A CATEGORICAL p-ADIC LANGLANDS CORRESPONDENCE FOR $\mathbf{GL_2(Q}_p\mathbf{)}$

ANDREA DOTTO, MATTHEW EMERTON, AND TOBY GEE

Abstract. Let $p \geq 5$ be a prime. We construct a fully faithful functor from the derived category of all smooth p-adic representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ (with a fixed central character) to a derived category of Ind-coherent sheaves on a stack of (φ, Γ)-modules.

Contents

AD was supported in various stages of this project by the Engineering and Physical Sciences Research Council [EP/L015234/1] (The London School of Geometry and Number Theory), the James D. Wolfensohn Fund at the Institute for Advanced Study, and a Royal Society University Research Fellowship. ME was supported in part by the NSF grants DMS-1601871, DMS-1902307, DMS-1952705, and DMS-2201242. TG was supported in part by an ERC Advanced grant. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 884596). ME and TG were both supported in part by the Simons Collaboration on Perfection in Algebra, Geometry and Topology.

1. Introduction

1.1. **Overview of our results.** We fix a prime $p \geq 5$. Our goal in this paper is to generalise the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. The existing form of the correspondence (as initiated by Breuil, established in full generality by Colmez, and then promoted to an equivalence of categories by Paškūnas) relates locally admissible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ to finitely generated modules over various kinds of deformation rings for two-dimensional mod p representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Motivated by the "categorical" Langlands program, our generalisation relates arbitrary (p-power torsion) smooth representations to complexes of (Ind-) coherent sheaves on a moduli stack of (φ,Γ)-modules.

Our results were announced in the survey [EGH25], and we refer the reader to that paper for an extensive introduction and motivation; here, we content ourselves with emphasizing that the category of smooth representations is significantly larger than that of locally admissible representations: for example, it contains the compactly supported inductions of finite-dimensional representations of any open and compact (modulo centre) subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$, and these are never locally admissible.

To state our results more precisely, we introduce some notation. Let $\mathcal{O}$ denote the ring of integers in a finite extension E of $\mathbf{Q}_p$, let $\mathbf{F}$ be the residue field of $\mathcal{O}$, and let $\mathcal{A}$ denote the category of smooth representations of $G := \mathrm{GL}_2(\mathbf{Q}_p)$ on $\mathcal{O}$-modules on which p acts locally nilpotently, and which have a central character equal to some fixed character $\zeta : \mathbf{Q}_p^\times \to \mathcal{O}^\times$. Let $D^b_{\mathrm{fp}}(\mathcal{A})$ denote the full sub-∞-category of the bounded derived category $D^b(\mathcal{A})$ of $\mathcal{A}$ consisting of objects with finitely presented cohomology.

Let $\mathcal{X}$ be the formal algebraic stack over $\mathrm{Spf}\,\mathcal{O}$ which parameterizes rank 2 projective étale (φ,Γ)-modules with fixed determinant $\zeta\varepsilon^{-1}$, where ε denotes the p-adic cyclotomic character. Finally, let $D^b_{\mathrm{coh}}(\mathcal{X})$ denote the bounded derived category of Ind-coherent sheaves on $\mathcal{X}$ with coherent cohomologies. (We refer to Section B.2 for a precise definition of $D^b_{\mathrm{coh}}(\mathcal{X})$.)

Our main theorem is then the following.

Theorem 1.1.1 (Definition 5.1.33 and Theorem 5.5.1)**.** *There exists an $\mathcal{O}$-linear fully faithful exact functor* $\mathrm{F} : D^b_{\mathrm{fp}}(\mathcal{A}) \hookrightarrow D^b_{\mathrm{coh}}(\mathcal{X})$.

Using our assumption that $p \geq 5$, we show that the unbounded derived category is compactly generated by $D^b_{\mathrm{fp}}(\mathcal{A})$, so for formal reasons, any functor F as in the statement of Theorem 1.1.1 extends to a continuous (i.e. colimit-preserving) fully faithful functor

$$\mathrm{F} : D(\mathcal{A}) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}).$$

Thus Theorem 1.1.1 in fact gives a correspondence that takes account of all smooth representations (and not just the finitely presented ones).

Remark 1.1.2. Although Theorem 1.1.1 is only phrased as an existence result, the functor F that we construct in this paper enjoys several additional properties. For example, Proposition 5.3.23 proves that F has amplitude $[-1, 0]$, and Proposition 5.3.20 describes some properties of the image under F of a natural set of compact generators of $D(\mathcal{A})$ (the compact inductions of Serre weights). These results are in line with the general expectations described in [EGH25, Section 6].

The functor F is not an equivalence, and a sketch of the computation of its essential image is given in [EGH25, Section 7.5.5]. We intend for the full details of this computation to appear elsewhere.

1.1.3. *Constructing the functor.* We now describe our construction of the functor F. We let $\mathcal{O}[\![G]\!]_\zeta$ denote the quotient of the completed group ring $\mathcal{O}[\![G]\!]$ on which the centre acts via ζ. Then general principles [EGH25, Rem. 6.1.23] suggest that F must be of the form

$$\mathrm{F}(\pi) = L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} \pi, \tag{1.1.4}$$

for a pro-coherent sheaf L_∞ of right $\mathcal{O}[\![G]\!]_\zeta$-modules on $\mathcal{X}$. The first step in proving our theorem is to directly construct a suitable pro-coherent sheaf L_∞, and then use the formula (1.1.4) to define the functor F.

Defined in this way, F *a priori* takes values in $\operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X})$, and so one of our problems is to show that F actually takes values in $D^b_{\mathrm{coh}}(\mathcal{X})$. We then furthermore have to show that F is fully faithful. Two key techniques used in the solution of these problems are those of *localization* and *completion*, both in the geometric sense of localizing or completing at points of $\mathcal{X}$, and in a more categorical sense, in which we pass from $\mathcal{A}$ to various of its quotient categories, or to blocks of its full subcategory $\mathcal{A}^{\mathrm{l.adm}}$ of locally admissible representations.

1.1.5. *Constructing the kernel.* Our construction of L_∞ uses Colmez's constructions from [Col10c], extended to the context of (φ, Γ)-modules with coefficients in an arbitrary Noetherian $\mathcal{O}/\varpi^a$-algebra A. This extension is not formal, and it is the subject of Section 4, which includes a systematic study of the base change properties of Colmez's functor $D \mapsto D^\natural \boxtimes \mathbf{P}^1$.

We then construct L_∞ by descent in Section 5.1. To be precise, if $\operatorname{Spf} A \to \mathcal{X}$ is a smooth chart, where A is an I-adically complete Noetherian $\mathcal{O}$-algebra, then we may form the corresponding pro-system of étale (φ, Γ)-modules D_{A/I^n}. Via our extension of Colmez's constructions, we obtain a pro-system $D^\natural_{A/I^n} \boxtimes \mathbf{P}^1$ of $\mathcal{O}[\![G]\!]$-modules on $\operatorname{Spf} A$ which admits descent data from $\operatorname{Spf} A$ to $\mathcal{X}$. Descending it (and twisting it appropriately), we obtain a pro-coherent sheaf of right $\mathcal{O}[\![G]\!]_\zeta$-modules L_∞ on $\mathcal{X}$.

1.1.6. *Localization.* The localization theory of [DEG23] can be briefly summarized as follows. In [Paš13], Paškūnas shows that the blocks of $\mathcal{A}^{\mathrm{l.adm}}$ are in bijection with $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$-conjugacy classes of 2-dimensional continuous $\overline{\mathbf{F}}$-valued pseudorepresentations $\overline{\theta}$ of $G_{\mathbf{Q}_p}$ having determinant $\zeta\varepsilon^{-1}$, and we typically denote the block of $\mathcal{A}^{\mathrm{l.adm}}$ corresponding to such a closed point by $\mathcal{A}_{\overline{\theta}}$. In [DEG23] we explain that these $\overline{\theta}$ are parameterized by (the closed points of) a scheme X which is a chain of projective lines over $\mathbf{F}$, and we show that the category $\mathcal{A}$ localizes in a natural way to a sheaf of abelian categories over X. Indeed, if Y is a closed subset of $|X|$, with open complement U, then in [DEG23] we define a certain localizing subcategory $\mathcal{A}_Y$ of $\mathcal{A}$ associated to Y, and set $\mathcal{A}_U := \mathcal{A}/\mathcal{A}_Y$. In the case when Y is finite, $\mathcal{A}_Y$ consists, by construction, of locally admissible representations. In particular, when $Y = \{\overline{\theta}\}$ is a single closed point, then (again by construction) $\mathcal{A}_Y$ is the block $\mathcal{A}_{\overline{\theta}}$.

By design, the process of passing to $\mathcal{A}_U$ is related to localization of sheaves on $\mathcal{X}$, because the closed points of $|\mathcal{X}|$ are *also* in bijection with the (Galois conjugacy classes of) residual pseudorepresentations $\overline{\theta}$, and the resulting bijection between the closed points of $|\mathcal{X}|$ and $|X|$ extends to a continuous morphism $\pi_{\mathrm{ss}} : |\mathcal{X}| \to |X|$, characterized by the following condition: for each pseudorepresentation $\overline{\theta}$, regarded as a closed point of $|X|$, the completion $\mathcal{X}_{\overline{\theta}}$ of $\mathcal{X}$ along its closed subset $\pi_{\mathrm{ss}}^{-1}(\overline{\theta})$ is the moduli stack of two-dimensional continuous Galois representations having determinant $\zeta\varepsilon^{-1}$ whose underlying pseudorepresentations deform $\overline{\theta}$. (The stack $\mathcal{X}_{\overline{\theta}}$ is an example of the moduli stacks of Galois representations constructed and studied by Wang-Erickson in [Wan18].)

1.1.7. *Establishing properties of* F. As already remarked, having defined L_∞, we are able to define a functor

$$\mathrm{F} := L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} : D^b_{\mathrm{fp}}(\mathcal{A}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}).$$

We then prove the following theorem, which collects a number of results from Section 5 of this paper.

Theorem 1.1.8.

(1) (Prop. 5.3.23.) *The functor* F *factors through the full subcategory* $D^b_{\mathrm{coh}}(\mathcal{X})$ *of* $\mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X})$*, and so may be regarded as a functor*

$$\mathrm{F} : D^b_{\mathrm{fp}}(\mathcal{A}) \to D^b_{\mathrm{coh}}(\mathcal{X}).$$

(2) (Prop. 5.3.25.) *For any closed subset* Y *of* X*, with open complement* U*, let* $\mathcal{X}_Y$ *denote the completion of* $\mathcal{X}$ *along* $\pi_{\mathrm{ss}}^{-1}(Y)$*, and let* $\mathcal{U} := \pi_{\mathrm{ss}}^{-1}(U)$*. Then the functor* F *induces a commutative diagram defining functors* F_Y *and* F_U

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^b_{\mathrm{fp}}(\mathcal{A}_Y) & \longrightarrow & D^b_{\mathrm{fp}}(\mathcal{A}) & \longrightarrow & D^b_{\mathrm{fp}}(\mathcal{A}_U) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{F}_Y} & & \big\downarrow{\scriptstyle \mathrm{F}} & & \big\downarrow{\scriptstyle \mathrm{F}_U} & & \\ 0 & \longrightarrow & D^b_{\mathrm{coh}}(\mathcal{X}_Y) & \longrightarrow & D^b_{\mathrm{coh}}(\mathcal{X}) & \longrightarrow & D^b_{\mathrm{coh}}(\mathcal{U}) & \longrightarrow & 0. \end{array} \tag{1.1.9}$$

(3) (Cor. 5.2.30.) *For any finite set of closed points* $Y \subset X$*, the functor* F_Y *is fully faithful.*

(4) (Thm. 5.4.25.) *There is a dense open subset* $U \subset X$ *such that the functor* F_U *is fully faithful.*

We deduce Theorem 1.1.1 from Theorem 1.1.8. More precisely, part (1) of the latter theorem is precisely the statement that F takes values in $D^b_{\mathrm{coh}}(\mathcal{X})$; while the full faithfulness of F follows by combining the full faithfulness results of parts (3) and (4) of Theorem 1.1.8 with general category-theoretic arguments. These arguments are developed in Appendix A.5; see especially Proposition A.5.3, which provides a criterion for full faithfulness of a functor that preserves a recollement-like decomposition of its source and target categories. Part (2) of Theorem 1.1.8 shows that F falls within the scope of Proposition A.5.3, and requires the completion theory of [DEG23] in its proof; in fact, we proceed by giving an independent definition of the functor F_Y as a completion of F, and then proving that (1.1.9) commutes.

The proof of part (3) of Theorem 1.1.8 immediately reduces to the case in which $Y = \{\overline{\theta}\}$ is a singleton, in which case $\mathcal{X}_Y$ coincides with $\mathcal{X}_{\overline{\theta}}$, and we write $\mathrm{F}_{\overline{\theta}}$ for F_Y. As explained in the next paragraph, the full faithfulness of $\mathrm{F}_{\overline{\theta}}$ is a consequence of the results of [JNW24]. This case of part (3) is then an input for the proof of part (4), which also requires an explicit quotient presentation of the underlying reduced $\mathcal{U}_{\mathrm{red}}$ of the stack $\mathcal{U}$, which we construct (for this specific choice of U) in Section 3.7.

1.1.10. *Relationship with other works.* For a closed point $\overline{\theta} \in X$, the functor $\mathrm{F}_{\overline{\theta}}$ discussed above turns out to be very closely related to one of the functors constructed by Johansson–Newton–Wang-Erickson in [JNW24]. Indeed, the only (relatively minor) difference between our functor $\mathrm{F}_{\overline{\theta}}$ and the corresponding functor of [JNW24] is that while our functor takes values in $D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$, where $\mathcal{X}_{\overline{\theta}}$ is the moduli stack of Galois representations introduced above, theirs takes values in $D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})$, for a certain canonical algebraization $\mathfrak{X}_{\overline{\theta}}$ of $\mathcal{X}_{\overline{\theta}}$; and our functor is then obtained as a completion of theirs. Given this fact, Theorem 1.1.8 (4) is essentially a restatement of the full faithfulness results of [JNW24].

However, establishing this relationship between $\mathrm{F}_{\overline{\theta}}$ and the corresponding functor of [JNW24] is non-trivial, since $\mathrm{F}_{\overline{\theta}}$ is defined in terms of Colmez's $D^\natural \boxtimes \mathbf{P}^1$ construction, while the functors of [JNW24] are defined by certain explicit kernels, using Morita-theoretic descriptions of the various blocks $\mathcal{A}_{\overline{\theta}}$ due to Paškūnas [Paš13]. The relationship between the two is given by Proposition 5.2.18, which in particular provides a conceptual explanation for the particular choice of kernels made in [JNW24].

More generally, we make full use of previously known results on the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, including those of [Col10c; Paš13; PT21; Paš15; JNW24; Kis09; HT15; San16]. We will often need to restate, or slightly extend, these results to fit our framework, and we claim no real novelty in doing so.

1.1.11. *Sheaf- and category-theoretic techniques.* As this introduction already indicates, we rely on Pro- and Ind-categorical techniques to deal with completions, and to pass from the coherent/finitely presented setting to the setting of sheaves and modules not satisfying any finiteness conditions. It seems likely that some, and perhaps all, of the resulting technicalities could be ameliorated by using the solid formalism of Clausen and Scholze; but for various reasons, including a lack of facility in those techniques on our part, we have decided to adopt the more traditional Ind/Pro perspective.

Our arguments also rely heavily on the theory of ∞-categories; in particular, *derived categories* are, for us, always understood in the stable ∞-categorical sense. The most substantial reason we require these techniques is that we prove our results via gluing arguments, and the traditional framework of triangulated categories is inadequate for making such arguments. We note, though, that we take advantage of this formalism at many other points of our arguments as well; in particular, we frequently exploit the flexibility it allows with regard to constructing and canonically characterizing derived functors.

While these techniques (Ind/Pro-categories and ∞-categories) are standard tools for experts in the categorical and geometric aspects of the Langlands program, they may be less familiar to readers who (like ourselves) are approaching the paper from the perspective of the traditional theory of the p-adic local Langlands correspondence. To this end, and in order to have a uniform phrasing of several results from the literature, we have included two appendices — Appendix A and Appendix B — which provide background on category-theory (especially ∞-category theory) and on coherent sheaf-theory on formal algebraic stacks.

1.1.12. *A guide to the paper.* Section 2 collects various background results from Galois representation theory, smooth representation theory of $\mathrm{GL}_2(\mathbf{Q}_p)$ in characteristic p, and the existing theory of the local p-adic Langlands correspondence for this group. We provide several complementary results, usually of a categorical nature, to use later in the paper. We then describe the localization theory of [DEG23] and generalize it to derived categories, and, using results of Heyer [Hey23; Hey24], we prove some finiteness results for Ext-groups between certain non-admissible representations.

Section 3 describes the Galois-theoretic side of our correspondence, building on the results in Appendix C, and on several previous works in the literature, including especially [EG23; Wan18; JNW24]. We then specialize the discussion of coherent, Ind-coherent, and Pro-coherent sheaves of Appendix B to the present context of moduli stacks of (φ,Γ)-modules, and carry out several explicit computations. We conclude by building a connection between these objects and the localization theory of [DEG23].

Section 4 contains our generalization of a significant part of Colmez's work [Col10c], to coefficients in arbitrary Noetherian $\mathcal{O}/\varpi^a$-algebras. We begin by proving some technical statements in Drinfeld's theory of Tate modules, which is the natural context for this generalization. We then work towards one of our main results, which is Theorem 4.8.8 on the $\mathrm{GL}_2(\mathbf{Q}_p)$-stability of $D^\natural \boxtimes \mathbf{P}^1$, and develop, along the way, analogues of Colmez's functors $D^+, D^{++}, D^{\mathrm{nr}}, D^\sharp, D^\natural$, and $D \boxtimes -$. We make significant use of novel base change results, which will also be important in Section 5 to construct L_∞; these results can be specialized to the classical context of coefficients in a complete Noetherian local $\mathcal{O}$-algebra, and yield new results even in that context (see Theorem 4.10.21).

Section 5 contains our construction of the functor F, and the proof of Theorems 1.1.1 and 1.1.8. We begin by constructing a version of $D^\natural \boxtimes \mathbf{P}^1$ for the universal object on the stack $\mathcal{X}$, which (as already explained) is done by descent, as an application of the results of Section 4; the pro-coherent sheaf L_∞ is then obtained as an appropriate twist of this universal version of $D^\natural \boxtimes \mathbf{P}^1$. We next make a detailed study of L_∞, starting from its (completed) pullbacks to the stacks $\mathcal{X}_{\overline{\theta}}$ of Galois representations, and of the functor $\mathrm{F} := L_\infty \otimes_{\mathcal{O}[\![G]\!]_\zeta} -$. Sections 5.3 and 5.4 establish the properties of F that are required to apply our full faithfulness criterion (Proposition A.5.3), which we do in Section 5.5, where we conclude the proof of our main result.

Appendix A assembles some background from 1-category and ∞-category theory, with a significant focus on Ind- and Pro-completions of categories, which are the context of our main full faithfulness criterion in Section A.5. One complication arises from the interaction between Pro-completions and the formation of left derived functors, which we study systematically in Sections A.8 and A.9. As already mentioned, much of this material is standard; but we have found it helpful to collect it here, to give a uniform phrasing to several results from the literature, and to have a uniform foundation for the less standard material in this appendix.

Appendix B develops the theory of coherent sheaves on formal algebraic stacks, both at the abelian and derived level, including various functorial operations on these objects, and recasting some of the coherent completeness results of [AHR23] and [AHL23] in our framework. Finally, Appendix C provides some complements to the results of [EG23] and [Wan18].

1.1.13. *A comparison with the proof outlined in* [EGH25]. The results of this paper were announced in [EGH25, Section 7], which also gave a sketch of the proofs, emphasizing the analogy between the results of [DEG23, Section 3.8], and Beauville–Laszlo gluing of sheaves on $\mathcal{X}$. This was presented as an important step in the proof. Furthermore, we indicated a proof of part (4) of Theorem 1.1.8 via an explicit Morita-theoretic argument.

However, in the course of writing this paper, we found a more efficient way of formulating these gluing results, avoiding explicit Beauville–Laszlo-type gluing, and proceeding instead via the notion of a recollement of ∞-categories. This resulted in the current proof of the full faithfulness of F, as well as a simplification of the proof of the Beauville–Laszlo-type results of [DEG23] (which has been incorporated into that paper). Furthermore, we found a more efficient proof of part (4) of Theorem 1.1.8, which avoids the explicit constructions and computations required in the Morita-theoretic approach, and instead uses the same results related to completions that are used to prove the full faithfulness of the functors F_Y for finite closed subsets Y of X.

In conclusion, although the overall structure of the argument sketched in [EGH25] has been preserved, and none of the statements of *loc. cit.* need to be corrected, several of the details of our arguments are different from those sketched there.

1.2. **Acknowledgements.** We would like to thank Bhargav Bhatt, George Boxer, Ana Caraiani, Pierre Colmez, Laurent Fargues, Michael Harris, Eugen Hellmann, Christian Johansson, Bao Le Hung, Brandon Levin, Jacob Lurie, Akhil Mathew, James Newton, Vytautas Paškūnas, David Savitt, Peter Scholze, and Carl Wang-Erickson for helpful correspondence and conversations. We are particularly grateful to Christian Johansson, James Newton, and Carl Wang-Erickson for sharing with us preliminary versions of the results of their paper [JNW24]. We would also like to thank Vytautas Paškūnas for comments on an earlier version of this paper.

1.3. **Notation and conventions.** We fix throughout the paper a prime $p \geq 5$. Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. If $K \subset \overline{\mathbf{Q}}_p$ is a finite extension of $\mathbf{Q}_p$, we write G_K for the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. Write I_K for the inertia subgroup of G_K. We normalise local class field theory so that a uniformizer corresponds to a geometric Frobenius element. Our convention for Hodge–Tate weights is that the p-adic cyclotomic character $\varepsilon : G_{\mathbf{Q}_p} \to \mathbf{Q}_p^\times$ has Hodge–Tate weight -1.

Let $\mathcal{O}$ denote the ring of integers in a fixed finite extension E of $\mathbf{Q}_p$, let ϖ be a fixed uniformizer of $\mathcal{O}$, and let $\mathbf{F} = \mathcal{O}/\varpi$ be the residue field of $\mathcal{O}$. Fix an algebraic closure $\overline{\mathbf{F}}_p$ of $\mathbf{F}$. Unless otherwise stated, all representations considered in this paper will be on $\mathcal{O}$-modules. We fix a continuous character $\zeta : \mathbf{Q}_p^\times \to \mathcal{O}^\times$, which we also regard as a character of $G_{\mathbf{Q}_p}$ via local class field theory. Throughout the paper we will work with representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ of central character ζ, and representations of $G_{\mathbf{Q}_p}$ with determinant $\zeta\varepsilon^{-1}$. We say that we are in the *even* (resp. *odd*) case if ζ is an even (resp. odd) character.

We write $G = \mathrm{GL}_2(\mathbf{Q}_p)$, $K = \mathrm{GL}_2(\mathbf{Z}_p)$, and $Z = \mathbf{Q}_p^\times$ for the centre of G. The diagonal torus in G is denoted by T, and we write B, resp. $\overline{B}$ for the upper-triangular, resp. lower-triangular Borel subgroup containing T, with unipotent radicals U and $\overline{U}$. The maximal compact subgroup of T is denoted by T_0. The nth congruence subgroup of K is denoted $K_n \coloneqq 1 + p^n M_2(\mathbf{Z}_p)$, for $n \geq 1$. The upper-triangular Iwahori subgroup of K is denoted Iw, and we write Iw_1 for its pro-p Sylow subgroup and $Z_1 = \mathrm{Iw}_1 \cap Z = K_1 \cap Z$ for the maximal pro-p subgroup of Z. Since $p \geq 3$, there are isomorphisms

$$\mathrm{Iw}_1 \xrightarrow{\sim} Z_1 \times (\mathrm{Iw}_1/Z_1) \text{ and } K_1 \xrightarrow{\sim} Z_1 \times (K_1/Z_1), \tag{1.3.1}$$

given by the determinant and the projection to the quotient. Since $p \geq 5$, the groups Iw_1 and K_1 are torsion-free, and so the same is true for their direct factors Iw_1/Z_1 and K_1/Z_1.

If $Z \subset H \subset G$ is a subgroup, we will write $\mathcal{O}[H]_\zeta$ for the quotient of the group algebra $\mathcal{O}[H]$ by the two-sided ideal generated by $\{z - \zeta(z) : z \in Z\}$.

Stacks. Our conventions on algebraic stacks and formal algebraic stacks are those of [Stacks] and [Eme]. The reader who is unfamiliar with this material may wish to refer to the overview of [Eme] in [EG23, App. A]. If A is a topological ring of the kind for which $\operatorname{Spf} A$ is defined (see [Stacks, Tag 0AIC] for an extensive discussion of this; we won't need any examples beyond I-adically complete Noetherian rings) and $\mathcal{C}$ is a stack then we write $\mathcal{C}(A)$ for $\mathcal{C}(\operatorname{Spf} A)$; if A has the discrete topology, then this is equal to $\mathcal{C}(\operatorname{Spec} A)$.

Module categories. We will consider several categories of modules over noncommutative rings R. Unless stated otherwise, we will work with left modules. We write $\mathrm{Mod}(R)$ for the category of left R-modules and R-linear maps, $\mathrm{Mod}^{\mathrm{f.l.}}(R)$ for the full subcategory of modules of finite length, and $\mathrm{Mod}^{\mathrm{fp}}(R)$ for the full subcategory of finitely presented modules, which coincides with the full subcategory of compact objects of $\mathrm{Mod}(R)$. We will introduce several variants in the course of the text, such as the category of profinite modules over a profinite topological ring, see Section A.1.30. Recall that the Jacobson radical of R, denoted $\mathrm{rad}(R)$, is the intersection of all maximal left ideals of R. It coincides with the intersection of all maximal right ideals, respectively two-sided ideals, of R. Unless stated otherwise, we write "Noetherian" as a shorthand for "left and right Noetherian".

Category theory and homological algebra. We employ the language of ∞-categories throughout the paper, and stable ∞-categories are the basic setting in which we undertake our homological algebra. We also liberally employ the language of Ind and Pro completion. Appendix A summarizes much of the material we use. Of course, most of what we use can be found (often in more general form) in [Lur09], [Lur17], and [Lur18]. On several occasions we cite [KS06], which provides an extensive survey of many topics in homological algebra and category theory (in the traditional language of derived categories). One caution we make is that its theory of Ind and Pro categories is developed in the context of large categories, rather than in context of small categories that we work with here.

We will be applying several of the constructions developed in [DEG23]. That paper has a more abelian categorical flavour, and we wish to translate its results into the derived context of this paper. The formation of left derived functors often involves passing to Pro-categories, but for the applications we have in mind, we do not wish to work in the derived category of the Pro-category, but rather in the Pro category of the derived category. An explanation of how to arrange this is given in § A.8.

2. Background

In this section we gather together a range of results related to Galois representations as well as to smooth representations which we will require later on in the paper.

2.1. Galois representations.

We introduce various notation and concepts related to two-dimensional representations and pseudorepresentations of $G_{\mathbf{Q}_p}$.

2.1.1. *Mod p Galois representations, Serre weights, and the weight part of Serre's conjecture.* We write $\omega = \overline{\varepsilon} : G_{\mathbf{Q}_p} \to \mathbf{F}_p^\times$ for the mod p cyclotomic character. We sometimes regard ω as a character of $\mathbf{Q}_p^\times$ via local class field theory (which, as noted above, is normalised so as to take a uniformizer to a geometric Frobenius element). Concretely, then, ω is the reduction mod p of the character $x|x|$ (i.e. trivial on p, and the reduction map to $\mathbf{F}_p^\times$ on $\mathbf{Z}_p^\times$).

For any $x \in \overline{\mathbf{F}}_p^\times$, we let
$$\mathrm{nr}_x : G_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^\times$$
be the unramified character taking a geometric Frobenius element Frob_p to x. Thus every character $G_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^\times$ is of the form $\mathrm{nr}_x\omega^i$ for some uniquely determined $x \in \overline{\mathbf{F}}_p^\times$ and $0 \le i < p-1$.

Definition 2.1.2. We say that an unordered pair of characters $\chi_1, \chi_2 : G_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^\times$ is *generic* if $\chi_1\chi_2^{-1} \neq 1, \omega^{\pm 1}$. Note that by Tate local duality, this implies that $H^0(G_{\mathbf{Q}_p}, \chi_1\chi_2^{-1})$ and $H^2(G_{\mathbf{Q}_p}, \chi_1\chi_2^{-1})$ both vanish, and Tate's local Euler characteristic formula then shows that $H^1(G_{\mathbf{Q}_p}, \chi_1\chi_2^{-1})$ is a one-dimensional $\overline{\mathbf{F}}_p$-vector space.

We work throughout with the ring of integers $\mathcal{O}$ in the finite extension E of $\mathbf{Q}_p$ as our base ring, and so its residue field $\mathbf{F}$ plays a special notational role. We often consider representations defined over $\mathbf{F}$, or $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-orbits of representations defined over $\overline{\mathbf{F}}_p$. However, all the results we discuss are insensitive to base-change from $\mathcal{O}$ to $\mathcal{O}'$ (an extension of rings of integers arising from an inclusion $E \subseteq E'$ of finite extensions of $\mathbf{Q}_p$), and so we are always free to enlarge $\mathbf{F}$ as necessary. Thus, although we have (of course) tried to be correct in our treatment of questions of fields of definition, we encourage the reader to not pay too much attention to this purely technical issue.

This being said, let $\mathbf{F}'$ now denote the unique quadratic extension of $\mathbf{F}$, and write $\omega_2 : I_{\mathbf{Q}_p} \to (\mathbf{F}')^\times$ for a choice of fundamental character of niveau 2; writing $\mathbf{Q}_{p^2}$ for the quadratic unramified extension of $\mathbf{Q}_p$, we can extend ω_2 to a character $G_{\mathbf{Q}_{p^2}} \to (\mathbf{F}')^\times$ in such a way that $\omega_2^{p+1} = \omega|_{G_{\mathbf{Q}_{p^2}}}$. Then the 2-dimensional absolutely irreducible representations $G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{F})$ are precisely those of the form $\mathrm{nr}_x \otimes \mathrm{Ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \omega_2^k$ for some $k \in \mathbf{Z}/(p^2-1)\mathbf{Z}$ with $(p+1) \nmid k$ and some $x \in (\mathbf{F}')^\times$ with $x^2 \in \mathbf{F}^\times$.[1] In this case x is uniquely determined up to multiplication by -1, and k is uniquely determined up to multiplication by p.

Definition 2.1.3. A *Serre weight* is an irreducible representation $\sigma_{a,b} \coloneqq \det{}^a \otimes \mathrm{Sym}^b\, \mathbf{F}_p^2$ of $\mathrm{GL}_2(\mathbf{F}_p)$, where $0 \le a < p-1$ and $0 \le b \le p-1$. It is sometimes convenient to view a as an element of $\mathbf{Z}/(p-1)\mathbf{Z}$, and we will do so without further comment. We say that $\sigma_{a,b}$ is *Steinberg* if $b = p-1$, and *non-Steinberg* otherwise.

We now explain what it means to say that "σ is a Serre weight for $\overline{\rho}$". We do not attempt to motivate this description, which goes back to [Ser87], but we remark that one possible motivation is given by Theorem 3.2.4 (5).

Definition 2.1.4.

(1) If $\overline{\rho} = \mathrm{nr}_x \otimes \mathrm{Ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \omega_2^k$ is irreducible, then $\sigma_{a,b}$ is a Serre weight for $\overline{\rho}$ if and only if $k \equiv (p+1)a + b - p \pmod{p^2-1}$ or $k \equiv (p+1)a + pb - 1 \pmod{p^2-1}$.

[1] Although this expression involves objects only defined over $\mathbf{F}'$, the representation itself is defined over $\mathbf{F}$. Since, as already noted, we allow ourselves to enlarge $\mathbf{F}$ as convenient, questions of precise fields of definition are of little consequence to us in any case.

(2) If $b \neq 0$, then $\sigma_{a,b}$ is a Serre weight for a reducible representation $\overline{\rho}$ if and only if

$$\overline{\rho} \cong \begin{pmatrix} \mathrm{nr}_x \omega^{a+b} & * \\ 0 & \mathrm{nr}_y \omega^{a-1} \end{pmatrix} \tag{2.1.5}$$

for some $x, y \in \mathbf{F}^\times$.

(3) If $b = 0$, then $\sigma_{a,0}$ is a Serre weight for a reducible representation $\overline{\rho}$ if and only if $\overline{\rho}$ can be written as an extension as in (2.1.5), and if $\overline{\rho}$ is furthermore finite flat. If $x \neq y$ then this further condition actually holds automatically, while if $x = y$ it is equivalent to requiring that the extension be peu ramifiée (which by definition amounts to asking that the corresponding Kummer class is given by an integral unit).

We now note a compatibility between central characters and determinants.

Definition 2.1.6. A Serre weight $\sigma_{a,b}$ is *compatible with* ζ, or ζ-*compatible*, if $\omega^{2a+b} = \overline{\zeta}|_{I_{\mathbf{Q}_p}}$.

It is immediate from the definitions above that if $\det \overline{\rho} = \overline{\zeta}\omega^{-1}$, and σ is a Serre weight for $\overline{\rho}$, then σ is compatible with ζ. Accordingly, we assume throughout the paper that all Serre weights σ that we consider are compatible with ζ. We will also sometimes think of a ζ-compatible Serre weight as being an $\mathcal{O}[KZ]_\zeta$-module, letting K act through its quotient $K/K_1 = \mathrm{GL}_2(\mathbf{F}_p)$ and letting Z act via ζ.

Remark 2.1.7. Via local class field theory, we can view the condition of Definition 2.1.6 as saying that $\overline{\zeta}$ (restricted to $\mathbf{F}_p^\times$) agrees with the central character of σ. In particular, b is even (resp. odd) if and only if ζ is even (resp. odd); furthermore, for each choice of b of the appropriate parity, there are two corresponding choices of a for which $\sigma_{a,b}$ is a compatible Serre weight.

The following definition will be used throughout the paper. Again, we do not attempt to motivate this here, but we remark that it goes back to [Gro90], and we point to Corollary 3.7.4 as one possible motivation.

Definition 2.1.8.

(1) The *companion* of a Serre weight $\sigma = \sigma_{a,b}$ with $0 \leq b \leq p-3$ is the Serre weight $\sigma^{\mathrm{co}} := \sigma_{a+b+1,p-3-b}$. A Serre weight of the form $\sigma_{a,p-2}$ is defined to be its own companion. (We do not define a companion for a Steinberg Serre weight.) We refer to the unordered pair of $\sigma, \sigma^{\mathrm{co}}$ as a *companion pair*. We say that the unordered pair $\sigma, \sigma^{\mathrm{co}}$ is of type (scalar) if it is of the form $\sigma_{a,p-2}$, of type (St) if it is of the form $\sigma_{a,0}, \sigma_{a+1,p-3}$, and otherwise we say that it is of type (gen).

(2) Let $\{\sigma, \sigma^{\mathrm{co}}\}$ be a companion pair of Serre weights, and write $\sigma = \sigma_{a,b}$. We write $\Theta(\sigma|\sigma^{\mathrm{co}})$ for the cuspidal $E[\mathrm{GL}_2(\mathbf{F}_p)]$-representation denoted $\Theta(\chi)$ in [CDT99, §3], where $\chi : \mathbf{F}_{p^2}^\times \to \mathcal{O}^\times$ lifts $x \mapsto x^{(b+2)+(a-1)(p+1)}$. Note that by [CDT99, Lem. 3.1.1(4)], the Jordan–Hölder factors of $\Theta(\chi) \otimes_{\mathcal{O}} \mathbf{F}$ are $\{\sigma, \sigma^{\mathrm{co}}\}$. If σ is ζ-compatible, we will often regard $\Theta(\sigma|\sigma^{\mathrm{co}})$ as an $E[KZ]_\zeta$-module, by letting Z act via ζ, and K via its quotient $K/K_1 = \mathrm{GL}_2(\mathbf{F}_p)$.

2.1.9. *Pseudorepresentations.* We will make use of the notion of a pseudorepresentation of $G_{\mathbf{Q}_p}$ on an $\mathcal{O}$-algebra A. We will assume throughout, without further comment, that all our pseudorepresentations are 2-dimensional and have determinant $\zeta\varepsilon^{-1}$. Since $p > 2$, a 2-dimensional pseudorepresentation is equivalent to the data of a 2-dimensional determinant in the sense of [Che14, Defn. 1.5], so such a pseudorepresentation is equivalent (see [Che14, Lem. 1.9, Prop. 1.29]) to a continuous function $\theta : G_{\mathbf{Q}_p} \to A$ satisfying

- $\theta(1) = 2$.
- $\theta(gh) = \theta(hg)$.
- $(\zeta\varepsilon^{-1})(g)\theta(g^{-1}h) - \theta(g)\theta(h) + \theta(gh) = 0$.

Given a continuous representation $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(A)$, the function $\operatorname{tr}\rho : G_{\mathbf{Q}_p} \to A$ is a pseudorepresentation in this sense. If A is a finite field, or an algebraically closed field, then any pseudorepresentation comes from a representation, by [Wan18, Corollary 2.9]. So we can and do identify $\mathbf{F}$-valued pseudorepresentations with semisimple representations with coefficients in $\mathbf{F}$.

Definition 2.1.10. We classify the 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentations $\overline{\theta} : G_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p$ into five "types" as follows; we explain the labels in Remark 2.4.9.

(ssg) $\overline{\theta} : G_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p$ is the trace of an absolutely irreducible two-dimensional $\mathbf{F}$-representation.

(gen) $\overline{\theta} = \chi + \zeta\omega^{-1}\chi^{-1}$ for some character $\chi : G_{\mathbf{Q}_p} \to \mathbf{F}^\times$ such that the pair $\{\chi, \zeta\omega^{-1}\chi^{-1}\}$ is generic in the sense of Definition 2.1.2.

(scalar) $\overline{\theta} = \chi + \chi$ for some character $\chi : G_{\mathbf{Q}_p} \to \mathbf{F}^\times$ such that $\chi^2 = \zeta\omega^{-1}$. These only exist if ζ is odd and $\zeta(p)$ is a square.

(St) $\overline{\theta} = \chi + \chi\omega^{-1}$ for some character $\chi : G_{\mathbf{Q}_p} \to \mathbf{F}^\times$ such that $\chi^2 = \zeta$. These only exist if ζ is even and $\zeta(p)$ is a square.

(gen+) $\overline{\theta} = \chi + \zeta\omega^{-1}\chi^{-1}$ for some character $\chi : G_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^\times$ not defined over $\mathbf{F}$.

Remark 2.1.11. The reason for the choice of terminology in Definition 2.1.8 is that the semisimple Galois representations of type (scalar) (resp. of type (St)) have Serre weights of the form $\sigma_{a,p-2}$ (respectively of the form $\sigma_{a,0}, \sigma_{a+1,p-3}, \sigma_{a,p-1}$).

Given any $\overline{\mathbf{F}}_p$-valued pseudorepresentation $\overline{\theta}$ we write $\mathbf{F}_{\overline{\theta}}$ for the (finite) subfield of $\overline{\mathbf{F}}_p$ generated by $\mathbf{F}$ and the values of $\overline{\theta}$. It only depends on the $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy class of $\overline{\theta}$. We write $(R^{\mathrm{ps}}_{\overline{\theta}}, \theta^{\mathrm{univ}})$ for the universal deformation of $\overline{\theta}$ to complete Noetherian local $\mathcal{O} \otimes_{W(\mathbf{F})} W(\mathbf{F}_{\overline{\theta}})$-algebras, where by a *deformation* of $\overline{\theta}$ to a complete local Noetherian $\mathcal{O}$-algebra A with residue field $\mathbf{F}_{\overline{\theta}}$, we mean a 2-dimensional pseudorepresentation $\theta : G_{\mathbf{Q}_p} \to A$, with determinant $\zeta\varepsilon^{-1}$, such that the composition $G_{\mathbf{Q}_p} \xrightarrow{\theta} A \to \mathbf{F}_{\overline{\theta}}$ equals $\overline{\theta}$; the universal deformation exists by [Che14, Prop. 3.3]. We sometimes refer to $R^{\mathrm{ps}}_{\overline{\theta}}$ as the (universal) pseudodeformation ring for $\overline{\theta}$.

