_P$ is a smooth quadric for any P in U_0. Then Y_P can be given by equation (3.2) for any $i = 1, \ldots, r$. By the implicit function theorem, Y_P has $\mathbb{Q}_p$-points close to N_p for $p \in S$ and a real point close to N_∞. We claim that

$$(3.4) \qquad Y(\mathbb{Q}_p) \neq \emptyset \quad \text{for all } p \notin S.$$

Once achieved this will show that Y_P is everywhere locally soluble over $\mathbb{Q}$ and hence has a $\mathbb{Q}$-point and satisfies weak approximation (by the theorem of Hasse and the rationality of a smooth quadric with a $\mathbb{Q}$-point). This, in turn, implies that $Y \times_U \mathscr{W}$ also has a $\mathbb{Q}$-point and satisfies weak approximation, as required to complete the proof of Theorem 1.4.

Let $\mathscr{W}_0$ be the inverse image of U_0 in $\mathscr{W}$. To establish (3.4) it is enough to show that the natural projection $(Y \times_U \mathscr{W}_0)(\mathbb{Q}_p) \to \mathscr{W}_0(\mathbb{Q}_p)$ is surjective for all $p \notin S$. We can assume that a point in $\mathscr{W}_0(\mathbb{Q}_p)$ has coordinates $(x_0, y_0, \dots, x_r, y_r) \in \mathbb{Z}_p^{2r+2}$, not all divisible by p. It maps to the point $P = (u : v) \in U_0(\mathbb{Q}_p)$, where u, $v \in \mathbb{Z}_p$, and $t = u/v \in \mathbb{Q}_p$ is such that $t \neq e_i$, for any $i = 1, \dots, r$. Let us denote by $x \mapsto \overline{x}$ the map $\mathbb{Q}_p \to \mathbb{F}_p \cup \{\infty\}$ such that $\overline{x} \equiv x \pmod p$ if $x \in \mathbb{Z}_p$ and $\overline{x} = \infty$ if $x \in \mathbb{Q}_p \setminus \mathbb{Z}_p$. We have three possible cases:

(a) $\bar{t}$ is not equal to any of the points $\bar{e}_i$ for $i = 1, \dots, r$;
(b) $\bar{t} = \bar{e}_i$ for some $i \in \{1, \dots, r\}$ and $\mathrm{val}_p(v)$ is even;
(c) $\bar{t} = \bar{e}_i$ for some $i \in \{1, \dots, r\}$ and $\mathrm{val}_p(v)$ is odd.

In case (a) the quadric Y_P reduces to a geometrically integral quadric over $\mathbb{F}_p$. Such a quadric has smooth $\mathbb{F}_p$-points, and any smooth $\mathbb{F}_p$-point lifts to a $\mathbb{Q}_p$-point on Y_P by Hensel's lemma. Thus (3.4) holds in this case.

Now suppose that we are in case (b) or case (c). Then the reduction of Y_P modulo p is the same as that of Y_{e_i}. If a_i is a square modulo p, the reduction of Y_P modulo p is a union of two projective planes defined over $\mathbb{F}_p$. Any $\mathbb{F}_p$-point not on the common line of the two planes is smooth and hence lifts to a $\mathbb{Q}_p$-point in Y_P by Hensel's lemma. Now assume that a_i is not a square modulo p. Since $P = (t : 1) \in U_i(\mathbb{Q})$, we can evaluate (3.2) at P and obtain an equation for $Y_P = X'_P$. From (3.3) we see that $\mathrm{val}_p(u - e_i v)$ must be even.

In case (b) we deduce that $\mathrm{val}_p(t - e_i)$ is also even. But then Y_P can be given by a quadratic form over $\mathbb{Z}_p$ that reduces to a rank 4 quadratic form over $\mathbb{F}_p$. This implies that Y_P has a $\mathbb{Q}_p$-point, as required for (3.4).

