"SCHEME-THEORETIC IMAGES" OF MORPHISMS OF STACKS

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ABSTRACT. We give criteria for certain morphisms from an algebraic stack to a (not necessarily algebraic) stack to admit an (appropriately defined) scheme-theoretic image. We apply our criteria to show that certain natural moduli stacks of local Galois representations are algebraic (or Ind-algebraic) stacks.

CONTENTS

1. INTRODUCTION

The goal of this paper is to prove an existence theorem for "scheme-theoretic images" of certain morphisms of stacks. We have put scheme-theoretic images in quotes here because, generally, the objects whose existence we prove will be certain algebraic spaces or algebraic stacks, rather than schemes. Like the usual scheme-theoretic images of morphisms of schemes, though, they will be closed substacks of the target, minimal with respect to the property that the given morphism factors through them. This explains our terminology.

In the case of morphisms of algebraic stacks (satisfying appropriate mild finiteness conditions), the existence of a scheme-theoretic image in the preceding sense follows directly from the basic results about scheme-theoretic images for morphisms of schemes. Our interest will be in more general contexts, namely, those in which the source of the morphism is assumed to be an algebraic stack, but the target is not; in particular, we will apply our results in one such situation to construct moduli stacks of Galois representations.

1.1. **Scheme-theoretic images.** We put ourselves in the setting of stacks in groupoids defined on the big étale site of a locally Noetherian scheme S all of whose local rings $\mathcal{O}_{S,s}$ at finite type points $s \in S$ are G-rings. (See Section 1.5 for any unfamiliar terminology.) Recall that in this context, Artin's representability

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theorem gives a characterisation of algebraic stacks which are locally of finite presentation over S among all such stacks: namely, algebraic stacks which are locally of finite presentation over S are precisely those stacks in groupoids $\mathcal{F}$ on the big étale site of S that satisfy:

[1] $\mathcal{F}$ is limit preserving;
[2] (a) $\mathcal{F}$ satisfies the Rim–Schlessinger condition (RS), and
   (b) $\mathcal{F}$ admits effective Noetherian versal rings at all finite type points;
[3] the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable by algebraic spaces;
[4] openness of versality.

See Section 2 below for an explanation of these axioms, and Theorem 2.8.4 for Artin's theorem. (In Subsection 2.4 we also introduce two further axioms, labelled [4a] and [4b], that are closely related to [4]. These are used in discussing the examples of Section 4, but not in the proof of our main theorem.)

We will be interested in *quasi-compact* morphisms $\xi : \mathcal{X} \to \mathcal{F}$ of stacks on the big étale site of S, where $\mathcal{X}$ is algebraic and locally of finite presentation over S, and $\mathcal{F}$ is assumed to have a diagonal that is representable by algebraic spaces and is locally of finite presentation (i.e. $\mathcal{F}$ satisfies [3], and a significantly weakened form of [1]). In this context, we are able to define a substack $\mathcal{Z}$ of $\mathcal{F}$ which we call the *scheme-theoretic image* of ξ. (The reason for assuming that ξ is quasi-compact is that this seems to be a minimal requirement for the formation of scheme-theoretic images to be well-behaved even for morphisms of schemes, e.g. to be compatible with *fpqc*, or even Zariski, localisation.) If $\mathcal{F}$ is in fact an algebraic stack, locally of finite presentation over S, then $\mathcal{Z}$ will coincide with the usual scheme-theoretic image of ξ. In general, the substack $\mathcal{Z}$ will itself satisfy Axioms [1] and [3].

Our main result is the following theorem (see Theorem 3.2.34).

1.1.1. **Theorem.** *Suppose that* $\xi : \mathcal{X} \to \mathcal{F}$ *is a* proper *morphism, where* $\mathcal{X}$ *is an algebraic stack, locally of finite presentation over* S, *and* $\mathcal{F}$ *is a stack over* S *satisfying* [3], *and whose diagonal is furthermore locally of finite presentation. Let* $\mathcal{Z}$ *denote the scheme-theoretic image of* ξ *as discussed above, and suppose that* $\mathcal{Z}$ *satisfies* [2]. *Suppose also that* $\mathcal{F}$ *admits* (*not necessarily Noetherian*) *versal rings at all finite type points* (*in the sense of Definition* 2.2.9 *below*).

Then $\mathcal{Z}$ *is an algebraic stack, locally of finite presentation over* S; *the inclusion* $\mathcal{Z} \hookrightarrow \mathcal{F}$ *is a closed immersion; and the morphism* ξ *factors through a proper, scheme-theoretically surjective morphism* $\bar{\xi} : \mathcal{X} \to \mathcal{Z}$. *Furthermore, if* $\mathcal{F}'$ *is a substack of* $\mathcal{F}$ *for which the monomorphism* $\mathcal{F}' \hookrightarrow \mathcal{F}$ *is representable by algebraic spaces and of finite type* (*e.g. a closed substack*) *with the property that* ξ *factors through* $\mathcal{F}'$, *then* $\mathcal{F}'$ *contains* $\mathcal{Z}$.

We show that if $\mathcal{F}$ satisfies [2], then the assumption that $\mathcal{Z}$ satisfies [2] follows from the other assumptions in Theorem 1.1.1 (see Lemma 3.2.20 below). By applying the theorem in the case when $\mathcal{Z} = \mathcal{F}$ (in which case we say that ξ is *scheme-theoretically dominant*), and taking into account this remark, we obtain the following corollary.

1.1.2. **Corollary.** *If* $\mathcal{F}$ *is an étale stack in groupoids over* S, *satisfying* [1], [2] *and* [3], *for which there exists a scheme-theoretically dominant proper morphism* $\xi : \mathcal{X} \to \mathcal{F}$ *whose domain is an algebraic stack locally of finite presentation over* S, *then* $\mathcal{F}$ *is an algebraic stack.*

1.1.3. *Some remarks on Theorem 1.1.1 and its proof.* The proof of Theorem 1.1.1 occupies most of the first three sections of the paper. One way for the reader to get an idea of the argument is to read Lemma 2.6.4, and then to turn directly to the proof of Theorem 3.2.34, taking the various results referenced in the argument (including even the definition of the scheme-theoretic image $\mathcal{Z}$) on faith.

It is often the case (and it is the case in the proof of Theorem 3.2.34) that the main problem to be overcome when using Artin's axiomatic approach to proving that a certain stack is algebraic is the verification of axiom [4] (openness of versality). Lemma 2.6.4 (which is a stacky version of [Art69a, Lem. 3.10]) shows that one can automatically get a slightly weaker version of [4], where "smooth" is replaced by "unramified", if the axioms [1], [2], and [3] are satisfied. Morally, this suggests that anything satisfying axioms [1], [2], and [3] is already close to an Ind-algebraic stack, because it admits unramified maps (which are at least morally quite close to immersions) from algebraic stacks that are even formally smooth at any particular point. So to prove Theorem 1.1.1, one has to build on this idea, and then use the extra hypothesis (namely, that there is a proper surjection from an algebraic stack onto the stack $\mathcal{Z}$) to prove axiom [4]. The argument is ultimately topological, using properness to eliminate the possibility of having more and more components building up Zariski locally around a point.

1.2. **Moduli of finite height Galois representations.** The results in this paper were developed with a view to applications to the theory of Galois representations, and in particular to constructing moduli stacks of mod p and p-adic representations of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ of a finite extension $K/\mathbb{Q}_p$. These applications will be developed more fully in the papers [EG19, CEGS19], and we refer the interested reader to those papers for a fuller discussion of our results and motivations.

Let $\bar{r} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{F}_p)$ be a continuous representation. The theory of deformations of $\bar{r}$ — that is, liftings of $\bar{r}$ to continuous representations $r : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_d(A)$, where A is a complete local ring with residue field $\mathbb{F}_p$ — is extremely important in the Langlands program, and in particular is crucial for proving automorphy lifting theorems via the Taylor–Wiles method. Proving such theorems often comes down to studying the moduli spaces of those deformations which satisfy various p-adic Hodge-theoretic conditions; see for example [Kis09b, Kis09a].

From the point of view of algebraic geometry, it seems unnatural to only study "formal" deformations of this kind, and Kisin observed about ten years ago that results on the reduction modulo p of two-dimensional crystalline representations suggested that there should be moduli spaces of p-adic representations (satisfying certain p-adic Hodge theoretic conditions, for example finite flatness) in which the residual representations $\bar{r}$ should be allowed to vary; in particular, the special fibres of these moduli spaces would be moduli spaces of (for example) finite flat representations of $\mathrm{Gal}(\overline{K}/K)$. Unfortunately, there does not seem to be any simple way of directly constructing such moduli spaces, and until now their existence has remained a mystery. (We refer the reader to the introduction to [EG19] for a further discussion of the difficulties of directly constructing moduli spaces of mod p representations of $\mathrm{Gal}(\overline{K}/K)$.)

Mod p and p-adic Galois representations are studied via integral p-adic Hodge theory; for example, the theories of (φ, Γ)-modules and Breuil–Kisin modules. Typically, one begins by analysing p-adic representations of the absolute Galois

group $\mathrm{Gal}(\overline{K}/K_\infty)$ of some highly ramified infinite extension K_∞/K. (In the theory of (φ, Γ)-modules this extension is the cyclotomic extension, but in the theory of Breuil–Kisin modules, it is a non-Galois extension obtained by extracting p-power roots of a uniformiser.) Having classified these representations in terms of semilinear algebra (modules over some ring, equipped with a Frobenius), one then separately considers the problem of descending the classification to representations of $\mathrm{Gal}(\overline{K}/K)$.

More precisely, by the theory of [Fon90], continuous mod p^a representations of $\mathrm{Gal}(\overline{K}/K_\infty)$ are classified by étale φ-modules, which are modules over a Laurent series ring, equipped with a Frobenius. Following the paper [PR09] of Pappas and Rapoport, we consider a moduli stack $\mathcal{R}$ of étale φ-modules, which, for appropriate choices of the Frobenius on the Laurent series ring can be thought of informally as a moduli stack classifying $\mathrm{Gal}(\overline{K}/K_\infty)$-representations. (To keep this paper at a reasonable length, we do not discuss the problem of descending our results to representations of $\mathrm{Gal}(\overline{K}/K)$; this is addressed in the papers [CEGS19, EG19].) Pappas and Rapoport prove various properties of the stack $\mathcal{R}$ (including that it *is* a stack, which they deduce from deep results of Drinfeld [Dri06] on the *fpqc* locality of the notion of a projective module over a Laurent series ring), including that its diagonal is representable by algebraic spaces. In the present paper we prove the following theorem about the geometry of $\mathcal{R}$ (see Theorem 5.4.20 below).

1.2.1. **Theorem.** *The stack $\mathcal{R}$ is an Ind-algebraic stack. More precisely, we may write $\mathcal{R} \cong \varinjlim_F \mathcal{R}_F$, where each $\mathcal{R}_F$ is a finite type algebraic stack over $\mathbb{Z}/p^a\mathbb{Z}$, and where each transition morphism in the inductive limit is a closed immersion.*

The theorem is proved by applying Theorem 1.1.1 to certain morphisms $\mathcal{C}_F \to \mathcal{R}$ (whose sources $\mathcal{C}_F$ are algebraic stacks) so as to prove that their scheme-theoretic images $\mathcal{R}_F$ are algebraic stacks; we then show that $\mathcal{R}$ is naturally identified with the inductive limit of the $\mathcal{R}_F$, and thus that it is an Ind-algebraic stack. (The index F is a certain element of a power series ring; replacing the indexing set with a cofinal subset given by powers of a fixed F, one can write $\mathcal{R}$ as an Ind-algebraic stack with the inductive limit being taken over the natural numbers.)

The stacks $\mathcal{C}_F$ were defined by Pappas and Rapoport in [PR09], where it is proved that they are algebraic, and that the morphisms $\mathcal{C}_F \to \mathcal{R}$ are representable by algebraic spaces and proper. The definition of the stacks $\mathcal{C}_F$ is motivated by the papers [Kis09b, Kis06, Kis08], in which Kisin (following work of Breuil and Fontaine) studied certain integral structures on étale φ-modules, in particular (what are now called) Breuil–Kisin modules of height at most F, where F is a power of an Eisenstein polynomial for a finite extension $K/\mathbb{Q}_p$. A Breuil–Kisin module is a module with Frobenius over a power series ring, satisfying a condition (depending on F) on the cokernel of the Frobenius. Inverting the formal variable u in the power series ring gives a functor from the category of Breuil–Kisin modules of height at most h to the category of étale φ-modules. We say that the Galois representations corresponding to the étale φ-modules in the essential image of this functor have *height at most F*.

With an eye to future applications, we work in a general context in this paper, and in particular we allow considerable flexibility in the choice of the polynomial F and the Frobenius on the coefficient rings. In particular our étale φ-modules do not obviously correspond to representations of some $\mathrm{Gal}(\overline{K}/K_\infty)$ (but the case that they do is the main motivation for our constructions and theorems). A key point

in the argument (since it is one of the hypotheses of Theorem 1.1.1) is that $\mathcal{R}$ admits versal rings at all finite type points, and that the scheme-theoretic images $\mathcal{R}_F$ satisfy [2]. In the setting of representations of $\mathrm{Gal}(\overline{K}/K_\infty)$, versal rings for $\mathcal{R}$ are given be the framed deformation rings associated to continuous mod p representations of $\mathrm{Gal}(\overline{K}/K_\infty)$, as we show in [EG19, CEGS19]; then the basic input to the verification of [2] for $\mathcal{R}_F$ is the theory of finite height framed Galois deformation rings (which are proved to be Noetherian by Kim in [Kim11], reflecting the fact that the scheme-theoretic images $\mathcal{R}_F$ turn out to be finite-type algebraic stacks). In our more general setting we work instead with lifting rings for φ-modules.

1.3. **Further remarks on the contents and organisation of the paper.** In the remainder of Section 1, we describe our notation and conventions (Subsection 1.5), and also record some simple lemmas in local algebra that will be needed later in the paper (Subsection 1.6).

In Section 2, we explain Artin's axioms (as listed above) in some detail, and present many related definitions and results. We do not expect the reader experienced in the theory of stacks to find much that is novel in this section, and indeed, many of the results that we have included are simple variants of results that are already in the literature. In light of this, it might be worthwhile to offer some justification for the length of this section. Primarily, we have been guided by the demands of the arguments presented in Section 3; these demands have largely dictated the choice of material presented in Section 2, and its organisation. Additionally, we anticipate that the typical reader of this paper interested in the application of our results to the moduli of Galois representations will not already be completely familiar with the foundational results discussed in this section, and so we have made an effort to include a more careful discussion of these results, as well as more references to the literature, than might strictly be required for the typical reader interested only in Theorem 1.1.1 and Corollary 1.1.2.

Moreover, the basic idea of our argument came from a careful reading of [Art69a, §3], especially the proof of Theorem 3.9 therein, which provided a model argument for deducing Axiom [4] (openness of versality) from a purely geometric assumption on the object to be represented. We also found the several (counter)examples that Artin presents in [Art69b, §5] to be illuminating. For these reasons, among others, we have chosen to discuss Artin's representability theorem in terms that are as close as possible to Artin's treatment in [Art69a, §3] and [Art69b, §5], making the minimal changes necessary to adapt the statement of the axioms, and of the theorem, to the stacky situation. Of course, such adaptations have been presented by many authors, including Artin himself, but these works have tended to focus on developments of the theory (such as the use of obstruction theories to verify openness of versality) which are unnecessary for our purposes. Ultimately, we found the treatment of Artin representability in the Stacks Project [Sta, Tag 07SZ] to be closest in spirit to the approach we wanted to take, and it forms the basis for our treatment of the theorem here. However, for the reason discussed above, of wanting to follow as closely as possible Artin's original treatment, we have phrased the axioms in different terms to the way they are phrased in the Stack Project, terms which are closer to Artin's original phrasing.

One technical reason for preferring Artin's phrasing is the emphasis that it places on the role of pro-representability (or equivalently, versality). As already noted, the main intended application of Theorem 1.1.1 is to the construction of moduli

of Galois representations, and phrasing the axioms in a way which emphasises pro-representability makes it easy to incorporate the formal deformation theory of Galois representations into our arguments (one of the main outputs of that formal deformation theory being various pro-representability statements of the kind that Theorem 1.1.1 requires as one of its inputs. In fact, in the interests of generality we work with φ-modules that do not evidently correspond to Galois representations, so we do not directly invoke results from Galois deformation theory, but rather adapt some arguments from that theory to our more general setting.)

On a few occasions it has seemed sensible to us to state and prove a result in its natural level of generality, even if this level of generality is not strictly required for the particular application we have in mind. We have also developed the analogue of Artin's axioms [4a] and [4b] of [Art69a, §3] (referred to as [4] and [5] in [Art69b, §5]) in the stacky context; while not necessary for the proof of Theorem 1.1.1, thinking in terms of these axioms helps to clarify some of the foundational results of Section 2 (e.g. the extent to which the unramifiedness condition of Lemma 2.6.4 can be promoted to the condition of being a monomorphism, as in Corollary 2.6.12).

Just to inventory the contents of Section 2 a little more precisely: in Subsections 2.1 through 2.4 we discuss each of Artin's axioms in turn. In Subsection 2.5 we develop a partial analogue of [Art70, Prop. 3.11], which allows us to construct stacks satisfying [1] by defining their values on algebras of finite presentation over the base and then taking appropriate limits. In Subsections 2.6 and 2.7 we develop various further technical consequences of Artin's axioms. Of particular importance is Lemma 2.6.4, which is a generalisation to the stacky context of one of the steps appearing in the proof of [Art69a, Lem. 3.10]: it provides unramified algebraic approximations to stacks satisfying Axioms [1], [2], and [3], and so is the key to establishing openness of versality (i.e. Axiom [4]) in certain contexts in which it is not assumed to hold *a priori*. In Subsection 2.8 we explain how our particular formulation of Artin's axioms does indeed imply his representability theorem for algebraic stacks.

In Section 3, after a preliminary discussion in Subsection 3.1 of the theory of scheme-theoretic images in the context of morphisms of algebraic stacks, in Subsection 3.2, we present our main definitions, and give the proof of Theorem 1.1.1. In Subsection 3.3, we investigate the behaviour of our constructions with respect to base change (both of the target stack $\mathcal{F}$, and of the base scheme S); as well as being of intrinsic interest, this will be important in our applications to Galois representations in [CEGS19, EG19].

In Section 4, we give various examples of stacks and Ind-stacks, which illustrate the results of Section 3, and the roles of the various hypotheses of Section 2 in the proofs of our main results. We also prove some basic results about Ind-stacks which are used in the proof of Theorem 1.2.1.

The paper concludes with Section 5, in which we define the moduli stacks of étale φ-modules and prove (via an application of Theorem 1.1.1) that they are Ind-algebraic stacks.

the moduli spaces of Galois representations over the years. We are also indebted to Brian Conrad, Michael Harris, and Florian Herzig for their close readings of, and many helpful comments on and corrections to, various versions of this manuscript. The first author would like to thank the members of his working group at the Stacks Project Workshop — Eric Ahlqvist, Daniel Bragg, Atticus Christensen, Ariyan Javanpeykar, Julian Rosen — for their helpful comments. Finally, we thank the anonymous referee for their careful reading of the manuscript, and their many corrections and suggestions for improvement.

1.5. **Notation and conventions.** We follow the conventions of [Sta] except where explicitly noted, and we refer to this reference for background material. We note that the references to [Sta] in the electronic version of this paper are clickable, and will take the reader directly to the web page of the corresponding entry. We use Roman letters $X, Y, \ldots$ for schemes and algebraic spaces, and calligraphic letters $\mathcal{X}, \mathcal{Y}, \ldots$ for (possibly non-algebraic) stacks (and more generally, for categories fibred in groupoids). We elide the difference between commutative diagrams and 2-commutative diagrams, referring to both as simply commutative diagrams.

Since we follow [Sta], we do not assume (unless otherwise stated) that our algebraic spaces and algebraic stacks have quasi-compact or quasi-separated diagonals. This is in contrast to references such as [Knu71, Art74, LMB00], and occasionally requires us to make some simple additional arguments; the reader interested only in our applications to moduli stacks of Galois representations should feel free to impose the additional hypotheses on the diagonal that are common in the stacks literature, and will lose nothing by doing so.

1.5.1. *Choice of site.* One minor difference between our approach and that taken in [Sta] is that we prefer to only assume that the stacks that we work with are stacks in groupoids for the étale topology, rather than the *fppf* topology. This ultimately makes no difference, as the definition of an algebraic stack can be made using either the étale or *fppf* topologies [Sta, Tag 076U]. In practice, this means that we will sometimes cite results from [Sta] that apply to stacks in groupoids for the *fppf* topology, but apply them to stacks in groupoids for the étale topology. In each such case, the proof goes over unchanged to this setting.

To ease terminology, from now on we will refer to a stack in groupoids for the étale topology (on some given base scheme S) simply as a stack (or a stack over S). (On a few occasions in the manuscript, we will work with stacks in sites other than the étale site, in which case we will be careful to signal this explicitly.)

1.5.2. *Finite type points.* If S is a scheme, and $s \in S$ is a point of S, we let $\kappa(s)$ denote the residue field of s. A *finite type point* $s \in S$ is a point such that the morphism $\operatorname{Spec} \kappa(s) \to S$ is of finite type. By [Sta, Tag 01TA], a morphism $f : \operatorname{Spec} k \to S$ is of finite type if and only if there is an affine open $U \subseteq S$ such that the image of f is a closed point $u \in U$, and $k/\kappa(u)$ is a finite extension. In a Jacobson scheme, the finite type points are precisely the closed points; more generally, the finite type points of any scheme S are dense in every closed subset of S by [Sta, Tag 02J4]. If $X \to S$ is a finite type morphism, then a morphism $\operatorname{Spec} k \to X$ is of finite type if and only if the composite $\operatorname{Spec} k \to S$ is of finite type, and so in particular a point $x \in X$ is of finite type if and only if the composite $\operatorname{Spec} \kappa(x) \to X \to S$ is of finite type.

1.5.3. *Points of categories fibred in groupoids.* If $\mathcal{X}$ is a category fibred in groupoids, then a *point* of $\mathcal{X}$ is an equivalence class of morphisms from spectra of fields $\operatorname{Spec} K \to \mathcal{X}$, where we say that $\operatorname{Spec} K \to \mathcal{X}$ and $\operatorname{Spec} L \to \mathcal{X}$ are equivalent if there is a field M and a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} M & \longrightarrow & \operatorname{Spec} L \\
\downarrow & & \downarrow \\
\operatorname{Spec} K & \longrightarrow & \mathcal{X}.
\end{array}
$$

(This is an equivalence relation by [Sta, Tag 04XF]; strictly speaking, this proves the claim in the case that $\mathcal{X}$ is an algebraic stack, but the proof goes over identically to the general case that $\mathcal{X}$ is a category fibred in groupoids.) If $\mathcal{X}$ is furthermore an algebraic stack, then the set of points of $\mathcal{X}$ is denoted $|\mathcal{X}|$; by [Sta, Tag 04XL] there is a natural topology on $|\mathcal{X}|$, which has in particular the property that if $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then the induced map $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous.

If $\mathcal{X}$ is a category fibred in groupoids, then a *finite type point* of $\mathcal{X}$ is a point that can be represented by a morphism $\operatorname{Spec} k \to \mathcal{X}$ which is locally of finite type. If $\mathcal{X}$ is an algebraic stack, then by [Sta, Tag 06FX], a point $x \in |\mathcal{X}|$ is of finite type if and only if there is a scheme U, a smooth morphism $\varphi : U \to \mathcal{X}$ and a finite type point $u \in U$ such that $\varphi(u) = x$. The set of finite type points of an algebraic stack $\mathcal{X}$ is dense in any closed subset of $|\mathcal{X}|$ by [Sta, Tag 06G2].

If X is an *algebraic space* which is locally of finite type over a locally Noetherian base scheme S, then any finite type point of X may be represented by a *monomorphism* $\operatorname{Spec} k \to X$ which is locally of finite type; this representative is unique up to unique isomorphism, and any other morphism $\operatorname{Spec} K \to X$ representing x factors through this one. (See Lemma 2.2.14 below.)

1.5.4. *pro-categories.* We will make several uses of the formal pro-category pro-$\mathcal{C}$ associated to a category $\mathcal{C}$, in the sense of [Gro95]. Recall that an object of pro-$\mathcal{C}$ is a projective system $(\xi_i)_{i \in I}$ of objects of $\mathcal{C}$, and the morphisms between two pro-objects $\xi = (\xi_i)_{i \in I}$ and $\nu = (\eta_j)_{j \in J}$ are by definition

$$
\operatorname{Mor}(\xi, \eta) = \varprojlim_{j \in J} \varinjlim_{i \in I} \operatorname{Mor}(\xi_i, \eta_j).
$$

We will apply this definition in particular to categories of Artinian local rings with fixed residue fields in Section 2.2, and to the category of affine schemes locally of finite presentation over a fixed base scheme in Section 2.5, as well as to categories (co)fibred in groupoids over these categories.

1.5.5. *G-rings.* Recall ([Sta, Tag 07GH]) that a Noetherian ring R is a G-ring if for every prime $\mathfrak{p}$ of R, the (flat) map $R_{\mathfrak{p}} \to (R_{\mathfrak{p}})^\wedge$ is regular. By [Sta, Tag 07PN], this is equivalent to demanding that for every pair of primes $\mathfrak{q} \subseteq \mathfrak{p} \subset R$ the algebra $(R/\mathfrak{q})_{\mathfrak{p}}^\wedge \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$ (where $\kappa(\mathfrak{q})$ denotes the residue field of $\mathfrak{q}$; recall [Sta, Tag 0382] that if k is a field, a Noetherian k-algebra A is geometrically regular if and only if $A \otimes_k k'$ is regular for every finitely generated field extension k'/k). Excellent rings are G-rings by definition.

In our main results we will assume that our base scheme S is locally Noetherian, and that its local rings $\mathcal{O}_{S,s}$ at all finite type points $s \in S$ are G-rings. This is a replacement of Artin's assumption that S be of finite type over a field or an excellent DVR; this more general setting is permitted by improvements in Artin

approximation, due essentially to Popescu ([Pop85, Pop86, Pop90]; see also [CdJ02] and [Sta, Tag 07GB]). However, since this assumption will not always be in force, we will indicate when it is assumed to hold.

1.5.6. *Groupoids.* We will make use of groupoids in algebraic spaces, and we will use the notation for them which is introduced in [Sta, Tag 043V], which we now recall. A groupoid in algebraic spaces over a base algebraic space B is a tuple (U, R, s, t, c) where U and R are algebraic spaces over B, and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms of algebraic spaces over B whose T-points form a groupoid for any scheme $T \to B$. (The maps s, t, c give the source, target and composition laws for the arrows of the groupoid.) Given such a groupoid in algebraic spaces, there are unique morphisms $e : U \to R$ and $i : R \to R$ of algebraic spaces over B which give the identity and inverse maps of the groupoid, and we sometimes denote the groupoid in algebraic spaces by the tuple (U, R, s, t, c, e, i).

1.5.7. *Properties of morphisms.* In most cases, we follow the terminology and conventions for properties of morphisms of stacks introduced in [Sta]. We recall some of the general framework of those conventions here.

An important concept, defined for morphisms of categories fibred in groupoids, and so in particular for morphisms of stacks, is that of being *representable by algebraic spaces*. Following [Sta, Tag 04SX], we say that a morphism $\mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids is representable by algebraic spaces if for any morphism $T \to \mathcal{Y}$ with T a scheme, the fibre product $\mathcal{X} \times_{\mathcal{Y}} T$ is (representable by) an algebraic space. (This condition then continues to hold whenever T is an algebraic space [Sta, Tag 0300].) A morphism of algebraic stacks is representable by algebraic spaces if and only if the associated diagonal morphism is a monomorphism [Sta, Tag 0AHJ].

If P is a property of morphisms of algebraic spaces which is preserved under arbitrary base-change, and which is *fppf* local on the target, then [Sta, Tag 03YK] provides a general mechanism for defining the property P for morphisms of categories fibred in groupoids that are representable by algebraic spaces: namely, such a morphism $f : \mathcal{X} \to \mathcal{Y}$ is defined to have property P if and only if for any morphism $T \to \mathcal{Y}$ with T a scheme, the base-changed morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ (which is a morphism of algebraic spaces, by assumption) has property P (and it is equivalent to impose the same condition with T being merely an algebraic space, because an algebraic space by definition has an étale (and therefore *fppf*) cover by a scheme, and P is *fppf* local on the target by assumption).

If P is a property of morphisms of algebraic spaces which is smooth local on the source-and-target, then [Sta, Tag 06FN] extends the definition of P to arbitrary morphisms of algebraic stacks (in particular, to morphisms that are not necessarily representable by algebraic spaces): a morphism $f : \mathcal{X} \to \mathcal{Y}$ is defined to have property P if it can be lifted to a morphism $h : U \to V$ having property P, where U is a smooth cover of $\mathcal{X}$ and V is a smooth cover of $\mathcal{Y}$. If P is furthermore preserved under arbitrary base-change and *fppf* local on the target, so that the definition of [Sta, Tag 03YK] applies, then the two definitions coincide in the case of morphisms that are representable by algebraic spaces [Sta, Tag 06FM].

Many additional properties of morphisms of algebraic stacks are defined in [Sta, Tag 04XM]. In Subsection 2.3 below, we further extend many of these definitions

to the case of morphisms of stacks whose source is assumed to be algebraic, but whose target is assumed only to satisfy condition [3] of Artin's axioms.

1.6. **Some local algebra.** In this subsection, we state and prove some results from local algebra which we will need in what follows.

1.6.1. **Lemma.** *If $B \to A$ and $C \to A$ are local morphisms from a pair of complete Noetherian local rings to an Artinian local ring, $C \to A$ is surjective, and the residue field of A is finite over the residue field of B, then the fibre product $B \times_A C$ is a complete Noetherian local ring, and the natural morphism $B \times_A C \to A$ is local.*

Proof. Write $R := B \times_A C$, and $\mathfrak{m}_R := \mathfrak{m}_B \times_{\mathfrak{m}_A} \mathfrak{m}_C$. Since $B \to A$ and $C \to A$ are local morphisms of local rings, we see that if $(b, c) \in R$, with both b and c lying over the element $a \in A$, then if $a \in \mathfrak{m}_A$, we have $(b, c) \in \mathfrak{m}_R$, while if $a \notin \mathfrak{m}_A$, then $(b, c) \notin \mathfrak{m}_R$. In the latter case, we find that (b, c) is furthermore a unit in R. Thus R is a local ring with maximal ideal $\mathfrak{m}_R$, and the natural morphism $R \to A$ is local.

If we choose $r \geq 0$ so that $\mathfrak{m}_A^r = 0$ (which is possible, since A is Artinian), then each of $\mathfrak{m}_B^r$ and $\mathfrak{m}_C^r$ has vanishing image in A, and so we see that

(1.6.2) $$\mathfrak{m}_R^r \subseteq \mathfrak{m}_B^r \times \mathfrak{m}_C^r \subseteq \mathfrak{m}_R.$$

From this inclusion, and the fact that $B \times C$ is $\mathfrak{m}_B \times \mathfrak{m}_C$-adically complete, it follows that R is $\mathfrak{m}_R$-adically complete. (Indeed, we see that R is open and closed as a topological subgroup of $B \times C$, and that the induced topology on R coincides with the $\mathfrak{m}_R$-adic topology.)

Finally, to see that R is Noetherian, we use the hypothesis that $C \to A$ is surjective, which implies that the residue fields k_C and k_A of C and A are isomorphic, as are the residue fields k_R and k_B of R and B, which are subfields of $k_C = k_A$.

Then the inclusion $\mathfrak{m}_R = \mathfrak{m}_B \times_{\mathfrak{m}_A} \mathfrak{m}_C \hookrightarrow \mathfrak{m}_B \times \mathfrak{m}_C$ induces an inclusion

$$\mathfrak{m}_R/\mathfrak{m}_R^2 \hookrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2 \times \mathfrak{m}_C/\mathfrak{m}_C^2,$$

and since B and C are Noetherian, and the extension degree $[k_R : k_B] = [k_C : k_A]$ is finite, the target of the inclusion is a finite-dimensional k_R-vector space. It follows that $\mathfrak{m}_R/\mathfrak{m}_R^2$ is also finite-dimensional, and therefore that R is Noetherian, as required. □

1.6.3. **Lemma.** *Let $B \to A$ be a local morphism from a complete Noetherian local ring to an Artinian local ring, which induces a finite extension of residue fields. Then this morphism admits a factorisation $B \to B' \to A$, where $B \to B'$ is a faithfully flat local morphism of complete local Noetherian rings, and $B' \to A$ is surjective (and so in particular induces an isomorphism on residue fields).*

Proof. Let $k_B \subseteq k_A$ be the embedding of residue fields induced by the given morphism $B \to A$. Let Λ_{k_A} denote a Cohen ring with residue field k_A, and choose (as we may) a surjection $\Lambda_{k_A}[[x_1, \ldots, x_d]] \twoheadrightarrow A$ (for some appropriate value of d). Let $\overline{B}$ denote the image of B in A, and let A' denote the fibre product $A' := \overline{B} \times_A \Lambda_{k_A}[[x_1, \ldots, x_d]]$; then Lemma 1.6.1 shows that A' is a complete Noetherian local subring of $\Lambda_{k_A}[[x_1, \ldots, x_d]]$ with residue field k_B. If Λ_{k_B} denotes a Cohen ring with residue field k_B, then we may find a local morphism $\Lambda_{k_B} \to A'$ inducing the identity on residue fields. The composite

(1.6.4) $$\Lambda_{k_B} \to A' \subseteq \Lambda_{k_A}[[x_1, \ldots, x_d]]$$

is flat.

By [Gro64, Thm. 0.19.8.6(i)], the composite morphism $\Lambda_{k_B} \to A' \to \overline{B}$ (the second arrow being the projection) may be lifted to a morphism $\Lambda_{k_B} \to B$. Now define $B' := B \widehat{\otimes}_{\Lambda_{k_B}} \Lambda_{k_A}[[x_1, \ldots, x_d]]$ (the completed tensor product). By [Gro64, Lem. 0.19.7.1.2], B' is a complete local Noetherian ring, flat over B.

The given morphisms $B \to A$ and $\Lambda_{k_A}[[x_1, \ldots, x_d]] \to A$ induce a surjection $B' \to A$, and $B \to B' \to A$ is the required factorisation of our given morphism $B \to A$. (Note that flat local morphisms of local rings are automatically faithfully flat.) □

2. STACKS IN GROUPOIDS AND ARTIN'S AXIOMS

Since Artin first introduced his axioms characterising algebraic spaces [Art69a], many versions of these axioms have appeared in the works of various authors. In this paper we have tried to follow Artin's original treatment closely, and the labelling of our four axioms is chosen to match the labelling in [Art69a].

In this section we will discuss each of the four axioms, explain why they imply representability (essentially, by relating them to the axioms given in [Sta, Tag 07SZ]) and also discuss some related foundational material.

As noted in the introduction, our basic setting will be that of stacks in groupoids on the big étale site of a scheme S. A general reference for the basic definitions and properties of such stacks is [Sta]. As remarked in the introduction, from now on we will refer to such a stack in groupoids simply as a stack. At times we will furthermore assume that S is locally Noetherian, and in Subsection 2.8, where we present Artin's representability theorem, we will additionally assume that the local rings $\mathcal{O}_{S,s}$ are G-rings, for each finite type point $s \in S$.

2.1. **Remarks on Axiom [1].** We begin by recalling the definition of limit preserving.

2.1.1. **Definition.** A category fibred in groupoids (e.g. an algebraic stack) $\mathcal{F}$ over S is *limit preserving* if, whenever we have a projective limit $T = \varprojlim T_i$ of affine S-schemes, the induced functor

$$(2.1.2) \qquad\qquad 2\text{-}\varinjlim \mathcal{F}(T_i) \to \mathcal{F}(T)$$

is an equivalence of categories.

More concretely, as in [Sta, Tag 07XK] this means that each object of $\mathcal{F}(T)$ is isomorphic to the restriction to T of an object of $\mathcal{F}(T_i)$ for some i; that for any two objects x, y of $\mathcal{F}(T_i)$, any morphism between the restrictions of x, y to T is the restriction of a morphism between the restrictions of x, y to $T_{i'}$ for some $i' \geq i$; and that for any two objects x, y of $\mathcal{F}(T_i)$, if two morphisms $x \rightrightarrows y$ coincide after restricting to T, than they coincide after restricting to $T_{i'}$ for some $i' \geq i$. (Since we are considering categories fibred in groupoids, it suffices to check this last condition when one of the morphisms is the identity.)

We have the following related definition [Sta, Tag 06CT].

2.1.3. **Definition.** A morphism $\mathcal{F} \to \mathcal{G}$ of categories fibred in groupoids (e.g. of algebraic stacks) over S is said to be *limit preserving on objects* if for any affine S-scheme T, written as a projective limit of affine S-schemes T_i, and any morphism $T \to \mathcal{F}$ for which the composite morphism $T \to \mathcal{F} \to \mathcal{G}$ factors through T_i for some i, there is a compatible factorisation of the morphism $T \to \mathcal{F}$ through $T_{i'}$, for some $i' \geq i$.

Somewhat more precisely, given a commutative diagram

$$
\begin{array}{ccc}
T & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
T_i & \longrightarrow & \mathcal{G}
\end{array}
$$

we may factor it in the following manner:

$$
\begin{array}{ccc}
T & & \\
\downarrow & \searrow & \\
T_{i'} & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
T_i & \longrightarrow & \mathcal{G}
\end{array}
$$

We also make the following variation on the preceding definition.

2.1.4. **Definition.** We say that a morphism $\mathcal{F} \to \mathcal{G}$ of categories fibred in groupoids (e.g. of algebraic stacks) over S is *étale locally limit preserving on objects* if for any affine S-scheme T, written as a projective limit of affine S-schemes T_i, and any morphism $T \to \mathcal{F}$ for which the composite morphism $T \to \mathcal{F} \to \mathcal{G}$ factors through T_i for some i, then there is an affine étale surjection $T'_{i'} \to T_{i'}$, for some $i' \geq i$, and a morphism $T'_{i'} \to \mathcal{F}$, such that, if we write $T' := T'_i \times_{T_i} T$, then the resulting diagram

$$
\begin{array}{ccc}
T' & & \\
\downarrow & \searrow & \\
T'_{i'} & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
T_i & \longrightarrow & \mathcal{G}
\end{array}
$$

commutes.

The following lemma relates Definitions 2.1.1 and 2.1.3.

2.1.5. **Lemma.** *If $\mathcal{F}$ is a category fibred in groupoids over S, then the following are equivalent:*

(1) *$\mathcal{F}$ is limit preserving.*
(2) *Each of the morphisms $\mathcal{F} \to S$, $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$, and $\Delta_\Delta : \mathcal{F} \to \mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} \mathcal{F}$ is limit preserving on objects.*

Proof. This is just a matter of working through the definitions. Indeed, the morphism $\mathcal{F} \to S$ being limit preserving on objects is equivalent to the functor (2.1.2) being essentially surjective (for all choices of $T = \varprojlim_i T_i$), the diagonal morphism being limit preserving on objects is equivalent to this functor being full, and the double diagonal being limit preserving on objects is equivalent to this functor being faithful.

More precisely, by definition the morphism $\mathcal{F} \to S$ is limit preserving on objects if and only if for every $T = \varprojlim_i T_i$ as above, any morphism $T \to \mathcal{F}$ factors through some T_i; equivalently, if and only if every object of $\mathcal{F}(T)$ is isomorphic to the restriction to T of an object of $\mathcal{F}(T_i)$ for some i; equivalently, if and only if the functor (2.1.2) is essentially surjective. Similarly, the morphism $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is limit preserving on objects if and only if for any pair of objects of $\mathcal{F}(T_i)$ (for some value of i), a morphism between their images in $\mathcal{F}(T)$ arises as the restriction of a morphism between their images in $\mathcal{F}(T_{i'})$, for some $i' \geq i$; or equivalently, if and only if the functor (2.1.2) is full. The claim about the double diagonal is similar, and is left to the interested reader. $\qquad\square$

We can strengthen the preceding lemma when the category fibred in groupoids involved is actually a stack.

2.1.6. **Lemma.** *If $\mathcal{F}$ is a stack over S, then the following are equivalent:*
   (1) *$\mathcal{F}$ is limit preserving.*
   (2) *The morphism $\mathcal{F} \to S$ is étale locally limit preserving on objects, while each of $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ and $\Delta_\Delta : \mathcal{F} \to \mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} \mathcal{F}$ is limit preserving on objects.*

Proof. Taking into account Lemma 2.1.5, we see that the lemma will follow if we show that the assumptions of (2) imply that $\mathcal{F} \to S$ is limit preserving on objects. Thus we put ourselves in the situation described in Definition 2.1.3 (taking $\mathcal{G} = S$), namely we give ourselves an affine S-scheme T, written as a projective limit $T = \varprojlim_i T_i$ of affine S-schemes, and we suppose given a morphism $T \to \mathcal{F}$; we must show that this morphism factors through T_i for some i. Applying the assumption that $\mathcal{F} \to S$ is étale locally limit preserving on objects, we find that, for some i, we may find an étale cover T_i' of T_i and a morphism $T_i' \to \mathcal{F}$, for some value of i, through which the composite $T' := T_i' \times_{T_i} T \to T \to \mathcal{F}$ factors; our goal is then to show that, for some $i' \geq i$, we may find a morphism $T_{i'} \to \mathcal{F}$ through which the morphism $T \to \mathcal{F}$ itself factors.

For any $i' \geq i$, we let $T_{i'}' := T_i' \times_{T_i} T_{i'}$. In order to find the desired morphism $T_{i'} \to \mathcal{F}$, it suffices to equip the composite $T_{i'}' \to T_i' \to \mathcal{F}$ with descent data to $T_{i'}$, in a manner compatible with the canonical descent data of the composite $T' \to T \to \mathcal{F}$ to T. That this is possible follows easily from the assumptions on the diagonal and double diagonal of $\mathcal{F}$ (*cf.* the proof of Lemma 2.5.5 (2) below). $\qquad\square$

We will have use for the following finiteness results.

2.1.7. **Lemma.** *If $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ are morphisms between categories fibred in groupoids over S, and if both the composite morphism $\mathcal{F} \to \mathcal{H}$ and the diagonal morphism $\Delta : \mathcal{G} \to \mathcal{G} \times_{\mathcal{H}} \mathcal{G}$ are limit preserving on objects, then the morphism $\mathcal{F} \to \mathcal{G}$ is also limit preserving on objects.*

Proof. Let $T = \varprojlim T_i$ be a projective limit of affine S-schemes, and suppose that we are given a morphism $T \to \mathcal{F}$ such that the composite $T \to \mathcal{F} \to \mathcal{G}$ factors through T_i for some i. We must show that there is a compatible factorisation of $T \to \mathcal{F}$ through $T_{i'}$ for some $i' \geq i$.

Since the composite $\mathcal{F} \to \mathcal{H}$ is limit preserving on objects, we may factor $T \to \mathcal{F}$ through some T_j, in such a way that the composites $T \to T_j \to \mathcal{F} \to \mathcal{H}$ and $T \to T_i \to \mathcal{G} \to \mathcal{H}$ coincide. Replacing i, j by some common $i'' \geq i, j$, we have two

morphisms $T_{i''} \to \mathcal{G}$ (one coming from the given morphism $T_i \to \mathcal{G}$, and one from the composite $T_j \to \mathcal{F} \to \mathcal{G}$) which induce the same morphism to $\mathcal{H}$, and which agree when pulled-back to T. Since Δ is limit preserving on objects, they agree over some $T_{i'}$, for some $i' \geq i''$, as required. $\qquad\square$

2.1.8. Corollary. *If $\mathcal{F}$ and $\mathcal{G}$ are categories fibred in groupoids over S, both of which are limit preserving, then any morphism $\mathcal{F} \to \mathcal{G}$ is limit preserving on objects.*

Proof. This follows directly from Lemmas 2.1.5 and 2.1.7 (taking $\mathcal{H} = S$ in the latter). $\qquad\square$

The next lemma (which is essentially [LMB00, Prop. 4.15(i)]) explains why the condition of being limit preserving is sometimes referred to as being locally of finite presentation.

2.1.9. Lemma. *If $\mathcal{F}$ is an algebraic stack over S, then the following are equivalent:*

  (1) *$\mathcal{F}$ is limit preserving.*
  (2) *$\mathcal{F} \to S$ is limit preserving on objects.*
  (3) *$\mathcal{F}$ is locally of finite presentation over S.*

Proof. Note that (1) $\implies$ (2) by definition (since (2) is just the condition that the functor (2.1.2) be essentially surjective), while the equivalence of (2) and (3) is a special case of Lemma 2.3.16 below. (The reader may easily check that the present lemma is not used in the proof of that result, and so there is no circularity in appealing to it.)

It remains to show that (3) $\implies$ (1). Assuming that (3) holds, we claim that the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is locally of finite presentation. To see this, choose a smooth surjection $U \to \mathcal{F}$ whose source is a scheme (which exists because $\mathcal{F}$ is assumed algebraic). Since $U \to \mathcal{F}$ and $\mathcal{F} \to S$ are locally of finite presentation by assumption, we see that $U \to S$ is locally of finite presentation by [Sta, Tag 06Q3]. Then the diagonal $U \to U \times_S U$ is locally of finite presentation by [Sta, Tag 0464] and [Sta, Tag 084P], and therefore the composite

$$U \to U \times_S U \to \mathcal{F} \times_S U \to \mathcal{F} \times_S \mathcal{F}$$

is locally of finite presentation (note that the last two morphisms are base changes of the smooth morphism $U \to \mathcal{F}$). Factoring this morphism as

$$U \to \mathcal{F} \xrightarrow{\Delta} \mathcal{F} \times_S \mathcal{F},$$

(where Δ is representable by algebraic spaces, because $\mathcal{F}$ is algebraic), we see from [Sta, Tag 06Q9] that Δ is locally of finite presentation, as claimed. A similar argument shows that the double diagonal $\Delta_\Delta : \mathcal{F} \to \mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} \mathcal{F}$ is locally of finite presentation. Applying Lemma 2.3.16 below (or [Sta, Tag 06CX]), we find that each of Δ, Δ_Δ and the structure map $\mathcal{F} \to S$ is limit preserving on objects. It now follows from Lemma 2.1.5 that $\mathcal{F}$ is limit preserving, as required. $\qquad\square$

2.2. Remarks on Axiom [2]. Throughout this subsection, we assume that S is locally Noetherian, and that $\mathcal{F}$ is a category fibred in groupoids over S, which is a stack for the Zariski topology. We denote by $\widehat{\mathcal{F}}$ the restriction of $\mathcal{F}$ to the category of finite type Artinian local S-schemes.

We begin by discussing Axiom [2](a), which is the Rim–Schlessinger condition (RS). Consider pushout diagrams

$$
\begin{array}{ccc}
Y & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Z'
\end{array}
$$

of S-schemes, with the horizontal arrows being closed immersions. We have an induced functor

(2.2.1) $$\mathcal{F}(Z') \to \mathcal{F}(Y') \times_{\mathcal{F}(Y)} \mathcal{F}(Z).$$

2.2.2. **Definition.** We say that $\mathcal{F}$ satisfies condition (RS) if the functor (2.2.1) is an equivalence of categories whenever Y, Y', Z, Z' are finite type local Artinian S-schemes.

2.2.3. **Lemma.** *If $\mathcal{F}$ is an algebraic stack, then $\mathcal{F}$ satisfies* (RS).

Proof. This is immediate from [Sta, Tag 07WN]. □

The same condition appears under a different name, and with slightly different phrasing, in [HR19, Lem. 1.2]. We recall this, and some closely related notions that we will occasionally use.

2.2.4. **Definition.** Following [HR19], we say that $\mathcal{F}$ is $\mathbf{Art^{fin}}$-homogeneous (resp. $\mathbf{Art^{triv}}$-homogeneous, resp. $\mathbf{Art^{sep}}$-homogeneous, resp. $\mathbf{Art^{insep}}$-homogeneous) if the functor (2.2.1) is an equivalence of categories whenever Y, Z are local Artinian S-schemes of finite type over S (resp. with the induced extension of residue fields being trivial, resp. separable, resp. purely inseparable), and $Y \to Y'$ is a nilpotent closed immersion.

2.2.5. **Lemma.** *$\mathcal{F}$ satisfies* (RS) *if and only if it is $\mathbf{Art^{fin}}$-homogeneous.*

Proof. This is just a matter of comparing the two definitions. Precisely: a closed immersion of local Artinian schemes is automatically nilpotent. Conversely, a finite type nilpotent thickening of a local Artinian scheme is local Artinian, and the pushout of local Artinian schemes of finite type over S is also local Artinian of finite type over S. (Recall from [Sta, Tag 07RT] that if above we write $Y = \operatorname{Spec} A$, $Y' = \operatorname{Spec} A'$, $Z = \operatorname{Spec} B$, then the pushout $Z' = \operatorname{Spec} B'$ is just given by $B' = B \times_A A'$.) □

The following lemma relates the various conditions of Definition 2.2.4.

2.2.6. **Lemma.** *The condition of $\mathbf{Art^{fin}}$-homogeneity of Definition 2.2.4 implies each of $\mathbf{Art^{sep}}$-homogeneity and $\mathbf{Art^{insep}}$-homogeneity, and these conditions in turn imply $\mathbf{Art^{triv}}$-homogeneity. If $\mathcal{F}$ is a stack for the étale site, then conversely, $\mathbf{Art^{triv}}$-homogeneity implies $\mathbf{Art^{sep}}$-homogeneity, while $\mathbf{Art^{insep}}$-homogeneity implies $\mathbf{Art^{fin}}$-homogeneity. If $\mathcal{F}$ is furthermore a stack for the fppf site, then $\mathbf{Art^{triv}}$-homogeneity implies $\mathbf{Art^{fin}}$-homogeneity.*

Proof. This follows immediately from [HR19, Lem. 1.6] and its proof (see also the proof of [HR19, Lem. 2.6]). □

We now discuss Axiom [2](b). To begin, we recall some definitions from [Sta, Tag 06G7].

Fix a Noetherian ring Λ, and a finite ring map $\Lambda \to k$, whose target is a field. Let the kernel of this map be $\mathfrak{m}_\Lambda$ (a maximal ideal of Λ). We let $\mathcal{C}_\Lambda$ be the category whose objects are pairs (A, ϕ) consisting of an Artinian local Λ-algebra A and a Λ-algebra isomorphism $\phi : A/\mathfrak{m}_A \to k$, and whose morphisms are given by local Λ-algebra homomorphisms compatible with ϕ. Note that any such A is finite over Λ, and that the morphism $\Lambda \to A$ factors through $\Lambda_{\mathfrak{m}_\Lambda}$, so that we have $\mathcal{C}_\Lambda = \mathcal{C}_{(\Lambda_{\mathfrak{m}_\Lambda})}$ in an evident sense.

There are some additional categories, closely related to $\mathcal{C}_\Lambda$, that we will also consider. We let $\widehat{\mathcal{C}}_\Lambda$ denote the category of complete Noetherian local Λ-algebras A equipped with a Λ-algebra isomorphism $A/\mathfrak{m}_A \xrightarrow{\sim} k$, while we let pro-$\mathcal{C}_\Lambda$ denote the category of formal pro-objects from $\mathcal{C}_\Lambda$ in the sense of Section 1.5.4. If $(A_i)_{i \in I}$ is an object of pro-$\mathcal{C}_\Lambda$, then we form the actual projective limit $A := \varprojlim_{i \in I} A_i$, thought of as a topological ring (endowed with the projective limit topology, each A_i being endowed with its discrete topology). In this manner we obtain an equivalence of categories between pro-$\mathcal{C}_\Lambda$ and the category of topological pro-(discrete Artinian) local Λ-algebras equipped with a Λ-algebra isomorphism between their residue fields and k [Gro95, §A.5]. We will frequently identify an object of pro-$\mathcal{C}_\Lambda$ with the associated topological local Λ-algebra A. There is a fully faithful embedding of $\widehat{\mathcal{C}}_\Lambda$ into pro-$\mathcal{C}_\Lambda$ given by associating to any object A of the former category the pro-object $(A/\mathfrak{m}_A^i)_{i \geq 1}$. In terms of topological Λ-algebras, this amounts to regarding A as a topological Λ-algebra by equipping it with its $\mathfrak{m}_A$-adic topology.

2.2.7. *Remark.* We note that objects of pro-$\mathcal{C}_\Lambda$, when regarded as topological rings, are examples of *pseudo-compact* rings, in the sense of [Gab62]. In particular, any morphism of such rings has closed image, and induces a topological quotient map from its source onto its image; consequently, a homomorphism $A \to B$ of such rings is surjective if and only if it is induced by a compatible collection of surjective morphisms $A_i \to B_i$ for projective systems $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ of objects in $\mathcal{C}_\Lambda$. (See the discussion beginning on [Gab62, p. 390], especially the statement and proof of Lem. 1 and of Thm. 3.)

We have the usual notion of a category cofibred in groupoids over $\mathcal{C}_\Lambda$, for which see [Sta, Tag 06GJ]. The particular choices of $\mathcal{C}_\Lambda$ and categories cofibred in groupoids over $\mathcal{C}_\Lambda$ that we are interested in arise as follows (see [Sta, Tag 07T2] for more details). Let $\mathcal{F}$ be a category fibred in groupoids over S, let k be a field and let $\operatorname{Spec} k \to S$ be a morphism of finite type.

Let x be an object of $\mathcal{F}$ lying over $\operatorname{Spec} k$, and let $\operatorname{Spec} \Lambda \subseteq S$ be an affine open so that $\operatorname{Spec} k \to S$ factors as $\operatorname{Spec} k \to \operatorname{Spec} \Lambda \to S$, where $\Lambda \to k$ is finite. (For the existence of such a Λ, and the independence of $\mathcal{C}_\Lambda$ of the choice of Λ up to canonical equivalence, see [Sta, Tag 07T2].) Write $p : \mathcal{F} \to S$ for the tautological morphism. We then let $\widehat{\mathcal{F}}_x$ be the category whose:

   (1) objects are morphisms $x \to x'$ in $\mathcal{F}$ such that $p(x') = \operatorname{Spec} A$, with A an Artinian local ring, and the morphism $\operatorname{Spec} k \to \operatorname{Spec} A$ given by $p(x) \to p(x')$ corresponds to a ring homomorphism $A \to k$ identifying k with the residue field of A, and

(2) morphisms $(x \to x') \to (x \to x'')$ are commutative diagrams in $\mathcal{F}$ of the form

$$x' \longleftarrow x''$$
$$\nwarrow \quad \nearrow$$
$$x$$

Note that the ring A in (1) is an object of $\mathcal{C}_\Lambda$. Under the assumption that $\mathcal{F}$ satisfies (RS), $\widehat{\mathcal{F}}_x$ is a deformation category by [Sta, Tag 07WU]. By definition, this means that $\widehat{\mathcal{F}}_x(\mathrm{Spec}\, k)$ is equivalent to a category with a single object and morphism, and that $\widehat{\mathcal{F}}_x$ is cofibred in groupoids and satisfies a natural analogue of (RS) (more precisely, an analogue of $\mathbf{Art^{triv}}$-homogeneity).

The category $\widehat{\mathcal{F}}_x$ naturally extends to its completion, which by definition is the pro-category pro-$\widehat{\mathcal{F}}_x$, which is a category cofibred in groupoids over pro-$\mathcal{C}_\Lambda$. (Note that this is a more general definition than that of [Sta, Tag 06H3], which restricts the definition to $\widehat{\mathcal{C}}_\Lambda$.) There is a fully faithful embedding of $\widehat{\mathcal{F}}_x$ into its completion, which attaches to any object of $\widehat{\mathcal{F}}_x$ lying over an Artinian Λ-algebra the corresponding pro-object, and this embedding induces an equivalence between $\widehat{\mathcal{F}}_x$ and the restriction of its completion to $\mathcal{C}_\Lambda$. We therefore also denote the completion of $\widehat{\mathcal{F}}_x$ by $\widehat{\mathcal{F}}_x$. If A is an object of pro-$\mathcal{C}_\Lambda$, then we will usually denote an object of the completion of $\widehat{\mathcal{F}}_x$ lying over A by a morphism $\mathrm{Spf}\, A \to \widehat{\mathcal{F}}_x$.

We also introduce the notation $\mathcal{F}_x$ to denote the following category cofibred in groupoids over pro-$\mathcal{C}_\Lambda$: if A is an object of pro-$\mathcal{C}_\Lambda$, then $\mathcal{F}_x(A)$ denotes the groupoid consisting of morphisms $\mathrm{Spec}\, A \to \mathcal{F}$, together with an isomorphism between the restriction of this morphism to the closed point of $\mathrm{Spec}\, A$ and the given morphism $x : \mathrm{Spec}\, k \to \mathcal{F}$. If A is Artinian, then $\mathcal{F}_x(A) = \widehat{\mathcal{F}}_x(A)$. In general, there is a natural functor $\mathcal{F}_x(A) \to \widehat{\mathcal{F}}_x(A)$ (the functor of *formal completion*); a morphism $\mathrm{Spf}\, A \to \mathcal{F}$ lying in the essential image of this functor is said to be *effective*.

2.2.8. *Remark.* If A is an object of pro-$\mathcal{C}_\Lambda$, then we may consider the formal scheme $\mathrm{Spf}\, A$ as defining a sheaf of sets on the étale site of S, via the definition $\mathrm{Spf}\, A := \varinjlim_i \mathrm{Spec}\, A_i$ (writing A as the projective limit of its discrete Artinian quotients A_i, and taking the inductive limit in the category of étale sheaves; this is a special case of the Ind-constructions considered in Subsection 4.2 below). Of course, we may also regard the resulting sheaf $\mathrm{Spf}\, A$ as a stack (in setoids).

Giving a morphism $\mathrm{Spf}\, A \to \widehat{\mathcal{F}}_x$ in the sense described above is then equivalent to giving a morphism of stacks $\mathrm{Spf}\, A \to \mathcal{F}$ which induces the given morphism $x : \mathrm{Spec}\, k \to \mathcal{F}$ when composed with the natural morphism $\mathrm{Spec}\, k \to \mathrm{Spf}\, A$. This view-point is useful on occasion; for example, we say that $\mathrm{Spf}\, A \to \widehat{\mathcal{F}}_x$ is a *formal monomorphism* if the corresponding morphism of stacks $\mathrm{Spf}\, A \to \mathcal{F}$ is a monomorphism. (Concretely, this amounts to the requirement that the induced morphism $\mathrm{Spec}\, A_i \to \mathcal{F}$ is a monomorphism for each discrete Artinian quotient A_i of A.)

We now introduce the notion of a versal ring at the morphism x, which will be used in the definition of Axiom [2](b). (See Remark 2.2.10 for a discussion of why we speak of a versal ring at a morphism, rather than at a point.) As above, fix the finite type morphism $\mathrm{Spec}\, k \to S$, an affine open subset $\mathrm{Spec}\, \Lambda \to S$ through which this morphism factors, and the lift of this morphism to a morphism $x : \mathrm{Spec}\, k \to \mathcal{F}$.

2.2.9. **Definition.** Let A_x be a topological local Λ-algebra corresponding to an element of pro-$\mathcal{C}_\Lambda$. We say that a morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is *versal* if it is smooth, in the sense of [Sta, Tag 06HR], i.e. satisfies the infinitesimal lifting property with respect to morphisms in $\mathcal{C}_\Lambda$. More precisely, given a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} B \\
\downarrow & \nearrow & \downarrow \\
\operatorname{Spf} A_x & \longrightarrow & \widehat{\mathcal{F}}_x
\end{array}
$$

in which the upper arrow is the closed immersion corresponding to a surjection $B \to A$ in $\mathcal{C}_\Lambda$, and the left hand vertical arrow corresponds to a morphism in pro-$\mathcal{C}_\Lambda$ (equivalently, it is continuous when A_x is given its projective limit topology, and A is given the discrete topology) we can fill in the dotted arrow (with a morphism coming from a morphism in pro-$\mathcal{C}_\Lambda$) so that the diagram remains commutative.

We refer to A_x as a *versal ring* to $\mathcal{F}$ at the morphism x. We say that A_x is an *effective versal ring* to $\mathcal{F}$ at the morphism x if the morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is effective.

We say that $\mathcal{F}$ *admits versal rings at all finite type points* if there is a versal ring for every morphism $x : \operatorname{Spec} k \to \mathcal{F}$ whose source is a finite type $\mathcal{O}_S$-field. We say that $\mathcal{F}$ *admits effective versal rings at all finite type points* if there is an effective versal ring for every morphism $x : \operatorname{Spec} k \to \mathcal{F}$ whose source is a finite type $\mathcal{O}_S$-field.

Then Axiom [2](b) is by definition the assertion that $\mathcal{F}$ admits Noetherian effective versal rings at all finite type points.

2.2.10. *Remark.* One complication in verifying Axiom [2](b) is that, at least *a priori*, it does not depend simply on the finite type point of $\mathcal{F}$ represented by a given morphism $x : \operatorname{Spec} k \to \mathcal{F}$ (for a field k of finite type over $\mathcal{O}_S$), but on the particular morphism. More concretely, if we are given a finite type morphism $\operatorname{Spec} l \to \operatorname{Spec} k$, and if we let x' denote the composite $\operatorname{Spec} l \to \operatorname{Spec} k \to \mathcal{F}$, then it is not obvious that validity of [2](b) for either of x or x' implies the validity of [2](b) for the other. One of the roles of Axiom [2](a) in the theory is to bridge the gap between different choices of field defining the same finite type point of $\mathcal{F}$, and indeed one can show that in the presence of [2](a), the property of $\widehat{\mathcal{F}}_x$ admitting a Noetherian versal ring depends only on the finite type point of $\mathcal{F}$ underlying x; see part (1) of Lemma 2.8.7 below.

The problem of showing that effectivity is independent of the choice of representative of the underlying finite type point of $\mathcal{F}$ seems slightly more subtle, and to obtain a definitive result we have to make additional assumptions on $\mathcal{F}$, and possibly on S. This is the subject of the parts (2), (3), and (4) of Lemma 2.8.7.

In the case when the morphism $\operatorname{Spec} l \to \operatorname{Spec} k$ is separable, it is possible to make a softer argument to pass the existence of a versal ring from from x to x', without making any Noetherianness assumptions. We do this in Lemma 2.2.13; we firstly recall from [Sta, Tag 07W7,Tag 07WW] a useful formalism for considering such a change of residue field.

2.2.11. *Remark.* Let $\mathcal{F}$ be a category fibred in groupoids over S, and fix a morphism $x : \operatorname{Spec} k \to \mathcal{F}$, where k is a finite type $\mathcal{O}_S$-field. Suppose that we are given a finite

type morphism $\operatorname{Spec} l \to \operatorname{Spec} k$, so that l/k is a finite extension of fields, and let x' denote the composite $\operatorname{Spec} l \to \operatorname{Spec} k \to \mathcal{F}$. Write $\mathcal{C}_{\Lambda,k}$ for the category of local Artinian Λ-algebras with residue field k, and similarly write $\mathcal{C}_{\Lambda,l}$ for the category of local Artinian Λ-algebras with residue field l.

We let $(\widehat{\mathcal{F}}_x)_{l/k}$ denote the category cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$ defined by setting $(\widehat{\mathcal{F}}_x)_{l/k}(B) := \widehat{\mathcal{F}}_x(B \times_l k)$, for any object B of $\mathcal{C}_{\Lambda,l}$. If $\mathcal{F}$ satisfies [2](a), it follows from [Sta, Tag 07WX] that there is a natural equivalence of categories cofibred in groupoids $(\widehat{\mathcal{F}}_x)_{l/k} \xrightarrow{\sim} \widehat{\mathcal{F}}_{x'}$; an examination of the proof shows that if l/k is separable, the same conclusion holds if $\mathcal{F}$ is only assumed to be $\mathbf{Art^{sep}}$-homogeneous.

2.2.12. *Remark.* Recall [Sta, Tag 06T4] that we say that a Noetherian versal morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is *minimal* if whenever we can factor this morphism through a morphism $\operatorname{Spf} A \to \widehat{\mathcal{F}}_x$, the underlying map $A \to A_x$ is surjective. This notion is closely related to conditions on the tangent space of $\widehat{\mathcal{F}}_x$, in the following way.

By definition, the *tangent space* $T\widehat{\mathcal{F}}_x$ of $\widehat{\mathcal{F}}_x$ is the k-vector space $\widehat{\mathcal{F}}_x(k[\epsilon])$. As explained in [Sta, Tag 06I1], there is a natural action of $\operatorname{Der}_\Lambda(k,k)$ on $T\widehat{\mathcal{F}}_x$, and the versal morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ gives rise to a $\operatorname{Der}_\Lambda(k,k)$-equivariant morphism $d : \operatorname{Der}_\Lambda(A_x, k) \to T\widehat{\mathcal{F}}_x$.

By [Sta, Tag 06IR], a Noetherian versal morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is minimal provided that the morphism d is bijective on $\operatorname{Der}_\Lambda(k,k)$-orbits. Conversely, if $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous, then it follows from the proof of [Sta, Tag 06J7] that $\widehat{\mathcal{F}}_x$ satisfies the condition (S2) of [Sta, Tag 06HW], and it then follows from [Sta, Tag 06T8] that if $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is minimal, then d is bijective on $\operatorname{Der}_\Lambda(k,k)$-orbits.

2.2.13. **Lemma.** *Let $\mathcal{F}$ be a category fibred in groupoids over S which is $\mathbf{Art^{sep}}$-homogeneous. Suppose given $x : \operatorname{Spec} k \to \mathcal{F}$, with k a finite type $\mathcal{O}_S$-field, let l/k be a finite separable extension, and let x' denote the composite $\operatorname{Spec} l \to \operatorname{Spec} k \xrightarrow{x} \mathcal{F}$. Suppose also that $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is a versal ring to $\mathcal{F}$ at the morphism x, so that in particular A_x is a pro-Artinian local $\mathcal{O}_S$-algebra with residue field k. Let $A_{x'}/A_x$ denote the finite étale extension of A_x corresponding (via the topological invariance of the étale site) to the finite extension l of k, so that in particular $A_{x'}$ is a pro-Artinian local $\mathcal{O}_S$-algebra with residue field l.*

Then the induced morphism $\operatorname{Spf} A_{x'} \to \widehat{\mathcal{F}}_{x'}$ realises $A_{x'}$ as a versal ring to $\mathcal{F}$ at the morphism x'. If A_x is Noetherian, and the morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is minimal, then so is the morphism $\operatorname{Spf} A_{x'} \to \widehat{\mathcal{F}}_{x'}$.

Proof. By Remark 2.2.11, since $\mathcal{F}$ is $\mathbf{Art^{sep}}$-homogeneous and l/k is separable, we have a natural equivalence of groupoids $(\widehat{\mathcal{F}}_x)_{l/k} \xrightarrow{\sim} \widehat{\mathcal{F}}_{x'}$. Suppose given a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} B \\
\downarrow & \nearrow & \downarrow \\
\operatorname{Spf} A_{x'} & \longrightarrow & \widehat{\mathcal{F}}_{x'}
\end{array}
$$

in which the upper arrow is the closed immersion corresponding to a surjection $B \to A$ in $\mathcal{C}_{\Lambda,l}$; we wish to show that we can fill in the dotted arrow in such a way that resulting diagram remains commutative. Passing to the pushout with k over l,

and noting that the morphism $A_x \to A_{x'}$ factors through $A_{x'} \times_l k$, we obtain a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A \times_l k & \longrightarrow & \operatorname{Spec} B \times_l k \\
\downarrow & & \vdots \quad \searrow \\
\operatorname{Spf} A_{x'} \times_l k & \longrightarrow & \operatorname{Spf} A_x \longrightarrow \widehat{\mathcal{F}}_x
\end{array}
$$

where the dotted arrow exists by the versality of the morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$. Now consider the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} B \\
\downarrow & \nearrow & \downarrow \\
& \operatorname{Spec} B \times_l k & \\
\downarrow & & \downarrow \\
\operatorname{Spf} A_{x'} & \longrightarrow & \operatorname{Spf} A_x.
\end{array}
$$

Since $\operatorname{Spf} A_{x'} \to \operatorname{Spf} A_x$ is formally étale, we may fill in the dotted arrow so as to make the resulting diagram commutative. This dotted arrow also makes the original diagram commutative, so $A_{x'}$ is a versal ring to $\mathcal{F}$ at the morphism x'.

