Patching and the p-adic Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$

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Abstract

We present a new construction of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ via the patching method of Taylor–Wiles and Kisin. This construction sheds light on the relationship between the various other approaches to both the local and global aspects of the p-adic Langlands program; in particular, it gives a new proof of many cases of the second author's local-global compatibility theorem, and relaxes a hypothesis on the local mod p representation in that theorem.

1. Introduction

The primary goal of this paper is to explain how (under mild technical hypotheses) the patching construction of [CEG+16], when applied to the group $\mathrm{GL}_2(\mathbb{Q}_p)$, gives rise to the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, as constructed in [Col10b], and as further analyzed in [Paš13] and [CDP14]. As a by-product, we obtain a new proof of many cases of the local-global compatibility theorem of [Eme11] (and of some cases not treated there).

1.1 Background

We start by recalling the main results of [CEG+16] and the role we expect them to play in the (hypothetical) p-adic local Langlands correspondence. Let F be a finite extension of $\mathbb{Q}_p$, and let G_F be its absolute Galois group. One would like to have an analogue of the local Langlands correspondence for all finite-dimensional, continuous, p-adic representations of G_F. Let E be another finite extension of $\mathbb{Q}_p$, which will be our field of coefficients, assumed large enough, with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field $\mathbb{F}$. To a continuous Galois representation $r : G_F \to \mathrm{GL}_n(E)$ one would like to attach an admissible unitary E-Banach space representation $\Pi(r)$ of $G := \mathrm{GL}_n(F)$ (or possibly a family of such Banach space representations). Ideally, such a construction should be compatible with deformations, should encode the classical local Langlands correspondence and should be compatible with a global p-adic correspondence, realized in the completed cohomology of locally symmetric spaces.

It is expected that the Banach spaces $\Pi(r)$ should encode the classical local Langlands

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correspondence in the following way: if r is potentially semi-stable with regular Hodge–Tate weights, then the subspace of locally algebraic vectors $\Pi(r)^{\mathrm{l.alg}}$ in $\Pi(r)$ should be isomorphic to $\pi_{\mathrm{sm}}(r) \otimes \pi_{\mathrm{alg}}(r)$ as a G-representation, where $\pi_{\mathrm{sm}}(r)$ is the smooth representation of G corresponding via classical local Langlands to the Weil–Deligne representation obtained from r by Fontaine's recipe, and $\pi_{\mathrm{alg}}(r)$ is an algebraic representation of G, whose highest weight vector is determined by the Hodge–Tate weights of r.

1.2 Example. If $F = \mathbb{Q}_p$, $n = 2$ and r is crystalline with Hodge–Tate weights $a < b$, then $\pi_{\mathrm{sm}}(r)$ is a smooth unramified principal series representation, whose Satake parameters can be calculated in terms of the trace and determinant of Frobenius on $D_{\mathrm{cris}}(r)$, and $\pi_{\mathrm{alg}}(r) = \mathrm{Sym}^{b-a-1} E^2 \otimes \det^{1-a}$. (We note that in the literature different normalisations lead to different twists by a power of det.)

Such a correspondence has been established in the case of $n = 2$ and $F = \mathbb{Q}_p$ by the works of Breuil, Colmez and others, see [Bre08], [Col10a] as well as the introduction to [Col10b]. Moreover, when $n = 2$ and $F = \mathbb{Q}_p$, this correspondence has been proved (in most cases) to satisfy local-global compatibility with the p-adically completed cohomology of modular curves, see [Eme11]. However, not much is known beyond this case. In [CEG$^+$16] we have constructed a candidate for such a correspondence using the Taylor–Wiles–Kisin patching method, which has been traditionally employed to prove modularity lifting theorems for Galois representations. We now describe the end product of the paper [CEG$^+$16].

Let $\bar{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation and let $R_p^{\square}$ be its universal framed deformation ring. Under the assumption that p does not divide $2n$ we construct an $R_\infty[G]$-module M_∞, which is finitely generated as a module over the completed group algebra $R_\infty[[\mathrm{GL}_n(\mathcal{O}_F)]]$, where R_∞ is a complete local noetherian $R_p^{\square}$-algebra with residue field $\mathbb{F}$. If $y \in \operatorname{Spec} R_\infty$ is an E-valued point then

$$\Pi_y := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M_\infty \otimes_{R_\infty, y} \mathcal{O}, E)$$

is an admissible unitary E-Banach space representation of G. The composition $R_p^{\square} \to R_\infty \xrightarrow{y} E$ defines an E-valued point $x \in \operatorname{Spec} R_p^{\square}$ and thus a continuous Galois representation $r_x : G_F \to \mathrm{GL}_n(E)$. We expect that the Banach space representation Π_y depends only on x and that it should be related to r_x by the hypothetical p-adic Langlands correspondence; see §6 of [CEG$^+$16] for a detailed discussion. We show in [CEG$^+$16, Theorem 4.35] that if $\pi_{\mathrm{sm}}(r_x)$ is generic and x lies on an automorphic component of a potentially crystalline deformation ring of $\bar{r}$ then $\Pi_y^{\mathrm{l.alg}} \cong \pi_{\mathrm{sm}}(r_x) \otimes \pi_{\mathrm{alg}}(r_x)$ as expected; moreover, the points x such that $\pi_{\mathrm{sm}}(r_x)$ is generic are Zariski dense in every irreducible component of a potentially crystalline deformation ring. (It is expected that every irreducible component of a potentially crystalline deformation ring is automorphic; this expectation is motivated by the Fontaine–Mazur and Breuil–Mézard conjectures. However, it is intrinsic to our method that we would not be able to access these non-automorphic components even if they existed.)

However, there are many natural questions regarding our construction for $\mathrm{GL}_n(F)$ that we cannot answer at the moment and that appear to be genuinely deep, as they are intertwined with questions about local-global compatibility for p-adically completed cohomology, with the Breuil–Mézard conjecture on the geometry of local deformation rings and with the Fontaine–Mazur conjecture for global Galois representations. For example, it is not clear that Π_y depends only on x, it is not clear that Π_y is non-zero for an arbitrary y, and that furthermore $\Pi_y^{\mathrm{l.alg}}$ is non-zero if r_x is potentially semistable of regular weight, and it is not at all clear that M_∞ does not depend on the different choices made during the patching process.

1.3 The present paper

In this paper, we specialize the construction of [CEG+16] to the case $F = \mathbb{Q}_p$ and $n = 2$ (so that $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and $K := \mathrm{GL}_2(\mathbb{Z}_p)$ from now on) to confirm our expectation that, firstly, M_∞ does not depend on any of the choices made during the patching process and, secondly, that it does recover the p-adic local Langlands correspondence as constructed by Colmez.

We achieve the first part *without* appealing to Colmez's construction (which relies on the theory of (φ, Γ)-modules). The proof that M_∞ is uniquely determined highlights some key features of the $\mathrm{GL}_2(\mathbb{Q}_p)$ setting beyond the use of (φ, Γ)-modules: the classification of irreducible mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ in terms of Serre weights and Hecke operators, and the fact that the Weil–Deligne representation and the Hodge–Tate weights determine a (irreducible) 2-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ uniquely (up to isomorphism).

When combined with the results of [Paš13] (which *do* rely on Colmez's functor $\check{\mathbf{V}}$), we obtain that M_∞ realizes the p-adic Langlands correspondence as constructed by Colmez.

We also obtain a new proof of local-global compatibility, which helps clarify the relationship between different perspectives and approaches to p-adic local Langlands.

1.4 Arithmetic actions

In the body of the paper we restrict the representations $\bar{r}$ we consider by assuming that they have only scalar endomorphisms, so that $\mathrm{End}_{G_{\mathbb{Q}_p}}(\bar{r}) = \mathbb{F}$, and that $\bar{r} \not\cong \left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$ for any character $\chi : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$. For simplicity, let us assume in this introduction that $\bar{r}$ is irreducible and let R_p be its universal deformation ring. Then $R_p^\square$ is formally smooth over R_p. Moreover, (as $F = \mathbb{Q}_p$ and $n = 2$) we may also assume that R_∞ is formally smooth over $R_p^\square$, and thus over R_p.

The following definition is meant to axiomatize the key properties of the patched module M_∞.

1.5 Definition. Let d be a non-negative integer, let $R_\infty := R_p[[x_1, \ldots, x_d]]$ and let M be a non-zero $R_\infty[G]$-module. We say that the action of R_∞ on M is *arithmetic* if the following conditions hold:

(AA1) M is a finitely generated module over the completed group algebra $R_\infty[[K]]$;

(AA2) M is projective in the category of pseudo-compact $\mathcal{O}[[K]]$-modules;

(AA3) for each pair of integers $a < b$, the action of R_∞ on

$$M(\sigma^\circ) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M, (\sigma^\circ)^d)^d$$

factors through the action of $R_\infty(\sigma) := R_p(\sigma)[[x_1, \ldots, x_d]]$. Here $R_p(\sigma)$ is the quotient of R_p constructed by Kisin, which parameterizes crystalline representations with Hodge–Tate weights (a, b), σ° is a K-invariant $\mathcal{O}$-lattice in $\sigma := \mathrm{Sym}^{b-a-1} E^2 \otimes \det^{1-a}$ and $(*)^d := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(*, \mathcal{O})$ denotes the Schikhof dual.
Moreover, $M(\sigma^\circ)$ is maximal Cohen–Macaulay over $R_\infty(\sigma)$ and the $R_\infty(\sigma)[1/p]$-module $M(\sigma^\circ)[1/p]$ is locally free of rank 1 over its support.

(AA4) for each σ as above and each maximal ideal y of $R_\infty[1/p]$ in the support of $M(\sigma^\circ)$, there is a non-zero G-equivariant map

$$\pi_{\mathrm{sm}}(r_x) \otimes \pi_{\mathrm{alg}}(r_x) \to \Pi_y^{\mathrm{l.alg}}$$

where x is the image of y in $\mathrm{Spec}\, R_p$.

The last condition says that M encodes the classical local Langlands correspondence. This is what motivated us to call such actions arithmetic. (In fact in the main body of the paper we use a reformulation of condition (AA4), see §3.1 and Remark 3.3.) To motivate (AA3), we note for the sake of the reader familiar with Kisin's proof of the Fontaine–Mazur conjecture [Kis09] that the modules $M(\sigma^\circ)$ are analogues of the patched modules denoted by M_∞ in [Kis09], except that Kisin patches algebraic automorphic forms for definite quaternion algebras and in this paper we will ultimately be making use of patching arguments for algebraic automorphic forms on forms of $U(2)$.

1.6 Uniqueness of M_∞

As already mentioned, the patched module M_∞ of [CEG$^+$16] carries an arithmetic action of R_∞ for some d. In order to prove that M_∞ is uniquely determined, it is enough to show that for any given d, any $R_\infty[G]$-module M with an arithmetic action of R_∞ is uniquely determined. The following is our main result, which for simplicity we state under the assumption that $\bar{r}$ is irreducible.

1.7 Theorem. *Let M be an $R_\infty[G]$-module with an arithmetic action of R_∞.*

(i) If π is any irreducible G-subrepresentation of the Pontryagin dual $M^\vee$ of M then π is isomorphic to the representation of G associated to $\bar{r}$ by the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

(ii) Let $\pi \hookrightarrow J$ be an injective envelope of the above π in the category of smooth locally admissible representations of G on $\mathcal{O}$-torsion modules. Let $\widetilde{P}$ be the Pontryagin dual of J. Then $\widetilde{P}$ carries a unique arithmetic action of R_p and, moreover,

$$M \cong \widetilde{P}\,\widehat{\otimes}_{R_p} R_\infty$$

as $R_\infty[G]$-modules.

The theorem completely characterizes modules with an arithmetic action and shows that M_∞ does not depend on the choices made in the patching process. A further consequence is that the Banach space Π_y depends only on the image of y in $\operatorname{Spec} R_p$, as expected.

Let us sketch the proof of Theorem 1.7 assuming for simplicity that $d = 0$. The first step is to show that $M^\vee$ is an injective object in the category of smooth locally admissible representations of G on $\mathcal{O}$-torsion modules and that its G-socle is isomorphic to π. This is done by computing $\mathrm{Hom}_G(\pi', M^\vee)$ and showing that $\mathrm{Ext}^1_G(\pi', M^\vee_\infty)$ vanishes for all irreducible $\mathbb{F}$-representations π' of G; see Proposition 4.2 and Theorem 4.15. The arguments here use the foundational results of Barthel–Livné [BL94] and Breuil [Bre03a] on the classification of irreducible mod p representations of G, arguments related to the weight part of Serre's conjecture, and the fact that the rings $R_p(\sigma)$ are formally smooth over $\mathcal{O}$, whenever σ is of the form $\mathrm{Sym}^{b-a-1}E^2 \otimes \det^{1-a}$ with $1 \leqslant b - a \leqslant p$.

This first step allows us to conclude that $M^\vee$ is an injective envelope of π, which depends only on $\bar{r}$. Since injective envelopes are unique up to isomorphism, we conclude that any two modules with an arithmetic action of R_p are isomorphic as G-representations. Therefore, it remains to show that any two arithmetic actions of R_p on $\widetilde{P}$ coincide. As R_p is $\mathcal{O}$-torsion free, it is enough to show that two such actions induce the same action on the unitary E-Banach space $\Pi := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M, E)$. Since M is a projective $\mathcal{O}[[K]]$-module by (AA2) one may show using the "capture" arguments that appear in [CDP14, §2.4], [Eme11, Prop. 5.4.1] that the subspace of K-algebraic vectors in Π is dense. Since the actions of R_p on Π are continuous it is enough to show that they agree on this dense subspace. Since the subspace of K-algebraic vectors is semi-simple as a K-representation, it is enough to show that the two actions agree on σ-isotypic subspaces in Π for all irreducible algebraic K-representations σ. These are precisely the representations σ in axiom (AA3). Taking duals one more time, we are left with showing that any two arithmetic actions induce the same action of R_p on $M(\sigma^\circ)[1/p]$ for all σ as above.

At this point we use another special feature of 2-dimensional crystalline representations of $G_{\mathbb{Q}_p}$: the associated Weil–Deligne representation together with Hodge–Tate weights determine a 2-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ up to isomorphism. Using this fact and axioms (AA3) and (AA4) for the arithmetic action we show that the action of the Hecke algebra $\mathcal{H}(\sigma) := \mathrm{End}_G(\text{c-Ind}_K^G \sigma)$ on $M(\sigma^\circ)[1/p]$ completely determines the action of $R_p(\sigma)$ on $M(\sigma^\circ)[1/p]$; see the proof of Theorem 4.30 as well as the key Proposition 2.13. Since the action of $\mathcal{H}(\sigma)$ on $M(\sigma^\circ)[1/p]$ depends only on the G-module structure of M, we are able to conclude that the two arithmetic actions are the same. The reduction from the case when d is arbitrary to the case when $d = 0$ is carried out in §4.16.

1.8 Remark. As we have already remarked, the arguments up to this point make no use of (φ, Γ)-modules. Indeed the proof of Theorem 1.7 does not use them. One of the objectives of this project was to find out how much of the p-adic Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$ correspondence can one recover from the patched module M_∞ without using Colmez's functors, as these constructions are not available for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$, while our patched module M_∞ is. Along the same lines, in section §5 we show that to a large extent we can recover a fundamental theorem of Berger–Breuil [BB10] on the uniqueness of unitary completions of locally algebraic principal series without making use of (φ, Γ)-modules, see Theorem 5.1 and Remark 5.3.

1.9 Remark. As already explained, Theorem 1.7 implies that Π_y depends only on the image of y in $\mathrm{Spec}\, R_p$. However, we are still not able to deduce using only our methods that Π_y is non-zero for an arbitrary $y \in \text{m-Spec}\, R_\infty[1/p]$. Since M_∞ is not a finitely generated module over R_∞ theoretically it could happen that $\Pi_y \neq 0$ for a dense subset of $\text{m-Spec}\, R_\infty[1/p]$, but $\Pi_y = 0$ at all other maximal ideals. We can only prove that this pathological situation does not occur after combining Theorem 1.7 with the results of [Paš13].

In §6 we relate the arithmetic action of R_p on $\widetilde{P}$ to the results of [Paš13], where an action of R_p on an injective envelope of π in the subcategory of representations with a fixed central character is constructed using Colmez's functor; see Theorem 6.18. Then by appealing to the results of [Paš13] we show that Π_y and r_x correspond to each other under the p-adic Langlands correspondence as defined by Colmez [Col10b], for all $y \in \text{m-Spec}\, R_\infty[1/p]$, where x denotes the image of y in $\text{m-Spec}\, R_p[1/p]$.

It follows from the construction of M_∞ that after quotienting out by a certain ideal of R_∞ we obtain a dual of completed cohomology, see [CEG+16, Corollary 2.11]. This property combined with Theorem 1.7 and with the results in §6 enables us to obtain a new proof of local-global compatibility as in [Eme11] as well as obtaining a genuinely new result, when $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is isomorphic to $\left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$, where ω is the cyclotomic character modulo p. (See Remark 7.7.)

1.10 Prospects for generalization

Since our primary goal in this paper is to build some new connections between various existing ideas related to the p-adic Langlands program, we have not striven for maximal generality, and we expect that some of our hypotheses on $\bar{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ could be relaxed. In particular, it should be possible to prove results when $p = 2$ by using results of Thorne [Tho] to redo the patching in [CEG$^+$16]. It may also be possible to extend our results to cover more general $\bar{r}$ (recall that we assume that $\bar{r}$ has only scalar endomorphisms, and that it is not a twist of an extension of the trivial character by the mod p cyclotomic character). In Section 6.27 we discuss the particular case where $\bar{r}$ has scalar semisimplification; as this discussion (and the arguments of [Paš13]) show, while it may well be possible to generalise our arguments, they will necessarily be considerably more involved in cases where $\bar{r}$ does not satisfy the hypotheses that we have imposed.

Since the patching construction in [CEG$^+$16] applies equally well to the case of $\mathrm{GL}_2(F)$ for any finite extension $F/\mathbb{Q}_p$, or indeed to $\mathrm{GL}_n(F)$, it is natural to ask whether any of our arguments can be extended to such cases (where there is at present no construction of a p-adic local Langlands correspondence). As explained in Remark 3.2, the natural analogues of our axioms (AA1)-(AA4) hold, even in the generality of $\mathrm{GL}_n(F)$. Unfortunately, the prospects for proving analogues of our main theorems are less rosy, as it seems that none of the main inputs to our arguments will hold. Indeed, already for the case of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ there is no analogue available of the classification in [Bre03a] of the irreducible $\mathbb{F}$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, and it is clear from the results of [BP12] that any such classification would be much more complicated.

Furthermore, beyond the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ it is no longer the case that crystalline representations are (essentially) determined by their underlying Weil–Deligne representations, so there is no possibility of deducing that a p-adic correspondence is uniquely determined by the classical correspondence in the way that we do here, and no hope that an analogue of the results of [BB10] could hold. Finally, it is possible to use the constructions of [Paš04] to show that for $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ the patched module M_∞ is not a projective $\mathrm{GL}_2(\mathbb{Q}_{p^2})$-module.

1.11 Outline of the paper

In Section 2 we recall some well-known results about Hecke algebras and crystalline deformation rings for $\mathrm{GL}_2(\mathbb{Q}_p)$. The main result in this section is Proposition 2.15, which describes the crystalline deformation rings corresponding to Serre weights as completions of the corresponding Hecke algebras. In Section 3 we explain our axioms for a module with an arithmetic action, and show how the results of [CEG$^+$16] produce patched modules M_∞ satisfying these axioms.

Section 4 proves that the axioms determine M_∞ (essentially) uniquely, giving a new construction of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. It begins by showing that M_∞ is a projective $\mathrm{GL}_2(\mathbb{Q}_p)$-module (Theorem 4.15), before making a category-theoretic argument that allows us to "factor out" the patching variables (Proposition 4.22). We then use the "capture" machinery to complete the proof.

In Section 5 we explain how our results can be used to give a new proof (not making use of (φ, Γ)-modules) that certain locally algebraic principal series representations admit at most one unitary completion. Section 6 combines our results with those of [Paš13] to show that our construction is compatible with Colmez's correspondence, and as a byproduct extends some results of [Paš13] to a situation where the central character is not fixed.

Finally, in Section 7 we explain how our results give a new proof of the second author's local-global compatibility theorem, and briefly explain how such results can be extended to quaternion algebras over totally real fields (Remark 7.8).

1.12 Notation

We fix an odd prime p, an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and a finite extension $E/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, which will be our coefficient field. We write $\mathcal{O} = \mathcal{O}_E$ for the ring of integers in E, $\varpi = \varpi_E$ for a uniformiser, and $\mathbb{F} := \mathcal{O}/\varpi$ for the residue field. We will assume without comment that E and $\mathbb{F}$ are sufficiently large, and in particular that if we are working with representations of the absolute Galois group of a p-adic field K, then the images of all embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$ are contained in E.

1.12.1 *Galois-theoretic notation* If K is a field, we let G_K denote its absolute Galois group. Let ε denote the p-adic cyclotomic character, and $\overline{\varepsilon} = \omega$ the mod p cyclotomic character. If K is a finite extension of $\mathbb{Q}_p$ for some p, we write I_K for the inertia subgroup of G_K. If R is a local ring we write $\mathfrak{m}_R$ for the maximal ideal of R. If F is a number field and v is a finite place of F then we let Frob_v denote a geometric Frobenius element of G_{F_v}.

If $K/\mathbb{Q}_p$ is a finite extension, we write $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ for the Artin map normalized to send uniformizers to geometric Frobenius elements. To avoid cluttering up the notation, we will use $\mathrm{Art}_{\mathbb{Q}_p}$ to regard characters of $\mathbb{Q}_p^\times$, $\mathbb{Z}_p^\times$ as characters of $G_{\mathbb{Q}_p}$, $I_{\mathbb{Q}_p}$ respectively, without explicitly mentioning $\mathrm{Art}_{\mathbb{Q}_p}$ when we do so.

If K is a p-adic field and ρ a de Rham representation of G_K over E and if $\tau : K \hookrightarrow E$ then we will write $\mathrm{HT}_\tau(\rho)$ for the multiset of Hodge–Tate numbers of ρ with respect to τ. By definition, the multiset $\mathrm{HT}_\tau(\rho)$ contains i with multiplicity $\dim_E(\rho \otimes_{\tau,K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example $\mathrm{HT}_\tau(\varepsilon) = \{-1\}$. If ρ is moreover crystalline then we have the associated filtered φ-module $D_{\mathrm{cris}}(\rho) := (\rho \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K}$, where B_{cris} is Fontaine's crystalline period ring.

1.12.2 *Local Langlands correspondence* Let $n \in \mathbb{Z}_{\geqslant 1}$, let K be a finite extension of $\mathbb{Q}_p$, and let rec denote the local Langlands correspondence from isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ to isomorphism classes of n-dimensional Frobenius semisimple Weil–Deligne representations of W_K defined in [HT01]. Fix an isomorphism $\imath : \overline{\mathbb{Q}}_p \to \mathbb{C}$. We define the local Langlands correspondence rec_p over $\overline{\mathbb{Q}}_p$ by $\imath \circ \mathrm{rec}_p = \mathrm{rec} \circ \imath$. Then $r_p(\pi) := \mathrm{rec}_p(\pi \otimes |\det|^{(1-n)/2})$ is independent of the choice of $\imath$. In this paper we are mostly concerned with the case that $n = 2$ and $K = \mathbb{Q}_p$.

1.12.3 *Notation for duals* If A is a topological $\mathcal{O}$-module, we write $A^\vee := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, E/\mathcal{O})$ for the Pontryagin dual of A. We apply this to $\mathcal{O}$-modules that are either discrete or profinite, so that the usual formalism of Pontryagin duality applies.

If A is a pseudocompact $\mathcal{O}$-torsion free $\mathcal{O}$-module, we write $A^d := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, \mathcal{O})$ for its Schikhof dual.

If F is a free module of finite rank over a ring R, then we write $F^* := \mathrm{Hom}_R(F, R)$ to denote its R-linear dual, which is again a free R-module of the same rank over R as F.

If R is a commutative $\mathcal{O}$-algebra, and if A is an R-module that is pseudocompact and $\mathcal{O}$-torsion free as an $\mathcal{O}$-module, then we may form its Schikhof dual A^d, which has a natural R-module structure via the transpose action, extending its $\mathcal{O}$-module structure. If F is a finite rank free R-module, then $A \otimes_R F$ is again an R-module that is pseudocompact as an $\mathcal{O}$-module (if F has rank n then it is non-canonically isomorphic to a direct sum of n copies of A), and there is a canonical isomorphism of R-modules $(A \otimes_R F)^d \xrightarrow{\sim} A^d \otimes_R F^*$.