Finally, the case (c) is not compatible with the condition that a_i is not a square modulo p. Indeed, if $\mathrm{val}_p(v)$ is odd, then $\mathrm{val}_p(t - e_i) > 0$ is also odd. Take any $j \in \{1, \dots, r\}$ with $j \neq i$. Since $e_i - e_j \in \mathbb{Z}_S^*$, we see that $t - e_j \in \mathbb{Z}_S^*$, so that $u - e_j v$ has odd valuation. Now (3.3) implies that a_j is a square modulo p. Since $v = x_0^2 - a_0 y_0^2$ has odd valuation, a_0 must also be a square modulo p. This is a contradiction to the fact that $a_0 \dots a_r$ is a square. This finishes the proof of (3.4) and so completes the proof of the theorem. □

4. **Proof of Theorem 1.5**

The purpose of this section is to establish Theorem 1.5. By [8, Prop. 3.9] it suffices to assume that $n = 3$ or $n = 4$. On multiplying (1.2) and each of the variables X_i by an appropriate nonzero rational function in $\mathbf{t} = (t_1, \dots, t_m)$, it suffices to replace (1.2) by a $\mathbb{Q}$-birationally equivalent variety that is given by an equation of the same form satisfying the following additional conditions. There exist pairwise nonproportional, nonconstant polynomials $l_1, \dots, l_r \in \mathbb{Q}[\mathbf{t}]$ of degree 1, which are not necessarily homogeneous, such that for $j = 1, \ldots, n$ we can write $f_j = c_j \prod_{i \in I_j} l_i$ where $c_j \in \mathbb{Q}^*$ and $I_j \subseteq \{1, \ldots, r\}$. Moreover, for $n = 3$ (resp. $n = 4$), each l_i divides exactly one of f_1, f_2, f_3 (resp. one or two of f_1, f_2, f_3, f_4). Finally, we may assume that

$$l_i(\mathbf{t}) = t_1 + d_{i,2}t_2 + \cdots + d_{i,m}t_m + d_{i,0}, \quad i = 1, \ldots, r,$$

for appropriate constants $d_{i,0}, d_{i,2}, \ldots, d_{i,m} \in \mathbb{Q}$. Indeed, for $i = 1, \ldots, r$, we can write $l_i(\mathbf{t}) = L_i(\mathbf{t}) + l_i(0)$, where $L_i(\mathbf{t})$ is homogeneous of degree 1. There is a nonzero vector $\mathbf{a} \in \mathbb{Q}^m$ such that $L_i(\mathbf{a}) \neq 0$ for $i = 1, \ldots, r$. Assuming without loss of generality that $a_1 \neq 0$, one achieves the claim by making the change of variables $t_1 = a_1 t_1'$ and $t_i = t_i' + a_i t_1'$ for $2 \leqslant i \leqslant m$ and then replacing c_j by $c_j \prod_{i \in I_j} L_i(\mathbf{a})$. The case when (1.2) is a quadric over $\mathbb{Q}$ being a subject of the Hasse–Minkowski theorem, we can assume without loss of generality that f_1 is not constant and is divisible by $l_1(\mathbf{t})$.

Let us denote the variety in (1.2) by V. When $m = 1$, the statement of the theorem follows from Theorems 1.1 and 1.4. We assume for the remainder of the proof that $m \geqslant 2$. The map $p : V \to \mathbb{A}^{m-1}_{\mathbb{Q}}$ sending $(X_1, \ldots, X_n, \mathbf{t})$ to $(t_2, \ldots, t_m)$ is a surjective morphism. The fibre $V_{\mathbf{b}} = p^{-1}(\mathbf{b})$ above a point $\mathbf{b} = (b_2, \ldots, b_m)$ of $\mathbb{A}^{m-1}_{\mathbb{Q}}$ is given by the following equation with coefficients in the residue field $\mathbb{Q}(\mathbf{b})$:

$$\sum_{j=1}^{n} \tilde{f}_j(t) X_j^2 = 0,$$

where $\tilde{f}_j(t) = f_j(t, \mathbf{b})$. We note that the morphism p has a section s that sends $(t_2, \ldots, t_m)$ to the point of V with coordinates $X_1 = 1$, $X_2 = \cdots = X_n = 0$, $t_1 = -l_1(0, t_2, \ldots, t_m)$.