Finally, suppose that the versal morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is minimal. By Remark 2.2.12, the natural morphism $\operatorname{Der}_\Lambda(A_x, k) \to T\widehat{\mathcal{F}}_x$ is a bijection on $\operatorname{Der}_\Lambda(k, k)$-orbits, and we need to show that the natural morphism $\operatorname{Der}_\Lambda(A_{x'}, l) \to T\widehat{\mathcal{F}}_{x'}$ is a bijection on $\operatorname{Der}_\Lambda(l, l)$-orbits.

By [Sta, Tag 06I0, Tag 07WB] there is a natural isomorphism of l-vector spaces $T\widehat{\mathcal{F}}_x \otimes_k l \xrightarrow{\sim} T\widehat{\mathcal{F}}_{x'}$. Since $A_{x'}/A_x$ is étale (and l/k is separable), restriction induces isomorphisms $\operatorname{Der}_\Lambda(A_{x'}, k) \cong \operatorname{Der}_\Lambda(A_x, k)$ and $\operatorname{Der}_\Lambda(l, k) \cong \operatorname{Der}_\Lambda(k, k)$, and thus we also have natural isomorphisms of l-vector spaces $\operatorname{Der}_\Lambda(A_x, k) \otimes_k l \xrightarrow{\sim} \operatorname{Der}_\Lambda(A_{x'}, l)$ and $\operatorname{Der}_\Lambda(l, l) \xrightarrow{\sim} \operatorname{Der}_\Lambda(k, k) \otimes_k l$. One easily checks that the base-change by l over k of the morphism $d : \operatorname{Der}_\Lambda(A_x, k) \to T\widehat{\mathcal{F}}_x$ coincides, with respect to these identifications, with the morphism $d : \operatorname{Der}_\Lambda(A_{x'}, l) \to T\widehat{\mathcal{F}}_{x'}$, in a manner which identifies, under the isomorphism $\operatorname{Der}_\Lambda(k, k) \otimes_k l \cong \operatorname{Der}_\Lambda(l, l)$, the action of $\operatorname{Der}_\Lambda(k, k) \otimes_k l$ with the action of $\operatorname{Der}_\Lambda(l, l)$. The result follows. $\qquad \square$

We now engage in a slight digression; namely, we use the theory of versal rings in order to define the notion of the complete local ring at a finite type point of an algebraic space locally of finite type over S.

In order to motivate this concept, we recall first that quasi-separated algebraic spaces are decent, in the sense of [Sta, Tag 03I8], which is to say that any point of such an algebraic space X is represented by a quasi-compact monomorphism $x : \operatorname{Spec} k \to X$. Given a point x of a quasi-separated algebraic space, thought of as such a quasi-compact monomorphism, Artin and Knutson ([Art69b, Defn. 2.5], [Knu71, Thm. 6.4]) define a Henselian local ring of X at x; completing this local ring gives our desired complete local ring in this case. Rather than imposing a quasi-separatedness hypothesis at this point and appealing to these results, we adopt a slightly different approach, which will allow us to define a complete local ring at any finite type point x of a locally finite type algebraic space X over S.

(Note that throughout the discussion, we will maintain our assumption that S is locally Noetherian.)

To begin with, we note that finite type points always admit representatives that are monomorphisms (regardless of any separatedness hypothesis); indeed, we have the following lemma.

2.2.14. **Lemma.** *Any finite type point of an algebraic space X, locally of finite type over the locally Noetherian scheme S, admits a representative $\operatorname{Spec} k \to X$ which a monomorphism. This representative is unique up to unique isomorphism, the field k is a finite type $\mathcal{O}_S$-field, and any other representative $\operatorname{Spec} K \to X$ of the given point factors through this monomorphic representative in a unique fashion.*

Proof. We apply the criterion of part (1) of [Sta, Tag 03JU]. More precisely, we choose an étale morphism $U \to X$ with U a scheme over S, which will again be locally of finite type over S (since smooth morphisms are locally of finite type, and X is assumed to be locally of finite type over S). We choose a pair of finite type points $u, u' \in U$ lying over the given finite type point of X; as noted in the proof of the lemma just cited, we must verify that the underlying topological space of the scheme $u \times_X u'$ is finite. But since the diagonal map $X \to X \times_S X$ is a monomorphism (as X is an algebraic space), the fibre product $u \times_X u'$ maps via a monomorphism into $u \times_S u'$. This latter scheme has a finite underlying topological space (since u and u' are each the Spec of some finite type $\mathcal{O}_S$-field). Since monomorphisms induce embeddings on underlying topological spaces, we see that $u \times_X u'$ also has a finite underlying topological space. By the above cited lemma, this implies the existence of the desired monomorphism $\operatorname{Spec} k \to X$ representing the given point. Denote this monomorphism by x.

If $x' : \operatorname{Spec} K \to X$ is any other representative of the same point, then we may consider the base-changed morphism $\operatorname{Spec} k \times_X \operatorname{Spec} K \to \operatorname{Spec} K$, which is again a monomorphism. Since its source is non-empty (as x and x' represent the same point of X), and as its target is the Spec of a field, it must be an isomorphism; equivalently, the morphism x' must factor through x. Since x is a monomorphism, this factorisation is unique. Since K can be chosen to be a finite type $\mathcal{O}_S$-field, we see that k must in particular be a finite type $\mathcal{O}_S$-field. If x' is also a monomorphism, then we may reverse the roles of x and x', and so conclude that x is determined uniquely up to unique isomorphism. This completes the proof of the lemma. $\square$

The following proposition then constructs complete local rings at finite type points of locally finite type algebraic spaces over S.

2.2.15. **Proposition.** *If X is an algebraic space, locally of finite type over the locally Noetherian scheme S, and if $x : \operatorname{Spec} k \to X$ is a monomorphism, for some field k of finite type over $\mathcal{O}_S$, then there is an effective Noetherian versal ring A_x to X at the morphism x with the property that the corresponding morphism $\operatorname{Spf} A_x \to X$ is a formal monomorphism. Furthermore, the ring A_x, equipped with its morphism $\operatorname{Spec} A_x \to X$ inducing the given morphism x, is unique up to unique isomorphism. Finally, if A is any object of $\mathcal{C}_\Lambda$, then any morphism $\operatorname{Spec} A \to X$ factors uniquely through the morphism $\operatorname{Spec} A_x \to X$.*

Proof. Let A_x be a minimal versal ring to X at the morphism x, in the sense of [Sta, Tag 06T4] (which exists, and is Noetherian, by virtue of [Sta, Tag 06T5] and the fact that X, being an algebraic space, admits a Noetherian versal ring at

the morphism x; see e.g. Theorem 2.8.4 below). We will show that the morphism $\operatorname{Spf} A_x \to X$ is a formal monomorphism. To this end, we choose an étale surjective morphism $U \to X$ whose source is a scheme (such a morphism exists, since X is an algebraic space). It suffices (by [Sta, Tag 042Q], and the definition of a formal monomorphism in Remark 2.2.8) to show that the base-changed morphism $U \times_X \operatorname{Spf} A_x \to U$ is a formal monomorphism. We begin by describing this morphism more explicitly.

Abusing notation slightly, we write x to denote the point $\operatorname{Spec} k$, as well its monomorphism into X. The pull-back of U over the monomorphism $x \to X$ is then an étale morphism $U_x \to x$ with non-empty source. We may write U_x as a disjoint union of points u_i, each of which is of the form $u_i = \operatorname{Spec} l_i$, for some finite separable extension l_i of k. Since $U_x \to U$ is a monomorphism (being the base-change of a monomorphism), each of the composites $u_i \to U_x \to U$ is also a monomorphism; in other words, for each i, the field l_i is also the residue field of the image of u_i in U; in light of this, we identify u_i with its image in U.

For each i, let A_i denote the finite étale extension of A_x corresponding, via the topological invariance of the étale site, to the finite extension l_i/k. The projection $U \times_X \operatorname{Spf} A_x \to \operatorname{Spf} A_x$ is formally étale (being the pull-back of an étale morphism), and thus admits a natural identification with the morphism $\coprod_{i \in I} \operatorname{Spf} A_i \to \operatorname{Spf} A_x$. Since the points u_i of U are distinct for distinct values of i, in order to show that $\coprod_{i \in I} \operatorname{Spf} A_i \cong U \times_X \operatorname{Spf} A_x \to U$ is a formal monomorphism, it suffices to show that each of the individual morphisms $\operatorname{Spf} A_i \to U$ is a formal monomorphism; and we now turn to doing this.

For each i, we let $\mathcal{C}_{\Lambda, l_i}$ denote the category of local Artinian $\mathcal{O}_S$-algebras with residue field l_i. We write x_i to denote the composite $\operatorname{Spec} l_i = u_i \to U \to X$. The infinitesimal lifting property for the étale morphism $U \to X$ shows that the induced morphism $\widehat{U}_{u_i} \to \widehat{X}_{x_i}$ is an equivalence of categories cofibred in groupoids over $\mathcal{C}_{\Lambda, l_i}$. Lemma 2.2.13 shows that A_i is a versal ring to X at x_i, and, by the noted equivalence, it is thus also a versal ring to U at u_i. In fact, since A_x is a minimal versal ring at x, each of the rings A_i is a minimal versal ring at u_i.

On the other hand, since u_i is a point of the scheme U, the complete local ring $\widehat{\mathcal{O}}_{U, u_i}$ is a minimal versal ring to U at u_i; thus we may identify A_i with $\widehat{\mathcal{O}}_{U, u_i}$, so that the morphism $\operatorname{Spf} A_i \to U$ is identified with the canonical morphism $\operatorname{Spf} \widehat{\mathcal{O}}_{U, u_i} \to U$. This latter morphism is a formal monomorphism, and thus so is the former.

The fact that the morphism $\operatorname{Spf} A_x \to X$ is effective is a consequence of X being an algebraic space; see [Sta, Tag 07X8]. The uniqueness of this morphism, up to unique isomorphism, follows from Lemma 2.2.16 below; the fact that its effectivisation is unique up to unique isomorphism again follows from [Sta, Tag 07X8].

Since A_x is versal to X at x, any morphism $\operatorname{Spec} A \to X$, for A an object of $\mathcal{C}_\Lambda$, factors through the morphism $\operatorname{Spf} A_x \to X$. Of course, it then factors through the induced morphism $\operatorname{Spec} B \to X$, for some Artinian quotient B of A_x, and hence also through the morphism $\operatorname{Spec} A_x \to X$. The uniqueness of this factorisation again follows from Lemma 2.2.16 below. $\qquad\square$

2.2.16. **Lemma.** *Let $\mathcal{F}$ be a category fibred in groupoids, and suppose that $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is a versal morphism at the morphism $x : \operatorname{Spec} k \to \mathcal{F}$, where k is a finite type $\mathcal{O}_S$-field. Suppose also that $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is a formal monomorphism.*

Then if A is any object of $\mathcal{C}_\Lambda$, any morphism $\operatorname{Spec} A \to \widehat{\mathcal{F}}_x$ factors uniquely through the morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$. Furthermore, the ring A_x, together with the morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$, is uniquely determined up to unique isomorphism by the property of being a versal formal monomorphism.

Proof. Since the morphism $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is versal by assumption, any morphism $\operatorname{Spec} A \to \widehat{\mathcal{F}}_x$ factors through this morphism; that this factorisation is unique is immediate from the definition of a formal monomorphism. If $\operatorname{Spf} A'_x \to \widehat{\mathcal{F}}_x$ is another versal formal monomorphism, then applying this property to the discrete Artinian quotients of A'_x, and then reversing the roles of A_x and A'_x, we find the required unique isomorphisms. $\qquad\square$

2.2.17. **Definition.** We refer to the ring A_x of Proposition 2.2.15 as the *complete local ring of X at the point x.*

In Subsection 4.2 below, we generalise the notion of the complete local ring at a point to certain Ind-algebraic spaces; see Definition 4.2.13. We now state and prove a result which will be used in Section 3.3, and which uses this generalisation.

2.2.18. **Lemma.** *If $\mathcal{F}$ admits versal rings at all finite type points (in the sense of Definition 2.2.9), and if $\mathcal{F}' \to \mathcal{F}$ is representable by algebraic spaces and locally of finite presentation, then $\mathcal{F}'$ admits versal rings at all finite type points. In fact, if $x' : \operatorname{Spec} k \to \mathcal{F}'$ is a morphism from a finite type $\mathcal{O}_S$-field to $\mathcal{F}'$, inducing the morphism $x : \operatorname{Spec} k \to \mathcal{F}$, and if $\operatorname{Spf} A \to \mathcal{F}$ is a versal ring to $\mathcal{F}$ at x, then $X := \mathcal{F}' \times_{\mathcal{F}} \operatorname{Spf} A$ is defined as an Ind-locally finite type algebraic space over S (in the sense of Definition 4.2.11 below), the morphism x' induces a lift of x to X, and the complete local ring of X at x is a versal ring to $\mathcal{F}'$ at x'.*

Proof. If we write $A \cong \varprojlim_{i \in I} A_i$ as a projective limit of finite type local Artinian $\mathcal{O}_S$-algebras, then we define $X := \varinjlim_{i \in I} \mathcal{F}' \times_{\mathcal{F}} \operatorname{Spec} A_i$; thus X is an Ind-locally finite type algebraic space over S, which is clearly well-defined (as a sheaf of setoids on the étale site of S) independently of the choice of description of A as a projective limit. The only claim, then, that is not immediate from the definitions is that the composite morphism $\operatorname{Spf} \mathcal{O}_{X,x} \to X \to \mathcal{F}'$ is versal. We leave this as an easy exercise for the reader; it is essentially immediate from the versality of $\operatorname{Spf} A$. $\quad\square$

We now introduce the notion of a presentation of a deformation category by an effectively Noetherianly pro-representable smooth groupoid in functors, which is closely related to the notion of admitting an effective Noetherian versal ring. Our reason for introducing this notion is to prove Lemma 2.2.24 and Corollary 2.7.3; under an appropriate hypothesis on the diagonal of $\mathcal{F}$, these will enable us to deduce Axiom [2](a) from [2](b).

We say that a set-valued functor on $\mathcal{C}_\Lambda$ is *pro-representable* if it representable by an object of pro-$\mathcal{C}_\Lambda$. If A is the associated topological ring to the representing pro-object, then we will frequently denote this functor by $\operatorname{Spf} A$. We say that a functor is *Noetherianly pro-representable* if it is pro-representable by an object A of $\widehat{\mathcal{C}}_\Lambda$.[1]

---

[1] In [Sta], what we call *Noetherian pro-representability* is called simply *pro-representability*; see [Sta, Tag 06GX]. However, we will need to consider more general pro-representable functors, and so we need to draw a distinction between the general case and the case of pro-representability by an object of $\widehat{\mathcal{C}}_\Lambda$.

We refer to [Sta, Tag 06K3] for the definition of a groupoid in functors over $\mathcal{C}_\Lambda$, and then make the following related definitions.

2.2.19. **Definition.** (1) We say that a groupoid in functors over $\mathcal{C}_\Lambda$, say (U, R, s, t, c), is *smooth* if $s, t : R \to U$ are smooth[2] ; equivalently, if the quotient morphism $U \to [U/R]$ is smooth.

(2) We say that (U, R, s, t, c) is (*Noetherianly*) *pro-representable* if U and R are each (*Noetherianly*) *pro-representable*.

A *presentation* of $\widehat{\mathcal{F}}_x$ is an equivalence $[U/R] \xrightarrow{\sim} \widehat{\mathcal{F}}_x$ of categories cofibred in groupoids over $\mathcal{C}_\Lambda$, where (U, R, s, t, c) is a groupoid in functors over $\mathcal{C}_\Lambda$. Suppose given such a presentation by a groupoid in functors that is Noetherianly pro-representable, in the sense of Definition 2.2.19, and let $A_x \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ be an object that pro-represents U. We then obtain an induced morphism

(2.2.20) $$\mathrm{Spf}\, A_x = U \to [U/R] \to \widehat{\mathcal{F}}_x.$$

2.2.21. **Definition.** We say that the given presentation is *effectively* Noetherianly pro-representable if the morphism (2.2.20) is effective, i.e. arises as the formal completion of a morphism $\mathrm{Spec}\, A_x \to \mathcal{F}$.

The existence of an effectively Noetherianly pro-representable presentation by a smooth groupoid in functors is closely related to the property of having effective versal rings, as we will now see.

2.2.22. **Lemma.** *If $[U/R] \xrightarrow{\sim} \widehat{\mathcal{F}}_x$ is a presentation of $\widehat{\mathcal{F}}_x$ by a smooth groupoid in functors, for which U is pro-representable by a topological local Λ-algebra A, then the morphism $\mathrm{Spf}\, A = U \to \widehat{\mathcal{F}}_x$ is versal. Conversely, if A is the topological local Λ-algebra corresponding to some element of pro-$\mathcal{C}_\Lambda$, and if $\mathrm{Spf}\, A \to \widehat{\mathcal{F}}_x$ is versal, then if we write $U := \mathrm{Spf}\, A$ and $R = U \times_{\widehat{\mathcal{F}}_x} U$, and s, t for the two projections $R \to U$, then (U, R, s, t) is a smooth groupoid in functors, and the natural morphism $[U/R] \to \widehat{\mathcal{F}}_x$ is an equivalence, and thus equips $\widehat{\mathcal{F}}_x$ with a presentation by a smooth groupoid in functors.*

Proof. Essentially by definition, if $[U/R] \xrightarrow{\sim} \widehat{\mathcal{F}}_x$ is a presentation of $\widehat{\mathcal{F}}_x$ by a smooth groupoid in functors, then the induced morphism $U \to [U/R] \xrightarrow{\sim} \widehat{\mathcal{F}}_x$ is smooth (in the sense of [Sta, Tag 06HR]), and so by definition is versal. The converse statement follows from [Sta, Tag 06L1]). $\square$

2.2.23. *Remark.* Suppose that $\widehat{\mathcal{F}}_x$ admits a presentation by a smooth Noetherianly pro-representable groupoid in functors. Then, if $U = \mathrm{Spf}\, A \to \widehat{\mathcal{F}}_x$ is a versal morphism with A an object of $\widehat{\mathcal{C}}_\Lambda$ (so that U is Noetherianly pro-representable), one finds that $R := U \times_{\widehat{\mathcal{F}}_x} U$ is also Noetherianly pro-representable (*cf.* the proof of [Sta, Tag 06L8]), and so the equivalence $[U/R] \xrightarrow{\sim} \widehat{\mathcal{F}}_x$ of Lemma 2.2.22 gives a particular smooth Noetherianly pro-representable presentation of $\widehat{\mathcal{F}}_x$.

In Lemma 2.7.2 below we will show that if the diagonal of $\mathcal{F}$ satisfies an appropriate hypothesis, and if we are given a versal morphism $U = \mathrm{Spf}\, A \to \widehat{\mathcal{F}}_x$ from a (not necessarily Noetherianly) pro-representable functor U, then (without any

---

[2]The term *smooth* is used here in the sense of [Sta, Tag 06HG]; i.e. we require the infinitesimal lifting property with respect to morphisms in $\mathcal{C}_\Lambda$. Other authors might use the term *versal* here, because the residue field is being held fixed.

a priori hypothesis that $\widehat{\mathcal{F}}_x$ admits a presentation by a smooth pro-representable groupoid in functors) the fibre product $R := U \times_{\widehat{\mathcal{F}}_x} U$ is also pro-representable, and thus (taking into account the isomorphism $[U/R] \xrightarrow{\sim} \widehat{\mathcal{F}}_x$ of Lemma 2.2.22) the existence of a versal ring to $\mathcal{F}$ at the morphism x will imply that $\widehat{\mathcal{F}}_x$ in fact admits a presentation by a pro-representable smooth groupoid in functors.

We close our discussion of Axiom [2] by stating a lemma that relates the existence of presentations by smooth pro-representable groupoids in functors to the conditions of Definition 2.2.4.

2.2.24. Lemma. *Suppose, for every morphism $x : \operatorname{Spec} k \to \mathcal{F}$, with k a finite type $\mathcal{O}_S$-field, that $\widehat{\mathcal{F}}_x$ admits a presentation by a pro-representable smooth groupoid in functors. Then $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous (in the sense of Definition 2.2.4).*

Proof. By [Sta, Tag 06KT], we need only check that a pro-representable functor is $\mathbf{Art^{triv}}$-homogeneous (or in the language of [Sta], satisfies (RS)). In the case of Noetherianly pro-representable functors, this is [Sta, Tag 06JB], and the proof in the general case is identical. $\qquad\square$

2.3. Remarks on Axiom [3]. Recall that a category fibred in groupoids $\mathcal{F}$ satisfies Axiom [3] if and only if the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable by algebraic spaces; equivalently (by [Sta, Tag 045G]), if and only if $X \times_{\mathcal{F}} Y$ is an algebraic space whenever $X \to \mathcal{F}, Y \to \mathcal{F}$ are morphisms from algebraic spaces X, Y. We begin with the following lemma.

2.3.1. Lemma. *Let $\mathcal{F}$ be a category fibred in groupoids satisfying Axiom [3]. If $\mathcal{X}$ and $\mathcal{Y}$ are categories fibred in groupoids satisfying [3], then for any morphisms of categories fibred in groupoids $\mathcal{X}, \mathcal{Y} \to \mathcal{F}$, the fibre product $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y}$ is again a category fibred in groupoids satisfying [3]. If $\mathcal{X}$ and $\mathcal{Y}$ are furthermore (algebraic) stacks, then the fibre product is also an (algebraic) stack.*

Proof. The claim for algebraic stacks is proved in [Sta, Tag 04TF]. (Note that, as stated, that result actually deals with stacks in the *fppf* topology; here, as explained in Section 1.5, we are applying the analogous result for the étale topology.) An examination of the proof of that result also gives the claim for categories fibred in groupoids satisfying [3]. The claim for stacks then follows from [Sta, Tag 02ZL]. $\quad\square$

Our next goal in this section is to extend the definition of certain properties of morphisms of algebraic stacks to morphisms of stacks $\mathcal{X} \to \mathcal{F}$ whose source is assumed algebraic, but whose target is merely assumed to satisfy [3]. To this end, we first note the following lemma.

2.3.2. Lemma. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{F}$ be morphisms of stacks, with $\mathcal{X}$ and $\mathcal{Y}$ algebraic stacks, and with $\mathcal{F}$ assumed to satisfy [3]. If P is a property of morphisms of algebraic stacks that is preserved under arbitrary base-change (by morphisms of algebraic stacks), then the morphism $\mathcal{X} \to \mathcal{Y}$ has the property P if and only if, for every morphism of stacks $\mathcal{Z} \to \mathcal{F}$ with $\mathcal{Z}$ being algebraic, the base-changed morphism $\mathcal{X} \times_{\mathcal{F}} \mathcal{Z} \to \mathcal{Y} \times_{\mathcal{F}} \mathcal{Z}$ has property P.*

Proof. The indicated base-change can be rewritten as the base-change of the morphism $\mathcal{X} \to \mathcal{Y}$ via the morphism $(\mathcal{Y} \times_{\mathcal{F}} \mathcal{Z}) \to \mathcal{Y}$. Since P is assumed to be preserved under arbitrary base-changes, we see that if $\mathcal{X} \to \mathcal{Y}$ has property P, so does the base-change $\mathcal{X} \times_{\mathcal{F}} \mathcal{Z} \to \mathcal{Y} \times_{\mathcal{F}} \mathcal{Z}$.

Conversely, suppose that all such base-changes have property P; then, in particular, the morphism $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{F}} \mathcal{Y}$ has property P. Thus so does the pull-back of this morphism via the diagonal $\Delta : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{F}} \mathcal{Y}$. This pull-back may be described by the usual "graph" Cartesian diagram (letting f denote the given morphism $\mathcal{X} \to \mathcal{Y}$)

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y} \\
{\scriptstyle \Gamma_f := \mathrm{id}_{\mathcal{X}} \times f} \downarrow & & \downarrow {\scriptstyle \Delta} \\
\mathcal{X} \times_{\mathcal{F}} \mathcal{Y} & \xrightarrow{\ f \times \mathrm{id}_{\mathcal{Y}}\ } & \mathcal{Y} \times_{\mathcal{F}} \mathcal{Y}
\end{array}
$$

from which we deduce that the original morphism f has property P. $\qquad\square$

2.3.3. *Example.* If $\mathcal{X} \to \mathcal{F}$ is a morphism of stacks, with $\mathcal{X}$ being algebraic and $\mathcal{F}$ satisfying [3], then we may apply the preceding lemma to the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$. (Note that the source of this morphism is an algebraic stack by assumption, and the target is an algebraic stack by Lemma 2.3.1.) If $\mathcal{Z} \to \mathcal{F}$ is any morphism of stacks with $\mathcal{Z}$ being algebraic, then (since the formation of diagonals is compatible with base-change), the base-change of the diagonal may be naturally identified with diagonal of the base-change

$$
\Delta : (\mathcal{X} \times_{\mathcal{F}} \mathcal{Z}) \to (\mathcal{X} \times_{\mathcal{F}} \mathcal{Z}) \times_{\mathcal{Z}} (\mathcal{X} \times_{\mathcal{F}} \mathcal{Z}).
$$

In particular, since the the properties of being representable by algebraic spaces, and of being locally of finite type, are preserved under any base-change, and hold for the diagonal of any morphism of algebraic stacks [Sta, Tag 04XS]), we see that $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$ is representable by algebraic spaces, and is locally of finite type. (Proposition 2.3.17 below generalises the first of these statements to the case when $\mathcal{X}$ is also assumed only to satisfy [3].)

We now make the following definition.

2.3.4. **Definition.** Assume that $\mathcal{F}$ is a stack satisfying Axiom [3]. Given an algebraic stack $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{F}$, and a property P of morphisms of algebraic stacks that is preserved under arbitrary base-change, then we say that $\mathcal{X} \to \mathcal{F}$ has property P, if and only if, for any algebraic stack $\mathcal{Y}$ equipped with a morphism $\mathcal{Y} \to \mathcal{F}$, the base-changed morphism $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y} \to \mathcal{Y}$ (which by Lemma 2.3.1 is a morphism of algebraic stacks) has property P.

2.3.5. *Remark.* In the spirit of [Sta, Tag 03YK], it might be better to restrict this definition to properties that are furthermore *fppf* local on the target. Since, in any case, we only apply the definition to such properties, we don't worry about this subtlety here.

2.3.6. *Remark.* We will apply the preceding definition to the following properties of morphisms of algebraic stacks:

- Locally of finite presentation (this property is smooth local on the source-and-target, so is defined by [Sta, Tag 06FN]).
- Locally of finite type (again, this is smooth local on the source-and-target, so is defined by [Sta, Tag 06FN]).
- Quasi-compact, [Sta, Tag 050U].
- Finite type, which as usual is defined to be locally of finite type and quasi-compact, [Sta, Tag 06FS].

- Universally closed, [Sta, Tag 0513].
- Surjective, [Sta, Tag 04ZS].
- Separated, i.e. having proper diagonal [Sta, Tag 04YW].
- Proper, which as usual we define to be separated, finite type, and universally closed.
- Representable by algebraic spaces, which is equivalent to the condition that the diagonal morphism be a monomorphism [Sta, Tag 0AHJ].
- Monomorphism [Sta, Tag 04ZW], which is equivalent to the condition that the diagonal be an isomorphism, or that the morphism, thought of as a functor between categories fibred in groupoids, is fully faithful [Sta, Tag 04ZZ]. Note that monomorphisms are necessarily representable by algebraic spaces.
- Closed immersion, [Sta, Tag 04YL]. (Closed immersions also admit an alternative characterisation as being the proper monomorphisms. To see this, note that by [Sta, Tag 045F] it is enough to prove the same statement for morphisms of algebraic spaces. Since closed immersions of algebraic spaces are representable by definition, and proper monomorphisms are representable by [Sta, Tag 0418], we reduce to the case of morphisms of schemes, which is [Gro67, 18.12.6].)
- Unramified, which is defined to be locally of finite type, with étale diagonal [Ryd11, Appendix B]. (Note that the diagonal morphism is representable by algebraic spaces [Sta, Tag 04XS], and so the notion of it being étale is defined by [Sta, Tag 04XB]. Note also that an unramified morphism is representable by algebraic spaces if and only if its diagonal is an open immersion, since open immersions of algebraic spaces are precisely the étale monomorphisms; see [Gro67, 17.9.1] for this statement in the context of morphisms of schemes, which implies the statement for algebraic spaces, because open immersions of algebraic spaces are representable by definition, and étale monomorphisms of algebraic spaces are representable by [Sta, Tag 0418].)

2.3.7. *Remark.* Some of the preceding properties, when interpreted via the mechanism of Definition 2.3.4, also admit a more direct interpretation. In particular, if $\mathcal{X} \to \mathcal{F}$ is a morphism of stacks with $\mathcal{X}$ being algebraic and $\mathcal{F}$ satisfying [3], then the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$ *is* a morphism of algebraic stacks, which is furthermore representable by algebraic spaces (as noted in Example 2.3.3), and so we know what it means for it to be proper, or étale (for example). The following lemma incorporates this, and some similar observations, to give more direct interpretations of some of the preceding properties.

2.3.8. **Lemma.** *Let $f : \mathcal{X} \to \mathcal{F}$ be a morphism of stacks, with $\mathcal{X}$ being algebraic and $\mathcal{F}$ satisfying [3].*

(1) *The morphism f is quasi-compact, in the sense of Definition 2.3.4, if and only if, for every morphism $\mathcal{Y} \to \mathcal{F}$ with $\mathcal{Y}$ a quasi-compact algebraic stack, the algebraic stack $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y}$ is quasi-compact.*

(2) *The morphism f is universally closed, in the sense of Definition 2.3.4, if and only if for every morphism $\mathcal{Y} \to \mathcal{F}$ with $\mathcal{Y}$ an algebraic stack, the induced morphism $|\mathcal{X} \times_{\mathcal{F}} \mathcal{Y}| \to |\mathcal{Y}|$ is closed.*

(3) *If P is a property which is preserved under arbitrary base-change, which is smooth local on the source-and-target, and which is fppf local on the*

target, then f satisfies P, in the sense of Definition 2.3.4, if and only if for some (or, equivalently, any) smooth cover $U \to \mathcal{X}$ of $\mathcal{X}$ by a scheme, the composite morphism $U \to \mathcal{F}$ (which is representable by algebraic spaces, since $\mathcal{F}$ satisfies [3]) satisfies condition P in the sense of [Sta, Tag 03YK].[3]

(4) *Let P' be a property of morphisms of algebraic stacks that are representable by algebraic spaces. Assume further that P' is preserved under arbitrary base-change, and let P be the property of morphisms of algebraic stacks defined by the requirement that the corresponding diagonal morphism should satisfy P'. Then f satisfies P, in the sense of Definition 2.3.4, if and only if the diagonal morphism $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$ satisfies P'.*

(5) *The morphism f is representable by algebraic spaces, in the sense of Definition 2.3.4 and Remark 2.3.6, if and only if it is representable by algebraic spaces in the usual sense, i.e. if and only if for any morphism $T \to \mathcal{F}$ with T a scheme, the base-change $\mathcal{X} \times_{\mathcal{F}} T$ is an algebraic space; and these conditions are equivalent in turn to the condition that the diagonal Δ_f be a monomorphism.*

(6) *If $f : \mathcal{X} \to \mathcal{F}$ is locally of finite type, in the sense of Definition 2.3.4, then $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$ is locally of finite presentation.*

Proof. Suppose that f is quasi-compact, in the sense of Definition 2.3.4. If $\mathcal{Z} \to \mathcal{F}$ is a morphism of stacks, with $\mathcal{Z}$ being algebraic and quasi-compact, then by definition $\mathcal{X} \times_{\mathcal{F}} \mathcal{Z} \to \mathcal{Z}$ is a quasi-compact morphism. Since $\mathcal{Z}$ is quasi-compact, it again follows by definition that $\mathcal{X} \times_{\mathcal{F}} \mathcal{Z}$ is a quasi-compact algebraic stack. Conversely, suppose that $\mathcal{X} \times_{\mathcal{F}} \mathcal{Z}$ is quasi-compact, for every morphism $\mathcal{Z} \to \mathcal{F}$ with $\mathcal{Z}$ being a quasi-compact algebraic stack. Let $\mathcal{Y} \to \mathcal{F}$ be any morphism of stacks with $\mathcal{Y}$ algebraic, and let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks with $\mathcal{Z}$ quasi-compact. Then the base-change

$$(\mathcal{X} \times_{\mathcal{F}} \mathcal{Y}) \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y} \times_{\mathcal{Y}} \mathcal{Z} \xrightarrow{\sim} \mathcal{Z}$$

may be naturally identified with the base-change $\mathcal{X} \times_{\mathcal{F}} \mathcal{Z} \to \mathcal{Z}$; in particular, $(\mathcal{X} \times_{\mathcal{F}} \mathcal{Y}) \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact. The base-changed morphism $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y} \to \mathcal{Y}$ is thus quasi-compact by definition. Since $\mathcal{Y} \to \mathcal{F}$ was arbitrary, we conclude that f is quasi-compact, in the sense of Definition 2.3.4; this proves (1).

The proofs of (2) and of the first claim of (5) proceed along identical lines to the proof of (1). To prove (3), suppose first that f has property P, in the sense of Definition 2.3.4. If $T \to \mathcal{F}$ is any morphism from a scheme to $\mathcal{F}$, and $U \to \mathcal{X}$ is any smooth surjection, then the base-changed morphism $U \times_{\mathcal{F}} T \to \mathcal{X} \times_{\mathcal{F}} T$ (which is naturally identified with the base-change of the morphism $\mathcal{U} \to \mathcal{X}$ by the morphism of algebraic stacks $\mathcal{X} \times_{\mathcal{F}} T \to \mathcal{X}$) is smooth, while the morphism $\mathcal{X} \times_{\mathcal{F}} T \to T$ satisfies P by assumption. Thus the composite $U \times_{\mathcal{F}} T \to T$ satisfies P (as P is assumed to be smooth local on the source-and-target), and so the morphism $U \to \mathcal{F}$ satisfies P in the sense of [Sta, Tag 03YK]. Conversely, if this latter condition holds, and if $T \to \mathcal{F}$ is a morphism whose source is a scheme, then the morphism $U \times_{\mathcal{F}} T \to T$ satisfies P. Since $U \times_{\mathcal{F}} T \to \mathcal{X} \times_{\mathcal{F}} T$ is smooth, and P is assumed to be smooth local on the source-and-target, we find that $\mathcal{X} \times_{\mathcal{F}} T \to T$ satisfies P. Now suppose that $\mathcal{Y} \to \mathcal{F}$ is any morphism of stacks whose source is algebraic, and let $T \to \mathcal{Y}$ be a smooth surjection. We have just seen that $\mathcal{X} \times_{\mathcal{F}} T \to T$ satisfies P.

---

[3]The assumption that P be *fppf* local on the target is included purely in order for the definition of [Sta, Tag 03YK] to apply to P.

Since P is smooth local on the source-and-target, and since each of the morphisms $\mathcal{X} \times_{\mathcal{F}} T \to \mathcal{X} \times_{\mathcal{F}} \mathcal{Y}$ and $T \to \mathcal{Y}$ are smooth and surjective (the latter by assumption and the former because it is naturally identified with the base-change of the latter by the projection $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y} \to \mathcal{Y}$), we find that $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y} \to \mathcal{Y}$ satisfies P, as required. Thus, by definition, the morphism f satisfies P, in the sense of Definition 2.3.4.

To prove (4), note that the formation of diagonals is compatible with base-change, after which (4) follows from Lemma 2.3.2. The second claim of (5) follows from (4), applied in the case when P' is the property of being a monomorphism (since [Sta, Tag 0AHJ] shows that the associated property P is then precisely the property of being representable by algebraic spaces).

Claim (6) also follows from Lemma 2.3.2, and the fact that the analogous claim holds for morphisms of algebraic stacks. (For lack of a reference, we now give a proof of this property; that is, we show that if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks which is locally of finite type, then Δ_f is locally of finite presentation. Choose a smooth surjection $V \to \mathcal{Y}$ from a scheme V. By [Sta, Tag 06Q8], it is enough to prove that the base change of Δ_f by $V \to \mathcal{Y}$ is locally of finite presentation. This base change is the diagonal of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$, so we may replace $\mathcal{X}$ by $\mathcal{X} \times_{\mathcal{Y}} V$ and $\mathcal{Y}$ by V, and therefore reduce to the case of a morphism $\mathcal{X} \to V$.

Now choose a smooth surjection from a scheme $U \to \mathcal{X}$. By [Sta, Tag 06Q9] it suffices to show that the composite $U \to \mathcal{X} \to \mathcal{X} \times_V \mathcal{X}$ is locally of finite presentation. Factoring this as the composite

$$U \to U \times_V U \to U \times_V \mathcal{X} \to \mathcal{X} \times_V \mathcal{X}$$

we see that it suffices to show that $U \to U \times_V U$ is locally of finite presentation; that is, we have reduced to the case of schemes, which is [Sta, Tag 0818].) $\square$

2.3.9. *Example.* Lemma 2.3.8 shows that a morphism of stacks $\mathcal{X} \to \mathcal{F}$, with $\mathcal{X}$ being algebraic and $\mathcal{F}$ satisfying [3], is proper, in the sense of Definition 2.3.4, if and only if the following properties hold: (i) for some (or, equivalently, any) smooth surjection $U \to \mathcal{X}$, with U a scheme, the composite morphism $U \to \mathcal{F}$ (which is a morphism of stacks representable by algebraic spaces) is locally of finite type in the sense of [Sta, Tag 03YK]; (ii) for any morphism of stacks $\mathcal{Y} \to \mathcal{F}$ with $\mathcal{Y}$ algebraic, the base-changed morphism $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y} \to \mathcal{Y}$ induces a closed morphism $|\mathcal{X} \times_{\mathcal{F}} \mathcal{Y}| \to |\mathcal{Y}|$; (iii) in the context of (ii), if $\mathcal{Y}$ is furthermore quasi-compact, then the base-changed algebraic stack $\mathcal{X} \times_{\mathcal{F}} \mathcal{Y}$ is quasi-compact; (iv) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$ is proper.

2.3.10. *Example.* Lemma 2.3.8 shows that a morphism of stacks $\mathcal{X} \to \mathcal{F}$, with $\mathcal{X}$ being algebraic and $\mathcal{F}$ satisfying [3], is unramified, in the sense of Definition 2.3.4, if and only if the following properties hold: (i) for some (or, equivalently, any) smooth surjection $U \to \mathcal{X}$, with U a scheme, the composite morphism $U \to \mathcal{F}$ (which is a morphism of stacks representable by algebraic spaces) is locally of finite type in the sense of [Sta, Tag 03YK]; (ii) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$ (which is *a priori* locally of finite presentation, by (i) and Lemma 2.3.8 (6)) is étale. The unramified morphism $\mathcal{X} \to \mathcal{F}$ is furthermore representable by algebraic spaces if the diagonal is in fact an open immersion (taking into account (5) of the preceding lemma, and the fact that open immersions are precisely the étale monomorphisms).

2.3.11. *Example.* Lemma 2.3.8 (4) shows that a morphism of stacks $f : \mathcal{X} \to \mathcal{F}$, with $\mathcal{X}$ being algebraic and $\mathcal{F}$ satisfying [3], is a monomorphism in the sense of

Definition 2.3.4, if and only if the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$ is an isomorphism. In turn, this is equivalent to the condition that f, thought of as a functor between categories fibred in groupoids, is fully faithful. Lemma 2.3.8 (5) shows that f is then in particular necessarily representable by algebraic spaces in the usual sense.

2.3.12. **Definition.** We say that a substack $\mathcal{F}'$ of $\mathcal{F}$ is a *closed* substack if the inclusion morphism $\mathcal{F}' \hookrightarrow \mathcal{F}$ is representable by algebraic spaces, and is a closed immersion in the sense of [Sta, Tag 03YK], i.e. has the property that for any morphism $T \to \mathcal{F}$ with T an affine scheme, the base-changed morphism $T \times_{\mathcal{F}} \mathcal{F}' \to T$ is a closed immersion of schemes.

2.3.13. **Lemma.** *Assume that S is locally Noetherian, and that $\mathcal{F}$ is a stack over S which satisfies [3], for which the morphism $\mathcal{F} \to S$ is limit preserving on objects. If $\mathcal{Z} \hookrightarrow \mathcal{F}$ is a closed immersion, then it is limit preserving on objects.*

Proof. By definition of what it means for $\mathcal{Z} \to \mathcal{F}$ to be a closed immersion, if $T \to \mathcal{F}$ is a morphism from an algebraic space, then the induced morphism $\mathcal{Z} \times_{\mathcal{F}} T \to T$ is a closed immersion. Since $\mathcal{F}$ satisfies [3], the source is an algebraic stack; but if T is a scheme, then in fact the source will be a scheme (being a closed substack of a scheme). In particular the morphism $\mathcal{Z} \to \mathcal{F}$ is representable by algebraic spaces, and so to check that it is limit preserving on objects, it suffices to check that it is locally of finite presentation [Sta, Tag 06CX]. For this, it suffices to check, in the preceding context, that if T is an affine scheme then $\mathcal{Z} \times_{\mathcal{F}} T \to T$ is locally of finite presentation.

If we write $T = \varprojlim_i T_i$ as the projective limit of finite type affine S-schemes T_i, then, since $\mathcal{F} \to S$ is limit preserving on objects and S is locally Noetherian, we may factor the morphism $T \to \mathcal{F}$ through one of the T_i, and hence reduce to the case when T is finite type over the locally Noetherian scheme S. But in this case T itself is Noetherian, and so the closed immersion $\mathcal{Z} \times_{\mathcal{F}} T \to T$ is in fact of finite presentation. This proves the lemma. $\qquad\square$

Our next goal is to state and prove a lemma which is a variant of [Sta, Tag 06CX] (which states that for a morphism between categories fibred in groupoids that is representable by algebraic spaces, being locally of finite presentation is equivalent to being limit preserving on objects). Before doing this, we recall some standard facts about descending finitely presented morphisms of affine schemes over S to morphisms of finitely presented affine schemes over S.

2.3.14. **Lemma.** *Suppose that T, T' are affine schemes over S, and that we are given a morphism of finite presentation $T' \to T$. Suppose that T may be written as a limit $\varprojlim T_i$ of affine schemes of finite presentation over S. Suppose also that we are given a morphism $T' \to T''$ with T'' a scheme locally of finite presentation over S.*

Then for some j we may find a factorisation of the given morphism $T' \to T''$ which fits into a commutative diagram

$$
\begin{array}{ccccc}
T' & \longrightarrow & T'_j & \longrightarrow & T'' \\
\downarrow & & \downarrow & & \\
T & \longrightarrow & T_j & &
\end{array}
$$

in which the square is Cartesian, T'_j is an affine scheme of finite presentation over S, and the morphism $T'_j \to T_j$ is of finite presentation. Furthermore, if $T' \to T$ is surjective, then $T'_j \to T_j$ can be taken to be surjective; and if $T' \to T$ is assumed to be étale (resp. fppf, resp. an open immersion), then $T'_j \to T_j$ can also be taken to be étale (resp. fppf, resp. an open immersion).

2.3.15. *Remark.* If S is quasi-separated, then the hypothesis that T may be written as a limit of affine schemes of finite presentation is automatically satisfied; see Theorem 2.5.1 below.

Proof of Lemma 2.3.14. By [Sta, Tag 01ZM], there is some j_0 and a morphism of finite presentation $T'_{j_0} \to T_{j_0}$ such that the pull-back of this morphism to T is the given morphism $T' \to T$. For $j \geq j_0$, set $T'_j := T_j \times_{T_{j_0}} T'_{j_0}$. Since products commute with limits, we have $T' = \varprojlim T'_j$.

Since T_{j_0} is affine, and $T'_{j_0} \to T_{j_0}$ is of finite presentation (hence quasi-compact and quasi-separated), it follows that T'_{j_0} is quasi-compact and quasi-separated. It then follows from [Sta, Tag 01ZN] that we may assume that T'_j is affine for all $j \geq j_0$. If $T' \to T$ is surjective (resp. étale, resp. *fppf*, resp. an open immersion), then by [Sta, Tag 07RR] (resp. [Sta, Tag 07RP], resp. [Sta, Tag 04AI,Tag 07RR], resp. [Sta, Tag 07RP,Tag 07RQ] (recall that open immersions are precisely the étale monomorphisms)) we may assume that $T'_j \to T_j$ is also surjective (resp. étale, resp. *fppf*, resp. an open immersion) for all $j \geq j_0$.

Since T'' is locally of finite presentation over S, it follows from [Sta, Tag 01ZC] that $T' \to T''$ factors through T'_j for some $j \geq j_0$, as required. $\square$

2.3.16. **Lemma.** *A morphism $\mathcal{X} \to \mathcal{F}$, where $\mathcal{X}$ is an algebraic stack and $\mathcal{F}$ is a stack satisfying* [3]*, is locally of finite presentation, in the sense of Definition* 2.3.4*, if and only if it is limit preserving on objects.*

Proof. Let $U \to \mathcal{X}$ be a smooth surjection from a scheme (which exists, since $\mathcal{X}$ is an algebraic stack). This morphism is representable by algebraic spaces (again, since $\mathcal{X}$ is an algebraic stack), and is locally of finite presentation (since it is smooth), and hence it is limit preserving on objects by [Sta, Tag 06CX]. Thus if the morphism $\mathcal{X} \to \mathcal{F}$ is limit preserving on objects, so is the composite $U \to \mathcal{F}$ [Sta, Tag 06CW]. Since $\mathcal{F}$ satisfies [3], this morphism is representable by algebraic spaces, and so we may apply [Sta, Tag 06CX] again to deduce that $U \to \mathcal{F}$ is locally of finite presentation. Hence the same is true of the morphism $\mathcal{X} \to \mathcal{F}$, by part (3) of Lemma 2.3.8.

Conversely, suppose that $\mathcal{X} \to \mathcal{F}$ is locally of finite presentation, in the sense of Definition 2.3.4. Suppose given a morphism $T \to \mathcal{X}$, for an affine S-scheme T, written as a projective limit $T = \varprojlim_i T_i$ of affine S-schemes T_i, and suppose further that the composite $T \to \mathcal{X} \to \mathcal{F}$ factors through one of the T_i. We must show that our given morphism $T \to \mathcal{X}$ factors in a compatible manner through $T_{i'} \to \mathcal{X}$ for some sufficiently large $i' \geq i$. Replacing $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{F}} T_i$ (which is an algebraic stack, since $\mathcal{F}$ satisfies [3]) and $\mathcal{F}$ by T_i, we may in fact assume that $\mathcal{F} = T_i$, which we do from now on.

Let $U \to \mathcal{X}$ be a smooth surjection, and write $R := U \times_{\mathcal{X}} U$, so that there is a natural isomorphism $[U/R] \xrightarrow{\sim} \mathcal{X}$. The assumption that $\mathcal{X} \to T_i$ is locally of finite presentation is by definition equivalent to supposing that $U \to T_i$ is locally of finite presentation.

Let $p_1, p_2 : R \rightrightarrows U$ denote the two projections. Consider the pull-backs U_T and R_T. There is a natural identification $R_T \xrightarrow{\sim} U_T \times_T U_T$, via which the base-changes of the natural projections p_i become identified with the natural projections $U_T \times_T U_T \rightrightarrows U_T$. There is a natural isomorphism of stacks $[U_T/R_T] \xrightarrow{\sim} T$.

We may find (for example by [Sta, Tag 055V]) an étale slice T' of U_T, i.e. a morphism $T' \to U_T$ for which the composite $T' \to U_T \to T$ is étale and surjective. We then define $T'' := T' \times_T T'$. We have the composite morphisms $T' \to U_T \to U$ and $T'' \to R_T \to R$, with respect to which the two projections $T'' \rightrightarrows T'$ are compatible with the two projections $p_i : R \to U$. Thus there is an induced morphism $[T'/T''] \to [U/R]$, which is naturally identified with our original morphism $T \to \mathcal{X}$.

Since étale morphisms are open, and since T is quasi-compact (being affine), we may replace T' by a quasi-compact open subscheme for which the induced morphism to T remains surjective. Finally, replacing T' by the disjoint union of the members of a finite affine open cover, we may in fact assume that T' is also affine. The morphism $T' \to T$ is then an affine morphism that is locally of finite presentation (being étale), and hence is actually of finite presentation. By Lemma 2.3.14 we may write the affine étale morphism $T' \to T$ as the projective limit of affine étale morphisms $T'_{i'} \to T_{i'}$ (starting from a sufficiently large value of i'). We write $T''_{i'} := T'_{i'} \times_{T_{i'}} T'_{i'}$.

Since $U \to T_i$ is locally of finitely presentation, as is $R \to T_i$, by applying [Sta, Tag 01ZM] and [Sta, Tag 07SJ] we find that we may factor the morphisms $T' \to U$ and $T'' \to R$ through $T'_{i'}$ and $T''_{i'}$, for some sufficiently large value of i', in such a manner that the projections $T''_{i'} \rightrightarrows T'_{i'}$ are compatible with the projections $R \rightrightarrows U$. Thus we obtain a morphism $[T'_{i'}/T''_{i'}] \to [U/R]$, which we may identify with the required morphism $T_{i'} \to \mathcal{X}$. $\square$

We close this section with some further propositions, the first of which generalises one of the observations of Example 2.3.3 to the case when both source and target are assumed merely to satisfy [3].

2.3.17. Proposition. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism between categories fibred in groupoids satisfying [3]. Then $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by algebraic spaces.*

Proof. The proof is essentially identical to that of [Sta, Tag 04XS]. Let $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism from a scheme T; by definition, this is the data of a triple (x, x', α) where x, x' are objects of $\mathcal{X}$ over T, and $\alpha : f(x) \to f(x')$ is a morphism in the fibre category of $\mathcal{Y}$ over T. Since $\mathcal{X}, \mathcal{Y}$ satisfy [3], the sheaves $\mathrm{Isom}_{\mathcal{X}}(x, x')$ and $\mathrm{Isom}_{\mathcal{Y}}(f(x), f(y))$ are algebraic spaces over T by [Sta, Tag 045G]. The morphism α corresponds to a section of the morphism $\mathrm{Isom}_{\mathcal{Y}}(f(x), f(x')) \to T$.

If $T' \to T$ is a morphism of schemes, then we see that a T'-valued point of $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} T$ is by definition an isomorphism $x|_{T'} \xrightarrow{\sim} x'|_{T'}$ whose image under f is $\alpha|_{T'}$. Putting this together, we see that there is a fibre product diagram of sheaves over T

(2.3.18)
$$
\begin{array}{ccc}
\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} T & \longrightarrow & \mathrm{Isom}_{\mathcal{X}}(x, x') \\
\downarrow & & \downarrow \\
T & \xrightarrow{\ \alpha\ } & \mathrm{Isom}_{\mathcal{Y}}(f(x), f(x'))
\end{array}
$$

Thus $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} T$ is an algebraic space, as required. $\square$

2.3.19. **Proposition.** *If $\mathcal{X}$ is a category fibred in groupoids satisfying* [3]*, and* $\Delta :$ $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ *is limit preserving on objects, then* $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X}$ *is representable by algebraic spaces and limit preserving on objects.*

Proof. By Proposition 2.3.17, Δ_Δ is representable by algebraic spaces. (It follows from Lemma 2.3.1 that $\mathcal{X} \times_S \mathcal{X}$ satisfies [3], since $\mathcal{X}$ does, so that the proposition does indeed apply.) By [Sta, Tag 06CX], we see that Δ is locally of finite presentation, and that to show that Δ_Δ is limit preserving on objects, it is equivalent to show that it is locally of finite presentation. Consider the diagram (2.3.18) in the case that f is Δ; we must show that for all choices of T, the left hand vertical arrow is locally of finite presentation.

It therefore suffices to show that the right hand vertical arrow is locally of finite presentation; but this is by definition the diagonal map

$$\mathrm{Isom}\,(x, x') \to \mathrm{Isom}\,(x, x') \times_S \mathrm{Isom}\,(x, x'),$$

which is locally of finite presentation by [Sta, Tag 084P] (note that since Δ is locally of finite presentation, so is $\mathrm{Isom}\,(x, x')$). $\qquad\square$

2.3.20. **Proposition.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids which is representable by algebraic spaces. Assume that $\mathcal{Y}$ satisfies* [3]*. Then:*

  (1) $\mathcal{X}$ satisfies [3]*.*
  (2) If $\mathcal{X} \to \mathcal{Y}$ and $\Delta_\mathcal{Y} : \mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$ are locally of finite presentation, then so is $\Delta_\mathcal{X} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$.
  (3) If $\mathcal{X} \to \mathcal{Y}$ is locally of finite presentation, and $\mathcal{Y}$ satisfies [1]*, then so does $\mathcal{X}$.*

Proof. Throughout the proof, we will freely make use of [Sta, Tag 06CX], the equivalence of being locally of finite presentation and being limit preserving on objects for morphisms representable by algebraic spaces. We begin with (1); we need to show that the morphism $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces. This may be factored as

$$\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X} \to \mathcal{X} \times_S \mathcal{X};$$

since the second morphism is a base change of $\mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$, it is enough to show that $\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is representable by algebraic spaces.

To this end, suppose that $T \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is a morphism whose source is an algebraic space. This induces a morphism $T \to \mathcal{Y}$, and we may consider the base change of our whole situation by this morphism. Writing $\mathcal{X}_T$ for $\mathcal{X} \times_\mathcal{Y} T$, we may reinterpret the given morphism $T \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ as a section of the morphism $\mathcal{X}_T \times_T \mathcal{X}_T \to T$, and it is then straightforward to see that we in fact have

(2.3.21) $$\mathcal{X} \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} T = \mathcal{X}_T \times_{\mathcal{X}_T \times_T \mathcal{X}_T} T.$$

Now, since $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, $\mathcal{X}_T$ is an algebraic space, so that $\mathcal{X}_T \times_{\mathcal{X}_T \times_T \mathcal{X}_T} T$ is an algebraic space, as required.

We now consider (2). We may factor $\Delta_\mathcal{X}$ as the composite

$$\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}.$$

Since the second arrow is a base change of $\Delta_\mathcal{Y}$, which is locally of finite presentation by assumption, it follows from [Sta, Tag 06CV,Tag 06CW] that it suffices to show that $\mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ is locally of finite presentation. By definition, we need to show that all the base changes of this morphism by morphisms $T \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ with source

an algebraic space are locally of finite presentation; examining (2.3.21), we see that we are reduced to the case that $\mathcal{Y} = T$ is an algebraic space, which is a special case of Lemma 2.3.8 (6).

To prove (3), by Lemma 2.1.5 we need to show that each of $\mathcal{X} \to S$, $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$, $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X}$ is limit preserving on objects; the same result also shows that both $\mathcal{Y} \to S$ and $\Delta_{\mathcal{Y}}$ are limit preserving on objects, so that $\Delta_{\mathcal{X}}$ is limit preserving on objects by (2). By Proposition 2.3.19, it is then enough to show that $\mathcal{X} \to S$ is limit preserving on objects; but this is immediate from [Sta, Tag 06CW], applied to the composite $\mathcal{X} \to \mathcal{Y} \to S$. □

2.3.22. **Lemma.** *If $\mathcal{F}' \to \mathcal{F}$ is a monomorphism of categories fibred in groupoids, and if $\Delta_{\mathcal{F}}$ is representable by algebraic spaces and locally of finite presentation, then $\Delta_{\mathcal{F}'}$ is also representable by algebraic spaces and locally of finite presentation.*

Proof. Since $\mathcal{F}' \to \mathcal{F}$ is a monomorphism, $\mathcal{F}' \to \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'$ is an equivalence, and therefore $\Delta_{\mathcal{F}'}$ is the base change of $\Delta_{\mathcal{F}}$ via the natural morphism $\mathcal{F}' \times_S \mathcal{F}' \to \mathcal{F} \times_S \mathcal{F}$. The result follows. □

2.4. **Remarks on Axiom [4].** We begin with some preliminary definitions and results related to the concept of smoothness of morphisms.

2.4.1. **Definition.** As in Section 2.3, we will use the notions of *unramified* and *étale* for morphisms between algebraic stacks that are not necessarily representable by algebraic spaces. We say that a morphism of algebraic stacks is unramified if it is locally of finite type and has étale diagonal, and that it is étale if it is unramified, flat, and locally of finite presentation [Ryd11, Appendix B]. Note that, although the definition of unramified morphism includes the condition that the diagonal be étale, there is no circularity, because diagonal morphisms are representable by algebraic spaces, and in this case the notation of étale is defined following [Sta, Tag 04XB]. By [Ryd11, Prop. B2] a morphism of algebraic stacks is étale if and only if it is smooth and unramified.

2.4.2. **Definition.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Then we say that f is *formally unramified* (resp. *formally smooth*, resp. *formally étale*) if for every affine $\mathcal{Y}$-scheme T, and every closed subscheme $T_0 \hookrightarrow T$ defined by a nilpotent ideal sheaf, the functor $\mathrm{Hom}_{\mathcal{Y}}(T, \mathcal{X}) \to \mathrm{Hom}_{\mathcal{Y}}(T_0, \mathcal{X})$ is fully faithful (resp. essentially surjective, resp. an equivalence of categories).

2.4.3. **Proposition.**   *(1) A morphism of algebraic stacks is smooth if and only if it is formally smooth and locally of finite presentation.*
   (2) A morphism of algebraic stacks is unramified if and only if it is formally unramified and locally of finite type.
   (3) A morphism of algebraic stacks is étale if and only if it is formally étale and locally of finite presentation.

Proof. See [Ryd11, Cor. B.9]. □

2.4.4. **Definition.** Assume that S is locally Noetherian, let $\mathcal{F}$ be a category fibred in groupoids over S, let U be a scheme locally of finite type over S equipped with a morphism $\varphi : U \to \mathcal{F}$, and let $u \in U$ be a point.

(1) We say that φ is *versal at u* if for any diagram

$$
\begin{array}{ccc}
Z_0 & \longrightarrow & U \\
\downarrow & & \downarrow{\scriptstyle\varphi} \\
Z & \longrightarrow & \mathcal{F},
\end{array}
$$

where Z_0 and Z are Artinian local schemes with the latter being a nilpotent thickening of the former, and where the closed point $z \in Z_0$ maps to u, inducing an isomorphism $\kappa(u) \cong \kappa(z)$, we may lift the morphism $Z \to \mathcal{F}$ to a morphism $Z \to U$.

(2) We say that φ is *formally smooth at u* if for any diagram as in (1) where Z_0 and Z are Artinian local schemes with the latter being a nilpotent thickening of the former, and where the closed point $z \in Z_0$ maps to u, inducing a finite extension $\kappa(u) \hookrightarrow \kappa(z)$, we may lift the morphism $Z \to \mathcal{F}$ to a morphism $Z \to U$.

(3) If φ is representable by algebraic spaces and locally of finite presentation, then we say that φ is *smooth at u* if for any finite type S-scheme X equipped with a morphism $X \to \mathcal{F}$ over S, there is an open neighbourhood U' of u in U such the base-change morphism $U' \times_{\mathcal{F}} X \to X$ is smooth.

(4) If φ is is representable by algebraic spaces and locally of finite presentation, then we say that φ is *smooth in a neighbourhood of u* if there exists a neighbourhood U' of u such that the restriction $\varphi_{|U'} : U' \to \mathcal{F}$ (which is again a morphism representable by algebraic spaces) is smooth.

2.4.5. *Remark.* If $u \in U$ is a finite type point, then φ is *versal at u* in the sense of (1) of the preceding definition if and only if the induced morphism $\mathrm{Spf}\,\widehat{\mathcal{O}}_{U,u} \to \mathcal{F}_{\varphi(u)}$ is versal in the sense discussed in Subsection 2.2 above. Note that we follow [Sta, Tag 07XF] in using the terminology *versal at u*, rather than *formally versal at u*, as some other sources (e.g. [HR19]) do.

Our definition of formal smoothness at a point follows that of [HR19, Def. 2.1]. In [Art69a, Def. 3.1], Artin defines the notion of formally étale at a point (for a morphism from a scheme to a functor), but his definition is not quite the obvious analogue of the definition of formal smoothness given here, in that he does not impose any condition on the degree of the extension of the residue field at the closed point of Z_0 over the residue field at u. The condition on the residue field that we impose (following [HR19]) makes it slightly easier to verify formal smoothness (see in particular Lemma 2.4.7 (2) below), and is harmless in practice. (Indeed, if $\mathcal{F}$ is a stack satisfying [1] and [3], then part (3) of Lemma 2.4.7 below shows that at finite type points the variant definition of formal smoothness, in which we impose no condition on the extension of residue fields, is equivalent to the definition given above.)

2.4.6. *Remark.* The notion of smoothness is defined for morphisms between algebraic stacks, or, more generally (via Definition 2.3.4), for morphisms from an algebraic stack to a stack satisfying [3]. Thus the notion of *smooth at a point* can naturally be extended to morphisms whose source is an algebraic stack, and whose target satisfies [3].

The notion of *versality at a point* or *formal smoothness at a point* of an algebraic stack is slightly more problematic to define, since a point of an algebraic stack is

defined as an equivalence class of morphisms from the spectrum of a field, and so we can't speak of the residue field at a point of a stack. In Definition 2.4.10, we *will* give a definition of formal smoothness at a point for a morphism whose source is an algebraic stack, under slightly restrictive conditions on the target of the morphism (which, however, will not be too restrictive for the applications of this notion that we have in mind). The key to making the definition work is Corollary 2.4.8, which shows (under suitable hypotheses) that formal smoothness at a point (for morphisms from a scheme) can be detected smooth locally.

The following lemma, which is essentially drawn from [HR19, §2], relates the various notions of Definition 2.4.4. We remark that part (3) of the lemma provides an analogue, for formal smoothness at a point, of Artin's [Art69a, Lem. 3.3], which provides a characterisation of morphisms from a scheme to a functor that are formally étale at a point.

2.4.7. **Lemma.** *Suppose that we are in the context of Definition 2.4.4, and that u is a finite type point.*

(1) In general, we have that $(4) \implies (3) \implies (2) \implies (1)$.
(2) If $\mathcal{F}$ satisfies (RS), *then* $(1) \iff (2)$; *i.e. φ is versal at u if and only if it is formally smooth at u.*
(3) Consider the following conditions:
(a) φ is formally smooth at u.
(b) For any finite type S-scheme X, and any morphism $X \to \mathcal{F}$ over S, the base-changed morphism $U \times_{\mathcal{F}} X \to X$ contains the fibre over u in its smooth locus.
(c) For any locally finite type algebraic stack $\mathcal{X}$ over S, and any morphism $\mathcal{X} \to \mathcal{F}$ over S, the base-changed morphism $U \times_{\mathcal{F}} \mathcal{X} \to \mathcal{X}$ contains the fibre over u in its smooth locus.
(d) For any algebraic stack $\mathcal{X}$ over S, and any morphism $\mathcal{X} \to \mathcal{F}$ over S, the base-changed morphism $U \times_{\mathcal{F}} \mathcal{X} \to \mathcal{X}$ contains the fibre over u in its smooth locus.
If φ is representable by algebraic spaces and locally of finite type, then conditions (a), (b), and (c) are equivalent. If furthermore $\mathcal{F}$ is a stack satisfying [1] *and* [3], *then all four of these conditions are equivalent.*
(4) If $\mathcal{F}$ is an algebraic stack which is locally of finite presentation over S, then conditions (1), (2), (3), and (4) of Definition 2.4.4 are equivalent.

Proof. That condition (2) of Definition 2.4.4 implies condition (1) is immediate, as is the fact that condition (4) implies condition (3). To see that condition (3) implies condition (2), note that if A is an Artinian local $\mathcal{O}_S$-algebra whose residue field is of finite type over $\mathcal{O}_S$, then $\operatorname{Spec} A$ is of finite type over S. Thus, if condition (3) holds and we are in the situation of condition (2), the base-changed morphism $U \times_{\mathcal{F}} Z \to Z$ is smooth in a neighbourhood of the image of the induced map $Z_0 \to U \times_{\mathcal{F}} Z$, and since smooth morphisms are formally smooth, we may find the desired lift $Z \to U$. Thus condition (3) of the definition implies condition (2). This completes the proof of part (1) of the present lemma.

Part (2) is [HR19, Lem. 2.3]; we recall the (short) argument. As explained in the previous paragraph, Z_0, Z are automatically of finite type over S. Let W_0 be the image of Z_0 in $\operatorname{Spec} \mathcal{O}_{U,u}$, and let W be the pushout of W_0 and Z over Z_0. Then

by (RS) we have a commutative diagram

$$
\begin{array}{ccccc}
Z_0 & \longrightarrow & W_0 & \longrightarrow & U \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} \\
Z & \longrightarrow & W & \longrightarrow & \mathcal{F},
\end{array}
$$

and versality at u allows us to lift the morphism $W \to \mathcal{F}$ to a morphism $W \to U$. The composite morphism $Z \to W \to U$ gives the required lifting.

We turn to proving (3). We first note that, if $\mathcal{F}$ is a stack satisfying [1] and [3], and $\varphi : U \to \mathcal{F}$ with U locally of finite type over S, then φ is representable by algebraic spaces, by [3], and is locally of finite type, by Lemma 2.6.3 (1) below.

Also, if $\mathcal{X} \to \mathcal{F}$ is a morphism from an algebraic stack to S, then $\mathcal{X}$ admits a smooth surjection $X \to \mathcal{X}$ whose source is a scheme (which is locally of finite type over S if $\mathcal{X}$ is, since smooth morphisms are locally of finite presentation), and, since smoothness can be tested smooth-locally on the source, we see that (c) or (d) for $\mathcal{X}$ is equivalent to the corresponding statement for X. Thus we need only consider the case when $\mathcal{X}$ is a scheme X from now on.

If $\mathcal{F}$ is a stack (on the étale site, and hence also on the Zariski site), and if $X \to \mathcal{F}$ is a morphism whose source is a scheme, then condition (d) may be checked Zariski locally on X, and thus the general case of (d) follows from the case when X is affine. If $\mathcal{F}$ furthermore satisfies [1], then we may factor $X \to \mathcal{F}$ through a finite type S-scheme, and thus assume that X is locally of finite type over S.

Thus we see that if $\mathcal{F}$ is a stack satisfying [1] and [3], then (d) follows from (c), while clearly (d) implies (c). Again, it is clear that (c) implies (b), and an argument essentially identical to the proof of (1) above shows that (b) implies (a). Thus it remains to show that (a) implies (c), for maps $X \to \mathcal{F}$ where X is a scheme locally of finite type over S, under the assumption that φ is representable by algebraic spaces and locally of finite type.

We must verify that the smooth locus of $U \times_{\mathcal{F}} X \to X$ contains the fibre over u. The projection $U \times_{\mathcal{F}} X \to X$ is locally of finite type, and hence $U \times_{\mathcal{F}} X$ is locally of finite type over S. As U is also locally of finite type over S, the projection $U \times_{\mathcal{F}} X \to U$ is locally of finite type as well. Since $U \times_{\mathcal{F}} X$ is an algebraic space, it admits an étale cover by a scheme V. Since smoothness may be checked smooth locally on the source, it suffices to show that the fibre of V over u is contained in the smooth locus of the composite $V \to U \times_{\mathcal{F}} X \to X$. Since $V \to U$ is locally of finite type, the fibre of V over u is a scheme locally of finite type over $\kappa(u)$, and so it suffices to show that every closed point of this fibre lies in the smooth locus of $V \to X$. By [Gro67, Prop. 17.14.2], it in fact suffices to show that this morphism is formally smooth at each of these points.