Remark 2.1.12. We will sometimes say that a pseudorepresentation $\overline{\theta}$ is $\mathbf{F}$-*rational* if $\mathbf{F}_{\overline{\theta}} = \mathbf{F}$. The first four items of Definition 2.1.10 are all $\mathbf{F}$-rational, and classify pseudorepresentations attached to semisimple representations $G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{F})$ all of whose irreducible summands are absolutely irreducible. The $\mathbf{F}$-rational pseudorepresentations of type (gen+) correspond to irreducible representations $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{F})$ that are not absolutely irreducible. These are precisely the twists of unramified representations such that the characteristic polynomial of Frobenius is irreducible quadratic.

Definition 2.1.13. We write $(\widetilde{R}_{\overline{\theta}}, \theta^{\mathrm{univ}})$ for the Cayley–Hamilton R^{ps}-algebra associated to θ^{univ}, with anti-involution $\dagger$. Explicitly, $\widetilde{R}_{\overline{\theta}}$ is the quotient $R^{\mathrm{ps}}_{\overline{\theta}}[\![G_{\mathbf{Q}_p}]\!]/J$, where $R^{\mathrm{ps}}_{\overline{\theta}}[\![G_{\mathbf{Q}_p}]\!]$ denotes the completed group ring, and J is the closure of the 2-sided ideal in $R^{\mathrm{ps}}_{\overline{\theta}}[\![G_{\mathbf{Q}_p}]\!]$ generated by the elements

$$g^2 - \theta^{\mathrm{univ}}(g)g + (\zeta\varepsilon^{-1})(g)$$

for all $g \in G_{\mathbf{Q}_p}$.

Note that the identity

$$\theta^{\mathrm{univ}}(g) = g + (\zeta\varepsilon^{-1})(g)g^{-1} \in \widetilde{R}_{\overline{\theta}} \tag{2.1.14}$$

for $g \in G_{\mathbf{Q}_p}$ shows that the natural map

$$\mathcal{O}[\![G_{\mathbf{Q}_p}]\!] \to \widetilde{R}_{\overline{\theta}} \tag{2.1.15}$$

is surjective, and uniquely determines the embedding $R^{\mathrm{ps}}_{\overline{\theta}} \hookrightarrow \widetilde{R}_{\overline{\theta}}$. The anti-involution $\dagger : g \mapsto (\zeta\varepsilon^{-1})(g)g^{-1}$ preserves the ideal J and descends to an $R^{\mathrm{ps}}_{\overline{\theta}}$-linear anti-involution of $\widetilde{R}_{\overline{\theta}}$.

Note that for any two pseudorepresentations in the $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy class of $\overline{\theta}$, the corresponding universal pseudodeformation rings and Cayley–Hamilton algebras are canonically identified (using the Galois action on coefficients).

We will need the following structural properties of $\widetilde{R}_{\overline{\theta}}$.

Proposition 2.1.16. *Let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation. The ring $\widetilde{R}_{\overline{\theta}}$ is Noetherian, of finite type as an $R^{\mathrm{ps}}_{\overline{\theta}}$-module, and p-torsion free. The natural map $R^{\mathrm{ps}}_{\overline{\theta}} \to Z(\widetilde{R}_{\overline{\theta}})$ is an isomorphism.*

Proof. All statements of the proposition can be checked after making a finite unramified extension $\mathcal{O} \to \mathcal{O}'$ (using that $Z(\widetilde{R}_{\overline{\theta}}) \otimes_{\mathcal{O}} \mathcal{O}' \to Z(\widetilde{R}_{\overline{\theta}} \otimes_{\mathcal{O}} \mathcal{O}')$ is an isomorphism). We can therefore assume without loss of generality that $\overline{\theta}$ does not have type (gen+).

Since $R^{\mathrm{ps}}_{\overline{\theta}}$ is Noetherian (by [Wan18, Proposition 3.2]), the first statement is a consequence of the second. The second statement is [Wan18, Proposition 3.6 (ii)]. There remains to prove the claim that $\widetilde{R}_{\overline{\theta}}$ is p-torsion free, and the map $R^{\mathrm{ps}}_{\overline{\theta}} \to Z(\widetilde{R}_{\overline{\theta}})$ is an isomorphism.

When $\overline{\theta}$ is residually multiplicity-free (i.e. always except in case (scalar)), the claim follows from the fact that $\widetilde{R}_{\overline{\theta}}$ is a generalized matrix algebra over $R^{\mathrm{ps}}_{\overline{\theta}}$, by [Wan18, Proposition 3.6 (5)], since $R^{\mathrm{ps}}_{\overline{\theta}}$ is a p-torsion free Noetherian integral domain. In more detail, $R^{\mathrm{ps}}_{\overline{\theta}}$ is formally smooth of dimension 3 over $\mathcal{O} \otimes_{W(\mathbf{F})} W(\mathbf{F}_{\overline{\theta}})$ in case (ssg), by e.g. [Paš13, Proposition 6.2, Remark 6.6], and in case (gen), by e.g. [Paš13, Proposition B.17, Remark B.28]. The structure of $R^{\mathrm{ps}}_{\overline{\theta}}$ in case (St) is described in [Paš13, Remark 10.88, Corollary B.5], using [Paš13, Remark 10.88] to identify $R^{\mathrm{ps}}_{\overline{\theta}}$ with the ring denoted R^{ψ} in *loc. cit.* This makes it clear that $R^{\mathrm{ps}}_{\overline{\theta}}$ is p-torsion free in this case as well.

Finally, when $\overline{\theta}$ is of type (scalar), our claim is a consequence of [Paš13, Corollary 9.24, Corollary 9.25], using [Paš13, Corollary 9.33] to identify $\widetilde{R}_{\overline{\theta}}$ with the ring denoted R in *loc. cit.* ∎

Lemma 2.1.17. *Let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation, and assume that $\overline{\theta}$ does not have type* (scalar) *or* (gen+). *Then every $R^{\mathrm{ps}}_{\overline{\theta}}$-algebra automorphism of $\widetilde{R}_{\overline{\theta}}$ that fixes all isomorphism classes of simple $\widetilde{R}_{\overline{\theta}}$-modules is inner.*

Proof. The assumptions on $\overline{\theta}$ mean that its irreducible summands are distinct and absolutely irreducible. As recalled during the proof of Proposition 2.1.16, $R^{\mathrm{ps}}_{\overline{\theta}}$ is an integral domain. Writing K for its field of fractions, it then follows from [BC09, Thm. 1.4.4(ii)] that $\widetilde{R}_{\overline{\theta}}$ is a generalized matrix $R^{\mathrm{ps}}_{\overline{\theta}}$-subalgebra of $M_2(K)$, in such a way that the pseudorepresentation $\theta^{\mathrm{univ}} : \widetilde{R}_{\overline{\theta}} \to R^{\mathrm{ps}}_{\overline{\theta}}$ coincides with the restriction of the trace function.

Each irreducible summand $\overline{\theta}_i$ of $\overline{\theta}$ is the pseudorepresentation associated to a unique simple $\widetilde{R}_{\overline{\theta}}$-module M_i. Part of the generalized matrix algebra structure on $\widetilde{R}_{\overline{\theta}}$ is an idempotent $e_{ii} \in \widetilde{R}_{\overline{\theta}}$ such that, for all $x \in \widetilde{R}_{\overline{\theta}}$, we have

$$\overline{\theta}(e_{ii}xe_{ii}) = \overline{\theta}_i(x).$$

Now we claim that the $\varphi(e_{ii})$ satisfy condition (3) in [BC09, Lemma 1.4.3], i.e.

$$\overline{\theta}(\varphi(e_{ii})x\varphi(e_{ii})) = \overline{\theta}_i(x)$$

for all $x \in \widetilde{R}_{\overline{\theta}}$. The assumption that φ fixes every isomorphism class of simple $\widetilde{R}_{\overline{\theta}}$-modules implies that $\overline{\theta}_i \circ \varphi = \overline{\theta}_i$ for all i (because if $\overline{\theta}_i$ is the trace of the simple $\widetilde{R}_{\overline{\theta}}$-module M_i, then $\overline{\theta}_i \circ \varphi$ is the trace of $\varphi^*(M_i) \cong M_i$). In turn, this implies that $\overline{\theta} \circ \varphi = (\sum_i \overline{\theta}_i) \circ \varphi = \sum_i (\overline{\theta}_i \circ \varphi) = \sum_i \overline{\theta}_i = \overline{\theta}$, and we can then compute that

$$\overline{\theta}(\varphi(e_{ii})x\varphi(e_{ii})) = (\overline{\theta} \circ \varphi)(e_{ii}\varphi^{-1}(x)e_{ii}) = \overline{\theta}(e_{ii}\varphi^{-1}(x)e_{ii}) = \overline{\theta}_i(\varphi^{-1}(x)) = \overline{\theta}_i(x).$$

Now the final statement of [BC09, Lemma 1.4.3] shows that φ is $\widetilde{R}^{\times}_{\overline{\theta}}$-conjugate to an automorphism that fixes all the e_{ii}. It thus suffices to prove that every R-automorphism φ of $\widetilde{R}_{\overline{\theta}}$ which fixes all the e_{ii} is inner. This follows by inspection from the presentation of $\widetilde{R}_{\overline{\theta}}$ described in Section 3.4.5—3.4.9 below; we give details in the case (St), which is the only case that we will need in the rest of the paper. Here $R := R^{\mathrm{ps}}_{\overline{\theta}} = \mathcal{O}[\![a_0, a_1, X_0, X_1]\!]/(a_0X_1 + a_1X_0)$, and

$$\widetilde{R}_{\overline{\theta}} \cong \begin{pmatrix} R & RX_0 + RX_1 \\ R & R \end{pmatrix} \subset M_2(R),$$

compare Remark 3.6.21. The automorphism φ fixes e_{11}, e_{22}, hence it preserves the direct sum decomposition of $\widetilde{R}_{\overline{\theta}}$ according to matrix entries, and it restricts to the identity on Re_{11} and Re_{22}. Since $\mathrm{Aut}_R(Re_{21}) = R^{\times}$, we can compose our automorphism with an inner automorphism and thereby assume that $\varphi(e_{21}) = e_{21}$. It must then also restrict to the identity on the remaining summand. Indeed, we may write $\varphi(X_ie_{12}) = x_ie_{12}$ for some $x_i \in R$, and we have

$$x_ie_{11} = x_ie_{12}e_{21} = \varphi(X_ie_{12})e_{21} = \varphi(X_ie_{12}e_{21}) = \varphi(X_i)\varphi(e_{11}) = X_ie_{11},$$

so that $x_i = X_i$, as required. $\square$

2.2. **Representation theory.** We introduce a range of notation and results related to the basics of the representation theory of $K\big(= \mathrm{GL}_2(\mathbf{Z}_p)\big)$ and $G\big(= \mathrm{GL}_2(\mathbf{Q}_p)\big)$.

2.2.1. *Abelian categories of representations.* We say that an $\mathcal{O}$-module is *locally torsion* if each element is annihilated by some power of the uniformizer ϖ of $\mathcal{O}$. As usual, if Γ is a locally profinite group, then we say that a representation on a locally torsion $\mathcal{O}$-module is *smooth* if every element is fixed by an open subgroup of Γ, and that a smooth representation is furthermore *admissible* if for any open subgroup of Γ, its submodule of invariants is finite over $\mathcal{O}$. Since we are working with representations on p-power torsion modules, it suffices to check this for a single open pro-p-subgroup of Γ (assuming that Γ admits such a subgroup). Following [Eme10, Defn. 2.2.15], we say that a smooth representation π of Γ is *locally admissible* if every vector $v \in \pi$ generates an admissible representation, and is *locally finite* if every vector $v \in \pi$ generates a representation of finite length.

We write $\mathrm{sm}.\,\Gamma$ for the category of smooth Γ-representations on locally torsion $\mathcal{O}$-modules. Then $\mathrm{sm}.\,\Gamma$ is a Grothendieck category. (See e.g. the proof of [DEG23, Lem. 2.2.3], which extends directly to any Γ.) We write $(\mathrm{sm}.\,\Gamma)^{\mathrm{adm}}$, resp. $(\mathrm{sm}.\,\Gamma)^{\mathrm{l.adm}}$, $(\mathrm{sm}.\,\Gamma)^{\mathrm{f.l.}}$, $(\mathrm{sm}.\,\Gamma)^{\mathrm{fg}}$, for the full subcategory of $\mathrm{sm}.\,\Gamma$ consisting of admissible objects, resp. locally admissible objects, finite length objects, finitely generated objects.

We will be most interested in the case when $\Gamma = G \coloneqq \mathrm{GL}_2(\mathbf{Q}_p)$. We note that, since the congruence subgroups K_n of G are open and pro-p, an object π of $\mathrm{sm}.\,G$ is admissible if and only if π^{K_n} is $\mathcal{O}$-finite for one (and hence every) $n \geq 1$.

Throughout the paper we have fixed a continuous character $\zeta : \mathbf{Q}_p^\times \to \mathcal{O}^\times$, and will focus on representations having fixed central character ζ. We therefore introduce notation for the various categories of representations having this fixed central character, which will be in force throughout the paper. Namely, we write $\mathcal{A}$ for the full subcategory of $\mathrm{sm}.\,G$ consisting of objects with central character ζ, and similarly $\mathcal{A}^{\mathrm{adm}}$, $\mathcal{A}^{\mathrm{l.adm}}$, $\mathcal{A}^{\mathrm{f.l.}}$, $\mathcal{A}^{\mathrm{fg}}$. Then $\mathcal{A}^{\mathrm{l.adm}}$ coincides with the category of locally finite objects of $\mathcal{A}$, by [Eme10, Thm. 2.2.17]. We have the following results about its structure; recall (see Section A.1.25) that a *locally finite category* is a Grothendieck category admitting a set of generators of finite length.

Lemma 2.2.2.

(1) $\mathcal{A}^{\mathrm{l.adm}}$ is a locally finite category.

(2) The inclusion $\mathcal{A}^{\mathrm{f.l.}} \to \mathcal{A}^{\mathrm{l.adm}}$ induces an equivalence

$$\operatorname{Ind} \mathcal{A}^{\mathrm{f.l.}} \xrightarrow{\sim} \mathcal{A}^{\mathrm{l.adm}}. \tag{2.2.3}$$

(3) Let E'/E be a finite extension with ring of integers $\mathcal{O}'$. Then extension and restriction of scalars define functors $\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}} \to \mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'}$ and $\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'} \to \mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}}$. If $\pi_1, \pi_2 \in \mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}}$, then

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}}}(\pi_1, \pi_2) \otimes_{\mathcal{O}} \mathcal{O}' \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'}}(\pi_1 \otimes_{\mathcal{O}} \mathcal{O}', \pi_2 \otimes_{\mathcal{O}} \mathcal{O}').$$

Proof. Since every object of $\mathcal{A}^{\mathrm{l.adm}}$ is locally finite, the set of objects of $\mathcal{A}^{\mathrm{l.adm}}$ of finite length is a set of generators of $\mathcal{A}^{\mathrm{l.adm}}$. Since $\mathcal{A}^{\mathrm{l.adm}}$ is an exact abelian subcategory of $\mathcal{A}$ (in the sense recalled in Appendix A.1.1), and $\mathcal{A}^{\mathrm{l.adm}}$ is closed under colimits in $\mathcal{A}$, and $\mathcal{A}$ is a Grothendieck category, we see that $\mathcal{A}^{\mathrm{l.adm}}$ is a Grothendieck category. Hence $\mathcal{A}^{\mathrm{l.adm}}$ is a locally finite category, proving part (1). Part (2) is then a consequence of Lemma A.1.26(2). We now prove part (3). The same argument as [Paš13, Lemma 5.1] shows that for all $\pi_1, \pi_2 \in \mathcal{A}_{\mathcal{O}}$ we have

$$\mathrm{Hom}_{\mathcal{A}_{\mathcal{O}}}(\pi_1, \pi_2) \otimes_{\mathcal{O}} \mathcal{O}' \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}_{\mathcal{O}'}}(\pi_1 \otimes_{\mathcal{O}} \mathcal{O}', \pi_2 \otimes_{\mathcal{O}} \mathcal{O}').$$

(Note that since $\mathcal{O}'$ is finite free over $\mathcal{O}$, we have $\mathrm{Hom}_{\mathcal{O}}(V,W)\otimes_{\mathcal{O}}\mathcal{O}' \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}'}(V\otimes_{\mathcal{O}}\mathcal{O}', W\otimes_{\mathcal{O}}\mathcal{O}')$ for all $\mathcal{O}$-modules V, W.) Hence, for all $\pi \in \mathcal{A}_{\mathcal{O}}$, we have $\pi^K \otimes_{\mathcal{O}} \mathcal{O}' = (\pi \otimes_{\mathcal{O}} \mathcal{O}')^K$. This immediately implies that $-\otimes_{\mathcal{O}} \mathcal{O}'$ preserves locally admissible representations. Finally, if $\pi \in \mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'}$, then π is also locally admissible when viewed as an object of $\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}}$ by restriction of scalars: in fact, we immediately reduce to the case in which π is admissible, and then π^K is a finitely generated $\mathcal{O}'$-module, hence a finitely generated $\mathcal{O}$-module. □

2.2.4. *Completed group rings.* If Γ° is a profinite group, we write $\mathcal{O}[\![\Gamma^\circ]\!]$ for the usual Iwasawa algebra (i.e. completed group ring) of Γ° with coefficients in $\mathcal{O}$. It is a profinite $\mathcal{O}$-algebra. If Γ° is a compact p-adic analytic group, then $\mathcal{O}[\![\Gamma^\circ]\!]$ is Noetherian [Laz65, Prop. V.2.2.4]. We also refer to [Eme10, Section 2.1] for a recollection of some of the basic theory of these rings.

In fact, we will frequently consider Iwasawa coefficients with coefficients, especially in the case when $\Gamma^\circ = K$. We recall the definition.

Definition 2.2.5. If A is an $\mathcal{O}$-algebra, then $A[\![\Gamma^\circ]\!]$ denotes the topological A-algebra

$$A[\![\Gamma^\circ]\!] \coloneqq \varprojlim_H A[\Gamma^\circ/H],$$

where H runs over a neighbourhood basis of normal open subgroups of Γ°, each of the group rings $A[\Gamma^\circ/H]$ is endowed with the discrete topology, and $A[\![\Gamma^\circ]\!]$ is endowed with the projective limit topology.

We next record some properties of $A[\![K]\!]$ (under the additional hypothesis that A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$). Before doing so, we recall from e.g. [LO96, Definition II.2.1.1] that a filtered $\mathcal{O}$-algebra $(R, \mathrm{Fil}^i R)_{i\in\mathbf{Z}}$ (possibly non-commutative, decreasingly filtered, and with $\mathrm{Fil}^0 = R$) is called *Zariskian* if $\mathrm{Fil}^1 R \subset \mathrm{rad}(R)$ and the Rees ring

$$\mathrm{Rees}(R) \coloneqq \bigoplus_{i\in\mathbf{Z}} \mathrm{Fil}^i R$$

is Noetherian. We will use [LO96] as our main reference for this material, with the caveat that we will work with decreasing filtrations, whereas *loc. cit.* uses increasing filtrations. A filtered R-module M is *good* if $\mathrm{Rees}(M)$ is finitely generated over $\mathrm{Rees}(R)$, which, by [LO96, Lem. I.5.4], is equivalent to the existence of generators $m_1, \ldots, m_n$ of M as an R-module, and integers $k_1, \ldots, k_n$, such that

$$\mathrm{Fil}^i M = (\mathrm{Fil}^{i-k_1} R)m_1 + \cdots + (\mathrm{Fil}^{i-k_n} R)m_n.$$

Lemma 2.2.6. *Let $\mathfrak{a}$ be the kernel of $\mathcal{O}[\![K_1]\!] \to \mathbf{F}$, i.e. the two-sided ideal generated by ϖ and the augmentation ideal of $\mathcal{O}[\![K_1]\!]$. Then the profinite topology on $\mathcal{O}[\![K_1]\!]$ and $\mathcal{O}[\![K]\!]$ is induced by the $\mathfrak{a}$-adic filtration, which is Zariskian.*

Proof. First, recall that since K_1 is a uniform pro-p group, the rings $\mathcal{O}[\![K_1]\!]$ and $\mathrm{gr}_{\mathfrak{a}}(\mathcal{O}[\![K_1]\!])$ are Noetherian: since the $\mathfrak{a}$-adic filtration on $\mathcal{O}[\![K_1]\!]$ is induced by a p-valuation on G, by [Ven02, Lem. 3.24, 3.25], this is a consequence of [Laz65, Thm. III.2.3.3]. Since $\mathfrak{a} = \mathrm{rad}(\mathcal{O}[\![K_1]\!])$, we now see that the topology on $\mathcal{O}[\![K_1]\!]$ is $\mathfrak{a}$-adic, by Lemma A.1.32 (8). Finally, since $\mathcal{O}[\![K_1]\!]$ is $\mathfrak{a}$-adically complete, [LO96, Prop. II.2.2.2] shows that $\mathcal{O}[\![K_1]\!]$ is Zariskian, because the associated graded ring $\mathrm{gr}_{\mathfrak{a}}\mathcal{O}[\![K_1]\!]$ is Noetherian. This concludes the proof for K_1.

Since $\mathcal{O}[\![K]\!]$ is a finite $\mathcal{O}[\![K_1]\!]$-module, its $\mathfrak{a}$-adic filtration induces the profinite topology, again by Lemma A.1.32 (8), and is a good filtration (by definition). By [LO96, Thm. I.5.7], we conclude that $\mathrm{gr}_{\mathfrak{a}}(\mathcal{O}[\![K]\!])$ is a finite $\mathrm{gr}_{\mathfrak{a}}(\mathcal{O}[\![K_1]\!])$-module. Hence $\mathrm{gr}_{\mathfrak{a}}(\mathcal{O}[\![K]\!])$ is a Noetherian ring, which by another application of [LO96, Prop. II.2.2.2] shows that $\mathcal{O}[\![K]\!]$ is Zariskian with respect to the $\mathfrak{a}$-adic filtration. □

Lemma 2.2.7. *Let A be a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$.*

(1) $A[\![K]\!]$ is topologically isomorphic to the $\mathfrak{a}$-adic completion of $A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]$.
(2) The rings $A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]$ and $A[\![K]\!]$ are Noetherian.
(3) $A[\![K]\!]$ is flat over $A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]$.
(4) The projective limit topology on $A[\![K]\!]$ coincides with its $\mathfrak{a}$-adic topology.
(5) Every morphism in $\mathrm{Mod}^{\mathrm{fp}}(A[\![K]\!])$ is strict with respect to the $\mathfrak{a}$-adic topology (in the sense that the quotient topology on the coimage coincides with the subspace topology on the image). The functor

$$\mathrm{Mod}^{\mathrm{fp}}(A[\![K]\!]) \to \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(A),\ M \mapsto \widetilde{M} := \lim_i M/\mathfrak{a}^i M \tag{2.2.8}$$

is exact.
(6) Let $A[\![K]\!]\text{-}\mathrm{Mod}^{\mathrm{fp}}(A)$ denote the abelian category of $A[\![K]\!]$-module objects in $\mathrm{Mod}^{\mathrm{fp}}(A)$, so that the functor (2.2.8) *factors through the forgetful functor $\mathrm{Pro}\,A[\![K]\!]\text{-}\mathrm{Mod}^{\mathrm{fp}}(A) \to \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(A)$. Then if*

$$X_1 \to X_2 \to X_3 \to X_4 \to X_5$$

is an exact sequence in $\mathrm{Pro}\,A[\![K]\!]\text{-}\mathrm{Mod}^{\mathrm{fp}}(A)$ and X_1, X_2, X_4, X_5 are in the essential image of (2.2.8)*, the same is true of X_3.*

Proof. Proof of (1): By Lemma 2.2.6, the sequences of ideals J_n and $\mathfrak{a}^n(\mathcal{O}/\varpi^a)[\![K]\!]$ are cofinal in $(\mathcal{O}/\varpi^a)[\![K]\!]$. So the pro-objects

$$\lim_n (\mathcal{O}/\varpi^a)[K/K_n] \quad \text{and} \quad \lim_n (\mathcal{O}/\varpi^a)[\![K]\!]/\mathfrak{a}^n(\mathcal{O}/\varpi^a)[\![K]\!]$$

are canonically isomorphic. Tensoring with A, we see that the same is true of

$$\lim_n A[K/K_n] \quad \text{and} \quad \lim_n A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]/\mathfrak{a}^n\mathcal{O}[\![K]\!].$$

Passing to the inverse limit, we see that $A[\![K]\!]$ is isomorphic to the $\mathfrak{a}$-adic completion of $A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]$, as desired. This concludes the proof of (1).

Proof of (2): Observe first that $\mathrm{Rees}_{\mathfrak{a}}(A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!])$ is Noetherian, since it is a quotient of $A \otimes_{\mathcal{O}} \mathrm{Rees}_{\mathfrak{a}}(\mathcal{O}[\![K]\!])$, which is Noetherian (because $\mathrm{Rees}_{\mathfrak{a}}(\mathcal{O}[\![K]\!])$ is Noetherian, by Lemma 2.2.6, and A has finite type over $\mathcal{O}$). Thus $A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]$ and $\mathrm{gr}_{\mathfrak{a}}(A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!])$ are also Noetherian, because they are quotients of $\mathrm{Rees}_{\mathfrak{a}}(A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!])$. Finally, [LO96, Prop. I.7.1.2] implies that $A[\![K]\!]$ is also Noetherian, since it is complete with respect to a filtration whose associated graded is $\mathrm{gr}_{\mathfrak{a}}(A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!])$.

Proof of (3): By [LO96, Cor. I.5.5, Rem. II.1.1.2(1)] and the already observed fact that $\mathrm{Rees}_{\mathfrak{a}}(A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!])$ is Noetherian, we deduce that the $\mathfrak{a}$-adic filtration on $A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]$ has the Artin–Rees property. The claim is then a consequence of [LO96, Thm. II.1.2.4], given that $\mathrm{gr}_{\mathfrak{a}}(A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!])$ is (as we have noted) also Noetherian.

Proof of (4): By part (1), it suffices to prove that

$$\mathfrak{a}^n A[\![K]\!] = \ker(A[\![K]\!] \to A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]/\mathfrak{a}^n\mathcal{O}[\![K]\!]).$$

Since $\mathfrak{a}^n$ is a finitely generated ideal of $\mathcal{O}[\![K]\!]$, a choice of (right) generators gives an exact sequence

$$(A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!])^{\oplus i} \to A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!] \to A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]/\mathfrak{a}^n\mathcal{O}[\![K]\!] \to 0$$

for some $i \geq 0$. This sequence stays exact after passing to the $\mathfrak{a}$-adic completion, because the $\mathfrak{a}$-adic filtration on $A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]$ has the Artin–Rees property (compare [LO96, Lem. II.1.2.5]). This concludes the proof of (4).

Proof of (5): By [LO96, Prop. II.2.2.1], the $\mathfrak{a}$-adic filtration on $A[\![K]\!]$ is Zariskian. By [LO96, Thm. 2.1.2(5)], every inclusion $N \to M$ of finitely generated $A[\![K]\!]$-modules is a topological embedding with respect to the $\mathfrak{a}$-adic topology on source and target. This immediately implies both statements of (5).

Proof of (6): By part (5), it suffices to prove this when $X_1 = X_5 = 0$, and so, writing $X \coloneqq X_3$, we have finitely presented $A[\![K]\!]$-modules $M \coloneqq X_2, M' \coloneqq X_4$, and an exact sequence in $\operatorname{Pro} A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$

$$0 \to \widetilde{M} \to X \to \widetilde{M}' \to 0. \tag{2.2.9}$$

Now both $\widetilde{M}$ and $\widetilde{M}'$ are countably indexed; indeed, $M = \lim_n M/\mathfrak{a}^n$, and similarly for M'. Lemma A.1.12 shows that (2.2.9) can be obtained as a countably indexed limit of short exact sequences in $A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$

$$0 \to M/\mathfrak{a}^n \to X_n \to M'_n \to 0.$$

(Here the projective system (M'_n) need not coincide with $(M'/\mathfrak{a}^n)$, but is isomorphic to it in $\operatorname{Pro} A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$.) We now pass to the actual limit over n of these short exact sequences, to obtain a short exact sequence

$$0 \to M \to Y \to M' \to 0$$

of $A[\![K]\!]$-modules. (The exactness on the right follows from the fact that the transition morphisms in $(M/\mathfrak{a}^n)$ are surjective.) We may equip Y with its inverse limit topology (each X_n being endowed with its discrete topology), and then (from its construction as a limit) we see that we have a strict short exact sequence of topological $A[\![K]\!]$-modules, in which each of M and M' are endowed with their $\mathfrak{a}$-adic topologies. Consequently, Y is an object of $\operatorname{Mod}^{\mathrm{fp}}(A[\![K]\!])$ (being an extension of two such objects). Furthermore, since Y is constructed as a countable limit of discrete spaces, the topology on Y is completely metrizable, and so [EG23, Prop. C.6] shows that it must coincide with the $\mathfrak{a}$-adic topology on Y (cf. the discussion of Remark 2.2.10 below). Thus we obtain a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M} \arrow[r] \arrow[d, equal] & \widetilde{Y} \arrow[r] \arrow[d] & \widetilde{M}' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{M} \arrow[r] & X \arrow[r] & \widetilde{M}' \arrow[r] & 0
\end{tikzcd}$$

in $\operatorname{Pro} A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$. The five lemma implies that $\widetilde{Y} \xrightarrow{\sim} X$, concluding the proof. $\square$

Remark 2.2.10. As in the statement of Lemma 2.2.7, assume that A is of finite type over $\mathcal{O}/\varpi^a$ for some $a \geq 1$. Then since the ring A is then countable, the completed group ring $A[\![K]\!]$, endowed with its $\mathfrak{a}$-adic topology, is Polish. The general theory of finitely generated modules over Noetherian Polish topological rings (recapitulated, for example, in [EG23, Prop. C.6]) shows that any finitely generated $A[\![K]\!]$-module is endowed with a canonical completely metrizable topology, and that morphisms between such modules are automatically continuous and strict with respect to this topology.

Since the topology on $A[\![K]\!]$ is its $\mathfrak{a}$-adic topology, this canonical topology on any finitely generated $A[\![K]\!]$-module is again the $\mathfrak{a}$-adic topology. One may then interpret Lemma 2.2.7 (5) as providing another proof of the strictness of morphisms with respect to canonical topologies.

Remark 2.2.11. Suppose that A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, that M is an $A[\![K]\!]$-module, and that M is finitely presented over A. Then M is also finitely presented over $A[\![K]\!]$, and its canonical topology is discrete. Indeed, we can write the complete metric space M as a countable union of closed subsets $M = \bigcup_{x \in M}\{x\}$, and if M were not discrete, each subset $\{x\}$ would have empty interior, contradicting the Baire category theorem.

Remark 2.2.12. Note that the forgetful functor induces an equivalence between $A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$ and the category $\mathcal{O}[\![K]\!]_{\mathfrak{a}\text{-tors}}\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$ of $\mathcal{O}[\![K]\!]$-module objects M in $\operatorname{Mod}^{\mathrm{fp}}(A)$ which are $\mathfrak{a}$-power torsion (in the sense that the $\mathcal{O}[\![K]\!]$-action factors through $\mathcal{O}[\![K]\!]/\mathfrak{a}^i$ for some i, depending on the module). Indeed, the $\mathfrak{a}$-adic topology on any $M \in A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$ is necessarily discrete, by Remark 2.2.11, hence M is $\mathfrak{a}$-power torsion. The inverse functor endows each $N \in \mathcal{O}[\![K]\!]_{\mathfrak{a}\text{-tors}}\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$ with its $A \otimes_{\mathcal{O}} \mathcal{O}[\![K]\!]$-module structure, which uniquely extends to an $A[\![K]\!]$-module structure, since N is $\mathfrak{a}$-power torsion. Hence, in the statement of Lemma 2.2.7 (6), we may replace $A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$ with the category of $\mathfrak{a}$-power torsion $\mathcal{O}[\![K]\!]$-modules in $\operatorname{Mod}^{\mathrm{fp}}(A)$.

Remark 2.2.13. Write $\mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A)$ for the category of $\mathcal{O}[\![K]\!]$-modules in $\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A)$. The forgetful functor $A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A) \to \operatorname{Mod}^{\mathrm{fp}}(A)$ is exact and faithful, so its Pro-extension factors through an exact and faithful functor

$$\operatorname{Pro} A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A) \to \mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A).$$

We claim that this functor is also full. Note that $\mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A)$ has cofiltered limits, and that the forgetful functor $\mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A) \to \operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A)$ preserves them; so it suffices to prove that if $M = \lim_i M_i \in \operatorname{Pro} A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$, and $N \in A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$, then every $\mathcal{O}[\![K]\!]$-linear morphism $\varphi : M \to N$ in $\mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A)$ factors through a morphism $M_i \to N$ in $A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$ for some i.

By definition, φ factors through an A-linear morphism $\varphi_j : M_j \to N$ for some j, which may not be $\mathcal{O}[\![K]\!]$-linear (i.e. it may not be a morphism in $A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$). However, the $\mathcal{O}[\![K]\!]$-linearity of φ shows that for all $x \in K$ there exists $i_x \geq j$ such that $x\varphi_j - \varphi_j x$ vanishes on M_{i_x}. Since M_j and N are $\mathfrak{a}$-power torsion, there exists n such that K_n acts trivially on both, and so φ_j is automatically K_n-linear. Choose now representatives $x_1, \ldots, x_t$ of K/K_n in K, as well as some $i \geq \sup\{i_{x_1}, \ldots, i_{x_t}\}$. Then $M_i \to M_j \xrightarrow{\varphi_j} N$ is $\mathcal{O}[\![K]\!]$-linear, as desired.

2.2.14. *Smooth representations of compact p-adic analytic groups.* If Γ° is a compact p-adic analytic group, then Pontrjagin duality induces an equivalence of categories

$$(-)^\vee : (\mathrm{sm.}\,\Gamma^\circ)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Mod}_c(\mathcal{O}[\![\Gamma^\circ]\!]^{\mathrm{op}}) \cong \operatorname{Mod}_c(\mathcal{O}[\![\Gamma^\circ]\!]). \tag{2.2.15}$$

More precisely, when we apply Pontrjagin duality to a smooth Γ°-representation, we typically convert the resulting right $\mathcal{O}[\![\Gamma^\circ]\!]$-module structure to a left $\mathcal{O}[\![\Gamma^\circ]\!]$-module structure by applying the continuous anti-involution of $\mathcal{O}[\![\Gamma^\circ]\!]$ induced by $g \mapsto g^{-1}$ (and this anti-involution is what provides the isomorphism between the second and third of the categories in (2.2.15)). However, in some contexts (particularly when forming tensor products, as we will do extensively below), it will be convenient to work directly with right $\mathcal{O}[\![\Gamma^\circ]\!]$-modules.

The equivalence (2.2.15) restricts to an equivalence

$$\big((\mathrm{sm.}\,\Gamma^\circ)^{\mathrm{adm}}\big)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{fg}}(\mathcal{O}[\![\Gamma^\circ]\!]), \tag{2.2.16}$$

where we regard the target as a full subcategory of $\mathrm{Mod}_c(\mathcal{O}[\![\Gamma^\circ]\!])$ via (A.1.36).

2.2.17. *Smooth Γ-representations as $\mathcal{O}[\![\Gamma]\!]$-modules in the non-compact case.* If Γ is a p-adic analytic group (compact or not), if $\Gamma^\circ \subset \Gamma$ is a compact open subgroup, and if π is an object of $\mathrm{sm.}\,\Gamma$, then the Γ°-action on π induces a unique continuous $\mathcal{O}[\![\Gamma^\circ]\!]$-action on π (where we endow π with its discrete topology). This action can be extended to a suitably completed group ring $\mathcal{O}[\![\Gamma]\!]$ of Γ. We recall its definition, in the more general context where we allow coefficients.

Definition 2.2.18. If A is an $\mathcal{O}$-algebra then we let

$$A[\![\Gamma]\!] \coloneqq \mathcal{O}[\Gamma] \otimes_{\mathcal{O}[\Gamma^\circ]} A[\![\Gamma^\circ]\!]$$

with its natural ring structure, as explained in [Sho20, §3].

Lemma 2.2.19. *If M is a Hausdorff topological A-module equipped with an action of Γ by continuous automorphisms, and with a continuous $A[\![\Gamma^\circ]\!]$-module structure, such that the two resulting $A[\Gamma^\circ]$-module structures on M coincide, then the induced $A[\Gamma]$-module structure on M extends uniquely to an $A[\![\Gamma]\!]$-module structure.*

Proof. This is easily verified; see e.g. [Tim23, Prop. 2.23], whose proof carries over verbatim. □

As a special case of Lemma 2.2.19, we deduce the claim made above, namely that if $\pi \in \mathrm{sm.}\,\Gamma$, then the action of $\mathcal{O}[\![\Gamma^\circ]\!]$ on π induces an $\mathcal{O}[\![\Gamma]\!]$-action on π. Thus we obtain a fully faithful embedding $\mathrm{sm.}\,\Gamma \hookrightarrow \mathrm{Mod}(\mathcal{O}[\![\Gamma]\!])$. The following lemma shows that this realizes $\mathrm{sm.}\,\Gamma$ as a localizing subcategory of $\mathrm{Mod}(\mathcal{O}[\![\Gamma]\!])$ (see Appendix A.1 for a brief recollection of this notion).

Lemma 2.2.20. *The fully faithful embedding* $\mathrm{sm.}\,\Gamma \hookrightarrow \mathrm{Mod}(\mathcal{O}[\![\Gamma]\!])$ *realizes* $\mathrm{sm.}\,\Gamma$ *as a localizing subcategory of* $\mathrm{Mod}(\mathcal{O}[\![\Gamma]\!])$.

Remark 2.2.21. The main result of [Hey21] shows that $\mathrm{sm.}\,\Gamma$ is even a localizing subcategory of $\mathrm{Mod}(\mathcal{O}[\Gamma])$, i.e. that extensions of smooth representations are automatically smooth. However, we will not need this more general fact, but will satisfy ourselves with recalling the (standard) proof of the easier result that we will be using.

Proof of Lemma 2.2.20. It is immediate that the essential image of $\mathrm{sm.}\,\Gamma$ is closed under subobjects, quotients, and colimits, and in particular under arbitrary direct sums. So it suffices to prove that if V is an extension in $\mathrm{Mod}(\mathcal{O}[\![\Gamma]\!])$ of two objects $\pi_1, \pi_2 \in \mathrm{sm.}\,\Gamma$, then V is smooth. Choose $v \in V$, let $\Gamma^\circ \subset \Gamma$ be a pro-p open subgroup that fixes the image of v in π_2, and let $n \geq 0$ be such that $\varpi^n v = 0$. Replacing π_2 by its cyclic $\mathcal{O}[\![\Gamma^\circ]\!]$-submodule generated by the image of v, which is isomorphic to $\mathcal{O}/\varpi^n$ with the trivial Γ°-action, it suffices to prove that

$$\mathrm{Ext}^1_{\mathrm{sm.}\,\Gamma^\circ}(\mathcal{O}/\varpi^n, \pi_1) \to \mathrm{Ext}^1_{\mathrm{Mod}\,\mathcal{O}[\![\Gamma^\circ]\!]}(\mathcal{O}/\varpi^n, \pi_1)$$

is an isomorphism (equivalently, a surjection); indeed, it follows that v is contained in a smooth Γ°-submodule of V.