Theorem 1.5 will follow from a variant of the fibration method with a section, which is a result of Harari [13, Thm. 4.3.1], once we check that

(1) the generic fibre V_η of p is geometrically integral and geometrically rational, and the section s defines a smooth point of V_η;
(2) there is a nonempty open subset $U \subset \mathbb{A}^{m-1}_{\mathbb{Q}}$ such that for any point $\mathbf{b} \in U(\mathbb{Q})$, the Brauer–Manin obstruction is the only obstruction to weak approximation on smooth and proper models of $V_{\mathbf{b}}$.

Let $U \subset \mathbb{A}^{m-1}_{\mathbb{Q}}$ be the open subset given by $l_{i_1}(0, \mathbf{b}) \neq l_{i_2}(0, \mathbf{b})$ for all $i_1 \neq i_2$. This set is not empty since no two polynomials l_{i_1} and l_{i_2} are equal for $i_1 \neq i_2$. The restriction of p to U has geometrically integral fibres, as follows from our assumption that if $n = 3$ (resp. $n = 4$), then each l_i divides exactly one of f_1, f_2, f_3 (resp. one or two of f_1, f_2, f_3, f_4). In the case $n = 3$ the fibre $V_{\mathbf{b}}$ is a smooth conic bundle over $\mathbb{A}^1_{\mathbb{Q}(\mathbf{b})}$ for any $\mathbf{b}$ in U. In particular, V_η is smooth, so the point of V_η defined by s is certainly smooth. In the case $n = 4$ the fibre $V_{\mathbf{b}}$ is an admissible quadric bundle over $\mathbb{A}^1_{\mathbb{Q}(\mathbf{b})}$. (See

Definition 3.2(2) above, with $\mathbb{P}^1_k$ replaced by $\mathbb{A}^1_k$.) An easy calculation (cf. the remarks after Definition 3.2) shows that at every point of the singular locus $(V_\eta)_{\rm sing}$ exactly two of the coordinates X_1, X_2, X_3, X_4 must vanish. Hence the point of V_η defined by s is also smooth in this case. Thus condition (1) is satisfied. Condition (2) follows from Theorems 1.1 and 1.4, so the proof of Theorem 1.5 is now complete.

5. Rational points on some del Pezzo surfaces of degrees 1 and 2

Theorem 1.1 can be used to study weak approximation for suitable del Pezzo surfaces of low degree, where our knowledge is still largely conditional. In this section we construct families of del Pezzo surfaces of degree 1 and 2 for which the failure of weak approximation is controlled by the Brauer–Manin obstruction. Recall that a smooth and projective surface V is called *minimal* if any birational morphism $V \to V'$, where V' is also smooth and projective, is an isomorphism. The surfaces that we construct will be minimal, so our results do not follow from earlier results for del Pezzo surfaces of higher degree.

We start with describing del Pezzo surfaces to which Theorem 1.1 can be applied, in terms of orbits of the Galois group action on the set of exceptional curves.

Let Γ_d be the graph whose vertices are the exceptional curves on a del Pezzo surface of degree d defined over an algebraically closed field; two vertices are connected by n edges if the intersection index of the corresponding curves is n. A del Pezzo surface X of degree d defined over $\mathbb{Q}$ induces an action of the Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Γ_d realised as the graph of exceptional curves on $\overline{X} = X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$.

Let $\Gamma(1)$ be the graph with two vertices joined by a single edge. For a positive integer r, we denote by $\Gamma(r)$ the disconnected union of r copies of $\Gamma(1)$. Recall that a subgraph Γ' of a graph Γ is *induced* if the vertices of Γ' are connected by exactly the same edges as in Γ.