Let v be such a point, and suppose given a commutative diagram

$$
\begin{array}{ccc}
Z_0 & \longrightarrow & V \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}
$$

as in the definition of formal smoothness at v. We may fit this into the larger diagram

$$\begin{CD} Z_0 @>>> V @>>> U \\ @VVV @VVV @VVV \\ Z @>>> X @>>> \mathcal{F} \end{CD}$$

Our assumption that φ is formally smooth at u implies that we may lift the composite of the lower horizontal arrows to a morphism $Z \to U$ (it is here that we use the assumption that v is a closed point of the fibre over u, so that the residue field at v is a finite extension of the residue field at u), and hence to a morphism $Z \to U \times_{\mathcal{F}} X$. Since V is étale, and so in particular formally smooth, over $U \times_{\mathcal{F}} X$, we may then further lift this morphism to a morphism $Z \to V$, as required. This completes the proof that (a) implies (c).

It remains to prove part (4) of the lemma. Lemma 2.2.3 and part (2) show that conditions (1) and (2) of Definition 2.4.4 are equivalent when $\mathcal{F}$ is an algebraic stack. Taking into account the statement of part (1), it suffices to show that condition (2) of Definition 2.4.4 implies condition (4). Thus we suppose that $\varphi : U \to \mathcal{F}$ is formally smooth at the point $u \in U$. The equivalence between conditions (a) and (d) of part (3) of the present lemma (which holds, since $\mathcal{F}$ is an algebraic stack, locally of finite presentation over S, and thus satisfies [1], by Lemma 2.1.9, and [3], by definition) shows (taking $\mathcal{X} = \mathcal{F}$) that φ is smooth in a neighbourhood of u, as required. $\qquad\square$

As a corollary of Lemma 2.4.7 (3), we next show (under mild assumptions on the morphism φ) that formal smoothness at a point can be checked smooth locally on U.

2.4.8. Corollary. *Let $\varphi : U \to \mathcal{F}$ be a morphism (over S) whose source U is a locally of finite type S-scheme, and whose target $\mathcal{F}$ is a category fibred in groupoid, and suppose that φ is representable by algebraic spaces, and is locally of finite type.[4] If $u \in U$ is a finite type point, then the following are equivalent:*

(1) *φ is formally smooth at u.*
(2) *There is a smooth morphism of schemes $V \to U$ and a finite type point $v \in V$ mapping to u such that the composite $V \to U \to \mathcal{F}$ is formally smooth at v.*
(3) *For any smooth morphism of schemes $V \to U$, and any finite type point $v \in V$ mapping to u, the composite $V \to U \to \mathcal{F}$ is formally smooth at v.*

Proof. If $V \to U$ is a smooth morphism, mapping the finite type point $v \in V$ to u, and $X \to \mathcal{F}$ is a morphism from a finite type S-scheme X, then we consider the following commutative diagram.

$$\begin{CD} V \times_{\mathcal{F}} X @>>> U \times_{\mathcal{F}} X @>>> X \\ @VVV @VVV @VVV \\ V @>>> U @>>> \mathcal{F} \end{CD}$$

---

[4] As seen in the proof of Lemma 2.4.7 (3), these conditions on φ hold automatically if $\mathcal{F}$ is a stack satisfying [1] and [3].

If (1) holds, then by the equivalence of (a) and (b) in part (3) of Lemma 2.4.7, the projection $U \times_{\mathcal{F}} X \to X$ contains the fibre over u in its smooth locus. The fibre over v (with the respect to the projection $V \times_{\mathcal{F}} X \to V$) is contained in the base-change to V of this fibre over u, and since the morphism $V \times_{\mathcal{F}} X \to U \times_{\mathcal{F}} X$ is smooth, the smooth locus of the projection $V \times_{\mathcal{F}} X \to X$ contains the fibre over v in its smooth locus. Employing Lemma 2.4.7 again, we see that (1) implies (3).

Considering the same diagram, and using the facts that smooth morphisms are open, and that smoothness can be tested smooth-locally on the source, we see in the same way that (2) implies (1).

Clearly (3) implies (2) (e.g. by taking $V = U$ and $v = u$), so we are done. $\qquad \square$

2.4.9. **Corollary.** *Suppose that $\mathcal{F}$ is a stack over S satisfying [1] and [3], that $\mathcal{U}$ is an algebraic stack, locally of finite type over S, and that $\mathcal{U} \to \mathcal{F}$ is a morphism of stacks over S. If $u \in |\mathcal{U}|$ is a finite type point of $\mathcal{U}$, then the following are equivalent:*

    (1) There exists a smooth morphism $V \to \mathcal{U}$ whose source is a scheme, and a finite type point $v \in V$ mapping to u, such that the composite $V \to \mathcal{U} \to \mathcal{F}$ is formally smooth at v.

    (2) For any smooth morphism $V \to \mathcal{U}$ whose source is a scheme, and any finite type point $v \in V$ mapping to u, the composite $V \to \mathcal{U} \to \mathcal{F}$ is formally smooth at v.

Proof. Since $\mathcal{U}$ is an algebraic stack, there does exist a smooth surjection $V \to \mathcal{U}$ whose source is a scheme, and the finite type points of V are dense in the fibre over u. Thus if (2) holds, so does (1). Conversely, suppose that (1) holds, for some choice of V and v, and suppose that $V' \to \mathcal{U}$ is a smooth morphism from a scheme to $\mathcal{U}$, and that $v' \in V'$ is a finite type point mapping to u. We must show that $V' \to \mathcal{F}$ is formally smooth at v'.

Consider the fibre product $V \times_{\mathcal{U}} V'$; this is an algebraic space (since $\mathcal{U}$ is an algebraic stack, and so satisfies [3]), and so we may find an étale (and in particular smooth) surjection $W \to V \times_{\mathcal{U}} V'$ whose source is a scheme. Since v and v' both map to u, and since $W \to V \times_{\mathcal{F}} V'$ is surjective, we may find a finite type point $w \in W$ lying over both v and v'.

Since $\mathcal{F}$ is a stack satisfying [1] and [3], and since V and V' are each of finite type over S, the morphisms $V \to \mathcal{F}$ and $V' \to \mathcal{F}$ are both representable by algebraic spaces and locally of finite type. Thus, since $W \to V$ and $W \to V'$ are both smooth morphisms, we conclude from Corollary 2.4.8, first that $W \to \mathcal{F}$ is formally smooth at w, and then that $V' \to \mathcal{F}$ is formally smooth at v'. This completes the proof that (2) implies (1). $\qquad \square$

2.4.10. **Definition.** If $\mathcal{U}$ is an algebraic stack, locally of finite type over S, and $\mathcal{F}$ is a stack over S satisfying [1] and [3], then we say that a morphism $\varphi : \mathcal{U} \to \mathcal{F}$ of stacks over S is formally smooth at a finite type point $u \in |\mathcal{U}|$ if the equivalent conditions of Corollary 2.4.9 hold.

2.4.11. *Remark.* Suppose we have a commutative diagram of morphisms of stacks over S

$$
\begin{array}{ccc}
\mathcal{U}'' & \xrightarrow{\varphi'} & \mathcal{U}' \\
\downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\psi} \\
\mathcal{U} & \xrightarrow{\varphi} & \mathcal{F}
\end{array}
$$

in which $\mathcal{U}$, $\mathcal{U}'$, and $\mathcal{U}''$ are algebraic stacks, locally of finite type over S, and $\mathcal{F}$ satisfies [1] and [3]. Suppose further that ψ' and φ' are smooth. Let $u'' \in |\mathcal{U}''|$ be a finite type point, with images $u \in |\mathcal{U}|$ and $u' \in |\mathcal{U}'|$. Then it follows directly from the definition (and Corollary 2.4.9) that φ is formally smooth at u if and only if ψ is formally smooth at u'. (Both conditions hold if and only if the composite $\varphi \circ \psi' = \psi \circ \varphi'$ is formally smooth at u''.)

The following lemma extends those parts of Lemma 2.4.7 dealing with formal smoothness at a point to the case of morphisms whose source is an algebraic stack.

2.4.12. **Lemma.** *Let $\mathcal{U}$ be an algebraic stack, locally of finite type over S, and let $\mathcal{F}$ be a stack over S satisfying [1] and [3]. Let $\varphi : \mathcal{U} \to \mathcal{F}$ be a morphism of stacks over S. Let $u \in |\mathcal{U}|$ be a point of $\mathcal{U}$.*

(1) *The morphism φ is formally smooth at u if and only if, for every morphism $\mathcal{X} \to \mathcal{F}$ whose source is an algebraic stack, the base-changed morphism $\mathcal{U} \times_{\mathcal{F}} \mathcal{X} \to \mathcal{X}$ contains the fibre over u in its smooth locus.*

(2) *If $\mathcal{F}$ is also an algebraic stack, then φ is formally smooth at u if and only if it is smooth in a neighbourhood of u.*

Proof. Given our assumptions on $\mathcal{F}$, conditions (3)(a) and (3)(d) of Lemma 2.4.7 are equivalent. The present lemma follows in a straightforward manner, taking into account Definition 2.4.10, Lemma 2.4.7 (4) (which we can apply, because $\mathcal{F}$ is locally of finite presentation by Lemma 2.1.9), and the fact that smoothness for morphisms from an algebraic stack to $\mathcal{F}$ can be checked smooth-locally on the source. □

The preliminaries being dealt with, we now state Axiom [4], the condition of *openness of versality.* We assume that S is locally Noetherian, and that $\mathcal{F}$ is a category fibred in groupoids over S.

Axiom [4]. If U is a scheme locally of finite type over S, and $\varphi : U \to \mathcal{F}$ is versal at some finite type point $u \in U$, then there is an open neighbourhood $U' \subseteq U$ of u such that φ is versal at every finite type point of U'.

2.4.13. *Alternatives to axiom* [4]. Following Artin [Art69a] we introduce a pair of axioms, labelled [4a] and [4b], which are closely related to Axiom [4]. These axioms are not needed for our main results (and in particular for our applications to Galois representations in Section 5), but as we have modelled our treatment of Artin's representability theorem on [Art69a], we have followed Artin in introducing and discussing these variants on Axiom [4]. We will also make use of these axioms when discussing the various examples in Section 4.

In order to state [4a], we first introduce some notation. Suppose that A is a DVR, with field of fractions K. If A'_K denotes an Artinian local thickening of K, i.e. an Artinian local ring equipped with a surjection $A'_K \to K$ (which then induces an isomorphism between the residue field of A'_K and K), then we let A' denote the

preimage of A under the surjection $A'_K \to K$; it is a subring of A'_K, equipped with a surjection $A' \to A$, whose kernel is nilpotent.

We now state Axiom [4a]. In the statement, we suppose that S is locally Noetherian, and that $\mathcal{F}$ is a category fibred in groupoids over S.

Axiom [4a]. Let A be a DVR over $\mathcal{O}_S$, whose residue field is of finite type over $\mathcal{O}_S$, and let K denote the field of fractions of A. Then for any morphisms $\operatorname{Spec} A \to \mathcal{F}$, $\operatorname{Spec} A'_K \to \mathcal{F}$, with A'_K being an Artinian local thickening of K, for which the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} K & \longrightarrow & \operatorname{Spec} A \\
\downarrow & & \downarrow \\
\operatorname{Spec} A'_K & \longrightarrow & \mathcal{F}
\end{array}
$$

commutes, there exists a morphism $\operatorname{Spec} A' \to \mathcal{F}$ making the enlarged diagram

$$
\begin{array}{ccccc}
\operatorname{Spec} K & \longrightarrow & \operatorname{Spec} A & & \\
\downarrow & & \downarrow & \searrow & \\
\operatorname{Spec} A'_K & \longrightarrow & \operatorname{Spec} A' & \longrightarrow & \mathcal{F}
\end{array}
$$

commute.

2.4.14. *Remark.* Our formulation of Axiom [4a] is slightly different to Artin's, who only requires the condition of [4a] to hold for A that are essentially of finite type over $\mathcal{O}_S$. To explain this, note that the key role of [4a] in the theory is to make Lemma 2.6.7 below true, and that, in the proof of that lemma, we apply [4a] after taking a certain integral closure. If S is a *Nagata scheme* (see e.g. [Sta, Tag 033R]) then taking this integral closure keeps us in the world of essentially finite type $\mathcal{O}_S$-algebras, and so in the case of a Nagata base (which holds in Artin's setting, since his base S is assumed to be excellent) the argument only requires Artin's more limited form of [4a].

We also observe that if A is essentially of finite type over $\mathcal{O}_S$, then A' may be written as the inductive limit $A' = \varinjlim_i A'_i$, where A'_i runs over the essentially finite type $\mathcal{O}_S$-subalgebras of A' with the properties that $A'_i \to A$ is surjective, and that the localisation of A'_i at its generic point is equal to A'_K. Thus, if $\mathcal{F}$ also satisfies [1], then if we can find a morphism $\operatorname{Spec} A' \to \mathcal{F}$ making the diagram of Axiom [4a] commute, we may similarly find a morphism $\operatorname{Spec} A'_i \to \mathcal{F}$ making the diagram

$$
\begin{array}{ccccc}
\operatorname{Spec} K & \longrightarrow & \operatorname{Spec} A & & \\
\downarrow & & \downarrow & \searrow & \\
\operatorname{Spec} A'_K & \longrightarrow & \operatorname{Spec} A'_i & \longrightarrow & \mathcal{F}
\end{array}
$$

commute, for some (sufficiently large) value of i. In Artin's context, this allows an alternate phrasing of [4a], e.g. in the form of Axiom [4] of [Art69b].

Artin proves in [Art69a, Thm. 3.7] that every locally separated algebraic space satisfies (his formulation of) Axiom [4a]. In fact this is true of arbitrary algebraic stacks (with the more expansive formulation of the axiom given here).

2.4.15. **Lemma.** *Every algebraic stack satisfies Axiom [4a].*

Proof. Since $\operatorname{Spec} A'$ may be thought of as the pushout of $\operatorname{Spec} A$ and $\operatorname{Spec} A'_K$ over $\operatorname{Spec} K$, this is a special case of [Sta, Tag 07WN]. □

We can now state Axiom [4b]. We assume that S is locally Noetherian, and that $\mathcal{F}$ is a category fibred in groupoids over S.

Axiom [4b]. If $\varphi : U \to \mathcal{F}$ is a morphism whose source is an S-scheme, locally of finite type, and φ is smooth at a finite type point $u \in U$, then φ is smooth in a neighbourhood of u.

2.4.16. *Remark.* If we assume that $\varphi : U \to \mathcal{F}$ is representable by algebraic spaces and locally of finite type, then Lemma 2.4.7 (3) shows that being *formally smooth* at the finite type point u is equivalent to requiring that for each morphism $X \to \mathcal{F}$ from a finite type S-scheme, there is an open set containing the fibre over u in $U \times_{\mathcal{F}} X$ at which the projection to X is smooth. The morphism is *smooth at u* if this open set can in fact be taken to be the preimage of a neighbourhood of $u \in U$. Finally, Axiom [4b] holds precisely when this neighbourhood of u can be chosen independently of X.

2.4.17. *Remark.* In Corollary 2.6.11 below, following Artin [Art69a, Lem. 3.10], we show that if $\mathcal{F}$ is a stack satisfying [1], [2](a), and [3], and whose diagonal is furthermore quasi-compact, then Axioms [4a] and [4b] for $\mathcal{F}$ together imply Axiom [4] for $\mathcal{F}$. This in turn implies that in Artin's representability theorem (Theorem 2.8.4) we may replace Axiom [4] by the combination of Axioms [4a] and [4b], provided that we add to [3] the condition that the diagonal be quasi-compact (see Theorem 2.8.5).

2.4.18. *Remark.* As Artin notes [Art69a, p. 39], the two axioms [4a] and [4b] are quite different in nature. Axiom [4a] is finitary; for example, the fact that it holds for algebraic stacks immediately implies that it also holds for Ind-algebraic stacks. Axiom [4b] is not finitary in nature, and although it holds for algebraic stacks, it will typically not hold for Ind-algebraic stacks. (See Subsection 4.2 below for a further discussion of Ind-algebraic stacks and their comportment with regard to Artin's axioms.)

2.5. **Imposing Axiom [1] via an adjoint construction.** We want to be able to apply our results to stacks that are not necessarily limit preserving, and we will need a way to pass from such a stack to one which is limit preserving without losing too much information.

Fix a quasi-separated base-scheme S. Let $\operatorname{Aff}_{/S}$ denote the category of affine S-schemes, and let $\operatorname{Aff}_{\mathrm{pf}/S}$ denote the full subcategory of finitely presented affine S-schemes. The category $\operatorname{Aff}_{/S}$ is closed under the formation of filtering projective limits (see [Sta, Tag 01YX]). Let $\operatorname{pro-Aff}_{\mathrm{pf}/S}$ denote the formal pro-category of $\operatorname{Aff}_{\mathrm{pf}/S}$.

2.5.1. **Theorem.** *The formation of projective limits induces an equivalence*

$$\operatorname{pro-Aff}_{\mathrm{pf}/S} \xrightarrow{\sim} \operatorname{Aff}_{/S}.$$

Proof. This is a consequence of the theory of absolute Noetherian approximation developed in [TT90, Appendix C]; for convenience, in this proof we will refer to the treatment of this material in [Sta, Tag 01YT], which develops the relative versions of the statements of [TT90, Appendix C] that we need.

Note firstly that if X is an affine S-scheme, then X is in particular quasi-compact and quasi-separated, so that by [Sta, Tag 09MU], we may write $X = \varprojlim X_i$ as the limit of a directed system of schemes X_i of finite presentation over S, such that the transition morphisms are affine over S. By [Sta, Tag 01Z6], we may assume that the X_i are all affine. This proves that the purported equivalence is essentially surjective.

Let $f : X \to Y$ be a morphism of S-schemes with Y also affine. Then we can also write $Y = \varprojlim Y_j$ with Y_j a directed system of affine schemes of finite presentation over S. Then we have

$$
\begin{aligned}
\mathrm{Mor}_{\mathrm{pro\text{-}Aff}_{\mathrm{pf}/S}}\big(\varprojlim_i X_i, \varprojlim_j X_j\big) &= \varprojlim_j \varinjlim_i \mathrm{Mor}_{\mathrm{Aff}_{/S}}(X_i, Y_j) \\
&= \varprojlim_j \mathrm{Mor}_{\mathrm{Aff}_{/S}}(X, Y_j) \\
&= \mathrm{Mor}_{\mathrm{Aff}_{/S}}(X, Y),
\end{aligned}
$$

where the first equality is the definition of a morphism in $\mathrm{pro\text{-}Aff}_{\mathrm{pf}/S}$, the second is [Sta, Tag 01ZB], and the third is by the universal property corresponding to the statement that $Y = \varprojlim Y_j$. Thus the purported equivalence is fully faithful, as required. $\qquad\square$

Now let $\mathcal{F} \to \mathrm{Aff}_{\mathrm{pf}/S}$ be a category fibred in groupoids. Passing to the corresponding pro-categories, we obtain a category fibred in groupoids

$$
\mathrm{pro\text{-}}\mathcal{F} \to \mathrm{pro\text{-}}(\mathrm{Aff}_{\mathrm{pf}/S}) \xrightarrow{\sim} \mathrm{Aff}_{/S}.
$$

On the other hand, given a category $\mathcal{F}'$ fibred in groupoids over $\mathrm{Aff}_{/S}$, we may always restrict it to $\mathrm{Aff}_{\mathrm{pf}/S}$, to obtain a category $\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}$ fibred in groupoids over $\mathrm{Aff}_{\mathrm{pf}/S}$.

Since $\mathrm{Aff}_{/S}$ is closed under the formation of filtering projective limits, we see that the same is true of any category fibred in groupoids $\mathcal{F}'$ over $\mathrm{Aff}_{/S}$, and so for any such $\mathcal{F}'$, evaluating projective limits induces a functor

$$
(2.5.2) \qquad\qquad \mathrm{pro\text{-}}(\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}) \to \mathcal{F}'
$$

over $\mathrm{Aff}_{/S}$.

2.5.3. **Lemma.** *If $\mathcal{F}$ is a category fibred in groupoids over $\mathrm{Aff}_{\mathrm{pf}/S}$, then the natural embedding $\mathcal{F} \to (\mathrm{pro\text{-}}\mathcal{F})|_{\mathrm{Aff}_{\mathrm{pf}/S}}$ is an equivalence.*

Proof. This is formal: the fully faithful embedding $\mathrm{Aff}_{\mathrm{pf}/S} \to \mathrm{pro\text{-}Aff}_{\mathrm{pf}/S} \xrightarrow{\sim} \mathrm{Aff}_{/S}$ identifies the essential image of $\mathrm{Aff}_{\mathrm{pf}/S}$ with the subcategory of pro-systems that are isomorphic to a pro-system which is eventually constant. Similarly, the essential image of $\mathcal{F}$ in $(\mathrm{pro\text{-}}\mathcal{F})$ consists of those pro-objects that are isomorphic to a pro-system which is eventually constant, which by the previous remark are precisely the pro-objects lying over elements of $\mathrm{Aff}_{\mathrm{pf}/S}$. $\qquad\square$

2.5.4. **Lemma.** *If $\mathcal{F}'$ is a category fibred in groupoids over $\mathrm{Aff}_{/S}$, then $\mathcal{F}'$ is limit preserving if and only if the functor (2.5.2) is an equivalence.*

Proof. This is almost formal. (The non-formal ingredient is supplied by [Sta, Tag 01ZC].) $\qquad\square$

The previous two lemmas show that the formation of pro-$\mathcal{F}$ from $\mathcal{F}$ gives an equivalence between the 2-category of categories fibred in groupoids over $\mathrm{Aff}_{\mathrm{pf}/S}$ and the full subcategory of the 2-category of categories fibred in groupoids over $\mathrm{Aff}_{/S}$ consisting of objects satisfying [1].

The following lemma establishes some additional properties of this construction.

2.5.5. **Lemma.** (1) *If $\mathcal{F}$ is a category fibred in groupoids over $\mathrm{Aff}_{\mathrm{pf}/S}$, and $\mathcal{F}'$ is a category fibred in groupoids over $\mathrm{Aff}_{/S}$, then there is an equivalence of categories*

$$\mathrm{Mor}_{\mathrm{Aff}_{\mathrm{pf}/S}}(\mathcal{F}, \mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}) \xrightarrow{\sim} \mathrm{Mor}_{\mathrm{Aff}_{/S}}(\text{pro-}\mathcal{F}, \mathcal{F}').$$

(2) *If $\mathcal{F}$ is a stack for the Zariski site (resp. the étale site, resp. the fppf site) on $\mathrm{Aff}_{\mathrm{pf}/S}$, then pro-$\mathcal{F}$ is a stack for the Zariski site (resp. the étale site, resp. the fppf site) on $\mathrm{Aff}_{/S}$.*

(3) *If $\mathcal{F}'$ is a category fibred in groupoids over $\mathrm{Aff}_{/S}$ whose diagonal and double diagonal are both limit preserving on objects, then the functor (2.5.2) is fully faithful.*

(4) *If $\mathcal{F}'$ is a category fibred in groupoids over $\mathrm{Aff}_{/S}$ whose diagonal is limit preserving on objects and representable by algebraic spaces, then the functor (2.5.2) is fully faithful, and the diagonal of pro-$(\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}})$ is also representable by algebraic spaces.*

(5) *If S is locally Noetherian, and if $\mathcal{F}'$ is a category fibred in groupoids over S which admits versal rings at all finite type points, then pro-$(\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}})$ also admits versal rings at all finite type points.*

Proof. (1) Passing to pro-categories, and then composing with the functor

$$\text{pro-}(\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}) \to \mathcal{F}'$$

of (2.5.2), induces a functor

$$\mathrm{Mor}_{\mathrm{Aff}_{\mathrm{pf}/S}}(\mathcal{F}, \mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}) \to \mathrm{Mor}_{\mathrm{Aff}_{/S}}(\text{pro-}\mathcal{F}, \text{pro-}(\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}))$$
$$\to \mathrm{Mor}_{\mathrm{Aff}_{/S}}(\text{pro-}\mathcal{F}, \mathcal{F}'),$$

which is the required equivalence. A quasi-inverse is given by the functor

$$\mathrm{Mor}_{\mathrm{Aff}_{/S}}(\text{pro-}\mathcal{F}, \mathcal{F}') \to \mathrm{Mor}_{\mathrm{Aff}_{\mathrm{pf}/S}}((\text{pro-}\mathcal{F})|_{\mathrm{Aff}_{\mathrm{pf}/S}}, \mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}})$$
$$\xrightarrow{\sim} \mathrm{Mor}_{\mathrm{Aff}_{\mathrm{pf}/S}}(\mathcal{F}, \mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}),$$

obtained by first restricting to $\mathrm{Aff}_{\mathrm{pf}/S}$, and then taking into account the equivalence of Lemma 2.5.3.

(2) This follows by a standard limiting argument, which we recall. We need to show that all descent data is effective, and that the presheaves $\mathrm{Isom}(x, y)$ are sheaves. We being with the argument for descent data.

We must show that if T is affine, and $T' \to T$ is a Zariski (resp. étale, resp. *fppf*) cover of T, equipped with a morphism $T' \to \text{pro-}\mathcal{F}$ with descent data, then there is a map $T \to \text{pro-}\mathcal{F}$ inducing the given map $T' \to \text{pro-}\mathcal{F}$.

By definition, the map $T' \to \mathrm{pro}\text{-}\mathcal{F}$ factors as $T' \to T'' \to \mathrm{pro}\text{-}\mathcal{F}$, with T'' of finite presentation over S. By Lemma 2.3.14, we can find a commutative diagram

$$
\begin{array}{ccccc}
T' & \longrightarrow & T'_j & \longrightarrow & T'' \\
\downarrow & & \downarrow & & \\
T & \longrightarrow & T_j & &
\end{array}
$$

in which the square is Cartesian, and T'_j is an affine scheme of finite presentation over S. Furthermore (by the same lemma), if $T' \to T$ is a Zariski (resp. étale, resp. *fppf*) covering, then we may assume that the morphism $T'_j \to T_j$ is a Zariski (resp. étale, resp. *fppf*) covering.

Since $\mathcal{F}$ is assumed to be a stack, it is enough to show that (after possibly increasing j) the descent data for the morphism $T' \to \mathrm{pro}\text{-}\mathcal{F}$ arises as the base change of descent data for the morphism $T'_j \to \mathcal{F}$. This descent data is given by an isomorphism between the two maps $T' \times_T T' \to \mathrm{pro}\text{-}\mathcal{F}$ given by the two projections, which satisfies the cocycle condition on the triple intersection.

Now, an isomorphism between the two maps $T' \times_T T' \to \mathrm{pro}\text{-}\mathcal{F}$ is equivalent to factoring the induced map $T' \times_T T' \to \mathrm{pro}\text{-}\mathcal{F} \times_S \mathrm{pro}\text{-}\mathcal{F}$ through $\Delta_{\mathrm{pro}\text{-}\mathcal{F}}$. So, we have a morphism $T'_j \times_{T_j} T'_j \to \mathrm{pro}\text{-}\mathcal{F} \times_S \mathrm{pro}\text{-}\mathcal{F}$ which factors through the diagonal after pulling back to T, and we want to show that it factors through the diagonal after pulling back to some $T_{j'}$. This will follow immediately provided that $\Delta_{\mathrm{pro}\text{-}\mathcal{F}}$ is limit preserving on objects. Similarly, to deal with the cocycle condition, it is enough to show that the double diagonal $\Delta_{\Delta_{\mathrm{pro}\text{-}\mathcal{F}}}$ is limit preserving on objects. Since $\mathrm{pro}\text{-}\mathcal{F}$ is limit preserving by definition, the required limit preserving properties of the diagonal and double diagonal follow from Lemma 2.1.5.

We now turn to proving that if $x, y : T \rightrightarrows \mathrm{pro}\text{-}\mathcal{F}$ are two morphisms, then $\mathrm{Isom}\,(x, y)$ is a sheaf on the Zariski (resp. étale, resp. *fppf*) site of T. By definition, we may find a morphism $T \to T_j$, with T_j affine of finite presentation over S, and morphisms $x_j, y_j : T_j \to \mathcal{F}$ such that x and y are obtained as the pull-backs of x_j and y_j. We may further find a projective system $\{T_{j'}\}$ of affine S-schemes of finite presentation, having T_j as final object, and an isomorphism $T \xrightarrow{\sim} \varprojlim_{j'} T_{j'}$, inducing the given morphism $T \to T_j$; we then write $x_{j'}$ and $y_{j'}$ for the composites $T_{j'} \to T_j \overset{x_j, y_j}{\rightrightarrows} \mathcal{F}$. We recall (see Lemma 2.3.14) that the Zariski (resp. étale, resp. *fppf*) site of T is then naturally identified with the projective limit of the corresponding sites of the $T_{j'}$; by [AGV72, Thm. VI.8.2.3], the same is true of the corresponding topoi. In particular, the various Isom presheaves $\mathrm{Isom}\,(x_j, y_j)$, which by assumption are in fact sheaves on the Zariski (resp. étale, resp. *fppf*) sites of the T_j, form a projective system whose projective limit can be identified with a sheaf on the Zariski (resp. étale, resp. *fppf*) sites of the T. Unwinding the definitions, one furthermore finds that this projective limit sheaf is naturally isomorphic to $\mathrm{Isom}\,(x, y)$. Thus we find that $\mathrm{Isom}\,(x, y)$ is indeed a sheaf.

(3) To check that (2.5.2) is fully faithful, it suffices to check that, for any affine S-scheme T, if we write T as a projective limit $T = \varprojlim_i T_i$ of finitely presented affine S-schemes, then the functor (2.1.2) is fully faithful. As was already noted in the proof of Lemma 2.1.5, this follows from the assumption that the diagonal and double diagonal of $\mathcal{F}'$ are limit preserving on objects.

(4) Proposition 2.3.19 shows that our assumptions on $\mathcal{F}'$ imply that its double diagonal is also limit preserving on objects, and so it follows from part (3) that (2.5.2) is fully faithful. This in turn implies that the diagram

$$
\begin{array}{ccc}
\mathrm{pro\text{-}}(\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}) & \xrightarrow{\ \Delta\ } & \mathrm{pro\text{-}}(\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}) \times_S \mathrm{pro\text{-}}(\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}) \\
\downarrow & & \downarrow \\
\mathcal{F}' & \xrightarrow{\qquad \Delta \qquad} & \mathcal{F}' \times_S \mathcal{F}'
\end{array}
$$

is 2-Cartesian, and thus if the bottom arrow is representable by algebraic spaces, the same is true of the top arrow.

(5) Since finite type Artinian $\mathcal{O}_S$-algebras are objects of $\mathrm{Aff}_{\mathrm{pf}/S}$, we see that the functors $\mathcal{F}'$ and $\mathrm{pro\text{-}}\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}$ induce equivalent groupoids when restricted to such algebras. Thus the finite type points of $\mathcal{F}'$ and $\mathrm{pro\text{-}}\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}$ are in natural bijection, in the strong sense that for each finite type $\mathcal{O}_S$-field k there is a natural bijection between the morphisms $x : \operatorname{Spec} k \to \mathcal{F}'$ and the morphisms $x : \operatorname{Spec} k \to$ $\mathrm{pro\text{-}}\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}$, and a versal ring to $\mathcal{F}'$ at such a morphism is also a versal ring to $\mathrm{pro\text{-}}\mathcal{F}'|_{\mathrm{Aff}_{\mathrm{pf}/S}}$ at the same morphism. $\qquad\square$

Finally, we note the following basic result.

2.5.6. Lemma. *If $\mathcal{F}$ is a category fibred in groupoids over $\mathrm{Aff}_{\mathrm{pf}/S}$, and $\mathcal{F}'$ is a full subcategory fibred in groupoids, then $\mathrm{pro\text{-}}\mathcal{F}'$ is a full subcategory fibred in groupoids of $\mathrm{pro\text{-}}\mathcal{F}$.*

Proof. This is immediate from the definitions. $\qquad\square$

2.6. Stacks satisfying [1] and [3]. In this subsection we discuss some properties of stacks satisfying Artin's axioms [1] and [3].

2.6.1. Lemma. *If $\mathcal{F}$ satisfies [1] and [3], and $\mathcal{X}$ is an algebraic stack locally of finite presentation over S, then any morphism of S-stacks $\mathcal{X} \to \mathcal{F}$ is locally of finite presentation (in the sense of Definition 2.3.4).*

Proof. Lemma 2.3.8 shows that we may verify this after composing the morphism $\mathcal{X} \to \mathcal{F}$ with a smooth surjection $U \to \mathcal{X}$ whose source is a scheme, and thus may assume that $\mathcal{X}$ is in fact an S-scheme X, locally of finite presentation. It follows from [Sta, Tag 06CX] that $X \to S$ is limit preserving on objects, so that X is limit preserving (as it is a stack in setoids). It follows from Corollary 2.1.8 that $X \to \mathcal{F}$ is limit preserving on objects. Since this morphism is representable by algebraic spaces (as $\mathcal{F}$ satisfies [3]), applying [Sta, Tag 06CX] again yields the lemma. $\qquad\square$

2.6.2. Lemma. *If $\mathcal{F}$ satisfies [1] and [3], then the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is locally of finite presentation.*

Proof. Since $\mathcal{F}$ is limit preserving by assumption, Lemma 2.1.5 shows that $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is limit preserving on objects. Since Δ is representable by algebraic spaces by assumption, it is locally of finite presentation by [Sta, Tag 06CX]. $\qquad\square$

We now establish some simple results related to finiteness conditions in the case when S is locally Noetherian. We first recall that a morphism $\mathcal{X} \to S$ from an algebraic stack $\mathcal{X}$ to the locally Noetherian scheme S is locally of finite type if and only if it is locally of finite presentation (since both conditions may be verified after

composing this morphism with a smooth surjection from a scheme to $\mathcal{X}$, which reduces us to the case when $\mathcal{X}$ is itself a scheme, in which case the claim follows immediately from the hypothesis that S is locally Noetherian).

2.6.3. **Lemma.** *Suppose that $\mathcal{F}$ satisfies* [1] *and* [3], *that S is locally Noetherian, and that $\mathcal{X} \to \mathcal{F}$ is a morphism whose source is an algebraic stack. Then the morphism $\mathcal{X} \to \mathcal{F}$ is locally of finite type if and only if it is locally of finite presentation, and these conditions are in turn equivalent to the composite $\mathcal{X} \to \mathcal{F} \to S$ being locally of finite type (or, equivalently, locally of finite presentation, as noted above).*

Proof. If $\mathcal{X} \to S$ is locally of finite type, and so locally of finite presentation, then Lemma 2.6.1 shows that $\mathcal{X} \to \mathcal{F}$ is locally of finite presentation. And certainly, if $\mathcal{X} \to \mathcal{F}$ is locally of finite presentation then it is locally of finite type.

Next, we want to show that if $\mathcal{X} \to \mathcal{F}$ is locally of finite type, then it is locally of finite presentation. To this end, let T be an affine S-scheme. Since $\mathcal{F} \to S$ is limit preserving, and so in particular limit preserving on objects, we may factor any S-morphism $T \to \mathcal{F}$ as $T \to T' \to \mathcal{F}$, where T' is of finite presentation over S. (It follows from [Sta, Tag 09MV] that T may be written as the limit of such T'.) The base-changed morphism $\mathcal{X}_T \to T$ is obtained by pulling back the base-changed morphism $\mathcal{X}_{T'} \to T'$. This latter morphism is locally of finite type, and its target is of finite presentation over the locally Noetherian scheme S (and hence locally Noetherian itself), and so it is in fact locally of finite presentation. Thus the morphism $\mathcal{X}_T \to T$ is also locally of finite presentation, and since T and the morphism $T \to \mathcal{F}$ were arbitrary, we conclude that the morphism $\mathcal{X} \to \mathcal{F}$ is locally of finite presentation, as claimed.

It remains to show that if $\mathcal{X} \to \mathcal{F}$ is locally of finite presentation, then $\mathcal{X} \to S$ is locally of finite presentation. Morally, we would like to prove this by arguing that since $\mathcal{F}$ satisfies [1], the morphism $\mathcal{F} \to S$ is locally of finite presentation, and so conclude that the composite $\mathcal{X} \to S$ is locally of finite presentation. Unfortunately, while this is a valid argument if $\mathcal{F}$ is an algebraic stack, it does not apply in the generality we are considering here, where $\mathcal{F}$ is assumed simply to satisfy [1] and [3]. Indeed, in this level of generality, we haven't defined what it means for $\mathcal{F} \to S$ to be locally of finite presentation.

Thus we are forced to make a slightly more roundabout argument. Since $\mathcal{X} \to \mathcal{F}$ is locally of finite presentation, it is limit preserving on objects, by Lemma 2.3.16. The morphism $\mathcal{F} \to S$ is also limit preserving on objects (since $\mathcal{F}$ satisfies [1]), and hence the composite $\mathcal{X} \to S$ is limit preserving on objects [Sta, Tag 06CW]. It then follows from Lemma 2.1.9 that $\mathcal{X} \to S$ is locally of finite presentation (or, equivalently, locally of finite type). $\square$

Our next results are inspired by a lemma of Artin [Art69a, Lem. 3.10]. Our first lemma isolates one of the steps in Artin's argument, and generalises it to the stacky context.

2.6.4. **Lemma.** *Suppose that S is locally Noetherian, that T is a locally finite type S-scheme, that $\mathcal{F}$ satisfies* [1] *and* [3], *that $t \in T$ is a finite type point, and that $T \to \mathcal{F}$ is an S-morphism which is formally smooth at t. Then, replacing T by an open neighbourhood of t if necessary, we may factor the morphism $T \to \mathcal{F}$ as*

$$T \to \mathcal{F}' \to \mathcal{F},$$

where $\mathcal{F}'$ is an algebraic stack, locally of finite presentation over S, the first arrow is a smooth surjection, and the second arrow is locally of finite presentation, unramified, representable by algebraic spaces, and formally smooth at the image t' of t in $|\mathcal{F}'|$.

Proof. Write $R := T \times_{\mathcal{F}} T$. By the assumption that $\mathcal{F}$ satisfies [3], this is a locally finite type algebraic space over S; indeed, it is the base-change of the morphism $T \times_S T \to \mathcal{F} \times_S \mathcal{F}$ via the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$, and Lemma 2.6.2 shows that this latter morphism is locally of finite presentation. The projections $R \rightrightarrows T$ endow R with the structure of a groupoid in algebraic spaces over T. Since the morphism $T \to \mathcal{F}$ is locally of finite presentation by Lemma 2.6.3 (1), each of these projections is also locally of finite presentation (or equivalently, locally of finite type, since the base S is locally Noetherian), and the formal smoothness of $T \to \mathcal{F}$ at t implies that both projections are smooth in a neighbourhood of (t, t) by Lemma 2.4.7 (3). Thus, applying Lemma 2.6.6 below, we see that, by shrinking T around t if necessary, we may find an open subgroupoid $V \subseteq R$ that is actually a smooth groupoid over T. We then define $\mathcal{F}' := [T/V]$, and let t' denote the image of t in $|\mathcal{F}'|$.

Certainly, since $V \subseteq T \times_{\mathcal{F}} T$, the map $T \to \mathcal{F}$ factors through $\mathcal{F}'$. Since $T \to \mathcal{F}'$ is smooth and T is locally of finite type over S, so is $\mathcal{F}'$. (The property of being locally of finite type is smooth local on the source; see [Sta, Tag 06FR]). The morphism $\mathcal{F}' \to \mathcal{F}$ is thus locally of finite presentation, by Lemma 2.6.3 (1). Since $T \to \mathcal{F}$ is formally smooth at t, and $T \to \mathcal{F}'$ is smooth, the morphism $\mathcal{F}' \to \mathcal{F}$ is also formally smooth at t' *by definition*. (See Definition 2.4.10.)

It remains to show that the morphism $\mathcal{F}' \to \mathcal{F}$ is unramified and representable by algebraic spaces. We have already seen that it is locally of finite presentation, and so, in particular, locally of finite type, and (as explained in Example 2.3.10) we must show furthermore that the diagonal morphism $\mathcal{F}' \to \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'$ is an open immersion. (Recall that since $\mathcal{F}$ satisfies [3] by assumption, the fibre product $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'$ is an algebraic stack.) Since the morphism $R = T \times_{\mathcal{F}} T \to \mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'$ is smooth and surjective (as $T \to \mathcal{F}'$ is), we can verify this after base-changing by this latter morphism. Since $R = T \times_{\mathcal{F}} T$, and $V = T \times_{\mathcal{F}'} T$, one verifies that the base-changed morphism

$$\mathcal{F}' \times_{\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'} R \to R$$

is precisely the open immersion $V \to R$. This establishes the claim. $\square$

2.6.5. *Remark.* In the case when $\mathcal{F}$ is simply a sheaf of sets (which is the context of [Art69a]), the algebraic stack $\mathcal{F}'$ will simply be an algebraic space, and if we replace it by an étale cover by a scheme X, we obtain an unramified morphism $X \to \mathcal{F}$ which is formally smooth at a point x above t. This is essentially the conclusion of the second paragraph of the proof of [Art69a, Lem. 3.10], and our argument is an adaptation to the stacky context of the argument given there.

2.6.6. **Lemma.** *Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces locally of finite type over a locally Noetherian scheme S, with U a scheme. If $u \in U$ is a finite type point such that s is smooth at the point $e(u)$ (or equivalently, such that t is smooth at the point $e(u)$), then there exists an open neighbourhood U' of u in U such that, if (R', s', t', c') denotes the restriction of the groupoid (R, s, t, c) to U', there is an open subgroupoid $V \subseteq R'$ such that $s'_{|V}$ and $t'_{|V}$ are smooth, i.e. such that V is a smooth groupoid over U'.*

Proof. Let V' denote the open subspace of R on which s is smooth. Write $U' = e^{-1}(V')$; then U' is an open subscheme of U containing u. If we replace U by U' and R by $R' := R_{|U'} = (s \times t)^{-1}(U' \times U')$, so that V' is replaced by $V' \cap R'$, then we may, and do, assume that $e(U) \subseteq V'$.

Recall that we have the commutative diagram

$$
\begin{array}{ccccc}
 & & U & & \\
 & \nearrow^{t} & & \nwarrow^{t} & \\
R & \xleftarrow{\text{pr}_0} & R \times_{s,U,t} R & \xrightarrow{c} & R \\
\downarrow^{s} & & \downarrow^{\text{pr}_1} & & \downarrow^{s} \\
U & \xleftarrow{t} & R & \xrightarrow{s} & U
\end{array}
$$

of [Sta, Tag 043Z], in which each square (including the top square) is Cartesian. Pulling back the right-hand square of this diagram via the morphism $V' \to R$, we form the Cartesian diagram

$$
\begin{array}{ccc}
R \times_{s,U,t} V' & \xrightarrow{c} & R \\
\downarrow^{\text{pr}_1} & & \downarrow^{s} \\
V' & \xrightarrow{s} & U.
\end{array}
$$

The bottom arrow is smooth by the definition of V', and is furthermore surjective, since we have put ourselves in a situation in which $e(U) \subseteq V'$. Thus the locus of smoothness of the left-hand vertical arrow is precisely the preimage under c of the locus of smoothness of the right-hand vertical arrow; i.e. the locus of smoothness of $\text{pr}_1 : R \times_{s,U,t} V' \to V'$ is equal to $c^{-1}(V')$. Now certainly $\text{pr}_1 : V' \times_{s,U,t} V' \to V'$ is smooth, since it is a base-change of the smooth morphism $s : V' \to U$. Thus $V' \times_{s,U,t} V' \subseteq c^{-1}(V')$, or equivalently, $c(V' \times_{s,U,t} V') \subseteq V'$.

Now define $V := V' \cap i(V')$. Clearly $c(V \times_{s,U,t} V) \subseteq V$. Also $e(U) \subseteq V$, and $i(V) = V$. Taken together, these properties show that V is an open subgroupoid of R. Since $V \subseteq V'$, we see that $s_{|V}$ is smooth. Since $t = si$ and $i(V) = V$, we see that $t_{|V}$ is smooth as well. Thus the lemma is proved. $\square$

In the remainder of the section, we return to Axioms [4a] and [4b]; none of this material is needed for our main theorems. The following lemma, and its corollary, provide an analogue in the stacky context of [Art69a, Lem. 3.10] itself. The argument is essentially the same as Artin's.

2.6.7. Lemma. *Suppose that S is locally Noetherian, that T is a locally finite type S-scheme, and that $\mathcal{F}$ satisfies [1], [3], and [4a]. If $\mathcal{X} \to \mathcal{F}$ is a morphism of finite type from an algebraic stack to $\mathcal{F}$ (in the sense of Definition 2.3.4), and if $T \to \mathcal{F}$ is an S-morphism which is formally smooth at a finite type point $t \in T$, then there exists a neighbourhood T' of t in T such that the base-changed morphism $T' \times_{\mathcal{F}} \mathcal{X} \to \mathcal{X}$ is smooth.*

Proof. Clearly we may replace T by an affine open neighbourhood of t, and thus suppose that T is quasi-compact. We may apply Lemma 2.6.4 to the morphism $T \to \mathcal{F}$, and so, replacing T by a neighbourhood of t in T if necessary, we factor $T \to \mathcal{F}$ as $T \to \mathcal{F}' \to \mathcal{F}$ as in the statement of that lemma. We also choose a

smooth surjection $U \to \mathcal{X}$ whose source is a scheme. We then consider the diagram

$$
\begin{array}{ccccc}
T \times_{\mathcal{F}} U & \longrightarrow & \mathcal{F}' \times_{\mathcal{F}} U & \longrightarrow & U \\
\downarrow & & \downarrow & & \downarrow \\
T \times_{\mathcal{F}} \mathcal{X} & \longrightarrow & \mathcal{F}' \times_{\mathcal{F}} \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
T & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}
\end{array}
$$

Since $\mathcal{F}$ satisfies [3], the fibre product $T \times_{\mathcal{F}} U$ is an algebraic space, while $\mathcal{F}' \times_{\mathcal{F}} U$, $T \times_{\mathcal{F}} \mathcal{X}$, and $\mathcal{F}' \times_{\mathcal{F}} \mathcal{X}$ are algebraic stacks. Since the morphism $\mathcal{X} \to \mathcal{F}$ is quasi-compact, and since T (and hence also $\mathcal{F}'$) is quasi-compact, the fibre-products $T \times_{\mathcal{F}} \mathcal{X}$ and $\mathcal{F}' \times_{\mathcal{F}} \mathcal{X}$ are furthermore quasi-compact.

Since the upper right horizontal arrow is an unramified morphism, and so (by definition) has an étale diagonal, it is in particular a DM morphism (i.e. has an unramified diagonal). Since its target is a scheme, we thus find that $\mathcal{F}' \times_{\mathcal{F}} U$ is in fact a DM stack. We may thus amplify the preceding diagram to a commutative diagram

$$
\begin{array}{ccccc}
V & \longrightarrow & V' & & \\
\downarrow & & \downarrow & & \\
T \times_{\mathcal{F}} U & \longrightarrow & \mathcal{F}' \times_{\mathcal{F}} U & \longrightarrow & U \\
\downarrow & & \downarrow & & \downarrow \\
T \times_{\mathcal{F}} \mathcal{X} & \longrightarrow & \mathcal{F}' \times_{\mathcal{F}} \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
T & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}
\end{array}
$$

in which V and V' are quasi-compact schemes, the vertical arrows with V and V' as their sources are étale, and the composites $V \to T \times_{\mathcal{F}} \mathcal{X}$ and $V' \to \mathcal{F}' \times_{\mathcal{F}} \mathcal{X}$, as well as the four horizontal arrows on the left half of the diagram, are smooth surjections. (Note that the upper-most square in this diagram is *not* assumed to be Cartesian.)

Let $C' \subseteq V$ denote the complement of the smooth locus of the composite $V \to U$. It is a closed subset of V, and so its image C in T is a constructible subset of T. (The morphism $V \to T$ is a locally finite type morphism between Noetherian schemes, and hence is finite type. Thus it maps constructible sets to constructible sets.)

Lemma 2.6.3 (1) shows that U is locally of finite type over S, and so, by Lemma 2.4.7 (3), the smooth locus of the morphism $T \times_{\mathcal{F}} U \to U$ contains the fibre over t, and thus the same is true of the smooth locus of the morphism $V \to U$. In other words, the point t does not lie in C. To prove the lemma we must show that there is an open neighbourhood of t disjoint from C; that is, we must show that t does not lie in the closure of C.

Suppose that t does lie in the closure of C. Then we claim that we may find a point $\tilde{t} \in C$ whose closure Y contains t, and such that $A := \mathcal{O}_{Y,t}$ is a one-dimensional domain. To see this, note firstly that C is a finite union of irreducible locally closed schemes, so t is in the closure of one of these, and we may replace

C by this irreducible component (with its induced reduced structure), and thereby assume that C is open in its closure. Replacing the closure of C by an affine open subset containing t, and C by a distinguished open subset of this affine open, we put ourselves in the situation of having a Noetherian domain A with a closed point $t \in \operatorname{Spec} A$, and an element $f \in A$ such that $\operatorname{Spec} A[1/f]$ is a proper non-empty subset. Replacing A by its localisation at the maximal ideal corresponding to t, we may assume that A is a local domain, and that f is in the maximal ideal of A.

Since A is local Noetherian, it is finite-dimensional, and we may choose a prime P not containing f so that $\dim A/P$ is as small as possible. Replacing A by A/P, we have a Noetherian local domain A containing a nonzero element $f \in A$, with the properties that f is not a unit, and f is contained in every non-zero prime ideal of A. It remains to check that A is one-dimensional. To see this, note that the height one primes of A are precisely the isolated associated primes of (f), so that A has only finitely many height one primes. Let Q be any prime ideal of A. Take an element $x \in Q$; then by Krull's principal ideal theorem, (x) is contained in some height one prime, so we see that Q is contained in the union of the height one primes. Since this is a finite union, we see by prime avoidance that Q is contained in some height one prime, as required.

Since C' is closed in V, and so of finite type over T, we may find a point $\tilde{v} \in C'$ lying over $\tilde{t}$ such that residue field extension $\kappa(\tilde{t}) \hookrightarrow \kappa(\tilde{v})$ is finite. We will obtain a contradiction by showing that the morphism $V \to U$ is in fact formally smooth at $\tilde{v}$ (and hence smooth at $\tilde{v}$, by Lemma 2.4.7 (3)).

Consider a diagram

$$
\begin{array}{ccc}
Z_0 & \longrightarrow & Z \\
\downarrow & & \downarrow \\
V & \longrightarrow & U
\end{array}
$$

as in the definition of formal smoothness at $\tilde{v}$ (Definition 2.4.4 (2)); so $Z_0 \to Z$ is a closed immersion of Artinian local $\mathcal{O}_S$-algebras, and the residue field L at the closed point of Z_0 is a finite extension of $\kappa(\tilde{v})$ (and thus also finite over $K = \kappa(\tilde{t})$.) We expand this diagram to the diagram

$$
\begin{array}{ccccc}
\operatorname{Spec} L & & & & \\
\downarrow & \searrow & & & \\
Z_0 & \longrightarrow & & & Z \\
\downarrow & \searrow & & & \downarrow \\
V & \longrightarrow & V' & \longrightarrow & U
\end{array}
$$

The morphism $V' \to U$ factors as $V' \to \mathcal{F}' \times_{\mathcal{F}} U \to U$, and hence, as the composite of an unramified and an étale morphism, is itself unramified.

We will show that the morphism $Z \to U$ lifts to a morphism $Z \to V'$, compatible with the given morphism $\operatorname{Spec} L \to V'$. Assuming this, we note that this morphism, when restricted to Z_0, must coincide with the given morphism $Z_0 \to V'$; this follows from the fact that $V' \to U$ is unramified, and thus formally unramified. Since $V \to V'$ is smooth, and hence formally smooth, we then see that we may further lift the morphism $Z \to V'$ to a morphism $Z \to V$ compatible with the given

morphism $Z_0 \to V$. This completes the proof that $V \to U$ is formally smooth at $\tilde{v}$, and thus completes the proof of the lemma.

It remains to prove the existence of the lifting $Z \to V'$. Since $V' \to \mathcal{F}' \times_{\mathcal{F}} U$ is étale, it suffices to obtain a lifting $Z \to \mathcal{F}' \times_{\mathcal{F}} U$. For this, it suffices in turn to obtain a morphism $Z \to \mathcal{F}'$ lifting the composite $Z \to U \to \mathcal{F}$, and, for this, it suffices to obtain a morphism $Z \to T$ lifting $Z \to \mathcal{F}$.

Write $Z = \operatorname{Spec} B'_L$, and let B denote the integral closure of A in L. Then by the Krull–Akizuki theorem ([Mat89, Thm. 11.7] and its Corollary) B is a semi-local Dedekind domain with field of fractions L, whose residue fields at its closed points are finite extensions of the residue field of A at its closed point, i.e. of $\kappa(t)$. Let B' be the inverse image of $B \subseteq L$ in B'_L. Since $\mathcal{F}$ satisfies [4a], we may extend the morphism $\operatorname{Spec} B'_L \to \mathcal{F}$ to a morphism $\operatorname{Spec} B' \to \mathcal{F}$, compatible with the given morphism

$$\operatorname{Spec} B \to \operatorname{Spec} A \to T \to \mathcal{F}.$$

Lemma 2.6.8 below then shows that we may lift the morphism $\operatorname{Spec} B' \to \mathcal{F}$ to a morphism $\operatorname{Spec} B' \to T$. Passing to the local ring at the generic point, this gives the required morphism $Z \to T$. $\square$

2.6.8. **Lemma.** *Suppose that S is locally Noetherian, that $\mathcal{F}$ is a stack over S satisfying [1] and [3], that T is a scheme, locally of finite type over S, and that $T \to \mathcal{F}$ is a morphism over S which is formally smooth at a finite type point $t \in T$. Suppose that B is a Noetherian local $\mathcal{O}_S$-algebra whose residue field is finite type over $\mathcal{O}_S$, that B' is an $\mathcal{O}_S$-algebra which is a nilpotent thickening of B, and that we have a commutative diagram of morphisms over S*

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} B' \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{F} \end{array}$$

such that the left-hand vertical arrow maps the closed point of $\operatorname{Spec} B$ to the given point $t \in T$. Then there is a lifting of the right-hand vertical arrow to a morphism $\operatorname{Spec} B' \to T$.

Proof. Consider the projection $T \times_{\mathcal{F}} \operatorname{Spec} B' \to \operatorname{Spec} B'$. The commutative diagram in the statement of the lemma gives rise to a morphism

$$(2.6.9) \qquad\qquad\qquad \operatorname{Spec} B \to T \times_{\mathcal{F}} \operatorname{Spec} B'$$

lifting the closed immersion $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} B'$. Since the closed point of $\operatorname{Spec} B$ maps to the point t of T, by assumption, it follows from Lemma 2.4.7 (3) that the image of the closed point of $\operatorname{Spec} B$ under this morphism lies in the smooth locus of the projection, and thus that the morphism (2.6.9) itself factors through this smooth locus. Since smooth morphisms are, in particular, formally smooth, we may thus lift this morphism to a section $\operatorname{Spec} B' \to T \times_{\mathcal{F}} \operatorname{Spec} B$. Composing this section with the projection onto T gives the morphism required by the statement of the lemma. $\square$

2.6.10. **Corollary.** *Suppose that S is locally Noetherian, that T is a locally finite type S-scheme, that $\mathcal{F}$ satisfies [1], [3], and [4a], and furthermore that the diagonal of $\mathcal{F}$ is quasi-compact. If $T \to \mathcal{F}$ is an S-morphism which is formally smooth at a finite type point $t \in T$, then this morphism is in fact smooth at t.*

Proof. Let $X \to \mathcal{F}$ be a morphism whose source is a scheme of finite type over S, and consider the usual "graph diagram"

$$
\begin{array}{ccc}
X & \xrightarrow{\Gamma} X \times_S \mathcal{F} & \longrightarrow \mathcal{F} \\
\downarrow & \downarrow & \\
\mathcal{F} & \xrightarrow{\Delta} \mathcal{F} \times_S \mathcal{F}
\end{array}
$$

in which the square is Cartesian (by construction). Lemma 2.1.5, along with [Sta, Tag 06CX], shows that the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is locally of finite presentation, and it is quasi-compact by assumption. Thus it is in particular of finite type, and so the same is true of the graph $\Gamma : X \to X \times_S \mathcal{F}$. The projection $X \times_S \mathcal{F} \to \mathcal{F}$ is also of finite type, being the base-change of the finite type morphism of Noetherian schemes $X \to S$. Thus the morphism $X \to \mathcal{F}$ is of finite type, and so Lemma 2.6.7 implies that we may find a neighbourhood U of t in T such that the base-changed morphism $U \times_\mathcal{F} X \to X$ is smooth. By definition, then, the morphism $T \to \mathcal{F}$ is smooth at t. $\qquad \square$

2.6.11. **Corollary.** *Suppose that S is locally Noetherian, and that $\mathcal{F}$ is a stack over S satisfying [1], [2](a),[3], [4a], and [4b], whose diagonal is quasi-compact. Then $\mathcal{F}$ satisfies [4].*

Proof. Suppose that $\varphi : U \to \mathcal{F}$ is a morphism from a scheme locally of finite type over S to $\mathcal{F}$ which is versal at a finite type point $u \in U$. Lemma 2.4.7 (2) then shows that φ is formally smooth at u. From Corollary 2.6.10 we conclude that φ is in fact smooth at u, and Axiom [4b] then implies that φ is smooth in a neighbourhood of u. Finally, Lemma 2.4.7 (1) shows that φ is versal at each finite type point in this neighbourhood, and thus $\mathcal{F}$ satisfies Axiom [4], as claimed. $\qquad \square$

Our final result in this subsection strengthens the conclusion of Lemma 2.6.4, in the presence of [4a].

2.6.12. **Corollary.** *Suppose, in the context of Lemma 2.6.4, that $\mathcal{F}$ furthermore satisfies [4a], and has quasi-compact diagonal. Then, in addition to the conclusions of that lemma, we may impose the condition that the morphism $\mathcal{F}' \to \mathcal{F}$ be a monomorphism.*

Proof. By Corollary 2.6.10, replacing T by a neighbourhood of t if necessary, we may suppose that the projections $R := T \times_\mathcal{F} T \rightrightarrows T$ are in fact smooth. Thus, in the proof of Lemma 2.6.4, we may take $V = R$. It is then easily verified that $\mathcal{F}' := [T/V] = [T/R] \to \mathcal{F}$ is in fact a monomorphism. Indeed, following the proof of Lemma 2.6.4, we deduce from the fact that $V = R$ that the diagonal $\mathcal{F}' \to \mathcal{F}' \times_\mathcal{F} \mathcal{F}'$ is an isomorphism, which is equivalent to $\mathcal{F}' \to \mathcal{F}$ being a monomorphism. $\qquad \square$

2.6.13. *Remark.* Example 4.3.4 shows that we cannot remove the assumption that $\mathcal{F}$ has quasi-compact diagonal from the preceding corollaries.

2.7. **Relationships between [2] and [3].** Throughout this subsection we suppose that $\mathcal{F}$ is a category fibred in groupoids over the locally Noetherian base scheme S.

2.7.1. **Lemma.** *Let $x : \operatorname{Spec} k \to \mathcal{F}$ be a finite type point of $\mathcal{F}$ (here k is a finite type field over $\mathcal{O}_S$).*

(1) If $\mathcal{F}$ admits an effective Noetherian versal ring at x, then, for each object A of $\widehat{\mathcal{C}}_\Lambda$, the functor $\mathcal{F}_x(A) \to \widehat{\mathcal{F}}_x(A)$ is essentially surjective. Conversely, if this functor is essentially surjective for each object A of $\widehat{\mathcal{C}}_\Lambda$, and if $\mathcal{F}$ admits a Noetherian versal ring at x, then $\mathcal{F}$ admits an effective Noetherian versal ring at x.

(2) If $\mathcal{F}$ satisfies [3], then, for each object A of $\widehat{\mathcal{C}}_\Lambda$, the functor $\mathcal{F}_x(A) \to \widehat{\mathcal{F}}_x(A)$ is fully faithful.

Proof. We begin by proving (1). Suppose first that $\mathcal{F}$ admits an effective Noetherian versal ring at x, and let $A_x \in \widehat{\mathcal{C}}_\Lambda$ be an effective versal ring at the morphism x; that is, we have an object η of $\mathcal{F}_x(A_x)$ whose image in $\widehat{\mathcal{F}}_x(A_x)$ is versal. If $\xi : \operatorname{Spf} B \to \mathcal{F}$ is any object of $\widehat{\mathcal{F}}_x(B)$, for some object B of $\widehat{\mathcal{C}}_\Lambda$, then by definition of versality we may find a morphism $f : A_x \to B$ and an isomorphism $\xi \cong f^*\eta$. Since η is effective, i.e. lies in the image of the functor $\mathcal{F}_x(A_x) \to \widehat{\mathcal{F}}_x(A_x)$, we see that ξ lies in the essential image of the functor $\mathcal{F}_x(B) \to \widehat{\mathcal{F}}_x(B)$; i.e. this functor is essentially surjective. On the other hand, if $\mathcal{F}$ admits a Noetherian versal ring A_x at x, and if furthermore the functor $\mathcal{F}_x(A_x) \to \widehat{\mathcal{F}}_x(A_x)$ is essentially surjective, then $\mathcal{F}$ admits an effective Noetherian versal ring at x by definition.

We turn to proving (2). Suppose that η and ξ are two objects of $\mathcal{F}_x(A)$ (for some object A of $\widehat{\mathcal{C}}_\Lambda$). Their product is a morphism $\eta \times \xi : \operatorname{Spec} A \to \mathcal{F} \times_S \mathcal{F}$. The morphisms η and ξ, when restricted to $\operatorname{Spec} k$ (the closed point of $\operatorname{Spec} A$), both induce the given morphism $x : \operatorname{Spec} k \to \mathcal{F}$, and hence we have the outer square in a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \\
\downarrow{\scriptstyle x} & \nearrow & \downarrow{\scriptstyle \eta \times \xi} \\
\mathcal{F} & \xrightarrow{\ \Delta\ } & \mathcal{F} \times_S \mathcal{F}
\end{array}
$$

The set of morphisms between η and ξ in the category $\mathcal{F}_x(A)$ may be identified with the set of morphisms $\operatorname{Spec} A \to \mathcal{F}$ which continue to make the diagram commute.

If we let $\widehat{\eta}$ and $\widehat{\xi}$ denote the images of η and ξ in $\widehat{\mathcal{F}}_x(A)$, then the set of morphisms between $\widehat{\eta}$ and $\widehat{\xi}$ may similarly be identified with the set of morphisms $\operatorname{Spf} A \to \mathcal{F}$ for which the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} k & \longrightarrow & \operatorname{Spf} A \\
\downarrow{\scriptstyle x} & \nearrow & \downarrow{\scriptstyle \widehat{\eta} \times \widehat{\xi}} \\
\mathcal{F} & \xrightarrow{\ \Delta\ } & \mathcal{F} \times_S \mathcal{F}
\end{array}
$$

commutes.

Since $\mathcal{F}$ satisfies [3], the fibre product $\mathcal{F} \times_{\Delta, \mathcal{F} \times_S \mathcal{F}, \eta \times \xi} \operatorname{Spec} A$ is an algebraic space over $\operatorname{Spec} A$. The claim of full faithfulness in (2) now follows from the following general fact [Sta, Tag 0AQH]: if A is a complete local ring, if $T \to \operatorname{Spec} A$ is a morphism from an algebraic space to $\operatorname{Spec} A$, and if $t : \operatorname{Spec} k \to T$ is a section of this morphism over the closed point, then the restriction map $\operatorname{Mor}_t(\operatorname{Spec} A, T) \to \operatorname{Mor}_t(\operatorname{Spf} A, T)$ from the set of sections over $\operatorname{Spec} A$ extending t to the set of sections over $\operatorname{Spf} A$ extending t is a bijection. $\qquad\square$

We now prove the lemma promised in Remark 2.2.23.

2.7.2. Lemma. *Suppose that $\mathcal{F}$ satisfies [3], and that the diagonal of $\mathcal{F}$ is furthermore locally of finite type. If, for some finite type point x of $\mathcal{F}$, there exists*

a versal morphism $U \to \widehat{\mathcal{F}}_x$ with U being (Noetherianly) pro-representable, then $R := U \times_{\widehat{\mathcal{F}}_x} U$ is also (Noetherianly) pro-representable, so that $\widehat{\mathcal{F}}_x$ admits a presentation by a (Noetherianly) pro-representable smooth groupoid in functors.

Proof. Note firstly that (U, R) is a smooth groupoid in functors by Lemma 2.2.22, so we only need to show that R is (Noetherianly) pro-representable. Suppose that U is pro-represented by $A = \varprojlim_{i \in I} A_i$. If $B \in \mathcal{C}_\Lambda$, then giving a morphism $\operatorname{Spec} B \to U \times_{\widehat{\mathcal{F}}_x} U$ is equivalent to giving a pair of morphisms $\operatorname{Spec} B \rightrightarrows U$ which induce the same morphism to $\mathcal{F}$. By definition these morphisms factor through $\operatorname{Spec} A_i$ for some i. Thus $U \times_{\widehat{\mathcal{F}}_x} U$ is the colimit over I of the fibre products $\operatorname{Spec} A_i \times_{\widehat{\mathcal{F}}_x} \operatorname{Spec} A_i$ (the fibre product denoting a fibre product of categories cofibred in groupoids on $\mathcal{C}_\Lambda$). Since a colimit of pro-representable functors is pro-representable (by the projective limit of the representing rings), it suffices to show that each of these fibre products is pro-representable.

If we let $\xi : \operatorname{Spec} A_i \to \mathcal{F}$ denote the composite $\operatorname{Spec} A_i \to \operatorname{Spf} A \to \widehat{\mathcal{F}}_x \hookrightarrow \mathcal{F}$, then $\operatorname{Spec} A_i \times_{\mathcal{F}} \operatorname{Spec} A_i$ is an algebraic space locally of finite type over S (because $\mathcal{F}$ satisfies [3], and its diagonal is locally of finite type) which represents the functor $\operatorname{Isom}(\xi, \xi)$. The fibre product $\operatorname{Spec} A_i \times_{\widehat{\mathcal{F}}_x} \operatorname{Spec} A_i$ is the subfunctor of the restriction of $\operatorname{Isom}(\xi, \xi)$ to $\mathcal{C}_\Lambda$ consisting of isomorphisms which induce the identity from x to itself. The identity from x to itself is a k-valued point of $\operatorname{Isom}(\xi, \xi)$, and this subfunctor may equally well be described as the formal completion of $\operatorname{Isom}(\xi, \xi)$ at this point. Since $\operatorname{Isom}(\xi, \xi)$ is represented by a locally Noetherian algebraic space, this formal completion is represented by the formal spectrum of the complete local ring of $\operatorname{Isom}(\xi, \xi)$ at x in the sense of Definition 2.2.17 and thus is pro-representable.

It remains to show that if A is Noetherian, then $U \times_{\widehat{\mathcal{F}}_x} U$ is Noetherianly pro-representable. By [Gro95, Prop. 5.1], we need to check that if k is the residue field at x, then the k-vector space of morphisms $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to U \times_{\widehat{\mathcal{F}}_x} U$ is finite-dimensional.

Now, any such morphism factors through a pair of morphisms $\operatorname{Spec} A/\mathfrak{m}_A^2 \to U$. Since A is Noetherian, $A/\mathfrak{m}_A^2$ is Artinian. Applying the argument of the previous paragraph to $\operatorname{Spec} A/\mathfrak{m}_A^2 \times_{\widehat{\mathcal{F}}_x} \operatorname{Spec} A/\mathfrak{m}_A^2$, we find that it is pro-represented by a Noetherian complete local ring (the completion of a locally Noetherian algebraic space at a closed point); any such ring admits only a finite-dimensional space of maps to $k[\epsilon]/\epsilon^2$, so we are done. $\qquad\square$

2.7.3. **Corollary.** *Suppose that $\mathcal{F}$ satisfies [3], that its diagonal is furthermore locally of finite type, and that $\mathcal{F}$ admits versal rings at all finite type points. Then $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous.*

Proof. This is immediate from Lemmas 2.2.24 and 2.7.2. $\qquad\square$

The next result shows that when $\mathcal{F}$ satisfies [2] and [3], we can strengthen condition [2](a) so as to allow Y' and Z to be complete Noetherian local rings, rather than merely Artinian.