We claim that it suffices to consider the case that π_1 is finitely generated in $\mathrm{sm}.\,\Gamma^\circ$ (and so is a finitely generated torsion $\mathcal{O}$-module); indeed, π_1 is the filtered colimit of its finitely generated subobjects in $\mathrm{sm}.\,\Gamma^\circ$, and $\mathrm{Ext}^1_{\mathrm{sm}.\,\Gamma^\circ}(\mathcal{O}/\varpi^n, -)$, $\mathrm{Ext}^1_{\mathrm{Mod}\,\mathcal{O}[\![\Gamma^\circ]\!]}(\mathcal{O}/\varpi^n, -)$ commute with filtered colimits, because the categories $\mathrm{sm}.\,\Gamma^\circ$ and $\mathrm{Mod}\,\mathcal{O}[\![\Gamma^\circ]\!]$ are locally Noetherian (see e.g. [DEG23, Proposition A.1.1]). If π_1 is finitely generated, then every extension of $\mathcal{O}/\varpi^n$ by π_1 in $\mathrm{Mod}(\mathcal{O}[\![\Gamma^\circ]\!])$ is a finitely generated torsion $\mathcal{O}$-module, and so its canonical topology as an $\mathcal{O}[\![\Gamma^\circ]\!]$-module is discrete, which implies that the action of Γ° is smooth, as required. □

2.2.22. *Smooth G-representations as $\mathcal{O}[\![G]\!]$-modules.* When $\Gamma = G = \mathrm{GL}_2(\mathbf{Q}_p)$ and A is an $\mathcal{O}$-algebra, we write $A[\![G]\!]_\zeta$ for the quotient of $A[\![G]\!]$ by the ideal generated by $z - \zeta(z)$ for $z \in Z(G) \cong \mathbf{Q}_p^\times$. The category $\mathcal{A}$ may thus be regarded as a full subcategory of $\mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$, and the inclusion $\mathcal{A} \subseteq \mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$ is exact and colimit-preserving (see e.g. [EGH25, §E.2]). We recall the following result about the structure of $\mathcal{O}[\![G]\!]$.

Lemma 2.2.23. *The ring $\mathcal{O}[\![G]\!]_\zeta$ is coherent.*

Proof. This is a consequence of [Tim23, Theorem 9.7] (see also [Sho20, Theorem 1.2]), which states that $\mathcal{O}[\![G]\!]$ is coherent, together with the fact that the quotient of a coherent ring by an ideal that is generated by finitely many central elements is coherent [Swa19, Corollary 3.6]. □

Remark 2.2.24. In the context of $\Gamma = G = \mathrm{GL}_2(\mathbf{Q}_p)$, an evident variant of Lemma 2.2.20 shows that the essential image of the fully faithful embedding $\mathcal{A} \hookrightarrow \mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$ is again a localizing subcategory of its target.

Remark 2.2.25. An object π of $\mathcal{A}$ is *finitely presented* if it is the cokernel of a map $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_1) \to c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_2)$ where σ_1, σ_2 are smooth KZ-representations on $\mathcal{O}$-finite modules. By [Sho20, Proposition 3.4], π is finitely presented if and only if the image of π under the embedding in Remark 2.2.24 is finitely presented as an $\mathcal{O}[\![G]\!]_\zeta$-module. Since $G = \mathrm{GL}_2(\mathbf{Q}_p)$, π is finitely presented if and only if it is finitely generated (see e.g. [DEG23, Theorem 2.2.1] for a proof of this fact). For this reason, and for consistency with our notation for modules, we will often write $\mathcal{A}^{\mathrm{fp}}$ for the category we have denoted $\mathcal{A}^{\mathrm{fg}}$. We can thus summarize our discussion by saying that the inclusion of $\mathcal{A}$ in $\mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$ induces an inclusion

$$\mathcal{A}^{\mathrm{fp}} \subseteq \mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta). \tag{2.2.26}$$

2.2.27. *Smooth G-representations and duality.* Pontrjagin duality induces an equivalence of $(\mathrm{sm}.\,G)^{\mathrm{op}}$ with the category $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$ introduced in [Eme10, Section 2.1]. The objects of $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$ are (left or right, compare Section 2.2.14) $\mathcal{O}[\![G]\!]$-modules endowed with a profinite topology that admits a neighborhood basis of the identity consisting of open topological $\mathcal{O}[\![K]\!]$-modules, while the morphisms in this category are the continuous $\mathcal{O}[\![G]\!]$-linear maps. The forgetful functor $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O}) \to \mathrm{Mod}(\mathcal{O}[\![G]\!])$ is exact. There are also forgetful functors $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O}) \to \mathrm{Mod}_c(\mathcal{O}[\![K]\!]) \to \mathrm{Mod}_c(\mathcal{O})$, which factor through the category of $\mathcal{O}[\![G]\!]$-modules in $\mathrm{Mod}_c(\mathcal{O}[\![K]\!])$, resp. $\mathrm{Mod}_c(\mathcal{O})$. When working with the category $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$, or one of its full subcategories, we will sometimes write $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![G]\!]}$ rather than $\mathrm{Hom}_{\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})}$.

We denote the image of $\mathcal{A}^{\mathrm{op}}$ in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$ by $\mathrm{Mod}_{G,\zeta}^{\mathrm{pro\,aug}}(\mathcal{O})$, and the image of $\mathcal{A}^{\mathrm{l.adm,op}}$ by $\mathfrak{C}$. This category is denoted $\mathfrak{C}(\mathcal{O})$ in [Paš13], but as we will not work with more general coefficients, we shorten the notation accordingly. It is a Serre subcategory of $\mathrm{Mod}^{\mathrm{pro\,aug}}_{G,\zeta}(\mathcal{O})$.

Remark 2.2.28. If we regard the objects of $\mathfrak{C}$ as left $\mathcal{O}[\![G]\!]$-modules, following the discussion in Section 2.2.14, then they have central character ζ^{-1}. If we regard them as right $\mathcal{O}[\![G]\!]$-modules (again as in that discussion), then they have central character ζ, i.e. they are right $\mathcal{O}[\![G]\!]_\zeta$-modules.

Since locally admissible representations of G are locally finite, every object $P \in \mathfrak{C}$ is the cofiltered limit of its finite length quotients in $\mathfrak{C}$. We refer to the image of $\mathcal{A}^{\mathrm{adm,op}}$ in $\mathfrak{C}$ as the (full) subcategory of *coadmissible* modules, and denote it by $\mathfrak{C}^{\mathrm{coadm}}$. Equivalently, this is the full subcategory of $\mathfrak{C}$ consisting of objects that are finitely generated over $\mathcal{O}[\![K]\!]$, or (again equivalently) over $\mathcal{O}[\![K_n]\!]$ for one, or any, $n \geq 1$. Note that the topology on an object of $\mathfrak{C}^{\mathrm{coadm}}$ coincides with the canonical compact topology it inherits as a finitely generated module over the compact Noetherian ring $\mathcal{O}[\![K]\!]$, by the uniqueness part of Lemma A.1.32 (5). By (2.2.3), we see that formation of projective limits (equipped with their projective limit topologies) induces an equivalence

$$\operatorname{Pro}\mathfrak{C}^{\mathrm{f.l.}} \xrightarrow{\sim} \mathfrak{C}, \tag{2.2.29}$$

where, as usual, $\mathfrak{C}^{\mathrm{f.l.}}$ denotes the full subcategory of finite length objects of $\mathfrak{C}$. The following "automatic continuity" result is often useful.

Lemma 2.2.30. *If M and N are objects of $\mathfrak{C}$, with M furthermore lying in $\mathfrak{C}^{\mathrm{coadm}}$, then the evident inclusion*

$$\mathrm{Hom}_{\mathfrak{C}}(M,N)\ \big(= \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![G]\!]}(M,N)\big) \subseteq \mathrm{Hom}_{\mathcal{O}[\![G]\!]}(M,N)$$

is an equality.

Proof. It suffices to prove that $\mathcal{O}[\![K]\!]$-linear morphisms between M and N are automatically continuous. Since M and N are objects of $\mathrm{Mod}_c(\mathcal{O}[\![K]\!])$, and M is finitely generated over $\mathcal{O}[\![K]\!]$, this is a consequence of [VV97, Proposition 3.5]. □

We also have the following related result.

Lemma 2.2.31. *The forgetful functor $\mathfrak{C}^{\mathrm{coadm}} \to \mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$ is fully faithful, and its essential image is a Serre subcategory of its target.*

Proof. The claimed full faithfulness is an immediate consequence of Lemma 2.2.30. If M is an object of $\mathfrak{C}^{\mathrm{coadm}}$, then (by definition) M is finitely generated over the Noetherian profinite $\mathcal{O}$-algebra $\mathcal{O}[\![K]\!]$, and (as noted above) is equipped with the canonical compact topology it inherits by virtue of this. Thus, any $\mathcal{O}[\![G]\!]_\zeta$-submodule of M is again finitely generated over $\mathcal{O}[\![K]\!]$, and so is an object of $\mathfrak{C}^{\mathrm{coadm}}$. Similar arguments in the case of quotients and extensions show that $\mathfrak{C}^{\mathrm{coadm}}$ is indeed a Serre subcategory of $\mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$. □

2.2.32. *Induced representations.* We write $\mathrm{Ind}_B^G : \mathrm{sm}.\,T \to \mathrm{sm}.\,G$ for the functor of parabolic induction with respect to B, and $c\text{-}\mathrm{Ind}_{KZ}^G : \mathrm{sm}.\,KZ \to \mathrm{sm}.\,G$ for the functor of compact induction. We adopt similar notation for other groups, and for the induction functors on the categories of smooth representations with fixed central character (note that Ind_B^G and $c\text{-}\mathrm{Ind}_{KZ}^G$ preserve central characters when these are defined). Then Ind_B^G is an exact functor, and $c\text{-}\mathrm{Ind}_{KZ}^G$ is an exact left adjoint to Res_{KZ}^G. The parabolic induction $\mathrm{Ind}_B^G(\mathbf{1}_T)$ of the trivial character of T is a reducible representation, and we write St for its unique irreducible quotient, the Steinberg representation of G. There is an exact sequence

$$0 \to \mathbf{1}_G \to \mathrm{Ind}_B^G(\mathbf{1}_T) \to \mathrm{St} \to 0.$$

If σ is a ζ-compatible Serre weight, and $\chi = \sigma^{\mathrm{Iw}_1}$ is the highest weight of σ viewed as a character of $T_0 Z$, then we write

$$\mathcal{H}_G(\sigma) \coloneqq \mathrm{End}_G(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$$
$$\mathcal{H}_T(\chi) \coloneqq \mathrm{End}_T(c\text{-}\mathrm{Ind}_{T_0 Z}^T \chi).$$

There are canonical isomorphisms

$$\mathcal{H}_T(\chi) \xrightarrow{\sim} \mathcal{H}_T(\mathbf{1}_{T_0 Z}) \xrightarrow{\sim} \mathbf{F}[T_p^{\pm 1}] \tag{2.2.33}$$

(so in particular $\mathcal{H}_T(\chi)$ is canonically independent of χ). We normalize our choice of generator T_p in such a way that the Satake map $\mathcal{H}_G(\sigma) \to \mathcal{H}_T(\chi)$ of [Her11] is an injection which sends the usual spherical Hecke operator to T_p, and determines an isomorphism

$$\mathcal{H}_G(\sigma) \xrightarrow{\sim} \mathbf{F}[T_p]. \tag{2.2.34}$$

If σ is not a twist of Sym^0, then [BL94, Theorem 25] gives an isomorphism

$$\left(c\text{-}\mathrm{Ind}_{KZ}^G \sigma\right)[1/T_p] \xrightarrow{\sim} \mathrm{Ind}_{\overline{B}}^G \left(c\text{-}\mathrm{Ind}_{T_0 Z}^T \chi\right), \tag{2.2.35}$$

where $\overline{B}$ is the lower-triangular Borel subgroup. This isomorphism is compatible with the Satake isomorphism, in the sense that it is an isomorphism of $\mathbf{F}[T_p^{\pm 1}]$-modules (acting on the right-hand side by $\mathrm{Ind}_{\overline{B}}^G$-functoriality), and so can be regarded as characterizing the Satake morphism (for σ not a twist of Sym^0).

If $\sigma \coloneqq \sigma_{a,0}$ is a twist of Sym^0, so that $\sigma_{a,p-1}$ is the corresponding twist of Sym^{p-1}, then there is a short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,0} \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,p-1} \to \left(\omega^a \mathrm{nr}_{T_p^2 - \zeta(p)} \circ \det\right) \otimes \mathrm{St} \to 0, \tag{2.2.36}$$

with the first embedding being $\mathbf{F}[T_p]$-linear. Similarly, there is a short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,p-1} \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,0} \to \left(\mathrm{nr}_{T_p^2 - \zeta(p)} \circ \det\right) \to\to 0, \tag{2.2.37}$$

where again the first embedding is $\mathbf{F}[T_p]$-linear. The composite of the injections $c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,0} \hookrightarrow c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,p-1}$ and $c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,p-1} \hookrightarrow c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,0}$, in either order, is a non-zero scalar multiple of multiplication by $T_p^2 - \zeta(p)$.

2.2.38. *Irreducible objects of $\mathcal{A}$.* Recall the following classification of the irreducible objects of $\mathcal{A}$.

Theorem 2.2.39.

(1) *Every irreducible object of $\mathcal{A}$ is isomorphic to one of the following representations:*
   - (a) *c-$\mathrm{Ind}_{KZ}^G \sigma_{a,b}/f(T_p)$ for some ζ-compatible $\sigma_{a,b}$ and monic irreducible $f \in \mathcal{H}_G(\sigma_{a,b})$, such that f is coprime to $T_p^2 - \zeta(p)$ if $b = 0, p-1$;*
   - (b) *an irreducible subquotient of c-$\mathrm{Ind}_{KZ}^G \sigma_{a,0}/f(T_p)$, or equivalently of c-$\mathrm{Ind}_{KZ}^G \sigma_{a,p-1}/f(T_p)$, for some monic irreducible factor of $T_p^2 - \zeta(p)$.*

(2) *The representations of part* (1a) *are pairwise non-isomorphic, with the exception of the isomorphisms*
$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,0}/f(T_p) \cong c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,p-1}/f(T_p)$$
when $f(T_p)$ is coprime to $T_p^2 - \zeta(p)$, and
$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,b}/T_p \cong c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a+b,p-1-b}/T_p.$$

(3) *A representation $c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,b}/f(T_p)$ as in* (1a) *is absolutely irreducible if and only if f has degree one.*

Proof. See [DEG23, Theorem 2.12, Corollary 2.14], which builds upon the results of [BL94; Bre03] (see also [Paš13, Section 5.3]). □

Remark 2.2.40. By (2.2.35), the representations $c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,b}/f(T_p)$ for $f(T_p) \neq T_p$ occurring in Theorem 2.2.39 can be written as parabolic inductions from the lower-triangular Borel subgroup $\overline{B}$, or equivalently (after a twist) from B. For example,
$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{a,b}/(T_p - \lambda) \cong \mathrm{Ind}_B^G(\mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^a \otimes \mathrm{nr}_\lambda \omega^{a+b})$$
for all $\lambda \in \mathbf{F}^\times$.

2.2.41. *Tensoring between $\mathfrak{C}$ and $\mathcal{A}$.* If M and N are respectively right and left $\mathcal{O}[\![G]\!]_\zeta$-modules, then we can form their tensor product $M \otimes_{\mathcal{O}[\![G]\!]_\zeta} N \in \mathrm{Mod}(\mathcal{O})$. There are the usual trifunctorial bijections
$$\mathrm{Hom}_{\mathcal{O}}(M \otimes_{\mathcal{O}[\![G]\!]_\zeta} N, W) \to \mathrm{Hom}_{\mathcal{O}[\![G]\!]_\zeta}\big(N, \mathrm{Hom}_{\mathcal{O}}(M, W)\big) \tag{2.2.42}$$
and
$$\mathrm{Hom}_{\mathcal{O}}(M \otimes_{\mathcal{O}[\![G]\!]_\zeta} N, W) \to \mathrm{Hom}_{\mathcal{O}[\![G]\!]_\zeta}\big(M, \mathrm{Hom}_{\mathcal{O}}(N, W)\big) \tag{2.2.43}$$
given by evaluation maps. On the right hand side of (2.2.43), contrary to our usual conventions, we are forming Hom of right $\mathcal{O}[\![G]\!]_\zeta$-modules. Of course, we can always use the anti-involution $g \mapsto g^{-1}$ on $\mathcal{O}[\![G]\!]$ to convert M and $\mathrm{Hom}_{\mathcal{O}}(N, W)$ to left $\mathcal{O}[\![G]\!]$-modules, and then write the target of (2.2.43) as $\mathrm{Hom}_{\mathcal{O}[\![G]\!]}\big(M, \mathrm{Hom}_{\mathcal{O}}(N, W)\big)$, now formed in the category of left $\mathcal{O}[\![G]\!]$-modules.

A key example of the tensor products over $\mathcal{O}[\![G]\!]_\zeta$ that we will consider are those of the form $M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$, where M is an object of $\mathfrak{C}$ (thought of as a right $\mathcal{O}[\![G]\!]_\zeta$-module; see Remark 2.2.28) and π is an object of $\mathcal{A}$. Since the formation of tensor products is compatible with filtered colimits, we will primarily focus on the case when π is an object of $\mathcal{A}^{\mathrm{fp}}$, and then it will be convenient to also consider the more general case when π is an object of $\mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)$. We are going to describe some general properties of this construction; see [JNW24, Section 6.1] for related results.

The functor $M \otimes_{\mathcal{O}[\![G]\!]_\zeta} - : \mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Mod}(\mathcal{O})$ is of course a special case of the general construction in Section A.1.46, viewing M as a right $\mathcal{O}[\![G]\!]_\zeta$-module in the abelian category $\mathrm{Mod}(\mathcal{O})$. Since $M \in \mathfrak{C}$, we can also regard it as a right $\mathcal{O}[\![G]\!]_\zeta$-module in $\mathrm{Mod}_c(\mathcal{O})$, and then Lemma A.1.55 constructs a lift of $M \otimes_{\mathcal{O}[\![G]\!]_\zeta} -$ through $\mathrm{Mod}_c(\mathcal{O}) \to \mathrm{Mod}(\mathcal{O})$. As explained in Remark A.1.49, for all $\pi \in \mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)$ and $W \in \mathrm{Mod}_c(\mathcal{O})$ there is an isomorphism
$$\mathrm{Hom}_{\mathrm{Mod}_c(\mathcal{O})}(M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}[\![G]\!]_\zeta}\big(\pi, \mathrm{Hom}_{\mathrm{Mod}_c(\mathcal{O})}(M, W)\big). \tag{2.2.44}$$
Furthermore, by Lemma A.1.51, the formation of $M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$, for M an object of $\mathfrak{C}$ and π an object of $\mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)$, is compatible with the formation of cofiltered

limits in $\mathfrak{C}$. We next describe the interaction of tensor products with the Pontrjagin duality functor $(-)^\vee$.

Lemma 2.2.45. *Let M be an object of $\mathfrak{C}$ and π be an object of $\mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)$.*

(1) There is a natural isomorphism of discrete $\mathcal{O}$-modules

$$(M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi)^\vee \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}[\![G]\!]_\zeta}(\pi, M^\vee),$$

or, equivalently, a natural isomorphism of compact $\mathcal{O}$-modules

$$M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}[\![G]\!]_\zeta}(\pi, M^\vee)^\vee.$$

(2) If M is coadmissible, then $M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$ is a discrete $\mathcal{O}$-module of finite cardinality.

(3) If π is of finite length, then there is a natural isomorphism of compact $\mathcal{O}$-modules

$$M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}}(M, \pi^\vee)^\vee$$

(where $\mathrm{Hom}_{\mathfrak{C}}(M, \pi^\vee)$ is endowed with its discrete topology, so its Pontrjagin dual is a compact $\mathcal{O}$-module).

Similarly, if $\pi \in \mathrm{Mod}^{\mathrm{f.l.}}(\mathcal{O}[\![KZ]\!]_\zeta)$, and $M \in \mathrm{Mod}_c(\mathcal{O}[\![KZ]\!]_\zeta)$, then there is a natural isomorphism of compact $\mathcal{O}$-modules

$$M \otimes_{\mathcal{O}[\![KZ]\!]_\zeta} \pi \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}_c(\mathcal{O}[\![KZ]\!]_\zeta)}(M, \pi^\vee)^\vee.$$

Proof. Part (1) follows from applying (2.2.44) with $W = \varpi^{-n}\mathcal{O}/\mathcal{O}$ and taking the colimit over n (using the assumption that π is a compact object of $\mathrm{Mod}\,\mathcal{O}[\![G]\!]_\zeta$).

For part (2), note that if M is coadmissible, then its Pontrjagin dual $M^\vee$ is an object of $\mathcal{A}^{\mathrm{adm}}$. Since furthermore π is finitely generated, we see that $\mathrm{Hom}_{\mathcal{O}[\![G]\!]_\zeta}(\pi, M^\vee)$ is a (literally) finite $\mathcal{O}$-module. Thus $M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$ has a finite Pontrjagin dual, by (1), and so is itself finite. This proves (2).

Finally we turn to (3), and so assume that π is finitely presented and of finite length. Write $M \xrightarrow{\sim} \varprojlim_i M_i$ as a cofiltered limit of objects $M_i \in \mathfrak{C}$ of finite length. Then the M_i have finite $\mathcal{O}[\![G]\!]_\zeta$-length, and so

$$\begin{aligned}
\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi, E/\mathcal{O}) &\xrightarrow{\sim} \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\varprojlim_i M_i \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi, E/\mathcal{O}) \\
&\xrightarrow{\sim} \varinjlim_i \mathrm{Hom}_{\mathcal{O}}(M_i \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi, E/\mathcal{O}) \xrightarrow{\sim} \varinjlim_i \mathrm{Hom}_{\mathcal{O}[\![G]\!]}(M_i, \pi^\vee) \\
&\xrightarrow{\sim} \varinjlim_i \mathrm{Hom}_{\mathfrak{C}}(M_i, \pi^\vee) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}}(M, \pi^\vee).
\end{aligned}$$

Here the first isomorphism follows from the compatibility with projective limits provided by Lemma A.1.51; the second isomorphism follows from the fact that the Pontrjagin dual of a projective limit of finite length $\mathcal{O}$-modules is the direct limit of their Pontrjagin duals (note that by part (2), the tensor products $M_i \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$ are finite length $\mathcal{O}$-modules, since the M_i are coadmissible, being of finite length); the third isomorphism is provided by (2.2.43); the fourth isomorphism is given by Lemma 2.2.30; and the fifth isomorphism follows from the fact that $\pi^\vee$ is of finite length (so that any homomorphism $M \to \pi^\vee$ in $\mathfrak{C}$ factors through some M_i). This proves (3), and the same proof works for $\mathcal{O}[\![KZ]\!]_\zeta$. $\square$

Lemma 2.2.46. *If P is a projective object of $\mathfrak{C}$, then:*

(1) *P is a projective object of* $\mathrm{Mod}_c(\mathcal{O}[\![KZ]\!]_\zeta^{\mathrm{op}})$ *(and so, in particular, it is topologically $\mathcal{O}[\![KZ]\!]_\zeta$-flat, and $\mathcal{O}[\![KZ]\!]_\zeta$-flat).*
(2) $\mathrm{Tor}_i^{\mathcal{O}[\![G]\!]_\zeta}(P,\pi) = 0$ *for any object π of $\mathcal{A}$ and any $i \geq 1$.*

Proof. Since $P^\vee$ is injective in the category $\mathcal{A}^{\mathrm{l.adm}}$, it follows from [EP10, Corollary 3.10] that $P^\vee$ is also injective in the category $(\mathrm{sm.}\,KZ)_\zeta$ of smooth KZ-representations with central character ζ. Hence P is projective in the dual category to $(\mathrm{sm.}\,KZ)_\zeta$, which is $\mathrm{Mod}_c(\mathcal{O}[\![KZ]\!]_\zeta^{\mathrm{op}})$, compare Remark 2.2.28. Lemma A.1.40 now implies that P is topologically $\mathcal{O}[\![KZ]\!]_\zeta$-flat and $\mathcal{O}[\![KZ]\!]_\zeta$-flat. This establishes the first part of the lemma.

We now prove the second part. Since Tor_i commutes with filtered colimits, we can assume without loss of generality that $\pi \in \mathcal{A}^{\mathrm{fp}}$. Since $\mathcal{A}^{\mathrm{fp}}$ is an abelian subcategory of $\mathcal{A}$, there exists a resolution

$$c\text{-}\mathrm{Ind}_{KZ}^G(V_\bullet) \to \pi,$$

where each V_i is a smooth $\mathcal{O}[\![KZ]\!]_\zeta$-module of finite $\mathcal{O}$-length. By the first part of the lemma, $c\text{-}\mathrm{Ind}_{KZ}^G V_i$ is acyclic for the functor $P \otimes_{\mathcal{O}[\![G]\!]_\zeta} (-)$, since

$$P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} c\text{-}\mathrm{Ind}_{KZ}^G V_i \cong P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} (\mathcal{O}[\![G]\!]_\zeta \otimes_{\mathcal{O}[\![KZ]\!]_\zeta} V_i) \cong P \otimes^L_{\mathcal{O}[\![KZ]\!]_\zeta} V_i.$$

Hence $\mathrm{Tor}_i^{\mathcal{O}[\![G]\!]_\zeta}(P,\pi)$ is the homology of

$$P \otimes_{\mathcal{O}[\![G]\!]_\zeta} c\text{-}\mathrm{Ind}_{KZ}^G V_\bullet,$$

which by Lemma 2.2.45 (1) is isomorphic to the homology of

$$\mathrm{Hom}_{\mathcal{O}[\![G]\!]_\zeta}(c\text{-}\mathrm{Ind}_{KZ}^G V_\bullet, P^\vee)^\vee.$$

This is concentrated in degree zero as a consequence of [Paš13, Corollary 5.18], which implies that $P^\vee$ is an injective object of $\mathcal{A}$. □

2.3. **Morita theory for blocks of $\mathcal{A}^{\mathrm{l.adm}}$ and $\mathfrak{C}$.** Unlike $\mathcal{A}$, the category $\mathcal{A}^{\mathrm{l.adm}}$ is locally finite, by Lemma 2.2.2. By the structure theory of locally finite categories, recalled in Section A.1.25, $\mathcal{A}^{\mathrm{l.adm}}$ admits a decomposition into blocks. An extensive analysis of the various blocks is made in [Paš13], and in what follows we recall, and slightly expand, some of these results. The arguments in [Paš13] rely crucially on Colmez's functor "V", but we postpone our discussion of this functor until the following section; in this section we focus on those results whose statements can be made without reference to Colmez's functor.

2.3.1. *Classification of blocks.* Recall that, by definition, a block of $\mathcal{A}^{\mathrm{l.adm}}$ is an equivalence class of (isomorphism classes of) irreducible objects[2] under the equivalence relation generated by

$$\pi_1 \sim \pi_2 \text{ if } \mathrm{Ext}^1_{\mathcal{A}}(\pi_1,\pi_2) \neq 0 \text{ or } \mathrm{Ext}^1_{\mathcal{A}}(\pi_2,\pi_1) \neq 0.$$

The blocks $\mathfrak{B}$ containing absolutely irreducible objects were classified in [Paš13]. The remaining blocks were classified in [DEG23, Proposition 2.4.8, Remark 2.4.9]. In summary, blocks come in five "types" as follows (we explain the labels in Remark 2.4.9 below):

[2]We note that it follows from the results of [BL94; Bre03] that the irreducible objects of $\mathcal{A}$ are automatically admissible, and hence lie in $\mathcal{A}^{\mathrm{l.adm}}$. Thus we can equally well regard this as an equivalence relation on the irreducible objects of $\mathcal{A}$.

(ssg) $\mathfrak{B} = \{\pi\}$ where $\pi = c\text{-}\mathrm{Ind}_{KZ}^G \sigma/T_p$, for some ζ-compatible Serre weight σ, is an irreducible supersingular representation.

(gen) $\mathfrak{B} = \{\mathrm{Ind}_B^G(\chi_1 \otimes \omega^{-1}\chi_2), \mathrm{Ind}_B^G(\chi_2 \otimes \chi_1\omega^{-1})\}$ for characters $\chi_1, \chi_2 : \mathbf{Q}_p^\times \to \mathbf{F}^\times$ such that $\chi_1\chi_2 = \overline{\zeta}\omega$ and $\chi_1\chi_2^{-1} \neq 1, \omega^{\pm 1}$.

(scalar) $\mathfrak{B} = \{\mathrm{Ind}_B^G(\chi \otimes \omega^{-1}\chi)\}$ for a character $\chi : \mathbf{Q}_p^\times \to \mathbf{F}^\times$ such that $\chi^2 = \overline{\zeta}\omega$.

(St) $\mathfrak{B} = \{\chi, \chi \otimes \mathrm{St}, \mathrm{Ind}_B^G(\omega\chi \otimes \omega^{-1}\chi)\}$ for a character $\chi : \mathbf{Q}_p^\times \to \mathbf{F}^\times$ such that $\chi^2 = \overline{\zeta}$.

(gen+) $\mathfrak{B}$ is the set of irreducible subquotients of

$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma/f_{\mathfrak{B}}(T_p) \oplus c\text{-}\mathrm{Ind}_{KZ}^G \sigma^{\mathrm{co}}/f_{\mathfrak{B}}^*(T_p)$$

for some companion pair $\sigma|\sigma^{\mathrm{co}}$ of ζ-compatible Serre weights, and some irreducible monic polynomial $f_{\mathfrak{B}} \in \mathbf{F}[T_p]$ of degree > 1. Here we have written $f_{\mathfrak{B}}^*(T_p) \coloneqq f_{\mathfrak{B}}(0)^{-1}T_p^{\deg f_{\mathfrak{B}}} f_{\mathfrak{B}}(\zeta(p)/T_p)$ for the irreducible monic polynomials whose roots are $\lambda^{-1}\zeta(p)$, where λ runs through the roots of $f_{\mathfrak{B}}$.

Remark 2.3.2. Note that blocks of type (gen) can also be written in the form $\{c\text{-}\mathrm{Ind}_{KZ}^G \sigma/(T_p - \lambda), c\text{-}\mathrm{Ind}_{KZ}^G \sigma^{\mathrm{co}}/(T_p - \lambda^{-1}\zeta(p))\}$ for an appropriately chosen companion pair of weights $\sigma|\sigma^{\mathrm{co}}$ and $\lambda \in \mathbf{F}^\times$. Thus the key point in the description of blocks of type (gen+) is that the polynomial $f_{\mathfrak{B}}$ be of degree strictly greater than 1.

2.3.3. *The full subcategory associated to a block.* If $\mathfrak{B}$ is a block of irreducible objects of $\mathcal{A}$, the discussion of Section A.1.25 yields a direct factor $\mathcal{A}_{\mathfrak{B}}$ of $\mathcal{A}$. By Lemma A.1.27, the category $\mathcal{A}_{\mathfrak{B}}$ is locally finite, and its subcategory $\mathcal{A}_{\mathfrak{B}}^{\mathrm{f.l.}}$ of objects of finite length coincides with its subcategory of compact objects, resp. Noetherian objects. These coincide furthermore with the finitely generated objects, by the following lemma.

Lemma 2.3.4. *The category $\mathcal{A}_{\mathfrak{B}}^{\mathrm{f.l.}}$ coincides with the full subcategory of finitely presented objects of $\mathcal{A}_{\mathfrak{B}}$.*

Proof. Lemma A.1.27 implies that $\mathcal{A}_{\mathfrak{B}}^{\mathrm{f.l.}}$ is the subcategory of Noetherian objects of $\mathcal{A}_{\mathfrak{B}}$. On the other hand, an object of $\mathcal{A}_{\mathfrak{B}}$ is Noetherian in $\mathcal{A}_{\mathfrak{B}}$ if and only if it is Noetherian in $\mathcal{A}$, and the Noetherian objects of $\mathcal{A}$ are precisely the finitely generated objects, by [DEG23, Corollary 2.2.4]. We conclude because finitely generated objects are finitely presented (see Remark 2.2.25). □

2.3.5. *Blocks of type* (gen+). If $\mathfrak{B}$ is a block of type (gen+), by [DEG23, Prop. 2.4.8, Rem. 2.4.9] we have the following (mutually exclusive and exhaustive) possibilities:

- $\sigma = \sigma^{\mathrm{co}} = \sigma_{a,p-2}$ and $f_{\mathfrak{B}} = f_{\mathfrak{B}}^*$. Then $\mathfrak{B}$ is a singleton.
- $\{\sigma, \sigma^{\mathrm{co}}\} = \{\sigma_{a,0}, \sigma_{a+1,p-3}\}$, and $f_{\mathfrak{B}} = T_p^2 - \zeta(p)$ (hence $\zeta(p)$ is not a square in $\mathcal{O}^\times$). Then $\mathfrak{B}$ has three elements.
- $\mathfrak{B}$ has two elements.

If $\mathbf{F}'$ is a splitting field of $f_{\mathfrak{B}}$ over $\mathbf{F}$, then the set

$$\mathfrak{B} \otimes_{\mathbf{F}} \mathbf{F}' \coloneqq \{\mathrm{JH}(\pi \otimes_{\mathbf{F}} \mathbf{F}') : \pi \in \mathfrak{B}\}$$

is a union of blocks of $\mathcal{A}_{\mathbf{F}'}^{\mathrm{l.adm}}$. The group $\mathrm{Gal}(\mathbf{F}'/\mathbf{F})$ acts transitively on $\mathfrak{B} \otimes_{\mathbf{F}} \mathbf{F}'$ and preserves its partition as a union of blocks; the stabilizer of each part is trivial, except in the following case:

- $\sigma = \sigma^{\mathrm{co}} = \sigma_{a,p-2}$, $f_{\mathfrak{B}} = f_{\mathfrak{B}}^*$, and $f_{\mathfrak{B}} \neq T_p^2 - \zeta(p)$.

In fact, in this case, the element of order two of $\mathrm{Gal}(\mathbf{F}'/\mathbf{F})$ sends each root λ of $f_{\mathfrak{B}}$ to $\lambda^{-1}\zeta(p) \neq \lambda$, and so it fixes each part. (This exception can be explained in Galois-theoretic terms: for example, when $\deg f_{\mathfrak{B}} = 2$, $f_{\mathfrak{B}} = f_{\mathfrak{B}}^*$, and $f_{\mathfrak{B}} \neq T_p^2 - \zeta(p)$, the block $\{c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_{a,p-2})/f_{\mathfrak{B}}\}$ will correspond under Definition 2.4.7 to an irreducible but not absolutely irreducible Galois pseudorepresentation. Note that these are all twists of unramified representations, hence have Serre weight $\sigma_{a,p-2}$, as explained in Remark 2.1.12.)

In keeping with Remark 2.1.12, for any block of $\mathcal{A}$ (of type (gen+) or not) we write $\mathbf{F}_{\mathfrak{B}}$ for the splitting field of $f_{\mathfrak{B}}$, except in the case just described, in which we write $\mathbf{F}_{\mathfrak{B}}$ for the index-two subextension of the splitting field (so that, in this case, $f_{\mathfrak{B}}$ splits in $\mathbf{F}_{\mathfrak{B}}[T_p]$ as a product of irreducible quadratic polynomials, each fixed by the involution $f \mapsto f^*$).

Definition 2.3.6. We say that a block $\mathfrak{B}$ of $\mathcal{A}^{\mathrm{l.adm}}$ is $\mathbf{F}$-rational if $\mathbf{F}_{\mathfrak{B}} = \mathbf{F}$.

The next result shows that every block $\mathfrak{B}$ of $\mathcal{A}$ is equivalent to an $\mathbf{F}_{\mathfrak{B}}$-rational block of $\mathcal{A}_{\mathbf{F}_{\mathfrak{B}}}$.

Lemma 2.3.7. *Let $\mathfrak{B}$ be a block of type* (gen+)*, and let $\mathfrak{B}'$ be a block of $\mathcal{A}_{\mathbf{F}_{\mathfrak{B}}}$ contained in $\mathfrak{B} \otimes_{\mathbf{F}} \mathbf{F}_{\mathfrak{B}}$. Write $\mathcal{O}'/\mathcal{O}$ for the unramified extension with residue field $\mathbf{F}_{\mathfrak{B}}$. Then $\mathfrak{B}'$ is $\mathbf{F}_{\mathfrak{B}}$-rational, and restriction of scalars from $\mathcal{O}'$ to $\mathcal{O}$ is an equivalence of categories $\mathcal{A}_{\mathcal{O}',\mathfrak{B}'} \to \mathcal{A}_{\mathcal{O},\mathfrak{B}}$.*

Proof. By definition, $\mathbf{F}_{\mathfrak{B}}$ is a splitting field of $f_{\mathfrak{B}}$, except when $\sigma = \sigma_{a,p-2}$, $f_{\mathfrak{B}} = f_{\mathfrak{B}}^*$, and $f_{\mathfrak{B}} \neq T_p^2 - \zeta(p)$. In this case, $f_{\mathfrak{B}}$ splits as a product of irreducible quadratic polynomial, which are fixed by the involution $f \mapsto f^*$. Thus we have the following mutually exclusive possibilities:

- $\sigma = \sigma_{a,p-2}$, and $f_{\mathfrak{B}} = T_p^2 - \zeta(p)$. Then $\mathfrak{B}'$ has type (scalar), and it has the form $\{\pi_0\}$ for some absolutely irreducible object π_0 of $\mathcal{A}_{\mathcal{O}'}$.
- $\sigma = \sigma_{a,p-2}$, $f_{\mathfrak{B}} = f_{\mathfrak{B}}^*$, and $f_{\mathfrak{B}} \neq T_p^2 - \zeta(p)$. Then $\mathfrak{B}'$ is an $\mathbf{F}_{\mathfrak{B}}$-rational block of type (gen+), and it has the form $\{\pi_0\}$ for some irreducible object π_0 of $\mathcal{A}_{\mathcal{O}'}$.
- $\sigma = \sigma_{a,0}$, and $f_{\mathfrak{B}} = T_p^2 - \zeta(p)$. Then $\mathfrak{B}'$ has type (St), and it has the form $\{\pi_0, \pi_1, \pi_2\}$ for some absolutely irreducible objects π_0, π_1, π_2 of $\mathcal{A}_{\mathcal{O}'}$.
- $\mathfrak{B}'$ has type (gen), and it has the form $\{\pi_0, \pi_1\}$ for some absolutely irreducible objects π_0, π_1 of $\mathcal{A}_{\mathcal{O}'}$.

In all of these cases, $\mathfrak{B}'$ is $\mathbf{F}_{\mathfrak{B}}$-rational, and for all $\alpha \neq \beta \in \mathrm{Gal}(\mathcal{O}'/\mathcal{O})$, we have $\alpha^*\mathfrak{B}' \neq \beta^*\mathfrak{B}'$. Furthermore, restriction of scalars preserves local admissibility, by Lemma 2.2.2 (3), and so we may also regard each π_i as a locally admissible object of $\mathcal{A}_{\mathcal{O}}$.

We now prove that π_i is an irreducible object of $\mathcal{A}_{\mathcal{O},\mathfrak{B}}$. Choose an irreducible subobject $\pi \to \pi_i$ of π_i in $\mathcal{A}_{\mathcal{O}}^{\mathrm{l.adm}}$. Then π is an irreducible subquotient of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma'/f'(T_p)$ for some irreducible monic $f' \in \mathbf{F}[T_p]$. Since $\pi \otimes_{\mathcal{O}} \mathcal{O}'$ is a subobject of $\pi_i \otimes_{\mathcal{O}} \mathcal{O}' = \oplus_{\gamma \in \mathrm{Gal}(\mathcal{O}'/\mathcal{O})} \gamma^* \pi_i$, we see that $c\text{-}\mathrm{Ind}_{KZ}^G \sigma'/f'(T_p)$ and

$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma/f_{\mathfrak{B}}(T_p) \oplus c\text{-}\mathrm{Ind}_{KZ}^G \sigma^{\mathrm{co}}/f_{\mathfrak{B}}^*(T_p)$$

have an irreducible subquotient in common after extending scalars to $\mathcal{O}'$. By [DEG23, Corollary 2.1.14] we deduce that $(\sigma', f') \in \{(\sigma, f_{\mathfrak{B}}), (\sigma^{\mathrm{co}}, f_{\mathfrak{B}}^*)\}$, and so π is an irreducible object of $\mathcal{A}_{\mathcal{O},\mathfrak{B}}$. Now, by construction, we know that $\pi \otimes_{\mathcal{O}} \mathcal{O}'$ has

$\mathcal{A}_{\mathcal{O}'}$-length equal to $[\mathcal{O}' : \mathcal{O}]$. This is the same as the length of $\pi_i \otimes_{\mathcal{O}} \mathcal{O}'$, and so the inclusion $\pi \to \pi_i$ is an isomorphism, as desired.