PROPOSITION 5.1. *Consider the family of del Pezzo surfaces of degree $d \leqslant 7$ over $\mathbb{Q}$ for which Γ_d has an induced subgraph $\Gamma(8-d)$ such that all the connected components of $\Gamma(8-d)$ are G-invariant. All surfaces in this family have the property that the Brauer–Manin obstruction is the only obstruction to weak approximation. Moreover, if $d \in \{1, 2, 4\}$, then the surfaces for which no vertex of $\Gamma(8-d)$ is fixed by G are minimal over $\mathbb{Q}$.*

Proof. Pick a connected component of $\Gamma(8-d)$, and let $C \in \mathrm{Pic}(\overline{X})$ be the class of the sum of corresponding exceptional curves. We have $(C, C) = 0$, and this implies that C is the class of a geometrically reducible fibre of a conic bundle morphism $\pi : X \to \mathbb{P}^1_{\mathbb{Q}}$. The curves orthogonal to C under the intersection pairing are components of the fibres of π. Thus the unions of exceptional curves corresponding to the connected components of $\Gamma(8-d)$ give rise to $8-d$ degenerate fibres of π, which are all defined over $\mathbb{Q}$. A del Pezzo surface of degree d that is a conic bundle has exactly $8-d$ degenerate fibres. Thus all the degenerate fibres of $\pi : X \to \mathbb{P}^1_{\mathbb{Q}}$ are defined over $\mathbb{Q}$ and Theorem 1.1 applies to X.

A conic bundle surface is called *relatively* minimal if all the fibres of the conic fibration are integral. For the surfaces considered in Theorem 1.1, this means that no component of a degenerate fibre is defined over $\mathbb{Q}$, or equivalently, no vertex of $\Gamma(8-d)$ is fixed by G. By a theorem of Iskovskikh [15, Thm. 4], if a del Pezzo surface of degree 1, 2 or 4 is a relatively minimal conic bundle, then it is a minimal surface. □

Let $f, g, h \in \mathbb{Q}[t]$ be polynomials such that $f(t)g(t)h(t) = c\prod_{i=1}^{r}(t-e_i)$ for $c \in \mathbb{Q}^*$ and pairwise different $e_1, \dots, e_r \in \mathbb{Q}$. Assume that $\ell = \deg f$, $m = \deg g$, $n = \deg h$ are integers of the same parity such that $\ell \leqslant m \leqslant n$. Consider the smooth surface in $\mathbb{P}^2_{\mathbb{Q}} \times \mathbb{A}^1_{\mathbb{Q}}$ defined by

$$f(t)x^2 + g(t)y^2 + h(t)z^2 = 0, \tag{5.1}$$

where t is a coordinate function on $\mathbb{A}^1_{\mathbb{Q}}$. We embed $\mathbb{A}^1_{\mathbb{Q}}$ into $\mathbb{P}^1_{\mathbb{Q}}$ as the complement to the point ∞. We may also take $\mathbb{A}^1_{\mathbb{Q}} \subset \mathbb{P}^1_{\mathbb{Q}}$ to be the complement to the point $t=0$, with the coordinate function $T = 1/t$. Let $F(T) = T^{\ell} f(1/T)$, $G(T) = T^m g(1/T)$, $H(T) = T^n h(1/T)$, and consider the smooth surface in $\mathbb{P}^2_{\mathbb{Q}} \times \mathbb{A}^1_{\mathbb{Q}}$ given by

$$F(T)X^2 + G(T)Y^2 + H(T)Z^2 = 0. \tag{5.2}$$

Let $\pi : V \to \mathbb{P}^1_{\mathbb{Q}}$ be the conic bundle obtained by gluing the surface (5.1) with the surface (5.2). For this we identify the restrictions of the two fibrations to $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\}$ by means of the isomorphism $t = T^{-1}$, $x = T^{\ell_1}X$, $y = T^{m_1}Y$, $z = T^{n_1}Z$, where $(\ell, m, n) = 2(\ell_1, m_1, n_1)$ or $(\ell, m, n) + (1,1,1) = 2(\ell_1, m_1, n_1)$. Since $F(0)G(0)H(0) \neq 0$, the fibre of π at $t = \infty$ is smooth, so π has precisely $r = \ell + m + n$ degenerate fibres.