2.7.4. **Lemma.** *Suppose that $\mathcal{F}$ satisfies [3] and [2](b), and that*

$$
\begin{array}{ccc}
Y & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Z'
\end{array}
$$

is a pushout diagram of S-schemes, with the horizontal arrows being closed immersions, Y being a finite type Artinian $\mathcal{O}_S$-scheme, each of Z and Y' being the spectrum of a complete Noetherian local $\mathcal{O}_S$-algebra whose residue field is finite type over $\mathcal{O}_S$, and the left-hand vertical arrow being closed (i.e. corresponding to a local morphism of local $\mathcal{O}_S$-algebras). If either $\mathcal{F}$ satisfies [2](a), or if $\mathcal{F}$ is an étale stack whose diagonal is locally of finite presentation and the extension of the residue field of Y over the residue field of Z is separable, then the induced functor

$$\mathcal{F}(Z') \to \mathcal{F}(Y') \times_{\mathcal{F}(Y)} \mathcal{F}(Z)$$

is an equivalence of categories.

Proof. Let x be the underlying closed point of Y. By Lemma 1.6.1, Z' is Noetherian (note that $Y \hookrightarrow Y'$ is assumed to be a closed immersion, and the residue fields of Y, Y', Z are all finite type $\mathcal{O}_S$-algebras, so that in particular the residue field of Y is a finite extension of that of Z), so by Lemma 2.7.1 and the assumption that $\mathcal{F}$ satisfies [3] and [2](b), we may replace $\mathcal{F}$ by $\widehat{\mathcal{F}}_x$, and by the definition of $\widehat{\mathcal{F}}_x$, we can then reduce to the case that Y' and Z are Artinian. In the case that $\mathcal{F}$ satisfies [2](a) we are then done by the very definition of that condition, and in the case that $\mathcal{F}$ is an étale stack whose diagonal is locally of finite presentation and the extension of the residue field of Y over the residue field of Z is separable the result follows from Corollary 2.7.3 and Lemma 2.2.6. □

2.8. Artin's representability theorem.

In this subsection, we continue to assume that S is locally Noetherian. In the statement of the next lemma, as well as for one direction of the main theorems, we will suppose further that, for each finite type point $s \in S$, the local ring $\mathcal{O}_{S,s}$ is a G-ring. As explained in Section 1.5.5, this assumption is needed in order to apply Artin approximation.

We begin with a lemma, which is essentially a rephrasing of [Sta, Tag 07XH]. It is the key application of Artin approximation which underlies Artin's representability theorem.

2.8.1. Lemma. *Suppose that, for each finite type point $s \in S$, the local ring $\mathcal{O}_{S,s}$ is a G-ring. If $\mathcal{F}$ is a category in groupoids satisfying [1] and [2](a), if k is a finite type $\mathcal{O}_S$-field, and if $x : \operatorname{Spec} k \to \mathcal{F}$ is a morphism representing the finite type point $t \in |\mathcal{F}|$, for which $\widehat{\mathcal{F}}_x$ admits an effective Noetherian versal ring, then there exists a morphism $\varphi : U \to \mathcal{F}$ whose source is a scheme locally of finite type over S, and a finite type point $u \in U$ such that $\varphi(u) = t$, and such that φ is formally smooth at u.*

Proof. It follows from [Sta, Tag 07XH] (i.e. Artin approximation), together with our assumptions, that we may find a morphism $\varphi : U \to \mathcal{F}$ with U a scheme locally of finite type over S, containing a point u, such that $\phi(u) = t$, and such that φ is versal at u. Lemma 2.4.7 (2), together with our assumption of [2](a), shows that in fact φ is formally smooth at u. □

2.8.2. Lemma. *Suppose that $\mathcal{F}$ is a category fibred in groupoids which satisfies [3], and that there exists a morphism $\varphi : T \to \mathcal{F}$ whose source is a scheme, locally of finite type over S, which is smooth in a neighbourhood of a finite type point $t \in T$. Then, if $\varphi' : T' \to \mathcal{F}$ is any morphism from a scheme, locally of finite type over S, to $\mathcal{F}$, which is formally smooth at a finite type point $t' \in T'$ whose image in $\mathcal{F}$ coincides with the image of t, there is a neighbourhood V of t' in T' such that the restriction of φ' to V is smooth.*

Proof. Replacing T by the hypothesised neighbourhood of t, we may assume that φ is smooth. Note that since $\mathcal{F}$ satisfies [3], the morphism φ is representable by algebraic spaces, and so this is to be understood in the sense of [Sta, Tag 03YK], i.e. the base change of this morphism over any morphism from a scheme to $\mathcal{F}$ is smooth. In particular, the base-changed morphism of algebraic spaces $T \times_{\mathcal{F}} T' \to T'$ is smooth. Since the morphism $T' \to \mathcal{F}$ is formally smooth at t', the projection $T \times_{\mathcal{F}} T' \to T$ is smooth in a neighbourhood U of the point (t, t'), by Lemma 2.4.7 (3). Now the the composite $U \to T \to \mathcal{F}$ is the composite of smooth morphisms, hence is smooth. Rewriting this morphism as $U \to T' \to \mathcal{F}$, we see that this composite is also smooth. If we let V denote the image of U in T', then V is an open subset of T' containing t', and $U \to V$ is a smooth surjection. Smoothness being a property that is smooth local on the source, we see that $V \to \mathcal{F}$ is a smooth morphism, as required. $\qquad\square$

2.8.3. *Remark.* The preceding lemma shows that in order to verify [4] for a category fibred in groupoids satisfying [2](a) and [3], it suffices to find, for each finite type point of $x \in |\mathcal{F}|$, a morphism $\varphi : T \to \mathcal{F}$ whose source is locally of finite type over S, and contains a finite type point t mapping to x, such that φ restricts to a smooth morphism on some neighbourhood of t. (The role of [2](a) is to ensure, via Lemma 2.4.7 (2), that versality at a point coincides with formal smoothness at a point.)

2.8.4. **Theorem.** *Suppose that S is locally Noetherian. Any algebraic stack, locally of finite presentation over S, satisfies Axioms [1], [2], [3] and [4]. Conversely, suppose further that for each finite type point $s \in S$, the local ring $\mathcal{O}_{S,s}$ is a G-ring. Then if $\mathcal{F}$ is an étale stack in groupoids over S satisfying [1], [2], [3] and [4], then $\mathcal{F}$ is an algebraic stack, locally of finite presentation over $\mathcal{F}$.*

Proof. If $\mathcal{F}$ is an algebraic stack, locally of finite presentation over S, then it follows from Lemma 2.1.9 that $\mathcal{F}$ satisfies [1], while Lemma 2.2.3, Lemma 2.7.1, [Sta, Tag 07WU], [Sta, Tag 07WV], [Sta, Tag 06IW], [Sta, Tag 07X8], and [Sta, Tag 07X1] show that $\mathcal{F}$ satisfies [2]. (More precisely, $\mathcal{F}$ satisfies [2](a) by Lemma 2.2.3. By [Sta, Tag 07X8] and Lemma 2.7.1, in order to prove that $\mathcal{F}$ satisfies [2](b), it suffices to show that $\mathcal{F}$ has Noetherian versal rings at each finite type point. [Sta, Tag 06IW] gives a criterion for the existence of such versal rings, which is satisfied by [Sta, Tag 07WU], [Sta, Tag 07WV], and [Sta, Tag 07X1].) By definition $\mathcal{F}$ satisfies [3]. Again by definition, we may find a smooth surjection $U \to \mathcal{F}$ with U a scheme. If $x \in |\mathcal{F}|$ is a point of finite type, then we may find a finite type point $u \in U$ lying over x, and Remark 2.8.3 then shows that $\mathcal{F}$ satisfies [4].

For the converse, we follow the proof of [Sta, Tag 07Y4]. By definition, we need to show that $\mathcal{X}$ admits a smooth surjection from a scheme. Taking the union of the morphisms obtained from Lemma 2.8.1 as we run over all finite type points of $\mathcal{X}$, we obtain a smooth map $U \to \mathcal{X}$ whose source is a scheme, whose image contains all finite type points of $\mathcal{X}$. It remains to show that this is surjective. As in the proof of [Sta, Tag 07Y4], this may be checked by pulling back to affine schemes of finite presentation over S, where it is immediate (as smooth maps are open, and the finite type points of a scheme are dense). $\qquad\square$

2.8.5. **Theorem.** *Any algebraic stack, locally of finite presentation over S, satisfies [1], [2], [3], [4a], and [4b]. Conversely, suppose further that for each finite type point $s \in S$, the local ring $\mathcal{O}_{S,s}$ is a G-ring. Then if $\mathcal{F}$ is an étale stack in groupoids*

over S satisfying [1], [2], [3], [4a], *and* [4b], *and if the diagonal of $\mathcal{F}$ is furthermore quasi-compact, then $\mathcal{F}$ is an algebraic stack, locally of finite presentation over S.*

Proof. For the first statement, in view of Theorem 2.8.4, we need to show that [4a] and [4b] are satisfied for algebraic stacks. Lemma 2.4.15 shows that [4a] is satisfied, while Lemma 2.4.7 (4) shows that [4b] is satisfied.

To prove the second statement, note that Corollary 2.6.11 shows, under the given hypotheses on $\mathcal{F}$, that $\mathcal{F}$ furthermore satisfies Axiom [4]. Theorem 2.8.4 then shows that $\mathcal{F}$ is an algebraic stack, locally of finite presentation over S. $\qquad\square$

Example 4.3.4 shows that the condition on the diagonal of $\mathcal{F}$ is necessary in order to deduce that a stack satisfying [1], [2], [3], [4a], and [4b] is algebraic.

2.8.6. *Remark.* In the proof of Artin representability, we don't require the full strength of Axiom [2](b); all we need is that $\widehat{\mathcal{F}}_x$ admits an effective Noetherian versal ring for at least one morphism x representing any given finite type point of $\mathcal{F}$. However, the following result, which describes the extent to which this hypothesis is independent of the choice of representative of a finite type point, implies in particular that, in the context of the preceding representability theorems, this hypothesis holds for at least one representative of a given finite type point if and only if it holds for every such representative.

2.8.7. **Lemma.** *Suppose that $\mathcal{F}$ satisfies* [2](a), *and suppose given $x : \operatorname{Spec} k \to \mathcal{F}$, with k a field which is a finite type $\mathcal{O}_S$-algebra.*

(1) If $\widehat{\mathcal{F}}_x$ admits a Noetherian versal ring, then for any other morphism $x' : \operatorname{Spec} l \to \mathcal{F}$, with l a field which is a finite type $\mathcal{O}_S$-algebra, representing the same point of $\mathcal{F}$, we have that $\widehat{\mathcal{F}}_{x'}$ again admits a Noetherian versal ring.

(2) If $\widehat{\mathcal{F}}_x$ admits an effective Noetherian versal ring, then for any finite extension l of k, if we let x' denote the composite $\operatorname{Spec} l \to \operatorname{Spec} k \xrightarrow{x} \mathcal{F}$, we have that $\widehat{\mathcal{F}}_{x'}$ again admits an effective Noetherian versal ring.

(3) Suppose that $\mathcal{F}$ satisfies [3] (in addition to [2](a)), and that, for some finite extension l of k, if we denote by x' the composite $\operatorname{Spec} l \to \operatorname{Spec} k \xrightarrow{x} \mathcal{F}$, we have that $\widehat{\mathcal{F}}_{x'}$ admits an effective Noetherian versal ring. Assume also either that l/k is separable, or that $\mathcal{F}$ is an fppf stack. Then $\widehat{\mathcal{F}}_x$ admits an effective Noetherian versal ring.

(4) Suppose either that (a) $\mathcal{F}$ satisfies [3] (in addition to [2](a)), and that either the residue field of the image of the composite $\operatorname{Spec} k \to \mathcal{F} \to S$ is perfect, or $\mathcal{F}$ is an fppf stack; or (b) that $\mathcal{F}$ satisfies [1] and [3] (in addition to [2](a)), and that the local rings of S at finite type points are G-rings. Suppose also that $\widehat{\mathcal{F}}_x$ admits an effective Noetherian versal ring. Then for any other morphism $x' : \operatorname{Spec} l \to \mathcal{F}$, with l a field which is a finite type $\mathcal{O}_S$-algebra, representing the same point of $\mathcal{F}$, we have that $\widehat{\mathcal{F}}_{x'}$ again admits an effective Noetherian versal ring.

Proof. Given l as in (1), we may find a common finite extension l' of k and l. After appropriate relabelling, then, to prove (1) we may assume that l is a finite extension of k and that x' is the composite $\operatorname{Spec} l \to \operatorname{Spec} k \to \mathcal{F}$, and we must show that $\widehat{\mathcal{F}}_x$ admits a Noetherian versal ring if and only if $\widehat{\mathcal{F}}_{x'}$ does.

We use the notation and results of Remark 2.2.11. Since $\mathcal{F}$ satisfies [2](a), there is a natural equivalence of categories cofibred in groupoids $(\widehat{\mathcal{F}}_x)_{l/k} \xrightarrow{\sim} \widehat{\mathcal{F}}_{x'}$. We will deduce part (1) of the lemma from from [Sta, Tag 06IW] (which gives a criterion for a category cofibred in groupoids over $\mathcal{C}_{\Lambda,k}$, or $\mathcal{C}_{\Lambda,l}$, to admit a Noetherian versal morphism), as follows. Since $\mathcal{F}$ satisfies [2](a), it follows from [Sta, Tag 07WA,Tag 06J7] that both $\widehat{\mathcal{F}}_x$ and $(\widehat{\mathcal{F}}_x)_{l/k}$ satisfy all of the hypotheses of [Sta, Tag 06IW], except possibly the condition of having a finite-dimensional tangent space. By [Sta, Tag 07WB], this condition holds for $\widehat{\mathcal{F}}_x$ if and only if it holds for $(\widehat{\mathcal{F}}_x)_{l/k}$, so it suffices to observe that the existence of a Noetherian versal ring implies the finite-dimensionality of the tangent space, by [Sta, Tag 06IU].

To prove part (2), we again use the equivalence $(\widehat{\mathcal{F}}_x)_{l/k} \xrightarrow{\sim} \widehat{\mathcal{F}}_{x'}$. Taking into account part (1), it suffices to show that if $A_{x'}$ denotes a versal ring at the morphism x', then the versal morphism $\operatorname{Spf} A_{x'} \to \widehat{\mathcal{F}}_{x'}$ arises from a morphism $\operatorname{Spec} A_{x'} \to \mathcal{F}$. The above-mentioned equivalence of categories shows that this versal morphism arises as the composite of the natural morphism $\operatorname{Spf} A_{x'} \to \operatorname{Spf}(A_{x'} \times_l k)$ and a morphism $\operatorname{Spf}(A_{x'} \times_l k) \to \widehat{\mathcal{F}}_x$. Lemma 2.7.1 (1), and our effectivity assumption regarding $\widehat{\mathcal{F}}_x$, shows that this latter morphism is effective, i.e. is induced by a morphism $\operatorname{Spec}(A_{x'} \times_l k) \to \mathcal{F}$. The composite

$$\operatorname{Spec} A_{x'} \to \operatorname{Spec}(A_{x'} \times_l k) \to \operatorname{Spec} A_x \to \mathcal{F}$$

then induces the original versal morphism $\operatorname{Spf} A_{x'} \to \widehat{\mathcal{F}}_{x'}$, and so (2) is proved.

In order to prove (3), we first fix a versal morphism $\operatorname{Spf} A_x \to \mathcal{F}$ at x. If l/k is separable, then, by the topological invariance of the étale site, we may find a finite étale local extension B of A_x which induces the extension l/k on residue fields. Otherwise, let Λ_k, Λ_l be Cohen rings for k, l respectively, and set $B := A_x \widehat{\otimes}_{\Lambda_k} \Lambda_l$; then B/A_x is a finite extension by the topological version of Nakayama's lemma, and it is a faithfully flat local extension of complete local Noetherian rings by [Gro64, Lem. 0.19.7.1.2]. In either case, the composite $\operatorname{Spf} B \to \operatorname{Spf} A_x \to \mathcal{F}$ may be regarded as a morphism $\operatorname{Spf} B \to \widehat{\mathcal{F}}_{x'}$, and so, by our effectivity assumption regarding $\widehat{\mathcal{F}}_{x'}$, together with Lemma 2.7.1 (1), it is effective, i.e. is induced by a morphism $\operatorname{Spec} B \to \mathcal{F}$.

Consider the pull-backs of this morphism along the two projections $\operatorname{Spec} B \otimes_{A_x} B \rightrightarrows \operatorname{Spec} B$. Since the morphism $\operatorname{Spf} B \to \mathcal{F}$ factors through $\operatorname{Spf} A_x$, we see that the two pull-backs become isomorphic over $\operatorname{Spf}(B \otimes_{A_x} B)$. Since $\mathcal{F}$ satisfies [3], Lemma 2.7.1 (2) shows that the two pull-backs are themselves isomorphic. We may check the cocycle condition in a similar way, and hence obtain étale (or *fppf*, in the case that l/k is inseparable) descent data on the morphism $\operatorname{Spec} B \to \mathcal{F}$, which (since $\mathcal{F}$ is an étale stack, and an *fppf* stack if l/k is inseparable) allows us to descend it to a morphism $\operatorname{Spec} A_x \to \mathcal{F}$, as required.

We turn to proving (4). Thus we suppose given a morphism $x' : \operatorname{Spec} l \to \mathcal{F}$ representing the same point of $\mathcal{F}$ that the given morphism $x : \operatorname{Spec} k \to \mathcal{F}$ represents. We are assuming that $\widehat{\mathcal{F}}_x$ admits an effective Noetherian versal ring, and we wish to prove the corresponding fact for $\widehat{\mathcal{F}}_{x'}$. If we let l' be a common finite extension of k and l, and let x'' denote the composite $\operatorname{Spec} l' \to \operatorname{Spec} k \to \widehat{\mathcal{F}}_x$, then it follows from (2) that $\widehat{\mathcal{F}}_{x''}$ admits an effective Noetherian versal ring. So, relabelling l' as l and l as k, we are reduced to the following problem: we are given the morphisms $x : \operatorname{Spec} k \to \mathcal{F}$ and $x' : \operatorname{Spec} l \to \operatorname{Spec} k \xrightarrow{x} \mathcal{F}$, for which $\widehat{\mathcal{F}}_{x'}$

admits an effective Noetherian versal ring, and we would like to conclude that $\widehat{\mathcal{F}}_x$ also admits an effective Noetherian versal ring.

If either the residue field of the image of $\operatorname{Spec} k$ in S is perfect or $\mathcal{F}$ is an *fppf* stack, and if $\mathcal{F}$ satisfies [3], then this follows from part (3). Otherwise, we assume further that $\mathcal{F}$ satisfies [1], and that the local rings of S at finite type points are G-rings. This allows us to apply Artin approximation, in the form of Lemma 2.8.1, to the morphism x', so as to conclude that there exists a morphism $\varphi : U \to \mathcal{F}$ whose source is a scheme, and a finite type point u of U, lying over the image of x' (which is also the image of x), such that φ is formally smooth at u.

At this stage x' and l have done their job in the argument, and so we drop them from consideration; in fact, we will recycle them as notation, in a manner which we now explain. Pulling back U over x, we obtain a k-scheme whose smooth locus contains the fibre over U. This fibre is non-empty, by Lemma 2.4.7 (2), and thus contains a point defined over a finite separable extension l of k. If we let x' denote the resulting composite $\operatorname{Spec} l \to \operatorname{Spec} k \to \mathcal{F}$, then $\widehat{\mathcal{F}}_{x'}$ admits an effective Noetherian versal ring (given by the complete local ring of this fibre at this point). Part (3) now implies that the same is true for $\widehat{\mathcal{F}}_x$, as required. $\square$

2.8.8. *Remark.* A version of Lemma 2.8.7 for the condition of admitting a presentation by a smooth Noetherianly pro-representable groupoid in functors, rather than admitting a Noetherian versal ring, can be proved in an almost identical fashion.

2.8.9. *Remark.* Example 4.3.10 shows that parts (3) and (4) of Lemma 2.8.7 are not true without the assumption of [3].

3. Scheme-theoretic images

Suppose that $\xi : \mathcal{X} \to \mathcal{F}$ is a proper morphism, where $\mathcal{X}$ is an algebraic stack, and $\mathcal{F}$ is a stack whose diagonal is representable by algebraic spaces and locally of finite presentation. In Section 3.2 we define the scheme-theoretic image $\mathcal{Z}$ of ξ, which is initially a Zariski substack of $\mathcal{F}$. Our main aim in this section is to prove Theorem 1.1.1, giving a criterion for $\mathcal{Z}$ to be an algebraic stack, as well as to prove a number of related properties of $\mathcal{Z}$.

Interpreting Theorem 1.1.1 *as taking the quotient by a proper equivalence relation.* Whether or not $\mathcal{Z}$ satisfies [2], we can show (under mild hypotheses on $\mathcal{F}$) that the morphism $\xi : \mathcal{X} \to \mathcal{F}$ factors through a morphism $\bar{\xi} : \mathcal{X} \to \mathcal{Z}$ (see Lemma 3.2.23 below), and this morphism is "scheme-theoretically surjective". If we define $\mathcal{R} := \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$, then $\mathcal{R}$ is an algebraic stack (because $\mathcal{F}$ satisfies [3]) which defines a proper equivalence relation on $\mathcal{X}$. Thus, at least morally, we may regard $\mathcal{Z}$ as the quotient of $\mathcal{X}$ by the equivalence relation $\mathcal{R}$, and Theorem 1.1.1 may be regarded as providing a context in which the quotient $\mathcal{X}/\mathcal{R}$ may be defined as an algebraic stack.

Note that in general the quotient of an algebraic stack, or a scheme, or even a variety, by a proper equivalence relation, may not admit a reasonable interpretation as an object of algebraic geometry. (See e.g. Examples 4.1.1 and 4.1.2.) Our result shows that when the desired quotient admits an interpretation as the scheme-theoretic image of a morphism from $\mathcal{X}$ to some stack $\mathcal{F}$, the quotient does indeed have a chance to be of an algebro-geometric nature.

One well-known theorem which concerns taking a quotient by a proper equivalence relation is Artin's result [Art70] on the existence of contractions. We close

this discussion by briefly describing the relationship between [Art70] and the present note.

The relationship with [Art70]. In [Art70], Artin proves the existence of dilatations and contractions of an algebraic space along a closed algebraic subspace, given a formal model for the desired dilatation or contraction. (See also Remark 4.1.3 below.) In the case of contractions, his result can be placed in the framework of Corollary 1.1.2. Namely, taking $\mathcal{X}$ in that theorem to be the algebraic space on which one wishes to perform a contraction, one can define a functor $\mathcal{F}$ which is supposed to represent the result of the contraction, and a proper morphism $\xi : \mathcal{X} \to \mathcal{F}$. One can then show that $\mathcal{F}$ satisfies [1], [2], and [3], and hence conclude (via Corollary 1.1.2) that $\mathcal{F}$ is representable by an algebraic space. This is in fact essentially how Artin proceeds, although there are some slight differences between our approach and his: Artin defines $\mathcal{F}$ concretely on the category of *Noetherian* $\mathcal{O}_S$-algebra, and then extends it to arbitrary $\mathcal{O}_S$-algebras by taking limits. Our approach would be to apply Artin's concrete definition only to *finite type* $\mathcal{O}_S$-algebras, and then to extend to arbitrary $\mathcal{O}_S$-algebras by taking limits. The difference between the two approaches is that in Artin's approach, the fact that condition [2] is satisfied by $\mathcal{F}$ is rather automatic, whereas the proof that $\mathcal{F}$ satisfies [1] becomes the crux of the argument. In our approach, while the arguments remain the same, their interpretation differs: essentially by definition, $\mathcal{F}$ will satisfy [1], while the bulk of the argument can be seen as proving that it also satisfies [2]. However, this attempt to link Artin's result to ours is a little misleading, since in Artin's context, the verification that $\mathcal{F}$ satisfies [4] is straightforward. For some other $\mathcal{F}$ (such as those that appear in the theory of moduli of Galois representations discussed in Section 5), the verification of [4] seems to be less straightforward, however, and indeed the only approach we know is via the general arguments of the present paper.

3.1. **Scheme-theoretic images (part one).** We recall the following definition (see for example [Sta, Tag 01R8]; note that we do not include a quasi-separated hypothesis, as it is not needed for the basic properties of scheme-theoretic images that we use here).

3.1.1. **Definition.** Let $f : Y \to Z$ be a quasi-compact morphism of schemes. The kernel of the natural morphism $\mathcal{O}_Z \to f_*\mathcal{O}_Y$ is a quasi-coherent ideal sheaf $\mathcal{I}$ on Z ([Sta, Tag 01R8]) and we define the *scheme-theoretic image of f* to be the closed subscheme $V(\mathcal{I})$ of Z cut out by $\mathcal{I}$.

We say that f is *scheme-theoretically dominant* if the induced morphism $\mathcal{O}_Z \to f_*\mathcal{O}_Y$ is injective; that is, if the scheme-theoretic image of f is Z.

We say that f is *scheme-theoretically surjective* if it is scheme-theoretically dominant, and surjective on underlying topological spaces.

3.1.2. *Remark.* (1) The morphism f factors through $V(\mathcal{I})$, and $V(\mathcal{I})$ is the minimal closed subscheme of Z through which f factors. The induced morphism $f' : Y \to V(\mathcal{I})$ is scheme-theoretically dominant, and also has dense image [Sta, Tag 01R8].

Thus, for a closed morphism, the notions of scheme-theoretical dominance and of scheme-theoretical surjectivity are equivalent.

(2) The formation of scheme-theoretic images is compatible with arbitrary flat base change [Sta, Tag 081I].

(3) It follows easily from (2) that the formation of scheme-theoretic images is *fpqc* local on the target, so that the condition of being scheme-theoretically dominant, or surjective, may be checked *fpqc* locally on the target.

(4) If $g : X \to Y$ is quasi-compact then the scheme-theoretic image of fg is a closed subscheme of the scheme-theoretic image of f. If g is furthermore scheme-theoretically dominant (e.g. *fpqc*), then $\mathcal{O}_Z \to f_*\mathcal{O}_Y$ and $\mathcal{O}_Z \to (fg)_*\mathcal{O}_X$ have the same kernel, and hence the scheme-theoretic images of f and fg coincide.

In particular, if $g : X \to Y$ is quasi-compact and scheme-theoretically dominant (resp. surjective), then f is scheme-theoretically dominant (resp. surjective) if and only if the composite fg is scheme-theoretically dominant (resp. surjective).

(5) An *fpqc* morphism is scheme-theoretically surjective.

Points (3), (4), and (5) of the preceding remark allow us to extend the notion of scheme-theoretically dominant (resp. scheme-theoretically surjective) morphisms to the context of morphisms of algebraic stacks in the following way.

Recall (*cf.* Remark 2.3.6 and Lemma 2.3.8) that an algebraic stack $\mathcal{Y}$ is quasi-compact if its underlying topological space $|\mathcal{Y}|$ is quasi-compact, or equivalently, by [Sta, Tag 04YC], if there is a smooth surjection $U \to \mathcal{Y}$ with U a quasi-compact scheme. A morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{Z}$ is *quasi-compact* if for every morphism $\mathcal{V} \to \mathcal{Z}$ with $\mathcal{V}$ a quasi-compact algebraic stack, the fibre product $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{V}$ is also quasi-compact [Sta, Tag 050U].

3.1.3. **Definition.** Let $f : \mathcal{Y} \to \mathcal{Z}$ be a quasi-compact morphism of algebraic stacks. Let $V \to \mathcal{Z}$ be a smooth surjection from a scheme, and let $V = \cup_i T_i$ be a cover of V by quasi-compact open subschemes. For each T_i, the fibre product $\mathcal{Y} \times_{\mathcal{Z}} T_i$ is quasi-compact, so admits a smooth surjection $U_i \to \mathcal{Y} \times_{\mathcal{Z}} T_i$ from a quasi-compact scheme.

The composite morphism $U_i \to T_i$ is quasi-compact, and we say that $f : \mathcal{Y} \to \mathcal{Z}$ is *scheme-theoretically dominant* if for all i, the morphism $U_i \to T_i$ is scheme-theoretically dominant. We say that f is *scheme-theoretically surjective* if the morphisms $U_i \to T_i$ are all scheme-theoretically surjective.

It follows from Remark 3.1.2 (3)–(5) that this notion is well-defined, independently of the choices of V and the T_i and U_i, and that it agrees with Definition 3.1.1 if $\mathcal{Y}$ and $\mathcal{Z}$ are schemes.

Similarly, we can extend the definition of scheme-theoretic images to quasi-compact morphisms of algebraic stacks.

3.1.4. **Definition.** Let $f : \mathcal{Y} \to \mathcal{Z}$ be a quasi-compact morphism of algebraic stacks, and choose V, T_i and U_i as in Definition 3.1.3.

The composite morphism $U_i \to T_i$ is quasi-compact, hence admits a scheme-theoretic image T_i'. The smooth equivalence relation $R_{T_i} := T_i \times_{\mathcal{Z}} T_i$ on T_i restricts to a smooth equivalence relation $R_{T_i'}$ on T_i', and the quotient stack $[T_i'/R_{T_i'}]$ is a closed substack of the quotient stack $[T_i/R_{T_i}]$, itself an open substack of $\mathcal{Z}$. The substacks $[T_i'/R_{T_i'}]$ glue together to form a closed substack of $\mathcal{Z}$, which we define to be the *scheme-theoretic image* of f.

Again, it follows from Remark 3.1.2 (2)–(5) that this notion is well-defined, independently of the choices of V and the T_i and U_i, and that it agrees with Definition 3.1.1 if $\mathcal{Y}$ and $\mathcal{Z}$ are schemes.

Given this definition, the following remarks are an easy consequence of Remark 3.1.2.

3.1.5. *Remark.* (1) The morphism f factors through its scheme theoretic image $\mathcal{X}$, and $\mathcal{X}$ is the minimal closed substack of $\mathcal{Z}$ through which f factors. The induced morphism $f' : \mathcal{Y} \to \mathcal{X}$ is scheme-theoretically dominant, and the induced map $|\mathcal{Y}| \to |\mathcal{X}|$ has dense image.

(2) The formation of scheme-theoretic images is compatible with arbitrary flat base change.

(3) The formation of scheme-theoretic images is *fpqc* local on the target, so that the condition of being scheme-theoretically dominant, or surjective, may be checked *fpqc* locally on the target.

(4) If $g : \mathcal{X} \to \mathcal{Y}$ is quasi-compact then the scheme-theoretic image of fg is a closed substack of the scheme-theoretic image of f. If g is also scheme-theoretically dominant, then the scheme-theoretic images of f and fg coincide.

In particular, if $g : \mathcal{X} \to \mathcal{Y}$ is quasi-compact and scheme-theoretically dominant (resp. surjective), then f is scheme-theoretically dominant (resp. surjective) if and only if the composite fg is scheme-theoretically dominant (resp. surjective).

3.1.6. *Remark.* One can show that if $f : \mathcal{Y} \to \mathcal{Z}$ is a quasi-compact and quasi-separated morphism of algebraic stacks, then the kernel of $\mathcal{O}_\mathcal{Z} \to f_*\mathcal{O}_\mathcal{Y}$ is a quasi-coherent ideal sheaf, and cuts out the scheme-theoretic image of f as defined above. (In fact, presumably this is true even without the quasi-separatedness assumption, although, as in the scheme-theoretic context, such a statement would be slightly more delicate to prove, since then $f_*\mathcal{O}_\mathcal{Y}$ need not be quasi-coherent.) However, since we do not need to consider sheaves on stacks elsewhere in this paper, except very briefly in the proof of Lemma 3.2.1 below, we will not give the details here, but rather simply establish the few basic facts that we need as we use them.

The one special case of this theory that we need is that if $f : \mathcal{X} \to S$ is a quasi-compact algebraic stack over a Noetherian base scheme S, then f is scheme-theoretically dominant if and only if $\mathcal{O}_S \to f_*\mathcal{O}_\mathcal{X}$ is injective. To see this, let $g : U \to \mathcal{X}$ be a smooth cover by a quasi-compact scheme; then by Definition 3.1.3, f is scheme-theoretically dominant if and only if the composite $fg : U \to \mathcal{X} \to S$ is scheme-theoretically dominant, or equivalently if and only if $\mathcal{O}_S \to (fg)_*\mathcal{O}_U$ is injective. It remains to show that $f_*\mathcal{O}_\mathcal{X} \to (fg)_*\mathcal{O}_U$ is injective. Since f_* is left exact, it is enough to show that $\mathcal{O}_\mathcal{X} \to g_*\mathcal{O}_U$ is injective. By the definition of $\mathcal{O}_\mathcal{X}$, this can be checked after pulling back by a smooth cover of $\mathcal{X}$ by a scheme (for example, U itself), so we reduce to the case of algebraic spaces, which is immediate from [Sta, Tag 082Z].

We have the following simple lemma concerning scheme-theoretic dominance.

3.1.7. **Lemma.** *Suppose that $f : \mathcal{Y} \to \mathcal{Z}$ is a scheme-theoretically dominant quasi-compact morphism of algebraic stacks, and that $\mathcal{Z}' \hookrightarrow \mathcal{Z}$ is a closed immersion for which the base-changed morphism $\mathcal{Y}' \to \mathcal{Y}$ is an isomorphism. Then $\mathcal{Z}' \hookrightarrow \mathcal{Z}$ is itself an isomorphism.*

Proof. The assumption that $\mathcal{Y}' \xrightarrow{\sim} \mathcal{Y}$ can be rephrased as saying that the morphism $\mathcal{Y} \to \mathcal{Z}$ can be factored through the closed substack $\mathcal{Z}'$ of $\mathcal{Z}$. Since $\mathcal{Y} \to \mathcal{Z}$ is scheme-theoretically dominant, we see that necessarily $\mathcal{Z}' = \mathcal{Z}$. (This last statement is easily reduced to the case of schemes, where it is immediate.) $\square$

3.2. **Scheme-theoretic images (part two).** Our goal in this section is to generalise the construction of scheme-theoretic images to certain morphisms of stacks

whose domain is algebraic, but whose target is of a possibly more general nature. The general set-up, which will be in force throughout this section, will be as follows: we suppose given a morphism $\xi : \mathcal{X} \to \mathcal{F}$ of stacks over a locally Noetherian base-scheme S, whose domain $\mathcal{X}$ is assumed to be algebraic, and whose target $\mathcal{F}$ is assumed to have diagonal $\Delta_{\mathcal{F}}$ which is representable by algebraic spaces and locally of finite presentation (so, in particular we are assuming throughout this section that $\mathcal{F}$ satisfies [3]).

We will furthermore typically assume that either $\mathcal{F}$ satisfies [1], or else that $\mathcal{X}$ is locally of finite presentation. We will sometimes reduce the latter situation to the former by using the results of Section 2.5. (See Remark 3.2.13 below.) We will also frequently need to assume that ξ is proper. We work in maximal generality when we can, only introducing these hypotheses at the points that they are needed (but see Remark 3.2.10 (3) below).

We begin with a lemma whose intent is to capture the properties that will characterise when a morphism $\operatorname{Spec} A \to \mathcal{F}$, with A a finite type Artinian local $\mathcal{O}_S$-algebra, factors through the scheme-theoretic image of ξ.

3.2.1. **Lemma.** *If* $\operatorname{Spec} A$ *is a finite type Artinian local S-scheme, and* $\varphi : \operatorname{Spec} A \to \mathcal{F}$ *is a morphism over S, then the following conditions are equivalent:*

(1) *There exists a complete Noetherian local $\mathcal{O}_S$-algebra B, and a factorisation of φ into S-morphisms $\operatorname{Spec} A \to \operatorname{Spec} B \to \mathcal{F}$, such that the base-changed morphism $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant.*

(2) *There exists a complete Noetherian local $\mathcal{O}_S$-algebra B, and a factorisation of φ into S-morphisms $\operatorname{Spec} A \to \operatorname{Spec} B \to \mathcal{F}$, such that the base-changed morphism $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant, and such that the morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is a closed immersion.*

If ξ is furthermore proper, then these conditions are equivalent to the following further two conditions.

(3) *There exists an Artinian local $\mathcal{O}_S$-algebra B, and a factorisation of φ into S-morphisms $\operatorname{Spec} A \to \operatorname{Spec} B \to \mathcal{F}$, such that the base-changed morphism $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant.*

(4) *There exists an Artinian local $\mathcal{O}_S$-algebra B, and a factorisation of φ into S-morphisms $\operatorname{Spec} A \to \operatorname{Spec} B \to \mathcal{F}$, such that the base-changed morphism $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant, and such that the morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is a closed immersion.*

3.2.2. *Remark.* We remark that if A is a Artinian local $\mathcal{O}_S$-algebra, which is furthermore of finite type (or, equivalently, whose residue field is of finite type over $\mathcal{O}_S$), and if B is a local $\mathcal{O}_S$-algebra for which there exists a morphism of $\mathcal{O}_S$-algebras $B \to A$, then the residue field of B is necessarily also of finite type over $\mathcal{O}_S$. Thus, in the context of the preceding lemma, the residue field of the ring B appearing in any of the conditions of the lemma will necessarily be of finite type over $\mathcal{O}_S$.

The proof of Lemma 3.2.1 will make use of the theorem on formal functions for algebraic stacks in the form of the following theorem of Olsson. As in [Ols07], we work with sheaves on the lisse-étale site.

3.2.3. **Theorem.** *Let A be a Noetherian adic ring, and let I be an ideal of definition of A. Let $\mathcal{X}$ be a proper algebraic stack over* $\operatorname{Spec} A$. *Then the functor sending a sheaf to its reductions is an equivalence of categories between the category of*

coherent sheaves $\mathcal{G}$ on $\mathcal{X}$ and the category of compatible systems of coherent sheaves $\mathcal{G}_n$ on the reductions $\mathcal{X}_n := \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I^{n+1})$.

Furthermore, if $\mathcal{G}$ is a coherent sheaf on $\mathcal{X}$ with reductions $\mathcal{G}_n$, then the natural map

$$H^0(\mathcal{X}, \mathcal{G}) \to \varprojlim_n H^0(\mathcal{X}_n, \mathcal{G}_n)$$

is an isomorphism of topological A-modules, where the left hand side has the I-adic topology, and the right hand side has the inverse limit topology.

Proof. A proper morphism of stacks is separated, so has quasi-compact and quasi-separated diagonal, so that the more restrictive definition of an algebraic stack in [Ols07] is automatically satisfied. The claimed result is then a special case of [Ols07, Thm. 11.1]. $\qquad\square$

Proof of Lemma 3.2.1. Evidently (2) implies (1). Conversely, suppose (1) holds. Bearing in mind Remark 3.2.2, we see that the residue field of A is finite over that of B, so Lemma 1.6.3 shows that we may factor the morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ as $\operatorname{Spec} A \to \operatorname{Spec} B' \to \operatorname{Spec} B$, where B' is again a complete Noetherian local ring, such that the first morphism is a closed immersion, and such that the second morphism is faithfully flat. Since scheme-theoretic dominance is preserved under flat base-change, we see that $\mathcal{X}_{B'} \to \operatorname{Spec} B'$ is scheme-theoretically dominant, and so replacing B by B', we find that (2) is satisfied.

It is evident that (4) implies (3) implies (1), with no assumption on ξ. Thus, to complete the proof of the lemma, it suffices to show that (2) implies (4), when ξ is proper. In fact, we will show that if $\operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism as in (2), then we may find an Artinian quotient B' of B such that this morphism factors through the closed immersion $\operatorname{Spec} B' \to \operatorname{Spec} B$, and such that the base-changed morphism $\mathcal{X}_{B'} \to \operatorname{Spec} B'$ is scheme-theoretically dominant. Certainly $\operatorname{Spec} A \to \operatorname{Spec} B$ factors through the closed immersion $\operatorname{Spec} B' \to \operatorname{Spec} B$ for *some* Artinian quotient of B, since A itself is Artinian. Thus it suffices to show that B admits a cofinal collection of Artinian quotients B' for which $\mathcal{X}_{B'} \to \operatorname{Spec} B'$ is scheme-theoretically dominant. This is the content of Lemma 3.2.4 below. $\qquad\square$

3.2.4. Lemma. *Assume that ξ is proper, and suppose that $\operatorname{Spec} B \to \mathcal{F}$ is a morphism over S, where B is a complete Noetherian local $\mathcal{O}_S$-algebra. Then the base-changed morphism $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant if and only if B admits a cofinal collection of Artinian quotients B' for which $\mathcal{X}_{B'} \to \operatorname{Spec} B'$ is scheme-theoretically dominant.*

Proof. Suppose firstly that B admits a cofinal collection of Artinian quotients B' for which $\mathcal{X}_{B'} \to \operatorname{Spec} B'$ is scheme-theoretically dominant. Then the scheme-theoretic image of $\mathcal{X}_B \to \operatorname{Spec} B$ is a closed subscheme of $\operatorname{Spec} B$ containing each of the $\operatorname{Spec} B'$, and since the quotients B' are cofinal, it must in fact be $\operatorname{Spec} B$.

We now consider the converse. Since ξ is proper, so is the base-changed morphism $\mathcal{X}_B \to \operatorname{Spec} B$. If $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant, then the natural map $B \to H^0(\operatorname{Spec} B, \xi_* \mathcal{O}_{\mathcal{X}_B}) = H^0(\mathcal{X}_B, \mathcal{O}_{\mathcal{X}_B})$ is injective by Remark 3.1.6. Set $\mathcal{X}_n = \mathcal{X}_B \times_{\operatorname{Spec} B} \operatorname{Spec}(B/\mathfrak{m}_B^{n+1})$. Noting that the structure sheaf $\mathcal{O}_{\mathcal{X}_B}$ is coherent, Theorem 3.2.3 applies to show that we have an isomorphism of topological B-modules

$$H^0(\mathcal{X}, \mathcal{O}_X) \xrightarrow{\sim} \varprojlim_n H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}).$$

Let I_n be the kernel of the composite morphism

$$B \to H^0(\mathcal{X}, \mathcal{O}_X) \to H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}),$$

so that we have injections $B/I_n \to H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$. The natural map $\mathcal{X}_n \to \mathcal{X} \times_{\operatorname{Spec} B} \operatorname{Spec}(B/I_n)$ is then an isomorphism by construction, and so $\mathcal{X} \times_{\operatorname{Spec} B} \operatorname{Spec}(B/I_n) \to \operatorname{Spec} B/I_n$ is scheme-theoretically dominant (again by Remark 3.1.6; since $B/I_n \to H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is injective, the natural map $\mathcal{X}_n \to \operatorname{Spec}(B/I_n)$ is scheme-theoretically dominant). Thus the B/I_n are the sought-after cofinal collection of Artinian quotients of B. $\qquad\square$

We now define, in our present context, a subgroupoid $\widehat{\mathcal{Z}}$ of $\widehat{\mathcal{F}}$.

3.2.5. **Definition.** If $\operatorname{Spec} A$ is a finite type Artinian local S-scheme, then we let $\widehat{\mathcal{Z}}(A)$ denote the full subgroupoid of $\mathcal{F}(A)$ consisting of morphisms $\operatorname{Spec} A \to \mathcal{F}$ that satisfy the equivalent conditions (1) and (2) of Lemma 3.2.1. (If ξ is proper, then we note that the objects of $\widehat{\mathcal{Z}}(A)$ can equally well be characterised in terms of conditions (3) and (4) of that lemma.)

We now define the scheme-theoretic image $\mathcal{Z}$ of ξ, using the terminology and results of Section 2.5 (note that since S is assumed to be locally Noetherian, it is in particular quasi-separated, so the results of Section 2.5 apply to categories fibred in groupoids over S). As in Section 2.5, we let $\operatorname{Aff}_{/S}$ denote the category of affine S-schemes, and let $\operatorname{Aff}_{\mathrm{pf}/S}$ denote the full subcategory of finitely presented affine S-schemes.

To begin with, we will consider the restriction $\mathcal{F}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ to a category fibred in groupoids over $\operatorname{Aff}_{\mathrm{pf}/S}$, and define a full subcategory of $\mathcal{F}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ which (by an abuse of notation which we will justify below) we denote $\mathcal{Z}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$.

3.2.6. **Definition.** Let $\mathcal{Z}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ be the full subcategory of $\mathcal{F}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ defined as follows: if A is a finite type $\mathcal{O}_S$-algebra, then we let $\mathcal{Z}_{|\operatorname{Aff}_{\mathrm{pf}/S}}(A)$ denote the full subgroupoid of $\mathcal{F}(A)$ consisting of points η whose formal completion $\widehat{\eta}_t$, at each finite type point t of $\operatorname{Spec} A$, factors through $\widehat{\mathcal{Z}}$.

3.2.7. **Lemma.** $\mathcal{Z}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ *is a Zariski substack of* $\mathcal{F}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$.

Proof. To check that $\mathcal{Z}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ is a full subcategory fibred in groupoids of $\mathcal{F}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$, we need to check that if $T' \to T$ is a morphism in $\operatorname{Aff}_{\mathrm{pf}/S}$, and $T \to \mathcal{F}$ satisfies the condition to lie in $\mathcal{Z}(T)$, then the same is true of the pulled-back morphism $T' \to \mathcal{F}$; but this is immediate from the definition of $\mathcal{Z}$. To see that it is actually a substack of $\mathcal{F}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$, it is then enough (by [Sta, Tag 04TU]) to note that since the condition of a morphism $T \to \mathcal{F}$ factoring through $\mathcal{Z}$ is checked pointwise, it is in particular a Zariski-local condition. $\qquad\square$

We remind the reader that Theorem 2.5.1 gives an equivalence of categories $\operatorname{pro-Aff}_{\mathrm{pf}/S} \xrightarrow{\sim} \operatorname{Aff}_{/S}$. We now give our definition of the scheme-theoretic image of $\mathcal{X} \to \mathcal{F}$; see Remark 3.2.10 (2) below for the relationship of this definition to the existing one in the case that $\mathcal{F}$ is an algebraic stack.

3.2.8. **Definition.** We define $\mathcal{Z}$ to be the pro-category $\operatorname{pro-}\mathcal{Z}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$, which (by Lemmas 2.5.6 and 3.2.7) is a full subcategory in groupoids of $\operatorname{pro-}\mathcal{F}_{|\operatorname{Aff}_{\mathrm{pf}/S}}$. Via the equivalence of Theorem 2.5.1, we regard $\mathcal{Z}$ as a category fibred in groupoids over $\operatorname{Aff}_{/S}$. We also note that Lemma 2.5.3 yields an equivalence between the restriction

of $\mathcal{Z}$ to $\mathrm{Aff}_{\mathrm{pf}/S}$ and our given category $\mathcal{Z}_{|\mathrm{Aff}_{\mathrm{pf}/S}}$, thus justifying the notation for the latter.

Since $\Delta_{\mathcal{F}}$ is assumed to be representable by algebraic spaces and locally of finite presentation, it follows from Lemma 2.5.5 (4) that pro-$(\mathcal{F}|_{\mathrm{Aff}_{\mathrm{pf}/S}})$ is equivalent to a full subcategory of $\mathcal{F}$, so that we may regard $\mathcal{Z}$ as a full subcategory of $\mathcal{F}$ (the latter being thought of as a category fibred in groupoids over the category of affine S-schemes). We refer to $\mathcal{Z}$ as the *scheme-theoretic image* of the morphism $\xi : \mathcal{X} \to \mathcal{F}$.

3.2.9. Lemma. *$\mathcal{Z}$ is a Zariski substack of $\mathcal{F}$ over $\mathrm{Aff}_{/S}$, and satisfies [1].*

Proof. This follows immediately from Lemmas 2.5.5 (2), 2.5.4, and 3.2.7. □

3.2.10. *Remark.* (1) Since $\mathcal{Z}$ is a Zariski stack over $\mathrm{Aff}_{/S}$, it naturally extends to a Zariski stack over S (and is again a Zariski substack of $\mathcal{F}$). We will see in Lemma 3.2.20 below that under some relatively mild additional assumptions, $\mathcal{Z}$ is in fact an étale substack of $\mathcal{F}$.

(2) In the case when $\mathcal{F}$ is an algebraic stack which satisfies [1], then its substack $\mathcal{Z}$ coincides with the scheme-theoretic image of ξ (in the sense of Definition 3.1.4); see Lemma 3.2.29 below.

(3) If neither $\mathcal{X}$ nor $\mathcal{F}$ satisfy [1], then Definition 3.2.6 is not a sensible one. E.g. if $\mathcal{F}$ is an algebraic stack that does not satisfy [1], and $\xi : \mathcal{F} \to \mathcal{F}$ is the identity morphism, then evidently the scheme-theoretic image of ξ (in the usual sense, i.e. in the sense of Definition 3.1.4) is just $\mathcal{F}$, and so doesn't satisfy [1] (by assumption) — whereas we noted above that the substack $\mathcal{Z}$ of $\mathcal{F}$ given by Definition 3.2.6 does satisfy [1], by stipulation.

Here and on several occasions below, we will have cause to consider how scheme-theoretic images interact with monomorphisms of stacks $\mathcal{F}' \hookrightarrow \mathcal{F}$. In light of this, it will be helpful to recall that Lemma 2.3.22 implies that if $\mathcal{F}$ has a diagonal which is representable by locally algebraic spaces and locally of finite presentation, then the same is true of any stack $\mathcal{F}'$ admitting a monomorphism $\mathcal{F}' \hookrightarrow \mathcal{F}$ (i.e. any substack $\mathcal{F}'$ of $\mathcal{F}$).

3.2.11. Lemma. *Suppose that $\mathcal{X} \to \mathcal{F}$ factors as $\mathcal{X} \to \mathcal{F}' \hookrightarrow \mathcal{F}$, where $\mathcal{F}'$ is a stack which satisfies [3], and $\mathcal{F}' \hookrightarrow \mathcal{F}$ is a monomorphism. Then the scheme-theoretic image of $\mathcal{X} \to \mathcal{F}'$ is contained in the scheme-theoretic image of $\mathcal{X} \to \mathcal{F}$.*

Proof. A consideration of the definitions shows that we need only check that if $\mathrm{Spec}\, B \to \mathcal{F}'$ is such that $\mathcal{X} \times_{\mathcal{F}'} \mathrm{Spec}\, B \to \mathrm{Spec}\, B$ is scheme-theoretically dominant, then $\mathcal{X} \times_{\mathcal{F}} \mathrm{Spec}\, B \to \mathrm{Spec}\, B$ is scheme-theoretically dominant; but this is immediate, because $\mathcal{X} \times_{\mathcal{F}'} \mathrm{Spec}\, B = \mathcal{X} \times_{\mathcal{F}} \mathrm{Spec}\, B$. □

3.2.12. *Remark.* Throughout the rest of this section we prove several variants and refinements of Lemma 3.2.11. More specifically, Lemma 3.2.21 gives a criterion for the scheme-theoretic images of $\mathcal{X} \to \mathcal{F}'$ and $\mathcal{X} \to \mathcal{F}$ to actually be equal, and Lemma 3.2.22 gives a criterion for $\mathcal{X} \to \mathcal{F}$ to factor through $\mathcal{Z}$. Lemmas 3.2.25 and 3.2.26 relate the existence of a factorisation $\mathcal{X} \to \mathcal{F}' \hookrightarrow \mathcal{F}$ to the property of $\mathcal{F}'$ containing $\mathcal{Z}$, and Proposition 3.2.31 shows that $\mathcal{Z}$ can be characterised by a universal property if it is assumed to be algebraic.

3.2.13. *Remark.* As we explained at the beginning of this section, our main results assume either that $\mathcal{F}$ satisfies [1], or that $\mathcal{X}$ satisfies [1] (equivalently, $\mathcal{X}$ is locally

of finite presentation over S). In the proof of the main theorem, we will argue by reducing the second case to the first case in the following way:

We first note that it follows from Lemmas 2.5.3, 2.5.4, and 2.5.5 (2) and (4) that pro-$(\mathcal{F}|_{\mathrm{Aff}_{\mathrm{pf}/S}})$ is a substack of $\mathcal{F}$ which satisfies [1] and [3]. If we assume in addition that $\mathcal{F}$ admits versal rings at all finite type points, then so does pro-$(\mathcal{F}|_{\mathrm{Aff}_{\mathrm{pf}/S}})$, by Lemma 2.5.5 (5).

If $\mathcal{X}$ satisfies [1], then it follows from Lemma 2.5.4 that there is an equivalence pro-$\mathcal{X}|_{\mathrm{Aff}_{\mathrm{pf}/S}} \xrightarrow{\sim} \mathcal{X}$, and thus from Lemma 2.5.5 (1) that $\xi : \mathcal{X} \to \mathcal{F}$ can be factored through a morphism $\mathcal{X} \to \mathrm{pro}\text{-}\mathcal{F}_{\mathrm{Aff}_{\mathrm{pf}/S}}$. By construction the monomorphism $\mathcal{Z} \hookrightarrow \mathcal{F}$ also factors through pro-$\mathcal{F}_{\mathrm{Aff}_{\mathrm{pf}/S}}$, and an examination of the definitions shows that $\mathcal{Z}$ is also the scheme-theoretic image (in the sense of Definition 3.2.8) of the induced morphism $\mathcal{X} \to \mathrm{pro}\text{-}(\mathcal{F}|_{\mathrm{Aff}_{\mathrm{pf}/S}})$. Taken together, the previous remarks will allow us to simply replace $\mathcal{F}$ by pro-$(\mathcal{F}|_{\mathrm{Aff}_{\mathrm{pf}/S}})$, and thus assume that we are in the first case.

3.2.14. Lemma. *Assume that $\xi : \mathcal{X} \to \mathcal{F}$ is proper, and let $x : \operatorname{Spec} k \to \mathcal{F}$ be a finite type point. Then x is a point of $\mathcal{Z}$ if and only if the fibre $\mathcal{X}_x$ is non-empty.*

Proof. If $\mathcal{X}_x$ is non-empty, then $\mathcal{X}_x \to \operatorname{Spec} k$ is scheme-theoretically surjective, and x is a point of $\mathcal{Z}$ by definition. Conversely, if x is a point of $\mathcal{Z}$, then by Lemma 3.2.1 we can factor $x : \operatorname{Spec} k \to \mathcal{F}$ through a closed immersion $\operatorname{Spec} k \to \operatorname{Spec} B$ with B an Artin local ring and $\mathcal{X}_B \to \operatorname{Spec} B$ scheme-theoretically dominant. This implies that the fibre $\mathcal{X}_x$ is non-empty, as required. $\square$

3.2.15. Definition. If $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is a versal ring at x, then (by Remark 3.1.2 (4)) if we let A_i run over the discrete Artinian quotients of A_x, the scheme-theoretic images $\operatorname{Spec} R_i$ of the morphisms $\mathcal{X}_{A_i} \to \operatorname{Spec} A_i$ fit together to give a formal subscheme $\operatorname{Spf} R_x := \operatorname{Spf} \varprojlim_i R_i$ of $\widehat{\mathcal{F}}_x$, which we call the scheme-theoretic image of the base-changed morphism $\mathcal{X} \times_{\mathcal{F}} \operatorname{Spf} A_x \to \operatorname{Spf} A_x$.

It follows from Remark 2.2.7 that the natural morphism $A_x \to R_x$ is surjective, and is a quotient map of topological rings. Thus $\operatorname{Spf} R_x$ is even a *closed* formal subscheme of $\operatorname{Spf} A_x$.

The following lemma shows that, when ξ is proper, versal rings for $\mathcal{Z}$ can be constructed from versal rings for $\mathcal{F}$ by taking scheme-theoretic images.

3.2.16. Lemma. *Assume that $\xi : \mathcal{X} \to \mathcal{F}$ is proper. Let $x : \operatorname{Spec} k \to \mathcal{Z}$ be a finite type point, which we also consider as a finite type point of $\mathcal{F}$. Suppose that $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ is a versal ring at x, and let $\operatorname{Spf} R_x$ be the scheme-theoretic image of $\mathcal{X}_{\operatorname{Spf} A_x} \to \operatorname{Spf} A_x$. Then the morphism $\operatorname{Spf} R_x \to \widehat{\mathcal{F}}_x$ factors through a versal morphism $\operatorname{Spf} R_x \to \widehat{\mathcal{Z}}_x$.*

Proof. We claim that a morphism $\operatorname{Spec} A \to \operatorname{Spf} A_x$, with A an object of $\mathcal{C}_\Lambda$, factors through $\operatorname{Spf} R_x$ if and only if the composite $\operatorname{Spec} A \to \operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ factors through $\widehat{\mathcal{Z}}_x$. In the notation of Definition 3.2.15, if $\operatorname{Spec} A \to \widehat{\mathcal{F}}_x$ factors through $\operatorname{Spf} R_x$, then it in fact factors through $\operatorname{Spec} R_i$ for some i, and hence through $\widehat{\mathcal{Z}}_x$ by definition (as the morphism $\mathcal{X}_{R_i} \to \operatorname{Spec} R_i$ is scheme-theoretically dominant).

Conversely, if the composite $\operatorname{Spec} A \to \operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ factors through $\widehat{\mathcal{Z}}_x$, then by Lemma 3.2.1, we have a factorisation $\operatorname{Spec} A \to \operatorname{Spec} B \to \mathcal{F}$, where B is an object

of $\mathcal{C}_\Lambda$, the morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is a closed immersion, and $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant. By the versality of $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$, we may lift the morphism $\operatorname{Spec} B \to \widehat{\mathcal{F}}_x$ to a morphism $\operatorname{Spec} B \to \operatorname{Spf} A_x$, which furthermore we may factor as $\operatorname{Spec} B \to \operatorname{Spec} A_i \to \operatorname{Spf} A_x$, for some value of i. Since $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant, the morphism $\operatorname{Spec} B \to \operatorname{Spec} A_i$ then factors through $\operatorname{Spec} R_i$, and thus through $\operatorname{Spf} R_x$, as claimed.

In particular, we see that the composite $\operatorname{Spf} R_x \to \operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ factors through a morphism $\operatorname{Spf} R_x \to \widehat{\mathcal{Z}}_x$. It remains to check that this morphism is versal. This is formal. Suppose we are given a commutative diagram

$$
\begin{array}{ccccc}
\operatorname{Spec} A_0 & \longrightarrow & \operatorname{Spf} R_x & \longrightarrow & \operatorname{Spf} A_x \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} A & \longrightarrow & \widehat{\mathcal{Z}}_x & \longrightarrow & \widehat{\mathcal{F}}_x
\end{array}
$$

where the left hand vertical arrow is a closed immersion, and A_0, A are objects of $\mathcal{C}_\Lambda$. By the versality of $\operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$, we may lift the composite $\operatorname{Spec} A \to \widehat{\mathcal{F}}_x$ to a morphism $\operatorname{Spec} A \to \operatorname{Spf} A_x$. Since the composite $\operatorname{Spec} A \to \operatorname{Spf} A_x \to \widehat{\mathcal{F}}_x$ factors through $\widehat{\mathcal{Z}}_x$, the morphism $\operatorname{Spec} A \to \operatorname{Spf} A_x$ then factors through $\operatorname{Spf} R_x$, as required. $\qquad\square$

If $\mathcal{Z}$ were to behave like the scheme-theoretic image of a morphism of algebraic stacks, i.e. like a closed substack, then we would expect to be able to test the property of a morphism to $\mathcal{F}$ factoring through $\mathcal{Z}$ by precomposing with a scheme-theoretically dominant morphism. We cannot prove this general statement at this point of the development, but we begin our discussion of this general problem by establishing some special cases.

3.2.17. Lemma. *Let $Y \to Z \to \mathcal{F}$ be a composite of morphisms over S, with Y and Z being local Artinian schemes of finite type over S, for which the morphism $Y \to Z$ is scheme-theoretically surjective, and such that the composite $Y \to \mathcal{F}$ factors through $\mathcal{Z}$. Suppose that:*

    (a) either ξ is proper and $\mathcal{F}$ is $\mathbf{Art}^{\mathbf{triv}}$-homogeneous, or $\mathcal{F}$ satisfies [2](b); and

    (b) either $\mathcal{F}$ is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous, i.e. $\mathcal{F}$ satisfies (RS), or the residue field extension in $Y \to Z$ is separable.

Then the morphism $Z \to \mathcal{F}$ also factors through $\mathcal{Z}$.

Proof. Set $Y = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A'$; the scheme-theoretically surjective morphism $Y \to Z$ then corresponds to an injective morphism $A' \to A$ of $\mathcal{O}_S$-algebras. By assumption we may find a surjection of $\mathcal{O}_S$-algebras $B \to A$, whose source is a complete Noetherian local $\mathcal{O}_S$-algebra, such that the given morphism $\operatorname{Spec} A \to \mathcal{F}$ factors through a morphism $\operatorname{Spec} B \to \mathcal{F}$ for which the induced morphism $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically surjective.

Let $B' := A' \times_A B$. We have a commutative diagram

$$\begin{array}{ccc}
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A' \\
\downarrow & & \downarrow \\
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} B' \\
& & \searrow \\
& & \mathcal{F}
\end{array}$$
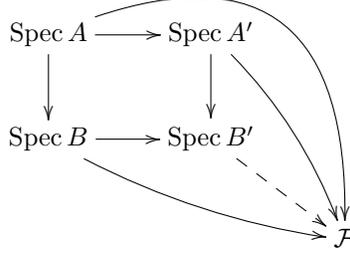
Suppose firstly that ξ is proper and $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous. By Lemma 3.2.1, we may assume that B is Artinian. We claim that we may fill in this diagram with a morphism $\operatorname{Spec} B' \to \mathcal{F}$; this follows since we are assuming that either $\mathcal{F}$ is $\mathbf{Art^{fin}}$-homogeneous, or that the residue field extension is separable, in which case this follows from Lemma 2.2.6. Otherwise, if ξ is not proper, then by assumption $\mathcal{F}$ satisfies [2](b). Lemma 2.7.4 and our hypotheses then show that we may once again fill in this diagram with a morphism $\operatorname{Spec} B' \to \mathcal{F}$.

It remains to show that $\mathcal{X}_{B'} \to \operatorname{Spec} B'$ is scheme-theoretically dominant. But this follows directly from the fact that each of $\mathcal{X}_B \to \operatorname{Spec} B$ and $\operatorname{Spec} B \to \operatorname{Spec} B'$ are scheme-theoretically dominant (the latter because $B' \to B$ is injective), and a consideration of the commutative diagram

$$\begin{array}{ccc}
\mathcal{X}_B & \longrightarrow & \mathcal{X}_{B'} \\
\downarrow & & \downarrow \\
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} B'
\end{array}$$

(This is a particular case of Remark 3.1.5 (4).)   □

3.2.18. **Lemma.** *Suppose that either ξ is proper and $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous, or that $\mathcal{F}$ satisfies [2](b). If $\mathcal{U} \to \mathcal{Y}$ is a smooth surjective morphism of algebraic stacks over S, and if $\mathcal{Y} \to \mathcal{F}$ is a morphism of stacks over S such that the composite $\mathcal{U} \to \mathcal{Y} \to \mathcal{F}$ factors through $\mathcal{Z}$, then the morphism $\mathcal{Y} \to \mathcal{F}$ itself factors through $\mathcal{Z}$.*

Proof. Let $T \to \mathcal{Y}$ be a morphism, with T a scheme over S. We must show that the composite $T \to \mathcal{Y} \to \mathcal{F}$ factors through $\mathcal{Z}$. By assumption the morphism $\mathcal{U} \times_{\mathcal{Y}} T \to T \to \mathcal{Y} \to \mathcal{F}$ (which admits the alternate factorisation $\mathcal{U} \times_{\mathcal{Y}} T \to \mathcal{U} \to \mathcal{Y} \to \mathcal{F}$) factors through $\mathcal{Z}$. Since $\mathcal{U} \times_{\mathcal{Y}} T$ is an algebraic stack, it admits a smooth surjection from a scheme U.

The composite $U \to T$ is a smooth morphism, and so we may apply [Sta, Tag 055V] to find a morphism $V \to U$ for which the composite $V \to U \to T$ is étale and surjective. Replacing U by V, we are reduced to showing that if $U \to T$ is a surjective étale morphism of S-schemes, and the composite $U \to T \to \mathcal{F}$ factors through $\mathcal{Z}$, then the morphism $T \to \mathcal{F}$ factors through $\mathcal{Z}$. (This is essentially the étale sheaf property of $\mathcal{Z}$; see Lemma 3.2.20 and its proof below.)

Since $\mathcal{Z}$ is a Zariski substack of $\mathcal{F}$, we may check this statement Zariski locally on T, and hence assume that T is affine. Since T is then quasi-compact, we may find a quasi-compact open subset of U which surjects onto T, and then replacing U by the disjoint union of the members of a finite affine cover of this quasi-compact open subset, we may further assume that U is affine. Since $\mathcal{Z}$ satisfies [1], we may further reduce to the case that T and U are locally of finite presentation over S, and hence of finite type over S (since S is locally Noetherian). (More precisely:

by Lemma 2.3.14, we may write $T = \varprojlim_i T_i$, $U = \varprojlim_i U_i$, with $U_i \to T_i$ an étale surjection of affine schemes locally of finite presentation over S, such that $U \to T$ is the pull-back of each $U_i \to T_i$. Since $\mathcal{Z}$ satisfies [1], the morphism $U \to \mathcal{Z}$ factors through U_i for some i, and thus so does the composite $U \to \mathcal{Z} \to \mathcal{F}$. Arguing as in the proof of Lemma 2.5.5 (2), the étale descent data for the morphism $U \to \mathcal{F}$ arises as the base change of étale descent data for the morphism $U_{i'} \to \mathcal{F}$ for some $i' \geq i$, and since $\mathcal{F}$ is an étale stack, it follows that $T \to \mathcal{F}$ factors through $T_{i'}$, as required.)

In this case we are reduced to checking that if $u : \operatorname{Spec} l \to U$ is a finite type point lying over the finite type point $t : \operatorname{Spec} k \to T$, then the morphism $\operatorname{Spf} \hat{\mathcal{O}}_{T,t} \to \widehat{\mathcal{F}}_t$ factors through $\widehat{\mathcal{Z}}_t$ provided that the morphism $\operatorname{Spf} \hat{\mathcal{O}}_{U,u} \to \widehat{\mathcal{F}}_u$ factors through $\widehat{\mathcal{Z}}_u$. That this is true follows from Lemma 3.2.17 and the fact that the local étale (and hence faithfully flat) morphism $\hat{\mathcal{O}}_{T,t} \to \hat{\mathcal{O}}_{U,u}$ can be written as the inverse limit of faithfully flat (and hence injective) morphisms of Artinian local rings with separable residue field extensions (to which the cited lemma applies). $\qquad\square$

The following lemma gives a useful criterion for when a morphism $\mathcal{T} \to \mathcal{F}$ factors through $\mathcal{Z}$.

3.2.19. **Lemma.** *Assume that either ξ is proper and $\mathcal{F}$ is $\mathbf{Art}^{\mathbf{triv}}$-homogeneous, or that $\mathcal{F}$ satisfies [2](b). Let $\mathcal{T}$ be an algebraic stack over S, and let $\eta : \mathcal{T} \to \mathcal{F}$ be a morphism for which the base-changed morphism $\mathcal{X}_{\mathcal{T}} \to \mathcal{T}$ is scheme-theoretically dominant. Assume further that either $\mathcal{F}$ satisfies [1], or that $\mathcal{T}$ is locally of finite type over S. Then η factors through $\mathcal{Z}$.*

Proof. Since $\mathcal{T}$ is an algebraic stack, we may find a smooth surjection $U \to \mathcal{T}$ with U a scheme. Since the formation of scheme-theoretic images commutes with flat base-change, we see that $\mathcal{X}_U \to U$ must be scheme-theoretically dominant. On the other hand, Lemma 3.2.18 shows that $\mathcal{T} \to \mathcal{F}$ factors through $\mathcal{Z}$ if and only if $U \to \mathcal{T} \to \mathcal{F}$ does. Thus we reduce to the case that $\mathcal{T}$ is a scheme T.