It follows from the previous paragraph that restriction of scalars defines a functor $\mathcal{A}_{\mathcal{O}',\mathfrak{B}'} \to \mathcal{A}_{\mathcal{O},\mathfrak{B}}$, and there remains to prove that it is an equivalence. We will do this by an application of Morita theory. Let $\mathcal{I}$ be an injective object of $\mathcal{A}_{\mathcal{O}',\mathfrak{B}'}$ with socle equal to the direct sum of the objects of $\mathfrak{B}'$. Then $\mathcal{I}$ is an injective object of $\mathcal{A}_{\mathcal{O},\mathfrak{B}}$: in fact, $\mathrm{Hom}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}}}(-,\mathcal{I})\otimes_{\mathcal{O}}\mathcal{O}'$ is exact, because it is naturally isomorphic to $\mathrm{Hom}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'}}(-\otimes_{\mathcal{O}}\mathcal{O}',\mathcal{I}\otimes_{\mathcal{O}}\mathcal{O}')$ (by Lemma 2.2.2 (3)) and $\mathcal{I}\otimes_{\mathcal{O}}\mathcal{O}' = \oplus_{\gamma\in\mathrm{Gal}(\mathcal{O}'/\mathcal{O})}\gamma^*\mathcal{I}$ is injective. Furthermore, $\mathcal{I}$ is an injective cogenerator of $\mathcal{A}_{\mathcal{O},\mathfrak{B}}$: this is a consequence of the fact that $\mathfrak{B}$ and $\mathfrak{B}'$ have the same number of elements, and non-isomorphic irreducible objects π_i, π_j of $\mathfrak{B}'$ are not isomorphic in $\mathcal{A}_{\mathcal{O}}$ (as can be seen by computing $\mathrm{Hom}_{\mathcal{A}_{\mathcal{O}}}(\pi_i,\pi_j)\otimes_{\mathcal{O}}\mathcal{O}' = \mathrm{Hom}_{\mathcal{A}_{\mathcal{O}'}}(\pi_i\otimes_{\mathcal{O}}\mathcal{O}',\pi_j\otimes_{\mathcal{O}}\mathcal{O}') = 0$, using that $\pi_i \neq \gamma^*\pi_j$ for any $\gamma\in\mathrm{Gal}(\mathcal{O}'/\mathcal{O})$). By Morita theory (see e.g. Section A.1.61) there remains to prove that the restriction map

$$\mathrm{End}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'}}(\mathcal{I}) \to \mathrm{End}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}}}(\mathcal{I}) \tag{2.3.8}$$

is an isomorphism, or equivalently, is surjective. We can verify this after applying $-\otimes_{\mathcal{O}}\mathcal{O}'$. Now

$$\mathrm{End}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}}}(\mathcal{I})\otimes_{\mathcal{O}}\mathcal{O}' = \bigoplus_{\alpha,\beta\in\mathrm{Gal}(\mathcal{O}'/\mathcal{O})}\mathrm{Hom}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'}}(\alpha^*\mathcal{I},\beta^*\mathcal{I}) = \bigoplus_{\gamma\in\mathrm{Gal}(\mathcal{O}'/\mathcal{O})}\mathrm{End}_{\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'}}(\gamma^*\mathcal{I})$$

since $\alpha^*\mathcal{I}$ and $\beta^*\mathcal{I}$ are in different blocks of $\mathcal{A}^{\mathrm{l.adm}}_{\mathcal{O}'}$ if $\alpha\neq\beta$, and $(2.3.8)\otimes_{\mathcal{O}}\mathcal{O}'$ is the diagonal inclusion. Since the image of $(2.3.8)\otimes_{\mathcal{O}}\mathcal{O}'$ also contains the idempotents in the target (because it contains $\mathcal{O}'\otimes_{\mathcal{O}}\mathcal{O}'$), this concludes the proof. $\square$

Because of Lemma 2.3.7, one can often assume without loss of generality that $\mathfrak{B}$ is $\mathbf{F}$-rational. As already mentioned, these blocks have been thoroughly studied in [Paš13], with the exception of $\mathbf{F}$-rational blocks $\mathfrak{B}$ of type (gen+); note that these are not equivalent to blocks of type (gen), e.g. because they contain a unique irreducible object. However, many of the invariants of $\mathfrak{B}$ that we define in the following sections become isomorphic to invariants of blocks of type (gen) after a base extension, which will be enough for our intended applications.

2.3.9. *The dual category associated to a block.* We write $\mathfrak{C}_{\mathfrak{B}}$ for the image of $\mathcal{A}_{\mathfrak{B}}$ under Pontrjagin duality. By Proposition A.1.28, there exists a pseudocompact $\mathcal{O}$-algebra $E_{\mathfrak{B}}$ such that $\mathfrak{C}_{\mathfrak{B}}$ is equivalent to the category of pseudocompact right $E_{\mathfrak{B}}$-modules. The ring $E_{\mathfrak{B}}$ is only well-defined up to its category of pseudocompact modules, and one of the main results of Paškūnas in [Paš13] is the computation of an explicit choice of $E_{\mathfrak{B}}$, and of its centre. We will recall these results in detail in the next section. For now, we content ourselves by noting the following particular lemma, which follows directly from those more precise results, and then deriving some consequences of it. We do this because [Paš13] only considers blocks that contain absolutely irreducible representations (i.e. not of type (gen+)) whereas it will be important for us to have a uniform statement for all blocks. This will recur throughout our discussion of "classical" p-adic local Langlands for $\mathrm{GL}_2(\mathbf{Q}_p)$ (for example, the absolutely irreducible case of Lemma 2.3.11 (3) can be found in [PT21, Corollary 6.7]), and as we already intimated in Section 1.1.10, these amplifications of the existing literature are straightforward consequences of those existing results.

Lemma 2.3.10. *Let M be an object of $\mathfrak{C}_{\mathfrak{B}}$ with cosocle of finite length. Then $\mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(M)$ is finitely generated over the Bernstein centre $\mathcal{Z}_{\mathfrak{B}}$ of $\mathfrak{C}_{\mathfrak{B}}$, which is a Noetherian, local, profinite $\mathcal{O}$-algebra. Hence $\mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(M)$ is a Noetherian profinite $\mathcal{O}$-algebra.*

Proof. It suffices to construct a projective generator $P_{\mathfrak{B}}$ of $\mathfrak{C}_{\mathfrak{B}}$, with cosocle of finite length, such that $E_{\mathfrak{B}} := \mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}})$ is finitely generated over its centre $Z(E_{\mathfrak{B}})$, and $Z(E_{\mathfrak{B}})$ is Noetherian and local. In fact, the equivalence (A.1.64) implies that $\mathcal{Z}_{\mathfrak{B}} \cong Z(E_{\mathfrak{B}})$, and so $\mathcal{Z}_{\mathfrak{B}}$ is Noetherian, local and profinite. Finally, if M has cosocle of finite length, then $\mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(M)$ is an $E_{\mathfrak{B}}$-submodule of $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}^{\oplus n}, M)$ for some $n \geq 0$, which is a quotient of $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}^{\oplus n}, P_{\mathfrak{B}}^{\oplus n}) \cong E_{\mathfrak{B}}^{\oplus n^2}$. Hence $\mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(M)$ is a subquotient of a finitely generated $\mathcal{Z}_{\mathfrak{B}}$-module, and so it is finitely generated over $\mathcal{Z}_{\mathfrak{B}}$.

If $\mathfrak{B}$ contains absolutely irreducible representations, then the existence of $P_{\mathfrak{B}}$ with these properties is established in [Paš13]. Hence the lemma is true when $\mathfrak{B}$ contains absolutely irreducible representations. By Lemma 2.3.7, there remains to prove the lemma in the case that $\mathfrak{B} = \{\pi_0\}$ is $\mathbf{F}$-rational of type (gen+). If $P_{\mathfrak{B}} \to \pi_0^\vee$ is a projective envelope, and $\mathcal{O} \to \mathcal{O}'$ is unramified quadratic, then $P_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}'$ is a projective generator of a block of $\mathfrak{C}_{\mathcal{O}'}$ of type (gen). Hence $\mathrm{End}(P_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}')$ is finitely generated over its centre, which is Noetherian and local. Since $\mathrm{End}(P_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}') = \mathrm{End}(P_{\mathfrak{B}}) \otimes_{\mathcal{O}} \mathcal{O}'$ and $Z(\mathrm{End}(P_{\mathfrak{B}}) \otimes_{\mathcal{O}} \mathcal{O}') = Z(\mathrm{End}(P_{\mathfrak{B}})) \otimes_{\mathcal{O}} \mathcal{O}'$, we deduce that $Z(\mathrm{End}(P_{\mathfrak{B}}))$ is Noetherian and local, and that $\mathrm{End}(P_{\mathfrak{B}})$ is finitely generated over $Z(\mathrm{End}(P_{\mathfrak{B}}))$, as desired. $\square$

Lemma 2.3.11. *Let P be a projective object of $\mathfrak{C}_{\mathfrak{B}}$ with cosocle of finite length.*

1. *The endomorphism ring $E := \mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(P)$ is a Noetherian profinite $\mathcal{O}$-algebra, and P is a Noetherian object of $\mathfrak{C}$.*
2. *The module P, with its natural topology as an object of $\mathfrak{C}_{\mathfrak{B}}$, is an object of* $\mathrm{Mod}_c(E)$.
3. *For all $i > 0$, the quotient $P/\mathrm{rad}(E)^i P$ is a finite length, hence coadmissible, object of $\mathfrak{C}_{\mathfrak{B}}$.*
4. *If P is furthermore a projective generator of $\mathfrak{C}_{\mathfrak{B}}$, then P is topologically flat over E (hence projective in* $\mathrm{Mod}_c(E)$, *by Lemma* A.1.44) *and E has finite global dimension.*

Proof. Proof of (1): An application of Lemma 2.3.10 shows that E is a (left and right) Noetherian profinite $\mathcal{O}$-algebra. To see that P is Noetherian in $\mathfrak{C}$, or equivalently in $\mathfrak{C}_{\mathfrak{B}}$, note that there exists a projective generator $P_{\mathfrak{B}}$ of $\mathfrak{C}_{\mathfrak{B}}$, with finite cosocle and Noetherian endomorphism ring $E_{\mathfrak{B}}$, such that P is a quotient of $P_{\mathfrak{B}}$. Under the equivalence $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, -) : \mathfrak{C}_{\mathfrak{B}} \to \mathrm{Mod}_c(E_{\mathfrak{B}}^{\mathrm{op}})$, P goes to a quotient of $E_{\mathfrak{B}}$, which is a Noetherian $E_{\mathfrak{B}}^{\mathrm{op}}$-module. Hence P is Noetherian in $\mathfrak{C}_{\mathfrak{B}}$.

Proof of (2): Since the natural topology on P is profinite, it suffices to prove that every open $\mathcal{O}$-submodule of P of finite index in P contains an open left E-submodule, which is exactly what is proved in [Paš13, Lemma 2.7]. (Strictly speaking, this reference works under the additional assumption that $\mathrm{cosoc}(P)$ is multiplicity free, but this assumption is only used to ensure that $E/\mathrm{rad}(E)$ has finite $\mathbf{F}$-dimension, which follows from the fact that E is compact and has the $\mathrm{rad}(E)$-adic topology, by Lemma A.1.30 (8)).

Proof of (3): By Lemma 2.3.10, E is finitely generated over the Bernstein centre $\mathcal{Z}_{\mathfrak{B}}$, which is a Noetherian local ring. Writing $\mathfrak{m} := \mathrm{rad}(\mathcal{Z}_{\mathfrak{B}})$, we deduce that E has the $\mathfrak{m}$-adic topology, and so the sequences $\mathrm{rad}(E)^i$ and $\mathfrak{m}^i E$ are cofinal in E. It thus suffices to prove that $P/\mathfrak{m}^i P$ has finite length. Choose a projective generator $P_{\mathfrak{B}}$ of $\mathfrak{C}_{\mathfrak{B}}$ with finite cosocle, and write $E_{\mathfrak{B}}$ for its endomorphism algebra. Since P has finite cosocle, there is a surjection $P_{\mathfrak{B}}^{\oplus n} \to P/\mathfrak{m}^i P$ for some n, and so $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, P/\mathfrak{m}^i P)$ is an $E_{\mathfrak{B}}^{\mathrm{op}}$-quotient of $E_{\mathfrak{B}}^{\oplus n}$. Hence $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, P/\mathfrak{m}^i P)$ is a finite $E_{\mathfrak{B}}/\mathfrak{m}^i E_{\mathfrak{B}}$-module. Since $\mathfrak{m}^i E_{\mathfrak{B}}$ is open in $E_{\mathfrak{B}}$, and $E_{\mathfrak{B}}$ is profinite, the quotient $E_{\mathfrak{B}}/\mathfrak{m}^i E_{\mathfrak{B}}$ is a finite set, and so we conclude that $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, P/\mathfrak{m}^i P)$ has finite $E_{\mathfrak{B}}^{\mathrm{op}}$-length. Since $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, -) : \mathfrak{C}_{\mathfrak{B}} \to \mathrm{Mod}_c(E_{\mathfrak{B}}^{\mathrm{op}})$ is an equivalence, we conclude that $P/\mathfrak{m}^i P$ has finite $\mathfrak{C}_{\mathfrak{B}}$-length, as desired.

Proof of (4): Assume that P is a projective generator of $\mathfrak{C}_{\mathfrak{B}}$. Then, by Proposition A.1.65, P is a complete left E-module in $\mathfrak{C}_{\mathfrak{B}}$, and the corresponding functor

$$-\widehat{\otimes}_E P : \mathrm{Mod}_c(E^{\mathrm{op}}) \to \mathfrak{C}_{\mathfrak{B}}$$

is an equivalence. Since the forgetful functor $i : \mathfrak{C}_{\mathfrak{B}} \to \mathrm{Mod}_c(\mathcal{O})$ is exact and cofiltered limit-preserving, the composition $i \circ (-\widehat{\otimes}_E P)$ is an exact and cofiltered limit-preserving functor $\mathrm{Mod}_c(E^{\mathrm{op}}) \to \mathrm{Mod}_c(\mathcal{O})$. By Lemma A.1.55, this composition is naturally isomorphic to the completed tensor product associated to $i(P)$. By Lemma A.1.59, this is the usual completed tensor product $-\widehat{\otimes}_E P$, which is therefore exact, as desired.

Finally, the claim that E has finite global dimension can be verified as follows. By [JNW24, Proposition 4.2.2] and Lemma 2.3.10, it suffices to prove that every simple left E^{op}-module M has a finite projective resolution (since there are finitely many isomorphism classes of simple modules). For this, it suffices to prove that M has a finite projective resolution in $\mathrm{Mod}_c(E^{\mathrm{op}})$, all of whose terms have finite length E^{op}-cosocle, or equivalently, that M has a finite injective resolution in $\mathrm{Mod}_c(E^{\mathrm{op}})^{\mathrm{op}}$, all of whose terms have socle of finite length. Since $\mathrm{Mod}_c(E^{\mathrm{op}})^{\mathrm{op}}$ is a locally finite category, general theory shows that M has an injective resolution $M \to J^\bullet$ such that, for all simple $N \in \mathrm{Mod}_c(E^{\mathrm{op}})$, we have

$$\mathrm{length}_{\mathcal{O}} \mathrm{Hom}_{\mathrm{Mod}_c(E^{\mathrm{op}})^{\mathrm{op}}}(N, J^t) = \mathrm{length}_{\mathcal{O}} \mathrm{Ext}^t_{\mathrm{Mod}_c(E^{\mathrm{op}})^{\mathrm{op}}}(N, M)$$

(see the explanation in [Paš13, Remark 10.11]). It thus suffices to prove that for any two simple modules M, N, the Ext-group $\mathrm{Ext}^t_{\mathrm{Mod}_c(E^{\mathrm{op}})^{\mathrm{op}}}(N, M)$ is $\mathcal{O}$-finite, and vanishes for large enough t. Since $\mathrm{Mod}_c(E^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathcal{A}_{\mathfrak{B}}$, this is a consequence of known properties of Ext-groups between irreducible objects of $\mathcal{A}_{\mathfrak{B}}$. $\square$

In Section A.1.16 we have defined socle and radical filtrations for objects of a complete and cocomplete abelian category, such as $\mathcal{A}$, $\mathfrak{C}$, or $\mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$. The next lemma shows that, for finite length objects, the natural inclusions $\mathcal{A} \to \mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$ and $\mathfrak{C} \to \mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$ preserve these filtrations.

Lemma 2.3.12. *If $M \in \mathfrak{C}$ has finite length, then the $\mathfrak{C}$-socle filtration and the $\mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$-socle filtration of M coincide. The same statement is true with $\mathfrak{C}$ replaced by $\mathcal{A}$, or "socle" replaced by "radical".*

Proof. This is an immediate consequence of Lemma 2.2.31, resp. Lemma 2.2.20. $\square$

We now prove two finiteness results for the radical filtration of objects of $\mathfrak{C}$, or equivalently, the socle filtration of objects of $\mathcal{A}$.

Lemma 2.3.13. *If M is an object of $\mathfrak{C}$ with cosocle of finite length, then M is Noetherian in $\mathfrak{C}$, and $M/\operatorname{rad}_n M$ is of finite length for each $n \geq 0$. Furthermore, $M \xrightarrow{\sim} \varprojlim_n (M/\operatorname{rad}_n M)$.*

Proof. If $\operatorname{cosoc}(M)$ has finite length, then M is a quotient of a projective envelope P of $\operatorname{cosoc}(M)$, and P is Noetherian by Lemma 2.3.11 (1). Hence M is Noetherian. If $\operatorname{rad}^i M/\operatorname{rad}^{i+1} M$ has infinite length for some i, then (by the dual to Corollary A.1.21 (1)) it is an infinite direct product of simple objects, and so it contains an infinite ascending chain of subobjects. This contradicts the fact that M is Noetherian and concludes the proof of the first statement of the lemma.

We now prove that $M \xrightarrow{\sim} \varprojlim_n M/\operatorname{rad}^n M$. Since $\mathfrak{C}$ is dual to $\mathcal{A}^{\mathrm{l.adm}}$, we know that

$$M \xrightarrow{\sim} \varprojlim_{M' \subseteq M} M/M'$$

where the limit is over the set of closed submodules $M' \subseteq M$ of finite $\mathfrak{C}$-colength. Since we have just proved that $\operatorname{rad}^n M$ has finite colength for all n, it suffices to prove that if $q : M \to N$ is a surjection, and N has finite length, then there exists $n \geq 0$ such $\operatorname{rad}^n M \subseteq \ker(q)$. However, $\operatorname{rad}^n N = 0$ for n large enough, and then Lemma A.1.23(2) implies that $\operatorname{rad}^n M \subseteq \ker(q)$, as desired. This concludes the proof. $\square$

Lemma 2.3.14. *Let $\pi \in \mathcal{A}^{\mathrm{fp}}$, let $\mathfrak{B}$ be a block of $\mathcal{A}^{\mathrm{l.adm}}$, and let $n \geq 0$. Then there exists a unique subobject $\pi_{\mathfrak{B}}^{(n)} \subseteq \pi$ such that*

(1) $\pi/\pi_{\mathfrak{B}}^{(n)} \in \mathcal{A}_{\mathfrak{B}}^{\mathrm{fp}}$ and has Loewy length $\leq n$.

(2) if $\pi' \subseteq \pi$, and $\pi/\pi' \in \mathcal{A}_{\mathfrak{B}}^{\mathrm{fp}}$ and has Loewy length $\leq n$, then $\pi_{\mathfrak{B}}^{(n)} \subseteq \pi'$.

The set $\{\pi_{\mathfrak{B}}^{(n)} : n \geq 0\}$ is cofinal in the set of quotients of π contained in $\mathcal{A}_{\mathfrak{B}}^{\mathrm{fp}}$.

Proof. The uniqueness part is immediate. For the existence part, let $\mathcal{I}$ be a direct sum of injective envelopes of the simple objects of $\mathcal{A}_{\mathfrak{B}}$. The dual of Lemma 2.3.13 implies that $\operatorname{soc}_{\mathcal{A},n} \mathcal{I}$ has finite $\mathcal{A}$-length. Hence $\operatorname{Hom}_{\mathcal{A}}(\pi, \operatorname{soc}_{\mathcal{A},n} \mathcal{I})$ is finitely generated over $\mathcal{O}$. Choose generators $\varphi_1, \ldots, \varphi_r$ of this $\mathcal{O}$-module, and let

$$\pi_{\mathfrak{B}}^{(n)} = \bigcap_i \ker(\varphi_i).$$

Then

$$\pi/\pi_{\mathfrak{B}}^{(n)} \subseteq \bigoplus_i \operatorname{soc}_{\mathcal{A},n} \mathcal{I},$$

and the right-hand side is an object of $\mathcal{A}_{\mathfrak{B}}^{\mathrm{fp}}$ and has Loewy length $\leq n$. Hence (1) holds, by Lemma A.1.23. To see that (2) holds, we need to prove that $\pi_{\mathfrak{B}}^{(n)}$ maps to zero in π/π'. Since $\mathcal{I}$ is an injective cogenerator of $\mathcal{A}_{\mathfrak{B}}$, it suffices to prove that for all $\varphi \in \operatorname{Hom}_{\mathcal{A}}(\pi/\pi', \mathcal{I})$, the composite $\pi_{\mathfrak{B}}^{(n)} \to \pi \to \pi/\pi' \xrightarrow{\varphi} \mathcal{I}$ is zero. Now Lemma A.1.23 implies that φ factors through $\operatorname{soc}_{\mathcal{A},n} \mathcal{I}$, and all elements of $\operatorname{Hom}_{\mathcal{A}}(\pi, \operatorname{soc}_{\mathcal{A},n} \mathcal{I})$ are zero on $\pi_{\mathfrak{B}}^{(n)}$, by construction. This concludes the proof of the existence of $\pi_{\mathfrak{B}}^{(n)}$.

Finally, the cofinality claim follows from the second property of $\pi_{\mathfrak{B}}^{(n)}$, since any quotient of π contained in $\mathcal{A}_{\mathfrak{B}}^{\mathrm{fp}}$ has finite $\mathcal{A}$-length, hence finite Loewy length. $\square$

We now consider the case of a projective generator with finite cosocle.

Lemma 2.3.15. *Let $\mathfrak{B}$ be a block of $\mathcal{A}^{\mathrm{l.adm}}$, and let $P_{\mathfrak{B}}$ be a projective generator of $\mathfrak{C}_{\mathfrak{B}}$ with cosocle of finite length. Let $E_{\mathfrak{B}} := \mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}})$. Then:*

(1) the functor $M \mapsto \mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, M) = \mathrm{Hom}_G^{\mathrm{cont}}(P_{\mathfrak{B}}, M)$ gives an equivalence

$$\mathfrak{C}_{\mathfrak{B}} \xrightarrow{\sim} \mathrm{Mod}_c(E_{\mathfrak{B}}^{\mathrm{op}}),$$

with quasi-inverse given by $N \mapsto N \widehat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}}$.

(2) $\pi \mapsto P_{\mathfrak{B}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$ gives an equivalence

$$\mathcal{A}_{\mathfrak{B}}^{\mathrm{fp}} \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{f.l.}}(E_{\mathfrak{B}}),$$

with quasi-inverse given by $M \mapsto \mathrm{Hom}_{E_{\mathfrak{B}}}^{\mathrm{cont}}(P_{\mathfrak{B}}, M)$.

(3) If M is an object of $\mathfrak{C}_{\mathfrak{B}}$ with cosocle of finite length, then the module $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, M)$ is finitely presented over $E_{\mathfrak{B}}^{\mathrm{op}}$ and over $\mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(M)$.

(4) If M is an object of $\mathfrak{C}_{\mathfrak{B}}$ with cosocle of finite length, then there is a natural isomorphism

$$M \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, M) \otimes_{E_{\mathfrak{B}}} P_{\mathfrak{B}}. \tag{2.3.16}$$

Proof. Part (1) is Proposition A.1.65, whose assumptions are met because of Lemma 2.3.11 (1). We now prove part (2). Restricting the equivalences in part (1) to the subcategories of finite length objects, and recalling that

$$\mathrm{Mod}_c(E_{\mathfrak{B}}^{\mathrm{op}})^{\mathrm{f.l.}} \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{f.l.}}(E_{\mathfrak{B}}^{\mathrm{op}})$$

by Lemma A.1.32 (8) (which applies since $E_{\mathfrak{B}}$ is Noetherian, by Lemma 2.3.11 (1)), we obtain inverse anti-equivalences

$$\mathcal{A}_{\mathfrak{B}}^{\mathrm{f.l.}} \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{f.l.}}(E_{\mathfrak{B}}^{\mathrm{op}}),\ \pi \mapsto \mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, \pi^\vee)$$

and

$$\mathrm{Mod}^{\mathrm{f.l.}}(E_{\mathfrak{B}}^{\mathrm{op}}) \to \mathcal{A}_{\mathfrak{B}}^{\mathrm{f.l.}}, N \mapsto (N \widehat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}})^\vee.$$

Now Pontrjagin duality gives an anti-equivalence $\mathrm{Mod}^{\mathrm{f.l.}}(E_{\mathfrak{B}}^{\mathrm{op}}) \to \mathrm{Mod}^{\mathrm{f.l.}}(E_{\mathfrak{B}})$, and we have

$$\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, \pi^\vee)^\vee \cong P_{\mathfrak{B}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$$

by Lemma 2.2.45. On the other hand, if $M \in \mathrm{Mod}^{\mathrm{f.l.}}(E_{\mathfrak{B}})$ then $N := M^\vee$ is finitely presented over $E_{\mathfrak{B}}^{\mathrm{op}}$, and so $N \widehat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}} = N \otimes_{E_{\mathfrak{B}}} P_{\mathfrak{B}}$, and

$$(N \otimes_{E_{\mathfrak{B}}} P_{\mathfrak{B}})^\vee \cong \mathrm{Hom}_{E_{\mathfrak{B}}^{\mathrm{op}}}(N, P_{\mathfrak{B}}^\vee) = \mathrm{Hom}_{E_{\mathfrak{B}}}^{\mathrm{cont}}(P_{\mathfrak{B}}, M),$$

where the isomorphism is because both sides are right exact functors of N, and send $N = E_{\mathfrak{B}}^{\mathrm{op}}$ to $P_{\mathfrak{B}}^\vee$. This concludes the proof of part (2).

We now prove part (3). By assumption, M is a quotient of $P_{\mathfrak{B}}^{\oplus n}$ for some n, and so the module $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, M)$ is a quotient of $(E_{\mathfrak{B}}^{\mathrm{op}})^{\oplus n}$, i.e. it is a finitely generated $E_{\mathfrak{B}}^{\mathrm{op}}$-module. By Lemma 2.3.10, it follows that $\mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, M)$ is finitely generated over the Bernstein centre of $\mathfrak{C}_{\mathfrak{B}}$, and so also over $\mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(M)$. This concludes the proof because $E_{\mathfrak{B}}$ and $\mathrm{End}_{\mathfrak{C}_{\mathfrak{B}}}(M)$ are Noetherian, again by Lemma 2.3.10.

Finally, part (4) follows because part (1) implies that

$$M \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}_{\mathfrak{B}}}(P_{\mathfrak{B}}, M) \widehat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}},$$

and we can use part (3) to identify the completed tensor product with the uncompleted tensor product. $\square$

2.4. **The p-adic local Langlands correspondence for locally admissible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations.** We now recall the results of Paškūnas from [Paš13] more precisely. A key ingredient in both the statement and proof of these results is Colmez's functor from $\mathrm{GL}_2(\mathbf{Q}_p)$-representations to $G_{\mathbf{Q}_p}$-representations, and so we begin by briefly recalling some of the basic properties of this functor.

2.4.1. *Colmez's functor V.* We refer to the fundamental work [Col10c] for the definition of a covariant exact functor

$$(\mathrm{sm.}\, G)^{\mathrm{f.l.}} \to (\mathrm{sm.}\, G_{\mathbf{Q}_p})^{\mathrm{f.l.}} \tag{2.4.2}$$

whose values on absolutely irreducible objects can be found in [Col10c, Section VII.4], see also [Paš13, Section 5.7]. This functor is usually denoted V. Since we wish to work with Galois representations of determinant $\zeta\varepsilon^{-1}$, we will use the notation V to denote the twist of this functor by the Galois character ε^{-1}, and so its values on absolutely irreducible objects (as classified in Theorem 2.2.39 and Remark 2.2.40 above) are as follows. Note that if E'/E is a finite extension with ring of integers $\mathcal{O}'$, then V commutes with $-\otimes_{\mathcal{O}} \mathcal{O}'$, and with restriction of scalars from $\mathcal{O}'$ to $\mathcal{O}$: this allows us to compute V on all irreducible objects.

Lemma 2.4.3.

(1) $V(\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2\omega^{-1})) \cong \chi_2\omega^{-1}$ *for any smooth characters* $\chi_i : \mathbf{Q}_p^\times \to \mathbf{F}^\times$.

(2) $V(c\text{-}\mathrm{Ind}_{KZ}^G\, \sigma_{a,b}/T_p) \cong \mathrm{Ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}}(\mathrm{nr}_{-\zeta(p)}\omega_2^{b+1}) \otimes \omega^{a-1}$.

(3) $V(\chi \circ \det) = 0$ *and* $V((\chi \circ \det) \otimes \mathrm{St}) = \chi$ *for any smooth character* $\chi : \mathbf{Q}_p^\times \to \mathbf{F}^\times$.

Proof. See [Paš13, Section 5.7] and [Col10c, Section VII.4]. □

Recall that $\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]$ has an anti-involution † induced by $g \mapsto (\zeta\varepsilon^{-1})(g)g^{-1}$. We write

$$V^\dagger : \mathcal{A}^{\mathrm{f.l.}} \to \mathrm{Mod}^{\mathrm{f.l.}}(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]^{\mathrm{op}})$$

for the functor obtained by precomposing the $\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]$-action on V with †. Following [Paš13, Section 5.7], we also introduce a covariant exact functor

$$\check{\mathrm{V}} : \mathfrak{C} \to \mathrm{Mod}_c(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!])$$

as follows. We first introduce an autoequivalence $W \mapsto W^*$ of $\mathrm{Mod}^{\mathrm{f.l.}}(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!])$ by letting W^* be the Pontrjagin dual of W, with the left $\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]$-module structure obtained by composing the natural right $\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]$-module structure with †. In other words, W^* is the contragredient of W, twisted by $\zeta\varepsilon^{-1}$. If M is an object of $\mathfrak{C}$ of finite length, we then define

$$\check{\mathrm{V}}(M) = V(M^\vee)^* \in \mathrm{Mod}^{\mathrm{f.l.}}(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]),$$

Then we extend $\check{\mathrm{V}}$ to $\mathfrak{C}$ by imposing compatibility with cofiltered limits, i.e. we use the identification $\mathfrak{C} = \mathrm{Pro}(\mathfrak{C}^{\mathrm{f.l.}})$ of Lemma A.1.26 and the fact that $\mathrm{Mod}_c(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!])$ is complete.

Remark 2.4.4. If $M \in \mathfrak{C}$ has finite length, and we regard $\check{\mathrm{V}}(M)^\vee$ as a right $\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]$-module via contravariant Pontrjagin functoriality, then it is naturally isomorphic to $V^\dagger(M^\vee)$.

Remark 2.4.5. Similarly to V, our functor $\check{\mathrm{V}}$ also differs from the $\check{\mathrm{V}}$ in [Paš13, Section 5.7] by a twist by ε^{-1}. We also caution the reader that on [Paš15, p. 320], Paškūnas also writes $\check{\mathrm{V}}$ to denote the contravariant functor on admissible Banach space representations obtained by first dualizing to land in the isogeny category associated to $\mathfrak{C}$, and then applying the functor $\check{\mathrm{V}}$ as we've recalled it above. We will not do this.

2.4.6. *Blocks and pseudorepresentations.* The functor V induces a bijection between the blocks of $\mathcal{A}^{\mathrm{l.adm}}$, and the $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy classes of two-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentations $\overline{\theta}$ of $G_{\mathbf{Q}_p}$ with determinant $\overline{\zeta}\omega^{-1}$. This bijection is given as follows.

Definition 2.4.7.

(1) Assume that $\mathfrak{B}$ is an $\mathbf{F}$-rational block. If $\mathfrak{B} = \{\pi\}$ has type (ssg) or type (gen+), we set $\overline{\theta}(\mathfrak{B}) := V(\pi)$. In all other cases, we set
$$\overline{\theta}(\mathfrak{B}) := V(\pi) + V(\pi)^*$$
for any infinite-dimensional irreducible $\pi \in \mathfrak{B}$.

(2) Assume that $\mathfrak{B}$ is not an $\mathbf{F}$-rational block. By Lemma 2.3.7, there exists an $\mathbf{F}_{\mathfrak{B}}$-rational block $\mathfrak{B}'$ of $\mathcal{A}_{\mathbf{F}_{\mathfrak{B}}}$ equivalent to $\mathfrak{B}$. We define $\overline{\theta}(\mathfrak{B})$ to be the $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy class of $\overline{\theta}(\mathfrak{B}')$.

Note that when $\mathfrak{B} = \{\pi\}$ is an $\mathbf{F}$-rational block of type (gen+), the compatibility of V with extension of scalars, together with Lemma 2.4.3 (1), shows that $V(\pi)$ is an irreducible, not absolutely irreducible representation $G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{F})$. This implies that $\mathbf{F}_{\overline{\theta}(\mathfrak{B})} = \mathbf{F}_{\mathfrak{B}}$ for all blocks $\mathfrak{B}$. Lemma 2.4.3 also implies that $\overline{\theta}(-)$ is a bijection, with the following explicit description on $\mathbf{F}$-rational objects.

Lemma 2.4.8.

(1) $\overline{\theta}\{c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_{a,b}/T_p\} = \omega^{a-1} \otimes \mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p} \mathrm{nr}_{-\zeta(p)}\omega_2^{b+1}$.

(2) $\overline{\theta}\{\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2\omega^{-1}), \mathrm{Ind}_B^G(\chi_2 \otimes \chi_1\omega^{-1})\} = \omega^{-1}(\chi_1 + \chi_2)$.

(3) $\overline{\theta}\{\mathrm{Ind}_B^G(\chi \otimes \omega^{-1}\chi)\} = 2\omega^{-1}\chi$.

(4) $\overline{\theta}\{\chi \circ \det, (\chi \circ \det) \otimes \mathrm{St}, \mathrm{Ind}_B^G(\omega\chi \otimes \omega^{-1}\chi)\} = (1 + \omega^{-1})\chi$.

(5) $\overline{\theta}\{c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_{a,b})/(T_p^2 - tT_p + \overline{\zeta}(p))\} = \omega^{a-1} \otimes \mathrm{nr}_{T_p^2 - tT_p + \overline{\zeta}(p)}$.

Proof. This is a direct computation. In case (5), we have written $\mathrm{nr}_{T_p^2 - tT_p + \overline{\zeta}(p)}$ for the two-dimensional unramified representation with characteristic polynomial of Frobenius given by $T_p^2 - tT_p + \overline{\zeta}(p)$. □

From now on we write $\mathfrak{B}_{\overline{\theta}}$ for the block corresponding to $\overline{\theta}$. Slightly more informally, we will often label data associated to blocks in terms of pseudorepresentations $\overline{\theta}$, rather than using the notation $\mathfrak{B}$ or $\mathfrak{B}_{\overline{\theta}}$. For example, from now on we'll typically write $\mathfrak{C}_{\overline{\theta}}$ in place of $\mathfrak{C}_{\mathfrak{B}_{\overline{\theta}}}$. Note that since $\mathbf{F}_{\mathfrak{B}_{\overline{\theta}}} = \mathbf{F}_{\overline{\theta}}$, this will not lead to ambiguity when discussing fields of definition.

Remark 2.4.9. Our labelling of the blocks and pseudorepresentations is now seen to be justified as follows:

(ssg) is short for "supersingular", because these blocks contain the supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$.

(gen) is short for "generic", because of Definition 2.1.2. Note that these pseudorepresentations are the traces of the generic representations on the Emerton–Gee stack for $G_{\mathbf{Q}_p}$.

(scalar) is for the scalar pseudorepresentations.

(St) is short for "Steinberg", because these blocks contain a twist of the Steinberg representation of G.

(gen+) becomes "gen" after extension of scalars.

2.4.10. *Construction of projective objects.* We now exhibit for every block $\mathfrak{B}_{\overline{\theta}}$ a particular projective object $P_{\overline{\theta}}$ of $\mathfrak{C}_{\overline{\theta}}$ with finite cosocle. If $\overline{\theta}$ is not of type (St), then $P_{\overline{\theta}}$ will furthermore be a generator of $\mathfrak{C}_{\overline{\theta}}$.

Definition 2.4.11. We define a projective object $P_{\overline{\theta}}$ of $\mathfrak{C}_{\overline{\theta}}$ in the following way:

(1) If $\overline{\theta}$ has type (ssg), let π be the unique irreducible object of $\mathfrak{B}_{\overline{\theta}}$, and let $P_{\pi^\vee}$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}_{\overline{\theta}}$. Then $P_{\overline{\theta}} := P_{\pi^\vee}^{\oplus 2}$.

(2) If $\overline{\theta}$ has type (St), let $\pi_1 = (\chi \circ \det) \otimes \pi_\alpha$ and $\pi_2 = (\chi \circ \det) \otimes \mathrm{St}$ be the irreducible, infinite-dimensional objects of $\mathfrak{B}_{\overline{\theta}}$. Let $P_{\pi_1^\vee}$ and $P_{\pi_2^\vee}$ be their projective envelopes in $\mathfrak{C}_{\overline{\theta}}$. Then $P_{\overline{\theta}} := P_{\pi_1^\vee} \oplus P_{\pi_2^\vee}$.

(3) Otherwise, $P_{\overline{\theta}} := \oplus_\pi P_{\pi^\vee}$ is the direct sum of projective envelopes of the duals of the irreducible objects π of $\mathfrak{B}_{\overline{\theta}}$.

We also write

$$\begin{aligned} E_{\overline{\theta}} &:= \mathrm{End}^{\mathrm{cont}}_{\mathcal{O}[\![G]\!]_\zeta}(P_{\overline{\theta}}) = \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P_{\overline{\theta}}) \\ \mathfrak{T}_{\overline{\theta}} &:= \text{ right orthogonal to } P_{\overline{\theta}} \text{ in } \mathfrak{C}_{\overline{\theta}} \\ \mathfrak{Q}_{\overline{\theta}} &:= \mathfrak{C}_{\overline{\theta}}/\mathfrak{T}_{\overline{\theta}}. \end{aligned}$$

2.4.12. *The functor* $\check{\mathrm{V}}$ *and Cayley–Hamilton modules.* In this subsection we study the restriction of $\check{\mathrm{V}}$ to $\mathfrak{C}_{\overline{\theta}}$ for all pseudorepresentations $\overline{\theta}$. When $\overline{\theta}$ has type (gen+), and $\mathcal{O}'/\mathcal{O}$ is the finite unramified extension with residue field $\mathbf{F}_{\overline{\theta}}$, Lemma 2.3.7 produces an $\mathbf{F}_{\overline{\theta}}$-rational block $\mathfrak{C}_{\overline{\theta}'}$ of $\mathfrak{C}_{\mathcal{O}'}$, and shows that restriction of scalars $\mathfrak{C}_{\overline{\theta}'} \to \mathfrak{C}_{\overline{\theta}}$ is an equivalence. Since V commutes with restriction of scalars, the diagram

$$\begin{tikzcd} \mathfrak{C}_{\overline{\theta}'} \arrow[r, "\check{\mathrm{V}}"] \arrow[d, "\sim"] & \mathrm{Mod}_c(\mathcal{O}'[\![G_{\mathbf{Q}_p}]\!]) \arrow[d] \\ \mathfrak{C}_{\overline{\theta}} \arrow[r, "\check{\mathrm{V}}"] & \mathrm{Mod}_c(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]) \end{tikzcd}$$

commutes. In the rest of this subsection, we will therefore be able to assume without loss of generality that $\overline{\theta}$ is $\mathbf{F}$-rational.