1.12.4 *Group-theoretic notation* Throughout the paper we write $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $K = \mathrm{GL}_2(\mathbb{Z}_p)$, and let $Z = Z(G)$ denote the centre of G. We also let B denote the Borel subgroup of G consisting of upper triangular matrices, and T denote the diagonal torus contained in B.

If $\chi : T \to E^\times$ is a continuous character, then we define the continuous induction $(\mathrm{Ind}_B^G \chi)_{\mathrm{cont}}$ to be the E-vector space of continuous functions $f : G \to E$ satisfying the condition $f(bg) = \chi(b)f(g)$ for all $b \in B$ and $g \in G$; it forms a G-representation with respect to the right regular action. If χ is in fact a smooth character, then we may also form the smooth induction $(\mathrm{Ind}_B^G \chi)_{\mathrm{sm}}$; this is the E-subspace of $(\mathrm{Ind}_B^G \chi)_{\mathrm{cont}}$ consisting of smooth functions, and is a G-subrepresentation of the continuous induction.

If χ_1 and χ_2 are continuous characters of $\mathbb{Q}_p^\times$, then the character $\chi_1 \otimes \chi_2 : T \to E^\times$ is defined via $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(d)$. Any continuous E-valued character χ of T is of this form, and χ is smooth if and only if χ_1 and χ_2 are.

2. Galois deformation rings and Hecke algebras

2.1 Galois deformation rings

Recall that we assume throughout the paper that p is an odd prime. Fix a continuous representation $\bar{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$, where as before $\mathbb{F}/\mathbb{F}_p$ is a finite extension. Possibly enlarging $\mathbb{F}$, we fix a sufficiently large extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$.

We will make the following assumption from now on.

2.2 Assumption. Assume that $\mathrm{End}_{G_{\mathbb{Q}_p}}(\bar{r}) = \mathbb{F}$, and that $\bar{r} \not\cong \left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$, for any character $\chi : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$.

In particular this assumption implies that $\bar{r}$ has a universal deformation $\mathcal{O}$-algebra R_p, and that either $\bar{r}$ is (absolutely) irreducible, or $\bar{r}$ is a non-split extension of characters.

We begin by recalling the relationship between crystalline deformation rings of $\bar{r}$, and the representation theory of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and $K := \mathrm{GL}_2(\mathbb{Z}_p)$. Given a pair of integers $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geqslant 0}$, we let $\sigma_{a,b}$ be the absolutely irreducible E-representation $\det^a \otimes \mathrm{Sym}^b E^2$ of K. Note that this is just the algebraic representation of highest weight $(a+b, a)$ with respect to the Borel subgroup given by the upper-triangular matrices in G.

We say that a representation $r : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is crystalline of Hodge type $\sigma = \sigma_{a,b}$ if it is crystalline with Hodge–Tate weights $(1-a, -a-b)$,[1] we write $R_p(\sigma)$, for the reduced, p-torsion free quotient of R_p corresponding to crystalline deformations of Hodge type σ.

2.3 The morphism from the Hecke algebra to the deformation ring

We briefly recall some results from [CEG+16, §4], specialised to the case of crystalline representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Set $\sigma = \sigma_{a,b}$, and let $\mathcal{H}(\sigma) := \mathrm{End}_G(\text{c-Ind}_K^G \sigma)$. The action of K on σ extends to the action of G. This gives rise to the isomorphism of G-representations:

$$(\text{c-Ind}_K^G \mathbf{1}) \otimes \sigma \cong \text{c-Ind}_K^G \sigma, \quad f \otimes v \mapsto [g \mapsto f(g)\sigma(g)v].$$

The map

$$\mathcal{H}(\mathbb{1}) \to \mathcal{H}(\sigma), \quad \phi \mapsto \phi \otimes \mathrm{id}_\sigma \tag{2.4}$$

[1] Note that this convention agrees with those of [CEG+16] and [Eme11].

is an isomorphism of E-algebras by Lemma 1.4 of [ST06]. Using the above isomorphism we will identify elements of $\mathcal{H}(\sigma)$ with E-valued K-biinvariant functions on G, supported on finitely many double cosets.

2.5 Proposition. *Let $S \in \mathcal{H}(\sigma)$ be the function supported on the double coset of $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, with value p^{2a+b} at $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, and let $T \in \mathcal{H}(\sigma)$ be the function supported on the double coset[2] of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, with value p^{a+b} at $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\mathcal{H}(\sigma) = E[S^{\pm 1}, T]$ as an E-algebra.*

Proof. This is immediate from (2.4) and the Satake isomorphism. □

Let r, s be integers with $r < s$, and let $t, d \in E$ with $d \in E^\times$. We let $D := D(r, s, t, d)$ be the two-dimensional filtered φ-module that has e_1, e_2 as a basis of its underlying E-vector space, has its E-linear Frobenius endomorphism φ being given by

$$\varphi(e_1) = e_2,\ \varphi(e_2) = -de_1 + te_2,$$

and has its Hodge filtration given by

$$\mathrm{Fil}^i D = D \text{ if } i \leqslant r,\ \mathrm{Fil}^i D = Ee_1 \text{ if } r + 1 \leqslant i \leqslant s, \text{ and } \mathrm{Fil}^i D = 0 \text{ if } i > s.$$

We note that t is the trace and d is the determinant of φ on D, and both are therefore determined uniquely by D. The same construction works if E is replaced with an E-algebra A. We will still write $D(r, s, t, d)$ for the resulting φ-module with A-coefficients if the coefficient algebra is clear from the context.

2.6 Lemma. *If V is an indecomposable 2-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ over E with distinct Hodge–Tate weights (s, r) then there exists a unique pair $(t, d) \in E \times E^\times$ such that $D_{\mathrm{cris}}(V) \cong D(r, s, t, d)$. Moreover, $v_p(d) = r + s$ and $v_p(t) \geqslant r$.*

Proof. This is well known, and is a straightforward computation using the fact that $D_{\mathrm{cris}}(V)$ is weakly admissible. For the sake of completeness, we sketch the proof; the key fact one employs is that $V \mapsto D_{\mathrm{cris}}(V)$ is a fully faithful embedding of the category of crystalline representations of $G_{\mathbb{Q}_p}$ into the category of weakly admissible filtered φ-modules. (Indeed, it induces an equivalence between these two categories, but that more difficult fact isn't needed for this computation.) We choose e_1 to be a basis for $\mathrm{Fil}^s D_{\mathrm{cris}}(V)$; the assumption that V is indecomposable implies that $\mathrm{Fil}^s D_{\mathrm{cris}}(V)$ is not stable under φ, and so if we write $e_2 := \varphi(e_1)$ then e_1, e_2 is a basis for $D_{\mathrm{cris}}(V)$, and φ has a matrix of the required form for a uniquely determined t and d. The asserted relations between $v_p(t)$, $v_p(d)$, r, and s follow from the weak admissibility of $D_{\mathrm{cris}}(V)$. □

In fact, it will be helpful to state a generalisation of the previous result to the context of finite dimensional E-algebras. (Note that the definition of $D(r, s, t, d)$ extends naturally to the case when t and d are taken to lie in such a finite-dimensional algebra.)

2.7 Lemma. *If A is an Artinian local E-algebra with residue field E', and if V_A is a crystalline representation of rank two over A whose associated residual representation $V_{E'} := E' \otimes_A V_A$ is indecomposable with distinct Hodge–Tate weights (s, r), and if $D_{\mathrm{cris}}(V_A)$ denotes the filtered φ-module associated to V_A, then there exists a unique pair $(t, d) \in A \times A^\times$ such that $D_{\mathrm{cris}}(V_A) \cong D(r, s, t, d)$.*

[2] The function supported on the double coset KgK with value 1 at g, viewed as an element of $\mathcal{H}(\mathbf{1})$, acts on $v \in V^K$ by the formula $[KgK]v = \sum_{hK \subset KgK} hv$.

Proof. Choose a basis $\overline{e}_1$ for $\mathrm{Fil}^s D_{\mathrm{cris}}(V_{E'})$, and choose $e_1 \in \mathrm{Fil}^s D_{\mathrm{cris}}(V_A)$ lifting $\overline{e}_1$. By Nakayama's lemma, e_1 generates $\mathrm{Fil}^s D_{\mathrm{cris}}(V_A)$, and by considering the length of $\mathrm{Fil}^s D_{\mathrm{cris}}(V_A)$ as an E-vector space, we see that $\mathrm{Fil}^s D_{\mathrm{cris}}(V_A)$ is a free A-module of rank one.

Let $e_2 = \phi(e_1)$, and write $\overline{e}_2$ for the image of e_2 in $D_{\mathrm{cris}}(V_{E'})$. As in the proof of Lemma 2.6, $\overline{e}_1, \overline{e}_2$ is a basis of $D_{\mathrm{cris}}(V_{E'})$, and thus by another application of Nakayama's lemma, e_1, e_2 are an A-basis of $D_{\mathrm{cris}}(V_A)$. The matrix of ϕ in this basis is evidently of the required form. $\square$

2.8 Corollary. *If V is an indecomposable 2-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ over E with distinct Hodge–Tate weights (s, r), for which $D_{\mathrm{cris}}(V) \cong D(r, s, t_0, d_0)$, then the formal crystalline deformation ring of V is naturally isomorphic to $E[[t - t_0, d - d_0]]$.*

Proof. This is immediate from Lemma 2.7, and the fact that $V \mapsto D_{\mathrm{cris}}(V)$ is an equivalence of categories. $\square$

Suppose that $\bar{r}$ has a crystalline lift of Hodge type σ. By [CEG+16, Thm. 4.1] there is a natural E-algebra homomorphism $\eta : \mathcal{H}(\sigma) \to R_p(\sigma)[1/p]$ interpolating a normalized local Langlands correspondence r_p (introduced in Section 1.12). In order to characterise this map, one considers the composite

$$\mathcal{H}(\sigma) \xrightarrow{\eta} R_p(\sigma)[1/p] \hookrightarrow \big(R_p(\sigma)\big)^{\mathrm{an}},$$

where $\big(R_p(\sigma)\big)^{\mathrm{an}}$ denotes the ring of rigid analytic functions on the rigid analytic generic fibre of $\mathrm{Spf}\, R_p(\sigma)$. Over $\big(R_p(\sigma)\big)^{\mathrm{an}}$, we may consider the universal filtered φ-module, and the underlying universal Weil group representation (given by forgetting the filtration). The trace and determinant of Frobenius on this representation are certain elements of $\big(R_p(\sigma)\big)^{\mathrm{an}}$ (which in fact lie in $R_p(\sigma)[1/p]$), and η is characterised by the fact that it identifies appropriately chosen generators of $\mathcal{H}(\sigma)$ with this universal trace and determinant.

It is straightforward to give explicit formulas for these generators of $\mathcal{H}(\sigma)$, but we have found it interesting (in part with an eye to making arguments in more general contexts) to also derive the facts that we need without using such explicit formulas.

Regarding explicit formulas, we have the following result.

2.9 Proposition. *The elements $\eta(S), \eta(T) \in R_p(\sigma)[1/p]$ are characterised by the following property: if $x : R_p(\sigma)[1/p] \to \overline{\mathbb{Q}}_p$ is an E-algebra morphism, and V_x is the corresponding two-dimensional $\overline{\mathbb{Q}}_p$-representation of $G_{\mathbb{Q}_p}$, then $x(\eta(T)) = p^{a+b}t$, and $x(\eta(S)) = p^{2a+b-1}d$, where t, d are respectively the trace and the determinant of φ on $D_{\mathrm{cris}}(V_x)$.*

Proof. Lemma 2.7 implies that there are uniquely determined $t, d \in \overline{\mathbb{Q}}_p$, such that $D_{\mathrm{cris}}(V_x) \cong D(r, s, t, d)$, where $r = -a - b$ and $s = 1 - a$. The Weil–Deligne representation associated to $D(r, s, t, d)$ is an unramified 2-dimensional representation of $W_{\mathbb{Q}_p}$, on which the geometric Frobenius Frob_p acts by the matrix of crystalline Frobenius on $D(r, s, t, d)$, which is $\left(\begin{smallmatrix} 0 & -d \\ 1 & t \end{smallmatrix}\right)$. Thus

$$\mathrm{WD}(D_{\mathrm{cris}}(V_x)) = \mathrm{rec}_p(\chi_1) \oplus \mathrm{rec}_p(\chi_2),$$

where $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \overline{\mathbb{Q}}_p^\times$ are unramified characters, such that $\chi_1(p) + \chi_2(p) = t$ and $\chi_1(p)\chi_2(p) = d$.

If $\pi = (\mathrm{Ind}_B^G |\centerdot| \chi_1 \otimes \chi_2)_{\mathrm{sm}}$ then $\pi \otimes |\det|^{-1/2} \cong \iota_B^G(\chi_1 \otimes \chi_2)$, where ι_B^G denotes smooth normalized parabolic induction. Then

$$r_p(\pi) = \mathrm{rec}_p(\iota_B^G(\chi_1 \otimes \chi_2)) = \mathrm{rec}_p(\chi_1) \oplus \mathrm{rec}_p(\chi_2).$$

The action of $\mathcal{H}(\mathbf{1})$ on π^K is given by sending $[K\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)K]$ to $p|p|\chi_1(p)+\chi_2(p) = t$ and $[K\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)K]$ to $|p|\chi_1(p)\chi_2(p) = p^{-1}d$. By [CEG+16, Thm. 4.1] and the fact that the evident isomorphism between $\pi^K = \mathrm{Hom}_K(\mathbf{1}, \mathbf{1} \otimes \pi)$ and $\mathrm{Hom}_K(\sigma, \sigma \otimes \pi)$ is equivariant with respect to the actions by $\mathcal{H}(\mathbf{1})$ and $\mathcal{H}(\sigma)$ via the isomorphism (2.4), we see that

$$x(\eta(T)) = p^{-r}t = p^{a+b}t, \quad x(\eta(S)) = p^{-r-s}d = p^{2a+b-1}d. \tag{2.10}$$

Since $R_p(\sigma)[1/p]$ is a reduced Jacobson ring, the formulas determine $\eta(T)$ and $\eta(S)$ uniquely. □

2.11 COROLLARY. *$\eta(S)$ and $\eta(T)$ are contained in the normalisation of $R_p(\sigma)$ in $R_p(\sigma)[1/p]$.*

Proof. It follows from (2.10) and Lemma 2.6 that for all closed points $x : R_p(\sigma)[1/p] \to \overline{\mathbb{Q}}_p$, we have $x(\eta(S)), x(\eta(T)) \in \overline{\mathbb{Z}}_p$. The result follows from [dJ95, Prop. 7.3.6]. □

A key fact that we will use, which is special to our context of 2-dimensional crystalline representations of $G_{\mathbb{Q}_p}$, is that the morphism

$$(\mathrm{Spf}\, R(\sigma))^{\mathrm{an}} \to \big(\mathrm{Spec}\, \mathcal{H}(\sigma)\big)^{\mathrm{an}}$$

induced by η is an open immersion of rigid analytic spaces, where the superscript an signifies the associated rigid analytic space. We prove this statement (in its infinitesimal form) in the following result.

2.12 LEMMA. *Let $\sigma = \sigma_{a,b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\dim_{\kappa(y)} \kappa(y) \otimes_{\mathcal{H}(\sigma)} R_p(\sigma)[1/p] \leqslant 1, \quad \forall y \in \text{m-Spec}\, \mathcal{H}(\sigma).$$

Proof. Let us assume that $A := \kappa(y) \otimes_{\mathcal{H}(\sigma)} R_p(\sigma)[1/p]$ is non-zero. If $x, x' \in \text{m-Spec}\, A$ then Frobenius on $D_{\mathrm{cris}}(V_x)$ and $D_{\mathrm{cris}}(V_{x'})$ will have the same trace and determinant (since, by Proposition 2.9, these are determined by the images of T and S in $\kappa(y)$); denote them by t and d. It follows from Lemma 2.6 that $D_{\mathrm{cris}}(V_x) \cong D_{\mathrm{cris}}(V_{x'})$ and hence $x = x'$. Since $D(r,s,t,d)$ can be constructed over $\kappa(y)$ (as t and d lie in $\kappa(y)$), so can V_x and thus $\kappa(x) = \kappa(y)$. To complete the proof of the lemma it is enough to show that the map $\mathfrak{m}_y \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is surjective. Since we know that $R_p(\sigma)[1/p]$ is a regular ring of dimension 2 by [Kis08, Thm. 3.3.8], it is enough to construct a 2-dimensional family of deformations of $D_{\mathrm{cris}}(V_x)$ to the ring of dual numbers $\kappa(y)[\epsilon]$, which induces a non-trivial deformation of the images of S, T. That this is possible is immediate from Corollary 2.8 and Proposition 2.9. □

2.13 PROPOSITION. *Let $\sigma = \sigma_{a,b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geqslant 0}$. Let $y \in \text{m-Spec}\, \mathcal{H}(\sigma)$ be the image of $x \in \text{m-Spec}\, R_p(\sigma)[1/p]$ under the morphism induced by $\eta : \mathcal{H}(\sigma) \to R_p(\sigma)[1/p]$. Then η induces an isomorphism of completions:*

$$\widehat{\mathcal{H}(\sigma)}_{\mathfrak{m}_y} \xrightarrow{\cong} \widehat{R_p(\sigma)[1/p]}_{\mathfrak{m}_x}.$$

Proof. This can be proved by explicit computation, taking into account Corollary 2.8 and Proposition 2.9.

We can also deduce it in more pure thought manner as follows: Since $\mathcal{H}(\sigma) \cong E[T, S^{\pm 1}]$ by Proposition 2.5 and $R_p(\sigma)[1/p]$ is a regular ring of dimension 2 as in the preceding proof, both completions are regular rings of dimension 2. It follows from Lemma 2.12 that $\kappa(y) = \kappa(x)$ and the map induces a surjection on tangent spaces. Hence the map is an isomorphism. □

If $0 \leqslant b \leqslant p-1$, then $\sigma_{a,b}$ has a unique (up to homothety) K-invariant lattice $\sigma^{\circ}_{a,b}$, which is isomorphic to $\det^a \otimes \mathrm{Sym}^b \mathcal{O}^2$ as a K-representation. We let $\overline{\sigma}_{a,b}$ be its reduction modulo ϖ.

Then $\overline{\sigma}_{a,b}$ is the absolutely irreducible $\mathbb{F}$-representation $\det^a \otimes \mathrm{Sym}^b\mathbb{F}^2$ of $\mathrm{GL}_2(\mathbb{F}_p)$; note that every (absolutely) irreducible $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathbb{F}_p)$ is of this form for some uniquely determined a, b with $0 \leqslant a < p-1$. We refer to such representations as *Serre weights*.

If $\overline{\sigma} = \overline{\sigma}_{a,b}$ is a Serre weight with the property that $\bar{r}$ has a lift $r : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ that is crystalline of Hodge type $\sigma = \sigma_{a,b}$, then we say that $\overline{\sigma}$ is a Serre weight of $\bar{r}$.

Again we consider $\sigma = \sigma_{a,b}$ with any $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geqslant 0}$. Let

$$\sigma^\circ := \det^a \otimes \mathrm{Sym}^b\mathcal{O}^2, \quad \overline{\sigma} := \det^a \otimes \mathrm{Sym}^b\mathbb{F}^2$$

so that $\sigma^\circ/\varpi = \overline{\sigma}$. We let

$$\mathcal{H}(\sigma^\circ) := \mathrm{End}_G(\text{c-Ind}_K^G \sigma^\circ), \quad \mathcal{H}(\overline{\sigma}) := \mathrm{End}_G(\text{c-Ind}_K^G \overline{\sigma}).$$

Note that $\mathcal{H}(\sigma^\circ)$ is p-torsion free, since $\text{c-Ind}_K^G \sigma^\circ$ is.

2.14 Lemma. (1) *For any σ, there is a natural isomorphism $\mathcal{H}(\sigma) \cong \mathcal{H}(\sigma^\circ)[1/p]$, and a natural inclusion $\mathcal{H}(\sigma^\circ)/\varpi \hookrightarrow \mathcal{H}(\overline{\sigma})$. Furthermore, the $\mathcal{O}$-subalgebra $\mathcal{O}[S^{\pm 1}, T]$ of $\mathcal{H}(\sigma)$ is contained in $\mathcal{H}(\sigma^\circ)$.*

(2) *If, in addition $\sigma = \sigma_{a,b}$ with $0 \leqslant b \leqslant p-1$, then $\mathcal{O}[S^{\pm 1}, T] = \mathcal{H}(\sigma^\circ)$, and there is a natural isomorphism $\mathcal{H}(\sigma^\circ)/\varpi \cong \mathcal{H}(\overline{\sigma})$.*

Proof. The isomorphism of (1) follows immediately from the fact that $\text{c-Ind}_K^G \sigma^\circ$ is a finitely generated $\mathcal{O}[G]$-module. To see the claimed inclusion, apply $\mathrm{Hom}_G(\text{c-Ind}_K^G \sigma^\circ, -)$ to the exact sequence

$$0 \to \text{c-Ind}_K^G \sigma^\circ \xrightarrow{\varpi} \text{c-Ind}_K^G \sigma^\circ \to \text{c-Ind}_K^G \overline{\sigma} \to 0$$

so as to obtain an injective map

$$\mathcal{H}(\sigma^\circ)/\varpi \hookrightarrow \mathrm{Hom}_G(\text{c-Ind}_K^G \sigma^\circ, \text{c-Ind}_K^G \overline{\sigma}) \cong \mathcal{H}(\overline{\sigma}).$$

To see the final claim of (1), we recall that from (2.4) and Frobenius reciprocity we have natural isomorphisms

$$\mathcal{H}(\mathbb{1}) \simeq \mathcal{H}(\sigma) \simeq \mathrm{Hom}_K(\sigma, \text{c-Ind}_K^G \sigma);$$

the image of $\phi \in \mathcal{H}(\mathbb{1})$ under the composite map sends $v \in \sigma$ to the function $g \mapsto \phi(g^{-1})\sigma(g)v$. A direct computation of the actions of S, T on the standard basis of σ° then verifies that $S^{\pm 1}$ and T lie in the $\mathcal{O}$-submodule $\mathrm{Hom}_K(\sigma^\circ, \text{c-Ind}_K^G \sigma^\circ)$ of $\mathcal{H}(\sigma)$.

To prove (2), we note that it follows from [Bre03b, §2] and [BL94] that the composite $\mathbb{F}[S^{\pm 1}, T] \to \mathcal{H}(\sigma^\circ)/\varpi \to \mathcal{H}(\overline{\sigma})$ is an isomorphism. Since the second of these maps is injective, by (1), we conclude that each of these maps is in fact an isomorphism, confirming the second claim of (2). Furthermore, this shows that the inclusion $\mathcal{O}[S^{\pm 1}, T] \hookrightarrow \mathcal{H}(\sigma^\circ)$ of (1) becomes an isomorphism both after reducing modulo ϖ as well as after inverting ϖ (because $\mathcal{H}(\sigma)$ is generated by $S^{\pm 1}$ and T by Proposition 2.5). Thus it is an isomorphism, completing the proof of (2). $\square$

The following lemma is well known, but for lack of a convenient reference we sketch a proof.

2.15 Lemma. *Assume that $\bar{r}$ satisfies Assumption 2.2. Then $\bar{r}$ has at most two Serre weights. Furthermore, if we let $\overline{\sigma} = \overline{\sigma}_{a,b}$ be a Serre weight of $\bar{r}$ then the following hold:*

(i) The deformation ring $R_p(\sigma)$ is formally smooth of relative dimension 2 over $\mathcal{O}$.

(ii) The morphism of E-algebras $\eta : \mathcal{H}(\sigma) \to R_p(\sigma)[1/p]$ induces a morphism of $\mathcal{O}$-algebras $\mathcal{H}(\sigma^\circ) \to R_p(\sigma)$.

(iii) *The character $\omega^{1-2a-b}\det\bar{r}$ is unramified, and if we let*
- $\mu = (\omega^{1-2a-b}\det\bar{r})(\mathrm{Frob}_p)$, *and*
- *if $\bar{r}$ is irreducible, then $\lambda = 0$, and*
- *if $\bar{r}$ is reducible, then we can write $\bar{r} \cong \omega^{a+b} \otimes \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2\omega^{-b-1}\end{pmatrix}$ for unramified characters $\overline{\chi}_1$, $\overline{\chi}_2$, and let $\lambda = \overline{\chi}_1(\mathrm{Frob}_p)$,*

then the composition

$$\alpha : \mathcal{H}(\sigma^\circ) \to R_p(\sigma) \to \mathbb{F}$$

maps $T \mapsto \lambda$ and $S \mapsto \mu$.