Since $\mathcal{Z}$ is a Zariski substack of $\mathcal{F}$ by Lemma 3.2.9, we can check the assertion of the lemma Zariski locally on T, and hence we may assume that $T = \operatorname{Spec} A$ is affine. If $\mathcal{T}$ is locally of finite type over S, then we may assume that T is of finite type over S. In the case that we are assuming instead that $\mathcal{F}$ satisfies [1], write $A = \varinjlim_i A_i$ as the inductive limit of its finitely generated $\mathcal{O}_S$-subalgebras. Then for some i there exists $\eta_i : \operatorname{Spec} A_i \to \mathcal{F}$ such that η is obtained as the composite of η_i with $\operatorname{Spec} A \to \operatorname{Spec} A_i$; Remark 3.1.5 (4) then shows that $\mathcal{X}_{A_i} \to \operatorname{Spec} A_i$ is scheme-theoretically dominant. Thus we may replace A by A_i, and hence again assume that T is finite type over S.

Let t be a finite type point of T. Since $\mathcal{X}_T \to T$ is scheme-theoretically dominant, the same is true of the (flat) base-change $\mathcal{X}_{\hat{\mathcal{O}}_{T,t}} \to \operatorname{Spec} \hat{\mathcal{O}}_{T,t}$. By definition, then, the composite morphism $\operatorname{Spf} \hat{\mathcal{O}}_{T,t} \to T \to \mathcal{F}$ factors through $\widehat{\mathcal{Z}}$. Since this holds for all finite type points t, the morphism $T \to \mathcal{F}$ factors through $\mathcal{Z}$ by definition. $\quad\square$

Our next lemma spells out the basic properties satisfied by $\mathcal{Z}$.

3.2.20. **Lemma.** *Suppose either that ξ is proper and that $\mathcal{F}$ is $\mathbf{Art}^{\mathbf{triv}}$-homogeneous, or that $\mathcal{F}$ satisfies [2](b). Assume also either that $\mathcal{F}$ satisfies [1], or that $\mathcal{X}$ is locally of finite presentation over S. Then the scheme-theoretic image $\mathcal{Z}$ forms a substack of $\mathcal{F}$, and satisfies Axioms [1] and [3]. If $\mathcal{F}$ satisfies [2](a), then so does $\mathcal{Z}$. If ξ is proper and $\mathcal{F}$ satisfies [1] and [2](b), then $\mathcal{Z}$ satisfies [2](b).*

Proof. As already noted, by its very definition, we see that $\mathcal{Z}$ satisfies [1]. It then follows by standard limit arguments that in order to show that $\mathcal{Z}$ is a stack on the big étale site of S, it suffices to do so after restricting $\mathcal{Z}$ to the category of finite type $\mathcal{O}_S$-algebras; more precisely, this follows easily from Lemma 2.5.5 (3) and [Sta, Tag 021E].

Thus, since $\mathcal{F}$ is a stack for the étale topology, and since $\mathcal{Z}$ is defined to be a full subcategory of $\mathcal{F}$, in order to show that $\mathcal{Z}$ is a stack on the big étale site of S, it suffices to verify that the property of a morphism $\eta : T \to \mathcal{F}$ factoring through $\mathcal{Z}$ (for an S-scheme T) is étale local on T. This follows from Lemma 3.2.18.

Since $\mathcal{Z}$ (thought of as a category fibred in groupoids) is a full subcategory of $\mathcal{F}$, we see that the diagonal $\mathcal{Z} \to \mathcal{Z} \times_S \mathcal{Z}$ is the base-change under the morphism $\mathcal{Z} \times_S \mathcal{Z} \to \mathcal{F} \times_S \mathcal{F}$ of the diagonal $\mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$. Thus, since $\mathcal{F}$ satisfies [3], the same is true of $\mathcal{Z}$.

Suppose now that $\mathcal{F}$ satisfies [2](a). To verify [2](a) for $\mathcal{Z}$, suppose that we are given a pushout diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Z'
\end{array}
$$

of finite type Artinian S-schemes, with the horizontal arrows being closed immersions. By assumption, the induced functor

$$\mathcal{F}(Z') \to \mathcal{F}(Y') \times_{\mathcal{F}(Y)} \mathcal{F}(Z)$$

induces an equivalence of categories. Since $\mathcal{Z}$ is a substack of $\mathcal{F}$, we see that the induced functor

$$\mathcal{Z}(Z') \to \mathcal{Z}(Y') \times_{\mathcal{Z}(Y)} \mathcal{Z}(Z)$$

is fully faithful, and, furthermore, that in order to verify that it induces an equivalence of categories, it suffices to show that an element of $\mathcal{F}(Z')$ whose image under pull-back in $\mathcal{F}(Y') \times \mathcal{F}(Z)$ lies in $\mathcal{Z}(Y') \times \mathcal{Z}(Z)$ itself necessarily lies in $\mathcal{Z}(Z')$.

Suppose given such an element of $\mathcal{F}(Z')$. By hypothesis, we may find $\mathcal{O}_S$-algebras $\widetilde{Y}'$ and $\widetilde{Z}$, each the spectrum of a complete Noetherian local $\mathcal{O}_S$-algebra with finite type residue field, and closed immersions $Y' \hookrightarrow \widetilde{Y}'$ and $Z \hookrightarrow \widetilde{Z}$, and morphisms $\widetilde{Y}' \to \mathcal{F}$ and $\widetilde{Z} \to \mathcal{F}$ inducing the given morphisms $Y' \to \mathcal{F}$ and $Z \to \mathcal{F}$, such that the pulled-back morphisms $\mathcal{X}_{\widetilde{Y}'} \to \widetilde{Y}'$ and $\mathcal{X}_{\widetilde{Z}} \to \widetilde{Z}$ are scheme-theoretically dominant. If ξ is proper, then we can and do assume that $\widetilde{Y}'$ and $\widetilde{Z}$ are Artinian, and then $\widetilde{Y}' \coprod \widetilde{Z} \to \mathcal{F}$ factors through $\widetilde{Z}' := \widetilde{Y}' \coprod_Y \widetilde{Z}$ by our assumption that $\mathcal{F}$ satisfies [2](a). Otherwise, $\mathcal{F}$ satisfies [2](b) by assumption, and Lemma 2.7.4 shows that the induced morphism $\widetilde{Y}' \coprod \widetilde{Z} \to \mathcal{F}$ factors through $\widetilde{Z}' := \widetilde{Y}' \coprod_Y \widetilde{Z}$. Since the tautological morphism $\widetilde{Y}' \coprod \widetilde{Z} \to \widetilde{Z}'$ is scheme-theoretically dominant, the pulled-back morphism $\mathcal{X}_{\widetilde{Z}'} \to \widetilde{Z}'$ is scheme-theoretically dominant. Since the given morphism $Z' := Y' \coprod_Y Z \to \mathcal{F}$ factors through the natural morphism $Z' \to \widetilde{Z}'$, by definition the morphism $Z' \to \mathcal{F}$ does indeed factor through $\mathcal{Z}$.

Finally, suppose that ξ is proper, and that $\mathcal{F}$ satisfies [1] and [2](b), and let $\operatorname{Spec} A_x \to \mathcal{F}$ be an effective versal morphism at a finite type point x of $\mathcal{Z}$ (regarded as a finite type point of $\mathcal{F}$). Let $\operatorname{Spec} R_x$ be the scheme-theoretic image of $\mathcal{X}_{\operatorname{Spec} A_x} \to \operatorname{Spec} A_x$; then the morphism $\operatorname{Spec} R_x \to \mathcal{F}$ factors through $\mathcal{Z}$, by

Lemma 3.2.19. Lemma 3.2.4, applied with $B = R_x$, then shows that the scheme-theoretic image of $\mathcal{X}_{\operatorname{Spf} A_x} \to \operatorname{Spf} A_x$ is equal to $\operatorname{Spf} R_x$, and it now follows from Lemma 3.2.16 that the induced morphism $\operatorname{Spf} R_x \to \mathcal{Z}$ is versal at x. $\qquad \square$

The next lemma gives a refinement of Lemma 3.2.11.

3.2.21. **Lemma.** *Suppose that ξ admits a factorisation $\mathcal{X} \xrightarrow{\xi'} \mathcal{F}' \hookrightarrow \mathcal{F}$, with $\mathcal{F}'$ also satisfying* [3], *and with the second arrow being a monomorphism. Suppose furthermore that the monomorphism $\mathcal{Z} \hookrightarrow \mathcal{F}$ factors through $\mathcal{F}'$, and that either ξ is proper and $\mathcal{F}$ is* $\mathbf{Art^{triv}}$*-homogeneous, or that $\mathcal{F}$ satisfies* [1] *and* [2](b). *Then the scheme theoretic image of ξ' is equal to $\mathcal{Z}$.*

Proof. If we let $\mathcal{Z}'$ denote the scheme-theoretic image of ξ', then Lemma 3.2.11 shows that $\mathcal{Z}' \hookrightarrow \mathcal{Z}$. We must prove the reverse inclusion; that is, we must show that for any morphism $T \to \mathcal{Z}$, with T an affine scheme over S, the composite $T \to \mathcal{Z} \hookrightarrow \mathcal{F}$ factors through $\mathcal{Z}'$. By definition the morphism $T \to \mathcal{Z}$ factors as $T \to T' \to \mathcal{Z}$, with T' of finite type over S, and so it is no loss of generality to assume from the beginning that T is finite type over S. By assumption the composite $T \to \mathcal{Z} \hookrightarrow \mathcal{F}$ factors through $\mathcal{F}'$, and now by definition of $\mathcal{Z}'$, we see that it suffices to check that this composite factors through $\mathcal{Z}'$ under the additional hypothesis that $T = \operatorname{Spec} A$ for some finite type Artinian $\mathcal{O}_S$-algebra A.

Since the morphism $T \to \mathcal{F}$ factors through $\mathcal{Z}$, by definition there exists a morphism $\operatorname{Spec} A \to \operatorname{Spec} B \to \mathcal{F}$ as in Lemma 3.2.1 (1) such that $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant. In the case that $\mathcal{F}$ does not satisfy [1], we are assuming that ξ is proper, so by Lemma 3.2.1 we can assume that B is Artinian, which by Remark 3.2.2 implies in particular that in this case, we can assume that $\operatorname{Spec} B$ is of finite type over S. Lemma 3.2.19 then implies that $\operatorname{Spec} B \to \mathcal{F}$ factors through $\mathcal{Z}$, and so in particular through $\mathcal{F}'$. It now follows from the definition that $T \to \mathcal{F}'$ factors through $\mathcal{Z}'$, as required. $\qquad \square$

3.2.22. **Lemma.** *Assume either that $\mathcal{F}$ satisfies* [1] *and* [2](b), *or that $\mathcal{F}$ is* $\mathbf{Art^{triv}}$*-homogeneous, ξ is proper, and either $\mathcal{F}$ satisfies* [1], *or $\mathcal{X}$ is locally of finite presentation over S. Then the morphism $\xi : \mathcal{X} \to \mathcal{F}$ factors through a morphism $\bar{\xi} : \mathcal{X} \to \mathcal{Z}$, and the scheme-theoretic image of $\bar{\xi}$ is just $\mathcal{Z}$.*

Proof. Since the diagonal map $\mathcal{X} \to \mathcal{X} \times_{\mathcal{F}} \mathcal{X}$ gives a section to the projection $\mathcal{X} \times_{\mathcal{F}} \mathcal{X} \to \mathcal{X}$, it is immediate from Lemma 3.2.19 that ξ factors through $\mathcal{Z}$. That the scheme-theoretic image of $\bar{\xi}$ is $\mathcal{Z}$ is immediate from Lemma 3.2.21. $\qquad \square$

3.2.23. **Lemma.** *Assume either that $\mathcal{F}$ satisfies* [1], *or that $\mathcal{X}$ is locally of finite presentation over S. Suppose further that ξ is proper, and that $\mathcal{F}$ is* $\mathbf{Art^{triv}}$*-homogeneous. Then the morphism $\bar{\xi}$ is proper and surjective.*

3.2.24. *Remark.* Note that Lemma 3.2.20 shows that $\mathcal{Z}$ is a stack satisfying [3], and so it makes sense (following Definition 2.3.4) to assert that $\bar{\xi}$ is proper and surjective.

Proof of Lemma 3.2.23. Following Definition 2.3.4, we have to show that if $\mathcal{Y}$ is an algebraic stack, and if $\mathcal{Y} \to \mathcal{Z}$ is a morphism of stacks, then the base-changed morphism of algebraic stacks $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{Y}$ is proper and surjective. Since by definition $\mathcal{Z} \to \mathcal{F}$ is a monomorphism, the fibre product over $\mathcal{Z}$ is isomorphic to

the fibre product $\mathcal{Y} \times_\mathcal{F} \mathcal{X}$, taken over $\mathcal{F}$, and so the properness of ξ immediately implies the properness of $\overline{\xi}$.

To verify the surjectivity, we note that surjectivity can be checked after pulling back by a surjective morphism. Replacing $\mathcal{Y}$ by a cover of $\mathcal{Y}$ by a scheme, we may assume that $\mathcal{Y}$ is in fact a scheme. Since we may also check surjectivity locally on the target, we may in fact assume that $\mathcal{Y}$ is an affine scheme T. Since $\mathcal{Z}$ satisfies [1], we may factor the morphism $T \to \mathcal{Z}$ through an affine S-scheme of finite type, and hence (since surjectivity is preserved under base-change) may further assume that T is of finite type over S.

Now $T \times_\mathcal{Z} \mathcal{X} \to T$ is a proper morphism of algebraic stacks, and so to check that it is surjective, it suffices to check that its image contains each finite type point of T. Thus it suffices to show that if $t : \operatorname{Spec} k \to T$ is any finite type point, then the fibre of $\mathcal{X}$ over t is non-empty. Since ξ is proper, it follows from Lemma 3.2.14 that the fact that the composite $t : \operatorname{Spec} k \to T \to \mathcal{Z} \to \mathcal{F}$ factors through $\mathcal{Z}$ implies that the fibre $\mathcal{X}_t \neq \varnothing$. $\qquad\square$

3.2.25. **Lemma.** *If $\mathcal{F}'$ is a closed Zariski substack of $\mathcal{F}$, and if the morphism $\xi : \mathcal{X} \to \mathcal{F}$ factors through $\mathcal{F}'$, then $\mathcal{F}'$ contains $\mathcal{Z}$.*

Conversely, assume either that $\mathcal{F}$ satisfies [1] and [2](b), or that ξ is proper, $\mathcal{F}$ is $\mathbf{Art}^{\mathbf{triv}}$-homogeneous, and either $\mathcal{X}$ is locally of finite presentation over S, or $\mathcal{F}$ satisfies [1]. Then if $\mathcal{F}'$ is a closed substack of $\mathcal{F}$ which contains $\mathcal{Z}$, then the morphism $\xi : \mathcal{X} \to \mathcal{F}$ factors through $\mathcal{F}'$.

Proof. Suppose that $\mathcal{F}'$ is a closed Zariski substack of $\mathcal{F}$, and that ξ factors through $\mathcal{F}'$; we will show that $\mathcal{F}'$ contains $\mathcal{Z}$. Since $\mathcal{Z}$ is (by definition) a Zariski substack of $\mathcal{F}$, it suffices to show that if $T \to \mathcal{Z}$ is a morphism with T an affine S-scheme, then the composite $T \to \mathcal{Z} \to \mathcal{F}$ factors through $\mathcal{F}'$. By definition, given such a morphism $T \to \mathcal{Z}$, we may find a finite type affine S-scheme T' and a factorisation $T \to T' \to \mathcal{Z}$; thus we may and do assume that T is of finite type over S.

In order to show that $T \to \mathcal{F}$ factors through $\mathcal{F}'$, it suffices to show that the base-changed closed immersion $T \times_\mathcal{F} \mathcal{F}' \hookrightarrow T$ is an isomorphism. Since T is of finite type over S, for this it suffices to show that if $t \in T$ is a finite type point, then for any $n \geq 0$, the induced morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{T,t}/\mathfrak{m}_t^n \to T$ factors through $T \times_\mathcal{F} \mathcal{F}'$, or equivalently, that the composite $\operatorname{Spec} \widehat{\mathcal{O}}_{T,t}/\mathfrak{m}_t^n \to T \to \mathcal{F}$ factors through $\mathcal{F}'$. Thus we are reduced to the case when $T = \operatorname{Spec} A$, with A an Artinian $\mathcal{O}_S$-algebra of finite type.

By the definition of $\mathcal{Z}$, we may find a complete Noetherian local $\mathcal{O}_S$-algebra B, a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ over S, and a morphism $\operatorname{Spec} B \to \mathcal{F}$ inducing the given morphism $\operatorname{Spec} A \to \mathcal{F}$, such that the base-changed morphism $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant. Since ξ factors through $\mathcal{F}'$ by assumption, the morphism $\mathcal{X}_B \to \operatorname{Spec} B$ factors through the closed immersion $\operatorname{Spec} B \times_\mathcal{F} \mathcal{F}' \hookrightarrow \operatorname{Spec} B$. Since the former morphism is also scheme-theoretically dominant, we see that this closed immersion is necessarily an isomorphism, and hence that the morphism $\operatorname{Spec} B \to \mathcal{F}$ factors through $\mathcal{F}'$. Thus the morphism $\operatorname{Spec} A \to \mathcal{F}$ also factors through $\mathcal{F}'$, and we are done.

For the converse, we note that, under the additional hypotheses, the morphism ξ factors through a morphism $\overline{\xi} : \mathcal{X} \to \mathcal{Z}$, by Lemma 3.2.22; so if $\mathcal{F}'$ contains $\mathcal{Z}$, we see that ξ factors through $\mathcal{F}'$, as required. $\qquad\square$

Under the assumption that ξ is proper, we may strengthen Lemma 3.2.25 as follows.

3.2.26. **Lemma.** *Assume either that $\mathcal{F}$ satisfies* [1], *or that $\mathcal{X}$ is locally of finite presentation over S. Assume also that ξ is proper and that $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous. If $\mathcal{F}'$ is a substack of $\mathcal{F}$, and if the monomorphism $\mathcal{F}' \hookrightarrow \mathcal{F}$ is representable by algebraic spaces and of finite type, then the morphism $\xi : \mathcal{X} \to \mathcal{F}$ factors through $\mathcal{F}'$ if and only if $\mathcal{F}'$ contains $\mathcal{Z}$.*

Proof. The "if" direction was proved in Lemma 3.2.25. For the "only if" direction, set $\mathcal{Z}' := \mathcal{Z} \times_{\mathcal{F}} \mathcal{F}'$. We begin by showing that the finite type monomorphism $\mathcal{Z}' \to \mathcal{Z}$ is a closed immersion; it suffices to show that it is proper, and since monomorphisms are automatically separated, and since it is finite type by assumption, it in fact suffices to show that it is universally closed.

Firstly, note that since $\mathcal{X} \to \mathcal{Z}$ is proper and surjective by Lemma 3.2.23, the composite $|\mathcal{X}| \to |\mathcal{Z}'| \to |\mathcal{Z}|$ is surjective, while $|\mathcal{Z}'| \to |\mathcal{Z}|$ is injective. Thus $|\mathcal{X}| \to |\mathcal{Z}'|$ is surjective. Similarly, $|\mathcal{X}| \to |\mathcal{Z}|$ is closed, and a trivial topological argument then shows that $|\mathcal{Z}'| \to |\mathcal{Z}|$ is closed. Since both properness and surjectivity are preserved by arbitrary base changes, we conclude that $\mathcal{Z}' \to \mathcal{Z}$ is universally closed, as claimed, and thus a closed immersion.

By Lemma 3.2.23, the morphism $\bar{\xi} : \mathcal{X} \to \mathcal{Z}$ is proper and surjective; but it also factors through the closed substack $\mathcal{Z}'$ of $\mathcal{Z}$, so we must have $\mathcal{Z}' = \mathcal{Z}$. Thus $\mathcal{F}'$ contains $\mathcal{Z}$, as required. $\square$

3.2.27. *Remark.* If $\mathcal{F}$ is an algebraic stack that is locally of finite presentation over S, then in particular it satisfies Axioms [1], [2], and [3] (by Theorem 2.8.4), and so all of the previous results apply.

3.2.28. *Remark.* Note that at this point in the development of the theory, we do not know that the monomorphism $\mathcal{Z} \to \mathcal{F}$ is representable by algebraic spaces, which means for example that it is hard to use Lemma 3.2.26 to uniquely characterise $\mathcal{Z}$ by a universal property. However, once we have shown that $\mathcal{Z}$ is an algebraic stack, it is immediate from the assumption that $\mathcal{F}$ satisfies [3] that the monomorphism $\mathcal{Z} \to \mathcal{F}$ *is* representable by algebraic spaces, and Proposition 3.2.33 below can then be applied to deduce that $\mathcal{Z}$ is in fact a closed substack of $\mathcal{F}$.

3.2.29. **Lemma.** *If $\mathcal{F}$ is an algebraic stack, locally of finite presentation over S, then $\mathcal{Z}$ coincides with the scheme-theoretic image of ξ (in the sense of Definition* 3.1.4).

Proof. Let $\mathcal{Y} \hookrightarrow \mathcal{F}$ denote the scheme-theoretic image of ξ. Since $\mathcal{Y}$ is a closed substack of $\mathcal{F}$, and since the morphism ξ factors through $\mathcal{Y}$, Lemma 3.2.25 shows that $\mathcal{Y}$ contains $\mathcal{Z}$. On the other hand, since $\mathcal{Y} \to \mathcal{F}$ is a monomorphism and ξ factors through $\mathcal{Y}$, we see that the projection $\mathcal{X}_{\mathcal{Y}} := \mathcal{X} \times_{\mathcal{F}} \mathcal{Y} \to \mathcal{Y}$ is naturally identified with the canonical morphism $\mathcal{X} \to \mathcal{Y}$, and so in particular is scheme-theoretically dominant. Lemma 3.2.19 then implies that $\mathcal{Y}$ is contained in $\mathcal{Z}$. $\square$

3.2.30. *Remark.* As noted in Remark 3.2.10 (3), the conclusion of this lemma needn't hold if $\mathcal{F}$ doesn't satisfy [1].

The following result will be useful in Section 3.3.

3.2.31. **Proposition.** *Suppose that there is a closed substack $\mathcal{Y}$ of $\mathcal{F}$ such that $\mathcal{Y}$ is an algebraic stack, locally of finite presentation over S, and that the morphism*

$\mathcal{X} \to \mathcal{F}$ factors as a composite $\mathcal{X} \to \mathcal{Y} \to \mathcal{F}$. Then $\mathcal{Z}$ coincides with the scheme-theoretic image of $\mathcal{X}$ in $\mathcal{Y}$ (in the sense of Definition 3.1.4).

Proof. Replacing $\mathcal{Y}$ by the scheme-theoretic image of $\mathcal{X} \to \mathcal{Y}$, we may assume that it coincides with this scheme-theoretic image; we must then show that $\mathcal{Y}$ equals $\mathcal{Z}$. By Lemma 3.2.25, we see that $\mathcal{Y}$ contains $\mathcal{Z}$. To show the reverse inclusion, we note first that Lemma 3.2.29 implies that $\mathcal{Y}$ is the scheme-theoretic image of the morphism $\mathcal{X} \to \mathcal{Y}$; the desired inclusion now follows by Lemma 3.2.11.   □

The following lemma will be used to help prove openness of versality for $\mathcal{Z}$.

3.2.32. **Lemma.** *Assume that $\mathcal{F}$ satisfies [1], that ξ is proper, and that $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous. If $\mathcal{Y}$ is an algebraic stack, locally of finite type over S, and if $\mathcal{Y} \to \mathcal{Z}$ is a morphism over S which is formally smooth at a finite type point $y \in |\mathcal{Y}|$, then there exists an open neighbourhood $\mathcal{U}$ of y in $\mathcal{Y}$ such that the induced morphism $\mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ is scheme-theoretically dominant (where $\mathcal{X}_{\mathcal{U}}$ denotes the base-change of $\bar{\xi} : \mathcal{X} \to \mathcal{Z}$ via the composite $\mathcal{U} \to \mathcal{Y} \to \mathcal{Z}$; or equivalently, the base-change of $\xi : \mathcal{X} \to \mathcal{F}$ via the composite $\mathcal{U} \to \mathcal{Y} \to \mathcal{Z} \to \mathcal{F}$).*

Proof. By definition (see Definition 2.4.10) we may find a smooth morphism $V \to \mathcal{Y}$ from a scheme V to $\mathcal{Y}$, and a finite type point $v \in V$ mapping to y, such that the composite $V \to \mathcal{Y} \to \mathcal{Z}$ is formally smooth at v. Write $B := \widehat{\mathcal{O}}_{V,v}$, let $\mathcal{X}_B \to \operatorname{Spec} B$ denote the base-change of $\bar{\xi} : \mathcal{X} \to \mathcal{Z}$ via the composite $\operatorname{Spec} B \to V \to \mathcal{Y} \to \mathcal{Z}$ (or, equivalently, the base-change of $\xi : \mathcal{X} \to \mathcal{F}$ via the composite $\operatorname{Spec} B \to V \to \mathcal{Y} \to \mathcal{Z} \to \mathcal{F}$), and let $\operatorname{Spec} B/I$ denote the scheme-theoretic image of this base-change.

Let A be any Artinian quotient of B. We will show that the surjection $B \to A$ necessarily contains I in its kernel. This will show that $I = 0$ (since A was arbitrary) and hence that $\mathcal{X}_B \to \operatorname{Spec} B$ is scheme-theoretically dominant. From this one sees that there is a neighbourhood U of v in V such that $\mathcal{X}_U \to U$ is scheme-theoretically dominant (here $\mathcal{X}_U$ has the evident meaning). Letting $\mathcal{U}$ denote the image of U in $\mathcal{Y}$ (an open substack of $\mathcal{Y}$), we see that $\mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ is scheme-theoretically dominant, as required. (Scheme-theoretic dominance can be checked *fpqc* locally, and in particular smooth locally, on the target.)

The composite $\operatorname{Spec} A \to \operatorname{Spec} B \to V \to \mathcal{Y} \to \mathcal{Z} \to \mathcal{F}$ factors through $\mathcal{Z}$ by construction, and so by definition we may find a complete Noetherian local $\mathcal{O}_S$-algebra C with finite type residue field, a closed immersion of $\mathcal{O}_S$-schemes $\operatorname{Spec} A \to \operatorname{Spec} C$, and a morphism $\operatorname{Spec} C \to \mathcal{F}$ inducing the above morphism $\operatorname{Spec} A \to \mathcal{F}$, such that the base-changed morphism $\mathcal{X}_C \to \operatorname{Spec} C$ is scheme-theoretically surjective. The morphism $\operatorname{Spec} C \to \mathcal{F}$ factors through $\mathcal{Z}$, by Lemma 3.2.19. Since $V \to \mathcal{Z}$ is formally smooth at v, we may lift the morphism $\operatorname{Spec} C \to \mathcal{Z}$ to a morphism $\operatorname{Spec} C \to \operatorname{Spec} B$, extending the composite morphism $\operatorname{Spec} A \to \operatorname{Spec} B$. Since $\mathcal{X}_C \to \operatorname{Spec} C$ is surjective, we conclude that the morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ factors through $\operatorname{Spec} B/I$, as required.   □

3.2.33. **Proposition.** *Assume either that $\mathcal{F}$ satisfies [1], or that $\mathcal{X}$ is locally of finite presentation over S. Assume also that ξ is proper, that $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous, and that the monomorphism $\mathcal{Z} \to \mathcal{F}$ is representable by algebraic spaces. Then $\mathcal{Z}$ is a closed substack of $\mathcal{F}$.*

Proof. We must show (under either set of finiteness assumptions on $\mathcal{X}$ and $\mathcal{F}$) that $\mathcal{Z} \to \mathcal{F}$ is a closed immersion. Since it is a monomorphism, it suffices to show

that it is proper. As monomorphisms are separated, and as it is locally of finitely presentation (this follows from Lemma 2.1.7, applied to the composite $\mathcal{Z} \hookrightarrow \mathcal{F} \to S$, together with Lemma 2.1.5 and [Sta, Tag 06CX]) it suffices to show that it is universally closed and quasi-compact. These are properties that (by definition) can be checked after pulling back by a morphism $W \to \mathcal{F}$ from a scheme.

Pulling back the morphisms $\mathcal{X} \to \mathcal{Z} \to \mathcal{F}$ via such a morphism, we obtain morphisms $\mathcal{X}_W \to \mathcal{Z}_W \to W$ with the composite being proper, and the first being surjective (by Lemma 3.2.23). We must show that $\mathcal{Z}_W \to W$ is closed, and that, if W is quasi-compact, the same is true of $\mathcal{Z}_W$. These are properties that can be checked on the underlying topological spaces. So we consider the continuous morphisms $|\mathcal{X}_W| \to |\mathcal{Z}_W| \to |W|$, with the first being surjective (by [Sta, Tag 04XI]) and the composite being closed. Furthermore, if $|W|$ is quasi-compact, the same is true of $|\mathcal{X}_W|$. It follows immediately that the second arrow is closed, and that $|\mathcal{Z}_W|$ is quasi-compact if $|W|$ is. This completes the proof of the proposition. $\square$

We will now prove Theorem 1.1.1, as well as a variant where we assume that $\mathcal{F}$ satisfies [1], but make no finiteness assumption on $\mathcal{X}$. (In fact, as remarked above, we will reduce Theorem 1.1.1 to this case.)

3.2.34. **Theorem.** *Let S be a locally Noetherian scheme, all of whose local rings $\mathcal{O}_{S,s}$ at finite type points $s \in S$ are G-rings. Suppose that $\xi : \mathcal{X} \to \mathcal{F}$ is a proper morphism, where $\mathcal{X}$ is an algebraic stack and $\mathcal{F}$ is a stack over S satisfying [3], and that $\mathcal{F}$ admits versal rings at all finite type points. Assume also either that $\mathcal{F}$ satisfies [1], or that $\mathcal{X}$ is locally of finite presentation over S. Let $\mathcal{Z}$ denote the scheme-theoretic image of ξ as in Definition 3.2.6, and suppose that $\mathcal{Z}$ satisfies [2].*

Then $\mathcal{Z}$ is an algebraic stack, locally of finite presentation over S; the inclusion $\mathcal{Z} \hookrightarrow \mathcal{F}$ is a closed immersion; and the morphism ξ factors through a proper, scheme-theoretically surjective morphism $\bar{\xi} : \mathcal{X} \to \mathcal{Z}$. Furthermore, if $\mathcal{F}'$ is a substack of $\mathcal{F}$ for which the monomorphism $\mathcal{F}' \hookrightarrow \mathcal{F}$ is of finite type (e.g. a closed substack) with the property that ξ factors through $\mathcal{F}'$, then $\mathcal{F}'$ contains $\mathcal{Z}$.

Proof. Note firstly that by Lemma 2.6.2, the diagonal of $\mathcal{F}$ is locally of finite presentation, so it follows from Corollary 2.7.3 that $\mathcal{F}$ is $\mathbf{Art^{triv}}$-homogeneous. Taking into account Lemmas 3.2.22, 3.2.23 and 3.2.26, Remark 3.2.28, and Proposition 3.2.33, we see that we only need to prove that $\mathcal{Z}$ is an algebraic stack, locally of finite presentation over S. By Theorem 2.8.4, it is enough to show that it is an étale stack over S, and satisfies Axioms [1], [2], [3] and [4]. It follows from Lemma 3.2.20 that $\mathcal{Z}$ is an étale stack and satisfies Axioms [1] and [3], and we are assuming that it satisfies Axiom [2]. It remains to show that $\mathcal{Z}$ also satisfies Axiom [4], i.e. openness of versality. We will assume from now on that we are in the case that $\mathcal{F}$ satisfies [1], and explain at the end how to reduce the other case to this situation.

Consider an S-morphism $T \to \mathcal{Z}$, where T is a locally finite type S-scheme, and let $t \in T$ be a finite type point at which this morphism is versal. In fact this morphism is then formally smooth at t, by Lemma 2.4.7 (2). Lemma 2.6.4 allows us, shrinking T if necessary, to factor the morphism $T \to \mathcal{Z}$ as

$$T \to \mathcal{Z}' \to \mathcal{Z},$$

where $\mathcal{Z}'$ is an algebraic stack, locally of finite presentation over S, the first arrow is a smooth surjection, and the second arrow is locally of finite presentation, unramified, and representable by algebraic spaces, and formally smooth at the image $t' \in |\mathcal{Z}'|$ of t.

Consider the following diagram, both squares in which are defined to be Cartesian:

(3.2.35)
$$
\begin{array}{ccccc}
T \times_{\mathcal{Z}'} \mathcal{X}' & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
T & \longrightarrow & \mathcal{Z}' & \longrightarrow & \mathcal{Z}
\end{array}
$$

The vertical arrows are proper, and hence closed, the first of the horizontal arrows are smooth surjections, and the second of the horizontal arrows are unramified, locally of finite presentation, and representable by algebraic spaces. By Lemma 2.4.12 (1), there is an open substack $\mathcal{U}$ of $\mathcal{X}'$, containing the fibre over t', such that the induced morphism $\mathcal{U} \to \mathcal{X}$ is smooth.

Since $\mathcal{X}' \to \mathcal{Z}'$ is closed, the complement $\mathcal{U}'$ of the image in $\mathcal{Z}'$ of the complement in $\mathcal{X}'$ of $\mathcal{U}$ is an open substack of $\mathcal{Z}'$, containing t', whose preimage in $\mathcal{X}'$ maps smoothly to $\mathcal{X}$. Thus, replacing $\mathcal{Z}'$ by $\mathcal{U}'$, and T by its preimage, we may further assume that $\mathcal{X}' \to \mathcal{X}$ is smooth, and hence (being also unramified) étale.

Let $t_1 \in T$ be another finite type point. We will show that the morphism $T \to \mathcal{Z}$ is formally smooth (and so also versal) at t_1. This will establish [4] for $\mathcal{Z}$. Let t_1' denote the image of t_1 in $\mathcal{Z}'$. Since $T \to \mathcal{Z}'$ is smooth, it suffices to show that $\mathcal{Z}' \to \mathcal{Z}$ is formally smooth at t_1'.

Let $\bar{t}_1$ denote the image of t_1 in $\mathcal{Z}$. By Lemma 2.8.1, we may find an S-scheme T_1, locally of finite type, a morphism $T_1 \to \mathcal{Z}$, and a point $\tilde{t}_1 \in T_1$ mapping to $\bar{t}_1$, such that the morphism $T_1 \to \mathcal{Z}$ is formally smooth at $\tilde{t}_1$. We apply Lemma 2.6.4 once more, to obtain a factorisation $T_1 \to \mathcal{Z}'' \to \mathcal{Z}$ (possibly after shrinking T_1 around $\tilde{t}_1$), where $\mathcal{Z}''$ is an algebraic stack, locally of finite presentation over S, the first arrow is a smooth surjection, and the second arrow is locally of finite presentation, unramified, representable by algebraic spaces, and formally smooth at the image $t'' \in |\mathcal{Z}''|$ of $\tilde{t}_1$. If we let $\mathcal{X}''$ denote the base-change of $\bar{\xi} : \mathcal{X} \to \mathcal{Z}$ over $\mathcal{Z}''$, then Lemma 3.2.32 shows that, shrinking $\mathcal{Z}''$ around t'' if necessary, we may assume that the morphism $\mathcal{X}'' \to \mathcal{Z}''$ is scheme-theoretically dominant, and thus scheme-theoretically surjective (being a base-change of the morphism $\bar{\xi}$, which is surjective by Lemma 3.2.23).

Now consider the Cartesian square

$$
\begin{array}{ccc}
\mathcal{Z}' \times_{\mathcal{Z}} \mathcal{Z}'' & \longrightarrow & \mathcal{Z}'' \\
\downarrow & & \downarrow \\
\mathcal{Z}' & \longrightarrow & \mathcal{Z}
\end{array}
$$

Applying Lemma 2.4.12, we find that the morphism $\mathcal{Z}' \times_{\mathcal{F}} \mathcal{Z}'' \to \mathcal{Z}'$ contains the fibre over t'' in its smooth locus. Since t_1' and t'' map to the same point of $|\mathcal{F}|$, this fibre contains a point t''' lying over t_1' and t'', and so we may find an open substack $\mathcal{Z}''' \subseteq \mathcal{Z}' \times_{\mathcal{F}} \mathcal{Z}''$, such that $t''' \in |\mathcal{Z}'''|$, and such that the morphism $\mathcal{Z}''' \to \mathcal{Z}'$ is smooth, and hence (being also unramified) étale.

In summary, we have a diagram

$$
\begin{array}{ccc}
\mathcal{Z}''' & \longrightarrow & \mathcal{Z}'' \\
\downarrow & & \downarrow \\
\mathcal{Z}' & \longrightarrow & \mathcal{Z}
\end{array}
$$

in which the horizontal arrows are locally of finite presentation and unramified, the left-hand vertical arrow is étale, the right-hand vertical arrow is formally smooth at t'', and for which there is a point $t''' \in |\mathcal{Z}'''|$ lying over $t'_1 \in |\mathcal{Z}'|$ and $t'' \in |\mathcal{Z}''|$. Our goal is to show that the lower horizontal arrow is formally smooth at the point t'_1. For this, it suffices (by Remark 2.4.11) to show that the upper horizontal arrow is smooth (or equivalently, étale, since it is unramified).

Thus we turn to showing that $\mathcal{Z}''' \to \mathcal{Z}''$ is étale. Form the Cartesian diagram of algebraic stacks

$$
\begin{array}{ccc}
\mathcal{X}''' & \longrightarrow & \mathcal{X}'' \\
\downarrow & & \downarrow \\
\mathcal{Z}''' & \longrightarrow & \mathcal{Z}''
\end{array}
$$

Regarding the top arrow as being the composition of an open immersion with the base-change of the top arrow of (3.2.35), we see that it is étale. Recall that the right-hand vertical arrow $\mathcal{X}'' \to \mathcal{Z}''$ is scheme-theoretically surjective. Both vertical arrows are proper, thus universally closed, and the bottom arrow is locally of finite presentation and unramified. Lemma 3.2.36 below thus implies that the bottom arrow is étale, and so Axiom [4] for $\mathcal{Z}$ is proved.

We have therefore shown that if $\mathcal{F}$ satisfies [1], then $\mathcal{Z}$ is an algebraic stack, locally of finite presentation over S. Assume now that we are in the case that $\mathcal{X}$ is locally of finite presentation over S. Recall that by Lemma 2.6.2, the diagonal of $\mathcal{F}$ is locally of finite presentation. By Remark 3.2.13, ξ factors as

$$
\mathcal{X} \xrightarrow{\xi'} \mathrm{pro}\text{-}(\mathcal{F}|_{\mathrm{Aff}_{\mathrm{pf}/S}}) \hookrightarrow \mathcal{F},
$$

and $\mathrm{pro}\text{-}(\mathcal{F}|_{\mathrm{Aff}_{\mathrm{pf}/S}})$ is an étale stack which satisfies [1] and [3] and admits versal rings at all finite type points; and the scheme-theoretic image of ξ' is $\mathcal{Z}$. Regarding ξ' as the pull-back of ξ along the embedding $\mathrm{pro}\text{-}(\mathcal{F}|_{\mathrm{Aff}_{\mathrm{pf}/S}}) \hookrightarrow \mathcal{F}$, we see that ξ' is proper, and therefore satisfies the assumptions of the Theorem; so it follows from the case already proved that $\mathcal{Z}$ is an algebraic stack, locally of finite presentation over S. □

Although we have stated and proved the following lemma in what seems to be its natural level of generality, the only application of it that we make is in the case when the morphism $\mathcal{Z}' \to \mathcal{Z}$ is in fact representable by algebraic spaces.

3.2.36. **Lemma.** *Let $\mathcal{Y} \to \mathcal{Z}$ be a quasi-compact morphism of algebraic stacks that is scheme-theoretically surjective and universally closed, and let $\mathcal{Z}' \to \mathcal{Z}$ be a morphism of algebraic stacks that is locally of finite presentation and unramified. If the base-changed morphism $\mathcal{Y}' \to \mathcal{Y}$ is étale, then $\mathcal{Z}' \to \mathcal{Z}$ is also étale.*

Proof. Since $\mathcal{Z}' \to \mathcal{Z}$ is unramified, its diagonal is étale and in particular unramified, so $\mathcal{Z}' \to \mathcal{Z}$ is a DM morphism in the sense of [Sta, Tag 04YW]. Choose a scheme mapping Z via a smooth surjection to $\mathcal{Z}$, and pull everything back over Z; in this way we reduce to the case $\mathcal{Z} = Z$. As explained in [Sta, Tag 04YW], it now follows from [Sta, Tag 06N3] that there is a scheme Z' and a surjective étale morphism $Z' \to \mathcal{Z}'$.[5]

_____

[5]As noted above, the only application of this lemma that we will make is in the case when $\mathcal{Z}' \to \mathcal{Z}$ is representable by algebraic spaces. In this case, the pull-back of the algebraic stack $\mathcal{Z}'$ over the scheme Z is already an algebraic space, and so by definition admits an étale surjection

Replacing $\mathcal{Z}'$ by Z', we have reduced to the case that the morphism $\mathcal{Z}' \to \mathcal{Z}$ is the morphism of schemes $Z' \to Z$.

We must show that $Z' \to Z$ is étale at each point $z' \in Z'$. Fix such a point, with image $z \in Z$. Then we may replace Z by its base change to the strict henselisation $\operatorname{Spec} \mathcal{O}_{Z,z}^{\mathrm{sh}}$, and thus assume that Z is a local strictly henselian scheme with closed point z. Then the étale local structure of unramified morphisms [Sta, Tag 04HH] shows that after shrinking Z' around z', we can arrange that $Z' \to Z$ is a closed immersion, so that in particular Z' is also local with closed point $z' = z$. Then $Z' \to Z$ is étale if and only if $Z' = Z$.

Since $Z' \to Z$ is a closed immersion, so is $\mathcal{Y}' \to \mathcal{Y}$. Since $\mathcal{Y}' \to \mathcal{Y}$ is étale by assumption, it is an étale monomorphism, and therefore it is an open (as well as closed) immersion. The complement in $\mathcal{Y}$ of the image of $\mathcal{Y}' \to \mathcal{Y}$ is therefore closed, and has closed image in Z, as $\mathcal{Y} \to Z$ is closed by assumption; but this image has empty special fibre, and so this complement is empty, and $\mathcal{Y}' = \mathcal{Y}$. It now follows from Lemma 3.1.7 that $Z' = Z$, as required. $\square$

3.2.37. *Remark.* In the statement of Lemma 3.2.36, it would not suffice to assume that $\mathcal{Y} \to \mathcal{Z}$ is merely scheme-theoretically surjective. For example, if $\mathcal{Z} = Z$ is taken to be the scheme given as the union of two lines Z_1 and Z_2 crossing at the point y in the plane, if $\mathcal{Y} = Y$ is taken to be the scheme obtained as the disjoint union $Z_1 \coprod Z_2 \setminus \{y\}$, if $Y \to Z$ is taken to be the obvious morphism (which is scheme-theoretically surjective), and if $\mathcal{Z}' = Z'$ is taken to be Z_1, then the closed immersion $Z' \to Z$ is unramified, and the base-changed morphism $Y' \to Y$ is an open immersion, and so in particular étale. On the other hand, the closed immersion $Z' \to Z$ is certainly not étale.

3.3. **Base change.** In this section we study the behaviour of our scheme-theoretic images under base change by a morphism which is representable by algebraic spaces. We will only need to consider cases where we know (in applications, as a consequence of Theorem 1.1.1) that the scheme-theoretic image of the morphism being base-changed is an algebraic stack, and we have therefore restricted to this case, and have not investigated the compatibility with base change in more general situations.

We first of all consider the question of base change of the target stack $\mathcal{F}$.

3.3.1. **Proposition.** *Suppose that we have a commutative diagram of étale stacks in groupoids over a locally Noetherian base scheme S*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{F}' & \longrightarrow & \mathcal{F} \end{array}$$

(which is not assumed to be Cartesian), in which $\mathcal{X}, \mathcal{X}'$ are algebraic stacks, $\mathcal{F}' \to \mathcal{F}$ is representable by algebraic spaces and locally of finite presentation, and $\Delta_{\mathcal{F}}$ is representable by algebraic spaces and locally of finite presentation. Write $\mathcal{Z}$ and $\mathcal{Z}'$ for the scheme-theoretic images of $\mathcal{X} \to \mathcal{F}$ and $\mathcal{X}' \to \mathcal{F}'$ respectively, and assume further that $\mathcal{Z}$ is an algebraic stack, that $\mathcal{Z}$ is a closed substack of $\mathcal{F}$, and that the morphism $\mathcal{X} \to \mathcal{F}$ factors through a scheme-theoretically dominant morphism $\mathcal{X} \to \mathcal{Z}$.

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from a scheme. Thus, in this case, we can avoid appealing to the theory of DM morphisms and their relationship to Deligne–Mumford stacks.

Then $\mathcal{Z}'$ is an algebraic stack, locally of finite presentation over S. In fact, the morphism $\mathcal{X}' \to \mathcal{F}'$ factors through the algebraic stack $\mathcal{Z} \times_{\mathcal{F}} \mathcal{F}'$ (which is in turn a closed substack of $\mathcal{F}'$), and $\mathcal{Z}'$ is the scheme-theoretic image of the induced morphism of algebraic stacks $\mathcal{X}' \to \mathcal{Z} \times_{\mathcal{F}} \mathcal{F}'$.

Proof. Since $\mathcal{X} \to \mathcal{F}$ factors through $\mathcal{Z}$ by assumption, the composite $\mathcal{X}' \to \mathcal{X} \to \mathcal{F}$ factors through $\mathcal{Z}$, and so $\mathcal{X}' \to \mathcal{F}'$ factors through the algebraic stack $\mathcal{Z} \times_{\mathcal{F}} \mathcal{F}'$, which is a closed substack of $\mathcal{F}'$. By Proposition 2.3.20, $\Delta_{\mathcal{F}'}$ is representable by algebraic spaces and locally of finite presentation. The result then follows from Proposition 3.2.31. $\square$

3.3.2. **Corollary.** *Suppose that we have a commutative diagram of étale stacks in groupoids over a locally Noetherian base scheme S*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{F}' & \longrightarrow & \mathcal{F} \end{array}$$

(which is not assumed to be Cartesian), in which $\mathcal{X}, \mathcal{X}'$ are algebraic stacks, $\mathcal{F}' \to \mathcal{F}$ is representable by algebraic spaces and locally of finite presentation, and $\Delta_{\mathcal{F}}$ is representable by algebraic spaces and locally of finite presentation.

Suppose that $\mathcal{X} \to \mathcal{F}$ is proper, and that $\mathcal{X}$ is locally of finite presentation over S. Write $\mathcal{Z}$, $\mathcal{Z}'$ for the scheme-theoretic images of $\mathcal{X} \to \mathcal{F}$, $\mathcal{X}' \to \mathcal{F}'$ respectively Suppose that $\mathcal{Z}$ satisfies [2], and that $\mathcal{F}$ admits versal rings at all finite type points. Then $\mathcal{Z}$ and $\mathcal{Z}'$ are both algebraic stacks, locally of finite presentation over S. In fact, the morphism $\mathcal{X}' \to \mathcal{F}'$ factors through the algebraic stack $\mathcal{Z} \times_{\mathcal{F}} \mathcal{F}'$ (which is in turn a closed substack of $\mathcal{F}'$), and $\mathcal{Z}'$ is the scheme-theoretic image of the induced morphism of algebraic stacks $\mathcal{X}' \to \mathcal{Z} \times_{\mathcal{F}} \mathcal{F}'$.

If x is a finite type point of $\mathcal{Z}'$, let R_x be a versal ring for the corresponding finite type point of $\mathcal{F}$, and write $\operatorname{Spf} S_x$ for the complete local ring at x of $\operatorname{Spf} R_x \times_{\mathcal{F}} \mathcal{F}'$, in the sense of Definition 4.2.13 below. Then a versal ring for $\mathcal{Z}'$ at x is given by the scheme-theoretic image of the induced morphism $\mathcal{X}'_{\operatorname{Spf} S_x} \to \operatorname{Spf} S_x$.

Proof. Everything except for the claim about versal rings is immediate from Theorem 1.1.1 and Proposition 3.3.1. The description of the versal rings follows from Lemmas 2.2.18 and 3.2.16. $\square$

We now consider base changes $S' \to S$. It is of course unreasonable to expect that the formation of scheme-theoretic images is compatible with such base changes unless we make a flatness hypothesis; but under this condition, we are able to prove the following general result.

3.3.3. **Proposition.** *Suppose that $\mathcal{X} \to \mathcal{F}$ is a morphism of étale stacks over the locally Noetherian base scheme S, where $\mathcal{X}$ is algebraic and $\Delta_{\mathcal{F}}$ is representable by algebraic spaces. Suppose that the scheme-theoretic image $\mathcal{Z}$ of $\mathcal{X} \to \mathcal{F}$ is an algebraic stack, that $\mathcal{Z} \to \mathcal{F}$ is a closed immersion, and that $\mathcal{X} \to \mathcal{F}$ factors through a scheme-theoretically dominant morphism $\mathcal{X} \to \mathcal{Z}$.*

Let $S' \to S$ be a flat morphism of locally Noetherian schemes, and write $\mathcal{X}_{S'}$, $\mathcal{F}_{S'}$, and $\mathcal{Z}_{S'}$ for the base changes of $\mathcal{X}$, $\mathcal{F}$, and $\mathcal{Z}$ to S'. Write $\mathcal{Z}'$ for the scheme-theoretic image of $\mathcal{X}_{S'} \to \mathcal{F}_{S'}$.

Then $\mathcal{Z}' = \mathcal{Z}_{S'}$; so $\mathcal{Z}'$ is an algebraic stack, $\mathcal{Z}' \to \mathcal{F}_{S'}$ is a closed immersion, and $\mathcal{X}_{S'} \to \mathcal{F}_{S'}$ factors through a scheme-theoretically dominant morphism $\mathcal{X} \to \mathcal{Z}'$.

Proof. Note firstly that $\mathcal{Z}_{S'}$ is an algebraic stack, that $\mathcal{Z}_{S'} \to \mathcal{F}_{S'}$ is a closed immersion, and that $\mathcal{X}_{S'} \to \mathcal{F}_{S'}$ factors through $\mathcal{Z}_{S'}$. Since $S' \to S$ is flat, we see that $\mathcal{X}_{S'} \to \mathcal{Z}_{S'}$ is scheme-theoretically dominant by Remark 3.1.5 (2). Since the formation of the diagonal is compatible with base change, $\Delta_{\mathcal{F}_{S'}}$ is representable by algebraic spaces and locally of finite presentation. The result follows immediately from Proposition 3.2.31. □

4. Examples

4.1. **Quotients of varieties by proper equivalence relations.** We indicate some simple examples of quotients of varieties (over $\mathbb{C}$, so that we may form the quotients in the sense of topological spaces) by proper equivalence relations which are not algebraic objects.

4.1.1. *Example.* Consider the equivalence relation on $\mathbb{P}^2$ which contracts a line to a point. In the topological category, the quotient of $\mathbb{P}^2(\mathbb{C})$ by this equivalence relation is S^4 (a 4-sphere), which is not a Kähler manifold (since it has vanishing H^2). Thinking more algebraically, one can show that there is no proper morphism of algebraic spaces $\mathbb{P}^2 \to X$ for which $\mathbb{P}^2 \times_X \mathbb{P}^2$ coincides with this equivalence relation; indeed, if there were such a morphism, then the theorem on formal functions would show that the complete local ring of X at the point obtained as the image of the contracted $\mathbb{P}^1$ is equal to the ring of global sections of the structure sheaf of the formal completion of $\mathbb{P}^2$ along $\mathbb{P}^1$; but since the conormal bundle of $\mathbb{P}^1$ in $\mathbb{P}^2$ equals $\mathcal{O}(-1)$, this ring of global sections is just equal to $\mathbb{C}$, and so cannot arise as the complete local ring at a point of a two-dimensional algebraic space. Thus there isn't any reasonable way to take the quotient of $\mathbb{P}^2$ by this equivalence relation in the world of algebraic spaces over $\mathbb{C}$.

4.1.2. *Example.* Consider the space X of endomorphisms of $\mathbb{P}^1$ of degree ≤ 1. Such an endomorphism is described by a linear fractional transformation $x \mapsto \dfrac{ax + b}{cx + d}$ for which the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-zero, and so X is a quotient of $\mathbb{P}^3$ (thought of as the projectivisation of the vector space $M_2(\mathbb{C})$ of 2×2 matrices). However, the locus of singular matrices (i.e. the subvariety cut out by the vanishing of the determinant), which gives rise to the constant endomorphisms, is two-dimensional (indeed a quadric surface in $\mathbb{P}^3$), while the space of constant endomorphisms is obviously just equal to $\mathbb{P}^1$. Thus X can be thought of as a quotient of $\mathbb{P}^3$ in which a quadric is contracted to a projective line (via one of the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$). Similarly to the preceding example, the space X can't be realised in the world of algebraic spaces. (The first author would like to thank V. Drinfeld for explaining this example to him.)

4.1.3. *Remark.* It follows from [Art70] that in contexts like those considered in the preceding examples, in which we wish to contract a closed subvariety of a proper variety in some manner, *if* we have a formal model for neighbourhood of a contraction, then we *can* perform the contraction in the category of algebraic spaces. The computation with complete local rings on the (hypothetical) quotient

in Example 4.1.1 can be thought of as showing that, for this example, such a formal model doesn't exist.

A key intuition behind the main theorem of the present paper is that, as one of the hypotheses of the theorem, we do assume that we have well-behaved complete local rings at each finite type point of the quotient we are trying to analyse.

4.2. **Ind-algebraic stacks.** Ind-algebraic stacks give simple examples of stacks which satisfy some but not all of Artin's axioms, and which admit surjective morphisms from algebraic stacks. We refer the reader to [Eme] for the details of the theory of Ind-algebraic stacks, contenting ourselves here with recalling some general facts, and giving various examples (several of which are taken from [Art69b]) which illustrate the roles of the various axioms in the results of Section 2. Example 4.3.12 also illustrates the necessity of the properness hypothesis in the statement of Theorem 1.1.1, while Example 4.3.7 illustrates the necessity of the hypothesis of scheme-theoretic dominance (rather than mere surjectivity) in the statement of Corollary 1.1.2.

4.2.1. **Definition.** We say that a stack $\mathcal{X}$ over S is an *Ind-algebraic stack* if we may find a directed system of algebraic stacks $\{\mathcal{X}_i\}_{i \in I}$ over S and an isomorphism $\varinjlim_{i \in I} \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$, the inductive limit being computed in the 2-category of stacks. If $\mathcal{X}$ is a stack in setoids (i.e. is equivalent to a sheaf of sets), then we say that $\mathcal{X}$ is an *Ind-algebraic space*, resp. an *Ind-scheme*, if the $\mathcal{X}_i$ can in fact be taken to be algebraic spaces, resp. schemes.

4.2.2. *Remark.* More properly, the inductive limit is a 2-colimit, but as in [Eme], we find it more suggestive to use the usual notation for direct limits.

4.2.3. *Remark.* Our definitions of Ind-algebraic stacks, Ind-algebraic spaces, and Ind-schemes are broader than usual. Indeed, at least in the case of Ind-schemes, it is conventional to require the transition morphisms $\mathcal{X}_i \to \mathcal{X}_{i'}$ to be closed immersions. We have adopted a laxer definition simply because it provides a convenient framework in which to discuss many of the examples of Subsection 4.3 below.

4.2.4. *Remark.* Given a directed system $\{\mathcal{X}_i\}_{i \in I}$ as in Definition 4.2.1, then for any S-scheme T there is morphism of groupoids

$$(4.2.5) \qquad \varinjlim_{i \in I} \mathcal{X}_i(T) \to \big(\varinjlim_{i \in I} \mathcal{X}_i \big)(T).$$

Although this is not an equivalence in general, it *is* an equivalence if T is quasi-compact and quasi-separated (e.g. if T is an affine S-scheme). We record this fact in the following lemma (which is presumably well-known); see e.g. [Sta, Tag 0738] for the analogous statement in the context of sheaves.

Also, although in the above definition all stacks involved are (following our conventions) understood to be stacks for the étale site, in the statement of the lemma we consider stacks for other topologies as well.

4.2.6. **Lemma.** *Let $\{\mathcal{X}_i\}_{i \in I}$ be an inductive system of Zariski, étale, fppf, or fpqc stacks, and consider the inductive limit $\varinjlim_{i \in} \mathcal{X}_i$, computed as a stack for the topology under consideration. If T is an quasi-compact S-scheme, then the morphism (4.2.5) is faithful. If, in addition, either T is quasi-separated, or the transition maps in the inductive system are monomorphisms, then it is in fact an equivalence of groupoids.*

Proof. We write $\mathcal{X} := \varinjlim_{i \in I} \mathcal{X}_i$ (the inductive limit being taken as stacks). If ξ_i, η_i are a pair of objects in $\mathcal{X}_i(T)$, inducing objects $\xi_{i'}, \eta_{i'}$ in $\mathcal{X}_{i'}(T)$ for each $i' \geq i$, and objects ξ, η in $\mathcal{X}(T)$, then the set of morphisms between $\xi_{i'}$ and $\eta_{i'}$ is equal to the space of global sections of the sheaf $\mathcal{X}_{i'} \times_{\Delta, \mathcal{X}_{i'} \times_S \mathcal{X}_{i'}, \xi_{i'} \times \eta_{i'}} T$, while the set of morphisms between ξ and η is equal to the space of global sections of $\mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \xi \times \eta} T$, which is also equal to the set of global sections of the inductive limit of sheaves $\varinjlim_{i' \geq i} \mathcal{X}_{i'} \times_{\Delta, \mathcal{X}_{i'} \times_S \mathcal{X}_{i'}, \xi_{i'} \times \eta_{i'}} T$.

If T is quasi-compact and quasi-separated, then the inductive limit of global sections maps isomorphically to the global sections of the inductive limit [Sta, Tag 0738], and so we find that the morphism (4.2.5) is fully faithful for such T. The same reference shows that if T is merely assumed to be quasi-compact, then the morphism on global sections is injective, and hence that the morphism (4.2.5) is faithful. Finally, if the transition morphisms in the inductive system of sheaves under consideration are injective, then this reference again shows that the map on global sections is an isomorphism provided merely that T is quasi-compact. If the transition morphisms $\mathcal{X}_{i'} \to \mathcal{X}_{i''}$ are monomorphisms, then the corresponding transition morphisms $\mathcal{X}_{i'} \times_{\Delta, \mathcal{X}_{i'} \times_S \mathcal{X}_{i'}, \xi_{i'} \times \eta_{i'}} T \to \mathcal{X}_{i''} \times_{\Delta, \mathcal{X}_{i''} \times_S \mathcal{X}_{i''}, \xi_{i''} \times \eta_{i''}} T$ are indeed injective, and hence in this case the morphism (4.2.5) is again fully faithful.

We turn to considering the essential surjectivity of (4.2.5). If T is quasi-compact, then for any object of $\mathcal{X}(T)$, we may find a cover $T' \to T$ (in the appropriate topology: Zariski, étale, *fppf*, or *fpqc*, as the case may be) such that T' is again quasi-compact, and a morphism $T' \to \mathcal{X}_i$ for some i such that the diagram

$$
\begin{array}{ccc}
T' & \longrightarrow & \mathcal{X}_i \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathcal{X}
\end{array}
$$

commutes. We obtain an induced morphism $T' \times_T T' \to \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_i$. The target of this morphism may be written as $\varinjlim_{i' \geq i} \mathcal{X}_i \times_{\mathcal{X}_{i'}} \mathcal{X}_i$. If T is also quasi-separated, so that $T' \times_T T'$ is again quasi-compact, then by what we have already observed, this morphism arises from a morphism $T' \times_T T' \to \mathcal{X}_i \times_{\mathcal{X}_{i'}} \mathcal{X}_i$ for some sufficiently large value of i'. Consequently, we obtain a factorisation of the induced morphism $T' \times_T T' \to \mathcal{X}_{i'} \times_S \mathcal{X}_{i'}$ through the diagonal. When we pass from $\mathcal{X}_{i'}$ to $\mathcal{X}$, we obtain the descent data to T of the composite $T' \to T \to \mathcal{X}$, and so the faithfulness result already proved shows that, if we enlarge i' sufficiently, this factorisation provides descent data to T for the morphism $T' \to \mathcal{X}_{i'}$. Since $\mathcal{X}_{i'}$ is a stack, we obtain a morphism $T \to \mathcal{X}_{i'}$ inducing the original morphism $T \to \mathcal{X}$.

If the transition morphisms in the inductive system $\{\mathcal{X}_i\}$ are monomorphisms, then the various fibre products $\mathcal{X}_i \times_{\mathcal{X}_{i'}} \mathcal{X}_i$ are all isomorphic to $\mathcal{X}_i$ (via the diagonal), and hence the natural map $\mathcal{X}_i \to \varinjlim_{i' \geq i} \mathcal{X}_i \times_{\mathcal{X}_{i'}} \mathcal{X}_i$ is an isomorphism. Thus in this case, the morphism $T' \times_T T' \to \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_i$ necessarily arises from a morphism $T' \times_T T' \to \mathcal{X}_i$, without any quasi-compactness assumption on $T' \times_T T'$. Thus, in this case, we obtain the essential surjectivity of (4.2.5) under the assumption that T is merely quasi-compact. $\qquad\square$

4.2.7. *Remark.* The preceding lemma (in particular, the fact that the isomorphism (4.2.5) provides an explicit description of $\left(\varinjlim_{i \in I} \mathcal{X}_i\right)(T)$ which makes no reference to the site over which the inductive limit is computed, when T is an affine

S-scheme), shows that to form the stack $\varinjlim_{i \in I} \mathcal{X}_i$, for an inductive systems of stacks in the étale, *fppf*, or *fpqc* topologies, it in fact suffices to form the corresponding inductive limit as a stack for the Zariski topology.

4.2.8. *Remark.* If $\{\mathcal{X}_i\}_{i \in I}$ is an inductive system of stacks (for the étale topology), and if $\mathcal{Y}$ is a category fibred in groupoids over S, then, analogously to (4.2.5), we have a natural morphism of groupoids

$$\varinjlim_{i \in I} \mathrm{Mor}(\mathcal{Y}, \mathcal{X}_i) \to \mathrm{Mor}(\mathcal{Y}, \varinjlim_{i \in I} \mathcal{X}_i).$$

In particular if $\mathcal{Y}$ is an algebraic stack, then the obvious extension of Lemma 4.2.6 holds: namely, if $\mathcal{Y}$ is quasi-compact then this morphism is faithful, and if furthermore either $\mathcal{Y}$ is quasi-separated, or the transition morphisms in the inductive system are monomorphisms, then this morphism is an equivalence. (This can be checked by choosing a smooth surjection $T \to \mathcal{Y}$ from an S-scheme to $\mathcal{Y}$, whose domain can be taken to be affine if $\mathcal{Y}$ is quasi-compact, and using the fact that morphisms from $\mathcal{Y}$ to any stack on the étale site can be identified with morphisms from T to the same stack with appropriate descent data; we leave the details to the reader.)

4.2.9. *Remark.* We briefly discuss the comportment of Ind-algebraic stacks with regard to Artin's axioms. Suppose that $\mathcal{X} := \varinjlim_{i \in I} \mathcal{X}_i$ is an Ind-algebraic stack.

(1) If each $\mathcal{X}_i$ satisfies [1], then it follows from Remark 4.2.4 that $\mathcal{X}$ also satisfies [1].

(2) It follows from Remark 4.2.4 that $\mathcal{X}$ satisfies [2](a), since this is true of each $\mathcal{X}_i$. The question of whether or not a particular Ind-stack satisfies [2](b) is more involved, and we discuss it in some detail for Ind-algebraic spaces in 4.2.10 below.

(3) Typically, $\mathcal{X}$ need not satisfy [3]. If T is a quasi-compact and quasi-separated S-scheme, then any morphism $T \to \mathcal{X} \times_S \mathcal{X}$ factors through a morphism $T \to \mathcal{X}_i \times_S \mathcal{X}_i$ for some $i \in I$, and thus $\mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} T$ is the base-change with respect to this morphism of

$$\mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} (\mathcal{X}_i \times_S \mathcal{X}_i) = \mathcal{X}_i \times_\mathcal{X} \mathcal{X}_i = \varinjlim_{i' \geq i} \mathcal{X}_i \times_{\mathcal{X}_{i'}} \mathcal{X}_i$$

(equipped with its natural morphism to $\mathcal{X}_i \times_S \mathcal{X}_i$); consequently the diagonal of $\mathcal{X}$ is representable by Ind-algebraic spaces. If each of the transition morphisms $\mathcal{X}_i \to \mathcal{X}_{i'}$ is a monomorphism, then so is the morphism $\mathcal{X}_i \to \mathcal{X}$, and we find that $\mathcal{X}_i \xrightarrow{\sim} \mathcal{X}_i \times_\mathcal{X} \mathcal{X}_i$. Thus in this case we find that $\mathcal{X}$ does satisfy [3]. One can give examples in which the transition morphisms are *not* monomorphisms, but for which nevertheless the diagonal of $\mathcal{X}$ is representable by algebraic spaces; see e.g. Examples 4.3.4 and 4.3.12 below. However, certainly in general the diagonal of $\mathcal{X}$ is not representable by algebraic spaces, even for examples which seem tame in many respects; see e.g. Example 4.3.5.

(4) Typically an Ind-algebraic stack will not satisfy [4]. However, since algebraic stacks satisfy [4a] (Lemma 2.4.15), we see that Ind-algebraic stacks will satisfy [4a]. Typically they will not satisfy [4b], though. In some cases, however, they may satisfy both [4a] and [4b], but not [4]; Example 4.3.4 illustrates this possibility (and demonstrates that the quasi-separatedness hypotheses in Corollary 2.6.11 and Theorem 2.8.5 are necessary).

4.2.10. *Versal rings for Ind-algebraic spaces.* If x is a finite type point of the Ind-algebraic stack $\mathcal{X} := \varinjlim_{i \in I} \mathcal{X}_i$, then x arises from a finite type point x_i of $\mathcal{X}_i$ for some $i \in I$, and if we write $x_{i'}$ to denote the corresponding finite type point of $\mathcal{X}_{i'}$, for each $i' \geq i$, then one can attempt to construct a versal deformation ring at x by taking a limit over the versal rings of each $x_{i'}$. This limit process is complicated in general by the non-canonical nature of the versal deformation ring of a finite type point in a stack, and so we will restrict our precise discussion to the case of Ind-algebraic spaces, for which we can find canonically defined minimal versal rings.

To be precise, suppose that S is locally Noetherian. Note that an algebraic space X over S satisfies [1] if and only if it is locally of finite presentation over S (Lemma 2.1.9), which in turn holds if and only if it is locally of finite type over S [Sta, Tag 06G4].

4.2.11. **Definition.** We say that a sheaf of sets X on the étale site of S is an *Ind-locally finite type algebraic space over S* if there is an isomorphism $X \cong \varinjlim_{i \in I} X_i$, where $\{X_i\}_{i \in I}$ is a directed system of algebraic spaces, locally of finite type over S.

In this case, we will be able to show that each finite type point x of X admits a representative which is a monomorphism, unique up to unique isomorphism, and for such a monomorphism $\operatorname{Spec} k \to X$, we will construct a canonical minimal versal ring to X at x. Despite this, it doesn't follow that X satisfies [2](b). Loosely speaking, there are two possible obstructions to [2](b) holding: Firstly, it may be that the versal ring at a point x is not Noetherian. This happens when the dimension of the various $\mathcal{X}_i$ at the points x_i increases without bound, so that the Ind-algebraic space X is infinite dimensional; a typical example is given by infinite-dimensional affine space (Example 4.3.3). Secondly, even if the versal ring x is Noetherian, it may not be effective. This happens when the infinitesimal germs of X_i at the points x_i collectively fill out a space of higher dimension than each of the individual germs of X_i individually does; typical examples are given by the union of infinitely many lines passing through a fixed point (Example 4.3.8), or a curve with a cusp of infinite order (Example 4.3.9). Roughly speaking, X will satisfy [2](b) when the infinitesimal germs of the X_i at the points x_i eventually stabilise, or equivalently, when the projective limit defining the versal deformation ring at x eventually stabilises. (This is not quite true; for example, in the inductive system X_i, one could alternately add additional components, and then contract them to a point. It will be true in all the examples we give below for which [2](b) is satisfied.)