We begin by forming the profinite module $\check{\mathrm{V}}(P_{\overline{\theta}})$, noting that it has commuting $\mathcal{O}$-linear left actions of $E_{\overline{\theta}}$ (by functoriality) and $\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]$. By Lemma 2.3.11(2), $P_{\overline{\theta}}$ is a pseudocompact left $E_{\overline{\theta}}$-module, and so $\check{\mathrm{V}}(P_{\overline{\theta}})$ can be seen as an object of $\mathrm{Mod}_c(E_{\overline{\theta}})$. The following result is essentially a restatement of some of the results of [PT21], which in turn, since we are assuming $p \geq 5$, are essentially a restatement of results from [Paš13].

Proposition 2.4.13. *Let $\overline{\theta}$ be a two-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation of $G_{\mathbf{Q}_p}$.*

(1) *The $\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]$-action on $\check{\mathrm{V}}(P_{\overline{\theta}})$ factors through $\widetilde{R}_{\overline{\theta}}$, and induces an isomorphism*
$$\widetilde{R}_{\overline{\theta}} \xrightarrow{\sim} \mathrm{End}_{\mathrm{Mod}_c(E_{\overline{\theta}})}\big(\check{\mathrm{V}}(P_{\overline{\theta}})\big).$$
(2) *$\check{\mathrm{V}}(P_{\overline{\theta}})$ is free of rank one over each of $\widetilde{R}_{\overline{\theta}}$ and $E_{\overline{\theta}}$; in particular there is an isomorphism $\mathrm{End}_{\mathrm{Mod}_c(E_{\overline{\theta}})}\big(\check{\mathrm{V}}(P_{\overline{\theta}})\big) \cong E_{\overline{\theta}}^{\mathrm{op}}$, well-defined up to inner automorphism.*
(3) *The composition of the isomorphisms in parts (1) and (2) is an isomorphism $\widetilde{R}_{\overline{\theta}} \xrightarrow{\sim} E_{\overline{\theta}}^{\mathrm{op}}$, well-defined up to inner automorphism.*

Proof. Without loss of generality, $\overline{\theta}$ is $\mathbf{F}$-rational. Furthermore, all statements in the proposition may be checked after a finite unramified base extension $-\otimes_{\mathcal{O}}\mathcal{O}'$. We can therefore assume without loss of generality that $\overline{\theta}$ does not have type (gen+). The statement of (1) is then a consequence of [PT21, Theorem 6.3, Theorem 6.13], together with the assertion in Proposition 2.1.16 that $\widetilde{R}_{\overline{\theta}}$ is $\mathcal{O}$-torsion free. The statement in (2) concerning the $E_{\overline{\theta}}$-structure of $\check{\mathrm{V}}(P_{\overline{\theta}})$ is contained in the last three paragraphs of the proof of [PT21, Proposition 4.18].

If we choose an isomorphism $\imath : \check{\mathrm{V}}(P_{\overline{\theta}}) \xrightarrow{\sim} E_{\overline{\theta}}$ in $\mathrm{Mod}_c(E_{\overline{\theta}})$, then we find that
$$\mathrm{End}_{\mathrm{Mod}_c(E_{\overline{\theta}})}\big(\check{\mathrm{V}}(P_{\overline{\theta}})\big) \xrightarrow{\sim} E_{\overline{\theta}}^{\mathrm{op}}$$
(with the isomorphism depending upon the choice of $\imath$, and thus being well-defined up to an inner automorphism) and that $\check{\mathrm{V}}(P_{\overline{\theta}})$ is also free of rank one over its endomorphism algebra $\mathrm{End}_{\mathrm{Mod}_c(E_{\overline{\theta}})}\big(\check{\mathrm{V}}(P_{\overline{\theta}})\big)$. Combining this with the statement of (1) proves the remainder of the proposition. $\square$

Part (2) of Proposition 2.4.13 allows us to regard $\check{\mathrm{V}}(P_{\overline{\theta}})$ as an $(\widetilde{R}_{\overline{\theta}}, E_{\overline{\theta}}^{\mathrm{op}})$-bimodule, free of rank one with respect to the action of either ring. We now use Morita theory to give an alternative description of the functor $\check{\mathrm{V}}$. We recall that the functor $\check{\mathrm{V}}$ is trivial on $\mathfrak{T}_{\overline{\theta}}$: this is a vacuous statement except in case (St), in which case it amounts to the statement that $\check{\mathrm{V}}\big((\chi\circ\det)^\vee\big) = 0$. Thus $\check{\mathrm{V}}$ induces a functor $\mathfrak{Q}_{\overline{\theta}} \to \mathrm{Mod}_c(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!])$ (again exact and cofiltered limit-preserving, because so is $\check{\mathrm{V}}$) which we continue to denote by $\check{\mathrm{V}}$. Because our projective object $P_{\overline{\theta}}$ has been chosen precisely so that it detects the quotient $\mathfrak{Q}_{\overline{\theta}}$, we can then use Morita theory with respect to $P_{\overline{\theta}}$ to describe $\check{\mathrm{V}}$. This is the subject of the following theorem, which is essentially due to Paškūnas. In particular, the statement of (1), to the effect that $\check{\mathrm{V}}$ induces the indicated equivalence, is one of the main results of [Paš13].

Theorem 2.4.14. *Let $\mathfrak{B}_{\overline{\theta}}$ be a block of $\mathcal{A}^{\mathrm{l.adm}}$, and let $\overline{\theta}$ be the associated $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy class of pseudorepresentations. Then the following are true:*

(1) *The functor $\check{\mathrm{V}} : \mathfrak{C}_{\overline{\theta}} \to \mathrm{Mod}_c(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!])$ factors through $\mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}})$, is naturally isomorphic to*
$$M \mapsto \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\overline{\theta}}, M) \otimes_{E_{\overline{\theta}}} \check{\mathrm{V}}(P_{\overline{\theta}}), \tag{2.4.15}$$
and induces an equivalence $\mathfrak{Q}_{\overline{\theta}} \xrightarrow{\sim} \mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}})$.
(2) *The functor $V^\dagger : \mathcal{A}_{\overline{\theta}}^{\mathrm{fp}} \to \mathrm{Mod}^{\mathrm{f.l.}}(\mathcal{O}[\![G_{\mathbf{Q}_p}]\!])$ factors through $\mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{R}_{\overline{\theta}}^{\mathrm{op}})$, and is naturally isomorphic to*
$$\pi \mapsto \mathrm{Hom}_{E_{\overline{\theta}}}(\check{\mathrm{V}}(P_{\overline{\theta}}), E_{\overline{\theta}}) \otimes_{E_{\overline{\theta}}} (P_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi),$$

where $E_{\overline{\theta}}$ *acts by right multiplication on the* Hom*-space, and* $\widetilde{R}_{\overline{\theta}}$ *acts through* $\check{\mathrm{V}}(P_{\overline{\theta}})$.

Proof. As in Section A.1.61, we write $\overline{M}$ for the image of an object M of $\mathfrak{C}_{\overline{\theta}}$ in $\mathfrak{Q}_{\overline{\theta}}$. Then Lemma A.1.63(2) shows that $\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\overline{\theta}}, M) = \mathrm{Hom}_{\mathfrak{Q}_{\overline{\theta}}}(\overline{P}_{\overline{\theta}}, \overline{M})$ for all $M \in \mathfrak{C}_{\overline{\theta}}$. Hence $\mathrm{End}_{\mathfrak{Q}_{\overline{\theta}}}(\overline{P}_{\overline{\theta}}) \cong E_{\overline{\theta}}$ is a Noetherian profinite $\mathcal{O}$-algebra. By Lemma A.1.63 (3), $\overline{P}_{\overline{\theta}}$ is a projective generator of $\mathfrak{Q}_{\overline{\theta}}$, and it has finite cosocle in $\mathfrak{Q}_{\overline{\theta}}$. Hence Proposition A.1.65 applies, and shows that $\mathrm{Hom}_{\mathfrak{Q}_{\overline{\theta}}}(\overline{P}_{\overline{\theta}}, -)$ an equivalence $\mathfrak{Q}_{\overline{\theta}} \xrightarrow{\sim} \mathrm{Mod}_c(E_{\overline{\theta}}^{\mathrm{op}})$, with quasi-inverse given by $-\widehat{\otimes}_{E_{\overline{\theta}}} \overline{P}_{\overline{\theta}}$. For any object M of $\mathfrak{C}_{\overline{\theta}}$, we now see that

$$\check{\mathrm{V}}(M) = \check{\mathrm{V}}(\overline{M}) \xrightarrow{\sim} \check{\mathrm{V}}\big(\mathrm{Hom}_{\mathfrak{Q}_{\overline{\theta}}}(\overline{P}_{\overline{\theta}}, \overline{M}) \widehat{\otimes}_{E_{\overline{\theta}}} \overline{P}_{\overline{\theta}}\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\overline{\theta}}, M) \otimes_{E_{\overline{\theta}}} \check{\mathrm{V}}(P_{\overline{\theta}}).$$

For the second isomorphism, we use the fact that $\check{\mathrm{V}}$ is exact and cofiltered limit-preserving, and also the fact that $\check{\mathrm{V}}(P_{\overline{\theta}})$ is finitely generated (in fact, free of rank one) over $E_{\overline{\theta}}$, so that $-\widehat{\otimes}_{E_{\overline{\theta}}} \check{\mathrm{V}}(P_{\overline{\theta}})$ can be computed as the usual tensor product. This gives the factorization and the natural isomorphism of (1). Since, as we've already noted, $\mathrm{Hom}_{\mathfrak{Q}_{\overline{\theta}}}(\overline{P}_{\overline{\theta}}, -)$ is an equivalence, it then follows from Proposition 2.4.13 (2) that $\check{\mathrm{V}}$ induces an equivalence $\mathfrak{Q}_{\overline{\theta}} \to \mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}})$. This completes the proof of (1).

We now prove part (2). By Remark 2.4.4, the functor $V^{\dagger}$ is the Pontrjagin dual of $\check{\mathrm{V}}$. So it follows from part (1) that

$$V^{\dagger}(\pi) \cong \big(\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\overline{\theta}}, \pi^{\vee}) \otimes_{E_{\overline{\theta}}} \check{\mathrm{V}}(P_{\overline{\theta}})\big)^{\vee}$$

functorially in π. Since $\check{\mathrm{V}}(P_{\overline{\theta}})$ is a free left $E_{\overline{\theta}}$-module of finite rank, we deduce from this that

$$V^{\dagger}(\pi) \cong \mathrm{Hom}_{E_{\overline{\theta}}}\big(\check{\mathrm{V}}(P_{\overline{\theta}}), E_{\overline{\theta}}\big) \otimes_{E_{\overline{\theta}}} \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\overline{\theta}}, \pi^{\vee})^{\vee}.$$

The result now follows from Lemma 2.2.45 (1). ☐

Remark 2.4.16. Write $\mathcal{Z}_{\overline{\theta}}$ for the centre of the category $\mathfrak{C}_{\overline{\theta}}$. Then there exists a unique isomorphism $Z(\widetilde{R}_{\overline{\theta}}) \to \mathcal{Z}_{\overline{\theta}}$ such that the functor $\check{\mathrm{V}}$ is $Z(\widetilde{R}_{\overline{\theta}})$-linear. This is part of the results of [Paš13]; from our point of view, it is a direct consequence of Theorem 2.4.14, except in case (St), in which case we need the additional fact that $\mathfrak{C}_{\overline{\theta}} \to \mathfrak{Q}_{\overline{\theta}}$ induces an isomorphism of centres, which follows from [Paš13, Theorem 10.80, Corollary 10.77].

Note also that by Proposition 2.1.16, the natural map $R^{\mathrm{ps}}_{\overline{\theta}} \to Z(\widetilde{R}_{\overline{\theta}})$ is an isomorphism. Hence we can reformulate the above as the existence of a unique isomorphism $R^{\mathrm{ps}}_{\overline{\theta}} \to \mathcal{Z}_{\overline{\theta}}$ such that the functor $\check{\mathrm{V}}$ is $R^{\mathrm{ps}}_{\overline{\theta}}$-linear. It then follows that the functors V and $V^{\dagger}$ are also $R^{\mathrm{ps}}_{\overline{\theta}}$-linear. In fact, since $V(\pi) = \check{\mathrm{V}}(\pi^{\vee})^*$, to prove the $R^{\mathrm{ps}}_{\overline{\theta}}$-linearity of V it suffices to check that for all $z \in R^{\mathrm{ps}}_{\overline{\theta}} \subset \widetilde{R}_{\overline{\theta}}$ and all $M \in \mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{R}_{\overline{\theta}})$, we have the equality $z_{M^*} = z_M^*$ as central endomorphisms of M^*. The equality $z_{M^{\vee}} = z_M^{\vee}$ holds by definition (here $\vee$ denotes the Pontrjagin dual). Now, again by definition, we have $z_{M^*} = (z^{\dagger})_{M^{\vee}}$, and $z_M^* = z_M^{\vee}$ (since the functors $M \mapsto M^*$ and $M \mapsto M^{\vee}$ act in the same way on morphisms). Since the anti-involution $\dagger$ is trivial on the centre of $\widetilde{R}_{\overline{\theta}}$, we conclude that $z_{M^*} = z_M^*$, as desired.

Since $\check{\mathrm{V}} : \mathfrak{Q}_{\overline{\theta}} \to \mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}})$ is an equivalence of categories, it also admits a Morita-theoretic description. It is easy to make this explicit, since the natural isomorphism (2.4.15) shows that the equivalences $\mathrm{Hom}_{\mathfrak{Q}_{\overline{\theta}}}(\overline{P}_{\overline{\theta}}, -)$ and $\check{\mathrm{V}}(-)$ differ by tensoring with the free rank one bimodule $\check{\mathrm{V}}(P_{\overline{\theta}})$.

Definition 2.4.17. We define $\widetilde{P}_{\overline{\theta}} := \mathrm{Hom}_{E_{\overline{\theta}}}\big(\check{\mathrm{V}}(P_{\overline{\theta}}), E_{\overline{\theta}}\big) \otimes_{E_{\overline{\theta}}} P_{\overline{\theta}}$ and $\widetilde{E}_{\overline{\theta}} := \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(\widetilde{P}_{\overline{\theta}})$.

If $\overline{\theta}$ is not of type (St), then $\widetilde{P}_{\overline{\theta}}$ is a generator of $\mathfrak{C}_{\overline{\theta}}$ (since the same is true of $P_{\overline{\theta}}$). Since $\check{\mathrm{V}}(P_{\overline{\theta}})$ is an $(\widetilde{R}_{\overline{\theta}}, E^{\mathrm{op}}_{\overline{\theta}})$-bimodule which is free of rank one over both rings, we see that $\mathrm{Hom}_{E_{\overline{\theta}}}\big(\check{\mathrm{V}}(P_{\overline{\theta}}), E_{\overline{\theta}}\big)$ is an $(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}, E_{\overline{\theta}})$-bimodule which is again free of rank one over both rings. Thus $\widetilde{P}_{\overline{\theta}}$ is a projective object of $\mathfrak{C}_{\overline{\theta}}$, with a natural isomorphism

$$\widetilde{R}^{\mathrm{op}}_{\overline{\theta}} \xrightarrow{\sim} \widetilde{E}_{\overline{\theta}} = \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(\widetilde{P}_{\overline{\theta}}). \tag{2.4.18}$$

Hence we can regard the functor $V^{\dagger}$ as a functor

$$V^{\dagger} : \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}} \to \mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}}), \tag{2.4.19}$$

and the functor V as a functor

$$V : \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}} \to \mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{R}_{\overline{\theta}}). \tag{2.4.20}$$

Remark 2.4.21. The main reason to introduce $\widetilde{P}_{\overline{\theta}}$ is to obtain the canonical isomorphism (2.4.18), and therefore avoid making a choice of isomorphism as in Proposition 2.4.13 (3). Throughout the paper we will mostly work with $\widetilde{P}_{\overline{\theta}}$, rather than $P_{\overline{\theta}}$. As an exception, in Section 3.6 we will need an explicit presentation of $\widetilde{E}_{\overline{\theta}}$ when $\overline{\theta} = 1 + \omega$, and so it will be convenient to choose an isomorphism $P_{\overline{\theta}} \xrightarrow{\sim} \widetilde{P}_{\overline{\theta}}$. By construction, this choice is equivalent to the choice of an $E_{\overline{\theta}}$-basis of the free rank-one $E_{\overline{\theta}}$-module $\check{\mathrm{V}}(P_{\overline{\theta}})$. A convenient choice of basis is described in Lemma 3.6.24.

Lemma 2.4.22.

(1) There is a natural isomorphism

$$\check{\mathrm{V}}(-) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\widetilde{P}_{\overline{\theta}}, -)$$

of functors $\mathfrak{C}_{\overline{\theta}} \to \mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}})$. *In particular, the functor*

$$M \mapsto M \,\widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}} \overline{\widetilde{P}_{\overline{\theta}}}$$

provides a quasi-inverse to the equivalence $\check{\mathrm{V}} : \mathfrak{Q}_{\overline{\theta}} \to \mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}})$.

(2) There is a natural isomorphism

$$V^{\dagger}(-) \xrightarrow{\sim} \widetilde{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_{\zeta}} -$$

of functors $\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}} \to \mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}})$.

(3) If $\overline{\theta}$ is not of type (St), *then* $V^{\dagger} : \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}} \to \mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}})$ *is an equivalence.*

Proof. The claimed natural isomorphism in (1) follows from the definition of $\widetilde{P}_{\overline{\theta}}$ together with (2.4.15). The rest of part (1) is then a special case of Proposition A.1.65,

The statement of (2) is dual to the statement of (1) in the same way that the statement of part (2) of Theorem 2.4.14 is dual to the statement of part (1) of that theorem, and is proved in an identical manner.

The final part is immediate from Lemma 2.3.15 (2), since $\widetilde{P}_{\overline{\theta}}$ is a projective generator of $\mathfrak{C}_{\overline{\theta}}$. ∎

2.5. **Localization of smooth representations of** $\mathrm{GL}_2(\mathbf{Q}_p)$**.** In this and the next section we recall various results from our paper [DEG23], and explain how to translate them from the abelian categorical context (which is the focus of that paper) to the stable ∞-categorical context in which we are working. Along the way, we establish various relationships between the stable ∞-categories of representations that we will be considering.

2.5.1. *A chain of projective lines.* One of the main results of [DEG23] is a localization theory for $\mathcal{A}$ over the Zariski site of a scheme X over $\mathbf{F}$, which is a chain of projective lines of length $(p \pm 1)/2$, with ordinary double points (where the sign is equal to $-\zeta(-1)$). There is a bijection between companion pairs of ζ-compatible weights and the irreducible components[3] of X, and we write $X(\sigma|\sigma^{\mathrm{co}})$ for the component corresponding to the pair $\{\sigma, \sigma^{\mathrm{co}}\}$.

If $\sigma = \sigma_{a,b}$ is ζ-compatible, and $b \neq p-1$, there is a morphism ([DEG23, Definition 2.6.1])

$$f_\sigma : \operatorname{Spec} \mathcal{H}_G(\sigma) = \mathbf{A}^1 \to X(\sigma|\sigma^{\mathrm{co}}) \tag{2.5.2}$$

which is an open immersion when $b \neq p-2$ (given by $x \mapsto x^{\pm 1}$, the sign depending on σ), and is a degree two morphism when $b = p-2$ (given by $x \mapsto (x+x^{-1})^{\pm 1}$). The identification of $\operatorname{Spec} \mathcal{H}_G(\sigma)$ with $\mathbf{A}^1$ uses the isomorphism (2.2.34). When $b = p-1$, we furthermore define $f_{\sigma_{a,p-1}} = f_{\sigma_{a,0}}$.

There is a bijection $x \mapsto \mathfrak{B}_x$ between the set of closed points of X and the set of blocks of $\mathcal{A}^{\mathrm{l.adm}}$, given by

$$\mathfrak{B}_x = \bigcup_{\sigma, y} \mathrm{JH}\left(c\text{-}\mathrm{Ind}_{KZ}^G \sigma \otimes_{\mathcal{H}_G(\sigma)} y\right), \tag{2.5.3}$$

i.e. the union of the Jordan–Hölder factors of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma \otimes_{\mathcal{H}_G(\sigma)} y$ for those σ, y with $f_\sigma(y) = x$. Composing with the bijection in Definition 2.4.7, we see that there is also a bijection between the set of closed points of X, and the set of $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy classes of two-dimensional $\overline{\mathbf{F}}_p$-pseudorepresentations $\overline{\theta}$ of $G_{\mathbf{Q}_p}$, having determinant $\overline{\zeta}\omega^{-1}$. Accordingly, we will typically denote a closed point $x \in X$ by $\overline{\theta}$. Note that in the notation of Section 2.1.9, the residue field of $\overline{\theta}$ is $\mathbf{F}_{\overline{\theta}}$.

Remark 2.5.4. Using the identification of closed points of X with $\overline{\mathbf{F}}_p$-pseudorepresentations that we have just described, we have the following explicit formula for the morphism f_σ from (2.5.2): if $\sigma = \sigma_{a,b}$, then for $t \in \overline{\mathbf{F}}_p^\times$ (regarded as an $\overline{\mathbf{F}}_p$-point of $\operatorname{Spec} \mathcal{H}_G(\sigma)$ via the isomorphism (2.2.34)) we have

$$f_\sigma(t) = \mathrm{nr}_{t^{-1}} \omega^{a-1} + \mathrm{nr}_t \overline{\zeta} \omega^{-a}, \tag{2.5.5}$$

while

$$f_\sigma(0) = \mathrm{Ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} (\mathrm{nr}_{-\zeta(p)} \omega_2^{b+1}) \otimes \omega^{a-1}. \tag{2.5.6}$$

Hence the points $0, \infty$ of $X(\sigma|\sigma^{\mathrm{co}})$ correspond to irreducible pseudorepresentations.

[3]In [DEG23, Prop. 2.5.4], we label the components as $X(\tau)$, where τ runs over the set of isomorphisms classes of irreducible cuspidal representations of $\mathrm{GL}_2(\mathbf{F}_p)$ over E with central character ζ. The map $\tau \mapsto \mathrm{JH}(\overline{\tau})$ (i.e. passing to the Jordan–Hölder factors of the semisimplified mod p reduction) induces a bijection between the set of such τ and the set of companion pairs. This bijection is described in more detail in the proof of Proposition 3.7.8 below. Compare also [DEG23, Rem. 2.5.7].

2.5.7. *Localization of abelian categories.* We associate to every closed subset $Y \subseteq |X|$ the full Serre subcategory $\mathcal{A}_Y \subset \mathcal{A}$ of objects all of whose irreducible subquotients are contained in $\bigcup_{\overline{\theta}\in Y} \mathfrak{B}_{\overline{\theta}}$. We write $i_{Y,*} : \mathcal{A}_Y \to \mathcal{A}$ for the fully faithful inclusion functor. Write $U := X \setminus Y$, let $\mathcal{A}_U := \mathcal{A}/\mathcal{A}_Y$ denote the Serre quotient, and write $j_U^* : \mathcal{A} \to \mathcal{A}_U$ for the quotient functor. We write $\mathcal{A}_Y^{\mathrm{fp}}$ for the full subcategory of finitely generated objects of $\mathcal{A}_Y$, and $\mathcal{A}_U^{\mathrm{fp}}$ for the essential image of $\mathcal{A}^{\mathrm{fp}}$ in $\mathcal{A}_U$. When $Y = \{\overline{\theta}\}$ is a singleton, the category $\mathcal{A}_{\{\overline{\theta}\}}$ agrees with the category $\mathcal{A}_{\mathfrak{B}_{\overline{\theta}}}$ of (A.1.29), and we will denote it by $\mathcal{A}_{\overline{\theta}}$ from now on. On the other hand, if $Y = X$ then $\mathcal{A}_Y = \mathcal{A}$, so all the results we recall below about the categories $\mathcal{A}_Y$ apply in particular to $\mathcal{A}$ itself.

The following proposition summarizes some of the fundamental properties of the categories $\mathcal{A}_Y$ established in [DEG23] (see Appendix A.1 for a brief recollection of some material on locally Noetherian categories).

Proposition 2.5.8.

(1) The categories $\mathcal{A}_Y$ and $\mathcal{A}_U$ are locally Noetherian.
(2) The compact objects of $\mathcal{A}_Y$ (resp. $\mathcal{A}_U$) are the finitely generated objects $\mathcal{A}_Y^{\mathrm{fp}}$ (resp. $\mathcal{A}_U^{\mathrm{fp}}$).
(3) The inclusion $i_{Y,*} : \mathcal{A}_Y \to \mathcal{A}$ is exact, preserves colimits, preserves injectives, and preserves compact objects.
(4) The quotient functor $j_U^* : \mathcal{A} \to \mathcal{A}_U$ admits an exact and fully faithful right adjoint $j_{U,*} : \mathcal{A}_U \to \mathcal{A}$. In particular, $\mathcal{A}_Y$ is a localizing subcategory of $\mathcal{A}$.
(5) The assignment $U \mapsto \mathcal{A}_U$ defines a stack $\mathcal{A}_\bullet$ of abelian categories on X_{Zar}.

Proof. The first four points are immediate from [DEG23, Lem. 3.1.2, 3.1.7, Prop. 3.1.13 (2), Cor. 3.5.8], together with the observation that by [DEG23, Proposition A.1.1], the Noetherian objects of $\mathcal{A}$, resp. $\mathcal{A}_Y$, are precisely the compact objects, and since $\mathcal{A}_Y \subset \mathcal{A}$ is a Serre subcategory, the inclusion preserves Noetherian objects. The final point is [DEG23, Theorem 3.3.1]. □

The exact functor $i_{Y,*}$ induces an exact functor $\widehat{i}_{Y,*} : \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fp}}) \to \mathrm{Pro}(\mathcal{A}^{\mathrm{fp}})$, which by the adjoint functor theorem has a left adjoint $\widehat{i}_Y^* : \mathrm{Pro}(\mathcal{A}^{\mathrm{fp}}) \to \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fp}})$. This functor was denoted $\widehat{(-)}_Y$ in [DEG23].

Lemma 2.5.9. *The functor $\widehat{i}_Y^* : \mathrm{Pro}(\mathcal{A}^{\mathrm{fp}}) \to \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fp}})$ is exact.*

Proof. By [DEG23, Cor. 3.5.5], the functor

$$\widehat{i}_Y^*|_{\mathcal{A}^{\mathrm{fp}}} : \mathcal{A}^{\mathrm{fp}} \to \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fp}})$$

is exact. The lemma then follows immediately from Lemma A.1.6 (2). □

Remark 2.5.10. If $Y \subseteq X$ consists of a single closed point $\overline{\theta}$, then we will typically write $i_{\overline{\theta},*}$ and $\widehat{i}_{\overline{\theta}}^*$ rather than $i_{Y,*}$ and $\widehat{i}_Y^*$.

Remark 2.5.11. Passing to Ind-extensions, we obtain an adjoint pair of functors, still denoted $(\widehat{i}_Y^*, \widehat{i}_{Y,*})$, between $\mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}^{\mathrm{fp}}$ and $\mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}_Y^{\mathrm{fp}}$. The functor $\widehat{i}_Y^*$ was also denoted $\widehat{(-)}_Y$ in [DEG23].

If $\overline{\theta}$ is a closed point of U, then the functor $\widehat{i}_{\overline{\theta}}^* : \mathcal{A}^{\mathrm{fp}} \to \mathrm{Pro}\,\mathcal{A}_{\overline{\theta}}^{\mathrm{fp}}$ factors through $\mathcal{A}_U^{\mathrm{fp}}$, and we continue to denote the resulting functor $\mathcal{A}_U^{\mathrm{fp}} \to \mathrm{Pro}\,\mathcal{A}_{\overline{\theta}}^{\mathrm{fp}}$ by $\widehat{i}_{\overline{\theta}}^*$. Bearing in mind this slight abuse of notation, we have a natural isomorphism

$$\widehat{i}^*_{\overline{\theta}} \xrightarrow{\sim} \widehat{i}^*_{\overline{\theta}} j^*_U : \mathcal{A}^{\mathrm{fp}} \to \operatorname{Pro} \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}. \tag{2.5.12}$$

Similarly, the composite $\widehat{i}^*_{\overline{\theta}} j_{U,*}$, which is *a priori* a functor $\mathcal{A}^{\mathrm{fp}}_U \to \operatorname{Ind} \operatorname{Pro} \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}$, is actually valued in $\operatorname{Pro} \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}$, and there is a natural isomorphism

$$\widehat{i}^*_{\overline{\theta}} \xrightarrow{\sim} \widehat{i}^*_{\overline{\theta}} j_{U,*} : \mathcal{A}^{\mathrm{fp}}_U \to \operatorname{Pro} \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}; \tag{2.5.13}$$

to see this, it suffices to precompose the Ind-extension of (2.5.12) with $j_{U,*}$, and use the fact that the counit $j^*_U j_{U,*} \to 1$ is an isomorphism. See also [DEG23, Section 3.5.21] for related material.

The functors $\widehat{i}^*_Y$ and $j_{U,*}$ can be explicitly computed in some cases. By [DEG23, Lem. 3.5.2], if π is an object of $\mathcal{A}^{\mathrm{fp}}$, there is a natural isomorphism

$$\widehat{i}^*_Y \pi \xrightarrow{\sim} \lim \pi', \tag{2.5.14}$$

where π' runs over the cofiltered set of quotients of π lying in $\mathcal{A}^{\mathrm{fp}}_Y$. Furthermore, the unit of adjunction $\pi \to \widehat{i}_{Y,*}\widehat{i}^*_Y \pi$ corresponds under (2.5.14) to the inverse limit of the corresponding quotient maps. When σ is a Serre weight, and $\pi = c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma$, we can produce an explicit cofinal set in (2.5.14) in the following way.

Definition 2.5.15. Let σ be a Serre weight, and let $Y \subset X$ be a closed subset.

(1) If $f^{-1}_\sigma(Y)$ is finite, we define $f_Y \in \mathcal{H}(\sigma) = \mathbf{F}[T_p]$ to be the unique monic squarefree generator of the ideal of $f^{-1}_\sigma(Y)$. If $Y = \{\overline{\theta}\}$ is a singleton, we write $f_{\overline{\theta}}$ for f_Y.
(2) If $f^{-1}_\sigma(Y)$ is infinite, we define $f_Y \coloneqq 0$.

When $Y = \{\overline{\theta}\}$ is a single $\mathbf{F}$-rational point, the polynomial $f_{\overline{\theta}}$ has degree one, unless $\sigma = \sigma_{a,p-2}$, in which case $f_{\overline{\theta}}$ can have degree two (and can be reducible). Furthermore, by [DEG23, Lemma 3.1.8(2)], $(c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma)/f_{\overline{\theta}}(c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma)$ is the maximal multiplicity-free quotient of $c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma$ which is an object of $\mathcal{A}_{\overline{\theta}}$.

Lemma 2.5.16. *Let $\sigma \coloneqq \sigma_{a,b}$ be a Serre weight, and let $Y \subset X$ be a closed subset.*

(1) (a) The set $\{c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma / f^n_Y c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma : n \geq 1\}$ is cofinal in (2.5.14)*, and so we have*

$$\widehat{i}^*_Y c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma \xrightarrow{\sim} \lim_n (c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma)/f^n_Y c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma. \tag{2.5.17}$$

(b) The unit morphism

$$c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma \to j_{U,*}j^*_U c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma$$

is given by the natural map

$$c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma \to (c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma)[1/f_Y]. \tag{2.5.18}$$

(2) Let $\overline{\theta} \in X$ be a closed point, and let $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be a semisimple Galois representation with trace $\overline{\theta}$. Then $\widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma \neq 0$ if and only if σ is a Serre weight of $\overline{\rho}$.

Proof. Part (1a) is [DEG23, Cor. 3.5.3], while Part (1b) is [DEG23, Prop. 3.1.13(1)]. We now prove part (2). By part (1a), $\widehat{i}^*_{\overline{\theta}}(c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma) \neq 0$ if and only if $f_{\{\overline{\theta}\}}$ is not a unit in $\mathcal{H}(\sigma)$. By Definition 2.5.15, this occurs if and only if there exists $t \in \operatorname{Spec} \mathcal{H}(\sigma)$ such that $f_\sigma(t) = \overline{\theta}$. By (2.5.5) and (2.5.6), this occurs if and only if

$t \in \overline{\mathbf{F}}_p^\times$ and $\overline{\theta} = \mathrm{nr}_{t^{-1}}\omega^{a-1} + \mathrm{nr}_t\overline{\zeta}\omega^{-a}$, or $t = 0$ and $\overline{\theta} = \mathrm{Ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}}(\mathrm{nr}_{-\zeta(p)}\omega_2^{b+1})\otimes\omega^{a-1}$. Bearing in mind Definition 2.1.4, this is equivalent to σ being a Serre weight of $\overline{\rho}$. $\square$

Remark 2.5.19. In the setting of Lemma 2.5.16 (1a), if $f_Y = 0$ then $c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma$ is an object of $\mathcal{A}_Y^{\mathrm{fg}}$, and so (2.5.17) exhibits its completion as a constant pro-object. Similarly, the localization $j_U^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma$ is equal to zero, as is the right-hand side of (2.5.18), which is by definition the $\mathcal{H}_G(\sigma)$-module localization

$$c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \otimes_{\mathcal{H}_G(\sigma)} \mathcal{H}_G(\sigma)[1/f_Y].$$

2.5.20. *Tensor product and completion.* If P is a projective object of $\mathfrak{C}_{\overline{\theta}}$ with finite length cosocle, then we have seen in Lemma 2.3.11 that $E \coloneqq \mathrm{End}_{\mathfrak{C}}(P)$ is a compact Noetherian $\mathcal{O}$-algebra and that $P \in \mathrm{Mod}_c(E)$. Hence P is a right $\mathcal{O}[\![G]\!]_\zeta$-module in $\mathrm{Mod}_c(E)$, and so Lemma A.1.55 shows that the functor

$$P \otimes_{\mathcal{O}[\![G]\!]_\zeta} - : \mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Mod}_c(\mathcal{O})$$

can be lifted through the forgetful functor $\mathrm{Mod}_c(E) \to \mathrm{Mod}_c(\mathcal{O})$ to a functor

$$P \otimes_{\mathcal{O}[\![G]\!]_\zeta} - : \mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Mod}_c(E). \tag{2.5.21}$$

Since $\mathrm{Mod}_c(E) = \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(E)$, this is an instance of Lemma A.10.10 (2f). Since $\mathcal{A}^{\mathrm{fp}}$ is a full subcategory of $\mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)$, the restriction of (2.5.21) to $\mathcal{A}^{\mathrm{fp}}$ extends uniquely to a cofiltered limit-preserving functor

$$P\,\widehat{\otimes}_{\mathcal{O}[\![G]\!]_\zeta} - : \mathrm{Pro}(\mathcal{A}^{\mathrm{fp}}) \to \mathrm{Mod}_c(E). \tag{2.5.22}$$

This is an instance of Lemma A.10.10 (2g), which strictly speaking considers the Pro-extension to all of $\mathrm{Pro}\big(\mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)\big)$, but in what follows we will be focussed on its full subcategory $\mathrm{Pro}(\mathcal{A}^{\mathrm{fp}})$. In particular, in that context we have the following exactness result.

Lemma 2.5.23. *The functor $P \otimes_{\mathcal{O}[\![G]\!]_\zeta} (-)$ is exact on $\mathcal{A}^{\mathrm{fp}}$, and hence the Pro-extended functor $P\,\widehat{\otimes}_{\mathcal{O}[\![G]\!]_\zeta}(-) : \mathrm{Pro}(\mathcal{A}^{\mathrm{fp}}) \to \mathrm{Mod}_c(E)$ is also exact.*

Proof. The forgetful functors $\mathrm{Mod}_c(E) \to \mathrm{Mod}(E)$ and $\mathrm{Mod}(E) \to \mathrm{Mod}(\mathcal{O})$ are exact, and so it suffices to verify the claimed exactness after applying their composite, i.e. after forgetting the topology and E-module structure on $P \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$. Doing this, we are reduced to considering the tensor product with P and π thought of as abstract $\mathcal{O}[\![G]\!]_\zeta$-modules, and the claimed exactness follows from the Tor-vanishing of Lemma 2.2.46 (2). $\square$

Regarding $\overline{\theta}$ as a closed point of X, we may consider the completion functor $\widehat{i}^*_{\overline{\theta}}$; we will use it together with (2.5.22) to relate (2.5.21) to its restriction to $\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}$. We first note the following.

Lemma 2.5.24. *If π is an object of $\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}$, then $P \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$ is a finite length E-module.*

Proof. It suffices to show that $P \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi$ has finite $\mathcal{O}$-length, or equivalently that $(P \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi)^\vee$ has finite $\mathcal{O}$-length. By Lemma 2.2.45 (1), $(P \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi)^\vee$ is isomorphic to $\mathrm{Hom}_{\mathcal{O}[\![G]\!]_\zeta}(\pi, P^\vee) = \mathrm{Hom}_{\mathcal{A}}(\pi, P^\vee)$. By Lemma 2.3.4, π has finite $\mathcal{A}$-length, and so $\mathrm{Hom}_{\mathcal{A}}(\pi, P^\vee) = \mathrm{Hom}_{\mathcal{A}}(\pi, \mathrm{soc}_{\mathcal{A},n} P^\vee)$ for some $n \geq 0$, by Lemma A.1.23(1). This concludes the proof since $\mathrm{soc}_{\mathcal{A},n} P^\vee$ has finite $\mathcal{A}$-length, by the dual to Lemma 2.3.13. $\square$

Lemma 2.5.24 shows that (2.5.21) restricts to a functor

$$P \otimes_{\mathcal{O}[\![G]\!]_\zeta} (-) : \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}} \to \mathrm{Mod}^{\mathrm{f.l.}}(E). \tag{2.5.25}$$

Because of the equivalence (A.1.37), the restriction of (2.5.22) to $\mathrm{Pro}(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})$ therefore coincides with the Pro-extension

$$\mathrm{Pro}(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}) \to \mathrm{Pro}\big(\mathrm{Mod}^{\mathrm{f.l.}}(E)\big) \xrightarrow{\sim} \mathrm{Mod}_c(E)$$

of (2.5.25), which we will also denote $P \mathbin{\widehat{\otimes}}_{\mathcal{O}[\![G]\!]_\zeta} (-)$.

Lemma 2.5.26. *If P is a projective object of $\mathfrak{C}_{\overline{\theta}}$ with finite length cosocle and $E = \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P)$, then the natural transformation*

$$P \otimes_{\mathcal{O}[\![G]\!]_\zeta} (-) \to P \mathbin{\widehat{\otimes}}_{\mathcal{O}[\![G]\!]_\zeta} \widehat{i}_{\overline{\theta},*} \widehat{i}^*_{\overline{\theta}}(-)$$

of functors $\mathcal{A}^{\mathrm{fp}} \to \mathrm{Mod}_c(E)$ (induced by the unit of adjunction $\mathrm{id}_{\mathrm{Pro}\,\mathcal{A}^{\mathrm{fp}}} \to \widehat{i}_{\overline{\theta},*} \widehat{i}^*_{\overline{\theta}}$) is an isomorphism.