(iv) *Let $\widehat{\mathcal{H}(\sigma^\circ)}$ be the completion of $\mathcal{H}(\sigma^\circ)$ with respect to the kernel of α. Then the map $\mathcal{H}(\sigma^\circ) \to R_p(\sigma)$ induces an isomorphism of local $\mathcal{O}$-algebras $\widehat{\mathcal{H}(\sigma^\circ)} \xrightarrow{\cong} R_p(\sigma)$. In coordinates, we have $R_p(\sigma) = \mathcal{O}[[S - \widetilde{\mu}, T - \widetilde{\lambda}]]$, where the tilde denotes the Teichmüller lift.*

(v) *If we set $\pi := (\text{c-Ind}_K^G \overline{\sigma}) \otimes_{\mathcal{H}(\overline{\sigma}),\alpha} \mathbb{F}$, then π is an absolutely irreducible representation of G, and is independent of the choice of Serre weight $\overline{\sigma}$ of $\bar{r}$.*

Proof. The claim that $\bar{r}$ has at most two Serre weights is immediate from the proof of [BDJ10, Thm. 3.17], which explicitly describes the Serre weights of $\bar{r}$. Concretely, in the case at hand these weights are as follows (see also the discussion of [Eme11, §3.5], which uses the same conventions as this paper). If $\bar{r}$ is irreducible, then we may write

$$\bar{r}|_{I_{\mathbb{Q}_p}} \cong \omega^{m-1} \otimes \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)}\end{pmatrix},$$

where ω_2 is a fundamental character of niveau 2, and $0 \leqslant m < p-1$, $0 \leqslant n \leqslant p-2$. Then the Serre weights of $\bar{r}$ are $\overline{\sigma}_{m,n}$ and $\overline{\sigma}_{m+n,p-1-n}$ (with $m+n$ taken modulo $p-1$). If $\bar{r}$ is reducible, then we may write

$$\bar{r}|_{I_{\mathbb{Q}_p}} \cong \omega^{m+n} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega^{-n-1}\end{pmatrix},$$

where $0 \leqslant m < p-1$, $0 \leqslant n < p-1$. Then (under Assumption 2.2) if $n \neq 0$, the unique Serre weight of $\bar{r}$ is $\overline{\sigma}_{m,n}$, while if $n = 0$, then $\overline{\sigma}_{m,0}$ and $\overline{\sigma}_{m,p-1}$ are the two Serre weights of $\bar{r}$.

Part (2) follows from (1) by Lemma 2.14 (2) and Corollary 2.11. We prove parts (1), (3) and (4) simultaneously. If $\overline{\sigma}$ is not of the form $\overline{\sigma}_{a,p-1}$, the claims about $R_p(\sigma)$ are a standard consequence of (unipotent) Fontaine–Laffaille theory; for example, the irreducible case with $\mathcal{O} = \mathbb{Z}_p$ is [FM95, Thm. B2], and the reducible case follows in the same way. The key point is that the corresponding weakly admissible modules are either reducible, or are uniquely determined by the trace and determinant of φ, by Lemma 2.6. Concretely, if $\bar{r}$ is irreducible, then the crystalline lifts of $\bar{r}$ of Hodge type $\sigma_{a,b}$ correspond exactly to the weakly admissible modules $D(-(a+b), 1-a, t, d)$ where $v_p(t) > -a-b$ and $\overline{p^{2a+b-1}d} = \mu$. The claimed description of the deformation ring then follows.

Similarly, if $\bar{r}$ is reducible, then it follows from Fontaine–Laffaille theory (and Assumption 2.2) that any crystalline lift of Hodge type $\overline{\sigma}_{a,b}$ is necessarily reducible and indecomposable, and one finds that these crystalline lifts correspond precisely to those weakly admissible modules with $D(-(a+b), 1-a, t, d)$ where $v_p(t) = -a-b$, $\overline{p^{a+b}t} = \lambda$, and $\overline{p^{2a+b-1}d} = \mu$.

This leaves only the case that $\overline{\sigma}$ is of the form $\overline{\sigma}_{a,p-1}$. In this case the result is immediate from the main result of [BLZ04], which shows that the above description of the weakly admissible modules continues to hold.

Finally, (5) is immediate from the main results of [BL94, Bre03a], together with the explicit description of $\overline{\sigma}$, λ, established above. More precisely, in the case that $\bar{r}$ is irreducible, the absolute irreducibility of π is [Bre03a, Thm. 1.1], and its independence of the choice of $\overline{\sigma}$ is [Bre03a, Thm. 1.3]. If $\bar{r}$ is reducible and has only a single Serre weight, then the absolute irreducibility of π is [BL94, Thm. 33(2)]. In the remaining case that $\bar{r}$ is reducible and has two Serre weights, then Assumption 2.2 together with the explicit description of λ, μ above implies that $\lambda^2 \neq \mu$, and the absolute irreducibility of π is again [BL94, Thm. 33(2)]. The independence of π of the choice of $\overline{\sigma}$ is [BL94, Cor. 36(2)(b)]. □

2.16 Remark. It follows from the explicit description of π that it is either a principal series representation or supersingular, and neither one-dimensional nor an element of the special series. (This would no longer be the case if we allowed $\bar{r}$ to be a twist of an extension of the trivial character by the mod p cyclotomic character, when in fact π would be an extension of a one-dimensional representation and a special representation, which would also depend on the Serre weight if $\bar{r}$ is peu ramifié.)

2.17 Remark. If π has central character ψ, then $\det \bar{r} = \psi\omega^{-1}$.

3. Patched modules and arithmetic actions

We now introduce the notion of an arithmetic action of (a power series ring over) R_p on an $\mathcal{O}[G]$-module. It is not obvious from the definition that any examples exist, but we will explain later in this section how to deduce the existence of an example from the results of [CEG+16] (that is, from the Taylor–Wiles patching method). The rest of the paper is devoted to showing a uniqueness result for such actions, and thus deducing that they encode the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. We anticipate that the axiomatic approach taken here will be useful in other contexts (for example, for proving local-global compatibility in the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ in global settings other than those considered in [Eme11] or [CEG+16]).

3.1 Axioms

Fix an integer $d \geqslant 0$, and set $R_\infty := R_p \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[x_1, \dots, x_d]]$. Then an *$\mathcal{O}[G]$-module with an arithmetic action of* R_∞ is by definition a non-zero $R_\infty[G]$-module M_∞ satisfying the following axioms.

(AA1) M_∞ is a finitely generated $R_\infty[[K]]$-module.

(AA2) M_∞ is projective in the category of pseudocompact $\mathcal{O}[[K]]$-modules.

Let σ° be a K-stable $\mathcal{O}$-lattice in $\sigma = \sigma_{a,b}$. Set

$$M_\infty(\sigma^\circ) := \left(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ)^d)\right)^d,$$

where we are considering continuous homomorphisms for the profinite topology on M_∞ and the p-adic topology on $(\sigma^\circ)^d$. This is a finitely generated R_∞-module by (AA1) and [Paš15, Cor. 2.5].

(AA3) For any σ, the action of R_∞ on $M_\infty(\sigma^\circ)$ factors through $R_\infty(\sigma) := R_p(\sigma)[[x_1, \dots, x_d]]$. Furthermore, $M_\infty(\sigma^\circ)$ is maximal Cohen–Macaulay over $R_\infty(\sigma)$, and the $R_\infty(\sigma)[1/p]$-module $M_\infty(\sigma^\circ)[1/p]$ is locally free of rank one over its support.

For each σ°, we have a natural action of $\mathcal{H}(\sigma^\circ)$ on $M_\infty(\sigma^\circ)$, and thus of $\mathcal{H}(\sigma)$ on $M_\infty(\sigma^\circ)[1/p]$.

(AA4) For any σ, the action of $\mathcal{H}(\sigma)$ on $M_\infty(\sigma^\circ)[1/p]$ is given by the composite

$$\mathcal{H}(\sigma) \xrightarrow{\eta} R_p(\sigma)[1/p] \to R_p(\sigma)[[x_1, \ldots, x_d]][1/p],$$

where $\mathcal{H}(\sigma) \xrightarrow{\eta} R_p(\sigma)[1/p]$ is defined in [CEG+16, Thm. 4.1].

3.2 Remark. While these axioms may appear somewhat mysterious, as we will see in the next subsection they arise very naturally in the constructions of [CEG+16]. (Indeed, those constructions give modules M_∞ satisfying obvious analogues of the above conditions for $\mathrm{GL}_n(K)$ for any finite extension $K/\mathbb{Q}_p$; however, our arguments in the rest of the paper will only apply to the case of $\mathrm{GL}_2(\mathbb{Q}_p)$.)

In these examples, axioms (AA1) and (AA2) essentially follow from the facts that spaces of automorphic forms are finite-dimensional, and that the cohomology of zero-dimensional Shimura varieties is concentrated in a single degree (degree zero). Axioms (AA3) and (AA4) come from the existence of Galois representations attached to automorphic forms on unitary groups, and from local-global compatibility at p for automorphic forms of level prime to p.

The following remark explains how axiom (AA4) is related to Breuil's original formulation of the p-adic Langlands correspondence in terms of unitary completions of locally algebraic vectors; see also the proof of Proposition 6.17 below.

3.3 Remark. Axiom (AA4) is, in the presence of axioms (AA1)-(AA3), equivalent to an alternative axiom (AA4'), which expresses a pointwise compatibility with the classical local Langlands correspondence, as we now explain. Write $R_\infty(\sigma) := R_p(\sigma)[[x_1, \ldots, x_d]]$. If y is a maximal ideal of $R_\infty(\sigma)[1/p]$ which lies in the support of $M_\infty(\sigma^\circ)[1/p]$, then we write

$$\Pi_y := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}\big(M_\infty \otimes_{R_\infty, y} \mathcal{O}_{\kappa(y)}, E\big).$$

We write x for the corresponding maximal ideal of $R_p(\sigma)[1/p]$, r_x for the deformation of $\bar{r}$ corresponding to x, and set $\pi_{\mathrm{sm}}(r_x) := r_p^{-1}(\mathrm{WD}(r_x)^{F-\mathrm{ss}})$, which is the smooth representation of G corresponding to the Weil–Deligne representation associated to r_x by the classical Langlands correspondence r_p (normalised as in Section 1.12). We write $\pi_{\mathrm{alg}}(r_x)$ for the algebraic representation of G whose restriction to K is equal to σ.

(AA4') For any σ and for any y and x as above, there is a non-zero G-equivariant map

$$\pi_{\mathrm{sm}}(r_x) \otimes \pi_{\mathrm{alg}}(r_x) \to \Pi_y^{\mathrm{l.alg}}.$$

That (AA1)-(AA4) imply (AA4') is a straightforward consequence of the defining property of the map η. Conversely, assume (AA1)-(AA3) and (AA4'), and write

$$\overline{R}_\infty(\sigma) := R_\infty(\sigma)/\mathrm{Ann}(M_\infty(\sigma^\circ)).$$

It follows from (AA3) that the natural map $\overline{R}_\infty(\sigma)[1/p] \to \mathrm{End}_{R_\infty(\sigma)[1/p]}(M_\infty(\sigma^\circ)[1/p])$ is an isomorphism, as it is injective and the cokernel is not supported on any maximal ideal of $R_\infty(\sigma)[1/p]$. In particular the action of $\mathcal{H}(\sigma^\circ)$ on $M_\infty(\sigma^\circ)$ induces a homomorphism $\eta' : \mathcal{H}(\sigma) \to \overline{R}_\infty(\sigma)[1/p]$. We have to show that this agrees with the map induced by η.

It follows from (AA4') and the defining property of η that η and η' agree modulo every maximal ideal of $R_p(\sigma)[1/p]$ in the support of $M_\infty(\sigma^\circ)$. It follows from (AA3) that $\overline{R}_\infty(\sigma)[1/p]$ is a union of irreducible components of $R_\infty(\sigma)[1/p]$. Since $R_\infty(\sigma)[1/p]$ is reduced we conclude that $\overline{R}_\infty(\sigma)[1/p]$ is reduced, thus the intersection of all maximal ideals is equal to zero. Hence (AA4) holds.

3.4 Existence of a patched module M_∞

We now briefly recall some of main results of [CEG+16], specialised to the case of two-dimensional representations. We emphasise that these results use only the Taylor–Wiles–Kisin patching method, and use nothing about the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. (We should perhaps remark, though, that we do make implicit use of the results of [BLGGT14] in the globalisation part of the argument, and thus of the Taylor–Wiles–Kisin method for unitary groups of rank 4, and not just for $U(2)$.) We freely use the notation of [CEG+16].

Enlarging $\mathbb{F}$ if necessary, we see from [CEG+16, Lem. 2.2] that the hypotheses on $\bar{r}$ at the start of [CEG+16, §2.1] are automatically satisfied. We fix the choice of weight ξ and inertial type τ in [CEG+16, §2.3] in the following way: we take τ to be trivial, and we take ξ to be the weight corresponding to a Serre weight of $\bar{r}$, as in Lemma 2.15.

With this choice, the modification of the Taylor–Wiles–Kisin method carried out in [CEG+16, §2.6] produces for some $d > 0$ an R_∞-module M_∞ with an action of G. (Note that for our choice of $\bar{r}$, ξ and τ, the various framed deformation rings appearing in [CEG+16] are formal power series rings over $\mathcal{O}$, and the framed deformation ring of $\bar{r}$ is formally smooth over R_p, so all of these rings are absorbed into the power series ring $\mathcal{O}[[x_1, \dots, x_d]]$. The module M_∞ is patched from the cohomology of a definite unitary group over some totally real field in which p splits completely.)

This $R_\infty[G]$-module automatically satisfies the axioms (AA1)–(AA4) above. Indeed, (AA1) and (AA2) follow from [CEG+16, Prop. 2.10], and (AA3) follows from [CEG+16, Lem. 4.17(1), 4.18(1)]. Finally, (AA4) is [CEG+16, Thm. 4.19].

4. Existence and uniqueness of arithmetic actions

We fix an $\mathcal{O}[G]$-module M_∞ with an arithmetic action of R_∞ in the sense of Section 3.1.

4.1 Serre weights and cosocles

Now let $\overline{\sigma} = \overline{\sigma}_{a,b}$ be a Serre weight, and let σ° be a K-stable $\mathcal{O}$-lattice in $\sigma_{a,b}$, so that $\sigma^\circ/\varpi\sigma^\circ = \overline{\sigma}$. We define $M_\infty(\overline{\sigma}) = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M_\infty, (\overline{\sigma})^\vee)^\vee$, so that by (AA2) we have $M_\infty(\overline{\sigma}) = M_\infty(\sigma^\circ)/\varpi M_\infty(\sigma^\circ)$. By definition, the deformation ring $R_p(\sigma) = R_p(\sigma_{a,b})$ is non-zero if and only if $\overline{\sigma}$ is a Serre weight of $\bar{r}$. Set $R_\infty(\sigma) = R_p(\sigma)[[x_1, \dots, x_d]]$.

We let π denote the absolutely irreducible smooth $\mathbb{F}$-representation of G associated to $\bar{r}$ via Lemma 2.15 (5).

4.2 Proposition. *(i) We have $M_\infty(\sigma^\circ) \neq 0$ if and only if $\overline{\sigma}$ is a Serre weight of $\bar{r}$, in which case $M_\infty(\sigma^\circ)$ is a free $R_\infty(\sigma)$-module of rank one.*

(ii) If $\overline{\sigma}$ is a Serre weight of $\bar{r}$ then the action of $\mathcal{H}(\overline{\sigma})$ on $M_\infty(\overline{\sigma})$ factors through the natural map $R_p(\sigma)/\varpi \to R_\infty(\sigma)/\varpi$, and $M_\infty(\overline{\sigma})$ is a flat $\mathcal{H}(\overline{\sigma})$-module.

(iii) If π' is an irreducible smooth $\mathbb{F}$-representation of G then we have

$$\mathrm{Hom}_G(\pi', M_\infty^\vee) \neq 0$$

if and only if π' is isomorphic to π.

Proof. It follows from (AA3) that $M_\infty(\sigma^\circ) \neq 0$ only if $R_p(\sigma)[1/p] \neq 0$, which is equivalent to $\overline{\sigma}$ being a Serre weight of $\bar{r}$. In this case, since $\sigma = \sigma_{a,b}$ with $0 \leqslant b \leqslant p-1$, $R_\infty(\sigma)$ is formally smooth over $\mathcal{O}$ by Lemma 2.15, so it follows from (AA3) and the Auslander–Buchsbaum theorem that $M_\infty(\sigma^\circ)$ is a free $R_\infty(\sigma)$-module of finite rank, and that $M_\infty(\sigma^\circ)[1/p]$ is a locally free $R_\infty(\sigma)[1/p]$-module of rank 1. Thus $M_\infty(\sigma^\circ)$ is free of rank one over $R_\infty(\sigma)$. This proves the "only if" direction of (1).

For (2), note that $M_\infty(\sigma^\circ) \neq 0$ if and only if $M_\infty(\overline{\sigma}) \neq 0$, so we may assume that $M_\infty(\sigma^\circ) \neq 0$. The first part of (2) follows from (AA4) together with Lemmas 2.15 (2) and 2.14 (2). For the remaining part of (2), note that $R_p(\sigma)/\varpi$ is flat over $\mathcal{H}(\overline{\sigma})$ by Lemmas 2.14 and 2.15, and $M_\infty(\overline{\sigma})$ is flat over $R_p(\sigma)/\varpi$ by the only if part of (1), as required.

To prove (3) we first note that it is enough to prove the statement for absolutely irreducible π' as we may enlarge the field $\mathbb{F}$. Let us assume that π' is absolutely irreducible and let $\overline{\sigma}'$ be an irreducible representation of K contained in the socle of π'. It follows from [BL94, Bre03a] that the surjection $\text{c-Ind}_K^G \overline{\sigma}' \twoheadrightarrow \pi'$ factors through the map

$$(\text{c-Ind}_K^G \overline{\sigma}') \otimes_{\mathcal{H}(\overline{\sigma}'),\alpha'} \mathbb{F} \twoheadrightarrow \pi' \tag{4.3}$$

where $\alpha' : \mathcal{H}(\overline{\sigma}') \to \mathbb{F}$ is given by the action of $\mathcal{H}(\overline{\sigma}')$ on the one dimensional $\mathbb{F}$-vector space $\mathrm{Hom}_K(\overline{\sigma}', \pi')$. Moreover, (4.3) is an isomorphism unless π' is a character or special series. Since

$$M_\infty(\overline{\sigma}') \cong \mathrm{Hom}_K(\overline{\sigma}', M_\infty^\vee)^\vee \cong \mathrm{Hom}_G(\text{c-Ind}_K^G \overline{\sigma}', M_\infty^\vee)^\vee,$$

from (4.3) we obtain a surjection of $R_\infty(\overline{\sigma}')$-modules

$$M_\infty(\overline{\sigma}') \otimes_{\mathcal{H}(\overline{\sigma}'),\alpha'} \mathbb{F} \cong \mathrm{Hom}_G(\text{c-Ind}_K^G \overline{\sigma}' \otimes_{\mathcal{H}(\overline{\sigma}'),\alpha'} \mathbb{F}, M_\infty^\vee)^\vee \twoheadrightarrow \mathrm{Hom}_G(\pi', M_\infty^\vee)^\vee,$$

which moreover is an isomorphism if π' is not a character or special series. Thus if $\mathrm{Hom}_G(\pi', M_\infty^\vee)$ is non-zero then we deduce from the previous displayed expression that $M_\infty(\overline{\sigma}') \otimes_{\mathcal{H}(\overline{\sigma}'),\alpha'} \mathbb{F} \neq 0$. In particular, $M_\infty(\overline{\sigma}') \neq 0$ and hence $\overline{\sigma}'$ is a Serre weight for $\bar{r}$ by the only if part of (1).

We claim that α' coincides with the morphism α of Lemma 2.15 (3) (with $\overline{\sigma}'$ in place of σ). To see this, note that by the only if part of (1), we have that $\big(R_p(\sigma')/\varpi\big) \otimes_{\mathcal{H}(\overline{\sigma}'),\alpha'} \mathbb{F} \neq 0$, and hence by Lemma 2.15 (4), we find that $\widehat{\mathcal{H}(\overline{\sigma}')} \otimes_{\mathcal{H}(\overline{\sigma}'),\alpha'} \mathbb{F} \neq 0$, where $\widehat{\mathcal{H}(\overline{\sigma}')}$ denotes the completion of $\mathcal{H}(\overline{\sigma}')$ with respect to the kernel of the morphism α. This proves that α and α' coincide.

Part (5) of Lemma 2.15 now implies that $\pi \cong (\text{c-Ind}_K^G \overline{\sigma}') \otimes_{\mathcal{H}(\overline{\sigma}'),\alpha'} \mathbb{F}$. Hence, (4.3) gives us a G-equivariant surjection $\pi \twoheadrightarrow \pi'$, which is an isomorphism as π is irreducible.

Conversely, it follows from from (AA1) that there is an irreducible smooth $\mathbb{F}$-representation π' of G such that $\mathrm{Hom}_G(\pi', M_\infty^\vee)$ is non-zero; we have just seen that $\pi \cong \pi'$, so that $\mathrm{Hom}_G(\pi, M_\infty^\vee) \neq 0$, as required.

Finally, suppose that $\overline{\sigma}$ is a Serre weight of $\bar{r}$. Then as above we have an isomorphism of $R_\infty(\overline{\sigma})$-modules

$$M_\infty(\overline{\sigma}) \otimes_{\mathcal{H}(\overline{\sigma}),\alpha} \mathbb{F} \cong \mathrm{Hom}_G(\pi, M_\infty^\vee)^\vee \neq 0$$

so that $M_\infty(\overline{\sigma}) \neq 0$. This completes the proof of the "if" direction of (1). □

4.4 Smooth and admissible representations

We record a few definitions, following Section 2 of [Paš13]. Let $(R, \mathfrak{m})$ be a complete local noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$. Then $\mathrm{Mod}_G^{\mathrm{sm}}(R)$ is the full subcategory of the category of $R[G]$-modules consisting of *smooth* objects. More precisely, these are objects V such that

$$V = \bigcup_{H,n} V^H[\mathfrak{m}^n],$$

where the union is taken over open compact subgroups $H \subset G$ and over positive integers n.

We say that an object V of $\mathrm{Mod}_G^{\mathrm{sm}}(R)$ is *admissible* if $V^H[\mathfrak{m}^n]$ is a finitely generated R-module for every compact open subgroup $H \subset G$ and every $n \geqslant 1$. Moreover, V is called *locally admissible* if, for every $v \in V$, the smallest $R[G]$-submodule of V containing v is admissible. We let $\mathrm{Mod}_G^{\mathrm{l.adm}}(R)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(R)$ consisting of locally admissible representations.

The categories $\mathrm{Mod}_G^{\mathrm{sm}}(R)$ and $\mathrm{Mod}_G^{\mathrm{l.adm}}(R)$ are abelian (see [Eme10] for the second one) and have enough injectives.

4.5 Definition. (i) A monomorphism $\iota : N \hookrightarrow M$ in an abelian category is called *essential* if, for every non-zero subobject $M' \subset M$, $\iota(N) \cap M'$ is non-zero.

(ii) An *injective envelope* of an object N of an abelian category is an essential monomorphism $\iota : N \hookrightarrow I$ with I an injective object of the abelian category.

If they exist, injective envelopes are unique up to (non-unique) isomorphism. By Lemma 2.3 of [Paš13], the category $\mathrm{Mod}_G^{\mathrm{sm}}(R)$ admits injective envelopes. The category $\mathrm{Mod}_G^{\mathrm{l.adm}}(R)$ also admits injective envelopes. (This follows from [Paš10, Lem. 3.2] and the fact that the inclusion of $\mathrm{Mod}_G^{\mathrm{l.adm}}(R)$ into $\mathrm{Mod}_G^{\mathrm{sm}}(R)$ has a right adjoint, namely the functor to which any smooth G-representation associates its maximal locally admissible subrepresentation.)

4.6 Lemma. *If V is a locally admissible representation of G, then the inclusion $\mathrm{soc}_G(V) \hookrightarrow V$ is essential.*

Proof. Any non-zero subrepresentation of V contains a non-zero finitely generated subrepresentation. Thus it suffices to show that any non-zero finitely generated subrepresentation W of V has a non-zero intersection with $\mathrm{soc}_G(V)$. Since $\mathrm{soc}_G(V) \cap W = \mathrm{soc}_G(W)$, it suffices to show that any such subrepresentation has a non-zero socle. This follows from the fact that every finitely generated admissible representation of G is of finite length by [Eme10, Thm. 2.3.8]. □

4.7 Definition. (i) An epimorphism $q : M \twoheadrightarrow N$ in an abelian category is called *essential* if a morphism $s : M' \to M$ is an epimorphism whenever $q \circ s$ is an epimorphism.

(ii) A *projective envelope* of an object N of an abelian category is an essential epimorphism $q : P \twoheadrightarrow N$ with P a projective object in the abelian category.

Pontryagin duality reverses arrows, so it exchanges injective and projective objects as well as injective and projective envelopes.

4.8 Projectivity of M_∞

Our first aim is to show that $M_\infty^\vee$ is an injective locally admissible representation of G.