We now present the details of the preceding claims.

4.2.12. **Lemma.** *Any finite type point of an Ind-locally finite type algebraic space $X = \varinjlim_{i \in I} X_i$ over the locally Noetherian scheme S admits a representative $x : \operatorname{Spec} k \to X$ which a monomorphism. This representative is unique up to unique isomorphism, the field k is a finite type $\mathcal{O}_S$-field, and any other representative $\operatorname{Spec} K \to X$ of the given point factors through the morphism x in a unique fashion. Furthermore, if $i \in I$ is sufficiently large, then $x : \operatorname{Spec} k \to X$ factors in a unique manner as a composite $\operatorname{Spec} k \xrightarrow{x_i} X_i \to X$, and the morphism $x_i : \operatorname{Spec} k \to X_i$ is again a monomorphism.*

Proof. From the definitions, one sees that $|X| = \varinjlim_{i \in I} |X_i|$. Thus the given point of X arises from a point of X_i for some $i \in I$. Let $x_i : \operatorname{Spec} k \to X_i$ be the

monomorphic representative of this point whose existence is given by Lemma 2.2.14. For each $i' \geq i$, let $x_{i'} : \operatorname{Spec} k_{i'} \to X_{i'}$ be the monomorphic representative of the image of this point in $X_{i'}$. Let k' denote the residue field of the image of this point in S. Then we have natural containments $k \supseteq k_{i'} \supseteq k'$, for each $i' \geq i$. Thus, since k is finite over k', we see, replacing i by some sufficiently large $i' \geq i$ if necessary, that we may assume that $k_{i'} = k$ for all $i' \geq i$, and thus conclude that $x_{i'}$ is simply the composite $x_i : \operatorname{Spec} k \to X_i \to X_{i'}$, for all $i' \geq i$. If we define x to be the composite $x : \operatorname{Spec} k \to X_i \to X$, then, since $x_{i'}$ is a monomorphism for each $i' \geq i$, one easily verifies that x is a monomorphism.

The remaining claims of the lemma are proved identically to the analogous claims of Lemma 2.2.14. $\qquad\square$

If x is a finite type point of the Ind-locally finite type algebraic space $X = \varinjlim_{i \in I} X_i$ over S, and if (by abuse of notation) we also write $x : \operatorname{Spec} k \to X$ to denote the monomorphic representative of x provided by the preceding lemma, and (following the lemma) write $x_i : \operatorname{Spec} k \to X_i$ to denote the induced monomorphisms (for sufficiently large i), then we may consider the complete local rings $\widehat{\mathcal{O}}_{X_i, x_i}$ (for sufficiently large i), in the sense of Definition 2.2.17. If $i' \geq i$ (both sufficiently large), then we obtain a canonical local morphism of complete local $\mathcal{O}_S$-algebras $\widehat{\mathcal{O}}_{X_{i'}, x_{i'}} \to \widehat{\mathcal{O}}_{X_i, x_i}$.

4.2.13. **Definition.** In the above context, we write

$$\widehat{\mathcal{O}}_{X,x} := \varprojlim_{i \in I} \widehat{\mathcal{O}}_{X_i, x_i},$$

considered as a pro-Artinian ring, and we refer to $\widehat{\mathcal{O}}_{X,x}$ as the complete local ring to X at x. By the following lemma it is canonically defined, independent of the description of X as an Ind-locally finite type algebraic space.

4.2.14. **Lemma.** *If X is an Ind-locally finite type algebraic space over the locally Noetherian scheme S, and if $x : \operatorname{Spec} k \to X$ is the monomorphic representative of a finite type point of X, which (by abuse of notation) we also denote by x, then $\widehat{\mathcal{O}}_{X,x}$ is a versal ring for X at x. Furthermore, the morphism $\operatorname{Spf} \widehat{\mathcal{O}}_{X,x} \to X$ is a formal monomorphism, and therefore the ring $\widehat{\mathcal{O}}_{X,x}$, equipped with this morphism, is unique up to unique isomorphism.*

Proof. Any commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} B \\
\downarrow & & \downarrow \\
\operatorname{Spf} \widehat{\mathcal{O}}_{X,x} & \longrightarrow & \widehat{X}_x
\end{array}
$$

in which A and B are finite type Artinian local $\mathcal{O}_S$-algebras, is induced by a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} B \\
\downarrow & & \downarrow \\
\operatorname{Spf} \widehat{\mathcal{O}}_{X_i, x_i} & \longrightarrow & \widehat{X}_{i, x_i}
\end{array}
$$

for some sufficiently large value of i. Since the lower horizontal arrow of this diagram is versal, we may lift the right-hand vertical arrow to a morphism $\operatorname{Spec} B \to \operatorname{Spf} \widehat{\mathcal{O}}_{X_i,x_i}$, and hence obtain a lifting of the right-hand vertical arrow of the original diagram to a morphism $\operatorname{Spec} B \to \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$. This establishes the versality.

The property of being a formal monomorphism follows easily from the fact that (by Proposition 2.2.15) the morphisms $\operatorname{Spf} \widehat{\mathcal{O}}_{X_i,x_i} \to X_i$ are formal monomorphisms. That the data of $\widehat{\mathcal{O}}_{X,x}$ together with the morphism $\operatorname{Spf} \widehat{\mathcal{O}}_{X,x} \to X$ is unique up to unique isomorphism is then immediate from Lemma 2.2.16.   □

4.2.15. *Ind-algebraic stacks as scheme-theoretic images.* If $\mathcal{X} := \varinjlim_{i \in I} \mathcal{X}_i$ is an Ind-algebraic stack, then there is an evident morphism of stacks $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}$, whose source is an algebraic stack (being the disjoint union of a collection of algebraic stacks), and which can be verified to be representable by algebraic spaces precisely when $\mathcal{X}$ satisfies [3]. Thus Ind-stacks give us examples of maps from algebraic stacks to not-necessarily-algebraic stacks, to which we can try to apply the machinery of Section 3 (although the fact that the morphism $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}$ is not quasi-compact in general is an impediment to such applications).

As one illustration of this, we note that Example 4.3.12 below shows that our main results will not extend in any direct way to morphisms of finite type which are not assumed to be proper.

4.3. **Illustrative examples.** Here we present various illustrative examples, and explain how they relate to the general theory. Several of them are due originally to Artin [Art69b, §5].

4.3.1. *Example.* [Art69b, Ex. 5.11]: X is the sheaf of sets obtained by taking the union of the two schemes $\operatorname{Spec} k[x,y][1/x]$ and $\operatorname{Spec} k[x,y]/(y)$ in the (x,y)-plane. (More precisely, X is the sheaf obtained as the pushout of these schemes along their common intersection.) The sheaf X satisfies [1], [2], [3], and [4b], but doesn't satisfy [4a]. (And hence is not an algebraic space, and so doesn't satisfy [4].)
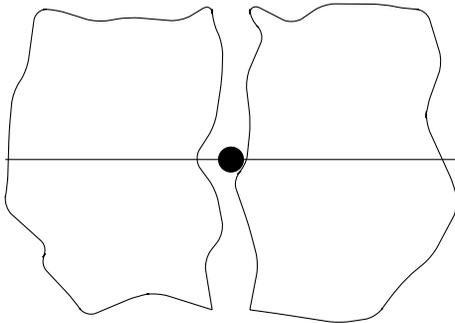
We now give a series of examples of various kinds of Ind-algebraic stacks over a field k. All the stacks we consider will be the inductive limit of stacks satisfying [1], and hence will satisfy [1], as well as [2](a) and [4a]. In fact, all the examples other than Example 4.3.2 will be Ind-locally finite algebraic spaces (in fact, even Ind-locally finite type schemes) over k, and so will admit versal rings at finite type points by Lemma 4.2.14. (As discussed above, though, this doesn't necessarily imply that they satisfy [2](b).)

4.3.2. *Example.* Let $\mathcal{X}_i$ be a directed system of algebraic stacks, locally of finite presentation over a locally Noetherian scheme S, for which the transition morphisms are smooth, and consider the Ind-algebraic stack $\mathcal{X} := \varinjlim \mathcal{X}_i$. Since each $\mathcal{X}_i$ satisfies [1], so does $\mathcal{X}$, and being an Ind-algebraic stack, it also satisfies [2](a) and [4a]. Since the transition morphisms $\mathcal{X}_i \to \mathcal{X}_{i'}$ (for $i' \geq i$) are smooth, and since each of the algebraic stacks $\mathcal{X}_i$ satisfies [4], we find that $\mathcal{X}$ also satisfies [4] (and hence also [4b]). If $x : \operatorname{Spec} k \to \mathcal{X}$ is a finite type point, then x factors as $\operatorname{Spec} k \xrightarrow{x_i} \mathcal{X}_i \to \mathcal{X}$ for some i, and one easily verifies (again using the smoothness of the transition morphisms) that if $\operatorname{Spf} R \to \mathcal{X}_i$ is a versal ring to x_i, then the composite $\operatorname{Spf} R \to \mathcal{X}_i \to \mathcal{X}$ is a versal ring to $\mathcal{X}$. Thus $\mathcal{X}$ satisfies [2](b). In conclusion, such an Ind-algebraic stack necessarily satisfies each of our axioms except possibly [3].

One example of such an Ind-algebraic stack is obtained by taking $\{G_i\}$ to be a directed system of smooth algebraic groups over a field k and setting $\mathcal{X}_i := [\cdot/G_i]$, so that $\mathcal{X} := \varinjlim [\cdot/G_i]$. (Here "$\cdot$" stands for "the point", i.e. $\operatorname{Spec} k$.) If we set $G := \varinjlim_{i \in I} G_i$, then $\mathcal{X}$ may be regarded as the classifying stack $[\cdot/G]$. The fibre product $\cdot \times_{\mathcal{X}} \cdot$ is then naturally identified with G, and so if the Ind-algebraic group G is not a scheme (as it typically will not be), then $\mathcal{X}$ does not satisfy [3].

4.3.3. *Example.* We take X to be "infinite dimensional affine space". More formally, we write $X := \varinjlim \mathbb{A}^n$, with the transition maps being the evident closed immersions:

$$\mathbb{A}^n \cong \mathbb{A}^n \times \{0\} \hookrightarrow \mathbb{A}^n \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}.$$

Since the transition maps are closed immersions, the Ind-scheme X satisfies [3] (and is quasi-separated). The complete local rings at points are non-Noetherian (they are power series rings in countably many variables), and so X does not satisfy [2](b). It does satisfy [4] (and so also [4b]) vacuously: because the complete local rings are so large, one easily verifies that a morphism from a finite type k-scheme to X is never versal at a finite type point.

4.3.4. *Example.* We take X to be the line with infinitely many nodes [Art69b, Ex. 5.8] (considered as an Ind-algebraic space in the evident way).
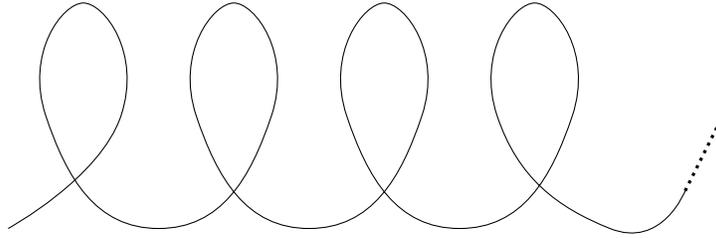
The Ind-scheme X satisfies [2](b), and, although the transition maps are not closed immersions, it also satisfies [3]; however, it is not quasi-separated, since the diagonal morphism $X \to X \times X$ is not quasi-compact. Concretely, if $\mathbb{A}^1 \to X$ is the obvious morphism, namely the one that identifies countably many pairs of points on $\mathbb{A}^1$ to nodes, then $\mathbb{A}^1 \times_X \mathbb{A}^1$ is the union of the diagonal copy of $\mathbb{A}^1$, and countably many discrete points (encoding the countably many identifications that are made to create the nodes of X).

The Ind-scheme X doesn't satisfy [4], but it does satisfy [4b], vacuously. (One can check that a morphism from a finite type k-scheme to X cannot be smooth at any point.) This example shows that the quasi-separatedness hypotheses are necessary in Corollaries 2.6.11 and 2.6.12 and Theorem 2.8.5. [6]

4.3.5. *Example.* We take X to be the line with infinitely many cusps.
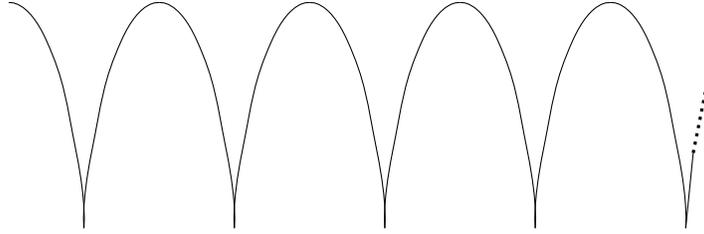
The Ind-scheme X satisfies [2](b). As in the previous example, the transition morphisms are not closed immersions, and in this case X does not satisfy [3]; indeed, if $\mathbb{A}^1 \to X$ is the natural morphism contracting a countable set of points to the cusps of X, then $\mathbb{A}^1 \times_X \mathbb{A}^1$ is the Ind-scheme obtained by adding non-reduced structure to the diagonal copy of $\mathbb{A}^1$ at each of the points that is contracted to a cusp.

The Ind-scheme X does not satisfy [4], but just as in the previous example, it does satisfy [4b] vacuously.

4.3.6. *Example.* We take X to be the line with infinitely many lines crossing it [Art69b, Ex. 5.10].
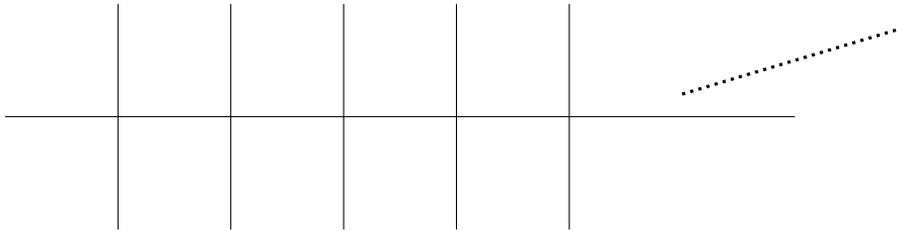
This example satisfies [2](b) and [3] (and it is quasi-separated, since the transition maps are closed immersions), but doesn't satisfy [4b] (and hence doesn't satisfy [4]).

4.3.7. *Example.* As a variation on the preceding example, we consider the Ind-scheme X given by adding infinitely many embedded points to a line.

As with the preceding example, this example satisfies [2](b) and [3], is quasi-separated, and doesn't satisfy [4b] (and hence doesn't satisfy [4]).

One point of interest related to this example is that the natural morphism $\mathbb{A}^1 \to X$ is a closed immersion (in particular it is both quasi-compact and proper), and is surjective; however it is not scheme-theoretically dominant. Since X is not an algebraic space, this shows the importance of scheme-theoretic dominance (rather than mere surjectivity) as a hypothesis in Corollary 1.1.2.

---

[6]Artin states that this example does not satisfy [4b], but seems to be in error on this point.

4.3.8. *Example.* We take X to be the union of infinitely many lines through the origin in the plane [Art69b, Ex. 5.9].
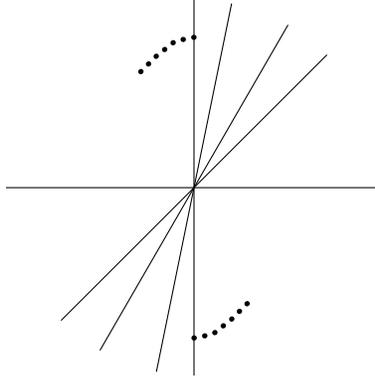
This example satisfies [3], but not [2](b): the complete local ring at the origin is equal to $k[[x, y]]$, and the corresponding morphism $\operatorname{Spf} k[[x, y]] \to X$ is not effective. It satisfies [4] (and hence also [4b]) vacuously: the non-effectivity of the complete local ring at the origin shows that one cannot find a morphism from a finite type k-scheme to X which is versal at a point lying over the origin in X.

4.3.9. *Example.* Let X_n be the plane curve cut out by the equation $y^{2^n} = x^{2^n+1}$, define the morphism $X_n \to X_{n+1}$ via $(x, y) \mapsto (x^2, xy)$, and let $X := \varinjlim X_n$; so X is a line with a cusp of infinite order.
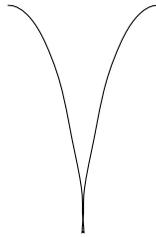
The Ind-scheme X does not satisfy [2](b): the complete local ring at the cusp is equal to $k[[x, y]]$, and the morphism $\operatorname{Spf} k[[x, y]] \to X$ is not effective. Also, X does not satisfy [3]: if $\mathbb{A}^1 = X_0 \to X$ is the natural morphism, then $\mathbb{A}_1 \times_X \mathbb{A}^1$ is a formal scheme, which is an infinite-order thickening up of the diagonal copy of $\mathbb{A}^1$ at the origin.

4.3.10. *Example.* In [Art69b, Ex. 5.3], Artin gives the example of two lines meeting to infinite order as an Ind-scheme satisfying [2](b) and [4], but not [3]. We give a variant of Artin's example here, which illustrates the necessity of [3] in Lemma 2.8.7 (3).

We take $k = \mathbb{R}$, and we define $X_n := \operatorname{Spec} \mathbb{R}[x, y]/(y^2 + x^{2n})$, with the transition morphism $X_n \to X_{n+1}$ given by $(x, y) \mapsto (x, xy)$, and set $X := \varinjlim_n X_n$.

complex conjugation
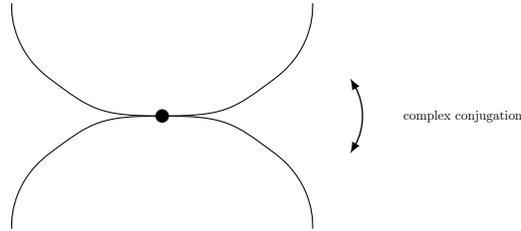
As with Artin's example, the Ind-scheme X satisfies [4]. The complete local ring of X at the origin is equal to $\mathbb{R}[[x]]$, but the natural morphism $\operatorname{Spf}\mathbb{R}[[x]] \to X$ is not effective; thus X does not satisfy [2](b). On the other hand, if we consider the map $\operatorname{Spec}\mathbb{C} \to X$ induced by the origin, then we obtain a versal morphism $\operatorname{Spf}\mathbb{C}[[x]] \to X$ which is effective, although the resulting morphism $\operatorname{Spec}\mathbb{C}[[x]] \to X$ is not unique; we can map $\operatorname{Spec}\mathbb{C}[[x]]$ along either of the branches through the origin.

4.3.11. *Example.* We take X to be the indicated Ind-scheme ("the zipper").
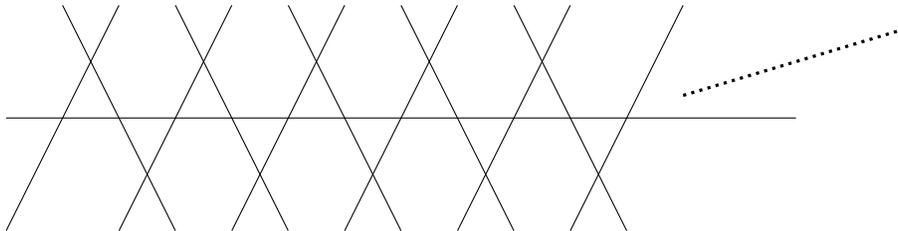
It has the same formal properties as the Ind-scheme of Example 4.3.6, namely it satisfies [2](b) and [3], but not [4] or [4b].

4.3.12. *Example.* We will give an example of an Ind-algebraic *surface* which contains
the zipper of the preceding example as a closed sub-Ind-scheme.
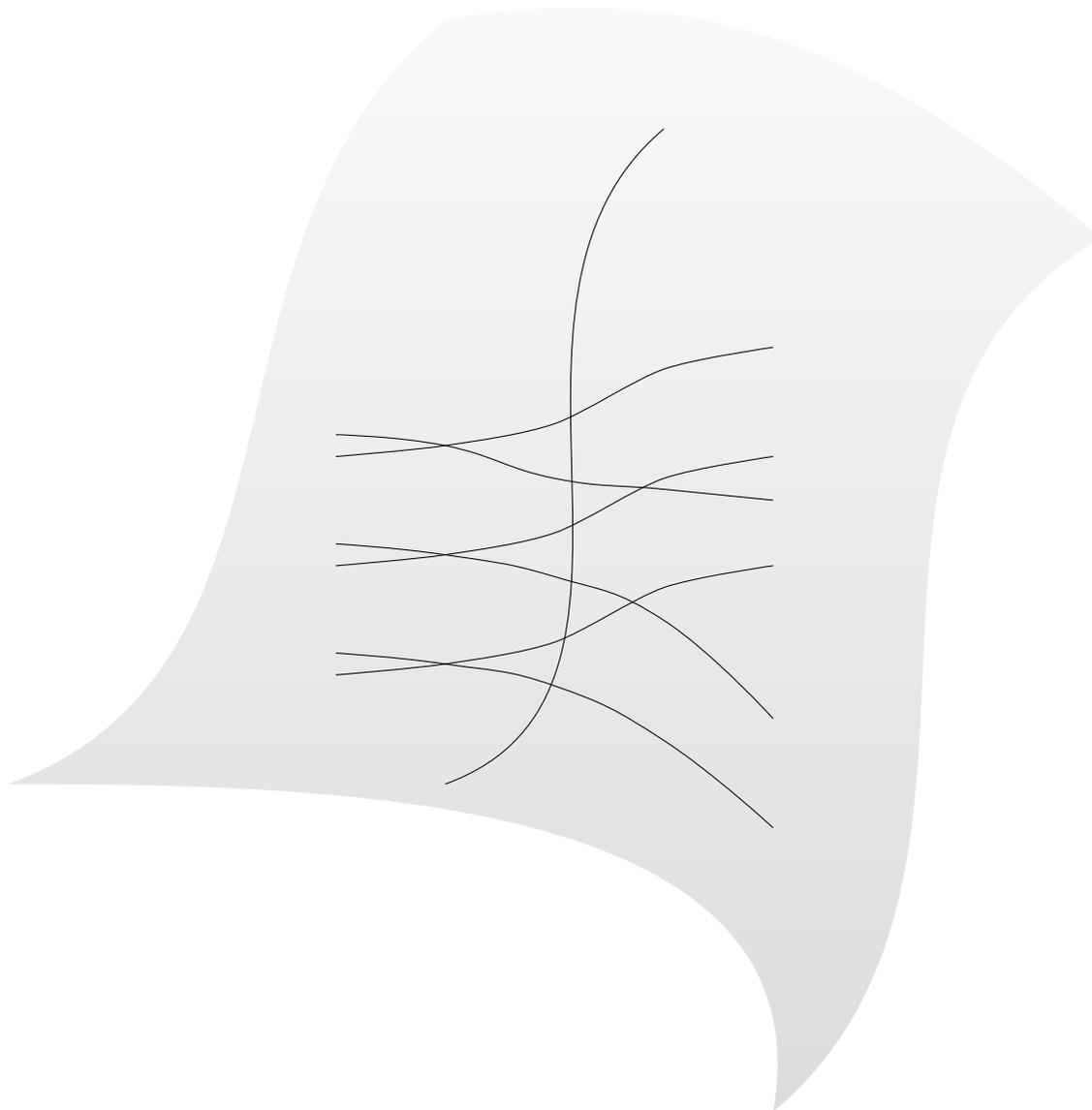
We begin by setting $\overline{X}_0 := \mathbb{A}^2$; we also choose a closed point $P_0 \in \overline{X}_0$. We let $\overline{X}_1$
be the blow-up of $\overline{X}_0$ at P_0; it contains an exceptional divisor E_1, and we choose a
closed point $P_1 \in E_1$. We proceed to construct surfaces $\overline{X}_n$ inductively: each $\overline{X}_n$
is a smooth surface obtained by blowing up $\overline{X}_{n-1}$ at a point P_{n-1}. The surface $\overline{X}_n$
contains an exceptional divisor E_n, as well as the strict transform of the exceptional
divisor E_{n-1} on $\overline{X}_{n-1}$. We choose a closed point $P_n \in E_n$ which does not lie in the
strict transform of E_{n-1}, and then define $\overline{X}_{n+1}$ to be the blow-up of $\overline{X}_n$ at P_n.

There are natural open immersions $\overline{X}_n \setminus P_n \subseteq \overline{X}_{n+1} \setminus P_{n+1}$; the Ind-scheme $\overline{X}$ obtained by taking the inductive limit of these open immersions is in fact a scheme, and is the standard example of a locally finite type smooth irreducible surface which is not of finite type. Inside $\overline{X}$ we have the union $E := \cup_{n \geq 1} E_n \setminus \{P_n\}$, which is an infinite chain of $\mathbb{P}^1$'s. We now form an Ind-scheme X by identifying a countable collection of points on $E_1 \setminus P_1$ with a point on each $E_n \setminus \{P_n\}$ (for $n \geq 2$); the image of E in X is then a copy of the zipper of Example 4.3.11.

The Ind-surface X satisfies [2](b) and [3], and is quasi-separated. It does not satisfy [4] or [4b].

We note that each of the composite morphisms $\overline{X}_n \setminus \{P_n\} \to \overline{X} \to X$ is of finite type, so that our formalism of scheme-theoretic images applies to it. Each of these morphisms is in fact scheme-theoretically dominant, in the sense that its scheme-theoretic image is all of X. (Morally, the Ind-surface X is irreducible.) Since X is not an algebraic space, this example shows that Theorem 1.1.1 and Corollary 1.1.2 don't extend in any direct way to the case of morphisms of finite type that are not proper.

5. Moduli of finite height φ-modules and Galois representations

In this section we will combine Theorem 1.1.1 with the results of [Kis09b, PR09] to construct moduli stacks of finite height and finite flat representations of the absolute Galois groups of p-adic fields.

We begin by proving some foundational results about φ-modules of finite height and étale φ-modules. We then introduce various moduli stacks of finite height φ-modules and étale φ-modules closely related to those considered by Pappas and Rapoport in [PR09], to which we apply the machinery of the earlier parts of the paper.

5.1. **Projective modules over power series and Laurent series rings.** We begin with a discussion of some foundational results concerning finitely generated modules over the power series ring $A[[u]]$ and the Laurent series ring $A((u))$, where A is an arbitrary (not necessarily Noetherian) commutative ring. In particular, we recall some deep results of Drinfeld [Dri06] on the *fpqc*-local nature of the projectivity of such modules. We are grateful to Drinfeld for sharing with us some of his unpublished notes on the subject; several arguments in this section are essentially drawn from these notes.

5.1.1. *Projective and locally free modules.* The basic objects we are interested in are finitely generated projective modules over the rings $A[[u]]$ and $A((u))$, where A is some given ground ring. Of course, a finitely generated projective module over $A[[u]]$ (resp. over $A((u))$) is Zariski locally free on $A[[u]]$ (resp. $A((u))$), and so we can speak of a finitely generated projective $A[[u]]$- or $A((u))$-module being *of rank d*: this just means that it is Zariski locally free of rank d on $\operatorname{Spec} A[[u]]$ (resp. $\operatorname{Spec} A((u))$). However, our point of view is that we want to regard the "base" of our modules as being $\operatorname{Spec} A$, and so we will be interested in understanding the behaviour of $A[[u]]$- or $A((u))$-modules locally on $\operatorname{Spec} A$.

This prompts the following definition, which is used in [PR09].

5.1.2. **Definition.** If $\mathfrak{M}$ is a finitely generated $A[[u]]$-module, then we say that that $\mathfrak{M}$ is *fpqc locally free of rank d over A* if there exists a faithfully flat A-algebra A' such that $\mathfrak{M} \otimes_{A[[u]]} A'[[u]]$ is free of rank d over $A'[[u]]$. Similarly, if M is a finitely

generated $A((u))$-module, then we say that that M is *fpqc locally free of rank d over A* if there exists a faithfully flat A-algebra A' such that $\mathfrak{M} \otimes_{A((u))} A'((u))$ is free of rank d over $A'[[u]]$.

We use analogous terminology for other topologies besides the *fpqc* topology. E.g. if A' can be chosen so that $\operatorname{Spec} A' \to \operatorname{Spec} A$ is an *fppf, étale, Nisnevich*, or *Zariski* cover (the last notion being understood in the sense that $\operatorname{Spec} A' \to \operatorname{Spec} A$ should be surjective, and locally on the source an open immersion), then we say that $\mathfrak{M}$ or M (as the case may be) is *fppf, étale, Nisnevich, or Zariski locally free of rank d over A*.

5.1.3. *Remark.* One of the main objects of our discussion in this section is to understand, as best we can, the relationship between the various local freeness properties just defined, and the property of being finitely generated and projective (over $A[[u]]$ or $A((u))$).

As we note in Lemma 5.1.7 below, if A is Noetherian and $A \to B$ is an *fppf* morphism, then the induced morphisms $A[[u]] \to B[[u]]$ and $A((u)) \to B((u))$ are themselves faithfully flat. Thus an $A[[u]]$- or $A((u))$- module which is *fppf* locally free of finite rank in the sense of Definition 5.1.2 is in fact finitely generated and projective (since being finitely generated and projective is a property of a module that can be checked *fpqc* locally).

In the case that A is not Noetherian, or that the morphism $A \to B$ is merely faithfully flat, but not of finite presentation, we aren't able to gain the same level of control over either of the morphisms $A[[u]] \to B[[u]]$ or $A((u)) \to B((u))$, and so the precise relationship between projectivity and the notions of local freeness introduced in Definition 5.1.2 is not completely clear to us.

The most general statement that we were able to prove in the context of $A[[u]]$-modules is given in Proposition 5.1.9 below, in which we show that if an $A[[u]]$-module is finitely generated and projective, then it is Zariski locally free (and thus also *fpqc* locally free); and that if an $A[[u]]$-module is *fpqc* locally free, u-torsion free and u-adically complete and separated, then it is finitely generated and projective.

In the context of $A((u))$-modules, the relationship between these notions is less clear. In Lemma 5.1.23 we show that a finitely generated projective $A((u))$-module is Nisnevich locally free as an $A((u^n))$-module for all n sufficiently large, but in Example 5.1.24 we give an example of a finitely generated projective $A((u))$-module of rank one which is not étale locally free as an $A((u))$-module (although it is Zariski locally free as an $A((u^n))$-module for all $n \geq 2$).

5.1.4. *Remark.* The second main object of our discussion is to describe the extent to which various notions of projectivity/local freeness for $A[[u]]$- and $A((u))$-modules are genuinely local notions, and (closely related) the descent properties of these notions.

It is obvious that the various local freeness notions presented in Definition 5.1.2 are local in the relevant topology. Below, we will recall Drinfeld's result that finitely generated projective $A((u))$-modules satisfy descent in the *fpqc* topology.

Before turning to these main points of our discussion, we note the following result.

5.1.5. **Lemma.** *Let A be a commutative ring. Each of the natural maps $\operatorname{Spec} A[[u]] \to \operatorname{Spec} A$ and $\operatorname{Spec} A((u)) \to \operatorname{Spec} A$ induces an isomorphism on the Boolean algebras of simultaneously open and closed subsets of its source and target.*

Proof. We need to show that the injections $A \hookrightarrow A[[u]] \hookrightarrow A((u))$ induce bijections on the sets of idempotents. Writing $\mathrm{Idem}(R)$ for the set of idempotents in the ring R, we note that the isomorphism $A[[u]] \xrightarrow{\sim} \varprojlim A[u]/(u^n)$ induces a bijection $\mathrm{Idem}(A[[u]]) \xrightarrow{\sim} \varprojlim \mathrm{Idem}(A[u]/u^n)$. Since idempotents lift uniquely through nilpotent ideals, each of the transition morphisms in this latter projective system is a bijection, and hence we find that the morphism $\mathrm{Idem}(A[[u]]) \to \mathrm{Idem}(A)$ (induced by the map $A[[u]] \to A[u]/(u) = A$) is a bijection. Its inverse is then given by the map $\mathrm{Idem}(A) \to \mathrm{Idem}(A[[u]])$ induced by the inclusion $A \hookrightarrow A[[u]]$; in particular, this map is also a bijection.

Now let $e \in A((u))$ be an idempotent, and write $e = \sum_{n=-\infty}^{\infty} a_n u^n$. We claim that a_i is nilpotent for each $i < 0$. Indeed, if A is reduced, then it is immediate from the equation $e^2 = e$ that $a_i = 0$ when $i < 0$. The claim in general then follows by considering the image of e in $A_{\mathrm{red}}((u))$. Let I denote the ideal of A generated by the a_i for $i < 0$. By the claim we have just proved, together with the fact that $a_i = 0$ for all but finitely many $i < 0$, we find that I is finitely generated by nilpotent elements, and so is a nilpotent ideal of A. Thus the kernel of the morphism $A((u)) \to (A/I)((u))$ is also nilpotent, and so we obtain a commutative square

$$
\begin{array}{ccc}
\mathrm{Idem}(A) & \longrightarrow & \mathrm{Idem}\big(A((u))\big) \\
\downarrow & & \downarrow \\
\mathrm{Idem}(A/I) & \longrightarrow & \mathrm{Idem}\big((A/I)((u))\big)
\end{array}
$$

in which the vertical arrows are bijections and the horizontal arrows are injections. By the definition of I, and the discussion above, we see that the image of e in $(A/I)((u))$ in fact lies in $(A/I)[[u]]$, and so, by what we have already proved in fact lies in A/I. A consideration of the preceding commutative square then shows that $e \in A$, as required. □

As a consequence of the preceding result, we have the following reassuring statement, which shows that in those contexts in which we have multiple ways to define the locally free rank of an $A[[u]]$ or $A((u))$-module, the definitions coincide.

5.1.6. Lemma. *If M is a finitely generated projective $A[[u]]$- (resp. $A((u))$-)module, and it is also fpqc locally free of rank d, then it is of rank d as a projective module over $A[[u]]$ (resp. $A((u))$).*

Proof. The rank of a finitely generated projective module over a ring is locally constant, and so we may find a partition of $\mathrm{Spec}\, A[[u]]$ (resp. $\mathrm{Spec}\, A((u))$) into disjoint open subsets on each of which the rank of M is constant. Lemma 5.1.5 shows that this partition of $\mathrm{Spec}\, A[[u]]$ is induced by a corresponding partition of $\mathrm{Spec}\, A$. Working separately over each of these open and closed subsets of A, we may assume that M is a projective $A[[u]]$- (resp. $A((u))$-)module of some fixed rank n.

By assumption there is a faithfully flat morphism $A \to B$ such that $B[[u]] \otimes_{A[[u]]} M$ (resp. $B((u)) \otimes_{A((u))} M$) is free of rank d over $B[[u]]$ (resp. $B((u))$). This module is also finitely generated projective of rank n, and thus we deduce that $n = d$, as required. □

We now develop those aspects of our discussion that don't require Drinfeld's theory of Tate modules.

5.1.7. Lemma. *If A is Noetherian and $A \to B$ is an fppf morphism, then each of the induced morphisms $A[[u]] \to B[[u]]$ and $A((u)) \to B((u))$ is faithfully flat.*

Proof. Note that $B[[u]]$ is flat over $B \otimes_A A[[u]]$ (being the u-adic completion of the latter ring, which is finitely presented over the Noetherian ring $A[[u]]$, and thus is itself Noetherian), which is in turn flat over $A[[u]]$. The maximal ideals of $A[[u]]$ are all of the form $(\mathfrak{m}, u)$, where $\mathfrak{m}$ is a maximal ideal of A. Given such a maximal ideal, since $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, we may find a prime ideal $\mathfrak{p}$ of B which maps to $\mathfrak{m}$, and then $(\mathfrak{p}, u)$ is a prime ideal of $B[[u]]$ which maps to the maximal ideal $(\mathfrak{m}, u)$ of $A[[u]]$.

Thus $A[[u]] \to B[[u]]$ is a flat morphism for which the induced map $\operatorname{Spec} B[[u]] \to \operatorname{Spec} A[[u]]$ contains all maximal ideals in its image. Since flat morphisms satisfy going-down, this morphism is in fact surjective, and thus $A[[u]] \to B[[u]]$ is faithfully flat. The morphism $A((u)) \to B((u))$ is obtained from this one via extending scalars from $A[[u]]$ to $A((u))$, and so is also faithfully flat. □

As already discussed in Remark 5.1.3, for $A[[u]]$-modules, the conditions of being finitely generated projective and of being *fpqc* locally free of finite rank are closely related, the precise nature of this relationship being the subject of the following results. The first part of the next proposition is closely related to [Kim09, Prop. 7.4.2].

5.1.8. Proposition. *Let $\mathfrak{M}$ be an $A[[u]]$-module. Then the following conditions are equivalent:*

(1) $\mathfrak{M}$ is a finitely generated projective $A[[u]]$-module.
(2) $\mathfrak{M}$ is u-torsion free and u-adically complete and separated, and $\mathfrak{M}/u\mathfrak{M}$ is a finitely generated projective A-module.

Moreover if these conditions hold then there is an isomorphism of $A[[u]]$-modules $(\mathfrak{M}/u\mathfrak{M}) \otimes_A A[[u]] \xrightarrow{\sim} \mathfrak{M}$, which reduces to the identity modulo u. In particular if furthermore $\mathfrak{M}/u\mathfrak{M}$ is a free A-module, then $\mathfrak{M}$ is a free $A[[u]]$-module.

Proof. If $\mathfrak{M}$ is projective, then it is a direct summand of a finite free $A[[u]]$-module, and is therefore u-torsion free and u-adically complete and separated; and certainly $\mathfrak{M}/u\mathfrak{M}$ is a projective A-module. For the reverse implication, note that by [GD71, Prop. 0.7.2.10(ii)], we need only show that for each integer $n \geq 1$, $\mathfrak{M}/u^n\mathfrak{M}$ is a projective $A[u]/u^n$-module.

To show this, note that firstly that since $\mathfrak{M}$ is u-torsion free, for each $m, n \geq 1$ we have a short exact sequence of A-modules

$$0 \to \mathfrak{M}/u^n\mathfrak{M} \xrightarrow{u^m} \mathfrak{M}/u^{m+n}\mathfrak{M} \to \mathfrak{M}/u^m\mathfrak{M} \to 0.$$

By the equivalence of conditions (1) and (4) of [Mat89, Thm. 22.3] (the local flatness criterion), we see that $\mathfrak{M}/u^n\mathfrak{M}$ is a flat $A[u]/u^n$-module for each n. It follows from the same short exact sequence and induction on n that $\mathfrak{M}/u^n\mathfrak{M}$ is an A-module of finite presentation; so by [Sta, Tag 0561], it is an $A[[u]]/u^n$-module of finite presentation, and is therefore projective, as required.

Finally, if these conditions hold, then since $\mathfrak{M}/u\mathfrak{M}$ is projective, we can choose an A-linear section to the natural surjection $\mathfrak{M} \to \mathfrak{M}/u\mathfrak{M}$. We therefore have a morphism of projective $A[[u]]$-modules $(\mathfrak{M}/u\mathfrak{M}) \otimes_A A[[u]] \to \mathfrak{M}$, which reduces to the identity modulo u. Write $\mathfrak{N} := (\mathfrak{M}/u\mathfrak{M}) \otimes_A A[[u]]$. Then the morphism $\mathfrak{N} \to \mathfrak{M}$ is surjective by the topological version of Nakayama's lemma, so we need only prove

that it is injective. For this, note that if K is the kernel of the morphism, then since $\mathfrak{M}$ is projective, the short exact sequence

$$0 \to K \to \mathfrak{N} \to \mathfrak{M} \to 0$$

splits, so that K is a finitely generated projective $A[[u]]$-module, and $K/uK = 0$. Since K if finitely generated projective, it is in particular u-adically separated, so we have $K = 0$, as required. $\qquad\square$

5.1.9. **Proposition.**

    (1) *If $\mathfrak{M}$ is a finitely generated projective $A[[u]]$-module, then $\mathfrak{M}$ is Zariski locally on $\operatorname{Spec} A$ free of finite rank as an $A[[u]]$-module.*

    (2) *If $\mathfrak{M}$ is u-torsion free and is u-adically complete and separated, and $\mathfrak{M}$ is fpqc locally on $\operatorname{Spec} A$ free of finite rank as an $A[[u]]$-module, then $\mathfrak{M}$ is a finitely generated projective $A[[u]]$-module.*

    (3) *If A is Noetherian, then $\mathfrak{M}$ is a finitely generated projective $A[[u]]$-module if and only if $\mathfrak{M}$ is fppf locally on $\operatorname{Spec} A$ free of finite rank as an $A[[u]]$-module.*

Proof. We begin with (1). If $\mathfrak{M}$ is a finitely generated projective $\mathfrak{S}_A$-module, then by Proposition 5.1.8, $\mathfrak{M}/u\mathfrak{M}$ is a finitely generated projective A-module. It is therefore Zariski locally free. It follows that it is enough to show that if $(\mathfrak{M}/u\mathfrak{M}) \otimes_A B$ is a free B-module of finite rank, then $\mathfrak{M} \otimes_{A[[u]]} B[[u]]$ is a free $B[[u]]$-module of finite rank. But $\mathfrak{M} \otimes_{A[[u]]} B[[u]]$ is a finitely generated projective $B[[u]]$-module, so it follows from Proposition 5.1.8 that $\mathfrak{M} \otimes_{A[[u]]} B[[u]]$ is isomorphic to $((\mathfrak{M}/u\mathfrak{M}) \otimes_A B) \otimes_B B[[u]]$, and is therefore free of finite rank.

For (2), if $\mathfrak{M}$ is *fpqc* locally free, then $\mathfrak{M}/u\mathfrak{M}$ is *fpqc* locally free of finite rank, and is in particular *fpqc* locally finitely generated projective. By [Sta, Tag 058S], the property of being finitely generated and projective is *fpqc* local, so we see that $\mathfrak{M}/u\mathfrak{M}$ is finitely generated projective. Thus $\mathfrak{M}$ is finitely generated projective, by Proposition 5.1.8.

For (3), one implication is immediate from (1) (which furthermore allows us to strengthen *fppf* locally free to Zariski locally free). The converse follows from Lemma 5.1.7, again using [Sta, Tag 058S]. $\qquad\square$

5.1.10. *Tate modules.* As discussed in Remark 5.1.3, for general faithfully flat morphisms $A \to B$ (i.e. outside the context of Lemma 5.1.7), we are not able to gain much direct control over the induced morphisms $A[[u]] \to B[[u]]$ or $A((u)) \to B((u))$, and so we are not able to apply standard descent results in the context of Definition 5.1.2. However, in [Dri06], Drinfeld is able to establish descent results for these morphisms, provided that one restricts attention to modules that are finitely generated and projective. Drinfeld's basic descent result is stated in the language of *Tate modules*, and we begin by recalling the definition of this notion from [Dri06, §3], as well as some related definitions.

5.1.11. **Definition.** Let A be a commutative ring. (In fact, [Dri06] defines Tate modules over not necessarily commutative rings, but we will only need the commutative case.) An *elementary Tate A-module* is a topological A-module which is isomorphic to $P \oplus Q^*$, where P, Q are discrete projective A-modules, and $Q^* := \operatorname{Hom}_A(Q, A)$ equipped with its natural projective limit topology (where we write Q^* as the projective limit of $(Q')^*$, where Q' is a finite direct summand of Q, and give

$(Q')^*$ the discrete topology). A *Tate A-module* is a direct summand of an elementary Tate A-module.

A morphism of Tate modules is a continuous morphism of the underlying A-modules.

5.1.12. Definition. A submodule L of a Tate module M is a *lattice* if it is open, and if furthermore for every open submodule $U \subseteq L$, the quotient L/U is a finitely generated A-module. We say that L is *coprojective* if M/L is a projective A-module (equivalently, a flat A-module; see [Dri06, Rem. 3.2.3(i)]). A Tate module contains a coprojective lattice if and only if it is elementary [Dri06, Rem. 3.2.3(ii)].

The most important example of these definitions for our purposes is the following.

5.1.13. Example. We endow $A[[u]]$ with its u-adic topology, and endow $A((u))$ with the unique topology in which $A[[u]]$ (equipped with its u-adic topology) is embedded as an open subgroup. Equivalently, we write $A((u)) = \varinjlim_n \frac{1}{u^n} A[[u]]$, and endow $A((u))$ with its inductive limit topology, where each term in the inductive limit is endowed with its u-adic topology. Or, again equivalently, we write $A((u)) = A[[u]] \oplus \frac{1}{u} A[1/u]$, in which the first factor is endowed with its u-adic topology, and the second factor is discrete.

Both $A[[u]]$ and $A((u))$ are then elementary Tate A-modules, and $A[[u]]$ is a coprojective lattice in $A((u))$. Furthermore, by [Dri06, Ex. 3.2.2], any finitely generated projective $A((u))$-module has a natural topology, making it a Tate A-module. (Indeed, we may write such a module M as a direct summand of $A((u))^n$ for some $n \geq 1$, where this latter module is endowed with its product topology.) In fact, we have the the following theorem [Dri06, Thm. 3.10].

5.1.14. Theorem. *There is a natural bijection between finitely generated projective $A((u))$-modules, and pairs (M, T) consisting of a Tate A-module M and a topologically nilpotent automorphism $T : M \to M$, by giving M the $A((u))$-module structure determined by $um := T(m)$. (Here T is topologically nilpotent if and only if for each pair of lattices $L, L' \subseteq M$, we have $T^n L \subseteq L'$ for all sufficiently large n.)*

5.1.15. Descent. Drinfeld's fundamental descent result is the following theorem [Dri06, Thm. 3.3], which shows that the notion of a Tate A-module is *fpqc*-local on $\operatorname{Spec} A$.

5.1.16. Theorem. *If A' is a faithfully flat A-algebra, then the functor $M \mapsto A' \widehat{\otimes}_A M$ induces an equivalence between the category of Tate A-modules and the category of Tate A'-modules with descent data to A. Furthermore, the identification of* Hom*-spaces given by this equivalence respects the natural topologies.*

The statement concerning topologies on Hom-spaces is left implicit in [Dri06], but is easily checked.[7]

5.1.17. Remark. It is a theorem of Raynaud–Gruson [RG71, Ex. 3.1.4, Seconde partie] that the property of a module being projective can be checked *fpqc* locally.

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[7]Any Tate module is a direct summand of one of the form $P \oplus Q^*$, where P and Q are both *free* A-modules. Since the formation of Homs is compatible with finite direct sums, this reduces us to verifying the claim in the case of $\operatorname{Hom}(M, N)$ where M and N are A-Tate modules which are either free or dual to free, in which case it is straightforward.

This implies that, for any faithfully flat morphism $A \to A'$, the functor $M \mapsto A' \otimes_A M$ induces an equivalence between the category of projective A-modules and the category of projective A'-modules equipped with descent data to A. Since projective modules are particular examples of Tate modules (they are precisely the discrete Tate modules), Drinfeld's Theorem 5.1.16 incorporates this descent result of Raynaud–Gruson as a special case.

The relationship between Tate modules and finitely generated projective $A((u))$-modules given by Theorem 5.1.14 then implies that the notion of a finitely generated projective $A((u))$-module is also *fpqc* local on $\operatorname{Spec} A$; this is [Dri06, Thm. 3.11], which we now recall. In addition, we prove some slight variants of this result that we will need below.

5.1.18. **Theorem.** *The following notions are local for the fpqc topology on* $\operatorname{Spec} A$.

(1) *A finitely generated projective* $A((u))$-*module.*
(2) *A projective* $A((u))$-*module of rank* d.
(3) *A finitely generated projective* $A((u))$-*module which is fpqc locally free of rank* d.
(4) *A finitely generated projective* $A[[u]]$-*module.*
(5) *A projective* $A[[u]]$-*module of rank* d.
(6) *A finitely generated projective* $A[[u]]$-*module which is fpqc locally free of rank* d.

5.1.19. *Remark.* More precisely, saying that the notion of a finitely generated projective $A((u))$-module is local for the *fpqc* topology on $\operatorname{Spec} A$ means the following (and the meanings of the other statements in Theorem 5.1.18 are entirely analogous):

If A' is any faithfully flat A-algebra, set $A'' := A' \otimes_A A'$. Then the category of finitely generated projective $A((u))$-modules is canonically equivalent to the category of finitely generated projective $A'((u))$-modules M' which are equipped with an isomorphism

$$M' \otimes_{A'((u)), a \mapsto 1 \otimes a} A''((u)) \xrightarrow{\sim} M' \otimes_{A'((u)), a \mapsto a \otimes 1} A''((u))$$

which satisfies the usual cocycle condition.

Proof of Theorem 5.1.18. Since the notion of being *fpqc* locally free of rank d is *fpqc* local by definition, and the rank of a finitely generated projective module can be computed *fpqc* locally by Lemma 5.1.6, it suffices to prove statements (1) and (4). The first of these is [Dri06, Thm. 3.11]. As noted above, it follows from Theorem 5.1.16 together with Theorem 5.1.14. (The fact that the property of an automorphism being topologically nilpotent satisfies descent follows from the compatibility with topologies on Hom-spaces stated in Theorem 5.1.16.)

For (4), let A' be a faithfully flat A-algebra, and let L' be a projective $A'[[u]]$-module equipped with descent data. Then $M' := L' \otimes_{A'[[u]]} A'((u))$ is a projective $A'((u))$-module equipped with descent data, and L' is a coprojective lattice in M' by Lemma 5.1.20 below. The short exact sequence of Tate A'-modules

$$0 \to L' \to M' \to M'/L' \to 0$$

is split, and admits descent data to A. Thus, by Theorem 5.1.16, we may descend this to a (split) short exact sequence of Tate A-modules. By (1) (or, perhaps better, by its proof), the endomorphism u of M' descends to an endomorphism of M, which

equips M with the structure of a finitely generated and projective $A((u))$-module. Since u preserves the submodule L' of M', it preserves the descended submodule L of M.

Since M'/L' is discrete (or, equivalently, a projective A'-module), the same is true of M/L, and thus we see that L is open in M, and coprojective (*cf.* Remark 5.1.17). In fact, L is a lattice in M. Indeed, since u is a topologically nilpotent automorphism of M, the submodules $u^n L$ ($n \geq 0$) form a neighbourhood basis of zero in L. Since

$$A' \otimes_A (L/u^n L) = (A' \widehat{\otimes}_A L)/u^n (A' \widehat{\otimes} L) = L'/u^n L'$$

is finitely generated, and since the property of being a finitely generated A-module is local for the *fpqc* topology on $\operatorname{Spec} A$, we find that $L/u^n L$ is finitely generated over A. To complete the proof, we note that L is a projective $A[[u]]$-module by another application of Lemma 5.1.20, as required.   □

We learned the following lemma from Drinfeld.

5.1.20. Lemma. *Let M be a finitely generated projective $A((u))$-module, and let L be an $A[[u]]$-submodule of M. Then the following are equivalent:*

   (1) L is a finitely generated projective $A[[u]]$-module with $A((u))L = M$.
   (2) L is a coprojective lattice in M.

Proof. If (1) holds, then L is certainly open in M, and each $L/u^n L$ is a finitely generated A-module, so L is a lattice in M. Since $M/L \cong L \otimes_{A[[u]]} \big(A((u))/A[[u]]\big)$, and $A((u))/A[[u]]$ is a free A-module, M/L is a projective A-module, so that L is a coprojective lattice, as required.

Conversely, suppose that L is a coprojective lattice. Since L is a lattice, we certainly have $A((u))L = M$. The short exact sequence of A-modules

$$0 \to L/uL \to M/L \overset{u}{\to} M/L \to 0$$

splits (because M/L is projective), so that L/uL is a direct summand of the projective A-module M/L, and is thus itself projective. Since L is a lattice, L/uL is finitely generated. It follows from Proposition 5.1.8 that L is a finitely generated projective $A[[u]]$-module, as required.   □

5.1.21. Remark. As we proved in Proposition 5.1.9, a finitely generated and projective $A[[u]]$-module is *fpqc* (indeed, even Zariski) locally free of finite rank; thus the only difference between the situations of parts (4), (5) and (6) of Theorem 5.1.18 is that, in parts (5) and (6), the locally free rank of the $A[[u]]$-module in question is prescribed.

5.1.22. Lemma. *Let M be a finitely generated $A((u))$-module, which is projective as an $A((u^n))$-module, for some n that is invertible in A. Then M is projective over $A((u))$.*

Proof. Identify $A((u))$ with $A((u^n))[X]/(X^n - u^n)$, so that we may regard M as a module over $A((u^n))[X]/(X^n - u^n)$ which is projective as an $A((u^n))$-module. Now consider the base-change $M' := A((u)) \otimes_{A((u^n))} M$; this is a module over $A((u))[X]/(X^n - u^n)$ which is projective as an $A((u))$-module. Since n and u are both invertible in $A((u))$, the quotient $A((u)) := A((u))[X]/(X - u)$ of $A((u))[X]/(X^n - u^n)$ is a direct summand of $A((u))[X]/(X^n - u^n)$. Thus

$M \cong M'/(X - u)M'$ is a direct summand of M', and hence is projective as an $A((u))$-module. $\square$

The following result relates the property of an $A((u))$-module being finitely generated and projective to the property of it being locally free, in the sense of Definition 5.1.2.

5.1.23. Lemma. *Let M be a finitely generated projective $A((u))$-module. Then there exists an $n_0 \geq 1$ such that for all $n \geq n_0$, M is Nisnevich locally free as an $A((u^n))$-module.*

Proof. By [Dri06, Thm. 3.4], we may make a Nisnevich localisation so that M is an elementary Tate A-module, i.e. contains a coprojective lattice L.

Since multiplication by u is topologically nilpotent, there is an integer $n_0 \geq 0$ such that $u^n L \subseteq L$ for all $n \geq n_0$; thus, for each such value of n, we see that L is naturally an $A[[u^n]]$-module, and that the natural morphism $A((u^n)) \otimes_{A[[u^n]]} L \to M$ is an isomorphism. By Lemma 5.1.20, L is a finitely generated projective $A[[u^n]]$-module. By Proposition 5.1.9 (1), after making a further Zariski localisation, we may suppose that L is free of finite rank as an $A[[u^n]]$-module, so that M is free of finite rank as an $A((u^n))$-module, as required. $\square$

The following example shows that, in the context of the preceding lemma, we can't necessarily take $n_0 = 1$.

5.1.24. Example. Let $A = k[x, y]/(y^2 - x^3)$, for some field k, and let $I \subseteq A((u))$ denote the ideal generated by $(u^2 - x, u^3 - y)$. One can check that I is freely generated over $A((u^2))$ by $u^2 - x$ and $u^3 - y$, and if the characteristic of k is different from 2, I is projective over $A((u))$ by Lemma 5.1.22.

Alternatively, and more conceptually, one can deduce this projectivity (with no assumption on the characteristic of k) by noting that (u^2, u^3) is a smooth point (over the complete non-archimedean field $k((u))$) of the rigid analytic curve $y^2 = x^3$ lying in the closed polydisk $|x|, |y| \leq 1$ over $k((u))$, and that I is ideal sheaf of (u^2, u^3) in the Tate algebra $A((u))$ of the curve.

One can check that the $A[[u^2]]$-submodule L of I generated by $u^2 - x$ and $u^3 - y$ is an $A[[u^2, u^3]]$-submodule of I, and hence is closed under multiplication by u^n for any $n \geq 2$. Since I is freely generated over $A((u^2))$ by $u^2 - x$ and $u^3 - y$, we see that I/L is a free A-module, so that L is a coprojective lattice in I. The proof of Lemma 5.1.23 shows that L is then Zariski locally free over $A[[u^n]]$ for any $n \geq 2$, and thus that I is Zariski locally free over $A((u^n))$, for any $n \geq 2$.

We claim that I is *not* an étale locally free $A((u))$-module. To see this, it suffices to show that if R denotes the strict Henselisation of A at the maximal ideal (x, y), then $I \otimes_{A((u))} R((u))$ is not free over $R((u))$, i.e. that the ideal $(u^2 - x, u^3 - y)$ is not a principal ideal in $R((u))$. We prove this in the following lemma.

5.1.25. Lemma. *If R denotes the strict Henselisation of $k[x, y]/(y^2 - x^3)$ at the maximal ideal (x, y), then the ideal $(u^2 - x, u^3 - y)$ of $R((u))$ is not principal.*

Proof. Let $f \in R[[u]]$ be non-zero, with non-zero constant term $f_0 \in R$. (Any non-zero element of $R((u))$ may be be multiplied by some power of u so as to satisfy this condition, and hence any principal ideal of $R((u))$ has a principal generator satisfying this condition.) We claim, then, that $fR((u)) \cap R[[u]] = fR[[u]]$.

Indeed, if $g \in fR((u)) \cap R[[u]]$, then $g = fh$ for some $h \in R((u))$. Choose N minimally so that $hu^N \in R[[u]]$, and suppose that $N > 0$. If we reduce the equation $gu^N = fhu^N$ modulo u, we find that

$$0 = f_0 \times \text{ the constant term of } hu^N$$

(an equation in R), and so (since R is a domain, as the cuspidal cubic $y^2 = x^3$ is geometrically unibranch at its singular point $(0,0)$) we find that the constant term of hu^N is zero. This contradicts the minimality of N, and thus shows that $N = 0$, so that in fact $h \in R[[u]]$ and $g \in fR[[u]]$, as claimed. Thus, if $fR((u))$ is any principal ideal of $R((u))$, with f chosen as above, then

$$R[[u]]/(fR((u)) \cap R[[u]] + uR[[u]]) = R[[u]]/(f, u)R[[u]] = R/f_0 R.$$

Since R (which is a one-dimensional local ring) is not regular, the quotient $R/f_0 R$ is necessarily of dimension > 1 over k.

On the other hand, we find that

$$R[[u]]/((u^2 - x, u^3 - y) \cap R[[u]] + uR[[u]]) = R/(x, y) = k.$$

Taking into account the result of the preceding paragraph, this shows that indeed $(u^2 - x, u^3 - y)$ is not a principal ideal in $R((u))$. $\qquad\square$

5.2. Modules of finite height and étale φ-modules.
In this section we discuss finite height φ-modules and étale φ-modules. With an eye to future applications (for example, the case of Lubin–Tate (φ, Γ)-modules), we work in a more general context than that usually considered.

5.2.1. *Definitions.* Fix a finite extension $k/\mathbb{F}_p$, and write $\mathfrak{S} := W(k)[[u]]$. Let q be some power of p, and let φ be a ring endomorphism of $\mathfrak{S}$ which is congruent to the q-power Frobenius endomorphism modulo p.

5.2.2. **Lemma.** *φ induces the usual q-power Frobenius on $W(k)$.*

Proof. It is enough to note that $W(k)$ is generated as a $\mathbb{Z}_p$-algebra by a primitive $(\#k - 1)$st root of unity, and that the $(\#k - 1)$st roots of unity are distinct modulo p. $\qquad\square$

5.2.3. **Lemma.** *For each $M, a \geq 1$, $\varphi(u^{M+a-1}) \in (u^{Mq}, p^a)$, and $u^{(M+a-1)q} \in (\varphi(u^M), p^a)$. In particular, φ is continuous with respect to the (p, u)-adic topology.*

Proof. Write $\varphi(u) = u^q + pY$. Then $\varphi(u^{M+a-1}) = (u^q + pY)^{M+a-1}$, and $u^{(M+a-1)q} = (\varphi(u) - pY)^{M+a-1}$, and the result follows from the binomial theorem. $\qquad\square$

We fix a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformiser ϖ. If A is an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, we write $\mathfrak{S}_A := (W(k) \otimes_{\mathbb{Z}_p} A)[[u]]$; we equip $\mathfrak{S}_A$ with its u-adic topology. Let $\mathcal{O}_{\mathcal{E},A}$ equal $\mathfrak{S}_A[1/u]$.

5.2.4. **Lemma.** *φ admits a unique continuous A-linear extension to $\mathfrak{S}_A$, which in turn admits a unique continuous A-linear extension to $\mathcal{O}_{\mathcal{E},A}$.*

Proof. This follows immediately from Lemma 5.2.3. $\qquad\square$

5.2.5. **Lemma.** *φ is faithfully flat and finite on $\mathfrak{S}_A$ and $\mathcal{O}_{\mathcal{E},A}$.*

Proof. It is enough to show this for $\mathfrak{S}_A$, as it then follows for the localisation $\mathcal{O}_{\mathcal{E},A}$. It then suffices to show that $\mathfrak{S}_A$ is a finite free $\varphi(\mathfrak{S}_A)$-module of rank q, with a basis given by $u^i, 0 \leq i \leq q-1$.

If A is an $\mathbb{F}_p$-algebra this is clear, and it follows from Nakayama's lemma that in general $\mathfrak{S}_A$ is generated as an $\varphi(\mathfrak{S}_A)$-module by $u^i, 0 \leq i \leq q-1$. It remains to check that the u^i are linearly independent. To this end, suppose that we have a relation $\sum_{i=0}^{q-1} u^i \varphi(a_i) = 0$, where the a_i are all contained in $p^n \mathfrak{S}_A$ for some $n \geq 0$. We will show that in fact the a_i are all contained in $p^{n+1} \mathfrak{S}_A$; since $p^n \mathfrak{S}_A = 0$ for n sufficiently large, it follows that $a_i = 0$ for all i, as required.

Write $\varphi_0 : \mathfrak{S}_A \to \mathfrak{S}_A$ for the particular lift of the q-power Frobenius on $\mathfrak{S}_{A/pA}$ defined via $\varphi_0(u) = u^q$. Then for any $x \in p^n \mathfrak{S}_A$, we have $\varphi(x) - \varphi_0(x) \in p^{n+1} \mathfrak{S}_A$, so our assumptions imply that $\sum_{i=0}^{q-1} u^i \varphi_0(a_i) \in p^{n+1} \mathfrak{S}_A$. Writing the a_i out as power series in u, and equating coefficients, we see immediately that $a_i \in p^{n+1} \mathfrak{S}_A$, as required. $\square$

We fix a polynomial $F \in (W(k) \otimes_{\mathbb{Z}_p} \mathcal{O})[u]$ that is congruent to a positive power of u modulo ϖ. The following elementary lemma will be useful below.

5.2.6. Lemma. *For all integers $a, h \geq 1$ there is an integer $n(a,h) \geq 1$ depending only on a, h and F such that if A is an $\mathcal{O}/\varpi^a$-algebra, then $u^{n(a,h)}$ is divisible by F^h in $\mathfrak{S}_A$, and $F^{n(a,h)}$ is divisible by u^h.*

Proof. Write $F = u^n - \varpi X$ for some $n \geq 1$ and $X \in \mathfrak{S}_A$. By the binomial theorem, F^{a+h-1} is divisible by u^{nh} in $\mathfrak{S}_A$, and thus by u^h; similarly, writing $u^n = F + \varpi X$, $u^{n(a+h-1)}$ is divisible by F^h. Putting these together, we see that we can take $n(a,h) := n(a+h-1)$. $\square$

We will also use the following result.

5.2.7. Lemma. *If A is a $\mathcal{O}/\varpi^a$-algebra, then for each $M \geq 1$, $u^{(M+a-1)q}$ is divisible by $\varphi(u^M)$, and $\varphi(u^{(M+a-1)})$ is divisible by u^{Mq}.*

Proof. This is immediate from Lemma 5.2.3. $\square$

If $\mathfrak{M}$ is an $\mathfrak{S}_A$-module (resp. M is an $\mathcal{O}_{\mathcal{E},A}$-module) then we write $\varphi^* \mathfrak{M}$ for $\mathfrak{M} \otimes_{\mathfrak{S}_A, \varphi} \mathfrak{S}_A$ (resp. $\varphi^* M$ for $M \otimes_{\mathcal{O}_{\mathcal{E},A}, \varphi} \mathcal{O}_{\mathcal{E},A}$). Since φ is faithfully flat, the functors $\mathfrak{M} \mapsto \varphi^* \mathfrak{M}$, $M \mapsto \varphi^* M$ are exact.

5.2.8. Corollary. *If A is a $\mathcal{O}/\varpi^a$-algebra, $\mathfrak{M}$ is an $\mathfrak{S}_A$-module, and $u^M \mathfrak{M} \neq 0$, then $u^{(M-a+1)q} \varphi^* \mathfrak{M} \neq 0$.*

Proof. If $u^{(M-a+1)q} \varphi^* \mathfrak{M} = 0$, then by Lemma 5.2.7 we have $\varphi^*(u^M \mathfrak{M}) = \varphi(u^M) \varphi^* \mathfrak{M} = 0$. Since φ is faithfully flat, this implies that $u^M \mathfrak{M} = 0$, a contradiction. $\square$

The following lemma is a straightforward generalisation (with a very similar proof) of [PR09, Prop. 2.2] to our setting. Let R be an $\mathcal{O}/\varpi^a$-algebra, and let $u \in R$ be a nonzerodivisor, such that R is u-adically complete and separated. (For example, we could take $R = \mathfrak{S}_A$ for some $\mathcal{O}/\varpi^a$-algebra A.) For $n \geq 0$, write

$$U_n = 1 + u^n M_d(R),$$

$$V_n = \{A \in \mathrm{GL}_d(R[1/u]) | A, A^{-1} \in u^{-n} M_d(R)\}.$$

5.2.9. Lemma. *Suppose that $n > (2m + (a-1)q)/(q-1)$.*

(1) *For each $g \in U_n$, $A \in V_m$, there is a unique $h \in U_n$ such that $g^{-1}A\varphi(g) = h^{-1}A$.*

(2) *For each $h \in U_n$, $A \in V_m$ there is a unique $g \in U_n$ such that $g^{-1}A\varphi(g) = h^{-1}A$.*

Proof. We follow the proof of [PR09, Prop. 2.2]. For the first part, note that we can solve for h^{-1}, namely $h^{-1} = g^{-1}A\varphi(g)A^{-1}$, so that the uniqueness of h is clear, and what we must show is that $h \in U_n$, or equivalently, that $h^{-1} \in U_n$. We can write $g^{-1} = I + u^n X$ with $X \in M_d(R)$, and by Lemma 5.2.7 we can write $\varphi(g) = I + u^{(n-a+1)q}Y$ with $Y \in M_d(R)$. Then $g^{-1}A\varphi(g)A^{-1} = (I + u^n X)(I + u^{(n-a+1)q}AYA^{-1})$. Since $A \in V_m$ we have $AYA^{-1} \in u^{-2m}M_d(R)$, so that $g^{-1}A\varphi(g)A^{-1} \in U_n$, as required.

For the second part we begin by showing uniqueness of g, for which it is enough to show that if $g^{-1}A\varphi(g) = A$ then $g = 1$. Write $g = I + X$, so that the preceding relation between A and g may be rewritten as $X = A\varphi(X)A^{-1}$. It is enough to check that we have $X \in u^s M_d(R)$ for all $s \geq n$. We prove this by induction on s, the case $s = n$ being by hypothesis. If $X \in u^s M_d(R)$, then as above we have $A\varphi(X)A^{-1} \in u^{(s-a+1)q-2m}M_d(R)$, and since $s \geq n$ we have $(s-a+1)q-2m > s$, as required.