Suppose $r = 5$, with $(\ell, m, n) = (1, 1, 3)$. Setting $z = 1$ in (5.1) and passing to homogeneous coordinates we obtain a smooth cubic surface in $\mathbb{P}^3_{\mathbb{Q}}$ with the equation

$$c_1(u - e_1 v)x^2 + c_2(u - e_2 v)y^2 + c_3(u - e_3 v)(u - e_4 v)(u - e_5 v) = 0.$$

It contains the line $u = v = 0$. If the conic bundle is relatively minimal, then, contracting this line, we obtain a minimal del Pezzo surface of degree 4 with a $\mathbb{Q}$-point by [14, Prop. 2.1].

Suppose next that $r = 6$, with $(\ell, m, n) = (2, 2, 2)$.

Proposition 5.2. *Let $f(t) = a(t-e_1)(t-e_2)$, $g(t) = b(t-e_3)(t-e_4)$, $h(t) = c(t-e_5)(t-e_6)$, where $e_1, \dots, e_6 \in \mathbb{Q}$ are pairwise different, and a, b, c*

are in $\mathbb{Q}^*$. *If* $f(t)$, $g(t)$ *and* $h(t)$ *are linearly independent over* $\mathbb{Q}$, *then* V *is a del Pezzo surface of degree* 2 *for which the Brauer–Manin obstruction is the only obstruction to weak approximation. If, moreover, the classes*

$$-1,\ a,\ b,\ c,\ e_i - e_j \text{ for } 1 \leqslant i < j \leqslant 6$$

are linearly independent in the $\mathbb{F}_2$*-vector space* $\mathbb{Q}^*/\mathbb{Q}^{*2}$, *then* V *is minimal.*

Proof. Let us write $f(t) = f_2t^2+f_1t+f_0$, with $f_0, f_1, f_2 \in \mathbb{Q}$, and similarly for $g(t)$ and $h(t)$. Using the equality of degrees of $f(t)$, $g(t)$ and $h(t)$ one checks immediately that the projection to the first factor $\mathbb{P}^2_{\mathbb{Q}} \times \mathbb{A}^1_{\mathbb{Q}} \to \mathbb{P}^2_{\mathbb{Q}}$ gives rise to a morphism $\varphi : V \to \mathbb{P}^2_{\mathbb{Q}}$ of (generic) degree 2. Linear independence of $f(t)$, $g(t)$ and $h(t)$ implies that φ has finite fibres, and so V is a double covering of $\mathbb{P}^2_{\mathbb{Q}}$ ramified in the quartic curve

$$(f_1x^2 + g_1y^2 + h_1z^2)^2 = 4(f_0x^2 + g_0y^2 + h_0z^2)(f_2x^2 + g_2y^2 + h_2z^2).$$

This curve is smooth because V is smooth. A double covering of $\mathbb{P}^2_{\mathbb{Q}}$ ramified in a smooth quartic is a del Pezzo surface of degree 2. The first statement now follows from Theorem 1.1. The final condition in the proposition implies that the conic bundle $\pi : V \to \mathbb{P}^1_{\mathbb{Q}}$ is relatively minimal, whence V is minimal by Proposition 5.1. □

The case $r = 7$ translates as $K_V^2 = 1$. This arises if and only if $(\ell, m, n) = (1, 1, 5)$ or $(\ell, m, n) = (1, 3, 3)$. We claim that neither of these surfaces can be isomorphic to a del Pezzo of degree 1. To see this we recall that del Pezzo surfaces are defined by the property that their anticanonical divisor is ample. It therefore suffices to find a geometrically integral curve C on V for which $(C, -K_V) \leqslant 0$. To do so we adapt an argument of Iskovskikh [14, Prop. 1.3 and Cor. 1.4]. In the case $(\ell, m, n) = (1, 1, 5)$ consider the curve C that is the Zariski closure in V of the closed subset of (5.1) given by $z = 0$. We claim that this is a smooth curve of genus 0 such that $(C, -K_V) = -1$. To see this we note that C is a smooth curve of genus 0 such that $(C, F) = 2$, where $F \in \mathrm{Pic}(V)$ is the class of a fibre. The divisor of the rational function z/x on V is $C + 2F_\infty - C'$, where F_∞ is the fibre at infinity and C' is the Zariski closure in V of the closed subset of (5.1) given by $x = 0$. Since $(C, C') = 1$, we see that $(C^2) = -3$, and then from the adjunction formula we find that $(C, -K_V) = -1$, as claimed. In the case $(\ell, m, n) = (1, 3, 3)$ we consider the pencil of genus 1 curves $E = E_{(\lambda:\mu)}$ cut out by $\lambda y + \mu z = 0$ on V. It is easy to see that $(E, E) = 1$, and hence adjunction gives $(E, -K_V) = 1$. It follows that $E = -K_V$. This pencil contains two reducible members, each consisting of the union of one component of the degenerate fibre at $f(t) = 0$ and a residual rational curve C. It follows that $(C, -K_V) = 0$.