4.9 Lemma. *$M_\infty^\vee$ is an admissible object of $\mathrm{Mod}_G^{\mathrm{sm}}(R_\infty)$, and thus lies in $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$.*

Proof. Dually it is enough by [Eme10, Lem. 2.2.11] to show that M_∞ is a finitely generated $R_\infty[[K]]$-module, which is (AA1). □

4.10 Lemma. *Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{H}(\overline{\sigma})$ with residue field $\kappa(\mathfrak{m})$. Then*

$$\mathrm{Tor}_i^{\mathcal{H}(\overline{\sigma})}(\text{c-Ind}_K^G \overline{\sigma}, \kappa(\mathfrak{m})) = 0, \quad \forall i > 0.$$

Proof. Since the map $\mathbb{F} \to \overline{\mathbb{F}}$ is faithfully flat, we can and do assume that $\mathbb{F}$ is algebraically closed. Since $\mathcal{H}(\overline{\sigma}) = \mathbb{F}[S^{\pm 1}, T]$, we have $\mathfrak{m} = (S - \mu, T - \lambda)$ for some $\mu \in \mathbb{F}^\times$, $\lambda \in \mathbb{F}$. Since the sequence $S - \mu, T - \lambda$ is regular in $\mathcal{H}(\overline{\sigma})$, the Koszul complex $K_\bullet$ associated to it is a resolution of $\kappa(\mathfrak{m})$ by free $\mathcal{H}(\overline{\sigma})$-modules, [Mat89, Thm. 16.5 (i)]. Thus the complex $K_\bullet \otimes_{\mathcal{H}(\overline{\sigma})} \text{c-Ind}_K^G \overline{\sigma}$ computes the Tor-groups we are after, and to verify the claim it is enough to show that the sequence $S-\mu, T-\lambda$ is regular on $\text{c-Ind}_K^G \overline{\sigma}$.

If $f \in \text{c-Ind}_K^G \overline{\sigma}$ then $(Sf)(g) = f(gz)$, where $z = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$. Since such an f is supported only on finitely many cosets $K\backslash G$, we deduce that the map

$$\text{c-Ind}_K^G \overline{\sigma} \xrightarrow{S-\mu} \text{c-Ind}_K^G \overline{\sigma}$$

is injective. The quotient is isomorphic to $\text{c-Ind}_{ZK}^G \overline{\sigma}$, where z acts on $\overline{\sigma}$ by μ. It follows from the proof of [BL94, Thm. 19] that $\text{c-Ind}_{ZK}^G \overline{\sigma}$ is a free $\mathbb{F}[T]$-module. Thus the map

$$\text{c-Ind}_{ZK}^G \overline{\sigma} \xrightarrow{T-\lambda} \text{c-Ind}_{ZK}^G \overline{\sigma}$$

is injective, and the sequence $S-\mu, T-\lambda$ is regular on $\text{c-Ind}_K^G \overline{\sigma}$, as required. $\square$

4.11 LEMMA. *Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{H}(\overline{\sigma})$. Then*

$$\mathrm{Ext}^i_G\big(\kappa(\mathfrak{m}) \otimes_{\mathcal{H}(\overline{\sigma})} \text{c-Ind}_K^G \overline{\sigma}, M_\infty^\vee\big) = 0, \quad \forall i \geqslant 1,$$

where the Ext*-groups are computed in* $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$.

Proof. We first prove that $\mathrm{Ext}^i_G\big(\text{c-Ind}_K^G \overline{\sigma}, M_\infty^\vee\big) = 0$, for all $i \geqslant 1$. Let $M_\infty^\vee \hookrightarrow J^\bullet$ be an injective resolution of $M_\infty^\vee$ in $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$. Since

$$\mathrm{Hom}_K(\tau, J|_K) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G \tau, J)$$

and the functor c-Ind_K^G is exact, the restriction of an injective object in $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ to K is injective in $\mathrm{Mod}_K^{\mathrm{sm}}(\mathcal{O})$. Thus $(J^\bullet)|_K$ is an injective resolution of $M_\infty^\vee|_K$ in $\mathrm{Mod}_K^{\mathrm{sm}}(\mathcal{O})$. Since $\mathrm{Hom}_G(\text{c-Ind}_K^G \overline{\sigma}, J^\bullet) \cong \mathrm{Hom}_K(\overline{\sigma}, (J^\bullet)|_K)$, we conclude that we have natural isomorphisms

$$\mathrm{Ext}^i_G\big(\text{c-Ind}_K^G \overline{\sigma}, M_\infty^\vee\big) \cong \mathrm{Ext}^i_K\big(\overline{\sigma}, M_\infty^\vee\big), \quad \forall i \geqslant 0.$$

Since M_∞ is a projective $\mathcal{O}[[K]]$-module by (AA2), $M_\infty^\vee$ is injective in $\mathrm{Mod}_K^{\mathrm{sm}}(\mathcal{O})$, and thus the Ext-groups vanish as claimed.

Let $F_\bullet \twoheadrightarrow \kappa(\mathfrak{m})$ be a resolution of $\kappa(\mathfrak{m})$ by finite free $\mathcal{H}(\overline{\sigma})$-modules. Lemma 4.10 implies that the complex $F_\bullet \otimes_{\mathcal{H}(\overline{\sigma})} \text{c-Ind}_K^G \overline{\sigma}$ is a resolution of $\kappa(\mathfrak{m}) \otimes_{\mathcal{H}(\overline{\sigma})} \text{c-Ind}_K^G \overline{\sigma}$ by acyclic objects for the functor $\mathrm{Hom}_G(*, M_\infty^\vee)$. We conclude that the cohomology of the complex

$$\mathrm{Hom}_G(F_\bullet \otimes_{\mathcal{H}(\overline{\sigma})} \text{c-Ind}_K^G \overline{\sigma}, M_\infty^\vee)$$

computes the groups $\mathrm{Ext}^i_G\big(\kappa(\mathfrak{m}) \otimes_{\mathcal{H}(\overline{\sigma})} \text{c-Ind}_K^G \overline{\sigma}, M_\infty^\vee\big)$. We may think of the transition maps in $F_\bullet$ as matrices with entries in $\mathcal{H}(\overline{\sigma})$. The functor $\mathrm{Hom}_G(*, M_\infty^\vee)^\vee$ transposes these matrices twice, thus we get an isomorphism of complexes:

$$\mathrm{Hom}_G(F_\bullet \otimes_{\mathcal{H}(\overline{\sigma})} \text{c-Ind}_K^G \overline{\sigma}, M_\infty^\vee)^\vee \cong F_\bullet \otimes_{\mathcal{H}(\overline{\sigma})} \mathrm{Hom}_G(\text{c-Ind}_K^G \overline{\sigma}, M_\infty^\vee)^\vee \cong F_\bullet \otimes_{\mathcal{H}(\overline{\sigma})} M_\infty(\overline{\sigma}).$$

The above isomorphism induces a natural isomorphism

$$\Big(\mathrm{Ext}^i_G\big(\kappa(\mathfrak{m}) \otimes_{\mathcal{H}(\overline{\sigma})} \text{c-Ind}_K^G \overline{\sigma}, M_\infty^\vee\big)\Big)^\vee \cong \mathrm{Tor}_i^{\mathcal{H}(\overline{\sigma})}\big(\kappa(\mathfrak{m}), M_\infty(\overline{\sigma})\big), \quad \forall i \geqslant 0.$$

The isomorphism implies the assertion, as $M_\infty(\overline{\sigma})$ is a flat $\mathcal{H}(\overline{\sigma})$-module by Proposition 4.2 (2). $\square$

4.12 LEMMA. *Let $y : \mathcal{H}(\overline{\sigma}) \to \mathbb{F}'$ be a homomorphism of $\mathbb{F}$-algebras, where $\mathbb{F}'$ is a finite field extension of $\mathbb{F}$. Let $\lambda' := \mathbb{F}' \otimes_{\mathcal{H}(\overline{\sigma}), y} \text{c-Ind}_K^G \overline{\sigma}$ and let λ be an absolutely irreducible $\mathbb{F}$-representation of G, which is either principal series or supersingular. If λ is a subquotient of λ' then λ' is isomorphic to a direct sum of finitely many copies of λ.*

Proof. In the course of the proof we will use the following fact repeatedly: if A and B are $\mathbb{F}$-representations of G and A is finitely generated as an $\mathbb{F}[G]$-module then:

$$\operatorname{Hom}_G(A,B)\otimes_{\mathbb{F}}\overline{\mathbb{F}}\cong \operatorname{Hom}_G(A\otimes_{\mathbb{F}}\overline{\mathbb{F}}, B\otimes_{\mathbb{F}}\overline{\mathbb{F}}), \tag{4.13}$$

where $\overline{\mathbb{F}}$ denotes the algebraic closure of $\mathbb{F}$, see [Paš13, Lem. 5.1]. Then

$$\overline{\mathbb{F}}\otimes_{\mathbb{F}}\lambda'\cong \overline{\mathbb{F}}\otimes_{\mathbb{F}}\mathbb{F}'\otimes_{\mathcal{H}(\overline{\sigma}),y}\text{c-Ind}_K^G\overline{\sigma}\cong \bigoplus_{\iota:\mathbb{F}'\to\overline{\mathbb{F}}}\overline{\mathbb{F}}\otimes_{\mathcal{H}(\overline{\sigma}),\iota\circ y}\text{c-Ind}_K^G\overline{\sigma}, \tag{4.14}$$

where the sum is taken over $\mathbb{F}$-algebra homomorphisms $\iota:\mathbb{F}'\to\overline{\mathbb{F}}$. By the classification theorems of Barthel–Livne [BL94] and Breuil [Bre03a], each representation $\overline{\mathbb{F}}\otimes_{\mathcal{H}(\overline{\sigma}),\iota\circ y}\text{c-Ind}_K^G\overline{\sigma}$ is either irreducible, an extension of a special series by a character, or an extension of a character by a special series. Since λ is a subquotient of λ' by assumption, $\lambda\otimes_{\mathbb{F}}\overline{\mathbb{F}}$ is a subquotient of $\lambda'\otimes_{\mathbb{F}}\overline{\mathbb{F}}$, and since λ is neither special series nor a character, we deduce that

$$\lambda\otimes_{\mathbb{F}}\overline{\mathbb{F}}\cong\overline{\mathbb{F}}\otimes_{\mathcal{H}(\overline{\sigma}),\iota\circ y}\text{c-Ind}_K^G\overline{\sigma},$$

for some embedding $\iota:\mathbb{F}'\hookrightarrow\overline{\mathbb{F}}$. For every $\tau\in\operatorname{Gal}(\overline{\mathbb{F}}/\mathbb{F})$, we have

$$\overline{\mathbb{F}}\otimes_{\mathcal{H}(\overline{\sigma}),\tau\circ\iota\circ y}\text{c-Ind}_K^G\overline{\sigma}\cong\overline{\mathbb{F}}\otimes_{\overline{\mathbb{F}},\tau}(\overline{\mathbb{F}}\otimes_{\mathcal{H}(\overline{\sigma}),\iota\circ y}\text{c-Ind}_K^G\overline{\sigma})\cong\overline{\mathbb{F}}\otimes_{\overline{\mathbb{F}},\tau}(\overline{\mathbb{F}}\otimes_{\mathbb{F}}\lambda)\cong\overline{\mathbb{F}}\otimes_{\mathbb{F}}\lambda.$$

Hence all the summands in (4.14) are isomorphic to $\lambda\otimes_{\mathbb{F}}\overline{\mathbb{F}}$. It follows from (4.13) that λ' is isomorphic to a direct sum of copies of λ, as required. □

4.15 Theorem. *$M_\infty^\vee$ is an injective object in* $\operatorname{Mod}_G^{\text{l.adm}}(\mathcal{O})$.

Proof. Let $M_\infty^\vee\hookrightarrow J$ be an injective envelope of $M_\infty^\vee$ in $\operatorname{Mod}_G^{\text{l.adm}}(\mathcal{O})$. Lemma 4.6 shows that the composition $\operatorname{soc}_G M_\infty^\vee\hookrightarrow M_\infty^\vee\hookrightarrow J$ is an essential monomorphism, and thus induces an isomorphism between $\operatorname{soc}_G M_\infty^\vee$ and $\operatorname{soc}_G J$. Proposition 4.2 (3) shows that $\operatorname{soc}_G M_\infty^\vee$, and thus also $\operatorname{soc}_G J$, is isomorphic to a direct sum of copies of the representation π associated to $\bar{r}$ via Lemma 2.15 (5).

Let us assume that the quotient $J/M_\infty^\vee$ is non-zero; then there is a smooth irreducible K-subrepresentation $\overline{\sigma}\subset J/M_\infty^\vee$. Let κ be the G-subrepresentation of $J/M_\infty^\vee$ generated by $\overline{\sigma}$. Since $J/M_\infty^\vee$ is locally admissible, and $\overline{\sigma}$ is finitely generated as a K-representation, κ is an admissible representation of G. Thus $\operatorname{Hom}_G(\text{c-Ind}_K^G\overline{\sigma},\kappa)\cong\operatorname{Hom}_K(\overline{\sigma},\kappa)$ is a finite dimensional $\mathbb{F}$-vector space.

Let m be any irreducible $\mathcal{H}(\overline{\sigma})$-submodule of $\operatorname{Hom}_G(\text{c-Ind}_K^G\overline{\sigma},\kappa)$. Since m is finite dimensional over $\mathbb{F}$, Schur's lemma implies that $\mathbb{F}':=\operatorname{End}_{\mathcal{H}(\overline{\sigma})}(\mathrm{m})$ is a finite dimensional division algebra over $\mathbb{F}$. Since $\mathbb{F}$ is a finite field we deduce that $\mathbb{F}'$ is a finite field extension of $\mathbb{F}$. Since $\mathcal{H}(\overline{\sigma})$ is commutative we further deduce that m is a one dimensional $\mathbb{F}'$-vector space and thus obtain a surjective homomorphism of $\mathbb{F}$-algebras $y:\mathcal{H}(\overline{\sigma})\twoheadrightarrow\mathbb{F}'$. Moreover, by the construction of m, we obtain a non-zero G-equivariant map:

$$\pi':=\mathbb{F}'\otimes_{\mathcal{H}(\overline{\sigma}),y}\text{c-Ind}_K^G\overline{\sigma}\to\kappa\subset J/M_\infty^\vee.$$

Since $\operatorname{Ext}_G^1(\pi',M_\infty^\vee)=0$ by Lemma 4.11, by applying $\operatorname{Hom}_G(\pi',*)$ to the exact sequence $0\to M_\infty^\vee\to J\to J/M_\infty^\vee\to 0$, we obtain a short exact sequence

$$0\to\operatorname{Hom}_G(\pi',M_\infty^\vee)\to\operatorname{Hom}_G(\pi',J)\to\operatorname{Hom}_G(\pi',J/M_\infty^\vee)\to 0.$$

Moreover, we know that $\operatorname{Hom}_G(\pi',J/M_\infty^\vee)$ is non-zero. Hence, $\operatorname{Hom}_G(\pi',J)$ is non-zero.

Fix a non-zero G-equivariant map $\varphi:\pi'\to J$; then $\varphi(\pi')\cap\operatorname{soc}_G J\neq 0$. Since $\operatorname{soc}_G J$ is isomorphic to a direct sum of copies of π, we find that π is an irreducible subquotient of π'. It follows from Lemma 4.12 that π' is then isomorphic to a finite direct sum of copies of π, and so in particular is semi-simple. As we've already noted, the map $M_\infty^\vee \hookrightarrow J$ induces an isomorphism $\mathrm{soc}_G M_\infty^\vee \cong \mathrm{soc}_G J$, and so the map $\mathrm{Hom}_G(\pi', M_\infty^\vee) \to \mathrm{Hom}_G(\pi', J)$ is an isomorphism. This implies $\mathrm{Hom}_G(\pi', J/M_\infty^\vee) = 0$, contradicting the assumption $J/M_\infty^\vee \neq 0$. Hence $M_\infty^\vee = J$ is injective, as required. □

4.16 Removing the patching variables

We now show that we can pass from M_∞ to an arithmetic action of R_p on a projective envelope of $\pi^\vee$, where as always π is the representation.associated to $\bar{r}$ via Lemma 2.15 (5). Let $(A, \mathfrak{m})$ be a complete local noetherian $\mathcal{O}$-algebra. Let $\mathfrak{C}(A)$ be the Pontryagin dual of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$, where, for the moment, we allow G to be any p-adic analytic group. There is a forgetful functor from $\mathfrak{C}(A)$ to $\mathfrak{C}(\mathcal{O})$. In this subsection we prove a structural result about objects P of $\mathfrak{C}(A)$ that are projective in $\mathfrak{C}(\mathcal{O})$. We will apply this result to $P = M_\infty$ and $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

4.17 LEMMA. *Let $(A, \mathfrak{m})$ be a complete local noetherian $\mathbb{F}$-algebra with residue field $\mathbb{F}$. Let $P \in \mathfrak{C}(A)$ be such that P is projective in $\mathfrak{C}(\mathbb{F})$ and the map $P \twoheadrightarrow \mathrm{cosoc}_{\mathfrak{C}(\mathbb{F})} P$ is essential. Assume that all irreducible subquotients of $\mathrm{cosoc}_{\mathfrak{C}(\mathbb{F})} P$ are isomorphic to some given object S, for which $\mathrm{End}_{\mathfrak{C}(\mathbb{F})}(S) = \mathbb{F}$. If $\mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(P, S)^\vee$ is a free A-module of rank 1 then there is an isomorphism $A \widehat{\otimes}_{\mathbb{F}} \mathrm{Proj}(S) \cong P$ in $\mathfrak{C}(A)$, where $\mathrm{Proj}(S) \twoheadrightarrow S$ is a projective envelope of S in $\mathfrak{C}(\mathbb{F})$.*

Proof. The assumption on the cosocle of P implies that $(\mathrm{cosoc}\, P)^\vee$ is isomorphic to a direct sum of copies of $\lambda := S^\vee$. This means that we have natural isomorphisms:

$$(\mathrm{cosoc}\, P)^\vee \cong \lambda \otimes_{\mathbb{F}} \mathrm{Hom}_G(\lambda, (\mathrm{cosoc}\, P)^\vee) \cong \lambda \otimes_{\mathbb{F}} \mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S).$$

Taking Pontryagin duals we get a natural isomorphism in $\mathfrak{C}(\mathbb{F})$:

$$\mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S)^\vee \widehat{\otimes}_{\mathbb{F}} S \cong \mathrm{cosoc}\, P.$$

Since the isomorphism is natural, it is an isomorphism in $\mathfrak{C}(A)$ with the trivial action of A on S. Hence, we get a surjection in $\mathfrak{C}(A)$:

$$P \twoheadrightarrow \mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S)^\vee \widehat{\otimes}_{\mathbb{F}} S.$$

The surjection $\mathrm{Proj}(S) \twoheadrightarrow S$ induces a surjection

$$\mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S)^\vee \widehat{\otimes}_{\mathbb{F}} \mathrm{Proj}(S) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S)^\vee \widehat{\otimes}_{\mathbb{F}} S,$$

with trivial A-action on $\mathrm{Proj}(S)$. The source of this surjection is projective in $\mathfrak{C}(A)$, since in general for a compact A-module m

$$\mathrm{Hom}_{\mathfrak{C}(A)}(\mathrm{m} \widehat{\otimes}_{\mathbb{F}} \mathrm{Proj}(S), -) \cong \mathrm{Hom}_A(\mathrm{m}, \mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{Proj}(S), -)), \tag{4.18}$$

and $\mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S)^\vee = \mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(P, S)^\vee$ is projective since it is a free A-module of rank 1.

Hence there is a map $\mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S)^\vee \widehat{\otimes}_{\mathbb{F}} \mathrm{Proj}(S) \to P$ in $\mathfrak{C}(A)$, such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S)^\vee \widehat{\otimes}_{\mathbb{F}} \mathrm{Proj}(S) \arrow[r] \arrow[d, two heads] & P \arrow[d, two heads] \\
\mathrm{Hom}_{\mathfrak{C}(\mathbb{F})}(\mathrm{cosoc}\, P, S)^\vee \widehat{\otimes}_{\mathbb{F}} S \arrow[r, "\cong"] & \mathrm{cosoc}\, P
\end{tikzcd}$$

commutes. If we forget the A-action then we obtain a map in $\mathfrak{C}(\mathbb{F})$ between projective objects, which induces an isomorphism on their cosocles. Hence the map is an isomorphism in $\mathfrak{C}(\mathbb{F})$, and hence also an isomorphism in $\mathfrak{C}(A)$. □

4.19 Proposition. *Let A be a complete local noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, that is $\mathcal{O}$-flat. Let P in $\mathfrak{C}(A)$ be such that P is projective in $\mathfrak{C}(\mathcal{O})$ and the map $P \twoheadrightarrow \operatorname{cosoc}_{\mathfrak{C}(\mathcal{O})} P$ is essential. Assume that all the irreducible subquotients of $\operatorname{cosoc}_{\mathfrak{C}(\mathcal{O})} P$ are isomorphic to S, and $\operatorname{End}_{\mathfrak{C}(\mathcal{O})}(S) = \mathbb{F}$. If $\operatorname{Hom}_{\mathfrak{C}(\mathcal{O})}(P, S)^\vee$ is a free $A/\varpi A$-module of rank 1 then there is an isomorphism in $\mathfrak{C}(A)$:*

$$A \widehat{\otimes}_{\mathcal{O}} \operatorname{Proj}(S) \cong P,$$

where $\operatorname{Proj}(S) \twoheadrightarrow S$ is a projective envelope of S in $\mathfrak{C}(\mathcal{O})$.

Proof. A special case of Lemma 4.17 implies that the reductions of $A \widehat{\otimes}_{\mathcal{O}} \operatorname{Proj}(S)$ and P modulo ϖ are isomorphic in $\mathfrak{C}(\mathbb{F})$. Arguing as in (4.18) we deduce that $A \widehat{\otimes}_{\mathcal{O}} \operatorname{Proj}(S)$ is projective in $\mathfrak{C}(A)$. Thus there is a map in $\mathfrak{C}(A)$,

$$A \widehat{\otimes}_{\mathcal{O}} \operatorname{Proj}(S) \to P,$$

which is an isomorphism modulo ϖ. If V is a cokernel of this map then $V/\varpi V = 0$, and Nakayama's lemma for compact $\mathcal{O}$-modules implies that $V = 0$. Since P is projective this surjection must split. Since the map is an isomorphism modulo ϖ, if U is the kernel of this map then the decomposition $A \widehat{\otimes}_{\mathcal{O}} \operatorname{Proj}(S) \cong U \oplus P$ implies that $U/\varpi U = 0$, and so $U = 0$, as required. □

4.20 Corollary. *Let A, P, and S be as in Proposition 4.19. Then for any $\mathcal{O}$-algebra homomorphism $x : A \to \mathcal{O}$, $P \widehat{\otimes}_{A,x} \mathcal{O}$ is a projective envelope of S in $\mathfrak{C}(\mathcal{O})$.*

Proof. It follows from Proposition 4.19 that

$$P \widehat{\otimes}_{A,x} \mathcal{O} \cong (\operatorname{Proj}(S) \widehat{\otimes}_{\mathcal{O}} A) \widehat{\otimes}_{A,x} \mathcal{O} \cong \operatorname{Proj}(S).$$ □

4.21 Remark. If we replace the assumption in Proposition 4.19 that $\operatorname{Hom}_{\mathfrak{C}(\mathcal{O})}(P, S)^\vee$ is a free $A/\varpi A$-module of rank 1 with the assumption that it is free of rank n, then we have an isomorphism $A^{\oplus n} \widehat{\otimes}_{\mathcal{O}} \operatorname{Proj}(S) \cong P$ in $\mathfrak{C}(A)$.

Indeed, a generalization of Lemma 4.17 to the rank n case gives an isomorphism in $\mathfrak{C}(\mathbb{F})$ of $A^{\oplus n} \widehat{\otimes}_{\mathcal{O}} \operatorname{Proj}(S)$ and P modulo ϖ. (In fact, the statement of Lemma 4.17 can be strengthened as follows: if $\operatorname{Hom}_{\mathfrak{C}(\mathbb{F})}(P, S)^\vee$ is a projective object in the category of compact A-modules, then we have an isomorphism $\operatorname{Hom}_{\mathfrak{C}(\mathbb{F})}(P, S)^\vee \widehat{\otimes}_{\mathbb{F}} \operatorname{Proj} S \simeq P$. The key is again the projectivity of the completed tensor product $\operatorname{Hom}_{\mathfrak{C}(\mathbb{F})}(P, S)^\vee \widehat{\otimes}_{\mathbb{F}} \operatorname{Proj} S$, which follows from our assumption and from (4.18).) We then upgrade the isomorphism modulo ϖ to an isomorphism in $\mathfrak{C}(A)$ as in the proof of Proposition 4.19, again relying on (4.18).