Finally we must show existence of g. For this, let $A' = h^{-1}A$, and set $A_0 = A, h_0 = h$. We inductively define sequences (h_i), (A_i) by setting $A_i = h_{i-1}^{-1}A_{i-1}\varphi(h_{i-1})$, $h_i = (A')^{-1}A_i$. These equalities imply that $h_i = A'\varphi(h_{i-1})(A')^{-1}$, so since $A' \in V_m$ and $h_0 \in U_n$, an easy induction as above shows that $h_i \in U_{n+i}$ for all i. If we now set $g_i = h_0 h_1 \cdots h_i$, then g_i tends to some limit $g \in U_n$. Since for all i we have $g_i^{-1}A\varphi(g_{i-1}) = h^{-1}A$, in the limit we have $g^{-1}A\varphi(g) = h^{-1}A$, as required. $\square$

5.2.10. **Definition.** Let h be a non-negative integer, and let A be an $\mathcal{O}/\varpi^a$-algebra. A *φ-module of height F with A-coefficients* is a pair $(\mathfrak{M}, \varphi_M)$ consisting of a finitely generated u-torsion free $\mathfrak{S}_A$-module $\mathfrak{M}$, and a φ-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$, with the further properties that if we write

$$\Phi_{\mathfrak{M}} := \varphi_{\mathfrak{M}} \otimes 1 : \varphi^*\mathfrak{M} \to \mathfrak{M},$$

then $\Phi_{\mathfrak{M}}$ is injective, and the cokernel of $\Phi_{\mathfrak{M}}$ is killed by F. A *φ-module of finite height with A-coefficients*, or a *finite height φ-module with A-coefficients*, is a φ-module with A-coefficients which is of height F for some F.

A *morphism* of finite height φ-modules is a morphism of the underlying $\mathfrak{S}_A$-modules which commutes with the morphisms $\Phi_{\mathfrak{M}}$.

We say that a finite height φ-module is *projective of rank d* if it is a finitely generated projective $\mathfrak{S}_A$-module of constant rank d.

5.2.11. *Remark.* We will primarily be interested in finite height φ-modules (resp. étale φ-modules) that are furthermore projective over $\mathfrak{S}_A$ (resp. over $\mathfrak{S}_A[1/u]$), to which the base-change and descent results of Drinfeld [Dri06] recalled in Subsection 5.1.15 are applicable. However, we sometimes need to make constructions that take us outside the category of projective modules, and in particular we need to consider finite height φ-modules which are not projective, but become projective after inverting u.

5.2.12. **Definition.** Let A be a $\mathcal{O}/\varpi^a$-algebra. An *étale φ-module with A-coefficients* is a pair (M, φ_M) consisting of a finitely generated $\mathcal{O}_{\mathcal{E},A}$-module M, and

a φ-semilinear map $\varphi_M : M \to M$ which induces an isomorphism of $\mathcal{O}_{\mathcal{E},A}$-modules $\Phi_M := \varphi_M \otimes 1 : \varphi^* M \to M$.

A morphism of étale φ-modules is a morphism of the underlying $\mathcal{O}_{\mathcal{E},A}$-modules which commutes with the morphisms Φ_M.

We say that M is projective (resp. free) of rank d if it is a finitely generated projective (resp. free) $\mathcal{O}_{\mathcal{E},A}$-module of constant rank d. If τ is any topology on the category of $\mathcal{O}/\varpi^a$-modules lying between the Zariski topology and the *fpqc* topology, then we say that M is τ-locally free of rank d if it is projective of rank d, and if τ-locally on $\operatorname{Spec} A$, it is free of rank d.

5.2.13. *Remark.* If $(\mathfrak{M}, \varphi)$ is a φ-module of height F with A-coefficients, then by Lemma 5.2.6, $(\mathfrak{M}[1/u], \varphi)$ is an étale φ-module with A-coefficients.

We will sometimes prove results about projective étale φ-modules by reducing to the free case, using the following lemma.

5.2.14. **Lemma.** *If M is a projective étale φ-module with A-coefficients, then M is a direct summand of a free étale φ-module with A-coefficients.*

Proof. Since M is projective, we may find another finitely generated projective $\mathcal{O}_{\mathcal{E},A}$-module P such $M \oplus P \xrightarrow{\sim} F$, for some finite rank free module F. Then

$$\varphi^* M \oplus \varphi^* P \xrightarrow{\sim} \varphi^* F,$$

and since M is an étale φ-module, we have $\varphi^* M \cong M$, while since F is free, we have $\varphi^* F \cong F$. Thus

$$M \oplus P \cong M \oplus \varphi^* P,$$

and so taking the direct sum with another copy of P, we find that

$$F \oplus P \cong F \oplus \varphi^* P \cong \varphi^*(F \oplus P).$$

In other words, the finitely generated projective module $Q := F \oplus P$ admits the structure of an étale φ-module, and

$$Q \oplus M \cong F \oplus F$$

is free of finite rank. $\square$

5.2.15. **Lemma.** *Let A be a Noetherian $\mathcal{O}/\varpi^a$-algebra, and let M be an étale φ-module with A-coefficients. Then for some F, there is a φ-module $\mathfrak{M}$ of height F with A-coefficients such that $\mathfrak{M}[1/u] = M$. If M is furthermore a free $\mathcal{O}_{\mathcal{E},A}$-module, then we may choose $\mathfrak{M}$ to be a free $\mathfrak{S}_A$-module.*

5.2.16. *Remark.* Note that in the case when M is projective but not necessarily free, we do not claim that the φ-module $\mathfrak{M}$ in Lemma 5.2.15 can be chosen to be projective.

Proof of Lemma 5.2.15. By definition M is finitely generated as an $\mathcal{O}_{\mathcal{E},A}$-module, so we may choose a generating set, and let $\mathfrak{M}$ be the $\mathfrak{S}_A$-span of this generating set; if M is free then we may and do also choose $\mathfrak{M}$ to be free. It follows easily from Lemma 5.2.7 that if we scale $\mathfrak{M}$ by a large enough power of u, we may assume that $\mathfrak{M}$ is φ-stable. Since M is u-torsion free, so is $\mathfrak{M}$, and since Φ_M is an isomorphism, $\Phi_{\mathfrak{M}}$ is injective and the cokernel of $\Phi_{\mathfrak{M}}$ is killed by some power of u. Thus $\mathfrak{M}$ is a finite height φ-module, as required. $\square$

5.2.17. *Remark.* If $q = p$, then for certain choices of φ and F, the theories of finite
height φ-modules and étale φ-modules with Artinian coefficients admit interpreta-
tions in terms of Galois representations; more precisely, in terms of representations
of the absolute Galois groups of certain perfectoid fields. We refer to [EG19, §2] for a
more thorough discussion of this; in Section 5.4.23 below we explain the connection
in a particular case, that of *Breuil–Kisin modules*.

5.3. **Lifting rings.** We now prove the existence of universal lifting rings for étale φ-
modules, as well as variants for finite height φ-modules. These will be used below
to show that our moduli stacks admit versal rings, and to verify the effectivity
hypothesis in our application of Theorem 1.1.1.

We remark that if we were in one of the settings mentioned in Remark 5.2.17,
then we could use the equivalence of categories between étale φ-modules and Galois
representations with Artinian coefficients to study the versal rings in which we are
interested using standard techniques from the formal deformation theory of Galois
representations; in particular, in the case of Breuil–Kisin modules, we could use the
results of [Kim11]. Even in the general framework that we have adopted, it seems
plausible that we could use the Artin–Schreier theory constructions that underlie
that equivalence to replace the formal deformation theory of étale φ-modules by
the formal deformation theory of some more finitistic objects. However, we have
found it more direct, and interesting in its own right, to argue with the formal
deformation theory of étale φ-modules. We caution the reader that this leads us,
in what follows, to consider some rather large pro-Artinian rings!

5.3.1. *Lifting rings.* The main results of this subsection are Proposition 5.3.6 and
Theorem 5.3.15.

We begin by studying the formal deformation theory of étale φ-modules.
Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformiser ϖ and residue
field $\mathbb{F}$, and let M be an étale φ-module with $\mathbb{F}$-coefficients which is free of rank d.
Fix an integer $a \geq 1$, and (following the notation of § 2.2) write $\mathcal{C}_{\mathcal{O}/\varpi^a}$ for the
category of Artinian local $\mathcal{O}/\varpi^a$-algebras for which the structure map induces an
isomorphism on residue fields.

Fix a choice of (ordered) $\mathcal{O}_{\mathcal{E},\mathbb{F}}$-basis of M, or equivalently, an identification of
$\mathcal{O}_{\mathcal{E},\mathbb{F}}$-modules

$$(5.3.2) \qquad\qquad M \xrightarrow{\sim} \mathcal{O}_{\mathcal{E},\mathbb{F}}^d.$$

5.3.3. **Definition.** A *lifting* of M to an object R of $\mathcal{C}_{\mathcal{O}/\varpi^a}$ is a triple consisting
of an étale φ-module M_R which is free of rank d, a choice of (ordered) $\mathcal{O}_{\mathcal{E},R}$-basis
of M_R, and an isomorphism $M_R \otimes_R \mathbb{F} \cong M$ of étale φ-modules which takes the
chosen basis of M_R to the fixed basis of M. Equivalently, a lifting M consists of an
étale φ-module M_R endowed with an isomorphism of $\mathcal{O}_{\mathcal{E},R}$-modules

$$(5.3.4) \qquad\qquad M_R \xrightarrow{\sim} \mathcal{O}_{\mathcal{E},\mathbb{F}}^d$$

such that the étale φ-module structure on $\mathcal{O}_{\mathcal{E},\mathbb{F}}^d$ which is obtained by reducing (5.3.4)
modulo $\mathfrak{m}_R$ coincides with the étale φ-module structure on $\mathcal{O}_{\mathcal{E},\mathbb{F}}^d$ induced by the
isomorphism (5.3.2).

By regarding the isomorphisms (5.3.2) and (5.3.4) as identifications, we see that
the liftings of M admit yet another equivalent description: namely, the identifica-
tion (5.3.2) allows us to regard M as simply a choice of matrix $\Phi \in \mathrm{GL}_d(\mathcal{O}_{\mathcal{E},\mathbb{F}})$,

which describes the étale φ-module structure

$$\mathcal{O}_{\mathcal{E},\mathbb{F}}^d = \varphi^* \mathcal{O}_{\mathcal{E},\mathbb{F}}^d = \varphi^* M \to M = \mathcal{O}_{\mathcal{E},\mathbb{F}}^d,$$

and the identification (5.3.4) allows us to regard the lifting M_R as a matrix $\Phi_R \in$ $\mathrm{GL}_d(\mathcal{O}_{\mathcal{E},R})$ lifting Φ; this is the matrix describing the étale φ-module structure

$$\mathcal{O}_{\mathcal{E},R}^d = \varphi^* \mathcal{O}_{\mathcal{E},R}^d = \varphi^* M_R \to M_R = \mathcal{O}_{\mathcal{E},R}^d.$$

We denote by $D^\square : \mathcal{C}_{\mathcal{O}/\varpi^a} \to \underline{Sets}$ the functor taking R to the set of isomorphism classes of liftings of M to R. (The functorial structure on $D^\square$ is the obvious one induced by extension of scalars.)

5.3.5. *Remark.* There is a natural action of the subgroup $1 + \mathfrak{m}_R M_d(\mathcal{O}_{\mathcal{E},R})$ of $\mathrm{GL}_d(\mathcal{O}_{\mathcal{E},R})$ on $D^\square(R)$, given by change of basis. In terms of the description of liftings via a choice of isomorphism (5.3.4), this is given by composing the isomorphism (5.3.4) with the automorphism of $\mathcal{O}_{\mathcal{E},R}^d$ induced by a matrix $g \in 1 + \mathfrak{m}_R M_d(\mathcal{O}_{\mathcal{E},R})$. In terms of the description of liftings in terms of a matrix Φ_R, this is given by the twisted conjugation action $\Phi_R \mapsto g\Phi_R\varphi(g)^{-1}$.

We begin our study of liftings by establishing the pro-representability of $D^\square$.

5.3.6. **Proposition.** *The functor $D^\square$ is pro-representable by an object $R^\square$ of* pro-$\mathcal{C}_{\mathcal{O}/\varpi^a}$.

Proof. We follow Dickinson's appendix to [Gou01], which uses Grothendieck's representability theorem to prove the existence of universal deformation rings. By [Gro95, Cor. to Prop. 3.1, §A], it is enough to show that $D^\square$ is left exact. Left exactness is equivalent to preserving fibre products and terminal objects. There is a unique terminal object of $\mathcal{C}_{\mathcal{O}/\varpi^a}$, namely $\mathbb{F}$, and since $D^\square(\mathbb{F})$ consists of the trivial lifting, it is also a terminal object. It remains to check that $D^\square$ preserves fibre products in $\mathcal{C}_{\mathcal{O}/\varpi^a}$; but this is obvious if we think of liftings as being a choice of Φ_R lifting Φ. $\qquad\square$

5.3.7. *Remark.* If M_R is a rank d free étale φ-module over the Artinian ring R lifting M, then Lemma 5.2.15 shows that we may write $M_R = \mathfrak{M}_R[1/u]$ for a free finite height φ-module $\mathfrak{M}_R$ of some height F (depending on M_R). The unicity statement of Lemma 5.2.9 (2) (with $h = 1$) then shows that if N is sufficiently large (depending on F), the elements of $1 + u^N \mathfrak{m}_R M_d(\mathfrak{S}_R)$ act freely (in the sense of the action described in Remark 5.3.5) on the element of $D^\square(R)$ represented by M.

Thus, for a non-trivial thickening R of k, if the set $D^\square(R)$ is non-empty (which e.g. necessarily will be the case if R is a k-algebra, since in that case we can simply base-change M from k to R), then it is quite enormous! In particular, the pro-representing ring $R^\square$ constructed in Proposition 5.3.6 does not admit a countable basis of neighbourhoods of zero.

We now consider liftings of bounded height. If M is an étale φ-module with A-coefficients, then we say that M admits a model of height F if there is a (not necessarily projective) φ-module $\mathfrak{M}$ of height F with A-coefficients such that $\mathfrak{M}[1/u] = M$.

5.3.8. **Definition.** Let $D_F^\square$ be the subfunctor of $D^\square$ whose elements are the liftings of M which admit a model of height F.

5.3.9. **Proposition.** *The functor $D_F^\square$ is pro-representable by a quotient $R_F^\square$ of $R^\square$.*

Proof. Let A be an Artinian quotient of $R^\square$. It follows from Lemma 5.3.10 below that there is a unique maximal quotient A^F of A for which the corresponding étale φ-module admits a model of height F. We then take $R_F^\square := \varprojlim_A A^F$. $\qquad\square$

5.3.10. Lemma. *Let A be Noetherian. The direct sum of two étale φ-modules with A-coefficients which have models of height F also has a model of height F. Any subquotient of an étale φ-module with A-coefficients with a model of height F also has a model of height F.*

Proof. The statement about direct sums is trivial. For subquotients, let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of étale φ-modules with A-coefficients, and suppose that $\mathfrak{M}$ is a finite height φ-module of height F with $\mathfrak{M}[1/u] = M$. Let $\mathfrak{M}', \mathfrak{M}''$ be respectively the kernel and image of the induced map $\mathfrak{M} \to M''$; then it is easy to check that $\mathfrak{M}', \mathfrak{M}''$ are φ-modules of height F, and that $\mathfrak{M}'[1/u] = M'$, $\mathfrak{M}''[1/u] = M''$. $\qquad\square$

5.3.11. *Remark.* As noted in Remark 5.3.7, there is an N (depending on F, and which we fix once and for all) such that for each object R of $\mathcal{C}_{\mathcal{O}/\varpi^a}$, the elements of $1 + u^N \mathfrak{m}_R M_d(\mathfrak{S}_R)$ act freely on $D_F^\square(R)$. This again implies that the rings $R_F^\square$ are huge.

However, since F is now fixed, we may also fix the integer N, and then systematically quotient out by the free action of $1 + u^N \mathfrak{m}_R M_d(\mathfrak{S}_R)$. This leads to the following definition.

5.3.12. Definition. We let $D_F : \mathcal{C}_{\mathcal{O}/\varpi^a} \to \underline{Sets}$ denote the functor defined by $D_F(R) := D_F^\square(R)/(1 + u^N \mathfrak{m}_R M_d(\mathfrak{S}_R))$.

In the rest of this section we develop the theory needed to prove Theorem 5.3.15, which shows that D_F is pro-representable by a Noetherian ring.

5.3.13. Lemma. *Let A be an Artinian $\mathcal{O}/\varpi^a$-algebra. Suppose that M is an étale φ-module with A-coefficients, and that $\mathfrak{M}, \mathfrak{M}'$ are two models of M of height F. If j is minimal such that $u^j \mathfrak{M} \subseteq \mathfrak{M}'$, and k is such that $u^k \varphi^* \mathfrak{M} \subseteq \varphi^* \mathfrak{M}'$, then $k > (j - a)q$.*

Proof. By assumption we have $u^{j-1}(\mathfrak{M}/\mathfrak{M} \cap \mathfrak{M}') \neq 0$, so by Corollary 5.2.8 we have $u^{(j-a)q} \varphi^* (\mathfrak{M}/\mathfrak{M} \cap \mathfrak{M}') \neq 0$. It is therefore enough to check that $\varphi^* \mathfrak{M} \cap \varphi \mathfrak{M}' = \varphi^* (\mathfrak{M} \cap \mathfrak{M}')$; but this follows from the flatness of φ (applied to the map $\mathfrak{M} \oplus \mathfrak{M}' \to M, (x, y) \mapsto x - y$). $\qquad\square$

The following lemma is a generalisation of [CL09, Cor. 3.2.6], and the proof is similar.

5.3.14. Lemma. *Let A be an Artinian $\mathcal{O}/\varpi^a$-algebra. If M is an étale φ-module with A-coefficients, and M has a model of height F, then it has both a minimal model $\mathfrak{M}_{\min}$ of height F and a maximal model $\mathfrak{M}_{\max}$ of height F. Furthermore, $\mathfrak{M}_{\max}/\mathfrak{M}_{\min}$ is an A-module of finite length.*

Proof. Let $\mathfrak{M}$ be a model of M of height F. We claim that there is an $i \geq 0$ such that any other model $\mathfrak{M}'$ of height F satisfies $u^i \mathfrak{M} \subseteq \mathfrak{M}' \subseteq u^{-i} \mathfrak{M}$.

To see this, we follow the proof of [Kis09b, Prop. 2.1.7], and choose r minimal such that $u^r \mathfrak{M} \subseteq \Phi_{\mathfrak{M}}(\varphi^* \mathfrak{M}) \subseteq u^{-r} \mathfrak{M}$ (note that r exists by Lemma 5.2.6), and choose j minimal such that $u^j \mathfrak{M} \subseteq \mathfrak{M}'$, and l minimal such that $\mathfrak{M}' \subseteq u^{-l} \mathfrak{M}$. We

must show that j, l are bounded independently of $\mathfrak{M}'$. It follows from Corollary 5.2.8 that if $u^k \varphi^* \mathfrak{M} \subseteq \varphi^* \mathfrak{M}'$ then $k > (j-a)q$. By Lemma 5.2.6 there is a constant $n(a)$ such that F divides $u^{n(a)}$ in $\mathfrak{S}_A$, so that

$$\Phi_{\mathfrak{M}}(\varphi^* \mathfrak{M}) \subseteq u^{-r} \mathfrak{M} \subseteq u^{-j-r} \mathfrak{M}' \subseteq u^{-n(a)-j-r} \Phi_{\mathfrak{M}'}(\varphi^* \mathfrak{M}').$$

It follows from Lemma 5.3.13 that $n(a) + j + r > (j-a)q$, so that j is bounded independently of $\mathfrak{M}'$.

Similarly, if we choose l minimal such that $\mathfrak{M}' \subseteq u^{-l} \mathfrak{M}$, then

$$\Phi_{\mathfrak{M}'}(\varphi^* \mathfrak{M}') \subseteq \mathfrak{M}' \subseteq u^{-l} \mathfrak{M} \subseteq u^{-r-l} \Phi_{\mathfrak{M}}(\varphi^* \mathfrak{M}),$$

so that by Lemma 5.3.13 again we have $r + l > (l-a)q$, and l is also bounded independently of $\mathfrak{M}'$, as required.

Since $u^{-i} \mathfrak{M} / u^i \mathfrak{M}$ has finite length as an A-module, it follows that the category of models of M of height F is Artinian and Noetherian. To see that maximal and minimal models of height F exist, it is now enough to note that if $\mathfrak{M}, \mathfrak{M}'$ are models of M of height F, then so are $\max(\mathfrak{M}, \mathfrak{M}') := \mathfrak{M} + \mathfrak{M}'$ and $\min(\mathfrak{M}, \mathfrak{M}') := \mathfrak{M} \cap \mathfrak{M}'$. $\qquad \square$

5.3.15. Theorem. *The functor D_F is pro-representable by a Noetherian ring R_F. Furthermore, we may choose a natural transformation $D_F \to D_F^\square$ which is a section to the natural transformation $D_F^\square \to D_F$; and thus for any R, we obtain a natural isomorphism*

$$D_F(R) \times \left(1 + u^N \mathfrak{m}_R M_d(\mathfrak{S}_R)\right) \xrightarrow{\sim} D_F^\square(R).$$

Proof. Define the group-valued functor $\mathcal{G}$ on $\mathcal{O}/\varpi^a$-algebras via $\mathcal{G}(A) := 1 + u^N M_d(\mathfrak{S}_A)$ (the group structure being given via multiplication of matrices). Then $\mathcal{G}$ is in fact an affine group scheme, equal to

$$\mathrm{Spec}(\mathcal{O}/\varpi^a)[\{x_{i,j,k,n}\}_{1 \le i \le f, 1 \le j,k \le d, N \le n < \infty}],$$

where we have written $f := [k : \mathbb{F}_p]$ and have chosen a basis for $W(k)$ over $\mathbb{Z}_p$, so that $\mathfrak{S}_A \xrightarrow{\sim} A[[u]]^f$ and $1 + u^N M_d(\mathfrak{S}_A) \xrightarrow{\sim} \left(1 + u^N M_d(A[[u]])\right)^f$; the variables $x_{i,j,k,n}$ then parameterize the coefficient of u^n in the (j,k) entries of the ith summand.

Now let S denote the pro-Artinian completion of

$$(\mathcal{O}/\varpi^a)[\{x_{i,j,k,n}\}_{1 \le i \le f, 1 \le j,k \le d, N \le n < \infty}]$$

at the closed point defined by setting ϖ, as well as all the $x_{i,j,k,n}$, equal to 0. (This point corresponds to the identity $1 \in \mathcal{M}_d(\mathfrak{S}_{\mathbb{F}})$.) If we set $G := \mathrm{Spf}\, S$, then for any object R of $\mathcal{C}_{\mathcal{O}/\varpi^a}$, we have $G(R) = 1 + u^N \mathfrak{m}_R M_d(\mathfrak{S}_R)$. The free action of $G(R)$ on $D_F^\square(R)$, for each R, induces an equivalence relation

$$G \times D_F^\square \rightrightarrows D_F^\square,$$

the quotient of $D_F^\square$ by which is of course just D_F. The product $G \times D_F^\square$ is pro-representable by $R_F^\square \widehat{\otimes}_{\mathcal{O}/\varpi^a} S$. In order to proceed, we need to show that this completed tensor product is *topologically flat* over $R_F^\square$, in the sense of [Gab65]. For this, we have have to give a more explict description of S, and so we make a small digression in order to do so.

Let V denote the free $\mathcal{O}/\varpi^a$-module spanned by the variables

$$\{x_{i,j,k,n}\}_{1 \le i \le f, 1 \le j,k \le d, N \le n < \infty},$$

endowed with its discrete topology. We let $U \mapsto U^*$ denote Pontrjagin duality on topological $\mathcal{O}/\varpi^a$-modules. If U is discrete, then U^* is profinite, and conversely. In particular, if U is a discrete finite $\mathcal{O}/\varpi^a$-module, then U^* is again both discrete and finite. Continuing to assume that U is discrete and finite, and writing V^*_δ to denote V^* endowed with its discrete topology, we find that

$$\mathrm{Hom}_{\mathcal{O}/\varpi^a}(V, U) = \mathrm{Hom}_{\mathcal{O}/\varpi^a, \mathrm{cont}}(U^*, V^*)$$
$$= \mathrm{Hom}_{\mathcal{O}/\varpi^a}(U^*, V^*) = \mathrm{Hom}_{\mathcal{O}/\varpi^a, \mathrm{cont}}((V^*_\delta)^*, U).$$

Thus the canonical embedding $V \hookrightarrow (V^*_\delta)^*$ identifies $(V^*_\delta)^*$ with the pro-Artinian (equivalently, pro-finite) completion of V. Thus $S = \mathcal{O}/\varpi^a[[(V^*_\delta)^*]]$, in the notation of [Gab65, §1.2.5]. In particular, S *is* topologically flat over $\mathcal{O}$ and $R^\square_F \widehat{\otimes}_{\mathcal{O}/\varpi^a} S = R^\square_F[[(V^*)_\delta]]$ is topologically flat over $R^\square_F$.

It follows from [Gab65, Thm. 1.4] that the kernel R_F of the corresponding pair of morphisms

$$R^\square_F \rightrightarrows R^\square_F \widehat{\otimes}_{\mathcal{O}/\varpi^a} S = R^\square_F[[(V^*)_\delta]].$$

pro-represents the quotient D_F. The same result also shows that $R^\square_F$ is topologically flat over R_F.

Since $R^\square_F \widehat{\otimes}_{\mathcal{O}/\varpi^a} S$ is a power series ring over $R^\square_F$, the morphism $G \times D^\square_F \to D^\square_F$ satisfies an infinitesimal lifting criterion of the type considered in Definition 2.2.9 above. Thus this morphism is *versal*, in the terminology employed in that definition, or *smooth*, in the terminology of [Sta, Tag 06HG]. It is then no doubt a matter of general principles that $D^\square_F \to D_F$ also satisfies this infinitesimal lifting property. In our particular case we can see this directly: since any surjection $R \to R'$ manifestly induces a surjection $G(R) \to G(R')$, one immediately confirms that $D^\square_F \to D_F$ satisfies the infinitesimal lifting property. An evident generalization of [Sta, Tag 06HL] to our pro-Artinian setting, a detailed proof of which may be be found at [uh], then shows that $R^\square_F \cong R_F[[\{x_i\}_{i \in I}]]$, for some index set I. The morphism $R_F \to R^\square_F$ obtained by mapping each x_i to 0 then determines a functorial section $D_F \to D^\square_F$ with the property stated in the theorem.

It remains to show that R_F is Noetherian. By [Gro95, Prop. 5.1, §A], this is equivalent to showing that $D_F(\mathbb{F}[\epsilon])$ is a finite-dimensional $\mathbb{F}$-vector space. Elements of $D_F(\mathbb{F}[\varepsilon])$ determine self-extensions $0 \to M \to M' \to M \to 0$ with the property that M' admits a model of height F. Let $\mathfrak{M}'$ be such a model; then the image of $\mathfrak{M}'$ in M is a model of M of height F, and thus contains $\mathfrak{M}_{\min}$, the minimal model of height F (whose existence is guaranteed by Lemma 5.3.14). We can replace $\mathfrak{M}'$ by the preimage of $\mathfrak{M}_{\min}$, and accordingly we can assume that the image of $\mathfrak{M}'$ in M is $\mathfrak{M}_{\min}$.

Similarly, the kernel of $\mathfrak{M}' \to M$ is a model of M, and is therefore contained in $\mathfrak{M}_{\max}$; replacing $\mathfrak{M}'$ by its sum with $\mathfrak{M}_{\max}$, we may assume that in fact $\mathfrak{M}'$ is an extension of $\mathfrak{M}_{\min}$ by $\mathfrak{M}_{\max}$. (Having made this replacement, $\mathfrak{M}'$ may only be of height F^2, rather than of height F, but this does not matter for our argument.) It suffices to show that the $\mathbb{F}$-vector space of such extensions, considered up to the equivalence relation induced by $(1 + u^N \varepsilon M_d(\mathfrak{S}_k))$-conjugacy, is finite-dimensional.

After choosing bases for $\mathfrak{M}_{\max}$ and $\mathfrak{M}_{\min}$, the possible matrices for $\varphi_{\mathfrak{M}'}$ are of the form

$$\begin{pmatrix} A_{\max} & B \\ 0 & A_{\min} \end{pmatrix}$$

where $A_{\min}$ and $A_{\max}$ are respectively the matrices of $\varphi_{\mathfrak{M}_{\min}}$ and $\varphi_{\mathfrak{M}_{\max}}$. Conjugating by matrices of the form $\begin{pmatrix} 1 & u^N \varepsilon X \\ 0 & 1 \end{pmatrix}$, we see that we are free to replace B by $B + u^N X A_{\min} - A_{\max} \varphi(u^N X)$. It therefore suffices to show that for some sufficiently large M, for every $Y \in M_d(\mathfrak{S}_k)$ we can write

$$u^M Y = u^N X A_{\min} - A_{\max} \varphi(u^N X)$$

for some $X \in M_d(\mathfrak{S}_k)$.

Since $\mathfrak{M}_{\min}$ has height F, it follows from Lemma 5.2.6 that for some $t \geq 0$ and some $Z_{\min} \in M_d(\mathfrak{S}_k)$ we can write $A_{\min} Z_{\min} = u^t \mathrm{Id}_d$. It therefore suffices to show that we can always solve the equation

$$(5.3.16) \qquad u^{M-N} Y = u^t X - u^{(q-1)N} A_{\max} \varphi(X) Z_{\min}$$

(as a solution to this equation with Y replaced by $Y Z_{\min}$ provides a solution to our original equation).

For any $V \in u^t M_d(\mathfrak{S}_k)$, write $\delta(V) := u^{(q-1)N-t} A_{\max} \varphi(V) Z_{\min} \in M_d(\mathfrak{S}_k)$. Note that if $V \in u^s M_d(\mathfrak{S}_k)$ for some $s \geq t+1$, then $\delta(V) \in u^{qs+(q-1)N-t} M_d(\mathfrak{S}_k)$, and in particular $\delta(V) \in u^{s+1} M_d(\mathfrak{S}_k)$. It follows that the sum $W := V + \delta(V) + \delta^2(V) + \cdots \in u^s M_d(\mathfrak{S}_k)$ converges, and $W - \delta(W) = V$. Therefore, if we set $M := N + 2t + 1$, take $V = u^{M-N-t} Y$, and write $W = X$, we have the required solution to (5.3.16). $\qquad\square$

If E'/E is a finite extension with ring of integers $\mathcal{O}'$, then we have an obvious map from φ-modules (and finite height φ-modules) with respect to $\mathcal{O}$ from those with respect to $\mathcal{O}'$, given by tensoring with $\mathcal{O}'$ over $\mathcal{O}$. We end this section with the following reassuring results.

5.3.17. Lemma. *If M is an étale φ-module with A-coefficients, then M admits a model of height F if and only if $M \otimes_{\mathcal{O}} \mathcal{O}'$ admits a model of height F.*

Proof. Since the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}'$ is split as an inclusion of $\mathcal{O}$-modules (e.g. because $\mathcal{O}'$ is a faithfully flat and finite $\mathcal{O}$-algebra), this is immediate from Lemma 5.3.10. $\qquad\square$

If $\mathbb{F}'$ is the residue field of $\mathcal{O}'$, then we have corresponding universal lifting rings $R^{\square}_{\mathcal{O}'}$, $R^{\square}_{\mathcal{O}',F}$ and $R_{\mathcal{O}',F}$ for liftings to $\mathcal{O}'/\varpi^a$-algebras of $M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$.

5.3.18. Corollary. *We have natural isomorphisms of $\mathcal{O}'/\varpi^a$-algebras $R^{\square}_{\mathcal{O}'} \cong R^{\square} \otimes_{\mathcal{O}} \mathcal{O}'$, $R^{\square}_{\mathcal{O}',F} \cong R^{\square}_F \otimes_{\mathcal{O}} \mathcal{O}'$, and $R_{\mathcal{O}',F} \cong R_F \otimes_{\mathcal{O}} \mathcal{O}'$*

Proof. This follows easily from Lemma 2.2.13. Alternatively, we can argue slightly more explicitly (but essentially equivalently) as follows. We give the argument for $R^{\square}$, the other cases being essentially identical.

Tensoring over $\mathcal{O}'$ with $\mathcal{O}$ and considering the universal property gives a natural map $R^{\square}_{\mathcal{O}'} \to R^{\square} \otimes_{\mathcal{O}} \mathcal{O}'$. Let $R^{\square}_{\mathcal{O}',\mathbb{F}}$ be the subring of $R^{\square}_{\mathcal{O}'}$ consisting of elements whose reductions modulo the maximal ideal lie in $\mathbb{F}$. Considering the matrix of the universal étale φ-module over $R^{\square}_{\mathcal{O}'}$ with respect to a basis which lifts (the extension of scalars of) a basis of $M_{\mathbb{F}}$, we see that this universal étale φ-module is defined over $R^{\square}_{\mathcal{O}',\mathbb{F}}$. It follows from the universal property that there is a natural map $R^{\square} \to R^{\square}_{\mathcal{O}',\mathbb{F}}$.

Considering the composites $R_{\mathcal{O}'}^{\square} \to R^{\square} \otimes_{\mathcal{O}} \mathcal{O}' \to R_{\mathcal{O}',\mathbb{F}}^{\square} \otimes_{\mathcal{O}} \mathcal{O}' \to R_{\mathcal{O}'}^{\square}$, and $R^{\square} \to R_{\mathcal{O}',\mathbb{F}}^{\square} \to R_{\mathcal{O}'}^{\square} \to R^{\square} \otimes_{\mathcal{O}} \mathcal{O}'$, we easily obtain the first claim. The second claim then follows from Lemma 5.3.17. $\qquad\square$

5.4. **Moduli of finite height φ-modules and of étale φ-modules.** In this subsection we will define the moduli stacks that we are interested in, and prove our key results regarding them. We begin by establishing some terminology, which will be important for all that follows.

We fix an integer $a \geq 1$, and proceed to define various categories fibred in groupoids (which will in fact be stacks, although some only in the Zariski topology) over $\mathcal{O}/\varpi^a$.

5.4.1. **Definition.** If $a, d \geq 1$ are positive integers, then for any $\mathcal{O}/\varpi^a$-algebra A, we define $\mathcal{R}_d^a(A)$ to be the groupoid of étale φ-modules with A-coefficients which are projective of rank d over $\mathcal{O}_{\mathcal{E},A}$. If $A \to B$ is a morphism of $\mathcal{O}/\varpi^a$-algebras, and if M is an object of $\mathcal{R}_d^a(A)$, then the pull-back of M to $\mathcal{R}^a(B)$ is defined to be the tensor product $\mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M$.

The resulting category fibred in groupoids $\mathcal{R}_d^a$ is in fact a stack in groupoids in the *fpqc* topology over $\mathcal{O}/\varpi^a$, as follows from the results of [Dri06], and more specifically from Theorem 5.1.18 above.

5.4.2. **Definition.** If $a, d \geq 1$ are positive integers, and if $F \in (W(k) \otimes_{\mathbb{Z}_p} \mathcal{O})[u]$ is a polynomial that is congruent to a positive power of u modulo ϖ, then for any $\mathcal{O}/\varpi^a$-algebra A, we define $\mathcal{C}_{d,F}^a(A)$ to be the groupoid of φ-modules of height F with A-coefficients which are projective of rank d over $\mathfrak{S}_A$. If $A \to B$ is a morphism of $\mathcal{O}/\varpi^a$-algebras, and if $\mathfrak{M}$ is an object of $\mathcal{C}_{d,F}^a(A)$, then the pull-back of $\mathfrak{M}$ to $\mathcal{C}_{d,F}^a(B)$ is defined to be the tensor product $\mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}$.

Again, it follows from Theorem 5.1.18 that the resulting category fibred in groupoids $\mathcal{R}_d^a$ is in fact a stack in groupoids in the *fpqc* topology over $\mathcal{O}/\varpi^a$. There is an obvious morphism $\mathcal{C}_{d,F}^a \to \mathcal{R}_d^a$, defined by sending $(\mathfrak{M}, \varphi)$ to $(\mathfrak{M}[1/u], \varphi)$.

5.4.3. *Remark.* Our notation, and the entire set-up that we have just introduced, is very much inspired by the work of Pappas and Rapoport [PR09]. Indeed, in the case when $q = p$, $\varphi(u) = u^p$, and $F \in W(k)[u]$ is an Eisenstein polynomial, our stack $\mathcal{C}_{d,F}^a$ coincides with the stack $\mathcal{C}_{h,W(k)[u]/F}^a$ defined in [PR09, §3.b]. However, our stack $\mathcal{R}_d^a$ is subtly different from the stack denoted in the same manner in [PR09]. In that reference, the étale φ-modules under consideration are not required to be projective, but are required to be *fpqc* locally free. However, it seems to us that it is necessary to impose this projectivity in order to obtain a stack, while the local freeness hypothesis seems unnatural from the point of view of our intended applications (which is that $\mathcal{R}_d^a$ should provide models for moduli stacks of local Galois representations — and a direct summand of a family of representations should again form such a family); also, at a technical level, the effectivity result of Theorem 5.4.19 (5) below depends on working with projective étale φ-modules that are not necessarily locally free over the coefficient ring.

In spite of the difference between our definition of $\mathcal{R}_d^a$ and that of [PR09], we nevertheless rely on many of the arguments of that reference. In order to make the connection between our set-up and that of [PR09], it is helpful to introduce the following auxiliary objects, in which we require projectivity of our étale φ-modules, but also impose freeness conditions, as in [PR09].

5.4.4. **Definition.** We define $\mathcal{R}^a_{d,\mathrm{free}}$ to be the full subgroupoid of $\mathcal{R}^a_d$ classifying free étale φ-modules of rank d.

If τ is any topology on the category of $\mathcal{O}/\varpi^a$-modules lying between the Zariski topology and the *fpqc* topology, then we define $\mathcal{R}^a_{d,\tau-\mathrm{free}}$ to be the full subgroupoid of $\mathcal{R}^a_d$ classifying projective étale φ-modules of rank d that are furthermore τ-locally free over the ring of coefficients.

Taking into account the fact that $\mathcal{R}^a_d$ is an *fpqc* stack, the category fibred in groupoids $\mathcal{R}^a_{d,\tau-\mathrm{free}}$ is evidently a stack in the topology τ. Indeed, one immediately checks that it is the τ-stackification of $\mathcal{R}^a_{d,\mathrm{free}}$.

5.4.5. *Remark.* As already noted, the category fibred in groupoids $\mathcal{R}^a_{d,\mathrm{free}}$, as well as the stacks $\mathcal{R}^a_{d,\tau-\mathrm{free}}$, will play a purely auxiliary role. Furthermore, we need only make one choice of topology τ and work with that particular choice throughout; e.g. we could simply take τ to be the Zariski topology.

From now on, for the duration of the paper, we fix a choice of $a \geq 1$, and omit it from the notation.

5.4.6. **Lemma.** *The morphism $\mathcal{C}_{d,F} \to \mathcal{R}_d$ factors through $\mathcal{R}_{d,\tau-\mathrm{free}}$ (for any choice of τ).*

Proof. This follows from Proposition 5.1.9 (1), which shows that a projective finite height φ-module is actually Zariski locally free. $\square$

5.4.7. **Proposition.** *If $\operatorname{Spec} A \to \mathcal{R}_d$ is a morphism with A a Noetherian $\mathcal{O}/\varpi^a$-algebra, then there exists a scheme-theoretically surjective morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ such that the composite morphism $\operatorname{Spec} B \to \mathcal{R}_d$ factors through $\mathcal{R}_{d,\mathrm{free}}$.*

Proof. Let M be the étale φ-module over A classified by the given A-valued point of $\mathcal{R}_d$, and let M_{red} denote the base-change of M over A_{red}. By Lemma 5.2.15, we may find a (not necessarily projective) finite height φ-module $\mathfrak{M} \subset M_{\mathrm{red}}$ with $\mathfrak{M}[1/u] = M_{\mathrm{red}}$. The quotient $\mathfrak{M}/u\mathfrak{M}$ is then a coherent sheaf on $\operatorname{Spec} A_{\mathrm{red}}$, and so we may find a dense open subset $U \subset \operatorname{Spec} A_{\mathrm{red}}$ such that $\mathfrak{M}/u\mathfrak{M}$ restricts to a free sheaf over U. Thus, by Proposition 5.1.8, $\mathfrak{M}$ restricts to a free finite height φ-module over U, and so M_{red} restricts to a free étale φ-module over U (necessarily of rank d).

We may regard U as an open subset of $\operatorname{Spec} A$, and without loss of generality we may in fact assume that $U = \operatorname{Spec} A_f$ for some $f \in A$. Because A_f is Noetherian, the nilradical of A_f is nilpotent, and so the kernel of the morphism $A_f((u)) \to (A_f)_{\mathrm{red}}((u))$ is also nilpotent. Thus the restriction of M to U is a projective $A_f((u))$-module which becomes free modulo a nilpotent ideal. By a standard Nakayama-type argument, we see that this restriction itself is a free étale φ-module over U.

We now note that we may choose a closed subscheme $Z \hookrightarrow \operatorname{Spec} A$ whose underlying closed subset is equal to $\operatorname{Spec} A \setminus U$, and for which the obvious morphism $U \coprod Z \to \operatorname{Spec} A$ is scheme-theoretically dominant, in addition to being surjective. (Since A is Noetherian, the kernel $A[f^\infty]$ of $A \to A_f$ is equal to $A[f^n]$ for some $n \geq 1$, and we may take $Z = \operatorname{Spec} A/f^n$.) The proposition now follows by an evident Noetherian induction. $\square$

5.4.8. **Proposition.** *Let A be an $\mathcal{O}/\varpi^a$-algebra, and let M, N be projective étale φ-modules of finite rank with A-coefficients. Then the functors on A-algebras taking B*

to $\mathrm{Hom}(M_B, N_B)$ *and* $\mathrm{Isom}\,(M_B, N_B)$ *are both represented by affine schemes of finite presentation over* A.

Proof. By Lemma 5.2.14, there are projective étale φ-modules P, Q of finite rank such that the étale φ-modules $F := M \oplus P$ and $G := N \oplus Q$ are both free of finite rank. We now follow the proof of [PR09, Cor. 2.6(b)]. Choosing bases of F, G as $\mathcal{O}_{\mathcal{E},A}$-modules, an element of $\mathrm{Hom}(F_B, G_B)$ is given by a matrix g with coefficients in $\mathcal{O}_{\mathcal{E},B}$. If the matrices of φ_M, φ_N with respect to the chosen bases are respectively X, Y then the condition that g respects φ is that $\varphi(g) = Y^{-1}gX$.

Choose an integer $n \geq 0$ such that X, X^{-1}, Y, Y^{-1} all have entries with poles of degree at most n, and let $s \geq 0$ be minimal such that g has poles of degree at most s. By Corollary 5.2.8, $\varphi(g)$ has poles of degree greater than $(s - a)q$. Since $\varphi(g) = Y^{-1}gX$, we see that we must have $(s - a)q < 2n + s$, whence $s < (2n + aq)/(q - 1)$.

Writing the matrix g as $\sum_{i=-s}^{\infty} g_i u^i$, $g_i \in M_d(W(k) \otimes_{\mathbb{Z}_p} B)$, the equation $g = Y\varphi(g)X^{-1}$ and Lemma 5.2.9 show that the g_i for $i \leq (2n + (a - 1)q)/(q - 1)$ determine all of the g_i. It follows that $\mathrm{Hom}(F_B, G_B)$ is represented by an affine scheme of finite presentation over A.

Let $e \in \mathrm{End}(F)$, $f \in \mathrm{End}(G)$ be the idempotents corresponding to M, N respectively. Since $\mathrm{Hom}(M_B, N_B) \subset \mathrm{Hom}(F_B, G_B)$ is given by those g satisfying $g(1 - e) = 0$ and $(1 - f)g = 0$, we see that it is represented by a closed subscheme of the scheme representing $\mathrm{Hom}(F_B, G_B)$, and is therefore of finite presentation (for example by Lemma 2.6.3). Finally, the result for $\mathrm{Isom}\,(M_B, N_B)$ follows by regarding it as the subfunctor of pairs $(\alpha, \beta) \in \mathrm{Hom}(M_B, N_B) \times \mathrm{Hom}(N_B, M_B)$ satisfying $\alpha\beta = \mathrm{Id}_{N_B}$, $\beta\alpha = \mathrm{Id}_{M_B}$. $\qquad\square$

The following theorem generalises some of the main results of [PR09] to our setting. The proofs are almost identical, and we content ourselves with explaining the changes that need to be made to the arguments of [PR09], rather than writing them out in full.

5.4.9. Theorem. *(1) The stack* $\mathcal{C}_{d,F}$ *is an algebraic stack of finite presentation over* $\mathrm{Spec}\,\mathcal{O}/\varpi^a$, *with affine diagonal.*

(2) The morphism $\mathcal{C}_{d,F} \to \mathcal{R}_{d,fpqc-\mathrm{free}}$ *is representable by algebraic spaces, proper, and of finite presentation.*

(3) The diagonal morphism $\Delta : \mathcal{R}_{d,fpqc-\mathrm{free}} \to \mathcal{R}_{d,fpqc-\mathrm{free}} \times_{\mathcal{O}/\varpi^a} \mathcal{R}_{d,fpqc-\mathrm{free}}$ *is representable by algebraic spaces, affine, and of finite presentation.*

Proof. In the case that $\mathcal{O} = \mathbb{Z}_p$, $q = p$, and $\varphi(u) = u^p$, it follows from the main results of [PR09] that $\mathcal{C}_{d,F}$ is an algebraic stack of finite type over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$, and that (2) holds. (Strictly speaking, [PR09] assume that F is an Eisenstein polynomial, but their arguments go through unchanged with our slightly more general choice of F.) In the case of general $\mathcal{O}$, q and φ, the arguments go over essentially unchanged provided that one replaces the use of [PR09, Prop. 2.2] with an appeal to Lemma 5.2.9, and that in [PR09, §3] one replaces *eah* by the quantity $n(a, h)$ appearing in Lemma 5.2.6.

Part (3) is immediate from Proposition 5.4.8. To prove the remaining claims of (1), we have to show that $\mathcal{C}_{d,F}$ is in fact of finite presentation over $\mathcal{O}/\varpi^a$, with affine diagonal. These facts are certainly implicit in the arguments of [PR09], but for the reader's convenience, we explain how they follow formally from the results already established. Since $\mathcal{O}/\varpi^a$ is Noetherian, and since we know already that $\mathcal{C}_{d,F}$ is finite type over $\mathcal{O}/\varpi^a$, the diagonal morphism $\mathcal{C}_{d,F} \to \mathcal{C}_{d,F} \times_{\mathcal{O}/\pi^a} \mathcal{C}_{d,F}$ is

automatically quasi-separated (being a representable morphism between finite type algebraic stacks over $\mathcal{O}/\varpi^a$), and so to show that $\mathcal{C}_{d,F}$ is of finite presentation over $\mathcal{O}/\varpi^a$, it suffices to show that this diagonal morphism is quasi-compact. Since affine morphisms are quasi-compact, this will follow once we show that $\mathcal{C}_{d,F}$ has affine diagonal. For this, we factor the diagonal of $\mathcal{C}_{d,F}$ as

$$\mathcal{C}_{d,F} \to \mathcal{C}_{d,F} \times_{\mathcal{R}_{d,fpqc-\text{free}}} \mathcal{C}_{d,F} \to \mathcal{C}_{d,F} \times_{\mathcal{O}/\varpi^a} \mathcal{C}_{d,F}.$$

The first of these morphisms is a closed immersion, since $\mathcal{C}_{d,F} \to \mathcal{R}_{d,fpqc-\text{free}}$ is representable and proper (by (2)), while the second is affine, being a base-change of the diagonal morphism of $\mathcal{R}_{d,fpqc-\text{free}} \to \mathcal{R}_{d,fpqc-\text{free}} \times_{\mathcal{O}/\varpi^a} \mathcal{R}_{d,fpqc-\text{free}}$ (which is affine, by (3)). Their composite is thus an affine morphism, as claimed.   □

In Theorem 5.4.11 we prove the analogue of Theorem 5.4.9 for $\mathcal{R}_d$. In order to do so we need the following Lemma.

5.4.10. Lemma. *Let A be an $\mathcal{O}/\varpi^a$-algebra, and let M be an étale φ-module with A-coefficients, which is free of rank d as an $\mathcal{O}_{\mathcal{E},A}$-module. Let T be an automorphism of M. Then the functor on A-algebras taking B to the set of T-invariant projective φ-modules $\mathfrak{M}_B \subset M_B$ of rank d and height F is representable by a projective A-scheme.*

Proof. Let Gr denote the affine Grassmannian classifying projective $\mathfrak{S}_A$-lattices in M; this is an Ind-projective A-scheme. We begin by showing that the subfunctor of Gr $\times$ Gr given by

$$\{(\mathfrak{M}, \mathfrak{N}) : \mathfrak{M} \subseteq \mathfrak{N}\}$$

is a closed Ind-subscheme of Gr $\times$ Gr. To see this, we have to show that for any A-algebra B, and any pair of lattices $\mathfrak{M}_B, \mathfrak{N}_B \in \text{Gr}(B)$, the locus in $\text{Spec}\, B$ over which $\mathfrak{M}_B \subseteq \mathfrak{N}_B$ is closed. Equivalently, we need to show that the locus over which the morphism $\mathfrak{M}_B \to M_B/\mathfrak{N}_B$ vanishes is closed. To see this, note that we may factor this map as $\mathfrak{M}_B \to P \to Q \to M_B/\mathfrak{N}_B$, where P is a finite rank (as a B-module) projective quotient (hence direct summand) of $\mathfrak{M}_B$, and Q is a finite rank projective direct summand of $M_B/\mathfrak{N}_B$. Since the maps $\mathfrak{M}_B \to P$ and $Q \to M_B/\mathfrak{N}_B$ are both split, we are in fact considering the locus over which the morphism $P \to Q$ vanishes, and this is obviously closed, as it is given by the vanishing of matrix entries.

The endomorphism T induces an automorphism T_* of Gr (taking $\mathfrak{M}$ to $T(\mathfrak{M})$), and we let $\Gamma_T := T_* \times \text{id} : \text{Gr} \to \text{Gr} \times \text{Gr}$ be the graph of T. Pulling back the closed locus considered above by Γ_T, we see that there is a closed Ind-subscheme Gr^T of Gr, classifying the lattices $\mathfrak{M}$ with $T(\mathfrak{M}) \subset \mathfrak{M}$.

The φ-module M corresponds to a morphism $\text{Spec}\, A \to \mathcal{R}_{d,fpqc-\text{free}}$, and for each F, the fibre product $\text{Gr}^F := \text{Spec}\, A \times_{\mathcal{R}_{d,fpqc-\text{free}}} \mathcal{C}_{d,F}$ is a closed subscheme of Gr (it is a scheme by Theorem 5.4.9 (2)). Then the intersection of Gr^F and Gr^T is closed in Gr^F, and is therefore projective over $\text{Spec}\, A$, as required.   □

We let $\text{End}(\mathcal{R}_{n,\text{free}})$ be the category fibred in groupoids over $\mathcal{O}/\varpi^a$ with $\text{End}(\mathcal{R}_{n,\text{free}})(A) = \{(M, f)\}$ where $M \in \mathcal{R}_{n,\text{free}}(A)$ and $f \in \text{End}(M)$. It contains a subcategory fibred in groupoids $\text{Proj}_{n,d}$, classifying those pairs (M, f) for which $\text{Im}\, f$ is projective of rank d. There are natural morphisms $\text{Proj}_{n,d} \subseteq \text{End}(\mathcal{R}_{n,\text{free}}) \to \mathcal{R}_{n,\text{free}}$ and $\text{Proj}_{n,d} \to \mathcal{R}_d$, which respectively take (M, f) to M

and to $\mathrm{Im}\, f$. These morphisms fit into the following commutative diagram.

$$
\begin{array}{ccccc}
\mathcal{C}_{n,F} & & \mathrm{Proj}_{n,d} & \hookrightarrow & \mathrm{End}(\mathcal{R}_{n,\mathrm{free}}) \\
\downarrow & & & & \\
\mathcal{R}_n & \longleftarrow & \mathcal{R}_{n,\mathrm{free}} & \longleftarrow & \mathcal{R}_d
\end{array}
$$

5.4.11. Theorem. *(1) The morphism $\mathcal{C}_{d,F} \to \mathcal{R}_d$ is representable by algebraic spaces, proper, and of finite presentation.*

(2) The diagonal morphism $\Delta : \mathcal{R}_d \to \mathcal{R}_d \times_{\mathcal{O}/\varpi^a} \mathcal{R}_d$ is representable by algebraic spaces, affine, and of finite presentation.

(3) $\mathcal{R}_d$ satisfies [1].

Proof. We begin with (1). Let B be an A-algebra, and let $\mathrm{Spec}\, B \to \mathcal{R}_d$ be a morphism, corresponding to a projective étale φ-module M_B of rank d. We need to show that $\mathrm{Spec}\, B \times_{\mathcal{R}_d} \mathcal{C}_{d,F}$ is representable by a proper algebraic space over $\mathrm{Spec}\, B$ of finite presentation. This can be checked étale locally, so in particular by Lemma 5.1.23 we can assume that M_B is free over $(W(k) \otimes_{\mathcal{O}/\varpi^a} B)((u^n))$ for some n. The claim follows from Lemma 5.4.10, applied with u replaced by u^n, and T being given by multiplication by u.

Part (2) is immediate from Proposition 5.4.8. For (3), by Proposition 2.3.19 and part (2), it is enough to show that $\mathcal{R}_d \to \mathrm{Spec}\, \mathcal{O}/\varpi^a$ is limit preserving on objects. To this end, suppose that we have a morphism $T \to \mathcal{R}_d$, where $T = \varprojlim T_i$ is a limit of affine schemes. By Lemma 5.2.14 we can lift the morphism $T \to \mathcal{R}_d$ to a morphism $T \to \mathrm{Proj}_{n,d}$ for some n. The composite morphism $T \to \mathrm{Proj}_{n,d} \to \mathcal{R}_{n,\mathrm{free}} \to \mathcal{R}_n$ lifts to a morphism $T \to \mathcal{C}_{n,F}$ for some F (because every free étale φ-module contains a free finite height φ-module, by Lemma 5.2.15) and since $\mathcal{C}_{n,F}$ is locally of finite presentation, this morphism factors through T_i for some i.

Consequently, the composite $T \to \mathrm{Proj}_{n,d} \to \mathcal{R}_{n,\mathrm{free}}$ factors through T_i, and it suffices to prove that the morphism $\mathrm{Proj}_{n,d} \to \mathcal{R}_{n,\mathrm{free}}$ is limit preserving on objects. This follows from Proposition 5.4.8, which shows that the morphism $\mathrm{Proj}_{n,d} \to \mathcal{R}_{n,\mathrm{free}}$ is representable by schemes of finite presentation (note that the condition that an endomorphism of a free module be idempotent is a closed condition, and is therefore of finite presentation by Lemma 2.6.3). $\qquad\square$

By Theorem 5.4.11, the running assumptions of Section 3.2 apply to the morphism $\mathcal{C}_{d,F} \to \mathcal{R}_d$; so we may use Definition 3.2.6 to define the scheme-theoretic image of $\mathcal{C}_{d,F} \to \mathcal{R}_d$, which we denote by $\mathcal{R}_{d,F}$. The main result of this section is Theorem 5.4.19 below, showing that $\mathcal{R}_{d,F}$ is an algebraic stack. Before proving it, we study the versal rings of $\mathcal{R}_d$ and $\mathcal{R}_{d,F}$.

Let $\mathbb{F}'/\mathbb{F}$ be a finite extension, and let $M_{\mathbb{F}'}$ be an étale φ-module with $\mathbb{F}'$-coefficients, corresponding to a finite type point $x : \mathrm{Spec}\, \mathbb{F}' \to \mathcal{R}_d$. Write $\mathcal{O}'$ for the ring of integers in the compositum of E and $W(\mathbb{F}')[1/p]$, so that $\mathcal{O}'$ has residue field $\mathbb{F}'$.

As in Section 5.3, we fix a choice of (ordered) $\mathcal{O}_{\mathcal{E},\mathbb{F}'}$-basis of $M_{\mathbb{F}'}$, and we let $D^\square : \mathcal{C}_{\mathcal{O}'/\varpi^a} \to \underline{Sets}$ be the functor taking R to the set of isomorphism classes of liftings of $M_{\mathbb{F}'}$ to R. By Remark 5.3.5, the group functor H defined via $H(R) :=$

$R^\times + \mathfrak{m}_R M_d(\mathcal{O}_{\mathcal{E},R})$ acts on $D^\square$ via change of basis. By Proposition 5.3.6, $D^\square$ is pro-representable by an object $R^\square$ of pro-$\mathcal{C}_{\mathcal{O}'/\varpi^a}$.

5.4.12. Lemma. *The natural morphism $D^\square \to \widehat{\mathcal{R}}_{d,x}$, defined by mapping any lift M_R of $M_{\mathbb{F}'}$ over some test object R to the underlying étale φ-module (i.e. forgetting the choice of basis of M_R, as well as the chosen isomorphism between $M_R/\mathfrak{m}_R$ and $M_{\mathbb{F}'}$), is versal, and is also H-equivariant, for the change-of-basis action of H on $D^\square$ and for the trivial action of H on $\widehat{\mathcal{R}}_{d,x}$.*

Proof. The claimed equivariance is clear, since the morphism is defined in part by forgetting the chosen bases. To see the claimed versality, it suffices to show that if M_A is an étale φ-module with A-coefficients, where A is a finite Artinian $\mathcal{O}'/\varpi^a$-algebra, and if M_B is an étale φ-module with B-coefficients, with B a finite Artinian $\mathcal{O}'/\varpi^a$-algebra admitting a surjection onto A, such that the base change $(M_B)_A$ of M_B to A is isomorphic to M_A, and we have compatible choices of basis for M_A and M_B, then we may find an M'_B (together with such a basis) which lifts M_A, and is isomorphic to M_B. The existence of such a lift amounts to showing that we can lift the choice of basis, and this is clear from the surjectivity of $\mathrm{GL}_d(\mathcal{O}_{\mathcal{E},B}) \to \mathrm{GL}_d(\mathcal{O}_{\mathcal{E},A})$. $\qquad\square$

The preceding lemma shows in particular that $\mathcal{R}_d$ admits versal rings at all finite type points.

Let $\mathcal{C}_{\mathrm{Spf}\,R^\square}$ denote the pull-back of $\mathcal{C}_{d,F} \to \mathcal{R}_d$ along the versal morphism $D^\square = \mathrm{Spf}\,R^\square \to \mathcal{R}_d$, and let $R^{\square,\mathcal{C}}$ be the scheme-theoretic image (in the sense of Definition 3.2.15) of the morphism $\mathcal{C}_{\mathrm{Spf}\,R^\square} \to \mathrm{Spf}\,R^\square$. By Lemmas 3.2.16 and 5.4.12, there is a versal morphism $\mathrm{Spf}\,R^{\square,\mathcal{C}} \to \widehat{\mathcal{R}}_{d,F,x}$.

5.4.13. Definition. We let $D^{\square,\mathcal{C}}$ denote the subfunctor of $D^\square$ represented by $\mathrm{Spf}\,R^{\square,\mathcal{C}}$.

5.4.14. *Remark.* The equivariance statement of Lemma 5.4.12 implies that the H-action on $D^\square$ restricts to an H-action on $D^{\square,\mathcal{C}}$, and the the morphism $D^{\square,\mathcal{C}} := \mathrm{Spf}\,R^{\square,\mathcal{C}} \to \widehat{\mathcal{R}}_{d,F,x}$ is H-equivariant, with respect to the induced H-action on $D^{\square,\mathcal{C}}$, and the trivial H-action on $\widehat{\mathcal{R}}_{d,F,x}$.

Our goal will be to show that an appropriately chosen subgroup functor of H acts freely on $D^{\square,\mathcal{C}}$, and that the corresponding quotient $D^{\mathcal{C}}$ of $D^{\square,\mathcal{C}}$ also admits a versal morphism to $\widehat{\mathcal{R}}_{d,F,x}$, and is *Noetherianly* pro-representable. To this end, we will relate $D^{\square,\mathcal{C}}$ to the subfunctor $D_F^\square$ of $D^\square$, where, as in Section 5.3, we let $D_F^\square$ be the subfunctor of $D^\square$ consisting of those M which have a model of height F. In fact, we will show that $D^{\square,\mathcal{C}}$ is a subfunctor of $D_F^\square$; equivalently, we will show that $R^{\square,\mathcal{C}}$, which is *a priori* a quotient of $R^\square$, is actually a quotient of $R_F^\square$ (the quotient of $R^\square$ that pro-represents $D_F^\square$, whose existence is proved in Proposition 5.3.9). We begin with a useful general criterion for such a factorisation of the map $R^\square \to R^{\square,\mathcal{C}}$ to exist.

We momentarily place ourselves in the general deformation-theoretic context of Subsection 2.2; that is, we fix a Noetherian ring Λ, and a finite ring map $\Lambda \to k$, whose target is a field. We will also allow ourselves to use the language of formal algebraic spaces from [Sta, Tag 0AHW], but not in a serious way. (If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathrm{Spf}\,R$ is simply the Ind-scheme $\varinjlim_i \mathrm{Spec}\,R/\mathfrak{m}^i$,

and giving a finite type morphism of formal algebraic spaces $X \to \operatorname{Spf} R$ amounts to giving a collection of compatible finite type morphisms of algebraic spaces $X_i \to \operatorname{Spec} R/\mathfrak{m}^i$.)

5.4.15. Lemma. *Let $R \to S$ be a continuous surjection of objects in* pro*-$\mathcal{C}_\Lambda$, let $X \to \operatorname{Spf} R$ be a finite type morphism of formal algebraic spaces, and make the following assumption: if A is any finite-type Artinian local R-algebra for which the canonical morphism $R \to A$ factors through a discrete quotient of R, and for which the canonical morphism $X \times_{\operatorname{Spf} R} \operatorname{Spec} A \to \operatorname{Spec} A$ admits a section, then the canonical morphism $R \to A$ furthermore factors through S.*

Then if A is any discrete Artinian quotient of R for which the base-changed morphism $X \times_{\operatorname{Spf} R} \operatorname{Spec} A \to \operatorname{Spec} A$ is scheme-theoretically dominant, the surjection $R \to A$ factors through S.

Proof. The desired conclusion is equivalent to the claim that the closed immersion

$$(5.4.16) \qquad \operatorname{Spf} S \times_{\operatorname{Spf} R} \operatorname{Spec} A \to \operatorname{Spec} A$$

is an isomorphism. By Yoneda's lemma it is enough to show that the following condition (*) holds whenever B is a finite type A-algebra:

(*) Any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ which can be factored through $X \times_{\operatorname{Spf} R} \operatorname{Spec} A$ necessarily factors through the closed immersion (5.4.16).

(Indeed, if (*) holds, then, since the morphism $X \times_{\operatorname{Spf} R} \operatorname{Spec} A \to \operatorname{Spec} A$ is of finite type, by assumption, we see that it factors through (5.4.16). On the other hand, this morphism is scheme-theoretically dominant, by assumption; thus (5.4.16) is an isomorphism, as required.)

By considering the product of the localisations of a finite type A-algebra B at all its maximal ideals, this will follow if we prove (*) when B is the localisation of a finite type A-algebra at one of its maximal ideals. Since such a localisation is Noetherian, this in turn will follow if we prove (*) when B is the completion of a finite type A-algebra at one of its maximal ideals. Considering the reduction of such a completion modulo the various powers of its maximal ideal, we then reduce further to proving (*) in the the case when B is a finite type Artinian local A-algebra. But in this case, condition (*) holds by assumption. $\qquad \square$

5.4.17. Proposition. *$D^{\square,\mathcal{C}}$ is a subfunctor of $D_F^\square$.*

Proof. The claim of the proposition amounts to showing that the surjection $R^\square \to R^{\square,\mathcal{C}}$ factors through $R_F^\square$. By Lemma 3.2.4, we may write $R^{\square,\mathcal{C}}$ as the inverse limit of Artinian quotients A, for each of which the base-changed morphism $\mathcal{C}_{d,F,a} \to \operatorname{Spec} A$ is scheme-theoretically dominant. It suffices to show that each of the composite surjections $R^\square \to R^{\square,\mathcal{C}} \to A$ factors through $R_F^\square$. This will follow from Lemma 5.4.15, taking $R = R^\square$, $S = R_F^\square$, and $X = \mathcal{C}_{d,F,\operatorname{Spf} R^\square}$, provided we show that the hypotheses of that lemma hold.

To this end, let A be a finite type Artinian local $R^\square$-algebra for which the canonical morphism $\mathcal{C}_{d,F,\operatorname{Spec} A} \to \operatorname{Spec} A$ admits a section, and let M_A denote the étale φ-module corresponding to the induced morphism $\operatorname{Spec} A \to \operatorname{Spf} R^\square$. The existence of the section to $\mathcal{C}_{d,F,\operatorname{Spec} A}$ is, by definition, equivalent to the existence of a projective φ-module $\mathfrak{M}_A$ of height F such that $\mathfrak{M}_A[1/u] = M_A$. Again by definition, we have $\mathfrak{M}_A \otimes_A \kappa(A) = M_{\mathbb{F}'} \otimes_{\mathbb{F}'} \kappa(A)$. By Corollary 5.3.18, the functor D_F' of liftings of $M_{\mathbb{F}'} \otimes_{\mathbb{F}'} \kappa(A)$ which have a model of type F is pro-represented

by $R_F^\square \otimes_{W(\mathbb{F}')} W(\kappa(A))$, so in particular the existence of $\mathfrak{M}_A$ implies that the morphism $\operatorname{Spec} A \to \operatorname{Spf} R^\square$ factors through $\operatorname{Spf} R_F^\square$, as required. $\square$

It follows from Proposition 5.4.17, together with Remark 5.3.11, that the action of $H(R) := R^\times + \mathfrak{m}_R M_d(\mathcal{O}_{\mathcal{E},R})$ on $D^{\square,\mathcal{C}}(R)$ (for any test object R), whose existence was noted in Remark 5.4.14, restricts to a free action of $G(R) := 1 + u^N \mathfrak{m}_R M_d(\mathfrak{S}_R)$. We then make the following definition (in analogy to Definition 5.3.12).

5.4.18. **Definition.** We let $D^\mathcal{C} : \mathcal{C}_{\mathcal{O}/\varpi^a} \to \underline{Sets}$ denote the functor defined by $D^\mathcal{C}(R) := D^{\square,\mathcal{C}}(R)/G(R)$.

By construction there is a Cartesian square

$$
\begin{array}{ccc}
D^{\square,\mathcal{C}} & \longrightarrow & D_F^\square \\
\downarrow & & \downarrow \\
D^\mathcal{C} & \longrightarrow & D_F
\end{array}
$$

and the section $D_F \to D_F^\square$ of Theorem 5.3.15 then restricts to a section $D^\mathcal{C} \to D^{\square,\mathcal{C}}$. An argument almost identical to that used in the proof of Theorem 5.3.15 shows that $D^\mathcal{C}(R)$ is pro-representable by some $R^\mathcal{C}$; since $D^\mathcal{C}$ is a subfunctor of D_F, this pro-representing object is a quotient of the ring R_F that pro-represents D_F. As the latter ring is Noetherian (by Theorem 5.3.15), so is $R^\mathcal{C}$.

With these various definitions and observations in place, we are now ready to prove our main theorem.

5.4.19. **Theorem.** *The hypotheses of Theorem 1.1.1 hold for the morphism* $\mathcal{C}_{d,F} \to \mathcal{R}_d$. *That is:*

(1) $\mathcal{C}_{d,F}$ is an algebraic stack, locally of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$.
(2) $\mathcal{R}_d$ satisfies [3], and its diagonal is locally of finite presentation.
(3) $\mathcal{C}_{d,F} \to \mathcal{R}_d$ is a proper morphism.
(4) $\mathcal{R}_d$ admits versal rings at all finite type points.
(5) $\mathcal{R}_{d,F}$ satisfies [2].

Accordingly, $\mathcal{R}_{d,F}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, and the morphism $\mathcal{C}_{d,F} \to \mathcal{R}_d$ factors as $\mathcal{C}_{d,F} \to \mathcal{R}_{d,F} \to \mathcal{R}_d$, with the first morphism being a proper surjection, and the second a closed immersion.