We can use some special conic bundles with *eight* degenerate fibres to construct del Pezzo surfaces of degree 1 to which Theorem 1.1 can be applied. Note that $r = 8$ gives $K_V^2 = 0$. Let $e_1, \ldots, e_8 \in \mathbb{Q}$ be pairwise distinct. Let $\pi : V \to \mathbb{P}^1_{\mathbb{Q}}$ be the conic bundle constructed as above from the surface given by the equation

$$x^2 = \prod_{i=1}^{4} \frac{t-e_i}{e_8-e_i} y^2 + \prod_{j=5}^{8} (t-e_j) z^2 \tag{5.3}$$

in $\mathbb{P}^2_{\mathbb{Q}} \times \mathbb{A}^1_{\mathbb{Q}}$. This conic bundle is not relatively minimal because the fibre at $t = e_8$ is a union of two components defined over $\mathbb{Q}$. Either of them can be smoothly contracted, thus producing a conic bundle surface $W \to \mathbb{P}^1_{\mathbb{Q}}$ with seven degenerate fibres.

Recall that the discriminant of the quartic polynomial $p(t) = \sum_{i=0}^4 p_i t^i$ is a homogeneous form $D_4(p_0, \ldots, p_4)$ of degree 6. Thus $D_4 = 0$ defines a hypersurface $Z \subset \mathbb{P}^4_{\mathbb{Q}}$ of degree 6. The space of projective lines in $\mathbb{P}^4_{\mathbb{Q}}$ is naturally identified with the Grassmannian $\mathrm{Gr}(2,5)$. The open subset of $\mathrm{Gr}(2,5)$ parametrising those lines that meet Z in six distinct complex points is nonempty. Let $\mathbb{A}^5_{\mathbb{Q}}$ be the space of polynomials of degree at most 4. Joining two points by a line gives a dominant rational map from $\mathbb{A}^5_{\mathbb{Q}} \times \mathbb{A}^5_{\mathbb{Q}}$ to $\mathrm{Gr}(2,5)$. It follows that the open subset of $\mathbb{A}^5_{\mathbb{Q}} \times \mathbb{A}^5_{\mathbb{Q}}$ consisting of pairs of polynomials $(p(t), q(t))$ such that the discriminant of $rp(t) + sq(t)$ vanishes for exactly six points $(r : s) \in \mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ is nonempty. These six points of Z are necessarily smooth in Z, and hence for each of them $rp(t) + sq(t)$ has exactly one double root. We conclude that there is a nonzero polynomial $f(p_0, \ldots, p_4, q_0, \ldots, q_4)$ with coefficients in $\mathbb{Q}$ such that if $f(p_0, \ldots, p_4, q_0, \ldots, q_4) \neq 0$, then $rp(t) + sq(t)$ has multiple roots for exactly six values of $(r : s) \in \mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$, and for each of these values, $rp(t) + sq(t)$ has exactly one double root. Writing the coefficients as symmetric functions of the roots and applying this to the polynomials

$$p(t) = \prod_{i=1}^{4} (t-e_i) \quad \text{and} \quad q(t) = \prod_{j=5}^{8} (t-e_j).$$

we obtain a nonzero polynomial $F(e_1, \ldots, e_8)$ with coefficients in $\mathbb{Q}$.