We now apply the results above in the special case of $P = M_\infty$ and $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

4.22 Proposition. *Let $A = \mathcal{O}[[x_1, \ldots, x_d]]$ and choose a homomorphism of local $\mathcal{O}$-algebras $A \to R_\infty$, that induces an isomorphism $R_p \widehat{\otimes}_{\mathcal{O}} A \cong R_\infty$. Then there is an isomorphism in $\mathfrak{C}(A)$:*

$$M_\infty \cong \widetilde{P} \widehat{\otimes}_{\mathcal{O}} A,$$

where $\widetilde{P} \twoheadrightarrow \pi^\vee$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$.

Proof. Theorem 4.15 implies that M_∞ is projective in $\mathfrak{C}(\mathcal{O})$. As we already noted in the proof of that theorem, since $M_\infty^\vee$ is locally admissible, it follows from Lemma 4.6 that $\operatorname{soc}_G M_\infty^\vee \hookrightarrow M_\infty^\vee$ is essential, and hence that $M_\infty \twoheadrightarrow \operatorname{cosoc}_{\mathfrak{C}(\mathcal{O})} M_\infty$ is essential. Proposition 4.2 (3) implies that all the irreducible subquotients of $\operatorname{cosoc}_{\mathfrak{C}(\mathcal{O})} M_\infty$ are isomorphic to $\pi^\vee$. It is therefore enough to show that

$$M_\infty(\pi) := \operatorname{Hom}_{\mathfrak{C}(\mathcal{O})}(M_\infty, \pi^\vee)^\vee \cong \operatorname{Hom}_G(\pi, M_\infty^\vee)^\vee$$

is a free A/ϖ-module of rank 1, since the assertion then follows from Proposition 4.19.

As in the proof of Proposition 4.2, we have

$$M_\infty(\pi) \cong \mathbb{F} \otimes_{\mathcal{H}(\overline{\sigma})} M_\infty(\overline{\sigma}) \cong \mathbb{F} \otimes_{\mathcal{H}(\sigma^\circ)} M_\infty(\sigma^\circ).$$

It follows from Proposition 4.2 (1) that $M_\infty(\pi)$ is a free $\mathbb{F} \otimes_{\mathcal{H}(\sigma^\circ)} R_\infty(\sigma)$-module of rank 1. Since $R_\infty(\sigma) \cong R_p(\sigma) \widehat{\otimes}_{\mathcal{O}} A$ and the map $\mathcal{H}(\overline{\sigma}) \to R_\infty(\sigma)/\varpi$ factors through $R_p(\sigma)/\varpi$ by Proposition 4.2 (2), we conclude that the map $A \to R_\infty$ induces an isomorphism

$$A/\varpi \cong \mathbb{F} \otimes_{\mathcal{H}(\sigma^\circ)} R_\infty(\sigma).$$

(Recall that $\mathbb{F} \otimes_{\mathcal{H}(\sigma^\circ)} R_p(\sigma)/\varpi = \mathbb{F}$, by Lemma 2.15 (4).) Thus $M_\infty(\pi)$ is a free A/ϖ-module of rank 1, as required. □

4.23 Corollary. *Let $A \to R_\infty$ be as in Proposition 4.22 and let $x : A \to \mathcal{O}$ be a homomorphism of local $\mathcal{O}$-algebras. Then $M_\infty \widehat{\otimes}_{A,x} \mathcal{O}$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$ with a continuous $R_p \cong R_\infty \otimes_{A,x} \mathcal{O}$-action, which commutes with the action of G.*

Proof. This follows from Corollary 4.20. □

4.24 Uniqueness of arithmetic actions

As in the statement of Proposition 4.22, we let $\widetilde{P} \twoheadrightarrow \pi^\vee$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$.

4.25 Proposition. *$\widetilde{P}$ can be endowed with an arithmetic action of R_p (in the sense of Section 3.1 when $d = 0$).*

Proof. Making any choice of morphism $x : A \to \mathcal{O}$ in Corollary 4.23, we obtain an action of R_p on $M_\infty \widehat{\otimes}_{A,x} \mathcal{O} \cong \widetilde{P}$. Since the action of R_∞ on M_∞ is an arithmetic action, it follows immediately from the definitions that this induced action of R_p on $\widetilde{P}$ is also an arithmetic action. □

4.26 Recapping capture

We now very briefly recall the theory of capture from [CDP14, §2.4], specialised to the case of interest to us. We note that analogues of these results are valid for general choices of G, and in particular do not use either Colmez's functor nor p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let M be a compact linear-topological $\mathcal{O}[[K]]$-module, and let $\{V_i\}_{i\in I}$ be a set of continuous K-representations on finite-dimensional E-vector spaces.

4.27 Definition. We say that $\{V_i\}_{i\in I}$ *captures* M if for any proper quotient $M \twoheadrightarrow Q$, we have $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(M, V_i^*) \neq \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(Q, V_i^*)$ for some $i \in I$.

This definition is used only in the proof of the following result.

4.28 Proposition. *Suppose that $\phi \in \mathrm{End}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P})$ kills each $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}, \sigma^*_{a,b})$ for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geqslant 0}$. Then $\phi = 0$.*

Proof. Since $\widetilde{P}$ is projective in $\mathfrak{C}(\mathcal{O})$, it follows from [CDP14, Prop. 2.12] that the set $\{\sigma_{a,b}\}$ captures $\widetilde{P}$. The result follows from [CDP14, Lem. 2.9] (that is, from an application of the definition of capture to the cokernel of ϕ). □

Set $M(\sigma^\circ) := \Big(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}, (\sigma^\circ)^d)\Big)^d$.

4.29 Lemma. *Let $\sigma = \sigma_{a,b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geqslant 0}$ and let $\mathfrak{m}_y$ be a maximal ideal of $\mathcal{H}(\sigma)$. Then $\kappa(y) \otimes_{\mathcal{H}(\sigma)} M(\sigma^\circ)[1/p] \neq 0$ if and only if $\mathfrak{m}_x := \eta(\mathfrak{m}_y) R_p(\sigma)[1/p]$ is a maximal ideal of $R_p(\sigma)[1/p]$ in the support of $M(\sigma^\circ)[1/p]$ for some (equivalently, any) arithmetic action of R_p on $\widetilde{P}$.*

Proof. Since $M(\sigma^\circ)$ is a finitely generated $R_p(\sigma)$-module and the action of $\mathcal{H}(\sigma)$ on $M(\sigma^\circ)[1/p]$ factors through the action of $R_p(\sigma)[1/p]$ via η, we deduce that $\kappa(y) \otimes_{\mathcal{H}(\sigma)} M(\sigma^\circ)[1/p]$ is a finitely generated $\kappa(y) \otimes_{\mathcal{H}(\sigma)} R_p(\sigma)[1/p]$-module. If $\kappa(y) \otimes_{\mathcal{H}(\sigma)} M(\sigma^\circ)[1/p] \neq 0$ then we deduce from Lemma 2.12 that $\mathfrak{m}_x$ is a maximal ideal of $R_p(\sigma)[1/p]$ in the support of $M(\sigma^\circ)$. Conversely, if y is the image of x then using Lemma 2.12 we obtain $\kappa(x) = \kappa(y) \otimes_{\mathcal{H}(\sigma)} R_p(\sigma)[1/p]$ and hence:

$$\kappa(x) \otimes_{R_p(\sigma)[1/p]} M(\sigma^\circ)[1/p] \cong \kappa(y) \otimes_{\mathcal{H}(\sigma)} M(\sigma^\circ)[1/p],$$

which implies that $\kappa(y) \otimes_{\mathcal{H}(\sigma)} M(\sigma^\circ)[1/p]$ is non-zero. □

4.30 Theorem. *There is a unique arithmetic action of R_p on $\widetilde{P}$.*

Proof. The existence of such an action follows from Proposition 4.25.

Let $\sigma = \sigma_{a,b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geqslant 0}$. Let $y \in \text{m-Spec}\, \mathcal{H}(\sigma)$, such that $\kappa(y) \otimes_{\mathcal{H}(\sigma)} M(\sigma^\circ)[1/p] \neq 0$. Lemma 4.29 implies that y is the image of $x \in \text{m-Spec}\, R_p(\sigma)[1/p]$, which lies in the support of $M(\sigma^\circ)$. Proposition 2.13 implies that

$$\widehat{M(\sigma^\circ)[1/p]}_{\mathfrak{m}_y} = \widehat{M(\sigma^\circ)[1/p]}_{\mathfrak{m}_x},$$

as $\widehat{R_p(\sigma)[1/p]}_{\mathfrak{m}_x}$-modules. Moreover, the action of $\widehat{R_p(\sigma)[1/p]}_{\mathfrak{m}_x}$ on $\widehat{M(\sigma^\circ)[1/p]}_{\mathfrak{m}_y}$ does not depend on a given arithmetic action of R_p on $\widetilde{P}$, as it acts via the isomorphism in Proposition 2.13. If M is a finitely generated module over a noetherian ring R then we have injections:

$$M \hookrightarrow \prod_{\mathfrak{m}} M_{\mathfrak{m}} \hookrightarrow \prod_{\mathfrak{m}} \widehat{M}_{\mathfrak{m}},$$

where the product is taken over all the maximal ideals in R. In fact it is enough to take the product over finitely many maximal ideals: if $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are minimal associated primes of M, just pick any maximal ideals $\mathfrak{m}_1 \in V(\mathfrak{p}_1), \dots, \mathfrak{m}_n \in V(\mathfrak{p}_n)$. (The second injection follows from [Mat89, Thm. 8.9].) This observation applied to $R = R_p(\sigma)[1/p]$ and $M = M(\sigma^\circ)[1/p]$ together with Lemma 4.29 implies that we have an injection of $R_p(\sigma)[1/p]$-modules:

$$M(\sigma^\circ)[1/p] \hookrightarrow \prod_{y \in \text{m-Spec}\, \mathcal{H}(\sigma)} \widehat{M(\sigma^\circ)[1/p]}_{\mathfrak{m}_y}.$$

Since the map and the action of $R_p(\sigma)[1/p]$ on the right hand side are independent of the arithmetic action of R_p on $\widetilde{P}$, we deduce that the action of $R_p(\sigma)[1/p]$ on $M(\sigma^\circ)[1/p]$ is also independent of the arithmetic action of R_p on $\widetilde{P}$.

If $\theta : R_p \to \operatorname{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$ and $\theta' : R_p \to \operatorname{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$ are two arithmetic actions and $r \in R_p$ then it follows from the above that $\theta(r) - \theta'(r)$ will annihilate $M(\sigma^\circ_{a,b})$ for all $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geqslant 0}$. Proposition 4.28 implies that $\theta(r) = \theta'(r)$. □

4.31 Remark. A different proof of the Theorem could be given using [CDP14, Prop. 2.19].

4.32 Theorem. *If $\widetilde{P} \twoheadrightarrow \pi^\vee$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$ equipped with an arithmetic action of R_p, then there is an isomorphism in $\mathfrak{C}(R_\infty)$*

$$\widetilde{P} \mathbin{\widehat{\otimes}}_{R_p} R_\infty \cong M_\infty.$$

Proof. Let $A = \mathcal{O}[[x_1, \dots, x_d]]$, and choose a homomorphism of local $\mathcal{O}$-algebras $A \to R_\infty$ which induces an isomorphism $R_p \widehat{\otimes}_{\mathcal{O}} A \cong R_\infty$. Proposition 4.22 implies that there is an isomorphism $\widetilde{P} \widehat{\otimes}_{\mathcal{O}} A \cong M_\infty$ in $\mathfrak{C}(A)$; it is therefore enough to show that this isomorphism is R_p-linear. Any $\mathcal{O}$-algebra homomorphism $x : A \to \mathcal{O}$ induces an isomorphism $(\widetilde{P} \widehat{\otimes}_{\mathcal{O}} A) \widehat{\otimes}_{A,x} \mathcal{O} \cong M_\infty \widehat{\otimes}_{A,x} \mathcal{O}$ in $\mathfrak{C}(\mathcal{O})$. We get two actions of R_p on $M_\infty \widehat{\otimes}_{A,x} \mathcal{O}$: one of them coming from the action of R_p on M_∞, the other transported by the isomorphism. Both actions are arithmetic: the first one by Proposition 4.25, the second one by assumption. Theorem 4.30 implies that the two actions coincide; thus the isomorphism $(\widetilde{P} \widehat{\otimes}_{\mathcal{O}} A) \widehat{\otimes}_{A,x} \mathcal{O} \cong M_\infty \widehat{\otimes}_{A,x} \mathcal{O}$ is R_p-linear.

We have a commutative diagram in $\mathfrak{C}(A)$:

$$\begin{tikzcd}
\widetilde{P} \widehat{\otimes}_{\mathcal{O}} A \arrow[r, "\cong"] \arrow[d] & M_\infty \arrow[d] \\
\prod_{x:A\to\mathcal{O}} (\widetilde{P} \widehat{\otimes}_{\mathcal{O}} A) \widehat{\otimes}_{A,x} \mathcal{O} \arrow[r, "\cong"] & \prod_{x:A\to\mathcal{O}} M_\infty \widehat{\otimes}_{A,x} \mathcal{O}
\end{tikzcd}$$

where the product is taken over all $\mathcal{O}$-algebra homomorphisms $x : A \to \mathcal{O}$. We know that both vertical and the lower horizontal arrows are R_p-linear. The map $A \to \prod_{x:A\to\mathcal{O}} \mathcal{O}$ is injective (see for example [Paš13, Lem. 9.22]). Since $\widetilde{P}$ is $\mathcal{O}$-torsion free the functor $\widetilde{P} \widehat{\otimes}_{\mathcal{O}} -$ is exact, hence the first vertical arrow is injective, which implies that the top horizontal arrow is R_p-linear. $\square$

4.33 Remark. The preceding result shows that, in particular, the construction of M_∞ in [CEG+16] is independent of the choices made in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. More precisely, the isomorphism

$$\widetilde{P} \widehat{\otimes}_{R_p} R_\infty \cong M_\infty$$

exhibits M_∞ as the extension of scalars of $\widetilde{P}$, and this latter object, with its arithmetic R_p-action, is independent of all choices by Theorem 4.30. Thus, the only ambiguity in the construction of M_∞ is in the number of power series variables in R_∞, and in their precise action. As one is free to choose as many Taylor–Wiles primes as one wishes in the patching construction, and as the presentations of global deformation rings as quotients of power series rings over local deformation rings are non-canonical, it is evident that this is the exact degree of ambiguity that the construction of M_∞ is forced to permit.

5. Unitary completions of principal series representations

In this section, we record some arguments related to the paper [BB10], which proves using (φ, Γ)-module techniques that the locally algebraic representations associated to crystabelline Galois representations admit a unique unitary completion. We will use the machinery developed in the previous sections, namely the projective envelope $\widetilde{P}$ and the purely local map $R_p \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$, to independently deduce (without using (φ, Γ)-module techniques) that the locally algebraic representations corresponding to crystalline types of regular weight admit *at most one* unitary completion satisfying certain properties, and that such a completion comes from some $\widetilde{P}$.

We will use this result in Section 7 below to show, assuming the existence results of [BB10], that certain of these representations occur in completed cohomology. This gives an alternative approach to proving modularity results in the crystalline case.

As in Section 2, we write $\sigma = \sigma_{a,b} = \det^a \otimes \mathrm{Sym}^b E^2$. Let $\theta : \mathcal{H}(\sigma) \to E$ be a homomorphism, and set $\Psi := (\text{c-Ind}_K^G \sigma) \otimes_{\mathcal{H}(\sigma),\theta} E$; so Ψ is a locally algebraic principal series representation of G.

5.1 Theorem. *If Ψ is irreducible, then Ψ admits at most one non-zero admissible unitary completion $\widehat{\Psi}$ with the following property: for an open bounded G-invariant lattice Θ in $\widehat{\Psi}$, $(\Theta/\varpi)\otimes_{\mathbb{F}}\overline{\mathbb{F}}_p$ contains no subquotient of the form $(\mathrm{Ind}_B^G \chi\otimes\chi\omega^{-1})_{\mathrm{sm}}$, for any character $\chi:\mathbb{Q}_p^\times\to\overline{\mathbb{F}}_p^\times$, and no special series or characters.*

If a completion satisfying this property exists, then it is absolutely irreducible.

Proof. Let Π be a non-zero admissible unitary completion of Ψ that satisfies the property in the statement of the theorem. We will first show that Π is absolutely irreducible (and, indeed, most of the work of the proof will be in showing this). We note that in the course of the proof we are allowed to replace E by a finite field extension E', since if $\Pi\otimes_E E'$ is an absolutely irreducible E'-Banach space representation of G then Π is an absolutely irreducible E-Banach space representation of G.

Since Π is admissible, it will contain an irreducible closed sub-Banach space representation Π_1. If we let Π' denote the quotient Π/Π_1, then we must show that Π' is zero. For the moment, we note simply that if Π' is non-zero, then since the composite $\Psi\to\Pi\to\Pi'$ has dense image, we see that Π' is another non-zero admissible unitary completion of Ψ.

Let Θ be an open bounded G-invariant lattice in Π, and let $\Theta_1:=\Pi_1\cap\Theta$. Since Π_1 is also admissible, Θ_1/ϖ will contain an irreducible subquotient π. Since we are allowed to enlarge E, we may assume that π is absolutely irreducible. Let $\widetilde{P}\twoheadrightarrow\pi^\vee$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$. By the assumption on the subquotients of Θ/ϖ, there is a Galois representation $\bar{r}$ satisfying Assumption 2.2 that corresponds to π via Lemma 2.15 (5). We equip $\widetilde{P}$ with the arithmetic action of R_p provided by Theorem 4.30.

We let (for any admissible unitary E-Banach space representation Π of G)

$$M(\Pi):=\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P},\Theta^d)^d[1/p]\cong\mathrm{Hom}_G^{\mathrm{cont}}(\Pi,\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\widetilde{P},E))^d.$$

The projectivity of $\widetilde{P}$ implies that $\Pi\mapsto M(\Pi)$ is an exact covariant functor from the category of admissible unitary E-Banach space representations of G to the category of $R_p[1/p]$-modules. In particular, we have an injection $M(\Pi_1)\hookrightarrow M(\Pi)$. Since $\widetilde{P}$ is projective and π is a subquotient of Θ_1/ϖ, we have $M(\Pi_1)\neq 0$ by [Paš13, Lem. 4.13], and hence $M(\Pi)\neq 0$.

Similarly, we let

$$M(\Psi):=\mathrm{Hom}_G(\Psi,\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\widetilde{P},E))^d\cong M(\sigma^\circ)[1/p]\otimes_{\mathcal{H}(\sigma),\theta}E.$$

Recall that $\mathcal{H}(\sigma)$ acts on $M(\sigma^\circ)[1/p]$ through the composite of the homomorphism $\mathcal{H}(\sigma)\xrightarrow{\eta}R_p(\sigma)[1/p]$ and the $R_p[1/p]$-action on $M(\sigma^\circ)[1/p]$, and that, by Lemma 4.29, either the fibre module $M(\sigma^\circ)[1/p]\otimes_{\mathcal{H}(\sigma),\theta}E$ vanishes, or else θ extends to a homomorphism $\widehat{\theta}:R_p(\sigma)[1/p]\to E$ (so that then $\theta=\widehat{\theta}\circ\eta$), in which case there is a natural isomorphism

$$M(\sigma^\circ)[1/p]\otimes_{\mathcal{H}(\sigma),\theta}E\xrightarrow{\sim}M(\sigma^\circ)[1/p]\otimes_{R_p(\sigma)[1/p],\widehat{\theta}}E$$

of E-vector spaces of dimension at most 1.

The map $\Psi\to\Pi$ induces a continuous homomorphism

$$\mathrm{Hom}_G^{\mathrm{cont}}(\Pi,\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\widetilde{P},E))\to\mathrm{Hom}_G(\Psi,\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\widetilde{P},E)).$$

Since the image of Ψ in Π is dense this map is injective, and by taking duals we obtain a surjection of $R_p[1/p]$-modules

$$M(\Psi)\twoheadrightarrow M(\Pi).$$

Since the target is non-zero, and the source is an E-vector space of dimension at most 1, this must be an isomorphism. Since $M(\Pi_1)$ is a non-zero subspace of $M(\Pi)$, we therefore have induced isomorphisms $M(\Pi_1) \xrightarrow{\sim} M(\Pi) \xrightarrow{\sim} M(\Psi)$. Since, as was noted above, M is an exact functor, we find that $M(\Pi') = 0$.

We digress for a moment in order to establish that Π_1 is in fact absolutely irreducible. Since it is irreducible and admissible, its endomorphism ring is a division algebra over E. On the other hand, since M is a functor, and since $M(\Pi_1)$ is one-dimensional over E, we see that this division algebra admits a homomorphism to E. Thus this division algebra is in fact equal to E, and this implies that Π_1 is absolutely irreducible by [Paš13, Lem. 4.2].

Suppose now that $\Pi' \neq 0$. We may then apply the above argument with Π' in place of Π, and find an absolutely irreducible subrepresentation Π_2 of Π', a $G_{\mathbb{Q}_p}$-representation $\bar{r}'$, an associated irreducible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation π', for which the projective envelope $\widetilde{P}_2$ of $(\pi')^\vee$ gives rise to an exact functor M' such that $M'(\Psi) = M'(\Pi') = M'(\Pi_2)$, with all three being one-dimensional.

Since both $M(\Psi)$ and $M'(\Psi)$ are non-zero, we find that each of $\bar{r}$ and $\bar{r}'$ admits a lift that is a lattice in a crystalline representation V of Hodge–Tate weights $(1-a, -(a+b))$ determined (via Lemma 2.6) by the homomorphism $\hat{\theta} : R_p[1/p] \to E$. (Strictly speaking, for this to make sense we need to know that V is indecomposable; but this is automatic, since each of $\bar{r}$ and $\bar{r}'$ is indecomposable and has non-scalar semisimplification.)

We note that V is reducible if and only if Ψ may be identified with the locally algebraic subrepresentation of a continuous induction $(\mathrm{Ind}_B^G \chi)_{\mathrm{cont}}$, for some unitary character $\chi : T \to E^\times$, where T denotes the diagonal torus contained in the upper triangular Borel subgroup B of G. In this case it is well-known that this continuous induction is the universal unitary completion of Ψ, see [BE10, Prop. 2.2.1], and the theorem is true in this case. Thus, for the remainder of the argument, we suppose that V is irreducible, or, equivalently, that Ψ does not admit an embedding into the continuous parabolic induction of a unitary character.

We now consider separately two cases, according to whether or not $\bar{r}$ and $\bar{r}'$ are themselves isomorphic. If they are, then M and M' are isomorphic functors, and we obtain a contradiction from the fact that $M(\Pi') = 0$ while $M'(\Pi')$ is one-dimensional, implying that in fact $\Pi' = 0$, as required.

If $\bar{r}$ and $\bar{r}'$ are not isomorphic, but have isomorphic semi-simplifications, then they must each consist of an extension of the same two characters, but in opposite directions. In this case $\pi \cong (\mathrm{Ind}_B^G \omega\chi_1 \otimes \chi_2)_{\mathrm{sm}}$, $\pi' \cong (\mathrm{Ind}_B^G \omega\chi_2 \otimes \chi_1)_{\mathrm{sm}}$, for some smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{F}^\times$. In the terminology of [Col10b], π and π' are the two constituents of an atome automorphe, which by definition is the unique (up to isomorphism) non-split extension between π and π'.

We first show that $\Pi_2 = \Pi'$. To this end, set $\Pi'' := \Pi'/\Pi_2$. If $\Pi'' \neq 0$, then, running through the above argument another time, we find $\bar{r}''$, etc., such that $(\bar{r}'')^{\mathrm{ss}} \cong (\bar{r}')^{\mathrm{ss}} \cong \bar{r}^{\mathrm{ss}}$, and such that $M''(\Pi'') \neq 0$. But either $\bar{r}'' \cong \bar{r}$ or $\bar{r}'' \cong \bar{r}'$. Thus the functor M'' is isomorphic to either M' or to M. On the other hand, $M'(\Pi'') = 0$, and also $M(\Pi'') = 0$ (since Π'' is a quotient of Π' and $M(\Pi') = 0$). This contradiction shows that $\Pi'' = 0$, and thus that $\Pi' = \Pi_2$ is absolutely irreducible.