Proof. Points (1), (2), (3) and (4) follow from Theorems 5.4.9 and 5.4.11 together with Lemma 5.4.12, so it only remains to check (5). For this, it follows from Lemma 2.2.5, Corollary 2.7.3, and Lemma 3.2.20 that we need only check that that $\mathcal{R}_{d,F}$ admits effective Noetherian versal rings at all finite type points.

The H-equivariance that was commented upon in Remark 5.4.14 implies that the versal morphism $D^{\square,\mathcal{C}} \to \widehat{\mathcal{R}}_{d,F,x}$ factors through the quotient $D^\mathcal{C}$ of $D^{\square,\mathcal{C}}$. The induced morphism $D^\mathcal{C} \to \widehat{\mathcal{R}}_{d,F,x}$ is again versal (as one immediately checks, using the chosen section $D^\mathcal{C} \to D^{\square,\mathcal{C}}$). As we have already observed, the functor $D^\mathcal{C}$ is pro-representable by a Noetherian ring $R^\mathcal{C}$.

To complete the verification of (5), we need to check that the morphism $\operatorname{Spf} R^\mathcal{C} = D^\mathcal{C} \to \widehat{\mathcal{R}}_{d,F,x}$ is effective. To this end, note that by Theorem 5.5.19 below, the morphism $\operatorname{Spf} R^\mathcal{C} \to \widehat{\mathcal{R}}_{d,x}$ is induced by a morphism $\operatorname{Spec} R^\mathcal{C} \to \mathcal{R}_d$. It remains to check that this morphism $\operatorname{Spec} R^\mathcal{C} \to \mathcal{R}_d$ factors through $\mathcal{R}_{d,F}$. By Lemma 3.2.19,

it suffices to show that the morphism $\mathcal{C}_{d,F,R^{\mathcal{C}}} \to \operatorname{Spec} R^{\mathcal{C}}$ is scheme-theoretically dominant. This is precisely the statement of Lemma 5.5.4.

The theorem now follows from Theorem 1.1.1, except that we have only proved that $\mathcal{R}_{d,F}$ is locally of finite presentation. In order to show that it is of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, we must show that it is quasi-compact and quasi-separated. Since $\mathcal{C}_{d,F}$ is quasi-compact and the map $\mathcal{C}_{d,F} \to \mathcal{R}_{d,F}$ is surjective, it follows from [Sta, Tag 050X] that $\mathcal{R}_{d,F}$ is quasi-compact. Since $\mathcal{R}_d$ has affine diagonal, by Theorem 5.4.9, and since $\mathcal{R}_{d,F}$ is a closed substack of $\mathcal{R}_d$, the diagonal of $\mathcal{R}_{d,F}$ is also affine. Thus $\mathcal{R}_{d,F}$ is quasi-separated, as required. $\square$

We now describe $\mathcal{R}_d$ as an Ind-stack. Note that the inductive limit in the statement of the following theorem could equivalently be computed with respect to any cofinal system of Fs (see [Eme, §2]).

5.4.20. Theorem. *If $F|F'$, then the natural morphism $\mathcal{R}_{d,F} \to \mathcal{R}_{d,F'}$ is a closed immersion. Furthermore, the natural morphism $\varinjlim_F \mathcal{R}_{d,F} \to \mathcal{R}_d$ (where, following Remark 4.2.7, the inductive limit is computed as a stack on any of the Zariski, étale, fppf, or fpqc sites) is an isomorphism of stacks. In particular, the stack $\mathcal{R}_d$ is an Ind-algebraic stack which satisfies [1].*

Proof. We first show that each of the morphisms $\mathcal{R}_{d,F} \hookrightarrow \mathcal{R}_{d,F'}$ is a closed immersion; indeed, this follows from the fact that in the chain of monomorphisms

$$\mathcal{R}_{d,F} \hookrightarrow \mathcal{R}_{d,F'} \hookrightarrow \mathcal{R}_d$$

both the composite and the second morphism are closed immersions.

It remains to be shown that the natural morphism $\varinjlim_F \mathcal{R}_{d,F} \to \mathcal{R}_d$ is an isomorphism. Since $\mathcal{R}_d$ satisfies [1], it suffices to show that if $\operatorname{Spec} A \to \mathcal{R}_d$ is a morphism with A a Noetherian $\mathcal{O}/\varpi^a$-algebra, then this morphism factors through the closed substack $\mathcal{R}_{d,F}$ for some F. Equivalently, we must show that for some F, the closed embedding $\mathcal{R}_{d,F} \times_{\mathcal{R}_d} \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A$ is an isomorphism. It therefore suffices to show that we may find a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ which is scheme-theoretically dominant (equivalently, so that the corresponding morphism $A \to B$ is injective) such that the induced morphism $\operatorname{Spec} B \to \mathcal{R}_d$ factors through $\mathcal{R}_{d,F}$. By Proposition 5.4.7, we can find a scheme-theoretically dominant morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ such that $\operatorname{Spec} B \to \mathcal{R}_d$ factors through $\mathcal{R}_{d,\mathrm{free}}$. Since a free étale φ-module contains a free finite height φ-module, by Lemma 5.2.15, we see that the morphism $\operatorname{Spec} B \to \mathcal{R}_{d,\mathrm{free}}$ factors through $\mathcal{C}_{d,F}$ for some F, and thus through $\mathcal{R}_{d,F}$, as required. $\square$

5.4.21. Remark. $\mathcal{R}_d$ is presumably not an algebraic stack. Indeed, since it is Ind-algebraic and satisfies [1], if it were algebraic, it would be locally finite dimensional. Since $\mathcal{R}_d$ is the inductive limit of its closed substacks $\mathcal{R}_{d,F}$, it would follow that for each finite type point x of $\mathcal{R}_d$, there would be a uniform bound on the dimension of $\mathcal{R}_{d,F}$ at x, independently of F. However, this dimension can be computed in terms of the versal rings at x, and it is presumably straightforward to use the arguments of [Kim11] to compute the dimensions of the rings R_F and $R_{\mathcal{C}}$ and thereby obtain a contradiction (for example, in the case considered in Section 5.4.23 below, the results of [Kim11] directly imply that the algebraic stacks $\mathcal{R}_{d,E^h}$ are equidimensional, with dimension growing linearly in h).

5.4.22. *An alternative approach.* As we now explain, by slightly altering the definitions of $\mathcal{C}_{d,F}$ and $\mathcal{R}_d$, we could avoid appealing to the descent results of [Dri06], without substantially altering our conclusions.

Namely, setting $S := \operatorname{Spec} \mathcal{O}/\varpi^a$, we define $\widetilde{\mathcal{C}}_{d,F} := \operatorname{pro-}\big((\mathcal{C}_{d,F})_{|\operatorname{Aff}_{\mathrm{pf}/S}}\big)$ and $\widetilde{\mathcal{R}}_d := \operatorname{pro-}\big((\mathcal{R}_d)_{|\operatorname{Aff}_{\mathrm{pf}/S}}\big)$. Without appealing to the results of [Dri06], we know that these are categories fibred in groupoids. Proposition 5.1.9 shows that $(\mathcal{C}_{d,F})_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ is in fact an *fppf* stack. Using faithfully flat descent results from rigid analytic geometry, one can similarly show that $(\mathcal{R}_d)_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ is an *fppf* stack. Furthermore, the arguments of [PR09], as adapted and modified in the present paper, show that $(\mathcal{C}_{d,F})_{|\operatorname{Aff}_{\mathrm{pf}/S}}$ is furthermore represented by an algebraic stack of finite type over S. Lemma 2.5.4 then implies that this same algebraic stack represents $\widetilde{\mathcal{C}}_{d,F}$, while Lemma 2.5.5 (2) implies that $\widetilde{\mathcal{R}}_d$ is an *fppf* stack, which satisfies axiom [1] by Lemma 2.5.4. The arguments of [PR09] are easily adapted to prove the analogue of Theorem 5.4.9 for $\widetilde{\mathcal{C}}_{d,F}$ and $\widetilde{\mathcal{R}}_d$. Furthermore, the proofs in the subsequent section immediately adapt to establish the analogue of Theorem 5.4.20.

Appealing to the results of [Dri06], as we do, we in fact prove that $\widetilde{\mathcal{C}}_{d,F} = \mathcal{C}_{d,F}$ and that $\widetilde{\mathcal{R}}_d = \mathcal{R}_d$. However, the primary appeal of this approach is aesthetic: it allows us to give straightforward and natural definitions of the stacks that we will study. In practice, and in applications, it seems that little would be lost by adopting the slightly weaker and more circumlocutious approach described here.

5.4.23. *Galois representations.* Let $K/\mathbb{Q}_p$ be a finite extension with residue field k. We now specialise to the case that $q = p$, $\varphi(u) = u^p$, and $\mathcal{O} = \mathbb{Z}_p$. Let E be the minimal polynomial over $W(k)$ of a fixed uniformiser π of K; then we refer to a φ-module of height at most E^h as a *Breuil–Kisin module of height at most h*. Fix a uniformiser π of K, and elements $\pi_n \in \overline{K}$, $n \geq 0$, such that $\pi_{n+1}^p = \pi_n$ and $\pi_0 = \pi$. Set $K_\infty = \cup_{n \geq 0} K(\pi_n)$, and $G_{K_\infty} := \operatorname{Gal}(\overline{K}/K_\infty)$.

The connections between Breuil–Kisin modules, étale φ-modules, and Galois representations are as follows. Let A be a finite[8] Artinian $\mathbb{Z}_p$-algebra. By [Kis09b, Lem. 1.2.7] (which is based on the results of [Fon94], and makes no use of the running hypothesis in [Kis09b] that $p \neq 2$), there is an equivalence of abelian categories between the category of continuous representations of G_{K_∞} on finite A-modules, and the category of étale φ-modules with A-coefficients. Furthermore, if we write $T(M)$ for the A-module with G_{K_∞}-action corresponding to the étale φ-module M, then M is a free $\mathfrak{S}_A[1/u]$-module of rank d if and only if $T(M)$ is a free A-module of rank d.

If M is an étale φ-module, and $V = T(M)$ is the corresponding representation of G_{K_∞}, then we say that M has height at most h if and only if it there is a Breuil–Kisin module $\mathfrak{M}$ of height at most h with $\mathfrak{M}[1/u] = M$, and we say that V has height at most h if and only if M has height at most h.

Suppose that $p \cdot A = 0$. We say that a continuous representation of G_K on a finite free A-module is *flat* if it arises as the generic fibre of a finite flat group scheme over $\mathcal{O}_K$ with an action of A. It follows from [Kis09b, Thm. 1.1.3, Lem. 1.2.5] (together with the results of [Kim12] in the case that $p = 2$) that restriction to G_{K_∞} induces

---

[8]Recall that an Artinian $\mathbb{Z}_p$-algebra that is finitely generated as a $\mathbb{Z}_p$-module is necessarily finite as a set, so that the word "finite" here can be interpreted either in the commutative algebra sense, or literally.

an equivalence of categories between the category of flat representations of G_K on finite free A-modules and the category of representations of G_{K_∞} of height at most 1 on finite free A-modules.

The above discussion shows that in the preceding context, we may (somewhat informally) regard $\mathcal{R}^a$ as a moduli space of d-dimensional continuous representations of G_{K_∞} over $\mathbb{Z}/p^a\mathbb{Z}$. Furthermore, if A is a reduced finite-$\mathbb{Z}/p^a$ algebra, and thus a product of finite fields, then any Breuil–Kisin module $\mathfrak{M}$ over A is necessarily free. Thus an A-valued point of $\mathcal{R}^a$ corresponds to a Galois representation of height h if and only if it factors through $\mathcal{R}^a_{d,E^h}$. Indeed, we have the following result.

5.4.24. Theorem. *The $\overline{\mathbb{F}}_p$-points of $\mathcal{R}^a_{d,E^h}$ naturally biject with isomorphism classes of Galois representations $G_{K_\infty} \to \mathrm{GL}_d(\overline{\mathbb{F}}_p)$ of height at most h.*

Proof. Since $\mathcal{R}_{d,E^h}$ is a finite type stack over $\mathbb{Z}/p^a\mathbb{Z}$, any $\overline{\mathbb{F}}_p$-point comes from an $\mathbb{F}$-point for some finite extension $\mathbb{F}/\mathbb{F}_p$, and by the definition of $\mathcal{R}_d$ (and the correspondence between étale φ-modules and continuous G_{K_∞}-representations explained above), we see that we need to prove that a morphism $\mathrm{Spec}\,\mathbb{F} \to \mathcal{R}_d$ factors through $\mathcal{R}_{d,E^h}$ if and only if the corresponding étale φ-module has height at most h (possibly after making a finite extension of scalars).

By the definition of $\mathcal{C}_{d,E^h}$, this latter condition is equivalent to the assertion that the morphism $\mathrm{Spec}\,\mathbb{F} \to \mathcal{R}_d$ factors through the morphism $\mathcal{C}_{d,E^h} \to \mathcal{R}_d$, while by Lemma 3.2.14, the former condition is equivalent to the assertion that the fibre $\mathrm{Spec}\,\mathbb{F} \times_{\mathcal{R}_d} \mathcal{C}_{d,E^h}$ be non-empty. Since this fibre product is a finite type $\mathbb{F}$-algebraic space (by Theorem 5.4.11 (1)), if it is non-empty it contains a point defined over a finite extension of $\mathbb{F}$. Thus these conditions are indeed equivalent, if we allow ourselves to replace $\mathbb{F}$ by an appropriate finite extension. $\square$

5.4.25. Corollary. *There is an algebraic stack of finite type over $\mathrm{Spec}\,\mathbb{F}_p$, whose $\overline{\mathbb{F}}_p$-points naturally biject with isomorphism classes of finite flat Galois representations $G_K \to \mathrm{GL}_d(\overline{\mathbb{F}}_p)$.*

Proof. In view of the equivalence between finite flat representations of G_K and representations of G_{K_∞} of height at most 1 explained above, this follows immediately from Theorem 5.4.24 (applied in the case $a = h = 1$). $\square$

5.5. Effectivity of projectivity. In this final subsection we prove Theorem 5.5.19, which was used in the proof of Theorem 5.4.19 in order to show that $\mathcal{R}_{d,F}$ admits *effective* Noetherian versal rings at all finite type points. The basic idea of the proof is to show that the universal family of finite height φ-modules on $\mathcal{C}_{d,F}$ can be pushed forward to give a finite height φ-module on $\mathrm{Spec}\,R^{\mathcal{C}}$, and the corresponding étale φ-module on $\mathrm{Spec}\,R^{\mathcal{C}}$ gives the required promotion of the morphism $\mathrm{Spf}\,R^{\mathcal{C}} \to \mathcal{R}_{d,F}$ to a morphism $\mathrm{Spec}\,R^{\mathcal{C}} \to \mathcal{R}_{d,F}$. It is, however, not obvious (at least to us) that this construction gives a *projective* étale φ-module on $\mathrm{Spec}\,R^{\mathcal{C}}$, and that the resulting morphism $\mathrm{Spec}\,R^{\mathcal{C}} \to \mathcal{R}_{d,F}$ agrees with the given morphism $\mathrm{Spf}\,R^{\mathcal{C}} \to \mathcal{R}_{d,F}$, and we have to do some work to establish both of these claims.

5.5.1. *Interpreting scheme-valued points of $\mathcal{C}_{d,F}$.* By definition, if A is an $\mathcal{O}/\varpi^a$-algebra, then giving an A-valued point of $\mathcal{C}_{d,F}$ is the same as giving a rank d projective $\mathfrak{S}_A$-module $\mathfrak{M}$ equipped with a φ-module structure of height F. Suppose more generally that T is a scheme over $\mathcal{O}/\varpi^a$, and let $\widehat{Z}$ denote the u-adic completion of $T \times_{\mathrm{Spec}\,\mathbb{Z}_p} \mathrm{Spec}\,\mathfrak{S}$, i.e. the formal scheme obtained by completing this

fibre product along its closed subscheme cut out by $u = 0$. Lemma 5.2.4 shows that the endomorphism φ of $\mathfrak{S}$ induces a morphism $\varphi_{\widehat{Z}} : \widehat{Z} \to \widehat{Z}$ of formal schemes over T.

Giving a T-valued point of $\mathcal{C}_{d,F}$ is then the same as giving a rank d locally free coherent sheaf $\widehat{\mathfrak{M}}$ on $\widehat{Z}$, equipped with an injective morphism $\Phi_{\widehat{\mathfrak{M}}} : \varphi_{\widehat{Z}}^* \widehat{\mathfrak{M}} \to \widehat{\mathfrak{M}}$ whose cokernel is killed by F.

In our discussion below, we will be in the following situation: the scheme T will be projective over a Noetherian affine scheme $\operatorname{Spec} A$. The morphism $T \to \operatorname{Spec} A$ then induces a morphism $Z := T \times_{\operatorname{Spec} A} \operatorname{Spec} \mathfrak{S}_A \to \operatorname{Spec} \mathfrak{S}_A$. The u-adic completion of the source is then equal to the formal scheme $\widehat{Z}$ introduced above (although the map $Z := T \times_{\operatorname{Spec} A} \operatorname{Spec} \mathfrak{S}_A \to T \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} \mathfrak{S}$ need not be an isomorphism in general). In this situation, the Grothendieck existence theorem shows that $\widehat{\mathfrak{M}}$ arises as the u-adic completion of a rank d locally free sheaf $\mathfrak{M}$ on Z. If $\pi : Z \to \operatorname{Spec} \mathfrak{S}_A$ denotes the projection, then $\pi_* \mathfrak{M}$ is a coherent sheaf on $\operatorname{Spec} \mathfrak{S}_A$, which is to say, a finite type $\mathfrak{S}_A$-module. Since multiplication by u on $\mathfrak{M}$ is injective (as $\mathfrak{M}$ is locally free), the same is true of multiplication by u on $\pi_* \mathfrak{M}$ (since π_* is left exact).

The endomorphism $\varphi_{\widehat{Z}}$ of $\widehat{Z}$ arises as the u-adic completion of an endomorphism φ_Z of Z; and in fact φ_Z is just the base-change to Z of the endomorphism φ of $\mathfrak{S}_A$. This latter endomorphism is finite and flat by Lemma 5.2.5, and thus so is φ_Z. Consequently, if $\pi : Z \to \operatorname{Spec} \mathfrak{S}_A$ denotes the projection, we find that there is a natural isomorphism $\varphi^* \pi_* \mathfrak{M} \xrightarrow{\sim} \pi_* \varphi_Z^* \mathfrak{M}$.

Now the morphism $\Phi_{\widehat{\mathfrak{M}}} : \varphi_{\widehat{Z}}^* \widehat{\mathfrak{M}} \to \widehat{\mathfrak{M}}$ algebraizes to a morphism $\Phi_{\mathfrak{M}} : \varphi_Z^* \mathfrak{M} \to \mathfrak{M}$, and since the cokernel of $\Phi_{\widehat{\mathfrak{M}}}$ is supported (set-theoretically) on the locus where $u = 0$ (since it is annihilated by F), it coincides with the cokernel of $\Phi_{\mathfrak{M}}$. Thus this latter cokernel is also killed by F.

Since π_* is left-exact, we may push forward the short exact sequence

$$0 \to \varphi_Z^* \mathfrak{M} \xrightarrow{\Phi_{\mathfrak{M}}} \mathfrak{M} \to \mathfrak{M}/\varphi_Z^* \mathfrak{M} \to 0$$

to obtain a left exact sequence

$$0 \to \pi_* \varphi_Z^* \mathfrak{M} \to \pi_* \mathfrak{M} \to \pi_*(\mathfrak{M}/\varphi_Z^* \mathfrak{M}),$$

which, using the previously noted isomorphism, we may rewrite as a left exact sequence

$$0 \to \varphi^* \pi_* \mathfrak{M} \to \pi_* \mathfrak{M} \to \pi_*(\mathfrak{M}/\varphi_Z^* \mathfrak{M}).$$

Since F annihilates $\mathfrak{M}/\varphi^* \mathfrak{M}$, it also annihilates $\pi_*(\mathfrak{M}/\varphi_Z^* \mathfrak{M})$, and thus we see that $\pi_* \mathfrak{M}$ is naturally endowed with the structure of a φ-module of height F with A-coefficients.

5.5.2. *The pushforward of the universal finite height φ-module.* To simplify notation, we write R for $R^{\mathcal{C}}$ from now on. We also fix a descending sequence of ideals I_n of R which is cofinal with the sequence $\mathfrak{m}_R^n$; equivalently, which determines the $\mathfrak{m}_R$-adic topology on R. (For the moment, the reader can just imagine that $I_n = \mathfrak{m}_R^n$; however, below we will make a different choice of I_n.) We write $R_n := R/I_n$, and let M_n denote the universal étale φ-module over $\operatorname{Spec} R_n$. We write $\widehat{\mathfrak{S}}_R[1/u]$ for the $\mathfrak{m}_R$-adic completion of $\mathfrak{S}_R[1/u]$, and write $\widehat{M} = \varprojlim_n M_n$; then $\widehat{M}$ is a finite projective $\widehat{\mathfrak{S}}_R[1/u]$-module, and we may recover the various M_n from $\widehat{M}$ via the natural isomorphism $R_n \otimes_R \widehat{M} \xrightarrow{\sim} M_n$.

We aim to prove the following proposition.

5.5.3. Proposition. *We may find a φ-module $\mathfrak{M}'$ of height F with R-coefficients such that $\widehat{\mathfrak{M}'[1/u]} = \widehat{M}$ (where the hat on $\widehat{\mathfrak{M}'[1/u]}$ denotes $\mathfrak{m}_R$-adic completion).*

The construction of $\mathfrak{M}'$ involves the pushforward of the universal finite height φ-module over R, and we begin by setting up some notation related to the study of this pushforward. Note that each of the morphisms

$$\mathcal{C}_{d,F} \times_{\mathcal{R}} \operatorname{Spec} R_n \to \operatorname{Spec} R_n$$

is in fact a projective morphism of schemes $\pi_n : X_n \to \operatorname{Spec} R_n$. These morphisms are compatible as n varies, and by Grothendieck existence give an algebraization of

$$\mathcal{C}_{d,F} \times_{\widehat{\mathcal{R}}_{d,x}} \operatorname{Spf} R \to \operatorname{Spf} R$$

to a projective morphism of schemes $\pi : X \to \operatorname{Spec} R$. (In the special case of Breuil–Kisin modules of height 1, this is [Kis09b, Prop. 2.1.10], and the general case can be proved in exactly the same way (see e.g. [Kis10, Prop. 1.3] for the case of Breuil–Kisin modules of arbitrary height); the key point is that $\mathcal{C}_{d,F} \times_{\widehat{\mathcal{R}}_{d,x}} \operatorname{Spf} R$ inherits a natural very ample (formal) line bundle from the affine Grassmannian.)

5.5.4. Lemma. *The morphism $X \to \operatorname{Spec} R$ is scheme-theoretically dominant.*

Proof. Recall that $\operatorname{Spf} R^{\square,\mathcal{C}}$, which is a formal closed subscheme of $\operatorname{Spf} R^{\square}$, is defined to be the scheme-theoretic image of the morphism

$$\mathcal{C}_{\operatorname{Spf} R^{\square}} := \mathcal{C}_{d,F} \times_{\mathcal{R}_d} \operatorname{Spf} R^{\square} \to \operatorname{Spf} R^{\square}.$$

We may rephrase this as the conjunction of the following two statements: the fibre product $\mathcal{C}_{\operatorname{Spf} R^{\square,\mathcal{C}}} := \mathcal{C}_{d,F} \times_{\mathcal{R}_d} \operatorname{Spf} R^{\square,\mathcal{C}}$ (which *a priori* is a closed formal subscheme of $\mathcal{C}_{\operatorname{Spf} R^{\square}}$) is equal to $\mathcal{C}_{\operatorname{Spf} R^{\square}}$; and the projection

$$(5.5.5) \qquad\qquad \mathcal{C}_{\operatorname{Spf} R^{\square,\mathcal{C}}} \to \operatorname{Spf} R^{\square,\mathcal{C}}$$

is scheme-theoretically dominant, in the sense that the target of the projection is equal to its scheme-theoretic image. It is from this latter statement that we will deduce the lemma.

As observed prior to the statement of Definition 5.4.18, there is a formal group G over $\operatorname{Spec} \mathcal{O}/\varpi^a$ acting freely on $\operatorname{Spf} R^{\square,\mathcal{C}}$, and (remembering that R is our abbreviated notation for $R^{\mathcal{C}}$), the closed formal subscheme $\operatorname{Spf} R$ of $\operatorname{Spf} R^{\square,\mathcal{C}}$ is constructed so that

$$(5.5.6) \qquad\qquad \operatorname{Spf} R \times_{\operatorname{Spec} \mathcal{O}/\varpi^a} G \xrightarrow{\sim} \operatorname{Spf} R^{\square,\mathcal{C}}.$$

(Compare Theorem 5.3.15 and its proof.) We write $\mathcal{C}_{\operatorname{Spf} R} := \mathcal{C}_{d,F} \times_{\mathcal{R}_d} \operatorname{Spf} R$. What we will show is that

$$(5.5.7) \qquad\qquad \mathcal{C}_{\operatorname{Spf} R} \to \operatorname{Spf} R$$

is scheme-theoretically dominant; the lemma follows directly from this. (Indeed, by definition this implies that the scheme-theoretic image of $X \to \operatorname{Spec} R$ contains $\operatorname{Spec} R_i$ for any discrete quotient R_i of R, and thus is equal to $\operatorname{Spec} R$; the reader should compare the first paragraph of the proof of Lemma 3.2.4. Indeed, as one sees from a consideration of the remainder of that proof, the theorem on formal functions implies that the scheme-theoretic dominance of the morphism $X \to \operatorname{Spec} R$ of *schemes* is equivalent to the scheme-theoretic dominance of the morphism (5.5.7) of *formal schemes*.)

We first note that the G-action on $\operatorname{Spf} R^{\square,\mathcal{C}}$ lifts to a G-action on $\mathcal{C}_{\operatorname{Spf} R^{\square,\mathcal{C}}}$. Indeed, the morphism $\operatorname{Spf} R^{\square,\mathcal{C}} \to \mathcal{R}_d$ which is used in the definition of $\mathcal{C}_{\operatorname{Spf} R^{\square,\mathcal{C}}}$ as a fibre product is invariant under the action of G on its source, and so this G-action indeed lifts to a G-action on the fibre product, i.e. on $\mathcal{C}_{\operatorname{Spf} R^{\square,\mathcal{C}}}$. The isomorphism (5.5.6) then induces a corresponding isomorphism

$$(5.5.8) \qquad \mathcal{C}_{\operatorname{Spf} R^{\mathcal{C}}} \times_{\operatorname{Spec} \mathcal{O}/\varpi^a} G \xrightarrow{\sim} \mathcal{C}_{\operatorname{Spf} R^{\square,\mathcal{C}}}.$$

As we observed in the proof of Theorem 5.3.15, we have an isomorphism $G = \operatorname{Spf} \mathcal{O}/\varpi^a[[(V_\delta^*)^*]]$ for a certain free $\mathcal{O}/\varpi^a$-module V. From this, from (5.5.8), and from the scheme-theoretic dominance of (5.5.5), we deduce that (5.5.7) is scheme-theoretically dominant, as required. $\qquad\square$

We set $S_n := \mathcal{O}_{X_n}(X_n)$; since X_n is proper over the affine scheme $\operatorname{Spec} R_n$, we see that S_n is a finite R_n-algebra. Let R_n' denote the image of R_n in S_n. By the scheme-theoretic dominance of (5.5.7), the R_n' are again a cofinal system of Artinian quotients of R and so, replacing each R_n by R_n' (which doesn't change X_n or S_n), we may, and indeed do, assume that each morphism $R_n \to S_n$ is injective.

We write $Y_n := X_n \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} \mathfrak{S} = X_n \times_{\operatorname{Spec} R_n} \operatorname{Spec} \mathfrak{S}_{R_n}$ (the equality holding because R_n is finite), and let $\widehat{Y}_n$ denote the completion of Y_n along the closed subscheme "$u = 0$". Equivalently, $\widehat{Y}_n := Y_n \times_{\operatorname{Spec} \mathfrak{S}_{R_n}} \operatorname{Spf} \mathfrak{S}_{R_n}$, where $\operatorname{Spf} \mathfrak{S}_{R_n}$ is taken with respect to the u-adic topology on $\mathfrak{S}_{R_n}$. Each of the schemes X_n is equipped with a tautological projection to $\mathcal{C}_{d,F}$, and we may apply the discussion of Subsection 5.5.1 to this morphism so as to obtain a locally free coherent sheaf $\widehat{\mathfrak{M}}_n$ on the formal scheme $\widehat{Y}_n$, which by the Grothendieck existence theorem can be promoted to a locally free coherent sheaf $\mathfrak{M}_n$ on Y_n itself.

If $\pi_{Y_n} : Y_n \to \operatorname{Spec} \mathfrak{S}_{R_n}$ denotes the projection, then the discussion of Subsection 5.5.1 shows that $(\pi_{Y_n})_* \mathfrak{M}_n$ is a φ-module of height F with coefficients in R_n. Since, in particular, $(\pi_{Y_n})_* \mathfrak{M}_n$ is u-torsion free, it naturally embeds into the étale φ-module $\big((\pi_{Y_n})_* \mathfrak{M}_n\big)[1/u]$. Our next goal is to compute this étale φ-module, and to compare it to M_n; we do this in (5.5.14) below.

Write

$$U_n := X_n \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_{\mathcal{E}} = X_n \times_{\operatorname{Spec} R_n} \operatorname{Spec} \mathcal{O}_{\mathcal{E},R_n}$$
$$= Y_n \times_{\operatorname{Spec} \mathfrak{S}_{R_n}} \operatorname{Spec} \mathcal{O}_{\mathcal{E},R_n}.$$

The projection $\pi_n : X_n \to \operatorname{Spec} R_n$ induces a projection $\pi_{U,n} : U_n \to \operatorname{Spec} \mathcal{O}_{\mathcal{E},R_n}$. The open immersion

$$(5.5.9) \qquad \operatorname{Spec} \mathcal{O}_{\mathcal{E},R_n} \to \operatorname{Spec} \mathfrak{S}_{R_n}$$

also induces an open immersion $j_n : U_n \to Y_n$.

In the statement of the following lemma, we regard the rank d projective $\mathcal{O}_{\mathcal{E},R_n}$-module M_n as a locally free coherent sheaf on $\operatorname{Spec} \mathcal{O}_{\mathcal{E},R_n}$.

5.5.10. **Lemma.** *There are canonical isomorphisms $j_n^* \mathfrak{M}_n \xrightarrow{\sim} \pi_{U,n}^* M_n$, compatible with the actions of φ, and with varying n.*

Proof. Choose an $\mathcal{O}_{\mathcal{E},R_n}$ basis for M_n, and let $\mathfrak{N}$ denote the free $\mathfrak{S}_{R_n}$-submodule of M_n generated by this basis. Then we have an isomorphism $\varinjlim_i \frac{1}{u^i} \mathfrak{N} \xrightarrow{\sim} M_n$.

Pulling back via π_n, we obtain an isomorphism of quasi-coherent sheaves

$$(5.5.11) \qquad \varinjlim_i \frac{1}{u^i} \pi_n^* \mathfrak{N} \xrightarrow{\sim} \pi_n^* M_n$$

on Y_n.

We pull the isomorphism (5.5.11) back to $\widehat{Y}_n$. On the one hand, if we let $\widehat{\pi_n^* \mathfrak{N}}$ denote the u-adic completion of $\mathfrak{N}$, then the left hand side pulls back to $\varinjlim_i \frac{1}{u^i} \widehat{\pi_n^* \mathfrak{N}}$. On the other hand, the defining relationship between M_n and the u-adic completion $\widehat{\mathfrak{M}}_n$ of $\mathfrak{M}_n$ (namely, that M_n is defined by the morphism $R_n \to \mathcal{R}_d$ that was itself used to construct the morphism $X_n \to \mathcal{C}_d$, which in turn defines the coherent sheaf $\widehat{\mathfrak{M}}_n$ on $\widehat{Y}_n$) shows that $\pi_n^* M_n$ pulls back to the sheaf $\varinjlim_i \frac{1}{u^i} \widehat{\mathfrak{M}}_i$ on $\widehat{Y}_n$. Since $(\frac{1}{u^i} \widehat{\pi_n^* \mathfrak{N}})_i$ and $(\frac{1}{u^i} \widehat{\mathfrak{M}}_n)_i$ are inductive systems of coherent sheaves on $\widehat{Y}_n$ with injective transition maps and having the same inductive limit on $\widehat{Y}_n$, we see that they are mutually cofinal. The equivalence of categories provided by Grothendieck existence then shows that $(\frac{1}{u^i} \pi_n^* \mathfrak{N})_i$ and $(\frac{1}{u^i} \mathfrak{M}_n)_i$ are mutually cofinal inductive systems of coherent sheaves on Y_n itself. Returning to (5.5.11), we find that

$$\varinjlim_i \frac{1}{u^i} \mathfrak{M}_n \xrightarrow{\sim} \pi_n^* M_n.$$

The left-hand side of this isomorphism is naturally identified with $(j_n)_* j_n^* \mathfrak{M}_n$. The natural adjunction between j_n^* and $(j_n)_*$ then shows that

$$j_n^* \mathfrak{M}_n \xrightarrow{\sim} j_n^* \pi_n^* M_n = \pi_{U,n}^* M_n,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since j_n is the base change of the open immersion (5.5.9), we find that

$$(5.5.12) \qquad (\pi_{U,n})_* j_n^* \mathfrak{M}_n = \mathcal{O}_{\mathcal{E},R_n} \otimes_{\mathfrak{S}_{R_n}} (\pi_{Y,n})_* \mathfrak{M}_n = \big((\pi_{Y,n})_* \mathfrak{M}_n\big)[1/u].$$

Since $\mathcal{O}_{\mathcal{E},R_n}$ is flat over R_n, we find that $(\pi_{U,n})_* \mathcal{O}_{U_n} = S_n \otimes_{R_n} \mathcal{O}_{\mathcal{E},R_n} = \mathcal{O}_{\mathcal{E},S_n}$, and so by the projection formula,

$$(\pi_{U,n})_* \pi_{U,n}^* M_n = \mathcal{O}_{\mathcal{E},S_n} \otimes_{\mathcal{O}_{\mathcal{E},R_n}} M_n = S_n \otimes_{R_n} M_n.$$

By Lemma 5.5.10, we may rewrite this as

$$(5.5.13) \qquad (\pi_{U,n})_* j_n^* \mathfrak{M}_n = S_n \otimes_{R_n} M_n.$$

Comparing (5.5.12) and (5.5.13), we find that

$$(5.5.14) \qquad \big((\pi_{Y,n})_* \mathfrak{M}_n\big)[1/u] = S_n \otimes_{R_n} M_n.$$

Since $R_n \to S_n$ is injective, and since M_n is flat over R_n (being projective over the flat R_n-algebra $\mathcal{O}_{\mathcal{E},R_n}$), we find that the natural map $M_n \to S_n \otimes_{R_n} M_n$ is injective. By the left exactness of pushforward, we see that $(\pi_{Y,n})_* \mathfrak{M}_n$ is u-torsion free, so the natural map $(\pi_{Y,n})_* \mathfrak{M}_n \to (\pi_{Y,n})_* \mathfrak{M}_n)[1/u] = S_n \otimes_{R_n} M_n$ is also injective. Consequently, we may define

$$\mathfrak{M}_n' := (\pi_{Y,n})_* \mathfrak{M}_n \cap M_n,$$

the intersection taking place in $S_n \otimes_{R_n} M_n$. This is a φ-stable $\mathfrak{S}_{R_n}$-submodule of M_n, whose formation is compatible with change in n.

5.5.15. **Lemma.** *The inclusion $\mathfrak{M}_n' \hookrightarrow M_n$ induces an isomorphism $\mathfrak{M}_n'[1/u] = M_n$.*

Proof. By virtue of the definition of $\mathfrak{M}'_n$, we find that

$$\mathfrak{M}'_n[1/u] = \big((\pi_{Y,n})_*\mathfrak{M}_n\big)[1/u] \cap M_n = (S_n \otimes_{R_n} M_n) \cap M_n = M_n,$$

the second-to-last equality following from (5.5.14). □

We now define $\mathfrak{M}' := \varprojlim_n \mathfrak{M}'_n$; this is an $\mathfrak{S}_R$-module, which we endow with its projective limit topology (in which each $\mathfrak{M}'_n$ is equipped with the discrete topology).

5.5.16. Lemma. *The $\mathfrak{S}_R$-module $\mathfrak{M}'$ is of finite type, and the projective limit topology on $\mathfrak{M}'$ coincides with the $\mathfrak{m}_R$-adic topology on $\mathfrak{M}'$.*

In order to prove this lemma, we revisit some of the preceding constructions, with the various X_n replaced by the projective R-scheme X. To be precise, we write $Y := X \times_{\operatorname{Spec} R} \operatorname{Spec} \mathfrak{S}_R$, and write $\widehat{Y}$ to denote the $\mathfrak{m}_R$-adic completion of Y. (Note that this notation is *not* compatible with the analogous notation $\widehat{Y}_n$, which denoted u-adic completion. If we similarly let $\widehat{X}$ denote the $\mathfrak{m}_R$-adic completion of X, then by the construction of X, there is an isomorphism $\widehat{X} \cong \mathcal{C}_{d,F} \times_{\widehat{\mathcal{R}}_{d,x}} \operatorname{Spf} R$, and consequently, an isomorphism $\widehat{Y} \cong \mathcal{C}_{d,F} \times_{\widehat{\mathcal{R}}_{d,x}} \operatorname{Spf} \mathfrak{S}_R$, where $\operatorname{Spf} \mathfrak{S}_R$ is formed with respect to the $\mathfrak{m}_R$-adic topology on $\mathfrak{S}_R$.)

We may regard $\widehat{Y}$ as the ind-scheme $\varinjlim_n Y_n$, and the compatible (in n) collection of coherent sheaves $\mathfrak{M}_n$ on the various Y_n give rise to a coherent sheaf $\widehat{\mathfrak{M}}$ on the formal scheme $\widehat{Y}$. By the Grothendieck existence theorem, this coherent sheaf arises as the completion of a coherent sheaf $\mathfrak{M}$ on the projective $\mathfrak{S}_R$-scheme Y.

If we write π_Y for the projection $Y \to \operatorname{Spec} \mathfrak{S}_R$, then the theorem on formal functions shows that $(\pi_Y)_*\mathfrak{M}$ is a finite type $\mathfrak{S}_R$-module, and that there is an isomorphism

$$(5.5.17) \qquad (\pi_Y)_*\mathfrak{M} \xrightarrow{\sim} \varprojlim_n (\pi_{Y,n})_*\mathfrak{M}_n,$$

which furthermore identifies the $\mathfrak{m}_R$-adic topology on the source with the projective limit topology on the target.

Proof of Lemma 5.5.16. Since each $\mathfrak{M}'_n$ is an $\mathfrak{S}_{R_n}$-submodule of $(\pi_{Y,n})_*\mathfrak{M}_n$, it follows from (5.5.17) and the left exactness of inverse limits that $\mathfrak{M}'$ $(= \varprojlim_n \mathfrak{M}'_n)$ is an $\mathfrak{S}_R$-submodule of $(\pi_Y)_*\mathfrak{M}$. Since $\mathfrak{S}_R$ is Noetherian, and $\mathfrak{M}'$ is an $\mathfrak{S}_R$-submodule of the finite type $\mathfrak{S}_R$-module $(\pi_Y)_*\mathfrak{M}$, we see that $\mathfrak{M}'$ is of finite type. The projective limit topology on $\mathfrak{M}'$ coincides with the topology induced by the projective limit topology on $(\pi_Y)_*\mathfrak{M}$ induced by the right-hand side of the isomorphism (5.5.17). Thus it also coincides with the topology induced by the $\mathfrak{m}_R$-adic topology on $(\pi_Y)_*\mathfrak{M}$, which by the Artin–Rees lemma is equal to the $\mathfrak{m}_R$-adic topology on $\mathfrak{M}'$. □

Proof of Proposition 5.5.3. We have already seen in Lemma 5.5.16 that $\mathfrak{M}'$ is a finite type $\mathfrak{S}_R$-module, and Lemmas 5.5.15 and 5.5.16 together show that

$$\widehat{\mathfrak{M}'[1/u]} = \varprojlim_n (\mathfrak{M}'/\mathfrak{m}_R^n\mathfrak{M}')[1/u] = \varprojlim_n \mathfrak{M}'_n[1/u] = \varprojlim_n M_n = \widehat{M}.$$

Since each $\mathfrak{M}'_n$ embeds into $M_n = \mathfrak{M}'_n[1/u]$, we see that each $\mathfrak{M}'_n$ is u-torsion free; thus $\mathfrak{M}'$ is also u-torsion free.

It remains to show that the φ-module structure on $\mathfrak{M}'$ makes it of height F. The endomorphism φ of $\mathfrak{S}_R$ is finite flat, and thus $\varphi^*\mathfrak{M}' = \varprojlim_n \varphi^*\mathfrak{M}'_n$. It hence suffices to show that each $\mathfrak{M}'_n$ is of height F.

We have already seen that each of the pushforwards $(\pi_{Y_n})_*\mathfrak{M}_n$ is of height F. Now let $x \in \mathfrak{M}'_n = (\pi_{Y_n})_*\mathfrak{M}_n \cap M_n$ be arbitrary. Then $Fx \in F(\pi_{Y,n})_*\mathfrak{M}_n$, so since $(\pi_{Y,n})_*\mathfrak{M}_n$ has height F, we see that there is some $y \in \varphi^*(\pi_{Y,n})_*\mathfrak{M}_n$ with $\Phi_{(\pi_{Y,n})_*\mathfrak{M}_n}(y) = Fx$. Since $\Phi_{M_n} : \varphi^*M_n \to M_n$ is a bijection, we have $y \in \varphi^*M_n$, so $y \in \varphi^*M_n \cap \varphi^*(\pi_{Y,n})_*\mathfrak{M}_n$. Since φ is flat, we have $\varphi^*M_n \cap \varphi^*(\pi_{Y,n})_*\mathfrak{M}_n = \varphi^*\mathfrak{M}'_n$, so $y \in \varphi^*\mathfrak{M}'_n$, and $\mathfrak{M}'_n$ is of height F, as required. $\qquad\square$

5.5.18. *Descent of projectivity.* We now come to the main result of this subsection.

5.5.19. **Theorem.** *There is a projective étale φ-module M over $R^{\mathcal{C}}$ whose $\mathfrak{m}_{R^{\mathcal{C}}}$-adic completion is isomorphic to the base change to $R^{\mathcal{C}}$ of the (formal) universal étale φ-module over $R^{\square}$.*

Proof. We write $M := \mathfrak{M}'[1/u]$, where $\mathfrak{M}'$ is the φ-module of height F over $R^{\mathcal{C}}$ constructed in Proposition 5.5.3. By construction, M is an étale φ-module with $R^{\mathcal{C}}$-coefficients, and the $\mathfrak{m}_{R^{\mathcal{C}}}$-adic completion of M is a finite projective module over $\widehat{\mathfrak{S}}_{R^{\mathcal{C}}}[1/u]$, and corresponds to the morphism $\mathrm{Spf}\, R^{\mathcal{C}} \to \mathcal{R}_d$. Theorem 5.5.20 below then shows that M itself is a projective étale φ-module, as required. $\qquad\square$

5.5.20. **Theorem.** *Let R be a complete Noetherian local $\mathcal{O}/\varpi^a$-algebra R with maximal ideal $\mathfrak{m}$, let M be an étale φ-module with R-coefficients, and suppose that the $\mathfrak{m}$-adic completion $\widehat{M}$ is projective, or equivalently, free (over $\widehat{R((u))}$, the $\mathfrak{m}$-adic completion of $R((u))$). Then M itself is projective (over $R((u))$).*

The remainder of this subsection is devoted to the proof of this theorem. However, before giving the proof in detail, we give an outline of it in the case that $R = K[[T_1, \ldots, T_n]]$, where K is a field of characteristic p. In this case, the ring $R((u))$ may be regarded as the ring of bounded holomorphic functions on the open unit n-dimensional polydisk $\mathbb{D}^n$ over the complete discretely valued field $K((u))$, and its $\mathfrak{m}$-adic completion $\widehat{R((u))}$ may be regarded as the formal completion of the local ring $\mathcal{O}_{\mathbb{D}^n,0}$, where 0 denotes the origin of $\mathbb{D}^n$. The extension $\mathcal{O}_{\mathbb{D}^n,0} \to \widehat{R((u))}$ is then faithfully flat, and since $\widehat{M} \xrightarrow{\sim} \widehat{R((u))} \otimes_{R((u))} M$ is projective, the same is true of $\mathcal{O}_{\mathbb{D}^n,0} \otimes_{R((u))} M$.

Writing the stalk as a direct limit of rings of holomorphic functions on a nested sequence of polydisks centred at 0 of shrinking radius, we find that the restriction of M to one of these polydisks is projective. The étale φ-module structure on M then allows us to employ Frobenius amplification ("Dwork's trick") to show that M itself gives rise to a projective module over the ring $\mathcal{O}(\mathbb{D}^n)$ of holomorphic functions on $\mathbb{D}^n$ — i.e. we find that $\mathcal{O}(\mathbb{D}^n) \otimes_{R((u))} M$ is projective. Finally, the inclusion $R((u)) \hookrightarrow \mathcal{O}(\mathbb{D}^n)$ of bounded holomorphic functions in all (i.e. not necessarily bounded) holomorphic functions is faithfully flat, and so M itself is projective over $R((u))$.

The proof in the general case follows the same outline: we regard $R((u))$ as being the ring of bounded holomorphic functions on a closed analytic subvariety of an open unit polydisk and make the same Frobenius amplification argument. We pass from the case of k-algebras to $\mathcal{O}/\varpi^a$-algebras via the usual graded techniques.

We now fill in the details of the preceding sketch.

Proof of Theorem 5.5.20. Assume to begin with that R is a complete Noetherian local k-algebra; we may then write $R = K[[T_1, \ldots, T_n]]/I$, for some field extension K of k, and some ideal I. For any integer $m \geq 0$, we write

$$A_m := \overline{R[x_1, \ldots, x_n, u]/(u^{p^m}x_1 - T_1, \ldots, u^{p^m}x_n - T_n)}[1/u],$$

and

$$B_m := \overline{R[x_1, \ldots, x_n, u]/(ux_1 - T_1^{p^m}, \ldots, ux_n - T_n^{p^m})}[1/u];$$

in both cases, the hat indicates u-adic completion. There are natural morphisms of $R((u))$-algebras

(5.5.21) $$A_m \hookrightarrow A_{m+1},$$

defined by mapping x_i in the source to $u^{p^m(p-1)}x_i$ in the target, and

$$B_{m+1} \hookrightarrow B_m,$$

defined by mapping x_i in the source to $u^{p-1}x_i^p$ in the target; Lemmas 5.5.33 and 5.5.36 below (together with Remark 5.5.34) show that all these morphisms are flat. There are also evident isomorphisms of $R((u))$-algebras

(5.5.22) $$R((u)) \otimes_{\varphi, R((u))} A_m \xrightarrow{\sim} A_{m+1},$$

defined by mapping $1 \otimes x_i$ in the source to x_i in the target, as well as isomorphisms

(5.5.23) $$R((u)) \otimes_{\varphi, R((u))} B_{m+1} \xrightarrow{\sim} B_m,$$

defined by mapping $1 \otimes x_i$ in the source to x_i^p in the target. (It is less evident that the morphisms (5.5.23) are isomorphisms, but this follows from Lemma 5.5.35 below.) Note also that $A_0 = B_0$.

We write $A := \varinjlim_m A_m$. Being the direct limit of a sequence of Noetherian rings with respect to flat transition morphisms, we see immediately that A is a coherent ring. In fact, something stronger is true: A is a Noetherian local $R((u))$-algebra, and there is an isomorphism

$$\widehat{A} \xrightarrow{\sim} \widehat{R((u))},$$

where now the hats indicate $\mathfrak{m}$-adic completion. To see this, it suffices to consider the case when the ideal I is zero. (Standard arguments with adic completions of finite type modules over Noetherian rings, and with direct limits, show that the constructions of the various rings A_m, A, $\widehat{A}$, and $\widehat{R((u))}$ are compatible with the passage from R to R/J, for any ideal J of R. Now apply this observation to the ideal I in the ring $K[[T_1, \ldots, T_n]]$.) In this case, we see that A is the ring of germs of holomorphic functions at the origin of the unit n-dimensional polydisk over the complete discretely valued field $K((u))$, and this ring is well-known to be Noetherian, and to have the formal power series ring $K((u))[[T_1, \ldots, T_n]]$ as its $\mathfrak{m}$-adic completion (see [BGR84, Prop. 7, §7.3.2]).

The discussion of the preceding paragraph shows that the embedding $A \hookrightarrow \widehat{R((u))}$ is faithfully flat, and since $\widehat{M}$ (which, by the Artin–Rees lemma, may be identified with $\widehat{R((u))} \otimes_{R((u))} M$) is projective, we conclude that

$$A \otimes_{R((u))} M$$

is also projective over A. This in turn implies that

$$A_m \otimes_{R((u))} M$$

is projective over A_m, for some sufficiently large value of m. (Indeed, this projectivity is witnessed by a split surjection from a finite free A-module, and both this surjection, and a splitting of the surjection, can be descended to some A_m.)

Composing the inclusion $A_{m-1} \to R((u)) \otimes_{\varphi,R((u))} A_{m-1}$, defined via $a \mapsto 1 \otimes a$, with the isomorphism (5.5.22), we may regard A_m as a faithfully flat A_{m-1}-algebra. (Note that this is not the same as the $R((u))$-linear map $A_{m-1} \to A_m$ considered in (5.5.21).) We then compute that

$$A_m \otimes_{A_{m-1}} (A_{m-1} \otimes_{R((u))} M) \xrightarrow{\sim} A_m \otimes_{\varphi,R((u))} M$$
$$\xrightarrow{\sim} A_m \otimes_{R((u))} (R((u)) \otimes_{\varphi,R((u))} M) \xrightarrow{\sim} A_m \otimes_{R((u))} M$$

(the last isomorphism following from the étale φ-module structure on M). Since the property of being finitely generated projective can be detected after a faithfully flat base-change, we find that $A_{m-1} \otimes_{R((u))} M$ is projective. Continuing via descending induction, we conclude that $A_0 \otimes_{R((u))} M$ is a projective A_0-module, or equivalently, a projective B_0-module. A completely analogous argument, taking into account (5.5.23), then shows that $B_m \otimes_{R((u))} M$ is projective for each $m \geq 0$.

Write $B := \varprojlim_m B_m$. Since the transition morphisms in this projective limit are flat morphisms of Noetherian Banach algebras over $K((u))$, we find that that B is a Fréchet–Stein algebra, in the sense of [ST03]. As a consequence, any finitely presented B-module N is coadmissible over B (by [ST03, Cor. 3.4]), so that the natural morphism $N \to \varprojlim_m B_m \otimes_B N$ is an isomorphism. In particular, if X is a finitely generated (and hence finitely presented) $R((u))$-module, then $B \otimes_{R((u))} X$ is a finitely presented B-module, and hence there is an isomorphism

$$(5.5.24) \qquad B \otimes_{R((u))} X \xrightarrow{\sim} \varprojlim_m B_m \otimes_{R((u))} X.$$

If we take X to be a finitely generated ideal I of $R((u))$, then (5.5.24) shows that the morphism

$$(5.5.25) \qquad B \otimes_{R((u))} I \to B$$

may be written as the projective limit of morphisms $B_m \otimes_{R((u))} I \to B_m$. By Lemma 5.5.32 below, each B_m is flat over $R((u))$, so we find that each of these latter morphisms is injective. Thus so is the morphism (5.5.25). It follows that B is flat over $R((u))$.

In fact B is faithfully flat over $R((u))$. To show this, it suffices (given the flatness that we have already proved, and the fact that flat maps satisfy *going down*) to show that each maximal ideal of $R((u))$ is obtained via restriction from a maximal ideal of B. Any such maximal ideal is the kernel of a surjection

$$(5.5.26) \qquad R((u)) \to L,$$

where L is a finite extension of $K((u))$ (see the proof of [dJ95, Lem. 7.1.9] for a proof of this fact), and each T_i maps to an element $a_i \in L$ satisfying $|a_i| < 1$ (i.e. each a_i lies in the maximal ideal of the ring of integers of L). If we choose m so that $|a_i| \leq |u|^{1/p^m}$ for each a_i, then the surjection (5.5.26) extends to a surjection $B_m \to L$. (This extension of the surjection $R((u)) \to L$ to a surjection $B_m \to L$ for some sufficiently large value of m is also explained carefully in the proof of [dJ95,

Lem. 7.1.9].) The kernel of the composite $B \to B_m \to L$ is then a maximal ideal of B that restricts to the kernel of (5.5.26).

Since M is finitely generated over $R((u))$, we obtain, from (5.5.24), an isomorphism

$$B \otimes_{R((u))} M \xrightarrow{\sim} \varprojlim_m B_m \otimes_{R((u))} M;$$

since each $B_m \otimes_{R((u))} M$ is projective over B_m, Lemma 5.5.28 below shows that $B \otimes_{R((u))} M$ is projective over B. Since $R((u)) \to B$ is faithfully flat, we find that M is projective, as required.

Finally, consider the general case of the theorem, in which R is assumed to be a complete Noetherian local $\mathcal{O}/\varpi^a$-algebra, without a necessarily equalling 1. Endow each of $\mathcal{O}/\varpi^a$, R, $R((u))$, $\widehat{R((u))}$, M, and $\widehat{M}$ with its ϖ-adic filtration, and let $\mathrm{Gr}^\bullet \mathcal{O}/\varpi^a$, etc., denote the corresponding associated graded object. We find that $\mathrm{Gr}^\bullet \mathcal{O}/\varpi^a = k[\varepsilon]/(\varepsilon^a)$, that $\mathrm{Gr}^\bullet R$ is a complete local Noetherian algebra over $k[\varepsilon]/(\varepsilon^a)$, that $\mathrm{Gr}^\bullet R((u)) = (\mathrm{Gr}^\bullet R)((u))$, and that $\mathrm{Gr}^\bullet M$ is an étale φ-module over $\mathrm{Gr}^\bullet R$.

Since $\mathfrak{m}$-adic completion of finitely generated modules over a Noetherian local ring (such as $R((u))$) is exact, we also see that the $\mathfrak{m}$-adic completion of $\mathrm{Gr}^\bullet R((u))$ is naturally isomorphic to $\mathrm{Gr}^\bullet \widehat{R((u))}$, and that the $\mathfrak{m}$-adic completion of $\mathrm{Gr}^\bullet M$ is naturally isomorphic to $\mathrm{Gr}^\bullet \widehat{M}$. The assumption that $\widehat{M}$ is finitely generated and projective over $\widehat{R((u))}$ then implies that $\widehat{\mathrm{Gr}^\bullet M} = \mathrm{Gr}^\bullet \widehat{M}$ is finitely generated and projective over $\widehat{\mathrm{Gr}^\bullet R((u))} = \mathrm{Gr}^\bullet \widehat{R((u))}$. (This is easily seen if one uses the equivalence between a module being finitely generated and projective and being a direct summand of a finite rank free module.) The case of the theorem already proved then shows that $\mathrm{Gr}^\bullet M$ is finitely generated and projective over $\mathrm{Gr}^\bullet R((u))$, or, equivalently, finitely generated and flat over $\mathrm{Gr}^\bullet R((u))$. Lemma 5.5.37 (applied with $R = \mathcal{O}/\varpi^a$ and $A = R((u))$) shows that M is finitely generated and flat — and thus projective — over $R((u))$. $\square$

The following lemmas, which were used in the proof of the preceding theorem, are presumably well-known to experts, but we include proofs, for lack of a reference.

5.5.27. Lemma. *If B is a commutative Fréchet–Stein algebra (over a complete discretely valued field K), if M is a finitely presented B-module, and if N is a coadmissible B-module, then $M \otimes_B N$ is again a coadmissible B-module.*

Proof. If we choose a presentation $B^r \to B^s \to M \to 0$, then we obtain a right exact sequence

$$N^r \to N^s \to M \otimes_B N \to 0.$$

Since N is coadmissible, so are each of N^r and N^s, and thus so is $M \otimes_B N$, being the cokernel of a morphism between coadmissible B-modules. $\square$

5.5.28. Lemma. *If B is a commutative Fréchet–Stein algebra over a complete discretely valued field K, say $B \xrightarrow{\sim} \varprojlim_n B_n$, where each B_n is a Noetherian Banach K-algebra, with the transition morphisms being flat, and if M is a finitely presented B-module with the property that each tensor product $B_n \otimes_B M$ is a projective B_n-module, then M is a projective B-module.*

Proof. Since M is a finitely presented B-module, it is furthermore projective if and only it is flat. To show that M is flat, it suffices to show that for each finitely generated ideal $I \subseteq B$, the induced morphism $M \otimes_B I \to M$ is injective.

Since I is finitely generated, it is the image of a morphism $B^r \to B$, for some $r \geq 0$, and thus is coadmissible. Lemma 5.5.27 then shows that $M \otimes_B I$ is coadmissible. The B-module M itself is also coadmissible (being finitely presented). Thus the morphism

$$(5.5.29) \qquad\qquad M \otimes_B I \to M$$

may be obtained as the projective limit of the morphisms

$$B_n \otimes_B \otimes M \otimes_B I \to B_n \otimes_B M.$$

We may rewrite each of these morphism as

$$(5.5.30) \qquad (B_n \otimes_B \otimes M) \otimes_{B_n} (B_n \otimes_B I) \to B_n \otimes_B M.$$

Since B_n is flat over B, we see that the inclusion of I in B induces a sequence of injections $B_n \otimes_B I \hookrightarrow B_n$. Since $B_n \otimes_B M$ is projective, and thus flat, over B_n, the morphisms (5.5.30) are then also injective. Hence so is their projective limit (5.5.29). $\qquad\square$

5.5.31. Lemma. *Let A be a ring, and a an element of A. If $M \to N$ is a morphism of A-modules whose kernel and cokernel are each annihilated by some power of a, then the induced morphism $\widehat{M}[1/a] \to \widehat{N}[1/a]$ (where $\widehat{}$ denotes a-adic completion) is an isomorphism.*

Proof. This is standard, and follows easily from the definitions. $\qquad\square$

5.5.32. Lemma. *Let A be a Noetherian ring, and let $a, b_1, \ldots, b_m$ be elements of A. If we write $B := A[x_1, \ldots, x_m]/(ax_1 - b_1, \ldots, ax_m - b_m)$, then the natural map $\widehat{A}[1/a] \to \widehat{B}[1/a]$ (where $\widehat{}$ denotes a-adic completion) is flat.*

Proof. We begin with the case $m = 1$, where we write x, y for x_1, y_1, and b for b_1. Note that each morphism in the sequence of natural morphisms

$$\widehat{A} \to \widehat{A}[x]/(ax - b) \to \widehat{A}[1/a]$$

becomes an isomorphism after inverting a. Indeed, this is evidently the case for their composite, and it is also evident that the first morphism becomes surjective after inverting a. Similarly, each morphism in the sequence of natural morphisms

$$B := A[x]/(ax - b) \to \widehat{A} \otimes_A B = \widehat{A}[x]/(ax - b) \to \widehat{B}$$

becomes an isomorphism after passing to a-adic completions. Thus, in order to prove the lemma in the case $m = 1$, it suffices to note that the natural morphism from $\widehat{A}[x]/(ax - b)$ to its a-adic completion is flat, as follows from the Artin–Rees lemma (and the fact that $\widehat{A}$ is Noetherian, as A is). (Note that by the discussion about, the natural map $\widehat{A}[1/a] \to \widehat{B}[1/a]$ is obtained from this map by inverting a.)

The general case follows by induction on m. Indeed, writing

$$C = A[x_1, \ldots, x_{m-1}]/(ax_1 - b_1, \ldots, ax_{m-1} - b_{m-1}),$$

we can factor $\widehat{A}[1/a] \to \widehat{B}[1/a]$ as $\widehat{A}[1/a] \to \widehat{C}[1/a] \to \widehat{B}[1/a]$, with the first map being flat by the inductive hypothesis, and the second being flat by the case $m = 1$, as $B = C[x_m]/(ax_m - b_m)$. $\qquad\square$

The following result is a somewhat technical modification of the preceding lemma.

5.5.33. Lemma. *Let A be a Noetherian ring, and let a, a', $b_1, \ldots, b_m$, $b'_1, \ldots, b'_m$ be elements of A. If we write $B := A[x_1, \ldots, x_m]/(ax_1 - b_1 b'_1, \ldots, ax_m - b_m b'_m)$ and $C := A[y_1, \ldots, y_m]/(aa'y_1 - b_1, \ldots, aa'y_m - b_m)$, then there is a morphism of A-algebras $B \to C$ defined by mapping each x_i to $a'b'_i y_i$, and the induced morphism $\widehat{B}[1/aa'] \to \widehat{C}[1/aa']$ (where $\widehat{}$ denotes aa'-adic completion) is flat.*

Proof. We factor the morphism $B \to C$ as

$$B \to B[y_1, \ldots, y_m]/(aa'y_1 - b_1, \ldots, aa'y_m - b_m)$$
$$\to B[y_1, \ldots, y_m]/(aa'y_1 - b_1, x_1 - a'b'_1 y_1, \ldots, aa'y_m - b_m, x_m - a'b'_m y_m) = C.$$

The second morphism is surjective, and its kernel, which is the ideal $(x_1 - a'b'_1 y_1, \ldots, x_n - a'b'_m y_m)$, is annihilated by a. Lemma 5.5.31 shows that this second morphism becomes an isomorphism after aa'-adically completing and then inverting aa'. Thus it suffices to show that the first morphism becomes flat after aa'-adically completing and inverting aa'; this follows from Lemma 5.5.32. $\square$

5.5.34. *Remark.* We note, in the context of the preceding lemma, that if $(a')^r = a^s$ for some $r, s \geq 1$, then aa'-adically completing is the same as a-adically completing, and inverting aa' is the same as inverting a.

We also have the following variations on the preceding results.

5.5.35. Lemma. *Let A be a Noetherian ring, and let $a, b_1, \ldots, b_m$ be elements of A, and let n be a positive integer. If we write $B := A[x_1, \ldots, x_n]/(a^n x_1 - b_1^n, \ldots, a^n x_n - b_m^n)$ and $C = A[y_1, \ldots, y_n]/(ay_1 - b_1, \ldots ay_n - b_m)$, then the morphism of A-algebras $B \to C$ defined by $x_i \mapsto y_i^n$ induces an isomorphism $\widehat{B}[1/a] \xrightarrow{\sim} \widehat{C}[1/a]$ (where $\widehat{}$ denotes a-adic completion).*

Proof. The morphism $B \to C$ is finite: C is generated as a B-module by the various monomials $y_1^{e_1} \cdots y_m^{e_m}$, for $1 \leq e_i \leq n - 1$. The image of y_i^j in the cokernel of this morphism is annihilated by a^j, so the entire cokernel is annihilated by $a^{m(n-1)}$.

Note that the morphisms $A[1/a] \to B[1/a] \to C[1/a]$ are all isomorphisms, so that the kernel of the morphism $B \to C$ is contained in the kernel of the morphism $B \to B[1/a]$. Each element of this kernel is annihilated by some power of a. Since this kernel is finitely generated (as B is Noetherian), we see that this entire kernel is annihilated by some power of a. Combining this with the conclusion of the preceding paragraph, and with Lemma 5.5.31, establishes the lemma. $\square$

5.5.36. Lemma. *Let A be a Noetherian ring, let $a, b_1, \ldots, b_m$ be elements of A, and let n be a positive integer. If we write $B := A[x_1, \ldots, x_m]/(ax_1 - b_1^n, \ldots, ax_m - b_m^n)$ and $C = A[y_1, \ldots, y_m]/(ay_1 - b_1, \ldots, ay_m - b_m)$, then the morphism of A-algebras $B \to C$ defined by $x_i \mapsto a^{n-1} y_i^n$ induces a flat morphism $\widehat{B}[1/a] \xrightarrow{\sim} \widehat{C}[1/a]$ (where $\widehat{}$ denotes a-adic completion).*

Proof. We factor the morphism $B \to C$ as

$$B = A[x_1, \ldots, x_m]/(ax_1 - b_1^n, \ldots, ax_m - b_m^n)$$
$$\to A[t_1, \ldots, t_m]/(a^n t_1 - b_1^n, \ldots, a^n t_m - b_m^n)$$
$$\to A[y_1, \ldots, y_m]/(ay_1 - b_1, \ldots, ay_m - b_m) = C,$$

where the first morphism is defined by $x_i \mapsto a^{n-1}t_i$, and the second morphism is defined by $t_i \mapsto y_i^n$. The present lemma then follows from Lemmas 5.5.33 and 5.5.35 (taking into account Remark 5.5.34). □

Let R be an Artinian local ring, with maximal ideal I and residue field k. If M is an R-module, then we let $\mathrm{Gr}^\bullet M$ denote the graded k-vector space associated to the I-adic filtration on M (so $\mathrm{Gr}^i M := I^i M / I^{i+1} M$).

5.5.37. Lemma. *If A is an R-algebra, then an A-module M is (faithfully) flat over A if and only if $\mathrm{Gr}^\bullet M$ is (faithfully) flat over $\mathrm{Gr}^\bullet A$. Furthermore, if any of these conditions holds, then the natural morphism $\mathrm{Gr}^\bullet A \otimes_{\mathrm{Gr}^0 A} \mathrm{Gr}^0 M \to \mathrm{Gr}^\bullet M$ is an isomorphism.*

Proof. If M is flat over A, then considering the result of tensoring M by the various short exact sequences

$$0 \to I^n A \to I^m A \to I^m A / I^n A \to 0$$

we find that the natural morphism $\mathrm{Gr}^i A \otimes_{\mathrm{Gr}^0 A} \mathrm{Gr}^0 M \to \mathrm{Gr}^i M$ is an isomorphism, for each i. This proves the final assertion of the lemma. Furthermore, since (faithful) flatness is preserved under base-change, we see first that $\mathrm{Gr}^0 M$ is flat over $\mathrm{Gr}^0 A$ (and faithfully flat if M is faithfully flat over A), and then (using the result already proved) that $\mathrm{Gr}^\bullet M$ is flat over $\mathrm{Gr}^\bullet A$ (and faithfully flat if M is).

It is not quite as obvious that flatness of $\mathrm{Gr}^\bullet M$ over $\mathrm{Gr}^\bullet A$ implies the flatness of M over A, but this is a standard fact in commutative algebra; e.g. it follows from [Sta, Tag 0AS8]. (If $i \geq 0$, then base-changing via the map $\mathrm{Gr}^\bullet A \to \mathrm{Gr}^{\leq i} A := A/IA \oplus I/I^2 \oplus \cdots \oplus I^i/I^{i+1}$, we find that $\mathrm{Gr}^{\leq i} M := M/IM \oplus \cdots \oplus I^i M / I^{i+1} M$ is flat over $\mathrm{Gr}^{\leq i} A$. In particular, the embedding $I^i/I^{i+1} A =: \mathrm{Gr}^i A \hookrightarrow \mathrm{Gr}^{\leq i} A$ induces an embedding

$$I^i/I^{i+1} \otimes_A M \xrightarrow{\sim} \mathrm{Gr}^i A \otimes_{\mathrm{Gr}^{\leq i} A} \mathrm{Gr}^{\leq i} M;$$

concretely, this means that the morphism

$$I^i/I^{i+1} \otimes_A M \to I^i M / I^{i+1} M$$

is an embedding. Letting i vary, and recalling that I is nilpotent, we deduce from [Sta, Tag 0AS8] that M is flat over A.)

If $\mathrm{Gr}^\bullet M$ is furthermore faithfully flat over $\mathrm{Gr}^\bullet A$, then (since faithful flatness is preserved under base-change) we see that $\mathrm{Gr}^0 M := M/IM$ is faithfully flat over $\mathrm{Gr}^0 A := A/IA$. Because I is nilpotent, an A-module vanishes if and only if its reduction mod I does, and we conclude that M is faithfully flat over A. □

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