Proof. Choose $\pi \in \mathcal{A}^{\mathrm{fp}}$, and let $P_n = P/\mathrm{rad}_n P$. By Lemma 2.3.13, P_n has finite $\mathfrak{C}$-length, and

$$P \xrightarrow{\sim} \varprojlim_n P_n.$$

Similarly, let $\pi_n := \pi^{(n)}_{\overline{\theta}}$ be the quotient of π defined in Lemma 2.3.14. By the cofinality statement in Lemma 2.3.14, the isomorphism (2.5.14) can be rewritten as

$$\widehat{i}_{\overline{\theta},*} \widehat{i}^*_{\overline{\theta}} \pi \xrightarrow{\sim} \lim_n \pi_n.$$

By Lemma 2.3.14 and Lemma A.1.23 respectively, we have isomorphisms of finite torsion $\mathcal{O}$-modules

$$\mathrm{Hom}_{\mathcal{A}}(\pi_n, \mathrm{soc}^n_{\mathcal{A}} P^\vee) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(\pi, \mathrm{soc}^n_{\mathcal{A}} P^\vee)$$

and

$$\mathrm{Hom}_{\mathcal{A}}(\pi_n, \mathrm{soc}^n_{\mathcal{A}} P^\vee) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(\pi_n, P^\vee)$$

(induced by pullback and pushforward, respectively). Applying $\mathrm{Hom}_{\mathcal{O}}(-, E/\mathcal{O})$, we obtain by Lemma 2.2.45 (1) a pair of isomorphisms

$$P_n \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi \xrightarrow{\sim} P_n \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi_n,$$

and

$$P \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi_n \xrightarrow{\sim} P_n \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi_n.$$

We thus find that

$$\begin{aligned} P \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi \xrightarrow{\sim} \varprojlim_n (P_n \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi) &\xrightarrow{\sim} \varprojlim_n (P_n \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi_n) \\ &\xrightarrow{\sim} \varprojlim_n (P \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi_n) \xrightarrow{\sim} P \mathbin{\widehat{\otimes}}_{\mathcal{O}[\![G]\!]_\zeta} \widehat{i}_{\overline{\theta},*} \widehat{i}^*_{\overline{\theta}} \pi \end{aligned}$$

(the first isomorphism being an application of Lemma A.1.51 in the category $\mathrm{Mod}_c(E)$, and the final isomorphism holding because $P \mathbin{\widehat{\otimes}}_{\mathcal{O}[\![G]\!]_\zeta} (-)$ is cofiltered limit-preserving by construction), as required. ∎

2.6. **Derived categories of smooth representations.** We now write $D(\mathcal{A}_Y)$ for the (unbounded) derived stable ∞-category of the Grothendieck category $\mathcal{A}_Y$. (See [EGH25, App. A.1] and Appendix A.2 for a brief recollection of the material on stable ∞-categories that we use below, and Appendix A.6 for a recollection of t-structures.)

Since the inclusion $i_{Y,*} : \mathcal{A}_Y \to \mathcal{A}$ is exact and preserves colimits, it induces (by Lemmas A.7.26 and A.7.22) a continuous t-exact functor

$$i_{Y,*} : D(\mathcal{A}_Y) \to D(\mathcal{A}). \tag{2.6.1}$$

(Here and below, we will use the same notation for exact functors between abelian categories, and their t-exact extensions to derived categories.) Recall that in the notation of Definition A.7.6 we write $D_{\mathcal{A}_Y}(\mathcal{A})$ for the full subcategory of $D(\mathcal{A})$ consisting of objects all of whose cohomologies lie in $\mathcal{A}_Y$, and similarly $D_{\mathcal{A}_Y}(\mathcal{O}[\![G]\!]_\zeta)$ (where $D(\mathcal{O}[\![G]\!]_\zeta)$ is the derived category of $\mathcal{O}[\![G]\!]_\zeta$-modules).

Lemma 2.6.2. *The t-exact continuous functor $i_{Y,*} : D(\mathcal{A}_Y) \to D(\mathcal{A})$ is fully faithful, and induces an equivalence $D(\mathcal{A}_Y) \xrightarrow{\sim} D_{\mathcal{A}_Y}(\mathcal{A})$. Furthermore we have a commutative diagram of t-exact fully faithful continuous functors*

$$\begin{tikzcd}
D(\mathcal{A}_Y) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & D_{\mathcal{A}_Y}(\mathcal{A}) \arrow[r, hook] \arrow[d, "\simeq"] & D(\mathcal{A}) \arrow[d, "\simeq"] \\
D_{\mathcal{A}_Y}(\mathcal{O}[\![G]\!]_\zeta) \arrow[r, "="] & D_{\mathcal{A}_Y}(\mathcal{O}[\![G]\!]_\zeta) \arrow[r, hook] & D_{\mathcal{A}}(\mathcal{O}[\![G]\!]_\zeta)
\end{tikzcd}$$

Proof. We begin by noting that since we have a colimit-preserving, exact and fully faithful functor $\mathcal{A} \to \mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$, we have a continuous t-exact functor $D(\mathcal{A}) \to D(\mathcal{O}[\![G]\!]_\zeta)$. By an identical argument to the proof of [EGH25, Prop. E.2.2], taking into account Remark 2.2.24, this functor satisfies the hypotheses of Theorem A.7.27, so it induces the right-most vertical equivalence in the commutative diagram above.

In turn, the right-most equivalence restricts to the middle equivalence. It therefore suffices to show that the composite

$$D(\mathcal{A}_Y) \to D(\mathcal{A}) \to D(\mathcal{O}[\![G]\!]_\zeta)$$

is fully faithful and induces an equivalence $D(\mathcal{A}_Y) \to D_{\mathcal{A}_Y}(\mathcal{O}[\![G]\!]_\zeta)$. Since $i_{Y,*} : \mathcal{A}_Y \to \mathcal{A}$ preserves injectives by [DEG23, Cor. 3.5.8], and its right adjoint has finite cohomological amplitude by [DEG23, Cor. 3.6.3], this follows from an application of Theorem A.7.27 to the fully faithful functor $\mathcal{A}_Y \to \mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$. □

We recall the following standard lemma.

Lemma 2.6.3. *If A is a coherent ring and M is a finitely generated A-module for which* $\mathrm{RHom}_A(M, –)$ *is of bounded amplitude, then M is compact as an object of $D(A)$.*

Proof. By for example [Wei94, Lem. 4.1.6], M has a finite length resolution by finitely generated projective A-modules, and is therefore compact in $D(A)$ (see for example [BN93, Prop. 4.6]). □

Proposition 2.6.4. *The objects of $\mathcal{A}^{\mathrm{fp}}$ are compact in $D(\mathcal{A})$.*

Proof. By Lemma 2.6.2, there is a fully faithful continuous functor $D(\mathcal{A}) \hookrightarrow D(\mathcal{O}[\![G]\!]_\zeta)$. By Lemma 2.2.23, the ring $\mathcal{O}[\![G]\!]_\zeta$ is coherent, so it follows from Lemma 2.6.3 that it suffices to show that for any finitely generated object π of $\mathcal{A}$, the functor $\mathrm{RHom}_{\mathcal{O}[\![G]\!]_\zeta}(\pi, -)$ has bounded amplitude.

To this end, recall that by for example [Dot25, Section 2.4.14] we have a short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_N^G \delta\pi \to c\text{-}\mathrm{Ind}_{KZ}^G \pi \to \pi \to 0$$

where N is the normalizer of the Iwahori subgroup Iw, and δ is the nontrivial quadratic character of $N/\mathrm{Iw}Z$. We can therefore reduce to showing that for any finitely generated π, and any subgroup H of G which contains $\mathrm{Iw}Z$ with finite coprime-to-p index, the functor $\mathrm{RHom}_{\mathcal{O}[\![G]\!]_\zeta}(c\text{-}\mathrm{Ind}_H^G \pi, -)$ has bounded amplitude.

Since π is finitely generated, and G/KZ is countable, π is countably generated over $\mathcal{O}$. Hence we can write π as a countably-indexed filtered colimit $\mathrm{colim}_i V_i$ of finitely generated H-subrepresentations. Then

$$\mathrm{RHom}_{\mathcal{O}[\![G]\!]_\zeta}(c\text{-}\mathrm{Ind}_H^G \pi, -) = \mathrm{Rlim}_i \mathrm{RHom}_{\mathcal{O}[\![H]\!]_\zeta}(V_i, -).$$

Using the fact that $R^j \lim_i = 0$ for $j \geq 2$ (since the limit is countably indexed) we are reduced to checking that $\mathrm{RHom}_{\mathcal{O}[\![H]\!]_\zeta}(V_i, -)$ has bounded amplitude. Since V_i is an H-linear direct summand of $c\text{-}\mathrm{Ind}_{\mathrm{Iw}_1 Z}^H(V_i)$ (since the index of $\mathrm{Iw}_1 Z$ in H is prime-to-p), it suffices to check that $\mathrm{RHom}_{\mathcal{O}[\![\mathrm{Iw}_1 Z]\!]_\zeta}(V_i, -)$ has bounded amplitude. This is a consequence of the fact that the ring $\mathcal{O}[\![\mathrm{Iw}_1 Z]\!]_\zeta$ has finite global dimension, which follows from [Ven02, Thm. 3.26]. More precisely, the decomposition (1.3.1) shows that we have an isomorphism of rings

$$\mathcal{O}[\![\mathrm{Iw}_1 Z]\!]_\zeta \xrightarrow{\sim} \mathcal{O}[\![\mathrm{Iw}_1/Z_1]\!]$$

and since $p \geq 5$, we see that Iw_1 is torsion-free, and thus so is Iw_1/Z_1 (being a direct factor of the former group). Thus [Ven02, Thm. 3.26] applies to the target of this isomorphism. $\square$

Remark 2.6.5. If $p = 2$ or 3 then Iw_1 has a nontrivial p-torsion element, namely $-1 \in \mathbf{Q}_p$ when $p = 2$, and a conjugate of $\zeta_3 \in \mathbf{Q}_p(\zeta_3) \subset M_2(\mathbf{Q}_p)$ when $p = 3$. So we anticipate that the analogue of Proposition 2.6.4 would fail in these cases.

Definition 2.6.6. We write $D^b_{\mathrm{fp}}(\mathcal{A}_Y)$ for $D^b_{\mathcal{A}_Y^{\mathrm{fp}}}(\mathcal{A}_Y)$, and similarly $D^b_{\mathrm{fp}}(\mathcal{A}_U)$ for $D^b_{\mathcal{A}_U^{\mathrm{fp}}}(\mathcal{A}_U)$.

Corollary 2.6.7. *Let $Y \subseteq X$ be a closed subset with open complement $U \subseteq X$. Then*

(1) The canonical functors $D^b(\mathcal{A}_Y^{\mathrm{fp}}) \to D^b_{\mathrm{fp}}(\mathcal{A}_Y)$ and $D^b(\mathcal{A}_U^{\mathrm{fp}}) \to D^b_{\mathrm{fp}}(\mathcal{A}_U)$ are equivalences.

(2) The categories $D(\mathcal{A}_Y)$ and $D(\mathcal{A}_U)$ are compactly generated, and we have $D(\mathcal{A}_Y)^c = D^b_{\mathrm{fp}}(\mathcal{A}_Y)$ and $D(\mathcal{A}_U)^c = D^b_{\mathrm{fp}}(\mathcal{A}_U)$. Hence the natural maps $\mathrm{Ind}\, D^b_{\mathrm{fp}}(\mathcal{A}_Y) \xrightarrow{\sim} D(\mathcal{A}_Y)$ and $\mathrm{Ind}\, D^b_{\mathrm{fp}}(\mathcal{A}_U) \to D(\mathcal{A}_U)$ are equivalences.

(3) $j_U^* : \mathcal{A} \to \mathcal{A}_U$ induces equivalences

$$D(\mathcal{A})/D(\mathcal{A}_Y) \xrightarrow{\sim} D(\mathcal{A}_U)$$

and

$$D^b_{\mathrm{fp}}(\mathcal{A})/D^b_{\mathrm{fp}}(\mathcal{A}_Y) \xrightarrow{\sim} D^b_{\mathrm{fp}}(\mathcal{A}_U).$$

(4) The functor $j_{U,*} : D(\mathcal{A}_U) \to D(\mathcal{A})$ is fully faithful, continuous, and is right adjoint to $j_U^* : D(\mathcal{A}) \to D(\mathcal{A}_U)$.

Proof. This is immediate from Proposition A.7.29 (whose hypotheses hold by Lemma 2.6.2, Proposition 2.5.8, and Proposition 2.6.4). □

We now give an explicit set of compact generators (in the sense recalled in Remark A.2.22) for the compactly generated categories $D(\mathcal{A})$ and $D(\mathcal{A}_U)$.

Corollary 2.6.8. *Letting σ range over the ζ-compatible Serre weights, the collection $\{c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma\}$ is a set of compact generators of $D(\mathcal{A})$, and the collection $\{j_U^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma\}$ is a set of compact generators of $D(\mathcal{A}_U)$.*

Proof. The second statement follows from the first. For the first, by Proposition A.7.13 it suffices to show that $\{c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma\}$ is a set of weak generators of $\mathcal{A}$, i.e. that for each $\pi \in \mathcal{A}$, there exists a non-zero map $c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \to \pi$ for some σ. This is standard; see e.g. the proof of [DEG23, Lem. 2.2.3]. □

Finally we turn to the exact functor $\widehat{i}_Y^* : \mathrm{Pro}(\mathcal{A}^{\mathrm{fp}}) \to \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fp}})$ from Lemma 2.5.9. Note that since $i_{Y,*}$ is exact, it induces a t-exact functor

$$\widehat{i}_{Y,*} : \mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to \mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}). \tag{2.6.9}$$

Lemma 2.6.10. *Let*

$$\widehat{i}_Y^* : \mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}) \to \mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_Y) \tag{2.6.11}$$

be the Pro-*extension of the t-exact functor $D^b_{\mathrm{fp}}(\mathcal{A}) \to \mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_Y)$ whose restriction to hearts is $\widehat{i}_Y^*$. Then $\widehat{i}_Y^*$ is left adjoint to $\widehat{i}_{Y,*}$.*

Proof. This is immediate from Lemma 2.5.9 and Proposition A.8.17 (applied with $\mathcal{C} = \mathcal{A}^{\mathrm{fp}}, \mathcal{C}' = \mathcal{A}_Y^{\mathrm{fp}}$, and $f = \widehat{i}_Y^*$). □

2.7. **Finiteness of some** Ext **groups.** In this subsection we prove some results on extension groups between parabolic inductions which we will use in the proof of our main theorem (see Proposition 5.4.21).

2.7.1. *Removing the central character.* We will make significant use of the results of Heyer [Hey23; Hey24] on parabolic induction and its adjoints. Since Heyer's papers work with the category $\mathrm{sm}.\,G$ of all smooth $\mathcal{O}$-representations of G, possibly without a central character, we begin by explaining how to relate Ext groups in $\mathcal{A}$ and $\mathrm{sm}.\,G$.

By Lemma 2.6.2, the inclusion $\mathcal{A} \to \mathrm{Mod}(\mathcal{O}[\![G]\!]_\zeta)$ induces a t-exact equivalence

$$D(\mathcal{A}) \to D_{\mathrm{sm}}(\mathcal{O}[\![G]\!]_\zeta),$$

where the right-hand side denotes the full subcategory of complexes with smooth cohomology. By [EGH25, Proposition E.2.2], we have a commutative diagram

$$\begin{tikzcd} D(\mathrm{sm}.\,G) \arrow[r, "\sim"] & D_{\mathrm{sm}}(\mathcal{O}[\![G]\!]) \\ D(\mathcal{A}) \arrow[u] \arrow[r, "\sim"] & D_{\mathrm{sm}}(\mathcal{O}[\![G]\!]_\zeta). \arrow[u] \end{tikzcd} \tag{2.7.2}$$

Writing $A \coloneqq \mathcal{O}[\![\mathbf{Q}_p^\times]\!]$ and $B \coloneqq \mathcal{O}$, we can thus study the relationship between $\mathrm{Ext}^i_{\mathcal{A}}$ and $\mathrm{Ext}^i_{\mathrm{sm}.\,G}$ as a particular case of the following problem: Suppose given a morphism $\zeta : A \to B$ of commutative rings and a (not necessarily commutative) A-algebra R, and write $S \coloneqq B \otimes_A R$. Then, if M and N are two S-modules, which can then also be regarded as R-modules via the canonical morphism $R \to S$, what is the relationship between $\mathrm{Ext}^\bullet_R(M,N)$ and $\mathrm{Ext}^\bullet_S(M,N)$?

To begin with, if M is an R-module and N is an S-module, then deriving the tensor–Hom adjunction gives

$$\mathrm{RHom}_R(M,N) \xrightarrow{\sim} \mathrm{RHom}_S(B \otimes^L_A M, N).$$

Now, if M is itself an S-module, then

$$B \otimes^L_A M \xrightarrow{\sim} (B \otimes^L_A B) \otimes^L_B M.$$

Thus we obtain, for S-modules M and N, an isomorphism

$$\mathrm{RHom}_R(M,N) \xrightarrow{\sim} \mathrm{RHom}_S\big((B \otimes^L_A B) \otimes^L_B M, N\big). \tag{2.7.3}$$

Lemma 2.7.4. *There exists an isomorphism of functors $D(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\mathcal{O})$:*

$$\mathrm{RHom}_{\mathrm{sm.}\,G}(\text{–},\text{–}) \cong \mathrm{RHom}_{\mathcal{A}}(\text{–},\text{–}) \oplus \mathrm{RHom}_{\mathcal{A}}(\text{–},\text{–})^{\oplus 2}[-1] \oplus \mathrm{RHom}_{\mathcal{A}}(\text{–},\text{–})[-2].$$

Proof. We apply the above discussion with $A = \mathcal{O}[\![\mathbf{Q}_p^\times]\!]$, $B = \mathcal{O}$, and the morphism $A \to B$ being $\zeta : \mathcal{O}[\![\mathbf{Q}_p^\times]\!] \to \mathcal{O}$; and we take $R = \mathcal{O}[\![G]\!]$ and $S = \mathcal{O}[\![G]\!]_\zeta$. Choosing a uniformizer of $\mathbf{Q}_p$ and an isomorphism $1 + p\mathbf{Z}_p \xrightarrow{\sim} \mathbf{Z}_p$ then induces an isomorphism

$$A \cong \mathcal{O}[\![U]\!][V^{\pm 1}][\mathbf{F}_p^\times],$$

and we note that the group ring (denoted $[\mathbf{F}_p^\times]$) of the cyclic group $\mathbf{F}_p^\times$ is semisimple in residue characteristic p. Computing with a Koszul complex in the variables U and V, we see that

$$B \otimes^L_A B \xrightarrow{\sim} B[2] \oplus B[1]^{\oplus 2} \oplus B.$$

Using (2.7.3) we have

$$\begin{aligned}&\mathrm{RHom}_{\mathcal{O}[\![G]\!]}(M,N) \xrightarrow{\sim} \\ &\quad \mathrm{RHom}_{\mathcal{O}[\![G]\!]_\zeta}(M,N) \oplus \mathrm{RHom}_{\mathcal{O}[\![G]\!]_\zeta}(M,N)^{\oplus 2}[-1] \oplus \mathrm{RHom}_{\mathcal{O}[\![G]\!]_\zeta}(M,N)[-2].\end{aligned}$$

The result now follows from (2.7.2). $\square$

2.7.5. *Extensions of universal parabolic inductions.* We begin by introducing some notation involving the connected components of X. We refer to Section 3.7 for motivation and some related material.

Definition 2.7.6. Let $\sigma|\sigma^{\mathrm{co}}$ be a companion pair of Serre weights.

(1) Define a polynomial $f \in \mathbf{F}[t]$ by

$$f(t) := \begin{cases} t & \text{if } \sigma|\sigma^{\mathrm{co}} \text{ is of type (gen)} \\ t(t^2 - \zeta(p)) & \text{if } \sigma|\sigma^{\mathrm{co}} \text{ is of type (scalar) or (St)}. \end{cases}$$

(2) Let $Y_{\mathrm{bad}} \subset X$ be the closed subset determined by the finite set of points which become of type (ssg), (scalar) or (St) after a finite extension of $\mathcal{O}$. Let U_{good} be the complement of Y_{bad}.
(3) If $\{\sigma, \sigma^{\mathrm{co}}\}$ is a pair of companion weights, let

$$U(\sigma|\sigma^{\mathrm{co}}) := U_{\mathrm{good}} \cap X(\sigma|\sigma^{\mathrm{co}}),$$

We fix for the rest of this section a companion pair of Serre weights $\{\sigma, \sigma^{\mathrm{co}}\}$, and a choice of $\sigma_1, \sigma_2 \in \{\sigma, \sigma^{\mathrm{co}}\}$ (possibly with $\sigma_1 = \sigma_2$). Recall from Section 2.2.32 that we write

$$\mathcal{H}_G(\sigma_i) = \mathrm{End}_G(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_i),$$
$$\mathcal{H}_T = \mathrm{End}_T(c\text{-}\mathrm{Ind}_{T_0 Z}^T 1) \xrightarrow{\sim} \mathbf{F}[X^{\pm 1}],$$

where $T_0 = T(\mathbf{Z}_p)$ is the maximal compact subgroup of the diagonal torus T. We let $\chi_i = (\sigma_i)^{\mathrm{Iw}_1}$, viewed as a character of $T_0 Z$. In Section 2.2.32, we have canonically identified the endomorphism algebras $\mathrm{End}_T(c\text{-}\mathrm{Ind}_{T_0 Z}^T \chi_i)$ with $\mathcal{H}_T$, and via the Satake morphisms, we have identified each of the $\mathcal{H}_G(\sigma_i)$ with $\mathbf{F}[T_p]$ (with T_p being identified with the spherical Hecke operator $T_p \in \mathcal{H}_G(\sigma_i)$).

In the rest of this section we will denote the element $f(T_p)$ of $\mathcal{H}_G(\sigma_i)$, where the polynomial f is defined in Definition 2.7.6, by f. In the notation of Definition 2.7.6, the vanishing set of f in $\mathrm{Spec}\, \mathcal{H}_G(\sigma_i)$ coincides with $f_{\sigma_i}^{-1}(Y_{\mathrm{bad}} \cap X(\sigma|\sigma^{\mathrm{co}}))$.

We now write j^*, j_* for the functors $j^*_{U(\sigma|\sigma^{\mathrm{co}})}, j_{U(\sigma|\sigma^{\mathrm{co}}),*}$ defined in Proposition 2.5.8. Then (2.5.18) implies that

$$j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_i = c\text{-}\mathrm{Ind}_{KZ}^G \sigma_i[1/f]. \tag{2.7.7}$$

Since

$$\left(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 \otimes \det{}^a\right)[1/(T_p^2 - \zeta(p))] \cong \left(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \otimes \det{}^a\right)[1/(T_p^2 - \zeta(p))],$$

in the case that σ_1 or σ_2 is a twist of Sym^0, we can and do replace it by the corresponding twist of Sym^{p-1}. (We do this in order that the isomorphism (2.2.35) is valid.)

Since (2.2.35) is equivariant for the Satake map, and Ind_B^G preserves colimits, we may combine (2.2.35) with (2.7.7) to obtain an isomorphism

$$j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_i \cong \mathrm{Ind}_B^G \left((c\text{-}\mathrm{Ind}_{T_0 Z}^T \chi_i)[1/f]\right). \tag{2.7.8}$$

In particular we see that $\mathcal{H}_T[1/f]$ acts on $j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2$ and thus on the Ext groups

$$\mathrm{Ext}^i_{\mathcal{A}}(j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2).$$

Proposition 2.7.9. *The $\mathcal{H}_T[1/f]$-modules $\mathrm{Ext}^i_{\mathcal{A}}(j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2)$ are finitely generated and torsion-free.*

Proof. We begin by noting that (2.7.7) induces an isomorphism of $\mathcal{H}_T[1/f]$-modules

$$\mathrm{Ext}^i_{\mathcal{A}}(j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2) \cong \mathrm{Ext}^i_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1[1/T_p], c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2[1/f]). \tag{2.7.10}$$

Indeed, it suffices to show that for all i we have

$$\mathrm{Ext}^i_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1[1/f]/c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1[1/T_p], c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2[1/f]) = 0,$$

and this is immediate from [DEG23, Corollary 3.1.15] (with Y taken to be the vanishing locus of f in X, as before). Furthermore, by (2.2.35) we have

$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1[1/T_p] \cong \mathrm{Ind}_B^G \left(c\text{-}\mathrm{Ind}_{T_0 Z}^T \chi_1\right). \tag{2.7.11}$$

We now use some results of Heyer [Hey23; Hey24]. By [Hey23, Theorem 4.1.1], the functor $\mathrm{Ind}_B^G : D(\mathrm{sm.}\, T) \to D(\mathrm{sm.}\, G)$ of parabolic induction has a left adjoint

$L(\overline{U},-)$. By [Hey23, Corollary 4.1.3] there is a spectral sequence of $\mathcal{H}_T[1/f]$-modules

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathrm{sm.}\,T}(L^{-j}(\overline{U}, \mathrm{Ind}_{\overline{B}}^G(c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_1)), (c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_2)[1/f]) \Rightarrow \mathrm{Ext}^{i+j}_{\mathrm{sm.}\,G}\left(\mathrm{Ind}_{\overline{B}}^G(c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_1), \mathrm{Ind}_{\overline{B}}^G\left((c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_2)[1/f]\right)\right). \tag{2.7.12}$$

As previously recalled, the target of this spectral sequence is isomorphic to

$$\mathrm{Ext}^{i+j}_{\mathrm{sm.}\,G}(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_1)[1/T_p], c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_2)[1/f]).$$

(In fact, this is (2.7.11) for the first term, and (2.7.7) and (2.7.8) for the second term.)

We now compute the E_2 page of (2.7.12). For all $w \in W(G,T)$, the intersection $T \cap w\overline{U}w^{-1}$ is trivial. Using this, the statement and proof of [Hey24, Example 4.2.1] go through unchanged to show that $L(\overline{U}, \mathrm{Ind}_{\overline{B}}^G(c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_1))$ is concentrated in degrees $[-1,0]$, and that there are natural isomorphisms

$$\begin{aligned} L^0(\overline{U}, \mathrm{Ind}_{\overline{B}}^G(c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_1)) &\cong c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_1, \\ L^{-1}(\overline{U}, \mathrm{Ind}_{\overline{B}}^G(c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_1)) &\cong \delta_s \otimes (c\text{-}\mathrm{Ind}_{T_0Z}^T \mathrm{ad}(s)^*\chi_1), \end{aligned} \tag{2.7.13}$$

where $s \in W(G,T)$ is the nontrivial element, and $\delta_s = \omega^{-1} \otimes \omega$ is the $\overline{B}$-positive root. Here we have used [Hey24, Lemma 4.1.6] to compute δ_s, and we have also used the natural isomorphism of objects of $\mathrm{sm.}\,T$

$$\mathrm{ad}(s)^*(c\text{-}\mathrm{Ind}_{T_0Z}^T)(\chi_1) \cong c\text{-}\mathrm{Ind}_{T_0Z}^T(\mathrm{ad}(s)^*(\chi_1)).$$

We now analyse the spectral sequence (2.7.12). By (2.7.13) and Lemma 2.7.15 below, the $E_2^{i,j}$-terms are finite free $\mathcal{H}_T[1/f]$-modules, so the Ext-groups

$$\mathrm{Ext}^i_{\mathrm{sm.}\,G}(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_1)[1/T_p], c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_2)[1/f]) \tag{2.7.14}$$

are finitely generated $\mathcal{H}_T[1/f]$-modules. By (2.7.10) and Lemma 2.7.4, the same is true of the $\mathrm{Ext}^i_{\mathcal{A}}(j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2)$. Furthermore, we see from Lemma 2.7.4 that if some $\mathrm{Ext}^j_{\mathcal{A}}(j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2)$ has $\mathcal{H}_T[1/f]$-torsion, then the same is true of the groups (2.7.14) for $j = i, i+1, i+2$. However, the vanishing result of Lemma 2.7.15 shows that (2.7.12) degenerates at E_3, and since all the objects on the E_2 page are free $\mathcal{H}_T[1/f]$-modules, it furthermore shows that the groups (2.7.14) could only have non-zero torsion in degrees 2 or 3. In particular, it is impossible for them to have non-zero torsion for a range of degrees $j = i, i+1, i+2$, so they are torsion-free, as required. □

Lemma 2.7.15. *For any ζ-compatible smooth character $\psi : T_0 \to \mathbf{F}^\times$, the* Ext-*groups*

$$\mathrm{Ext}^i_{\mathrm{sm.}\,T}(c\text{-}\mathrm{Ind}_{T_0Z}^T \psi, c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_2[1/f])$$

are finite free $\mathcal{H}_T[1/f]$-modules, and are zero for $i \geq 4$.

Proof. By Frobenius reciprocity, we have a natural isomorphism

$$\mathrm{Ext}^i_{\mathrm{sm.}\,T}(c\text{-}\mathrm{Ind}_{T_0Z}^T \psi, c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_2[1/f]) \cong \mathrm{Ext}^i_{\mathrm{sm.}\,T_0Z}(\psi, \mathrm{Res}_{T_0Z}^T c\text{-}\mathrm{Ind}_{T_0Z}^T \chi_2[1/f]).$$

Since the ring $\mathcal{O}[\![T_0Z]\!]$ is Noetherian (e.g. by [Tim23, Section 2.1]), the category $\mathrm{sm.}\,T_0Z$ is locally Noetherian (e.g. because it is a localizing subcategory of $\mathrm{Mod}\,\mathcal{O}[\![T_0Z]\!]$, by Lemma 2.2.20). Thus $\mathrm{Ext}^i_{\mathrm{sm.}\,T_0Z}(\psi,-)$ commutes with filtered colimits (e.g. by [DEG23, Proposition A.1.1(3)]). On the other hand, we have an isomorphism of $\mathcal{H}_T[1/f]$-modules

$$\mathrm{Res}^T_{T_0 Z}\, c\text{-}\mathrm{Ind}^T_{T_0 Z}\, \chi_2[1/f] \cong \chi_2 \otimes_{\mathbf{F}} \mathcal{H}_T[1/f],$$

and thus an identification

$$\mathrm{Ext}^i_{\mathrm{sm.}\,T}(c\text{-}\mathrm{Ind}^T_{T_0 Z}\, \psi, c\text{-}\mathrm{Ind}^T_{T_0 Z}\, \chi_2[1/f]) \cong \mathrm{Ext}^i_{\mathrm{sm.}\,T_0 Z}(\psi, \chi_2) \otimes_{\mathbf{F}} \mathcal{H}_T[1/f],$$

which shows that the Ext-groups are finite free over $\mathcal{H}_T[1/f]$. Since $T_0 Z \cong \mathbf{Q}_p^\times \times \mathbf{Z}_p^\times$, these Ext groups are furthermore concentrated in degrees $i \in [0, 3]$, as required. $\square$

Remark 2.7.16. Assume that $\chi_1 \neq \delta_s \otimes \mathrm{ad}(s)^* \chi_1$. There are two actions of $\mathcal{H}_T$ on $\mathrm{Ext}^i_{\mathcal{A}}(j_* j^* c\text{-}\mathrm{Ind}^G_{KZ}\, \sigma_1, j_* j^* c\text{-}\mathrm{Ind}^G_{KZ}\, \sigma_2)$, arising from the action of $\mathcal{H}_G(\sigma_i)$ on the two factors. The proof of Proposition 2.7.9 shows that they coincide when $\chi_2 = \chi_1$, and are related by the automorphism $T_p \mapsto T_p^{-1}$ of $\mathcal{H}_T$ when $\chi_2 = \delta_s \otimes \mathrm{ad}(s)^* \chi_1$.

2.7.17. Ext *and completion.* We now show the compatibility with completion of the formation of Ext-modules. Recall that we have fixed a companion pair of Serre weights $\{\sigma, \sigma^{\mathrm{co}}\}$ and a choice $\sigma_1 = \sigma$, $\sigma_2 = \sigma^{\mathrm{co}}$. We continue to write j_*, j^* for $j_{U(\sigma|\sigma^{\mathrm{co}}),*}, j^*_{U(\sigma|\sigma^{\mathrm{co}})}$. Fix a closed point $\overline{\theta}$ of $U(\sigma|\sigma^{\mathrm{co}})$, and let $f_{\overline{\theta}} \in \mathcal{H}_G(\sigma_2)$ be the squarefree monic polynomial associated to $\{\overline{\theta}\} \subset X$ in Definition 2.5.15.

Lemma 2.7.18. *Let* $\pi_i \coloneqq c\text{-}\mathrm{Ind}^G_{KZ}\, \sigma_i$. *Then the natural map*

$$\mathrm{Ext}^i_{\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \widehat{i^*_{\overline{\theta}}}(\pi_2)) \to \varprojlim_n \mathrm{Ext}^i_{\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \pi_2/f^n_{\overline{\theta}}\pi_2) \tag{2.7.19}$$

is an isomorphism.

Proof. By (2.5.17), we have

$$\widehat{i^*_{\overline{\theta}}}(\pi_2) = \lim_n \pi_2/f^n_{\overline{\theta}}\pi_2 \in \mathrm{Pro}(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}).$$

By Lemma A.2.12 (1), applied to the compactly generated ∞-category $\mathrm{Ind}\,\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})$, we have natural isomorphisms

$$\begin{aligned} &\mathrm{RHom}_{\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \widehat{i^*_{\overline{\theta}}}(\pi_2)) = \mathrm{RHom}_{\mathrm{Ind}\,\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \widehat{i^*_{\overline{\theta}}}(\pi_2)) \\ &= \lim_n \mathrm{RHom}_{\mathrm{Ind}\,\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \pi_2/f^n_{\overline{\theta}}\pi_2) = \lim_n \mathrm{RHom}_{\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \pi_2/f^n_{\overline{\theta}}\pi_2)\end{aligned}$$

(where $\lim_n$ is formed in the derived ∞-category $D(\mathbf{F})$ of $\mathbf{F}$-vector spaces, and we have used that $\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to \mathrm{Ind}\,\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})$ is fully faithful and limit-preserving). The map (2.7.19) is thus an edge map in a spectral sequence

$$E_2^{p,q} = R^p \lim_n \mathrm{Ext}^q_{\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \pi_2/f^n_{\overline{\theta}}\pi_2) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \widehat{i^*_{\overline{\theta}}}(\pi_2)). \tag{2.7.20}$$

We now claim that, for all $\tau \in \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}$ and $i \in \mathbf{Z}$, the $\mathbf{F}$-vector space

$$\mathrm{Ext}^i_{\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i^*_{\overline{\theta}}}(\pi_1), \tau) \tag{2.7.21}$$

is finite-dimensional. Assuming the claim, the Mittag-Leffler criterion shows that $E_2^{p,q} = 0$ whenever $p \geq 1$, and so (2.7.20) degenerates at E_2, and (2.7.19) is an isomorphism, as desired.

We now prove the claim. By Lemma 2.6.10, the functor

$$\widehat{i^*_{\overline{\theta}}} : \mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}) \to \mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})$$

is left adjoint to the inclusion, and so (2.7.21) is isomorphic to

$$\mathrm{Ext}^i_{\mathrm{Pro}\, D^b_{\mathrm{fp}}(\mathcal{A})}(\pi_1, \tau) = \mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A})}(\pi_1, \tau).$$

In turn, since $\mathcal{A} = \mathrm{Ind}(\mathcal{A}^{\mathrm{fp}})$, the equivalence (A.9.4) induces an isomorphism

$$\mathrm{Ext}^i_{D^b(\mathcal{A})}(\pi_1, \tau) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Ind}\, D^b_{\mathrm{fp}}(\mathcal{A})}(\pi_1, \tau) = \mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A})}(\pi_1, \tau),$$

and so we have reduced to proving that $\mathrm{Ext}^i_{D^b(\mathcal{A})}(\pi_1, \tau)$ has finite $\mathbf{F}$-dimension. By dévissage, we can assume that τ is irreducible. Then the claim is a consequence of [DEG23, Proposition 4.2.4]. ☐

Continue to write $\pi_i := c\text{-}\mathrm{Ind}_{KZ}^G\, \sigma_i$. By Remark 2.5.11, the exact functor $\widehat{i}^*_{\overline{\theta}}$ induces a map

$$\mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}(j^*\pi_1, j^*\pi_2) \to \mathrm{Ext}^i_{D^b(\mathrm{Pro}\, \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, \widehat{i}^*_{\overline{\theta}}\pi_2)$$

which can be composed with the t-exact functor $p : D^b(\mathrm{Pro}\, \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}) \to \mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})$ of (A.9.1) to produce a map

$$\widehat{i}^*_{\overline{\theta}} : \mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}(j^*\pi_1, j^*\pi_2) \to \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, \widehat{i}^*_{\overline{\theta}}\pi_2) \tag{2.7.22}$$

Proposition 2.7.23. *Let* $\pi_i := c\text{-}\mathrm{Ind}_{KZ}^G\, \sigma_i$. *Then* (2.7.22) *induces an isomorphism*

$$\mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}(j^*\pi_1, j^*\pi_2)^{\wedge}_{f_{\overline{\theta}}} \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, \widehat{i}^*_{\overline{\theta}}\pi_2). \tag{2.7.24}$$

Proof. By Remark 2.5.11, as well as the fact that j_* is fully faithful and t-exact, it suffices to prove that $\widehat{i}^*_{\overline{\theta}}$ induces an isomorphism

$$\mathrm{Ext}^i_{D^b(\mathcal{A})}(j_*j^*\pi_1, j_*j^*\pi_2)^{\wedge}_{f_{\overline{\theta}}} \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, \widehat{i}^*_{\overline{\theta}}\pi_2). \tag{2.7.25}$$

By Proposition 2.7.9, the group $\mathrm{Ext}^i_{D^b(\mathcal{A})}(j_*j^*\pi_1, j_*j^*\pi_2)$ is $\mathcal{H}_T[1/f]$-torsion free and in particular $f_{\overline{\theta}}$-torsion free, so that for any $n \geq 1$, the exact sequence

$$0 \to j_*j^*\pi_2 \xrightarrow{f^n_{\overline{\theta}}} j_*j^*\pi_2 \to \pi_2/f^n_{\overline{\theta}}\pi_2 \to 0 \tag{2.7.26}$$

induces an isomorphism

$$\mathrm{Ext}^i_{D^b(\mathcal{A})}(j_*j^*\pi_1, j_*j^*\pi_2)/f^n_{\overline{\theta}} \cong \mathrm{Ext}^i_{D^b(\mathcal{A})}(j_*j^*\pi_1, \pi_2/f^n_{\overline{\theta}}\pi_2). \tag{2.7.27}$$

By Lemma 2.6.10, we see that $\widehat{i}^*_{\overline{\theta}}$ induces an isomorphism

$$\mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A})}(\pi_1, \pi_2/f^n_{\overline{\theta}}\pi_2) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, \pi_2/f^n_{\overline{\theta}}\pi_2),$$

hence (bearing in mind Remark 2.5.11) an isomorphism

$$\widehat{i}^*_{\overline{\theta}} : \mathrm{Ext}^i_{D^b(\mathcal{A})}(j_*j^*\pi_1, \pi_2/f^n_{\overline{\theta}}\pi_2) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, \pi_2/f^n_{\overline{\theta}}\pi_2). \tag{2.7.28}$$

On the other hand, applying $\mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, -)$ to the exact sequence

$$0 \to \widehat{i}^*_{\overline{\theta}}\pi_2 \xrightarrow{f^n_{\overline{\theta}}} \widehat{i}^*_{\overline{\theta}}\pi_2 \to \pi_2/f^n_{\overline{\theta}}\pi_2 \to 0$$

obtained by applying $\widehat{i}^*_{\overline{\theta}}$ to (2.7.26), we obtain an injection

$$\mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, \widehat{i}^*_{\overline{\theta}}\pi_2)/f^n_{\overline{\theta}} \hookrightarrow \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}\pi_1, \pi_2/f^n_{\overline{\theta}}\pi_2). \tag{2.7.29}$$

We conclude from the discussion above that the composition of (2.7.27) and (2.7.28) is an isomorphism

$$\mathrm{Ext}^i_{D^b(\mathcal{A})}(j_*j^*\pi_1, j_*j^*\pi_2)/f_\theta^n \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \pi_2/f_\theta^n\pi_2),$$

which can also be factored as a composition

$$\begin{aligned}\mathrm{Ext}^i_{D^b(\mathcal{A})}(j_*j^*\pi_1, j_*j^*\pi_2)/f_\theta^n &\xrightarrow{\widehat{i}^*_\theta} \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \widehat{i}^*_\theta\pi_2)/f_\theta^n \\ &\overset{(2.7.29)}{\hookrightarrow} \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \pi_2/f_\theta^n\pi_2),\end{aligned}$$

It follows that both arrows in this composition are isomorphisms, and so

$$\widehat{i}^*_\theta : \mathrm{Ext}^i_{D^b(\mathcal{A})}(j_*j^*\pi_1, j_*j^*\pi_2) \to \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \widehat{i}^*_\theta\pi_2)$$

becomes an isomorphism after quotienting out by f_θ^n. Hence, to conclude the proof that (2.7.25) is an isomorphism, it suffices to prove that the natural map

$$\mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \widehat{i}^*_\theta\pi_2) \to \varprojlim_n \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \widehat{i}^*_\theta\pi_2)/f_\theta^n$$

is an isomorphism. Since (2.7.29) is an isomorphism, the right-hand side is isomorphic to

$$\varprojlim_n \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \pi_2/f_\theta^n\pi_2).$$

So it suffices to prove that

$$\mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \widehat{i}^*_\theta\pi_2) \to \varprojlim_n \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_\theta)}(\widehat{i}^*_\theta\pi_1, \pi_2/f_\theta^n\pi_2)$$

is an isomorphism. This is Lemma 2.7.18. ☐

Remark 2.7.30. Proposition 2.7.9 (and the fact that $\mathcal{H}_T[1/f]$ is a principal ideal domain) shows that the $\mathbf{F}[T_p^{\pm 1}, f(T_p)^{-1}]$-module

$$\mathrm{Ext}^i_{\mathcal{A}}\big(j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1, j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2\big)$$

is finite free. A further analysis of the spectral sequence (2.7.12) then allows one to determine its rank:

- If $i = 0$ then the rank is 1 if $\sigma_1 = \sigma_2$, and zero otherwise.
- If $i = 1$ then the rank is 1 unless $\{\sigma, \sigma^{\mathrm{co}}\}$ is of type (scalar) in which case the rank is 2.
- If $i = 2$ then the rank is 0 if $\sigma_1 = \sigma_2$ and $\{\sigma, \sigma^{\mathrm{co}}\}$ is not of type (scalar), and otherwise the rank is 1.
- If $i = 3$ then the rank is 0.