Since $M(\Pi') = 0$, we see that Θ'/ϖ does not contain a copy of π as a subquotient (here Θ' denotes some choice of G-invariant lattice in Π'). Since (as we have just shown) Π' is absolutely irreducible, it follows from [Paš13, Cor. 8.9] that $\Pi' \cong (\mathrm{Ind}_B^G \chi)_{\mathrm{cont}}$ for some unitary character $\chi : T \to E^\times$, where T denotes the diagonal torus contained in the upper triangular Borel subgroup B of G. (The proof of this result uses various Ext computations in $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$, but does not use either of Colmez's functors. The basic idea is that the extension of π' by π given by the atome automorphe induces an embedding $\widetilde{P} \hookrightarrow \widetilde{P}'$ whose cokernel is dual to the induction from B to G of a character; since $M(\Pi') = 0$, we see that Θ' embeds into this induction, and so is itself such an induction.) Thus Ψ admits an embedding into the continuous parabolic induction of a unitary character, contradicting our hypothesis that V is irreducible. Thus we conclude that in fact $\Pi' = 0$, as required.

If Ψ were to admit two non-isomorphic admissible irreducible unitary completions Π_1 and Π_2 satisfying the assumptions of the theorem, the image of the diagonal map $\Psi \to \Pi_1 \oplus \Pi_2$ would be dense in $\Pi_1 \oplus \Pi_2$. This yields a contradiction to what we have already proved, as $\Pi_1 \oplus \Pi_2$ is not irreducible. $\square$

5.2 Remark. It follows from the proof of Theorem 5.1 that if a completion $\widehat{\Psi}$ of the kind considered there exists, then there is a Galois representation $\bar{r}$ satisfying Assumption 2.2 such that θ extends to a homomorphism $\widehat{\theta} : R_p(\sigma)[1/p] \to E$. Furthermore, for any M_∞ as in Section 3, we have $M_\infty(\sigma^\circ)[1/p] \otimes_{\mathcal{H}(\sigma),\theta} E \neq 0$.

Conversely, if there is an $\bar{r}$ satisfying Assumption 2.2 such that θ extends to a homomorphism $\widehat{\theta} : R_p(\sigma)[1/p] \to E$, then the existence of a completion $\widehat{\Psi}$ is immediate from the main theorem of [BB10] (which shows that some completion exists) together with the main theorem of [Ber10] (which implies the required property of the subquotients).

5.3 Remark. As is true throughout this paper, the results of this section are equally valid for crystabelline representations, but for simplicity of notation we have restricted ourselves to the crystalline case.

6. Comparison to the local approach

We now examine the compatibility of our constructions with those of [Paš13], which work with fixed central characters. The arguments of [Paš13] make use of Colmez's functor, and the results of this section therefore also depend on this functor. In Section 6.24 we briefly discuss how to prove some of the more elementary statements in [Paš13] using the results of the previous section (and in particular not using Colmez's functor). We assume throughout this section that $p \geqslant 5$, as this assumption is made in various of the results of [Paš13] that we cite.

There are two approaches that we could take to this comparison. One would be to note that the axioms of Section 3.1 and arguments of Section 4 admit obvious analogues in the setting of a fixed central character, and thus show that if we pass to a quotient of M_∞ with fixed central character, we obtain a uniquely determined p-adic Langlands correspondence. These axioms are satisfied by the purely local object constructed in [Paš13] (which is a projective envelope of $\pi^\vee$ in a category of representations with fixed central character), and this completes the comparison.

While this route would be shorter, we have preferred to take the second approach, and go in the opposite direction: we promote the projective envelope from [Paš13] to a representation with non-constant central character by tensoring with the universal deformation of the trivial 1-dimensional representation (which has a natural Galois action by local class field theory), and show that this satisfies the axioms of Section 3.1. This requires us to make a careful study of various twisting constructions; the payoff is that we prove a stronger result than that which would follow from the first approach.

6.1 Deformation rings and twisting

Let Λ be the universal deformation ring of the trivial 1-dimensional representation of $G_{\mathbb{Q}_p}$ and let $\mathbf{1}^{\mathrm{univ}}$ be the universal deformation. Then $\mathbf{1}^{\mathrm{univ}}$ is a free Λ-module of rank 1 with a continuous $G_{\mathbb{Q}_p}$-action. We let G act on $\mathbf{1}^{\mathrm{univ}}$ via the inverse determinant (composed with the Artin map). We let Λ^{ur} be the quotient of Λ unramified deformations. We note that Λ and Λ^{ur} are formally smooth over $\mathcal{O}$ of relative dimensions 2 and 1, respectively, and in particular are $\mathcal{O}$-torsion free.

Let $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a continuous character such that $\psi\varepsilon^{-1}$ is congruent to $\det \bar{r}$ modulo ϖ. (By Remark 2.17, this implies that ψ modulo ϖ considered as a character of $\mathbb{Q}_p^\times$ coincides with the central character of π.) We let R_p^ψ denote the quotient of R_p parameterising deformations with determinant $\psi\varepsilon^{-1}$. Let $r^{\mathrm{univ},\psi}$ be the tautological deformation of $\bar{r}$ to R_p^ψ. Then $r^{\mathrm{univ},\psi} \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is a deformation of $\bar{r}$ to $R_p^\psi \widehat{\otimes}_{\mathcal{O}} \Lambda$. Since $p > 2$, this induces an isomorphism of local $\mathcal{O}$-algebras

$$R_p \xrightarrow{\cong} R_p^\psi \widehat{\otimes}_{\mathcal{O}} \Lambda. \tag{6.2}$$

Let $R_p^\psi(\sigma)$ denote the quotient of $R_p(\sigma)$ corresponding to deformations of determinant $\psi\varepsilon^{-1}$; note that $R_p^\psi(\sigma) = 0$ unless $\psi|_{\mathbb{Z}_p^\times}$ is equal to the central character of σ. If $\psi|_{\mathbb{Z}_p^\times}$ is equal to the central character of σ, then the isomorphism $R_p \xrightarrow{\cong} R_p^\psi \widehat{\otimes}_{\mathcal{O}} \Lambda$ induces an isomorphism

$$R_p(\sigma) \xrightarrow{\cong} R_p^\psi(\sigma) \widehat{\otimes}_{\mathcal{O}} \Lambda^{\mathrm{ur}}. \tag{6.3}$$

Let $\delta : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a character that is trivial modulo ϖ. Twisting by δ induces isomorphisms of $\mathcal{O}$-algebras

$$\mathrm{tw}_\delta : R_p \xrightarrow{\sim} R_p, \quad \mathrm{tw}_\delta : R_p^{\psi\delta^2} \xrightarrow{\sim} R_p^\psi.$$

(In terms of the deformation functor $D_{\bar{r}}$ pro-represented by R_p, and the deformation functors $D_{\bar{r}}^\psi$ and $D_{\bar{r}}^{\psi\delta^2}$ pro-represented by R_p^ψ and $R_p^{\psi\delta^2}$, these isomorphisms are induced by the natural bijections

$$D_{\bar{r}}(A) \xrightarrow{\sim} D_{\bar{r}}(A), \quad D_{\bar{r}}^\psi(A) \xrightarrow{\sim} D_{\bar{r}}^{\psi\delta^2}(A)$$

defined by $r_A \mapsto r_A \otimes \delta$.) Similarly we obtain isomorphisms

$$\mathrm{tw}_\delta : \Lambda \xrightarrow{\sim} \Lambda, \quad \mathrm{tw}_\delta : \Lambda(\delta)^{\mathrm{ur}} \xrightarrow{\sim} \Lambda^{\mathrm{ur}},$$

and we have a commutative diagram

$$\begin{tikzcd} R_p \arrow[r, "\cong", "(6.2)"'] \arrow[d, "\mathrm{id}"'] & R_p^{\psi\delta^2} \widehat{\otimes}_{\mathcal{O}} \Lambda \arrow[d, "\mathrm{tw}_\delta \widehat{\otimes} \mathrm{tw}_{\delta^{-1}}"] \\ R_p \arrow[r, "\cong", "(6.2)"'] & R_p^\psi \widehat{\otimes}_{\mathcal{O}} \Lambda \end{tikzcd} \tag{6.4}$$

6.5 Unfixing the central character

Let $\mathrm{Mod}_G^{\mathrm{l.adm},\psi}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$, consisting of those representations, where Z acts by the central character ψ. Let $\mathfrak{C}^\psi(\mathcal{O})$ be the Pontryagin dual of $\mathrm{Mod}_G^{\mathrm{l.adm},\psi}(\mathcal{O})$, so that we can identify $\mathfrak{C}^\psi(\mathcal{O})$ with a full subcategory of $\mathfrak{C}(\mathcal{O})$ consisting of those objects on which Z acts by ψ^{-1}.

Let $\widetilde{P}^\psi \twoheadrightarrow \pi^\vee$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}^\psi(\mathcal{O})$. By [Paš13, Prop. 6.3, Cor. 8.7, Thm.

10.71] there is a natural isomorphism

$$R_p^{\psi} \xrightarrow{\sim} \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}^{\psi}). \tag{6.6}$$

In Corollary 6.23 below we will prove a version of the isomorphism (6.6) for R_p.

6.7 Lemma. *Let $\delta : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a character that is trivial modulo ϖ. There is an isomorphism in $\mathfrak{C}(\mathcal{O})$:*

$$\varphi : \widetilde{P}^{\psi\delta^2} \xrightarrow{\sim} \widetilde{P}^{\psi} \otimes \delta^{-1} \circ \det .$$

Moreover, the following diagram commutes:

$$\begin{tikzcd} R_p^{\psi\delta^2} \arrow[r, "\mathrm{tw}_\delta", "\cong"'] \arrow[d, "(6.6)"', "\cong"] & R_p^{\psi} \arrow[d, "\cong"', "(6.6)"] \\ \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}^{\psi\delta^2}) \arrow[r, "\cong"] & \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}^{\psi}) \end{tikzcd}$$

where the lower horizontal arrow is given by $\alpha \mapsto \varphi \circ \alpha \circ \varphi^{-1}$.

Proof. The claimed isomorphism follows from the fact that twisting by $\delta^{-1} \circ \det$ induces an equivalence of categories between $\mathfrak{C}^{\psi}(\mathcal{O})$ and $\mathfrak{C}^{\psi\delta^2}(\mathcal{O})$, so that the twist of a projective envelope of $\pi^\vee$ in $\mathfrak{C}^{\psi}(\mathcal{O})$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}^{\psi\delta^2}(\mathcal{O})$. The commutativity of the diagram follows from the compatibility of the constructions of [Paš13], and in particular of the isomorphism (6.6), with twisting. (This comes down to the compatibility of the functor $\check{\mathbf{V}}$, discussed below, with twisting.) $\square$

Let $\delta : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a character that is trivial modulo ϖ. There is an evident isomorphism of pseudocompact $\mathcal{O}[[G_{\mathbb{Q}_p}]]$-modules:

$$\theta : \mathbf{1}^{\mathrm{univ}} \otimes \delta \xrightarrow{\sim} \mathbf{1}^{\mathrm{univ}},$$

and a commutative diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{tw}_{\delta^{-1}}", "\cong"'] \arrow[d, "\cong"] & \Lambda \arrow[d, "\cong"'] \\ \mathrm{End}^{\mathrm{cont}}_{G_{\mathbb{Q}_p}}(\mathbf{1}^{\mathrm{univ}}) \arrow[r, "\cong"] & \mathrm{End}^{\mathrm{cont}}_{G_{\mathbb{Q}_p}}(\mathbf{1}^{\mathrm{univ}}) \end{tikzcd} \tag{6.8}$$

where the lower horizontal arrow is given by $\alpha \mapsto \theta \circ \alpha \circ \theta^{-1}$. (In terms of the deformation functor $D_{\mathbf{1}}$ pro-represented by Λ, the isomorphism $\mathrm{tw}_{\delta^{-1}}$ is induced by the bijection $D_{\mathbf{1}}(A) \xrightarrow{\sim} D_{\mathbf{1}}(A)$ defined by $\chi \mapsto \chi\delta^{-1}$.)

6.9 Lemma. *Let $\delta : \mathbb{Q}_p^\times \to \mathcal{O}^\times$ be a character that is trivial modulo ϖ. Then there is an isomorphism*

$$\widetilde{P}^{\psi\delta^2} \mathbin{\widehat{\otimes}}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}} \xrightarrow{\sim} \widetilde{P}^{\psi} \mathbin{\widehat{\otimes}}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$$

in the category $\mathfrak{C}(R_p)$, where R_p acts on both sides by the isomorphism of (6.2).

Proof. Using Lemma 6.7 and (6.8) we obtain isomorphisms in $\mathfrak{C}(\mathcal{O})$:

$$\widetilde{P}^{\psi\delta^2} \mathbin{\widehat{\otimes}}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}} \xrightarrow{\varphi \widehat{\otimes} \mathrm{id}} (\widetilde{P}^{\psi} \otimes \delta^{-1} \circ \det) \mathbin{\widehat{\otimes}}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}} \xrightarrow{\sim}$$
$$\xrightarrow{\sim} \widetilde{P}^{\psi} \mathbin{\widehat{\otimes}}_{\mathcal{O}} (\mathbf{1}^{\mathrm{univ}} \otimes \delta \circ \det) \xrightarrow{\mathrm{id} \widehat{\otimes} \theta} \widetilde{P}^{\psi} \mathbin{\widehat{\otimes}}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}.$$

The composition of these isomorphisms is equal to $\phi \,\widehat{\otimes}\, \theta : \widetilde{P}^{\psi\delta^2} \,\widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}} \xrightarrow{\sim} \widetilde{P}^{\psi} \,\widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$, which is an isomorphism in $\mathfrak{C}(\mathcal{O})$. It follows from Lemma 6.7 and (6.8) that the following diagram commutes:

$$\begin{tikzcd}
R_p^{\psi\delta^2} \,\widehat{\otimes}_{\mathcal{O}} \Lambda \arrow[r, "\mathrm{tw}_\delta \,\widehat{\otimes}\, \mathrm{tw}_{\delta^{-1}}", "\cong"'] \arrow[d] & R_p^{\psi} \,\widehat{\otimes}_{\mathcal{O}} \Lambda \arrow[d] \\
\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}^{\psi\delta^2} \,\widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}) \arrow[r, "\cong"] & \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}^{\psi} \,\widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}})
\end{tikzcd}$$

where the lower horizontal arrow is given by $\alpha \mapsto (\varphi \,\widehat{\otimes}\, \theta) \circ \alpha \circ (\varphi \,\widehat{\otimes}\, \theta)^{-1}$. It follows from the above diagram and (6.4) that $\varphi \,\widehat{\otimes}\, \theta$ is an isomorphism in $\mathfrak{C}(R_p)$. □

6.10 Proposition. *$\widetilde{P}^{\psi} \,\widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is projective in the category of pseudocompact $\mathcal{O}[[K]]$-modules.*

Proof. Let $I_1 := \{g \in K : g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\varpi}\}$. Then I_1 is a pro-p Sylow subgroup of K and it is enough to show that $\widetilde{P}^{\psi} \,\widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is projective in the category of pseudocompact $\mathcal{O}[[I_1]]$-modules. Since I_1 is a pro-p group there is only one indecomposable projective object in the category, namely $\mathcal{O}[[I_1]]$, and thus a pseudocompact $\mathcal{O}[[I_1]]$-module is projective if and only if it is pro-free, which means that it is isomorphic to $\prod_{j\in J} \mathcal{O}[[I_1]]$ for some indexing set J.

Let $\Gamma := 1 + p\mathbb{Z}_p$. We identify Γ with the image in $G_{\mathbb{Q}_p}^{\mathrm{ab}}$ of the wild inertia subgroup of $G_{\mathbb{Q}_p}$. We may identify Λ with the completed group algebra of the pro-p completion of $G_{\mathbb{Q}_p}^{\mathrm{ab}}$, which is isomorphic to $\Gamma \times \mathbb{Z}_p$. There is an isomorphism of $\mathcal{O}$-algebras $\Lambda \cong \mathcal{O}[[\Gamma]][[x]]$, and in particular $\mathbf{1}^{\mathrm{univ}}$ is a pro-free and hence projective $\mathcal{O}[[\Gamma]]$-module.

The restriction of $\widetilde{P}^{\psi}$ to K is projective in the category of pseudocompact $\mathcal{O}[[K]]$-modules on which $Z \cap K$ acts by the central character ψ^{-1}, [Paš15, Corollary 5.3]. By restricting further to I_1, we deduce that $\widetilde{P}^{\psi}$ is projective in the category of pseudocompact $\mathcal{O}[[I_1]]$-modules, where $Z_1 := I_1 \cap Z$ acts by the central character ψ^{-1}.

Since $p > 2$ there is a character $\delta : \Gamma \to \mathcal{O}^{\times}$, such that $\delta^2 = \psi$. Twisting by characters preserves projectivity, so that $\mathbf{1}^{\mathrm{univ}} \otimes \delta$ is a projective in the category of pseudocompact $\mathcal{O}[[\Gamma]]$-modules and $\widetilde{P}^{\psi} \otimes (\delta \circ \det)$ is projective in the category of pseudocompact $\mathcal{O}[[I_1]]$-modules on which Z_1 acts trivially. We may identify this last category with the category of pseudocompact $\mathcal{O}[[I_1/Z_1]]$-modules.

We have an isomorphism of I_1-representations

$$\widetilde{P}^{\psi} \,\widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}} \cong (\widetilde{P}^{\psi} \otimes \delta \circ \det) \,\widehat{\otimes}_{\mathcal{O}} (\mathbf{1}^{\mathrm{univ}} \otimes_{\mathcal{O}} \delta),$$

where I_1 acts on $\mathbf{1}^{\mathrm{univ}} \otimes_{\mathcal{O}} \delta$ via the homomorphism $I_1 \to \Gamma$, $g \mapsto (\det g)^{-1}$. We may therefore assume that $\psi|_{Z_1}$ is trivial, so that $\widetilde{P}^{\psi}$ is projective in the category of pseudocompact $\mathcal{O}[[I_1/Z_1]]$-modules. Thus there are indexing sets J_1 and J_2 such that

$$\widetilde{P}^{\psi}|_{I_1} \cong \prod_{j_1 \in J} \mathcal{O}[[I_1/Z_1]], \quad \mathbf{1}^{\mathrm{univ}} \cong \prod_{j_2 \in J_2} \mathcal{O}[[\Gamma]]$$

where the first isomorphism is an isomorphism of pseudocompact $\mathcal{O}[[I_1]]$-modules, and the second isomorphism is an isomorphism of pseudocompact $\mathcal{O}[[\Gamma]]$-modules. Since completed tensor products commute with products, we obtain an isomorphism of pseudocompact $\mathcal{O}[[I_1]]$-modules:

$$(\widetilde{P}^{\psi} \,\widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}) \cong \prod_{j_1 \in J_1} \prod_{j_2 \in J_2} \mathcal{O}[[I_1/Z_1]] \,\widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\Gamma]].$$

Since $p > 2$, the determinant induces an isomorphism between Z_1 and Γ. Thus the map $I_1 \to I_1/Z_1 \times \Gamma$, $g \mapsto (gZ_1, (\det g)^{-1})$ is an isomorphism of groups. The isomorphism induces a natural isomorphism of $\mathcal{O}[[I_1]]$-modules, $\mathcal{O}[[I_1]] \cong \mathcal{O}[[I_1/Z_1]] \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\Gamma]]$. We conclude that $\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is a pro-free and hence a projective $\mathcal{O}[[I_1]]$-module. $\square$

6.11 Remark. We will see in Theorem 6.18 below that in fact $\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is a projective object of $\mathfrak{C}(\mathcal{O})$. It is not so difficult to prove this directly, but we have found it more convenient to deduce it as part of the general formalism of arithmetic actions.

6.12 Proposition. *Let $\sigma = \sigma_{a,b}$, and let $\delta : \mathbb{Q}_p^{\times} \to \mathcal{O}^{\times}$ be a character that is trivial modulo ϖ, chosen so that $\psi\delta^2|_{\mathbb{Z}_p^{\times}}$ is the central character of σ°. Then there is a natural isomorphism of R_p-modules*

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}, (\sigma^{\circ})^d)^d \xrightarrow{\cong} \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi\delta^2}, (\sigma^{\circ})^d)^d \widehat{\otimes}_{\mathcal{O}} \Lambda^{\mathrm{ur}},$$

where R_p acts on the left hand side via the isomorphism $R_p \cong R_p^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda$ and on the right hand side via the isomorphism $R_p \cong R_p^{\psi\delta^2} \widehat{\otimes}_{\mathcal{O}} \Lambda$.

Proof. Using Lemma 6.9 we may assume that $\psi|_{K\cap Z}$ is the central character of σ. We note that $\mathbf{1}^{\mathrm{univ}} \otimes_{\Lambda} \Lambda^{\mathrm{ur}}$ is the largest quotient of $\mathbf{1}^{\mathrm{univ}}$ on which $K\cap Z$ acts trivially. Since the central character of $(\sigma^{\circ})^d$ is $\psi^{-1}|_{\mathbb{Z}_p^{\times}}$, and the central character of $\widetilde{P}^{\psi}$ is ψ^{-1}, we have a natural isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}, (\sigma^{\circ})^d)^d \xrightarrow{\cong} \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}}(\mathbf{1}^{\mathrm{univ}} \otimes_{\Lambda} \Lambda^{\mathrm{ur}}), (\sigma^{\circ})^d)^d.$$

Since K acts trivially on $\mathbf{1}^{\mathrm{univ}} \otimes_{\Lambda} \Lambda^{\mathrm{ur}}$, we have an isomorphism:

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}}(\mathbf{1}^{\mathrm{univ}} \otimes_{\Lambda} \Lambda^{\mathrm{ur}}), (\sigma^{\circ})^d)^d \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda^{\mathrm{ur}}, (\sigma^{\circ})^d)^d,$$

where Λ^{ur} carries a trivial K-action and R_p acts by the isomorphism $R_p \cong R_p^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda$. To finish the proof we need to construct a natural isomorphism of $R_p^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda$-modules:

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda^{\mathrm{ur}}, (\sigma^{\circ})^d)^d \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi}, (\sigma^{\circ})^d)^d \widehat{\otimes}_{\mathcal{O}} \Lambda^{\mathrm{ur}}. \tag{6.13}$$

Both sides of (6.13) are finitely generated $R_p^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda^{\mathrm{ur}}$-modules. The $\mathfrak{m}$-adic topology on $R_p^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda^{\mathrm{ur}}$ induces a topology on them, and makes them into pseudo-compact, $\mathcal{O}$-torsion free $\mathcal{O}$-modules. It is therefore enough to construct a natural isomorphism between the Schikhof duals of both sides of (6.13). To ease the notation we let $A = \widetilde{P}^{\psi}$, $B = \Lambda^{\mathrm{ur}}$, $C = (\sigma^{\circ})^d$. Since for a pseudo-compact $\mathcal{O}$-module D, we have $D^d = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(D, \mathcal{O}) \cong \varprojlim_n \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(D, \mathcal{O}/\varpi^n)$, using the adjointness between $\widehat{\otimes}_{\mathcal{O}}$ and $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}$ (see [Bru66, Lem. 2.4]), we obtain natural isomorphisms

$$\begin{aligned}(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(A, C)^d \widehat{\otimes}_{\mathcal{O}} B)^d &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(B, (\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(A, C)^d)^d) \\ &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(B, \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(A, C)) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(A \widehat{\otimes}_{\mathcal{O}} B, C),\end{aligned}$$

and hence a natural isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(A \widehat{\otimes}_{\mathcal{O}} B, C)^d \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(A, C)^d \widehat{\otimes}_{\mathcal{O}} B.$$

Since the isomorphism is natural and K acts trivially on B we obtain a natural isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(A \widehat{\otimes}_{\mathcal{O}} B, C)^d \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(A, C)^d \widehat{\otimes}_{\mathcal{O}} B.$$

Since the action of R_p^{ψ} commutes with the action of K and the isomorphism is natural, we deduce that (6.13) holds. $\square$

6.14 Proposition. *For any σ, the action of $R_p \cong R_p^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda$ on*

$$M'(\sigma^{\circ}) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}, (\sigma^{\circ})^d)^d$$

factors through $R_p(\sigma)$. Moreover, $M'(\sigma^\circ)$ is a maximal Cohen–Macaulay $R_p(\sigma)$-module and $M'(\sigma^\circ)[1/p]$ is a locally free $R_p(\sigma)[1/p]$-module of rank 1.

6.15 Remark. Recall that $M(\sigma^\circ)$ was defined in the last section. Soon we will see from Theorem 6.18 below that $\widetilde{P} \xrightarrow{\sim} \widetilde{P}^\psi \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$, thus also that $M(\sigma^\circ) \simeq M'(\sigma^\circ)$.