Proposition 5.3. *If $e_1, \ldots, e_8 \in \mathbb{Q}$ satisfy $F(e_1, \ldots, e_8) \neq 0$, then W is a del Pezzo surface of degree 1 over $\mathbb{Q}$ for which the Brauer–Manin obstruction is the only obstruction to weak approximation. If, moreover, the classes of $e_i - e_j$, where $1 \leqslant i \leqslant 4$ and $5 \leqslant j \leqslant 8$, are linearly independent in the $\mathbb{F}_2$-vector space $\mathbb{Q}^*/\mathbb{Q}^{*2}$, then W is minimal.*

Proof. For $(\lambda : \mu) \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$, let $E_{(\lambda:\mu)} \subset V$ be the Zariski closure of the subset of (5.3) given by $\lambda y + \mu z = 0$. It has an affine equation $u^2 = \mu^2 c^{-1} p(t) + \lambda^2 q(t)$, where $u = \lambda x / z$ and $c = \prod_{i=1}^4 (e_8 - e_i)$. Since $\deg p(t) = \deg q(t) = 4$, the smooth curves in this family have genus 1. Let E be the class of $E_{(\lambda:\mu)}$ in $\mathrm{Pic}(\overline{V})$. Since $E_{(1:0)}$ and $E_{(0:1)}$ are disjoint, we have $(E,E)=0$. Thus V is an elliptic surface, with a morphism $\varepsilon : V \to \mathbb{P}^1_{\mathbb{Q}}$ such that the fibre above $(\lambda:\mu)$ is $E_{(\lambda:\mu)}$.

As was explained above, the condition $F(e_1,\dots,e_8)\neq 0$ guarantees that there are exactly six points $(r:s)\in\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ such that $rp(t)+sq(t)$ is not separable. Moreover, for each of these values of $(r:s)$ this polynomial has exactly one double root. Since $p(t)$ and $q(t)$ are separable, $\varepsilon : V\to\mathbb{P}^1_{\mathbb{Q}}$ has exactly twelve singular geometric fibres, and each of them is a geometrically irreducible rational curve with one node. In particular, the fibres of $\varepsilon : V\to\mathbb{P}^1_{\mathbb{Q}}$ do not contain exceptional curves, that is, smooth rational curves with self-intersection -1. In addition, the elliptic surface V is geometrically rational so we have $-K_V = E$ by [1, Cor. 12.3, p. 214]. It follows that if $C\subset\overline{V}$ is an irreducible curve that is not a fibre of ε, then $(-K_V,C)>0$.

Let L be the irreducible component of the fibre of $\pi : V\to\mathbb{P}^1_{\mathbb{Q}}$ at $t=e_8$ that is contracted to a point on W. Since $(L,E)=\frac{1}{2}(F,E)=1$, we see that L is a section of $\varepsilon : V\to\mathbb{P}^1_{\mathbb{Q}}$. Let $\sigma : V\to W$ be the contraction of L. It is easy to see that $-K_W=\sigma_*(E)$. Any two distinct curves $\sigma(E_{(\lambda:\mu)})$ and $\sigma(E_{(\lambda':\mu')})$ have exactly one common point $\sigma(L)$, whence $(K_W,K_W)=1$. It follows that every irreducible curve in $\overline{W}$ has positive intersection with $-K_W$. By the Nakai–Moishezon criterion, $-K_W$ is ample, and so W is a del Pezzo surface of degree 1. Theorem 1.1 can be applied to the conic bundle $W\to\mathbb{P}^1_{\mathbb{Q}}$. This proves our first statement.

The components of the degenerate fibre at e_i, for $i=1,2,3,4$, are defined over $\mathbb{Q}(\sqrt{a_i})$, where $a_i=\prod_{j=5}^{8}(e_i-e_j)$. The components of the degenerate fibre at e_j, for $j=5,6,7$, are defined over $\mathbb{Q}(\sqrt{a_j})$, where $a_j=\prod_{i=1}^{4}(e_j-e_i)/(e_8-e_i)$. Now the condition in the last sentence of the proposition implies the relative minimality of the conic bundle $W\to\mathbb{P}^1_{\mathbb{Q}}$, and hence, by Proposition 5.1, the minimality of the del Pezzo surface W of degree 1. □

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