3. The moduli stack of rank 2 étale (φ, Γ)-modules

In this section we will recall some of the main results of [EG23; EG22], specialised to the case of $\mathrm{GL}_2(\mathbf{Q}_p)$, and prove some additional results in this setting.

3.1. **Étale (φ, Γ)-modules.** We begin by recalling some definitions and notation from [EG23]. We write $\Gamma := \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)$, so that the cyclotomic character induces an isomorphism $\chi : \Gamma \xrightarrow{\sim} \mathbf{Z}_p^\times$. For each $a \in \mathbf{Z}_p^\times$, we write $\sigma_a \in \Gamma$ for the element with $\chi(\sigma_a) = a$. If we choose a compatible system of p^n-th roots of 1, then these give rise in the usual way to an element $\varepsilon \in (\widehat{\mathbf{Q}_p(\zeta_{p^\infty})})^\flat$. As usual, let $[\varepsilon]$ denote the Teichmüller lift of ε to an element of $W\big(\mathcal{O}^\flat_{\widehat{\mathbf{Q}_p(\zeta_{p^\infty})}}\big)$. There is then a continuous embedding

$$\mathbf{Z}_p[\![T]\!] \hookrightarrow W(\mathcal{O}^\flat_{\widehat{\mathbf{Q}_p(\zeta_{p^\infty})}})$$

(the source being endowed with its (p,T)-adic topology, and the target with its weak topology), defined via $T \mapsto [\varepsilon] - 1$. We denote the image of this embedding by $\mathbf{A}^+$. This embedding extends to an embedding

$$\widehat{\mathbf{Z}_p((T))} \hookrightarrow W\big((\widehat{\mathbf{Q}_p(\zeta_{p^\infty})})^\flat\big)$$

(here the source is the p-adic completion of the Laurent series ring $\mathbf{Z}_p((T))$), whose image we denote by $\mathbf{A}$.

We have commuting actions of φ and Γ on $\mathbf{A}$ which are given by the explicit formulae

$$\varphi(T) = (1+T)^p - 1,$$
$$\sigma_a(T) = (1+T)^a - 1.$$

Note that $\mathbf{A}^+$ is visibly (φ, Γ)-stable.

There is a left inverse ψ to φ defined as follows. We have a decomposition $\mathbf{A} = \oplus_{i=0}^{p-1}(1+T)^i\varphi(\mathbf{A})$, and $\psi : \mathbf{A} \to \mathbf{A}$ is defined to be the projection onto the $i = 0$ factor; so in particular $\psi(\varphi(x)) = x$. This restricts to a surjection $\psi : \mathbf{A}^+ \to \mathbf{A}^+$.

Remark 3.1.1. Our notation and conventions differ from that of [EG23], for two reasons. Firstly, since we are only working over $\mathbf{Q}_p$, and not over an extension $K/\mathbf{Q}_p$, we have dropped K from the notation. Secondly, and more significantly, we use (φ, Γ)-modules for the full cyclotomic extension $\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p$, and we make no use of (φ, Γ)-modules for the $\mathbf{Z}_p$-subextension.

The theories of (φ, Γ)-modules for the two extensions are equivalent (via taking invariants for the group of roots of unity in $\mathbf{Z}_p^\times$), but in most of [EG23] the $\mathbf{Z}_p$-theory was used, because it was convenient when proving the existence of various moduli stacks to work with a pro-cyclic pro-p group. However, for applications to the p-adic Langlands program (or more generally in arguments involving explicit formulas for the actions of φ and Γ), it is much more convenient to work with the full cyclotomic extension. Since we make no use of the $\mathbf{Z}_p$-subextension, and we want to avoid notational clutter where possible, we have written Γ for the group denoted $\widetilde{\Gamma}$ in [EG23], and $\mathbf{A}, \mathbf{A}^+$ for the rings denoted $\mathbf{A}'_{\mathbf{Q}_p}$ and $(\mathbf{A}')^+_{\mathbf{Q}_p}$ respectively in [EG23].

We now introduce coefficients. Recall that $\mathcal{O}$ is the ring of integers in our fixed finite extension $E/\mathbf{Q}_p$. Let A be a p-adically complete $\mathcal{O}$-algebra. Frequently, we will work modulo a fixed power ϖ^a of ϖ, and thus assume that A is actually an $\mathcal{O}/\varpi^a$-algebra (and sometimes we impose further conditions on A, such as that of being Noetherian, or even of finite type over $\mathcal{O}/\varpi^a$). We write

$$\mathbf{A}_A^+ = \mathbf{A}^+ \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} A := \varprojlim_n \mathbf{A}^+/(p,T)^n \otimes_{\mathbf{Z}_p} A = \varprojlim_m (\varprojlim_n \mathbf{A}^+/(p^m, T^n) \otimes_{\mathbf{Z}_p} A),$$

and

$$\mathbf{A}_A = \mathbf{A}\,\widehat{\otimes}_{\mathbf{Z}_p} A := \varprojlim_m((\varprojlim_n \mathbf{A}^+/(p^m, T^n)\otimes_{\mathbf{Z}_p} A)[1/T]).$$

If A is a Noetherian $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, then $\mathbf{A}_A^+ = A[\![T]\!]$ and $\mathbf{A}_A = A((T))$ are flat A-algebras, and the morphism $\mathbf{A}_A^+ \to \mathbf{A}_A$ is flat and injective. If $A \to B$ is a (faithfully) flat homomorphism of Noetherian $\mathcal{O}/\varpi^a$-algebras, then the morphisms $\mathbf{A}_A^+ \to \mathbf{A}_B^+$ and $\mathbf{A}_A \to \mathbf{A}_B$ are (faithfully) flat. (All of these statements follow easily from the facts that completions and localizations of Noetherian rings are flat, see [EG21, Lem. 5.1.7].)

The rings $\mathbf{A}_A^+$, $\mathbf{A}_A$ have natural topologies, which are discussed further in Section 4.1. By [EG23, Lem. 2.2.17], the actions of φ, Γ on $\mathbf{A}^+$ and $\mathbf{A}$ extend to continuous A-linear actions on $\mathbf{A}_A^+$ and $\mathbf{A}_A$. Similarly, the action of ψ extends to continuous A-linear surjections $\psi : \mathbf{A}_A \to \mathbf{A}_A$, $\psi : \mathbf{A}_A^+ \to \mathbf{A}_A^+$, satisfying $\psi(\varphi(x)) = x$.

By definition, a projective étale (φ, Γ)-module with A-coefficients is a finitely generated projective $\mathbf{A}_A$-module D, equipped with

- a φ-linear morphism $\varphi : D \to D$ with the property that the corresponding morphism $\Phi_D : \varphi^* D \to D$ is an isomorphism (i.e. D is given the structure of an étale φ-module over $\mathbf{A}_A$), and
- a continuous semi-linear action of Γ that commutes with φ.

Here the notion of continuity is with respect to the canonical topology on D inherited from that on $\mathbf{A}_A$ (see [EG23, Rem. D.2]). Our étale (φ, Γ)-modules will usually be assumed to be projective, and we will often write "étale (φ, Γ)-module" for "projective étale (φ, Γ)-module" when no confusion should arise. We will also sometimes write "étale (φ, Γ)-module over A" for "étale (φ, Γ)-module with A-coefficients".

Remark 3.1.2. In the case that A is Artinian, there is the usual equivalence of categories (see [Fon90]) between the category of (projective) étale (φ, Γ)-module with A-coefficients and the category of finite projective A-modules with an action of $G_{\mathbf{Q}_p}$, given by the functor $D \mapsto (W(\mathbf{C}^\flat)\otimes_{\mathbf{A}} D)^{\varphi=1}$.

3.2. **Moduli stacks of étale (φ, Γ)-modules.**

Definition 3.2.1. We write $\mathcal{X}$ for the stack of projective étale (φ, Γ)-modules of rank 2 with determinant $\zeta\varepsilon^{-1}$, as defined in Definition C.1.1. Explicitly, if A is a p-adically complete $\mathcal{O}$-algebra, then $\mathcal{X}(\operatorname{Spf} A)$ is the groupoid of pairs (D, θ) where D is a rank 2 projective étale (φ, Γ)-module with A-coefficients, and θ is an identification of $\wedge^2 D$ with (the (φ, Γ)-module corresponding to) $\zeta\varepsilon^{-1}$.

Theorem 3.2.4 below summarises the basic properties of $\mathcal{X}$. In order to state it, we make the following definitions. Our notation for Serre weights is as in Section 1.3; recall in particular that we assume throughout this paper that our Serre weights σ are compatible with ζ in the sense of Definition 2.1.6.

Definition 3.2.2. Let $\lambda \in \{(a,b)\in \mathbf{Z}^2 : a > b\}$ be a regular Hodge type, and let $\tau : I_{\mathbf{Q}_p} \to \mathrm{GL}_2(E)$ be an inertial type (i.e. a representation with open kernel that extends to the Weil group of $\mathbf{Q}_p$) such that the pair (λ, τ) is compatible with $\zeta\varepsilon^{-1}$, in the sense of Definition C.1.3. Then we write $\mathcal{X}^{\lambda,\tau}$ for the ϖ-adic formal algebraic substack of $\mathcal{X}$ denoted $\mathcal{X}_2^{\mathrm{crys},\underline{\lambda},\tau,\zeta\varepsilon^{-1}}$ in (C.1.2). If τ is the trivial representation, we will often write $\mathcal{X}^{\lambda,\mathrm{crys}}$ for $\mathcal{X}^{\lambda,\tau}$.

Definition 3.2.3. If $\sigma = \sigma_{a,b}$ is a Serre weight, then we define

$$\mathcal{Z}(\sigma) \coloneqq (\mathcal{X}^{(1-a,-a-b),\mathrm{crys}} \times_{\mathcal{O}} \mathbf{F})_{\mathrm{red}}.$$

This is a closed algebraic substack of $\mathcal{X}$ (since $\mathcal{X}^{(1-a,-a-b),\mathrm{crys}}$ is ϖ-adically formal).

The next theorem collects some of the basic properties of the moduli stacks we have just introduced. Recall that by [EG23, Theorem 6.6.3], the finite type points of $\mathcal{X}$ are in bijection with $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy classes of continuous representations $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ with determinant $\zeta\varepsilon^{-1}$. This bijection sends $\overline{\rho}$ to the image of the classifying map $\operatorname{Spec} \overline{\mathbf{F}}_p \to \mathcal{X}$ of the (φ, Γ)-module corresponding to $\overline{\rho}$ under the equivalence of Remark 3.1.2.

Theorem 3.2.4.

(1) $\mathcal{X}$ is a Noetherian formal algebraic stack.

(2) The underlying reduced substack $\mathcal{X}_{\mathrm{red}}$ is of finite type over $\mathbf{F}$, and is equidimensional of dimension 1.

(3) The irreducible components of $\mathcal{X}_{\mathrm{red}}$ admit a natural surjection to the set of ζ-compatible Serre weights. The fibres of this surjection are single irreducible components $\mathcal{X}(\sigma)$ unless $\sigma = \sigma_{a,p-1}$ for some a, in which case we have two irreducible components $\mathcal{X}(\sigma)^{\pm}$.

(4) The stack $\mathcal{Z}(\sigma)$ is algebraic, and coincides with $\mathcal{X}(\sigma)$ unless $\sigma = \sigma_{a,p-1}$ for some a, in which case it is the scheme-theoretic union of $\mathcal{X}(\sigma_{a,0})$ and the two irreducible components $\mathcal{X}(\sigma)^{\pm}$.

(5) Let $x \in |\mathcal{X}|$ be a finite type point, with corresponding representation $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$. Then $x \in |\mathcal{Z}(\sigma)|$ if and only if σ is a Serre weight of $\overline{\rho}$.

Proof. The first three parts are a special case of Theorem C.1.8. Part (4) is immediate from [EG23, Thm. 8.6.2]. Finally, Definition 2.1.4 was formulated in such a way that part (5) is true by definition; see e.g. [EG23, §8.5] for an elaboration of this point. ☐

We now describe a choice of (λ, τ) so that $\mathcal{X}^{\lambda,\tau} \otimes_{\mathcal{O}} \mathbf{F} = \mathcal{Z}(\sigma)$.

Lemma 3.2.5. *Let σ be a Serre weight.*

(1) If $\sigma = \mathrm{Sym}^b \otimes \det^a$, and $b \neq p-2$, we let $\lambda = (1-a, -a-b)$, and we let τ be trivial.

(2) If $\sigma = \mathrm{Sym}^{p-2} \otimes \det^a$, we let $\lambda = (1, 0)$, and we let $\tau = \omega_2^{p+(a-1)(p+1)} \oplus \omega_2^{p^2+(a-1)(p+1)}$.

Then the special fibre $\mathcal{X}^{\lambda,\tau} \otimes_{\mathcal{O}} \mathbf{F}$ is reduced, and $\mathcal{X}^{\lambda,\tau} \otimes_{\mathcal{O}} \mathbf{F} = \mathcal{Z}(\sigma)$.

Proof. The top-dimensional cycle of $\mathcal{X}^{\lambda,\tau} \otimes_{\mathcal{O}} \mathbf{F}$ is always $[\mathcal{Z}(\sigma)]$; this is immediate from the definitions in case (1), and from the geometric Breuil–Mézard conjecture [EG23, Theorem 8.6.2] in case (2). (In case (2), note that τ is the inertial type corresponding to the representation $\Theta(\sigma|\sigma^{\mathrm{co}})$ of Definition 2.1.8 (2), whose semisimplified mod ϖ reduction is σ.) It thus suffices to show that $\mathcal{X}^{\lambda,\tau} \otimes_{\mathcal{O}} \mathbf{F}$ is reduced.

In case (1), if furthermore $b \neq p-1$, the reducedness follows from Fontaine–Laffaille theory. If $b = p-1$, it suffices to prove that for any semisimple $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{F})$, there exists a versal ring to $\mathcal{X}^{\lambda,\mathrm{crys}} \otimes_{\mathcal{O}} \mathbf{F}$ at $\overline{\rho}$ which is reduced. (This follows for example from an application of Lemma B.3.34 to the nilradical of the structure sheaf of $\mathcal{X}^{\lambda,\tau} \otimes_{\mathcal{O}} \mathbf{F}$.) When $\overline{\rho}$ is not a twist of $1 \oplus \omega^{-1}$, we can take the versal ring to be $R_{\overline{\rho}}^{\square,\lambda,\mathrm{crys}}/\varpi$, i.e. the special fibre of the crystalline lifting ring of $\overline{\rho}$ with weight λ, which is reduced by [Kis09, Corollary 1.7.14]. When $\overline{\rho}$ is a twist of $1 \oplus \omega^{-1}$, we can apply e.g. Corollary 3.5.43 below. We thus conclude that $\mathcal{X}^{\lambda,\mathrm{crys}} \otimes_{\mathcal{O}} \mathbf{F} = \mathcal{Z}(\sigma)$, as desired.

In case (2), we know by [CEGS25, Theorem 1.3] that $\mathcal{X}^{\lambda,\tau} \otimes_{\mathcal{O}} \mathbf{F}$ is generically reduced. Furthermore, by [LMM23, Theorem 1.1.2], $\mathcal{X}^{\lambda,\tau}$ is normal, so that $\mathcal{X}^{\lambda,\tau} \otimes_{\mathcal{O}} \mathbf{F}$ is S_1, and thus reduced, as claimed. $\square$

3.3. **A continuous map from $|\mathcal{X}|$ to $|X|$.** We now construct a continuous map $\pi_{\mathrm{ss}} : |\mathcal{X}| \to |X|$ from the underlying topological space of the stack $\mathcal{X}$ to the underlying topological space of the chain X of projective lines constructed in Section 2.5.1.

3.3.1. *Soberization.* Recall that a topological space is *sober* if every irreducible closed subset has a unique generic point. The inclusion of the full subcategory of sober spaces into the category of all topological spaces (and continuous maps) admits a left adjoint, the *soberization* of a space, which we will denote sob(–). Concretely (see e.g. [Stacks, Tag 0A2N]), for any topological space S, we define $\mathrm{sob}(S)$ to be the set of irreducible closed subspaces of S. The topology on $\mathrm{sob}(S)$ is defined by letting its closed subsets be the collection of subsets $\mathrm{sob}(T)$, where T ranges over the closed subsets of S. Of course, the natural morphism (unit of adjunction) $S \to \mathrm{sob}(S)$ is a homeomorphism if and only if S itself is sober.

3.3.2. *Underlying topological spaces of stacks.* If $\mathcal{Z}$ is an algebraic stack of finite presentation over a field k (and so is in particular quasi-compact and quasi-separated), then we write (as usual) $|\mathcal{Z}|$ for the underlying topological space of $\mathcal{Z}$, $|\mathcal{Z}|_{\mathrm{ft}}$ for its subset of finite type points and $|\mathcal{Z}|_{\mathrm{cl}}$ for its subset of closed points. We endow each of these subsets with the subspace topology. We always have that $|\mathcal{Z}|_{\mathrm{cl}} \subseteq |\mathcal{Z}|_{\mathrm{ft}}$, with equality for schemes (or more generally Deligne–Mumford stacks), but not in general. We have that $|\mathcal{Z}|_{\mathrm{ft}}$ is dense in $|\mathcal{Z}|$, and that $|\mathcal{Z}|$ is sober [Stacks, Tag 0DQP]. This implies that the inclusion $|\mathcal{Z}|_{\mathrm{ft}} \hookrightarrow |\mathcal{Z}|$ induces a homeomorphism

$$\mathrm{sob}(|\mathcal{Z}|_{\mathrm{ft}}) \xrightarrow{\sim} |\mathcal{Z}|. \tag{3.3.3}$$

3.3.4. *Finite type points of $\mathcal{X}$.* As already recalled, by [EG23, Thm. 6.6.3], the finite type points of $\mathcal{X}$ are in bijection with $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy classes of continuous representations $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ with determinant $\zeta\varepsilon^{-1}$. Furthermore, the finite type point associated to $\overline{\rho}$ is closed if and only if $\overline{\rho}$ is semisimple; and the closure of $\{\overline{\rho}\}$ contains a unique closed point, which is associated to the semisimplification $\overline{\rho}^{\mathrm{ss}}$. Thus we obtain a map (of sets!)

$$|\mathcal{X}|_{\mathrm{ft}} \to |\mathcal{X}|_{\mathrm{cl}}, \overline{\rho} \mapsto \overline{\rho}^{\mathrm{ss}}. \tag{3.3.5}$$

We caution the reader that this map is *not* continuous.

3.3.6. *Constructing the morphism.* We let X denote the chain of projective lines over $\mathbf{F}$ recalled in Section 2.5.1, so that $|X|_{\mathrm{cl}}$ is in bijection with the set of blocks of $\mathcal{A}^{\mathrm{l.adm}}$, and hence is also in bijection with the set of residual pseudorepresentations. Thus we obtain a bijection $|\mathcal{X}|_{\mathrm{cl}} \xrightarrow{\sim} |X|_{\mathrm{cl}}$, which, when composed with (3.3.5), yields a morphism

$$|\mathcal{X}|_{\mathrm{ft}} \to |X|_{\mathrm{cl}}. \tag{3.3.7}$$

Lemma 3.3.8. *The map* (3.3.7) *is continuous.*

Proof. The closed subsets of $|X|_{\mathrm{cl}}$ consist of finite unions of points and irreducible components, and so it suffices to show that the preimage of each of these in $|\mathcal{X}|_{\mathrm{ft}}$ is closed.

To see that the preimage of a point is closed, it suffices to prove that it is a finite union of closed sets, and so it suffices to prove that if $\overline{\rho}, \overline{\rho}' \in |\mathcal{X}|_{\mathrm{ft}}$, and $\overline{\rho}' \in \overline{\{\overline{\rho}\}}$ (the closure in $|\mathcal{X}|$ of $\{\overline{\rho}\}$), then $(\overline{\rho}')^{\mathrm{ss}} = \overline{\rho}^{\mathrm{ss}}$. This is a consequence of [EG23, Thm. 6.6.3 (3)]. In more detail, we find that $(\overline{\rho}')^{\mathrm{ss}} \in \overline{\{\overline{\rho}'\}} \subset \overline{\{\overline{\rho}\}}$, and so $(\overline{\rho}')^{\mathrm{ss}}$ must coincide with the unique closed point of $\overline{\{\overline{\rho}\}}$, which is $\overline{\rho}^{\mathrm{ss}}$, as desired.

On the other hand, the preimage of an irreducible component of X, corresponding to the companion pair $\{\sigma, \sigma^{\mathrm{co}}\}$, is equal either to $|\mathcal{X}(\sigma)|_{\mathrm{ft}} \cup |\mathcal{X}(\sigma^{\mathrm{co}})|_{\mathrm{ft}}$, if the pair $\{\sigma, \sigma^{\mathrm{co}}\}$ is not of the form $\{\sigma_{a,0}, \sigma_{a+1,p-3}\}$, or to $|\mathcal{X}(\sigma_{a,0})|_{\mathrm{ft}} \cup |\mathcal{X}(\sigma_{a,p-1})^+|_{\mathrm{ft}} \cup |\mathcal{X}(\sigma_{a,p-1})^-|_{\mathrm{ft}} \cup |\mathcal{X}(\sigma_{a+1,p-3})|_{\mathrm{ft}}$ otherwise. In either case, this is a union of irreducible components of $|\mathcal{X}|_{\mathrm{ft}}$, and thus closed. $\square$

Note that $|\mathcal{X}| = |\mathcal{X}_{\mathrm{red}}|$ (essentially by definition), and so, since $\mathcal{X}_{\mathrm{red}}$ is an algebraic stack of finite presentation over $\mathbf{F}$, the discussion in Section 3.3.2 applies to $|\mathcal{X}|$. In particular, we have a canonical homeomorphism $\mathrm{sob}(|\mathcal{X}|_{\mathrm{ft}}) \xrightarrow{\sim} |\mathcal{X}|$. Since X is a finite type scheme over $\mathbf{F}$, we also have a canonical homeomorphism $\mathrm{sob}(|X|_{\mathrm{cl}}) \xrightarrow{\sim} |X|$. Soberizing the map (3.3.7), and post- and pre-composing with the second of these homeomorphisms and the inverse of the first, we obtain the continuous morphism

$$\pi_{\mathrm{ss}} : |\mathcal{X}| \xrightarrow{\sim} \mathrm{sob}(|\mathcal{X}|_{\mathrm{ft}}) \overset{\mathrm{sob}\,(3.3.7)}{\longrightarrow} \mathrm{sob}(|X|_{\mathrm{cl}}) \xrightarrow{\sim} |X| \tag{3.3.9}$$

that we wished to construct.

3.4. **Stacks of Galois representations.** As we recall in greater generality in Appendix C.2.9, Wang–Erickson has associated to each pseudorepresentation $\overline{\theta}$ of $G_{\mathbf{Q}_p}$ over $\overline{\mathbf{F}}_p$, with determinant $\overline{\zeta}\omega^{-1}$, a Noetherian formal algebraic stack $\mathcal{X}_{\overline{\theta}}$. If A is an $\mathcal{O}$-algebra in which p is nilpotent, then $\mathcal{X}_{\overline{\theta}}(\mathrm{Spf}\, A)$ is the groupoid of continuous representations of $G_{\mathbf{Q}_p}$ on rank two projective A-modules of determinant $\zeta\varepsilon^{-1}$, whose associated residual pseudorepresentation is $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugate to $\overline{\theta}$. Passage to the associated pseudorepresentation yields then a morphism

$$\mathcal{X}_{\overline{\theta}} \to \mathrm{Spf}\, R^{\mathrm{ps}}_{\overline{\theta}}. \tag{3.4.1}$$

On the other hand, passage to the associated étale (φ, Γ)-module yields a morphism

$$i'_{\overline{\theta}} : \mathcal{X}_{\overline{\theta}} \to \mathcal{X}. \tag{3.4.2}$$

Theorem 3.4.3.

(1) *The morphism* (3.4.1) *is representable by algebraic stacks.*
(2) *The morphism* (3.4.2) *is a completion of $\mathcal{X}$ at the closed subset $|\mathcal{X}_{\overline{\theta}}| \subset |\mathcal{X}|$, which coincides with $\pi_{\mathrm{ss}}^{-1}\{\overline{\theta}\}$.*

Proof. Part (1) is a special case of Theorem C.2.13. The first claim of part (2) is a special case of Theorem C.2.12 (bearing in mind Remark C.2.17). Finally, since $|\mathcal{X}_{\overline{\theta}}|$ and $\pi_{\mathrm{ss}}^{-1}\{\overline{\theta}\}$ are closed subsets of $|\mathcal{X}|$ with the same set of finite type points, the equality $|\mathcal{X}_{\overline{\theta}}| = \pi_{\mathrm{ss}}^{-1}\{\overline{\theta}\}$ follows from (3.3.3). $\square$

Furthermore, as explained in Appendix C.2.9, Wang-Erickson shows in [Wan18, Section 3.2] that there is an algebraic stack $\mathfrak{X}_{\overline{\theta}}$ of finite type over $\mathrm{Spec}\, R^{\mathrm{ps}}_{\overline{\theta}}$, such that $\mathcal{X}_{\overline{\theta}}$ can be identified with the completion of $\mathfrak{X}_{\overline{\theta}}$ at the maximal ideal of $R^{\mathrm{ps}}_{\overline{\theta}}$.

By definition, $\mathfrak{X}_{\overline{\theta}}$ represents the moduli problem of 2-dimensional "compatible representations" of the Cayley–Hamilton algebra $\widetilde{R}_{\overline{\theta}}$ (see Definition 2.1.13) with determinant $\zeta\varepsilon^{-1}$. In particular, there is a canonical rank two vector bundle $\mathfrak{V}_{\overline{\theta}}$ lying over $\mathfrak{X}_{\overline{\theta}}$, endowed with an action of $\widetilde{R}_{\overline{\theta}}$.

Following [JNW24], we now give descriptions of $\mathfrak{X}_{\overline{\theta}}$, together with the bundle $\mathfrak{V}_{\overline{\theta}}$ lying over it, and the $\widetilde{R}_{\overline{\theta}}$-action on this bundle. We will also define a versal ring to $\mathcal{X}_{\overline{\theta}}$ at its unique closed point, and make the following definitions.

Definition 3.4.4. Let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation.

(1) We write $R^{\mathrm{ver}}_{\overline{\theta}}$ for the versal ring to $\mathcal{X}_{\overline{\theta}}$ defined in Sections 3.4.5—3.4.13, and $V^{\mathrm{ver}}_{\overline{\theta}}$ for the versal object on $R^{\mathrm{ver}}_{\overline{\theta}}$.
(2) For all pairs (λ,τ) as in Definition 3.2.2, we will write $R^{\lambda,\tau}_{\overline{\theta}}$ for the quotient of $R^{\mathrm{ver}}_{\overline{\theta}}$ defined by $\operatorname{Spf} R^{\lambda,\tau}_{\overline{\theta}} := \operatorname{Spf} R^{\mathrm{ver}}_{\overline{\theta}} \times_{\mathcal{X}} \mathcal{X}^{\lambda,\tau}$.
(3) If σ is a Serre weight, and (λ,τ) is the pair associated to σ in Proposition 3.2.5, we will write $R^{\sigma}_{\overline{\theta}}$ for the ring $R^{\lambda,\tau}_{\overline{\theta}}$. By Lemma 3.2.5, we thus have $\operatorname{Spf} R^{\sigma}_{\overline{\theta}}/\varpi = \operatorname{Spf} R^{\mathrm{ver}}_{\overline{\theta}}/\varpi \times_{\mathcal{X}/\varpi} \mathcal{Z}(\sigma)$.

When $\overline{\theta}$ is clear from the context, we will sometimes omit it from this notation.

3.4.5. *The case* (ssg). In this case, $\overline{\theta}$ is the trace of an absolutely irreducible representation $\overline{\rho}: G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{F})$. We let $R^{\mathrm{ver}}_{\overline{\theta}}$ be the universal deformation ring of $\overline{\rho}$, to complete Noetherian local $\mathcal{O}$-algebras, with determinant $\zeta\varepsilon^{-1}$. Then $\mathfrak{X}_{\overline{\theta}} \cong [\operatorname{Spec} R^{\mathrm{ver}}_{\overline{\theta}}/\boldsymbol{\mu}_2]$ for the trivial action of $\boldsymbol{\mu}_2$, $\mathfrak{V}_{\overline{\theta}} = (R^{\mathrm{ver}}_{\overline{\theta}})^{\oplus 2}$ placed in degree -1, and $\widetilde{R}_{\overline{\theta}}$ is isomorphic to $M_2(R^{\mathrm{ver}}_{\overline{\theta}})$ with its natural action on $\mathfrak{V}_{\overline{\theta}}$. See for example [JNW24, Section 3.1].

3.4.6. *The case* (gen). After a twist (which possibly changes ζ) , we can assume without loss of generality that $\overline{\theta} = \mathrm{nr}_{\lambda}\omega^r + \mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^{-1}$ for some $\lambda \in \mathbf{F}^{\times}$ and some $0 \le r \le p-2$, such that $\lambda \neq \pm\zeta(p)^{1/2}$ if $r \in \{0, p-2\}$. As explained in [JNW24, Section 3.2], the stack $\mathfrak{X}_{\overline{\theta}}$ admits the following description as a torus quotient: Let

$$S = \mathcal{O}[\![a_0, a_1, X]\!][b,c]/(bc - X),$$

with grading

$$\deg(b) = 2, \deg(c) = -2, \deg(a_0) = \deg(a_1) = 0.$$

Then $\mathfrak{X}_{\overline{\theta}} \cong [\operatorname{Spec} S/\mathbf{G}_m]$ (the $\mathbf{G}_m$-action being determined by the grading on S). The map $R^{\mathrm{ps}}_{\overline{\theta}} \to S$, corresponding to the morphism $\mathfrak{X}_{\overline{\theta}} \to \operatorname{Spec} R^{\mathrm{ps}}_{\overline{\theta}}$, induces an isomorphism

$$R^{\mathrm{ps}}_{\overline{\theta}} \xrightarrow{\sim} \mathcal{O}[\![a_0, a_1, X]\!],$$

identifying $R^{\mathrm{ps}}_{\overline{\theta}}$ with the degree zero component of S.

This description of $\mathfrak{X}_{\overline{\theta}}$ allows us to identify the category $\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}})$ with the category of finitely generated graded S-modules. As usual, if M is a graded S-module, we write $M(i)$ for the graded shift given by

$$M(i)_n := M(n+i).$$

The universal rank two vector bundle $\mathfrak{V}_{\overline{\theta}}$ on $\mathfrak{X}_{\overline{\theta}}$ corresponds to the graded module $S(1) \oplus S(-1)$. The action of $\widetilde{R}_{\overline{\theta}}$ on $\mathfrak{V}_{\overline{\theta}}$ is described by identifying $\widetilde{R}_{\overline{\theta}}$ (as an

$R^{\mathrm{ps}}_{\overline{\theta}}$-algebra) with the matrix order

$$\mathrm{End}_{\mathrm{S\text{-}gr}}(\mathfrak{V}_{\overline{\theta}}) = \begin{pmatrix} R^{\mathrm{ps}}_{\overline{\theta}} & R^{\mathrm{ps}}_{\overline{\theta}}b \\ R^{\mathrm{ps}}_{\overline{\theta}}c & R^{\mathrm{ps}}_{\overline{\theta}} \end{pmatrix} \subseteq M_2(S) \tag{3.4.7}$$

acting by left multiplication on $\mathfrak{V}_{\overline{\theta}}$ (thought of as column vectors) (see [JNW24, Thm. 3.2.1]). We write e_{ij} for the matrix in $M_2(S)$ with 1 in the ij-th entry, and zero elsewhere, and we normalize our presentation of S in such a way that $\widetilde{R}_{\overline{\theta}}e_{11}$ (i.e. the first column of $\widetilde{R}_{\overline{\theta}}$) is the $\widetilde{R}_{\overline{\theta}}$-projective envelope of the simple module $\mathrm{nr}_{\lambda}\omega^r$. Then the completion $R^{\mathrm{ver}}_{\overline{\theta}}$ of S at the maximal homogeneous ideal is a versal ring to $\mathcal{X}$ at $\overline{\rho} \coloneqq \mathrm{nr}_{\lambda}\omega^r \oplus \mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^{-1}$.

3.4.8. *The case* (scalar). In this case, $\overline{\theta}$ is not multiplicity-free, and so $\mathfrak{X}_{\overline{\theta}}$ does not have a natural presentation as a $\mathbf{G}_m$-quotient of an affine scheme. Rather, $\mathfrak{X}_{\overline{\theta}}$ is the stack denoted $\mathrm{Rep}(E)$ in [JNW24, Section 3.3], where $E = \widetilde{R}_{\overline{\theta}}$, and so it is a global quotient $\mathrm{Rep}(E) = [\mathrm{Rep}^{\square}(E)/\mathrm{SL}_2]$ for some affine scheme $\mathrm{Rep}^{\square}(E) = \mathrm{Spec}\,A$. A presentation of A as an $R^{\mathrm{ps}}_{\overline{\theta}}$-algebra is given after the proof of [JNW24, Proposition 3.3.9]; we remark that

$$A \otimes_{R^{\mathrm{ps}}_{\overline{\theta}}} R^{\mathrm{ps}}_{\overline{\theta}}/\mathfrak{m} \cong \mathbf{F}[a, b, c_1, c_2, d_1, d_2]/(a^2 + c_1c_2, b^2 + d_1d_2, 2ab + (c_1d_2 + c_2d_1)),$$

which is not a reduced ring (since for example $c_1d_2 - c_2d_1$ is a nonzero element which squares to zero). There is a unique maximal SL_2-invariant ideal $\mathfrak{n}$ of A, and it is a maximal ideal; the corresponding morphism

$$\mathrm{Spec}\,A/\mathfrak{n} \hookrightarrow \mathrm{Spf}\,A \to \mathcal{X}_{\overline{\theta}} \hookrightarrow \mathcal{X}$$

classifies the unique scalar Galois representation $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(A/\mathfrak{n})$ having trace $\overline{\theta}$. We write $R^{\mathrm{ver}}_{\overline{\theta}}$ for the $\mathfrak{n}$-adic completion of A, which is a versal ring to $\mathcal{X}$ at the closed point corresponding to this $\overline{\rho}$.

3.4.9. *The case* (St). After a twist (which possibly changes ζ), we can assume without loss of generality that $\overline{\theta} = 1 + \omega^{-1}$. As explained in [JNW24, Section 3.4], the stack $\mathfrak{X}_{\overline{\theta}}$ admits the following description as a torus quotient: Let

$$S = \mathcal{O}[\![a_0, a_1, X_0, X_1]\!][b_0, b_1, c]/(X_0 - b_0c, X_1 - b_1c, a_1b_0 + a_0b_1),$$

with grading

$$\deg(b_0) = \deg(b_1) = 2, \deg(c) = -2, \deg(a_0) = \deg(a_1) = 0.$$

Then $\mathfrak{X}_{\overline{\theta}} \cong [\mathrm{Spec}\,S/\mathbf{G}_m]$ (the $\mathbf{G}_m$-action being determined by the grading on S). The map $R^{\mathrm{ps}}_{\overline{\theta}} \to S$, corresponding to the morphism $\mathfrak{X}_{\overline{\theta}} \to \mathrm{Spec}\,R^{\mathrm{ps}}_{\overline{\theta}}$, induces an isomorphism

$$R^{\mathrm{ps}}_{\overline{\theta}} \xrightarrow{\sim} \mathcal{O}[\![a_0, a_1, X_0, X_1]\!]/(a_0X_1 + a_1X_0),$$

identifying $R^{\mathrm{ps}}_{\overline{\theta}}$ with the degree zero component of S. Note that our a_1 is denoted $a_1' = a_1 + p$ in [JNW24], and our X_i is denoted b_ic.

The universal rank two vector bundle $\mathfrak{V}_{\overline{\theta}}$ on $\mathfrak{X}_{\overline{\theta}}$ corresponds to the graded module $S(1) \oplus S(-1)$. The action of $\widetilde{R}_{\overline{\theta}}$ on $\mathfrak{V}_{\overline{\theta}}$ is described by identifying $\widetilde{R}_{\overline{\theta}}$ (as an $R^{\mathrm{ps}}_{\overline{\theta}}$-algebra) with the matrix order

$$\mathrm{End}_{\mathrm{S\text{-}gr}}(\mathfrak{V}_{\overline{\theta}}) = \begin{pmatrix} R^{\mathrm{ps}}_{\overline{\theta}} & R^{\mathrm{ps}}_{\overline{\theta}}b_0 + R^{\mathrm{ps}}_{\overline{\theta}}b_1 \\ R^{\mathrm{ps}}_{\overline{\theta}}c & R^{\mathrm{ps}}_{\overline{\theta}} \end{pmatrix} \subseteq M_2(S) \tag{3.4.10}$$

acting by left multiplication on $\mathfrak{V}_{\overline{\theta}}$ (thought of as column vectors) (see [JNW24, Thm. 3.4.1]). Again we write e_{ij} for the matrix in $M_2(S)$ with 1 in the ij-th entry, and zero elsewhere. Then the modules $\widetilde{R}_{\overline{\theta}}e_{11}, \widetilde{R}_{\overline{\theta}}e_{22}$ are the $\widetilde{R}_{\overline{\theta}}$-projective envelopes of the simple modules 1, ω^{-1} respectively. We let R^{ver} denote the completion of S at its maximal homogeneous ideal; then R^{ver} is a versal ring to $\mathcal{X}$ at the closed point corresponding to $\overline{\rho} := 1 \oplus \omega^{-1}$.

It will also be useful to have some notation for the scheme-theoretic union of the (finitely many) substacks $\mathcal{X}_{\overline{\theta}} \subset \mathcal{X}$ where $\overline{\theta}$ is of type (St), and so we make the following definition.

Definition 3.4.11. We write $\mathcal{X}(\mathrm{St}) := \bigcup_{\overline{\theta} \text{ of type (St)}} \mathcal{X}_{\overline{\theta}}$.

Remark 3.4.12. For $\overline{\theta}$ of type (St), the stack $\mathcal{X}_{\overline{\theta},\mathrm{red}}$ coincides with one of the irreducible components $\mathcal{X}(\sigma_{a,p-1})^{\pm}$ (with a and the sign $\pm$ depending on $\overline{\theta}$). Thus $\mathcal{X}(\mathrm{St})_{\mathrm{red}}$ is the disjoint union of four irreducible components $\mathcal{X}(\sigma_{a,p-1})^{\pm}$ (assuming that it is non-empty, i.e. that ζ is even).