Proof. If the central character of σ is not congruent to $\psi|_{\mathbb{Z}_p^\times}$ modulo ϖ, then both $R_p(\sigma)$ and $M'(\sigma^\circ)$ are zero. Otherwise, there is a character $\delta : {\mathbb{Q}_p}^\times \to \mathcal{O}^\times$ trivial modulo ϖ such that $(\psi\delta^2)|_{\mathbb{Z}_p^\times}$ is equal to the central character of σ. Proposition 6.12 gives us an isomorphism of R_p-modules

$$M'(\sigma^\circ) \cong M^{\psi\delta^2}(\sigma^\circ) \,\widehat{\otimes}_{\mathcal{O}}\, \Lambda^{\mathrm{ur}}, \tag{6.16}$$

where $M^{\psi\delta^2}(\sigma^\circ) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[[K]]}(\widetilde{P}^{\psi\delta^2}, (\sigma^\circ)^d)^d$, and the action of R_p on the right hand side is given by $R_p \cong R_p^{\psi\delta^2} \,\widehat{\otimes}_{\mathcal{O}}\, \Lambda$. The action of $R_p^{\psi\delta^2}$ on $M^{\psi\delta^2}(\sigma^\circ)$ factors through the action of $R_p^{\psi\delta^2}(\sigma)$ and makes it into a maximal Cohen–Macaulay $R_p^{\psi\delta^2}(\sigma)$-module [Paš15, Cor. 6.4, 6.5]. Since $\Lambda^{\mathrm{ur}} \cong \mathcal{O}[[x]]$, we conclude that $M^{\psi\delta^2}(\sigma^\circ) \,\widehat{\otimes}_{\mathcal{O}}\, \Lambda^{\mathrm{ur}}$ is a maximal Cohen–Macaulay $R_p^{\psi\delta^2}(\sigma) \,\widehat{\otimes}_{\mathcal{O}}\, \Lambda^{\mathrm{ur}}$-module. Using (6.3) we see that $M'(\sigma^\circ)$ is a maximal Cohen–Macaulay $R_p(\sigma)$-module. Since $R_p(\sigma)[1/p]$ is a regular ring a standard argument using the Auslander–Buchsbaum theorem shows that $M'(\sigma^\circ)[1/p]$ is a locally free $R_p(\sigma)[1/p]$-module. It follows from [Paš15, Prop. 4.14, 2.22] that

$$\dim_{\kappa(x)} M^{\psi\delta^2}(\sigma^\circ) \otimes_{R_p^{\psi\delta^2}} \kappa(x) = 1, \quad \forall x \in \text{m-Spec}\, R_p^{\psi\delta^2}(\sigma)[1/p].$$

This together with (6.16) gives us

$$\dim_{\kappa(x)} M'(\sigma^\circ) \otimes_{R_p} \kappa(x) = 1, \quad \forall x \in \text{m-Spec}\, R_p(\sigma)[1/p].$$

Hence, $M'(\sigma^\circ)[1/p]$ is a locally free $R_p(\sigma)[1/p]$-module of rank 1. □

The natural action of $\mathcal{H}(\sigma^\circ)$ on $M'(\sigma^\circ)$ commutes with the action of $R_p(\sigma)$, and hence induces an action of $\mathcal{H}(\sigma)$ on $M'(\sigma^\circ)[1/p]$. Since $M'(\sigma^\circ)[1/p]$ is locally free of rank 1 over $R_p(\sigma)[1/p]$ by Proposition 6.14, we obtain a homomorphism of E-algebras:

$$\alpha : \mathcal{H}(\sigma) \to \mathrm{End}_{R_p(\sigma)[1/p]}(M'(\sigma^\circ)[1/p]) \cong R_p(\sigma)[1/p].$$

6.17 Proposition. *The map $\alpha : \mathcal{H}(\sigma) \to R_p(\sigma)[1/p]$ coincides with the map $\eta : \mathcal{H}(\sigma) \to R_p(\sigma)[1/p]$ constructed in [CEG+16, Thm. 4.1].*

Proof. It is enough to show that the specialisations of α and η at x coincide for x in a Zariski dense subset of m-Spec $R_p(\sigma)[1/p]$. The isomorphism $R_p \cong R_p^\psi \,\widehat{\otimes}_{\mathcal{O}}\, \Lambda$ maps x to a pair (y, z), where $y \in \text{m-Spec}\, R_p^\psi[1/p]$ and $z \in \text{m-Spec}\, \Lambda[1/p]$, so that if r_x^{univ}, $r_y^{\mathrm{univ},\psi}$ and $\mathbf{1}_z^{\mathrm{univ}}$ are Galois representations corresponding to x, y and z respectively then

$$r_x^{\mathrm{univ}} \cong r_y^{\mathrm{univ},\psi} \otimes \mathbf{1}_z^{\mathrm{univ}}.$$

Let

$$\Pi_x := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}((\widetilde{P}^\psi \,\widehat{\otimes}_{\mathcal{O}}\, \mathbf{1}^{\mathrm{univ}}) \,\widehat{\otimes}_{R_p}\, \mathcal{O}_{\kappa(x)}, E),$$

$$\Pi_y := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\widetilde{P}^\psi \,\widehat{\otimes}_{R_p^\psi}\, \mathcal{O}_{\kappa(y)}, E).$$

Then both are unitary G-Banach space representations and we have

$$\Pi_x \cong \Pi_y \otimes (\mathbf{1}_z^{\mathrm{univ}} \circ \det).$$

It follows from [Paš15, Prop. 2.22] that $\mathrm{Hom}_K(\sigma, \Pi_x)$ is a one-dimensional $\kappa(x)$-vector space, and the action of $\mathcal{H}(\sigma)$ on it coincides with the specialisation of α at x, which can be written as the composite

$$\mathcal{H}(\sigma) \to \mathrm{End}_{\kappa(x)}(M'(\sigma^\circ) \otimes_{R_p} \kappa(x)) \cong \kappa(x).$$

Since σ is an algebraic representation of K, we have

$$\mathrm{Hom}_K(\sigma, \Pi_x) \cong \mathrm{Hom}_K(\sigma, \Pi_x^{\mathrm{l.alg}}),$$

where $\Pi_x^{\mathrm{l.alg}}$ is the subspace of locally algebraic vectors in Π_x.

It follows from the main result of [Paš13] that the specialisation Π_x of $\widetilde{P}^\psi$ coincides with the Banach space representation attached to r_x^{univ} via the p-adic local Langlands correspondence. It is then a consequence of the construction of the appropriate cases of the p-adic local Langlands correspondence (that is, the construction of Π_x) that there is an embedding

$$\pi_{\mathrm{sm}}(r_x^{\mathrm{univ}}) \otimes \pi_{\mathrm{alg}}(r_x^{\mathrm{univ}}) \hookrightarrow \Pi_x^{\mathrm{l.alg}},$$

where $\pi_{\mathrm{sm}}(r_x^{\mathrm{univ}}) = r_p^{-1}(\mathrm{WD}(r_x^{\mathrm{univ}})^{F-\mathrm{ss}})$ is the smooth representation of G corresponding to the Weil–Deligne representation associated to r_x^{univ} by the classical Langlands correspondence r_p (normalised as in Section 1.12), and $\pi_{\mathrm{alg}}(r_x^{\mathrm{univ}})$ is the algebraic representation of G whose restriction to K is equal to σ. Indeed, if the representation r_x^{univ} is irreducible, then (for a Zariski dense set of x) Π_x is a completion of $\pi_{\mathrm{sm}}(r_x^{\mathrm{univ}}) \otimes \pi_{\mathrm{alg}}(r_x^{\mathrm{univ}})$ by the main result of [BB10] (see in particular [BB10, Thm. 4.3.1]), while in the case that r_x^{univ} is reducible, the result follows from the explicit description of Π_x in [BE10].

Since we have already noted that $\mathrm{Hom}_K(\sigma, \Pi_x)$ is one-dimensional, we find that in fact

$$\mathrm{Hom}_K(\sigma, \Pi_x) \cong \mathrm{Hom}_K\big(\sigma, \pi_{\mathrm{sm}}(r_x^{\mathrm{univ}}) \otimes \pi_{\mathrm{alg}}(r_x^{\mathrm{univ}})\big) \cong \pi_{\mathrm{sm}}(r_x^{\mathrm{univ}})^K,$$

and the right-hand side of this isomorphism is indeed a one-dimensional vector space on which $\mathcal{H}(\sigma)$ acts via the specialisation of η. □

Let R_p act on $\widetilde{P}^\psi \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ via the isomorphism $R_p \cong R_p^\psi \widehat{\otimes}_{\mathcal{O}} \Lambda$, where (as throughout this section) the action of R_p^ψ on $\widetilde{P}^\psi$ is via the isomorphism $R_p^\psi \cong \mathrm{End}_{\mathfrak{C}^\psi(\mathcal{O})}(\widetilde{P}^\psi)$ constructed in [Paš13].

6.18 Theorem. *$\widetilde{P}^\psi \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$, and the action of R_p on $\widetilde{P}^\psi \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is arithmetic.*

Proof. We will show that the action of R_p on $\widetilde{P}^\psi \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ satisfies the axioms (AA1) – (AA4) with $d = 0$; then the action is arithmetic by definition, and $\widetilde{P}^\psi \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$ by Theorem 4.32 (applied with $d = 0$ and with M_∞ taken to be $\widetilde{P}^\psi \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$).

It is shown in [Paš15, Propositition 6.1] that $\mathbb{F} \widehat{\otimes}_{R_p^\psi} \widetilde{P}^\psi$ is a finitely generated $\mathcal{O}[[K]]$-module, so the topological version of Nakayama's lemma implies that $\widetilde{P}^\psi$ is a finitely generated $R_p^\psi[[K]]$-module. Since $\mathbf{1}^{\mathrm{univ}}$ is a free Λ-module of rank 1, $\widetilde{P}^\psi \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$ is a finitely generated module over $(R_p^\psi \widehat{\otimes}_{\mathcal{O}} \Lambda)[[K]]$, and so (AA1) holds. Proposition 6.10 implies that (AA2) holds. Proposition 6.14 implies that (AA3) holds (indeed, it shows that the support of $M'(\sigma^\circ)$ is all of $R_p(\sigma)[1/p]$). Proposition 6.17 implies that (AA4) holds. □

Recall that for each fixed central character $\psi : Z \to \mathcal{O}^\times$, there is an exact functor $\check{\mathbf{V}}$ from $\mathfrak{C}^\psi(\mathcal{O})$ to the category of continuous $G_{\mathbb{Q}_p}$-representations on compact $\mathcal{O}$-modules. This is a modification of the functor introduced by Colmez in [Col10b], see [Paš13, §5.7] for details (we additionally have to twist the functor in [Paš13] by the inverse of the cyclotomic character to get the desired relationship between the determinant of the Galois representations and the central character of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations.) If Π is an admissible unitary E-Banach space representation of G with central character ψ, and Θ is an open bounded G-invariant lattice in Π then the Schikhof dual of Θ is an object of $\mathfrak{C}^{\psi}(\mathcal{O})$ and $\check{\mathbf{V}}(\Pi) := \check{\mathbf{V}}(\Theta^d)[1/p]$ does not depend on the choice of Θ.

The representation $\widetilde{P}^{\psi}$ satisfies the conditions (N0), (N1), (N2) of [Paš15, §4] by [Paš15, Prop. 6.1]. In particular, we have $\check{\mathbf{V}}(\widetilde{P}^{\psi}) \cong r^{\mathrm{univ},\psi}$ as $R_p^{\psi}[[G_{\mathbb{Q}_p}]]$-modules.

Let $R_\infty = R_p[[x_1, \ldots, x_d]]$ and let M_∞ be an $R_\infty[G]$-module satisfying the axioms (AA1)–(AA4). To $x \in \text{m-Spec}\, R_\infty[1/p]$ we associate a unitary $\kappa(x)$-Banach space representation of G, $\Pi_{\infty,x} := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M_\infty \widehat{\otimes}_{R_\infty} \mathcal{O}_{\kappa(x)}, E)$. The map $R_p \to R_\infty$ induces a map $R_p \to \kappa(x)$ and we let $r_x^{\mathrm{univ}} := r^{\mathrm{univ}} \otimes_{R_p} \kappa(x)$.

6.19 Corollary. *We have an isomorphism of Galois representations* $\check{\mathbf{V}}(\Pi_{\infty,x}) \cong r_x^{\mathrm{univ}}$. *In particular,* $\Pi_{\infty,x} \neq 0$ *for all* $x \in \text{m-Spec}\, R_\infty[1/p]$.

Proof. Theorem 4.32 allows us to assume that $R_\infty = R_p$ and $M_\infty = \widetilde{P}$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$. Since an arithmetic action of R_p on $\widetilde{P}$ is unique by Theorem 4.30, using Theorem 6.18 we may assume that $R_\infty = R_p^{\psi} \widehat{\otimes}_{\mathcal{O}} \Lambda$ and $M_\infty = \widetilde{P}^{\psi} \widehat{\otimes}_{\mathcal{O}} \mathbf{1}^{\mathrm{univ}}$. Then, with the notation introduced in the course of the proof of Proposition 6.17, x corresponds to a pair (y, z), $\Pi_{\infty,x} = \Pi_x \cong \Pi_y \otimes (\mathbf{1}_z^{\mathrm{univ}} \circ \det)$ as in the proof of Proposition 6.17. It follows from [Paš15, Lem. 4.3] that $\check{\mathbf{V}}(\Pi_y) \cong r_y^{\mathrm{univ},\psi}$. Since $\check{\mathbf{V}}$ is compatible with twisting by characters we have $\check{\mathbf{V}}(\Pi_x) \cong \check{\mathbf{V}}(\Pi_y) \otimes \mathbf{1}_z^{\mathrm{univ}} \cong r_x^{\mathrm{univ}}$, as required. □

6.20 Corollary. *R_∞ acts faithfully on M_∞.*

Proof. It follows from Corollary 6.19 that $M_\infty \otimes_{R_\infty} \kappa(x)$ is non-zero for all $x \in \text{m-Spec}\, R_\infty[1/p]$. Since R_p and hence R_∞ are reduced we deduce that the action is faithful. □

We now use the results of [Paš13] to describe $\mathbb{F} \widehat{\otimes}_{R_p} \widetilde{P}$. We will use this result in Corollary 7.5 below to describe the $\mathfrak{m}$-torsion in the completed cohomology of a modular curve, where $\mathfrak{m}$ is a maximal ideal in a Hecke algebra.

6.21 Proposition. *The representation $\pi^\vee$ occurs as a subquotient of $\mathbb{F} \widehat{\otimes}_{R_p} \widetilde{P}$ with multiplicity one. More precisely, if we let $\kappa(\bar{r}) := (\mathbb{F} \widehat{\otimes}_{R_p} \widetilde{P})^\vee$ then the G-socle filtration of $\kappa(\bar{r})$ is described as follows:*

(i) If $\bar{r}$ is irreducible then $\kappa(\bar{r}) \cong \pi$.

(ii) If $\bar{r}$ is a generic non-split extension of characters (so the ratio of the two characters is not $1, \omega^{\pm 1}$), the G-socle filtration of $\kappa(\bar{r})$ has length two, with graded pieces consisting of π and of the other principal series representation in the block of π.

(iii) If $\bar{r} \cong \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$ then the G-socle filtration of $\kappa(\bar{r})$ has length three and the graded pieces are π, the twist by $\chi \circ \det$ of the Steinberg representation, and two copies of the one-dimensional representation $\chi \circ \det$.

Proof. We choose any continuous character ψ such that $\psi\varepsilon^{-1}$ lifts $\det \bar{r}$. It follows from Theorem 6.18 that $\mathbb{F} \widehat{\otimes}_{R_p} \widetilde{P} \cong \mathbb{F} \widehat{\otimes}_{R_p^{\psi}} \widetilde{P}^{\psi}$. Since we can identify the endomorphism ring of $\widetilde{P}^{\psi}$ with R_p^{ψ}, see (6.6), it follows from Lemma 3.7 in [Paš13] applied with $S = \pi^\vee$ that any Q in $\mathfrak{C}^{\psi}(\mathcal{O})$ satisfying the hypotheses (H1)–(H4) of [Paš13, §3] is isomorphic to $\mathbb{F} \widehat{\otimes}_{R_p^{\psi}} \widetilde{P}^{\psi}$, so that $\kappa(\bar{r}) \cong Q^\vee$. (We leave the reader to check that (H5) is not used to prove this part of Lemma 3.7 in [Paš13].) In all the cases Q has been constructed explicitly in [Paš13] and it is immediate from the construction of Q that the assertions about the socle filtration hold, see [Paš13, Propositions 6.1, 8.3, Remark 10.33]. □

6.22 Remark. In the first two cases of Proposition 6.21, $\kappa(\bar{r})$ coincides with what Colmez calls the *atome automorphe* in [Col10b, §VII.4]. In the last case, $\kappa(\bar{r})$ has an extra copy of $\chi \circ \det$. This has to do with the fact that Colmez requires that his atome automorphe lift to irreducible unitary Banach space representations of G.

6.23 Corollary. *There is a natural isomorphism $R_p \xrightarrow{\sim} \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$.*

Proof. Theorems 4.30 and 6.18 yield an isomorphism $\widetilde{P} \xrightarrow{\sim} \widetilde{P}^{\psi}\widehat{\otimes}_{\mathcal{O}}\mathbf{1}^{\mathrm{univ}}$ as $R_p[G]$-modules via the isomorphism $R_p \xrightarrow{\sim} R_p^{\psi}\widehat{\otimes}_{\mathcal{O}}\Lambda$ given by (6.2). Now $\mathrm{End}_G^{\mathrm{cont}}(\mathbf{1}^{\mathrm{univ}}) = \Lambda$, while (6.6) gives a natural isomorphism $R_p^{\psi} \xrightarrow{\sim} \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}^{\psi})$. Thus the corollary amounts to proving that the natural homomorphism

$$\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}^{\psi})\widehat{\otimes}_{\mathcal{O}}\mathrm{End}_G^{\mathrm{cont}}(\mathbf{1}^{\mathrm{univ}}) \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}^{\psi}\widehat{\otimes}_{\mathcal{O}}\mathbf{1}^{\mathrm{univ}})$$

is an isomorphism. The map is an injection of pseudo-compact $\mathcal{O}$-algebras. This makes the ring $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$ into a compact R_p-module. By the topological version of Nakayama's lemma, in order to show that the map is surjective it is enough to show that $\mathbb{F}\widehat{\otimes}_{R_p}\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$ is a one dimensional $\mathbb{F}$-vector space. Since $\widetilde{P}$ is projective, we have an isomorphism:

$$\mathbb{F}\widehat{\otimes}_{R_p}\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}) \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \mathbb{F}\widehat{\otimes}_{R_p}\widetilde{P}).$$

By Proposition 6.21, $\pi^{\vee}$ occurs as a subquotient of $\mathbb{F}\widehat{\otimes}_{R_p}\widetilde{P}$ with multiplicity one. Since $\widetilde{P}$ is a projective envelope of $\pi^{\vee}$ and $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\pi^{\vee}) = \mathbb{F}$, this implies that $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \mathbb{F}\widehat{\otimes}_{R_p}\widetilde{P})$ is a one dimensional $\mathbb{F}$-vector space, as required. □

6.24 Endomorphism rings and deformation rings

We maintain the notation of the previous sections; in particular, $\widetilde{P}$ denotes a projective envelope of $\pi^{\vee}$ in $\mathfrak{C}(\mathcal{O})$. If we write $\widetilde{R} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$, then the arithmetic action of R_p on $\widetilde{P}$ provided by Theorem 4.30 gives a morphism $R_p \to \widetilde{R}$, which Corollary 6.23 shows is an isomorphism. The proof of that Corollary uses the analogous statement proved in [Paš13] (when the central character is fixed), a key input to the proof of which is Colmez's functor from $\mathrm{GL}_2(\mathbb{Q}_p)$-representation to Galois representations. It is natural to ask (especially in light of possible generalizations) whether this isomorphism can be proved using just the methods of the present paper, without appealing to Colmez's results. In this subsection we address this question, to the extent that we can.

We begin by noting that since $\widetilde{P}$ is a projective envelope of the absolutely irreducible representation $\pi^{\vee}$, the ring $\widetilde{R}$ is a local ring. We will furthermore give a proof that it is commutative, from the perspective of this paper. As already noted, this result is not new. Indeed, in addition to being a consequence of Corollary 6.23 (and thus, essentially, of the results of [Paš13]), another proof is given in [CDP14] (see Cor. 2.22 of that paper). This latter proof uses the capturing techniques that we are also employing in the present paper, and (since it is easy to do so) we present a slightly rephrased version of the argument here, in order to illustrate how it fits naturally into our present perspective.

6.25 Proposition. *The ring $\widetilde{R}$ is commutative.*

Proof. We first prove that the image of R_p in $\widetilde{R}$ lies in the centre of $\widetilde{R}$. To see this, suppose that $\phi \in \widetilde{R}$. By Proposition 4.28, to show that ϕ commutes with the action of R_p, it suffices to show that ϕ commutes with the action of $R_p(\sigma)[1/p]$ on $M(\sigma^\circ)[1/p]$ for each σ. Since the action of $\mathcal{H}(\sigma)$ on $M(\sigma^\circ)[1/p]$ depends only on the G-action on $\widetilde{P}$, we see that ϕ commutes with the $\mathcal{H}(\sigma)$-action on $M(\sigma^\circ)[1/p]$. The desired result then follows from Proposition 2.13.

To see that $\widetilde{R}$ is commutative, we again apply Proposition 4.28, by which it suffices to show that $\widetilde{R}$ acts on each $M(\sigma^\circ)[1/p]$ through a commutative quotient. This follows from the fact that each $M(\sigma^\circ)[1/p]$ is locally free of rank one over its support in $\operatorname{Spec} R_p(\sigma)[1/p]$, and the fact that (by the result of the previous paragraph) the $\widetilde{R}$-action commutes with the R_p-action. ☐

6.26 Remark. As for proving the stronger result that the canonical map $R_p \to \widetilde{R}$ is an isomorphism, in the forthcoming paper [EP] two of us (ME and VP) will establish the *injectivity* of the morphism $R_p \to \widetilde{R}$. (In fact we will prove a result in the more general context of [CEG+16]; in particular, our arguments won't rely on any special aspects of the $\mathrm{GL}_2(\mathbb{Q}_p)$ situation, such as the existence of Colmez's functors.) However, proving the *surjectivity* of this morphism seems to be more difficult, and we currently don't know a proof of this surjectivity that avoids appealing to the theory of Colmez's functor from $\mathrm{GL}_2(\mathbb{Q}_p)$-representations to $G_{\mathbb{Q}_p}$-representations.

6.27 Speculations in the residually scalar semi-simplification case

Suppose for the rest of this section that $\bar{r} \cong \left(\begin{smallmatrix} \chi & * \\ 0 & \chi \end{smallmatrix}\right)$ for some χ; so in particular $\bar{r}$ does not satisfy Assumption 2.2. It is natural to ask what the modules M_∞ constructed in [CEG+16] look like in this case; we give a speculative answer below. By twisting we may assume that χ is the trivial character. Let $\pi = (\operatorname{Ind}_B^G \omega \otimes \mathbf{1})_{\mathrm{sm}}$, and let $\widetilde{P}$ be a projective envelope of $\pi^\vee$. We first give a conjectural description of $\operatorname{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P})$, under the assumption $p > 2$.

Let D^{ps} be a functor from the category $\mathfrak{A}$ of complete local noetherian $\mathcal{O}$-algebras with residue field $\mathbb{F}$ to the category of sets, that assigns to $A \in \mathfrak{A}$ the set of pairs of functions $(t, d) : G_{\mathbb{Q}_p} \to A$, where:

- $d : G_{\mathbb{Q}_p} \to A^\times$ is a continuous group homomorphism, congruent to $\det \bar{r}$ modulo $\mathfrak{m}_A$,
- $t : G_{\mathbb{Q}_p} \to A$ is a continuous function with $t(1) = 2$, and,
- for all $g, h \in G_{\mathbb{Q}_p}$, we have:
  (i) $t(g) \equiv \operatorname{tr} \bar{r}(g) \pmod{\mathfrak{m}_A}$;
  (ii) $t(gh) = t(hg)$;
  (iii) $d(g)t(g^{-1}h) - t(g)t(h) + t(gh) = 0$.

(The "ps" is for "pseudocharacter". By [Che14, Lem. 1.9], $D^{\mathrm{ps}}(A)$ is the set of pseudocharacters deforming the pseudocharacter $(\operatorname{tr} \bar{r}, \det \bar{r})$ associated to $\bar{r}$.) This functor is representable by a complete local noetherian $\mathcal{O}$-algebra R^{ps}. Let $(t^{\mathrm{univ}}, d^{\mathrm{univ}}) : G_{\mathbb{Q}_p} \to R^{\mathrm{ps}}$ be the universal object. We expect that there is a natural isomorphism of $\mathcal{O}$-algebras

$$\widetilde{E} := \operatorname{End}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}) \cong (R^{\mathrm{ps}}[[G_{\mathbb{Q}_p}]]/J)^{\mathrm{op}}, \tag{6.28}$$

where J is the closed two sided ideal of $R^{\mathrm{ps}}[[G_{\mathbb{Q}_p}]]$ generated by all the elements of the form $g^2 - t^{\mathrm{univ}}(g)g + d^{\mathrm{univ}}(g)$ for all $g \in G_{\mathbb{Q}_p}$, and the superscript op indicates the opposite algebra. We note that such an isomorphism has been established in [Paš13, §9], when the central character is fixed, and we expect that one can deduce (6.28) from this using the twisting techniques of the previous subsection.

Let $R_p^\square$ be the framed deformation ring of $\bar{r}$ and let M_∞ be the patched module constructed in [CEG+16] (or the variant for the completed cohomology of modular curves that we briefly discuss in Section 7 below) and let R_∞ be the patched ring. Then R_∞ is an $R_p^\square$-algebra, and the map $R_p^\square \to R_\infty$ gives rise to a Galois representation $r_\infty : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R_\infty)$ lifting $\bar{r}$. The pair $(\operatorname{tr} r_\infty, \det r_\infty)$ gives us a point in $D^{\mathrm{ps}}(R_\infty)$ and hence a map $R^{\mathrm{ps}} \to R_\infty$. Hence we obtain a homomorphism of R^{ps}-algebras $R^{\mathrm{ps}}[[G_{\mathbb{Q}_p}]] \to M_{2\times 2}(R_\infty)$.

The Cayley–Hamilton theorem implies that this map is zero on J, so we obtain a left action of $R^{\mathrm{ps}}[[G_{\mathbb{Q}_p}]]/J$ on the standard module $R_\infty \oplus R_\infty$. If we admit (6.28) then we get a right action of $\widetilde{E}$ on $R_\infty \oplus R_\infty$. We expect that there are isomorphisms in $\mathfrak{C}(R_\infty)$

$$M_\infty \cong (R_\infty \oplus R_\infty) \widehat{\otimes}_{\widetilde{E}} \widetilde{P} \cong R_\infty \widehat{\otimes}_{R_p^\square} (R_p^\square \oplus R_p^\square) \widehat{\otimes}_{\widetilde{E}} \widetilde{P}. \tag{6.29}$$

We note that the representation appearing on the right hand side of this equation has been studied by Fabian Sander in his thesis [San16], in the setting where the central character is fixed. Motivated by [San16, Thm. 2] we expect $(R_\infty \oplus R_\infty) \widehat{\otimes}_{\widetilde{E}} \widetilde{P}$ to be projective in the category of pseudocompact $\mathcal{O}[[K]]$-modules. We do not expect $(R_\infty \oplus R_\infty) \widehat{\otimes}_{\widetilde{E}} \widetilde{P}$ to be projective in $\mathfrak{C}(\mathcal{O})$, so the methods of Section 4 cannot directly be applied to this case. However, it might be possible to prove (6.29) using Colmez's functor. This would show that M_∞ does not depend on the choices made in the patching process.

7. Local-global compatibility

In this final section, we briefly explain how the results of this paper give a simple new proof of the local-global compatibility theorem of [Eme11] (under the hypotheses that we have imposed in this paper, which differ a little from those of [Eme11]: locally at p, we have excluded the case of split $\bar{r}$, and have allowed a slightly different collection of indecomposable reducible $\bar{r}$'s; and in the global context we consider below, we exclude the possibility of so-called vexing primes). Applying these considerations to the patched modules constructed in [CEG+16] allows us to prove a local-global compatibility result for the completed cohomology of a compact unitary group, but for ease of comparison to [Eme11], we instead briefly discuss the output of Taylor–Wiles patching for modular curves.

Patching in this context goes back to [TW95], but the precise construction we need is not in the literature. It is, however, essentially identical to that of [CEG+16] (or the variant for Shimura curves presented in [Sch15]), so to keep this paper at a reasonable length we simply recall the output of the construction here.

Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be an absolutely irreducible odd (so modular, by Serre's conjecture) representation, and assume that $p \geqslant 5$ and that $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is irreducible. Write $\bar{r} := \overline{\rho}|_{G_{\mathbb{Q}_p}}$, and assume that $\bar{r}$ satisfies Assumption 2.2. Write $r^{\mathrm{univ}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R_p)$ for the universal deformation of $\bar{r}$. Let $N(\overline{\rho})$ be the prime-to-p conductor of $\overline{\rho}$; that is, the level of $\overline{\rho}$ in the sense of [Ser87]. We assume that if $q|N(\overline{\rho})$ with $q \equiv -1 \pmod p$ and $\overline{\rho}|_{G_{\mathbb{Q}_q}}$ is irreducible then $\overline{\rho}|_{I_{\mathbb{Q}_q}}$ is also irreducible.

7.1 Remark. The last condition we have imposed excludes the so-called *vexing primes* q. The assumption that there are no vexing primes means that the Galois representations associated to modular forms of level $N(\overline{\rho})$ are necessarily minimally ramified. This assumption can be removed by considering inertial types at such primes as in [CDT99]. Since the arguments using types are standard and are orthogonal to the main concerns of this paper, we restrict ourselves to this simple case.

We caution the reader that while the use of types would also allow us to work at certain non-minimal levels, the most naive analogues of Theorem 7.4 fail to hold at arbitrary tame levels. It seems that to formulate a clean statement, one should pass to infinite level at a finite set of primes, and formulate the compatibility statement in terms of the local Langlands correspondence in families of [EH14], as is done in [Eme11].

However, it does not seem to be easy to prove this full local-global compatibility statement using only the methods of the present paper; indeed, the proof in [Eme11] ultimately makes use of mod p multiplicity one theorems that rely on q-expansions, whereas in our approach, we are only using multiplicity one theorems that result from our patched modules being Cohen–Macaulay, and certain of our local deformation rings being regular (namely the minimal deformation rings at places not dividing p, and the deformation rings considered in Lemma 2.15). Note that in general the (non-minimal) local deformation rings at places away from p need not be regular (even after inverting p), so that carrying out the patching construction below would result in a ring R_∞ that was no longer formally smooth over R_p, to which the results of Section 4 would not apply.

Let $\mathbb{T}$ be the usual Hecke algebra acting on (completed) homology and cohomology of modular curves with (tame) level $\Gamma_1(N(\overline{\rho}))$ and $\mathcal{O}$-coefficients; so $\mathbb{T}$ is an $\mathcal{O}$-algebra, generated by the operators T_l, S_l with $l \nmid Np$. Let $\mathfrak{m}(\overline{\rho})$ be the maximal ideal of $\mathbb{T}$ corresponding to $\overline{\rho}$ (so that $T_l - \operatorname{tr}\overline{\rho}(\mathrm{Frob}_l)$ and $lS_l - \det\overline{\rho}(\mathrm{Frob}_l)$ are both zero in $\mathbb{T}/\mathfrak{m}(\overline{\rho})$). Let $R_{\mathbb{Q},N(\overline{\rho})}$ be the universal deformation ring for deformations of $\overline{\rho}$ that are minimally ramified at primes $l \neq p$, in the sense that they have the same conductor as $\overline{\rho}|_{G_{\mathbb{Q}_l}}$ (and in particular are unramified if $l \nmid N(\overline{\rho})$). Let $\rho^{\mathrm{univ}}_{\mathbb{Q},N(\overline{\rho})} : G_{\mathbb{Q}} \to \mathrm{GL}_2(R_{\mathbb{Q},N(\overline{\rho})})$ denote the corresponding universal deformation of $\overline{\rho}$.

We now use the notation introduced in Section 3, so that in particular we write $R_\infty := R_p\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[x_1,\ldots,x_d]]$ for some $d \geqslant 0$. Patching the completed étale homology of the modular curves $Y_1(N(\overline{\rho}))$ (and using an argument of Carayol [Car94], as in [Eme11, §5.5] and [EGS15, §§6.2, 6.3], to factor out the Galois action on the completed cohomology; see also [Sch15, §9] for the analogous patching construction for Shimura curves), we obtain (for some $d \geqslant 0$) an $R_\infty[G]$-module M_∞ with an arithmetic action, with the further property that there is an ideal $\mathfrak{a}_\infty$ of R_∞, an isomorphism of local $\mathcal{O}$-algebras $R_\infty/\mathfrak{a}_\infty \xrightarrow{\sim} R_{\mathbb{Q},N(\overline{\rho})}$, and an isomorphism of $R_{\mathbb{Q},N(\overline{\rho})}[G \times G_{\mathbb{Q}}]$-modules

$$(M_\infty/\mathfrak{a}_\infty) \otimes_{R_{\mathbb{Q},N(\overline{\rho})}} (\rho^{\mathrm{univ}}_{\mathbb{Q},N(\overline{\rho})})^* \xrightarrow{\sim} \widetilde{H}_{1,\text{ét}}\big(Y_1(N(\overline{\rho})),\mathcal{O}\big)_{\mathfrak{m}(\overline{\rho})}. \tag{7.2}$$

Here $\widetilde{H}_{1,\text{ét}}\big(Y_1(N(\overline{\rho})),\mathcal{O}\big)$ denotes completed étale homology, as described for example in [CE12]. The action of $G_{\mathbb{Q}}$ on the left hand side is via its action on $(\rho^{\mathrm{univ}}_{\mathbb{Q},N(\overline{\rho})})^*$, which as in (1.12.3) denotes the $R_{\mathbb{Q},N(\overline{\rho})}$-linear dual of $\rho^{\mathrm{univ}}_{\mathbb{Q},N(\overline{\rho})}$.

7.3 Remark. Here we have used implicitly that the minimally ramified local (framed) deformation rings are all smooth, which follows for example from [CHT08, Lem. 2.4.19]; this ensures that the ring R_∞ occurring in the patching argument is formally smooth over R_p.

7.4 THEOREM. *Let $p > 3$ be prime, and let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be an absolutely irreducible odd representation, with the property that $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is irreducible, $N(\overline{\rho})$ is not divisible by any vexing primes, and $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ satisfies Assumption 2.2. Then there is an isomorphism of $R_{\mathbb{Q},N(\overline{\rho})}[G \times G_{\mathbb{Q}}]$-modules*

$$\widetilde{H}_{1,\text{ét}}\big(Y_1(N(\overline{\rho})),\mathcal{O}\big)_{\mathfrak{m}(\overline{\rho})} \xrightarrow{\sim} \widetilde{P}\,\widehat{\otimes}_{R_p}(\rho^{\mathrm{univ}}_{\mathbb{Q},N(\overline{\rho})})^*,$$

where the completed tensor product on the right-hand side is computed by regarding $(\rho^{\mathrm{univ}}_{\mathbb{Q},N(\overline{\rho})})^*$ as a $G_{\mathbb{Q}}$-representation on an R_p-module via the natural morphism $R_p \to R_{\mathbb{Q},N(\overline{\rho})}$.

Proof. As noted before Remark 7.1, $\overline{\rho}$ is modular by Serre's conjecture, so in particular M_∞ is not zero. By Theorem 4.32 there is an isomorphism of $R_\infty[G]$-modules

$$M_\infty \cong \widetilde{P}\,\widehat{\otimes}_{\mathcal{O}}\,\mathcal{O}[[x_1,\dots,x_d]] \cong \widetilde{P}\,\widehat{\otimes}_{R_p}\,R_\infty.$$

Quotienting out by $\mathfrak{a}_\infty$ yields an isomorphism

$$M_\infty/\mathfrak{a}_\infty \cong \widetilde{P}\,\widehat{\otimes}_{R_p}\,R_{\mathbb{Q},N(\overline{\rho})}.$$

The result now follows by tensoring both sides with $(\rho^{\mathrm{univ}}_{\mathbb{Q},N(\overline{\rho})})^*$ and applying (7.2). □

We now show how to compute the $\mathfrak{m}(\overline{\rho})$-torsion in the completed étale cohomology of modular curves $\widetilde{H}^1_{\text{ét}}\big(Y_1(N(\overline{\rho})),\mathcal{O}\big)$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation.

7.5 Corollary. *Under the assumptions of Theorem* 7.4, *we have an isomorphism of* $\mathbb{F}[G\times G_{\mathbb{Q}}]$-*modules*

$$\widetilde{H}^1_{\text{ét}}\big(Y_1(N(\overline{\rho})),\mathbb{F}\big)[\mathfrak{m}(\overline{\rho})] \simeq \kappa(\overline{\rho}|_{G_{\mathbb{Q}_p}})\otimes_{\mathbb{F}}\overline{\rho},$$

where $\kappa(\overline{\rho}|_{G_{\mathbb{Q}_p}})$ *is the representation defined in Proposition* 6.21.

Proof. The Pontryagin dual of the left hand side is $\widetilde{H}_{1,\text{ét}}\big(Y_1(N(\overline{\rho})),\mathcal{O}\big)\,\widehat{\otimes}_{R_{\mathbb{Q},N(\overline{\rho})}}\,\mathbb{F}$. By Theorem 7.4, we have an isomorphism of $\mathbb{F}[G\times G_{\mathbb{Q}}]$-modules

$$\widetilde{H}_{1,\text{ét}}\big(Y_1(N(\overline{\rho})),\mathcal{O}\big)\,\widehat{\otimes}_{R_{\mathbb{Q},N(\overline{\rho})}}\,\mathbb{F} \simeq (\mathbb{F}\,\widehat{\otimes}_{R_p}\,\widetilde{P})\,\widehat{\otimes}_{\mathbb{F}}\,\overline{\rho}^* \cong \kappa(\overline{\rho}|_{G_{\mathbb{Q}_p}})^\vee\,\widehat{\otimes}_{\mathbb{F}}\,\overline{\rho}^*,$$

the last isomorphism following from the definition of $\kappa(\overline{\rho}|_{G_{\mathbb{Q}_p}})$. Passing to Pontryagin duals (and noting that since $\overline{\rho}$ is an $\mathbb{F}$-representation, we have $\overline{\rho}^*=\overline{\rho}^\vee$) gives the result. □

7.6 Remark. Corollary 7.5 together with Proposition 6.21 gives a description of the G-socle filtration of $\widetilde{H}^1_{\text{ét}}\big(Y_1(N(\overline{\rho})),\mathbb{F}\big)[\mathfrak{m}(\overline{\rho})]$. Even more is true.

Since in Corollary 6.23 we have identified the endomorphism ring of $\widetilde{P}$ with R_p and $\kappa(\bar{r})$ is by definition $\mathbb{F}\,\widehat{\otimes}_{R_p}\,\widetilde{P}$, a completely formal argument (see the proof of Proposition 2.8 in [Paš16]) shows that $\kappa(\bar{r})$ is up to isomorphism the unique representation in $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})$ that is maximal with respect to the following two properties:

(i) the socle of $\kappa(\bar{r})$ is π;

(ii) π occurs as a subquotient of $\kappa(\bar{r})$ with multiplicity 1.

(It is maximal in the sense that it cannot be embedded into any other strictly larger representation in $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})$ satisfying these two properties.)

Corollary 7.5 shows that (after factoring out the $G_{\mathbb{Q}}$-action) the same characterisation carries over to $\widetilde{H}^1_{\text{ét}}\big(Y_1(N(\overline{\rho})),\mathbb{F}\big)[\mathfrak{m}(\overline{\rho})]$. However, we warn the reader that a simple-minded application of this recipe will not work in general, and in particular it fails if $\overline{\rho}|_{G_{\mathbb{Q}_p}}\cong\left(\begin{smallmatrix}\omega & *\\ 0 & 1\end{smallmatrix}\right)\otimes\chi$.

More precisely, if Assumption 2.2 is in force then $\bar{r}$ is determined up to isomorphism by the data of its determinant and its unique irreducible subrepresentation. This information can be recovered from π, which in turn determines $\kappa(\bar{r})$. If on the other hand $\bar{r}\cong\left(\begin{smallmatrix}\omega & *\\ 0 & 1\end{smallmatrix}\right)\otimes\chi$ and $\mathrm{End}_{G_{\mathbb{Q}_p}}(\bar{r})=\mathbb{F}$ then the G-socle of the atome automorphe associated by Colmez in [Col10b, §VII.4] to $\bar{r}$, which we still call π, is the Steinberg representation twisted by $\chi\circ\det$. This representation still carries the information about the irreducible subrepresentation of $\bar{r}$ and the determinant of $\bar{r}$ but it does not determine $\bar{r}$ up to isomorphism, as it does not carry the information about the extension class in $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\chi,\chi\omega)$ corresponding to $\bar{r}$. The maximal representation satisfying (1) and (2) above will contain the atome automorphe corresponding to $\bar{r}$ as a subrepresentation, but it will be strictly bigger. In fact it can be shown that it is the smallest representation that contains all the atomes automorphes corresponding to different non-zero extensions in $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\chi, \chi\omega)$.

7.7 Remark. Theorem 7.4 and Corollary 7.5, when combined with Theorems 6.18 and 4.32 (which together show that $\widetilde{P}$ realises the usual p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$), prove a local-global compatibility result for completed cohomology. They are new in the case that $\overline{\rho}|_{G_{\mathbb{Q}_p}} \cong \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$. In particular, they answer a question raised in Remark 1.2.9 in [Eme11] by confirming the expectation of Remark 6.1.23 of *loc.cit.*. In the other cases, Theorem 7.4 can be deduced from Theorem 6.4.6 in [Eme11] with $\widetilde{P}$ replaced by a deformation of $\kappa(\bar{r})^\vee$ to R_p, such that one obtains the universal deformation of $\bar{r}$ after applying Colmez's functor $\check{\mathbf{V}}$ to it. If $\overline{\rho}|_{G_{\mathbb{Q}_p}} \cong \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$ then it can be shown that $\widetilde{P}$ is not flat over R_p, and that is why the approach of [Eme11] does not work in this case.

7.8 Remark. Theorem 7.4 and Corollary 7.5 have analogues in more general settings when the group at p is essentially $\mathrm{GL}_2(\mathbb{Q}_p)$ (or a product of copies of $\mathrm{GL}_2(\mathbb{Q}_p)$). For example, taking M_∞ to be the patched module of Section 3.4 (as constructed in [CEG$^+$16] for $n = 2$), we obtain statements about the completed cohomology of unitary groups that are compact at infinity.

Perhaps a case of greater interest is that of the completed cohomology of definite quaternion algebras over totally real fields. (One reason for this case to be of interest is its relationship to the cohomology of the Lubin-Tate tower as in [Sch15, Thm. 6.2].) We expect that our results can be extended to this setting, although there is one wrinkle: in order to carry out Taylor–Wiles patching, we need to fix a central character, and as a consequence, our patched module has a fixed central character, and no longer satisfies the axioms of Section 3. One approach to this difficulty would be to formulate analogues of those axioms with an arbitrary fixed central character, making use of the twisting constructions of Section 6 and “capture” arguments of [Paš16, §2.1], but this leads to ugly statements.

Instead, we content ourselves with considering the case that the fixed central character is the trivial character. In this case we can think of our patched modules as modules for $\mathrm{PGL}_2(\mathbb{Q}_p)$, and natural analogues of our axioms can be formulated in this setting; this is carried out in [GN16, §5], where an analogue of the results of Section 4 is proved. In fact, the arguments there allow us to consider modules for a product of copies of $\mathrm{PGL}_2(\mathbb{Q}_p)$, which is convenient when there is more than one place lying over p; accordingly, we work below with cohomology that is completed at all primes above p. (Of course, the case of cohomology that is completed at a single prime above p can be deduced from this by returning to finite level via taking appropriate locally algebraic vectors.)

As explained in Remark 7.1, one has to take some care with ramification at places away from p, and we therefore content ourselves with considering quaternion algebras that split at all finite places. The reader wishing to prove extensions of these results to more general quaternion algebras is advised to examine the patching arguments of [GK14, §4], which work in this setting. We further caution the reader that we have not attempted to check every detail of the expected result explained below.

Let F be a totally real field in which $p \geqslant 5$ splits completely, and let D be a quaternion algebra over F that is split at all finite places and definite at all infinite places (note that in particular this requires $[F : \mathbb{Q}]$ to be even).

Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$ be absolutely irreducible, and assume that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible, and that $\det \overline{\rho} = \omega^{-1}$. Suppose that $\overline{\rho}$ has no vexing primes; that is, if $v \nmid p$ is a finite place at which $\overline{\rho}$

is ramified and $\mathbf{N}v \equiv -1 \pmod p$, and $\overline{\rho}|_{G_{F_v}}$ is irreducible, then $\overline{\rho}|_{I_{F_v}}$ is also irreducible. Finally, suppose that for each place $v|p$, $\overline{\rho}|_{G_{F_v}}$ satisfies Assumption 2.2.

Since D splits at all finite places, we can consider the tame level subgroup $U_1(N(\overline{\rho})) \subset \mathrm{PGL}_1(D \otimes \mathbb{A}_F^{\infty,p}) \simeq \mathrm{PGL}_2(\mathbb{A}_F^{\infty,p})$ given by the image of those matrices in $\mathrm{GL}_2(\mathbb{A}_F^{\infty,p})$ that are unipotent and upper triangular modulo $N(\overline{\rho})$. Let $\widetilde{H}_0(U_1(N(\overline{\rho})), \mathcal{O})$ (resp. $\widetilde{H}^0(U_1(N(\overline{\rho})), \mathcal{O})$) denote the completed homology (resp. cohomology) of the tower of locally symmetric spaces associated to $\mathrm{PGL}_1(D)$ with tame level $U_1(N(\overline{\rho}))$. (Note that at finite level, the locally symmetric spaces are just finite sets of points.)

We assume that $\overline{\rho}$ is modular, in the following sense: $\overline{\rho}$ determines a maximal ideal $\mathfrak{m}(\overline{\rho})$ in the spherical Hecke algebra (generated by Hecke operators at places not dividing p at which $\overline{\rho}$ is unramified) acting on $\widetilde{H}_0(U_1(N(\overline{\rho})), \mathcal{O})$ and we assume that $\widetilde{H}_0(U_1(N(\overline{\rho})), \mathcal{O})_{\mathfrak{m}(\overline{\rho})} \neq 0$.

Let $R_{\overline{\rho}}$ be the universal deformation ring for deformations of $\overline{\rho}$ that are minimally ramified at places not dividing p, and which have determinant ε^{-1}. For each place $v|p$, let R_v be the universal deformation ring for deformations of $\overline{\rho}|_{G_{F_v}}$ with determinant ε^{-1}, and set $R_p := \widehat{\otimes}_{v|p} R_v$.

Set $G = \prod_{v|p} \mathrm{PGL}_1(D_v)$, which we identify with $\prod_{v|p} \mathrm{PGL}_2(\mathbb{Q}_p)$ via a fixed isomorphism. By patching the completed homology $\widetilde{H}_0(U_1(N(\overline{\rho})), \mathcal{O})_{\mathfrak{m}(\overline{\rho})}$ (with a variation of the argument in [Sch15, §9]), we obtain[3] a ring R_∞ which is a power series ring over R_p, and an $R_\infty[G]$-module M_∞ with an arithmetic action in the sense of [GN16, §5.2], together with an ideal $\mathfrak{a}_\infty \subset R_\infty$ such that $R_\infty/\mathfrak{a}_\infty \cong R_{\overline{\rho}}$ as local $\mathcal{O}$-algebras and $M_\infty/\mathfrak{a}_\infty \cong \widetilde{H}_0(U_1(N(\overline{\rho})), \mathcal{O})_{\mathfrak{m}(\overline{\rho})}$ as $R_{\overline{\rho}}[G]$-modules. (Note that we can ensure the smoothness of R_∞ over R_p since we are assuming that D splits at all finite places and that there are no vexing primes.)

Applying [GN16, Prop. 5.2.2] now gives a G-equivariant isomorphism

$$\widetilde{H}_0(U_1(N(\overline{\rho})), \mathcal{O})_{\mathfrak{m}(\overline{\rho})} \simeq (\widehat{\otimes}_{v|p} \widetilde{P}^1(\overline{\rho}|_{G_{F_v}}))\, \widehat{\otimes}_{R_p} R_{\overline{\rho}},$$

where $\widetilde{P}^1(\overline{\rho}|_{G_{F_v}})$ denotes the projective envelope considered in Section 6 in the case that $\bar{r} = \overline{\rho}|_{G_{F_v}}$ and $\psi = 1$. Arguing as in the proof of Corollary 7.5, we obtain a G-equivariant isomorphism

$$\widetilde{H}^0(U_1(N(\overline{\rho})), \mathbb{F})[\mathfrak{m}(\overline{\rho})] \simeq \widehat{\otimes}_{v|p}\, \kappa(\overline{\rho}|_{G_{F_v}}),$$

where $\kappa(\overline{\rho}|_{G_{F_v}})$ denotes the representation defined in Proposition 6.21.

One can obtain analogous results in the case where $[F : \mathbb{Q}]$ is odd and D is split at one infinite place of F and ramified at all the others. In this case, one works with a tower of Shimura curves. The main difference to the argument is to note that $\overline{\rho}$ only contributes to completed homology (or cohomology) in degree 1 (since the $D^\times(\mathbb{A}^\infty)$-action factors through the reduced norm in degree 0), and the G_F-action can be factored out by the same argument as for modular curves.

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[3]We caution the reader that one should carefully check this claim and we have not done so.

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