3.4.13. *The case* (gen+). If $\overline{\theta}$ is a $\overline{\mathbf{F}}_p$-pseudorepresentation of type (gen+), then there exists a smooth character $\chi : G_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^{\times}$ such that $\overline{\theta} = \chi + \zeta\omega^{-1}\chi^{-1}$, and χ does not factor through $\mathbf{F}^{\times}$. Writing $\mathbf{F}_{\overline{\theta}}/\mathbf{F}$ for the extension generated by the values of $\overline{\theta}$, we can identify $\overline{\theta}$ with an $\mathbf{F}_{\overline{\theta}}$-valued pseudorepresentation $\overline{\theta}'$, and by definition, we have equalities

$$R^{\mathrm{ps}}_{\overline{\theta}} = R^{\mathrm{ps}}_{\overline{\theta}'}, \widetilde{R}_{\overline{\theta}} = \widetilde{R}_{\overline{\theta}'}, \mathfrak{X}_{\overline{\theta}} = \mathfrak{X}_{\overline{\theta}'}, \mathcal{X}_{\overline{\theta}} = \mathcal{X}_{\overline{\theta}'}.$$

Up to natural isomorphism, these objects only depend on the $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy class of $\overline{\theta}$.

Note that $\mathbf{F}_{\overline{\theta}}$ almost always coincides with the extension $\mathbf{F}'_{\overline{\theta}}/\mathbf{F}$ generated by the values of χ, and then $\overline{\theta}'$ is the trace of a semisimple reducible Galois representation, classified by a morphism $\operatorname{Spec} \mathbf{F}_{\overline{\theta}} \to \mathcal{X}_{\overline{\theta}}$. We then let $R^{\mathrm{ver}}_{\overline{\theta}}$ be the versal ring at $\operatorname{Spec} \mathbf{F}_{\overline{\theta}} \to \mathcal{X}_{\overline{\theta}}$ defined in Section 3.4.6.

The only exception is when χ and $\zeta\omega^{-1}\chi^{-1}$ are Galois conjugates over $\mathbf{F}$, in which case $\overline{\theta}'$ is the trace of an irreducible Galois representation $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{F}_{\overline{\theta}})$, for which $\overline{\rho} \otimes_{\mathbf{F}_{\overline{\theta}}} \mathbf{F}'_{\overline{\theta}}$ is reducible. We have a morphism $\operatorname{Spec} \mathbf{F}'_{\overline{\theta}} \to \mathcal{X}_{\overline{\theta}}$ classifying $\overline{\rho} \otimes_{\mathbf{F}_{\overline{\theta}}} \mathbf{F}'_{\overline{\theta}}$, and we let

$$\operatorname{Spf} R^{\mathrm{ver}}_{\overline{\theta}} \to \mathcal{X}_{\overline{\theta}} \times_{W(\mathbf{F}_{\overline{\theta}})} W(\mathbf{F}'_{\overline{\theta}})$$

be the versal morphism to $\operatorname{Spec} \mathbf{F}'_{\overline{\theta}} \to \mathcal{X}_{\overline{\theta}} \times_{W(\mathbf{F}_{\overline{\theta}})} W(\mathbf{F}'_{\overline{\theta}})$ defined in Section 3.4.6. Note that the composite $\operatorname{Spf} R^{\mathrm{ver}}_{\overline{\theta}} \to \mathcal{X}_{\overline{\theta}}$ is versal to $\operatorname{Spec} \mathbf{F}'_{\overline{\theta}} \to \mathcal{X}_{\overline{\theta}}$.

3.5. **Coherent sheaves on stacks of Galois representations.**

We fix throughout this section a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation $\overline{\theta}$ of $G_{\mathbf{Q}_p}$. We will discuss some coherent sheaf theory on the stacks $\mathfrak{X}_{\overline{\theta}}$ and $\mathcal{X}_{\overline{\theta}}$ defined in Section 3.4. To ease notation, we write $R := R^{\mathrm{ps}}_{\overline{\theta}}$ and $\mathfrak{m} := \mathfrak{m}_R$. We write $\mathfrak{X}_0$ to denote the underlying reduced substack of $\mathcal{X}_{\overline{\theta}}$; this coincides with the underlying reduced substack of the base-change $\mathfrak{X}_{\overline{\theta}} \times_{\operatorname{Spec} R} \operatorname{Spec} R/\mathfrak{m}$.

Remark 3.5.1. We saw above (see in particular Section 3.4.8 that in case (scalar)) that in fact $\mathfrak{X}_{\overline{\theta}} \times_{\operatorname{Spec} R} \operatorname{Spec} R/\mathfrak{m}$ is reduced, except in case (scalar).

Adopting notation from Appendix C.2.9, we write $k_{\overline{\theta}} : \mathcal{X}_{\overline{\theta}} \to \mathfrak{X}_{\overline{\theta}}$ for the completion map, and

$$k_{\overline{\theta},*} : D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}) \to D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}),$$
$$\widehat{k}^*_{\overline{\theta}} : D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$$

for the pushforward and completed pullback functors. Their basic properties are summarized in Theorem C.2.15 (see also Remark C.2.17).

3.5.2. *Tensoring pro-coherent sheaves with compact modules.* We now fix a (not necessarily commutative) R-algebra E which is finite and flat as an R-module. We regard E as a topological R-algebra by endowing it with its $\mathfrak{m}$-adic topology. With this topology, E is a Noetherian profinite $\mathcal{O}$-algebra, and so we can apply the material of Section A.1.30. We also suppose given a coherent sheaf $\mathfrak{F}$ on $\mathfrak{X}_{\overline{\theta}}$ endowed with a right action of E extending the natural R-action on $\mathfrak{F}$. We write $\mathcal{F} := \widehat{k}^*_{\overline{\theta}}\mathfrak{F}$, so $\mathcal{F}$ is the $\mathfrak{m}$-adic completion of $\mathfrak{F}$; then $\mathcal{F}$ is an object of $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$ endowed with a right E-action extending the R-action on $\mathfrak{F}$.

Our goal in this subsection is to discuss the various functors we can obtain by tensoring $\mathfrak{F}$ and $\mathcal{F}$ with different flavours of E-modules. We first discuss things on the abelian level, before turning to the analogous constructions on the level of stable ∞-categories.

By Proposition A.1.48 we have a right exact functor

$$\mathfrak{F} \otimes_E - : \operatorname{Mod}^{\mathrm{fp}}(E) \to \operatorname{Coh}(\mathfrak{X}_{\overline{\theta}}), \tag{3.5.3}$$

characterized up to unique isomorphism by sending E to $\mathfrak{F}$. Similarly, we have a unique right exact functor

$$\mathcal{F} \otimes_E - : \operatorname{Mod}^{\mathrm{fp}}(E) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}}) \tag{3.5.4}$$

sending E to $\mathcal{F}$. Since $\mathcal{F} = \widehat{k}^*_{\overline{\theta}}\mathfrak{F}$, we thus have a natural isomorphism

$$\mathcal{F} \otimes_E - \xrightarrow{\sim} \widehat{k}^*_{\overline{\theta}} \circ (\mathfrak{F} \otimes_E -). \tag{3.5.5}$$

On the other hand, by definition, $\mathcal{F}$ is complete as an R-module object of $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$. Remark A.10.19 then shows that $\mathcal{F}$ is also derived complete, and in turn, by Lemma A.10.18, we see that $\mathcal{F}$ is complete and derived complete as an E-module. We thus obtain, by Lemma A.1.53, a unique right exact and cofiltered limit-preserving functor

$$\mathcal{F}\,\widehat{\otimes}_E - : \operatorname{Mod}_c(E) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}}) \tag{3.5.6}$$

sending E to $\mathcal{F}$.

Lemma 3.5.7. *With the notation of the previous paragraph, there exists a unique (up to isomorphism) right exact functor*

$$\mathcal{F} \otimes_E - : \operatorname{Mod}^{\mathrm{f.l.}}(E) \to \operatorname{Coh}(\mathcal{X}_{\overline{\theta}}) \tag{3.5.8}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Mod}^{\mathrm{f.l.}}(E) \arrow[r, hook] \arrow[d, "(3.5.8)"', "\mathcal{F}\otimes_E -"] & \operatorname{Mod}^{\mathrm{fp}}(E) \arrow[r, hook] \arrow[d, "(3.5.3)"', "\mathfrak{F}\otimes_E -"] & \operatorname{Mod}_c(E) \arrow[d, "(3.5.6)"', "\mathcal{F}\,\widehat{\otimes}_E -"] \\
\operatorname{Coh}(\mathcal{X}_{\overline{\theta}}) \arrow[r, hook, "k_{\overline{\theta},*}"] & \operatorname{Coh}(\mathfrak{X}_{\overline{\theta}}) \arrow[r, hook, "\widehat{k}^*_{\overline{\theta}}"] & \operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}}).
\end{tikzcd}$$

Furthermore, $\mathcal{F}\,\widehat{\otimes}_E -$ is the Pro*-extension of $\mathcal{F} \otimes_E -$.*

Proof. By Corollary A.1.54, the restriction of (3.5.6) through $\mathrm{Mod}^{\mathrm{fp}}(E) \to \mathrm{Mod}_c(E)$ is (3.5.4). Hence (3.5.5) shows that the rightmost square commutes.

We now construct the left-hand square. Each of the quotients

$$\mathfrak{F}/\mathfrak{m}^n\mathfrak{F} := \mathfrak{F} \otimes_E (E/\mathfrak{m}^n E) = \mathcal{F} \otimes_E (E/\mathfrak{m}^n E)$$

is an object of $\mathrm{Coh}_{\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}})$, since $\mathfrak{X}_0$ was defined to be the underlying reduced substack of the vanishing locus of $\mathfrak{m}$. If N is a finite length E-module, then it is annihilated by some power of $\mathfrak{m}$, so we find more generally that $\mathfrak{F} \otimes_E N$ lies in $\mathrm{Coh}_{\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}})$, which is to say that (3.5.3) restricts to a right exact functor

$$\mathfrak{F} \otimes_E - : \mathrm{Mod}^{\mathrm{f.l.}}(E) \to \mathrm{Coh}_{\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}}). \tag{3.5.9}$$

Theorem C.2.15 shows that $k_{\overline{\theta},*}$ is an equivalence $\mathrm{Coh}(\mathcal{X}_{\overline{\theta}}) \xrightarrow{\sim} \mathrm{Coh}_{\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}})$ We deduce the existence and uniqueness of (3.5.8), and the commutativity of the left-hand square. Finally, bearing in mind that $\mathrm{Mod}_c(E) = \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(E)$ and $\mathcal{F}\,\widehat{\otimes}_E-$ preserves cofiltered limits, we see that the commutativity of the outer rectangle implies that $\mathcal{F}\,\widehat{\otimes}_E-$ is the Pro-extension of $\mathcal{F} \otimes_E -$, as desired. □

We now apply Lemma A.10.10 to construct derived versions of the various tensor products just introduced. Namely, Lemma A.10.10 (1) shows that (3.5.3) induces a right t-exact functor

$$\mathfrak{F} \otimes^L_E - : D^b_{\mathrm{fp}}(E) \to D^-\big(\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}})\big), \tag{3.5.10}$$

while Lemma A.10.10 (2) shows that (3.5.4) induces a right t-exact functor

$$\mathcal{F} \otimes^L_E - : D^b_{\mathrm{fp}}(E) \to \mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}), \tag{3.5.11}$$

where we implicitly use the equivalence $D^b\big(\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})\big) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$. Finally, Lemma A.10.10 (3) shows that (3.5.8) induces a right t-exact limit preserving functor

$$\mathcal{F}\,\widehat{\otimes}^L_E- : \mathrm{Pro}\,D^b_{\mathrm{f.l.}}(E) \to \mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}). \tag{3.5.12}$$

For simplicity, from now on we will make the additional assumption that E has finite global dimension; in our applications, this will be true by Lemma 2.3.11 (4). The functor (3.5.10) then factors through $D^b(\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}}))$, by Lemma A.10.26.

Lemma 3.5.13. *Assume that E has finite global dimension. Then there exists a unique (up to isomorphism) right t-exact functor*

$$\mathcal{F} \otimes^L_E - : D^b_{\mathrm{f.l.}}(E) \to D^b\big(\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})\big) \tag{3.5.14}$$

such that the following diagram commutes:

$$\begin{tikzcd}
D^b_{\mathrm{f.l.}}(E) \arrow[r, hook, "(\mathrm{A.10.6})"] \arrow[d, "\mathcal{F}\otimes^L_E-"] & D^b_{\mathrm{fp}}(E) \arrow[r, hook, "(\mathrm{A.10.8})"] \arrow[d, "\mathfrak{F}\otimes^L_E-"] \arrow[dr, "\mathcal{F}\otimes^L_E-"] & \mathrm{Pro}\,D^b_{\mathrm{f.l.}}(E) \arrow[d, "\mathcal{F}\,\widehat{\otimes}^L_E-"] \\
D^b\big(\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})\big) \arrow[r, hook, "k_{\overline{\theta},*}"] & D^b\big(\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}})\big) \arrow[r, hook, "\widehat{k}^*_{\overline{\theta}}"] & \mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}).
\end{tikzcd}$$

Proof. The upper triangle commutes by Lemma A.10.15. The two directions in the lower triangle are right t-exact functors $D^b_{\mathrm{fp}}(E) \to \mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$, sending E to $\mathcal{F}$, resp. $\widehat{k}^*_{\overline{\theta}}\mathfrak{F} = \mathcal{F}$. Hence they are naturally isomorphic, by Lemma A.10.10 (2).

We now construct (3.5.14), and prove the commutativity of the left-hand square. Since $\widehat{k}_{\overline{\theta}}^* : D^b(\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})) \to D^b_{\mathrm{coh},\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}})$ is an equivalence, by Theorem C.2.15, it suffices to prove that the composite

$$D^b_{\mathrm{f.l.}}(E) \to D^b_{\mathrm{fp}}(E) \xrightarrow{\mathfrak{F}\otimes^L_E -} D^b(\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}}))$$

factors through $D^b_{\mathrm{coh},\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}})$, i.e. the full subcategory of $D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})$ whose objects are the bounded complexes of coherent sheaves whose cohomology is set-theoretically supported on the vanishing locus $\mathfrak{X}_0$ of $\mathfrak{m} := \mathrm{rad}(R)$ in $\mathfrak{X}$. By induction on the amplitude of objects in $D^b_{\mathrm{f.l.}}(E)$, it suffices to prove that $\mathfrak{F}\otimes^L_E N \in D^b_{\mathrm{coh},\mathfrak{X}_0}(\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}}))$ whenever N is an object of $\mathrm{Mod}^{\mathrm{f.l.}}(E)$, which is a consequence of the fact that N is annihilated by a power of $\mathfrak{m}$. $\square$

3.5.15. *Tensoring pro-coherent sheaves with G-representations.* We now fix a projective generator P of $\mathfrak{C}_{\overline{\theta}}$, chosen to have finite length cosocle, and write

$$E := \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P).$$

Recall from Lemma 2.3.10 and Lemma 2.3.11 (4) that E has finite type as a module over its centre (which is the Bernstein centre of $\mathfrak{C}_{\overline{\theta}}$), has finite global dimension, and is Noetherian. Hence, by Lemma A.1.32 (8), the natural topology on E (that it inherits as the endomorphism ring of the compact $\mathcal{O}[\![G]\!]_\zeta$-module P) coincides with its $\mathfrak{m}$-adic topology, and makes E a Noetherian profinite $\mathcal{O}$-algebra. Recall also from Remark 2.4.16 that there is a unique isomorphism $R \xrightarrow{\sim} Z(E) = \mathcal{Z}(\mathfrak{C}_{\mathfrak{B}})$ such that the functor $\check{\mathrm{V}}$ is R-linear. Furthermore, we assume given a coherent sheaf $\mathfrak{F}$ on $\mathfrak{X}_{\overline{\theta}}$ with a left E^{op}-action, so that we are in the situation of Section 3.5.2.

Applying the functor (3.5.6) to P, we obtain an object $\mathcal{F}\widehat{\otimes}_E P$ of $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})$. Since P is projective in $\mathrm{Mod}_c(E)$ (by Lemma 2.3.11 (4)), this coincides with the object $\mathcal{F}\widehat{\otimes}^L_E P$ obtained by applying (3.5.12) to P. The right $\mathcal{O}[\![G]\!]_\zeta$-action on P induces a right $\mathcal{O}[\![G]\!]_\zeta$-action on $\mathcal{F}\widehat{\otimes}_E P$. An application of Lemma A.10.10 (2) then produces a right t-exact functor

$$(\mathcal{F}\widehat{\otimes}_E P)\otimes^L_{\mathcal{O}[\![G]\!]_\zeta} - : D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}).$$

Another application of Lemma A.10.10 (2), this time to P itself, yields a right t-exact functor

$$P\otimes^L_{\mathcal{O}[\![G]\!]_\zeta} - : D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Pro}\, D^b_{\mathrm{f.l.}}(E).$$

In fact this functor may be factored as a composite

$$P\otimes^L_{\mathcal{O}[\![G]\!]_\zeta} - : D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to D^-\big(\mathrm{Mod}_c(E)\big) \to \mathrm{Pro}\, D^b_{\mathrm{f.l.}}(E), \tag{3.5.16}$$

where the first arrow is the restriction to

$$D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \xrightarrow{\sim} D^b\big(\mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)\big) \subset D^-\big(\mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)\big)$$

of the (right t-exact) derived functor of the functor

$$P\otimes_{\mathcal{O}[\![G]\!]_\zeta} - : \mathrm{Mod}^{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Mod}_c(E)$$

arising from an application of Theorem A.7.15; and the second arrow is the t-exact functor

$$D^-\big(\mathrm{Mod}_c(E)\big) \xrightarrow{\sim} D^-\big(\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(E)\big) \xrightarrow{p} \mathrm{Pro}\, D^b_{\mathrm{f.l.}}(E)$$

obtained as a special case of the diagram (A.8.5), taking $\mathcal{C}$ there to be $\mathrm{Mod}^{\mathrm{f.l.}}(E)$, and recalling the equivalence $\mathrm{Mod}_c(E) \xrightarrow{\sim} \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(E)$. That this composite coincides, up to natural isomorphism, with the functor $P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} -$ follows from Lemma A.10.10 (2), since both functors are right t-exact, and both take $\mathcal{O}[\![G]\!]_\zeta$ to the E-module object P in $\mathrm{Mod}_c(E) \xrightarrow{\sim} \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(E)$.

We will often make implicit use of the following lemmas involving these two functors.

Lemma 3.5.17. *The composite*

$$D^b_{\mathrm{fp}}(\mathcal{A}) \xrightarrow{(2.2.26)} D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \xrightarrow{P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} -} \mathrm{Pro}\,D^b_{\mathrm{f.l.}}(E)$$

is t-exact.

Proof. Let F denote the composite functor appearing in the statement of the lemma. By Corollary A.7.21, it suffices to prove that the restriction of F to $\mathcal{A}^{\mathrm{fp}}$ (which is the heart of $D^b(\mathcal{A}^{\mathrm{fp}}) \xrightarrow{\sim} D^b_{\mathrm{fp}}(\mathcal{A})$) takes values in the heart $\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(E)$ of $\mathrm{Pro}\,D^b_{\mathrm{f.l.}}(E)$.

By the alternative description of $P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} -$ provided by (3.5.16), it suffices to prove that the composite

$$\mathcal{A}^{\mathrm{fp}} \subset D^b_{\mathrm{fp}}(\mathcal{A}) \xrightarrow{(2.2.26)} D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to D^-(\mathrm{Mod}_c(E))$$

has essential image in $\mathrm{Mod}_c(E)$. To do so, it suffices to prove that the composite

$$\mathcal{A}^{\mathrm{fp}} \subset D^b_{\mathrm{fp}}(\mathcal{A}) \xrightarrow{(2.2.26)} D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to D^-(\mathrm{Mod}_c(E)) \to D^-(\mathrm{Mod}(E))$$

has essential image in $\mathrm{Mod}(E)$, where the final arrow is the t-exact extension of the forgetful functor $\mathrm{Mod}_c(E) \to \mathrm{Mod}(E)$ (which is exact and conservative, by Lemma A.1.32 (1)). Now the composite $D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to D^-(\mathrm{Mod}(E))$ is right t-exact and takes $\mathcal{O}[\![G]\!]_\zeta$ to P, hence coincides with the left derived functor of the usual tensor product $P \otimes_{\mathcal{O}[\![G]\!]_\zeta} -$, by Theorem A.7.15. Hence our claim follows from Lemma 2.2.46 (2). $\square$

Lemma 3.5.18. *There is a natural isomorphism of right t-exact functors $D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$*

$$(3.5.19)\qquad (\mathcal{F} \mathbin{\widehat{\otimes}}_E P) \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} - = (\mathcal{F} \mathbin{\widehat{\otimes}}^L_E P) \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} - \xrightarrow{\sim} \mathcal{F} \mathbin{\widehat{\otimes}}^L_E (P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} -),$$

and there is a natural isomorphism of right t-exact functors $\mathrm{Pro}\,D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$

$$(3.5.20)\qquad (\mathcal{F} \mathbin{\widehat{\otimes}}_E P) \mathbin{\widehat{\otimes}}^L_{\mathcal{O}[\![G]\!]_\zeta} - = (\mathcal{F} \mathbin{\widehat{\otimes}}^L_E P) \mathbin{\widehat{\otimes}}^L_{\mathcal{O}[\![G]\!]_\zeta} - \xrightarrow{\sim} \mathcal{F} \mathbin{\widehat{\otimes}}^L_E (P \mathbin{\widehat{\otimes}}^L_{\mathcal{O}[\![G]\!]_\zeta} -).$$

Proof. Recalling again that P is projective in $\mathrm{Mod}_c(E)$, Lemma A.10.10 (2) shows that to prove (3.5.19), it suffices to note that each functor takes $\mathcal{O}[\![G]\!]_\zeta$ to the object $\mathcal{F} \mathbin{\widehat{\otimes}}^L_E P = \mathcal{F} \mathbin{\widehat{\otimes}}_E P$ of $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})$. On the other hand, since $P \mathbin{\widehat{\otimes}}^L_{\mathcal{O}[\![G]\!]_\zeta} -$, resp. $(\mathcal{F} \mathbin{\widehat{\otimes}}^L_E P) \mathbin{\widehat{\otimes}}^L_{\mathcal{O}[\![G]\!]_\zeta} -$, is the Pro-extension of $P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} -$, resp. $(\mathcal{F} \mathbin{\widehat{\otimes}}^L_E P) \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} -$, and $\mathcal{F} \mathbin{\widehat{\otimes}}^L_E -$ is cofiltered limit-preserving, the isomorphism (3.5.20) can be taken to be the Pro-extension of (3.5.19). $\square$

3.5.21. *The universal vector bundle.* Recall from Section 3.4 that the universal object on $\mathfrak{X}_{\overline{\theta}}$ is denoted $\mathfrak{V}_{\overline{\theta}}$. It is a free $\mathcal{O}_{\mathfrak{X}_{\overline{\theta}}}$-module of rank two with a left action of $\widetilde{R}_{\overline{\theta}}$. Its pullback via $\widehat{k}^*_{\overline{\theta}}$ to $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$ is denoted

$$\mathcal{V}_{\overline{\theta}} := \widehat{k}^*_{\overline{\theta}}\mathfrak{V}_{\overline{\theta}}. \tag{3.5.22}$$

We may apply the discussion of Section 3.5.2 to $\mathfrak{V}_{\overline{\theta}}$ and $\mathcal{V}_{\overline{\theta}}$, viewing them as right $\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}$-modules. In particular, from the commutative diagram of Lemma 3.5.7, we obtain the commutative diagram

$$\begin{tikzcd} \operatorname{Mod}^{\mathrm{f.l.}}(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}) \arrow[r, "\mathcal{V}_{\overline{\theta}}\otimes_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}}-"] \arrow[dr, "\mathfrak{V}_{\overline{\theta}}\otimes_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}}-"'] & \operatorname{Coh}(\mathcal{X}_{\overline{\theta}}) \arrow[d, hook] \\ & \operatorname{Coh}(\mathfrak{X}_{\overline{\theta}}) \end{tikzcd} \tag{3.5.23}$$

We now describe the main properties of $\mathfrak{V}_{\overline{\theta}}$ and of $\mathcal{V}_{\overline{\theta}}$ that we will use in Section 5. These results are essentially due to Johansson–Newton–Wang-Erickson [JNW24].

Proposition 3.5.24. *Assume that $\overline{\theta}$ does not have type* (St)*. Then $\mathcal{V}_{\overline{\theta}}$, which is a complete right $\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}$-module object in* $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$*, is topologically flat* (*in the sense of Definition* A.10.23).

Proof. By the discussion in Section 3.4.13, we can assume without loss of generality that $\mathbf{F} = \mathbf{F}_{\overline{\theta}}$. We need to prove that the functor

$$\mathcal{V}_{\overline{\theta}}\,\widehat{\otimes}_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}}- : \operatorname{Mod}_c(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$$

is exact. Since $\operatorname{Mod}_c(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}) = \operatorname{Pro}\operatorname{Mod}^{\mathrm{f.l.}}(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}})$, it suffices to prove that its restriction to $\operatorname{Mod}^{\mathrm{f.l.}}(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}})$ is exact. By Lemma 3.5.7, this restricted functor is the same as the composite

$$\operatorname{Mod}^{\mathrm{f.l.}}(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}) \to \operatorname{Mod}^{\mathrm{fp}}(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}) \xrightarrow{\mathfrak{V}_{\overline{\theta}}\otimes_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}}-} \operatorname{Coh}(\mathfrak{X}_{\overline{\theta}}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}}).$$

The last arrow is exact, so it suffices to prove that the composite of the first two arrows is exact.

Assume first that $\overline{\theta}$ does not have type (scalar). We will prove that $\mathfrak{V}_{\overline{\theta}}$ is flat as a right $\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}$-module, or equivalently, flat as a left $\widetilde{R}_{\overline{\theta}}$-module. This can be verified after base change to a quadratic unramified extension $\mathcal{O}'/\mathcal{O}$, hence we can assume without loss of generality that $\overline{\theta}$ has type (ssg) or type (gen). For type (ssg), the flatness of $\mathfrak{V}_{\overline{\theta}}$ over $\widetilde{R}_{\overline{\theta}}$ is clear from Section 3.4.5. Indeed, we have $\mathfrak{X}_{\overline{\theta}} \cong [\operatorname{Spec} R^{\mathrm{ps}}_{\overline{\theta}}/\boldsymbol{\mu}_2]$, while $\mathfrak{V}_{\overline{\theta}} \cong R^{\mathrm{ps}}_{\overline{\theta}}(1) \oplus R^{\mathrm{ps}}_{\overline{\theta}}(1)$, and $\widetilde{R}_{\overline{\theta}} \cong M_2(R^{\mathrm{ps}}_{\overline{\theta}})$ with its standard left action on $\mathfrak{V}_{\overline{\theta}}$ (thought of as column vectors). Thus $\mathfrak{V}_{\overline{\theta}}$ is a direct summand of $\widetilde{R}_{\overline{\theta}}$ and is in particular $\widetilde{R}_{\overline{\theta}}$-flat, hence $\widetilde{R}_{\overline{\theta}}$-flat. For type (gen), the flatness of $\mathfrak{V}_{\overline{\theta}}$ over $\widetilde{R}_{\overline{\theta}}$ is [JNW24, Prop. 5.3.1].

Finally, assume that $\overline{\theta}$ has type (scalar). Let $p : \operatorname{Spec} A \to \mathfrak{X}_{\overline{\theta}}$ be the presentation described in [JNW24, Section 3.3] (and recalled in Section 3.4.8). Write $V_{\overline{\theta}} := p^*\mathfrak{V}_{\overline{\theta}}$. Since p^* is exact and faithful, it suffices to prove that the composite

$$V_{\overline{\theta}} \otimes_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}} - : \operatorname{Mod}^{\mathrm{f.l.}}(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}) \to \operatorname{Mod}^{\mathrm{fp}}(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}) \xrightarrow{\mathfrak{V}_{\overline{\theta}}\otimes_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}}-} \operatorname{Coh}(\mathfrak{X}_{\overline{\theta}}) \xrightarrow{p^*} \operatorname{Mod}^{\mathrm{fp}}(A)$$

is exact. This follows from the proof of [JNW24, Prop. 5.4.2], which shows that if N is the unique simple left $\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}$-module, then $\mathrm{Tor}_1^{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}}(V_{\overline{\theta}}, N) = 0$. (More precisely, *loc. cit.* writes E for $\widetilde{R}_{\overline{\theta}}$, and proves that $\mathrm{Tor}_1^E(p^*\mathcal{V}^*, M) = 0$, where M is the unique simple E-module, $\mathcal{V}$ is $\mathfrak{V}_{\overline{\theta}}$, and $\mathcal{V}^*$ is the coherent dual of $\mathfrak{V}_{\overline{\theta}}$, viewed as a right $\widetilde{R}_{\overline{\theta}}$-module via the dual action. Now [JNW24, Prop. 2.2.4] shows that $\mathcal{V}^*$ with the dual right $\widetilde{R}_{\overline{\theta}}$-action is isomorphic to the pullback of $\mathcal{V}$ under $\dagger : \widetilde{R}_{\overline{\theta}}^{\mathrm{op}} \xrightarrow{\sim} \widetilde{R}_{\overline{\theta}}$, and so we deduce that $\mathrm{Tor}_1^{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}}(p^*\mathcal{V}, M^\dagger) = 0$. Since $M^\dagger$ is the unique simple left $\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}$-module, this is what we wanted.) $\square$

Theorem 3.5.25. *Let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation of $G_{\mathbf{Q}_p}$.*

(1) The natural map

$$\widetilde{R}_{\overline{\theta}} \to \mathrm{REnd}_{\mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})}(\mathcal{V}_{\overline{\theta}})$$

is an isomorphism, and the functor

$$\mathcal{V}_{\overline{\theta}} \otimes^L_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} - : D^b_{\mathrm{fp}}(\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$$

defined in Lemma A.10.10 (2h)*, is fully faithful.*

(2) If $\overline{\theta}$ does not *have type* (St)*, the composite*

$$D^b_{\mathrm{f.l.}}(\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}) \xrightarrow{(\mathrm{A.10.6})} D^b_{\mathrm{fp}}(\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}) \xrightarrow{\mathcal{V}_{\overline{\theta}} \otimes^L_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} -} \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$$

is t-exact.

Proof. By Lemma A.10.21, the second statement of part (1) is a consequence of the first statement. By the discussion in Section 3.4.13, we can assume without loss of generality that $\mathbf{F} = \mathbf{F}_{\overline{\theta}}$.

We begin by proving the first part. By Theorem C.2.15, the completion map

$$\mathrm{REnd}_{D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})}(\mathfrak{V}_{\overline{\theta}}) \to \mathrm{REnd}_{\mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})}(\mathcal{V}_{\overline{\theta}})$$

is an isomorphism. So it suffices to prove that the natural map

$$\widetilde{R}_{\overline{\theta}} \to \mathrm{REnd}_{D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})}(\mathfrak{V}_{\overline{\theta}}) \tag{3.5.26}$$

is an isomorphism. If $\mathcal{O}'/\mathcal{O}$ is a quadratic unramified extension, then the formation of $\mathrm{Ext}^i_{D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})}$ commutes with base-change to $\mathcal{O}'$ (as can be seen e.g. by computing with free resolutions after pullback by a smooth surjection $p : \mathrm{Spec}\, S \to \mathfrak{X}_{\overline{\theta}}$). Thus we can assume without loss of generality that $\overline{\theta}$ does not have type (gen+). It then follows from [JNW24, Thms. 3.1.1, 3.2.3, 3.3.1, 3.5.1] that (3.5.26) is an isomorphism. This concludes the proof of the first part.

We now prove the second part. By Lemma A.10.15, it suffices to prove that the functor

$$\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} - : \mathrm{Pro}\, D^b_{\mathrm{f.l.}}(\widetilde{R}_{\overline{\theta}^{\mathrm{op}}}) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}),$$

defined in Lemma A.10.10 (3n), is t-exact. By Lemma A.10.25, it suffices to prove that $\mathcal{V}_{\overline{\theta}}$ is a topologically flat $\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}$-module. This is Proposition 3.5.24. $\square$

3.5.27. *Completion at closed points.* In Section 5.3 we will need the following results about coherent sheaves on versal rings to $\mathcal{X}$ at closed points.

Recall from Section 3.3.4 that the closed points of $\mathcal{X}$ are in bijection with $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy classes of semisimple representations $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ with determinant $\zeta\varepsilon^{-1}$, or equivalently, $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy classes of 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentations $\overline{\theta}$. Given a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation $\overline{\theta}$, we will adopt without further comments the notation in Definition 3.4.4 for versal rings.

Definition 3.5.28. For any Serre weight σ and 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation $\overline{\theta}$, we introduce the notation

$$M_{\sigma,\overline{\theta}} := (\mathrm{Pro}\, V)(\widehat{i^*_{\overline{\theta}}} c\text{-}\mathrm{Ind}_{KZ}^G\, \sigma) \in \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{R}_{\overline{\theta}}) \xrightarrow{\sim} \mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}}),$$

where we write $\mathrm{Pro}\, V : \mathrm{Pro}\, \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}} \to \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{R}_{\overline{\theta}})$ for the Pro-extension of (2.4.20). Similarly, we write

$$M^\dagger_{\sigma,\overline{\theta}} := (\mathrm{Pro}\, V^\dagger)(\widehat{i^*_{\overline{\theta}}} c\text{-}\mathrm{Ind}_{KZ}^G\, \sigma) \in \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}) \xrightarrow{\sim} \mathrm{Mod}_c(\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}).$$

The $\widetilde{R}_{\overline{\theta}}$-modules $M_{\sigma,\overline{\theta}}$ have been computed in [Kis09] following work of Berger–Breuil [BB10]; we recall these results in the next lemma.

Lemma 3.5.29. *Let $\sigma = \sigma_{a,b}$ be a Serre weight, and let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation. Assume that $M_{\sigma,\overline{\theta}} \neq 0$, and that $\overline{\theta}$ does not have type* (gen+) (*i.e. it is* $\mathbf{F}$*-valued with absolutely irreducible summands*)*. Then:*

(1) If $\overline{\theta}$ has type (ssg), *then $M_{\sigma,\overline{\theta}}$ is $\widetilde{R}_{\overline{\theta}}$-linearly isomorphic to the Galois representation $V^{\mathrm{ver}}_{\overline{\theta}} \otimes_{R^{\mathrm{ver}}_{\overline{\theta}}} R^{\sigma}_{\overline{\theta}}/\varpi$.*

(2) If $\overline{\theta} = \mathrm{nr}_\lambda \omega^{a+b} + \mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^{a-1}$, then $M_{\sigma,\overline{\theta}}$ is $\widetilde{R}_{\overline{\theta}}$-linearly isomorphic to a free $\mathbf{F}[\![S]\!]$-module of rank one on which $\widetilde{R}_{\overline{\theta}}$ acts via the character

$$\mathrm{nr}_{S+\lambda}\omega^{a+b} : \widetilde{R}_{\overline{\theta}} \to \mathbf{F}[\![S]\!]. \tag{3.5.30}$$

The restriction of (3.5.30) *to $R^{\mathrm{ps}}_{\overline{\theta}} = Z(\widetilde{R}_{\overline{\theta}})$ is a ring homomorphism*

$$\beta : R^{\mathrm{ps}}_{\overline{\theta}} \to \mathbf{F}[\![S]\!], \tag{3.5.31}$$

which is surjective whenever $\overline{\theta}$ does not have type (scalar)*.*

Proof. Part (1) follows from [Kis09, Lemma 1.5.3] (and its proof). Part (2) follows from [Kis09, Lemma 1.5.9, Lemma 1.5.11], where the morphism β is denoted θ. □

Remark 3.5.32. Using the description (2.1.14) of the central embedding $R^{\mathrm{ps}}_{\overline{\theta}} \to \widetilde{R}_{\overline{\theta}}$, one sees that β classifies the deformation

$$\mathrm{nr}_{S+\lambda}\omega^{a+b} + \mathrm{nr}_{(S+\lambda)^{-1}\zeta(p)}\omega^{a-1}$$

of $\overline{\theta}$ to $\mathbf{F}[\![S]\!]$.

We now deduce an alternative description of $M_{\sigma,\overline{\theta}}$. It will be useful to employ the following notation: if $J \subset \widetilde{R}_{\overline{\theta}}$ is a subset, we will write $\langle J \rangle$ for the two-sided ideal generated by J, and $J\widetilde{R}_{\overline{\theta}}$ for the right ideal generated by J. We also note that in part (2) of the following lemma, which encompasses the cases when $\overline{\theta}$ has type (gen) or (St), we employ the explicit presentation of $\widetilde{R}_{\overline{\theta}}$ given above in Section 3.4.6, resp. Section 3.4.9.

Lemma 3.5.33. *Let $\sigma = \sigma_{a,b}$ be a Serre weight, and let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation.*

(1) If $\overline{\theta}$ has type (ssg)*, and $M_{\sigma,\overline{\theta}} \neq 0$, then $M_{\sigma,\overline{\theta}}$ is $R^{\mathrm{ver}}_{\overline{\theta}}$-linearly isomorphic to $(R^{\sigma}_{\overline{\theta}}/\varpi)^{\oplus 2}$.*

(2) Suppose that $0 \leq b \leq p-2$, let $\lambda \in \mathbf{F}^{\times}$, and assume that $(b, \lambda) \neq (p-2, \pm\zeta(p)^{1/2})$. Let $\sigma = \sigma_{a,b}$ and $\overline{\theta} = \mathrm{nr}_{\lambda}\omega^{a+b} + \mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^{a-1}$. Then $M_{\sigma,\overline{\theta}} \cong \widetilde{R}_{\overline{\theta}}/\langle \ker\beta, e_{22}\rangle$, and $M_{\sigma^{\mathrm{co}},\overline{\theta}} \cong \widetilde{R}_{\overline{\theta}}/\langle \ker\beta, e_{11}\rangle$.

Proof. Recalling that when $\overline{\theta}$ has type (ssg) we have $R^{\mathrm{ps}}_{\overline{\theta}} = R^{\mathrm{ver}}_{\overline{\theta}}$, so that $\widetilde{R}_{\overline{\theta}}$ is in particular an $R^{\mathrm{ver}}_{\overline{\theta}}$-algebra, we see that part (1) follows from Lemma 3.5.29 (1).

We now prove part (2). By Lemma 3.5.29, we have $M_{\sigma,\overline{\theta}} = \widetilde{R}_{\overline{\theta}}/\ker(\mathrm{nr}_{S+\lambda}\omega^{a+b})$. Since $\mathbf{F}[\![S]\!]$ is a local ring, at least one of the orthogonal idempotents e_{11} and e_{22} goes to zero under $\mathrm{nr}_{S+\lambda}\omega^{a+b}$, and since the kernel is a two-sided ideal, it contains at most one of e_{11}, e_{22}. Hence there exists $i = 1$ or 2 such that $\langle \ker\beta, e_{ii}\rangle \subset \ker(\mathrm{nr}_{S+\lambda}\omega^{a+b})$. The inclusion must then be an equality, since $\widetilde{R}_{\overline{\theta}}/\langle \ker\beta, e_{ii}\rangle \cong R^{\mathrm{ps}}_{\overline{\theta}}/\ker(\beta) = \mathbf{F}[\![S]\!]$.

It remains to prove that $i = 2$. If $e_{22} : M_{\sigma,\overline{\theta}} \to M_{\sigma,\overline{\theta}}$ is not zero, then there exists a nonzero $\widetilde{R}_{\overline{\theta}}$-linear map $\widetilde{R}_{\overline{\theta}}e_{22} \to M_{\sigma,\overline{\theta}}$. Bearing in mind that our presentation of $\widetilde{R}_{\overline{\theta}}$ is such that $\widetilde{R}_{\overline{\theta}}e_{22}$ is a projective envelope of $\mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^{a-1}$, this contradicts our assumption that either $\omega^{a+b} \neq \omega^{a-1}$ (i.e. $b \neq p-2$) or $\lambda \neq \lambda^{-1}\zeta(p)$ (when $b = p-2$). □

Remark 3.5.34. If $\overline{\theta} = 1 + \omega^{-1}$, then the kernel of β is (ϖ, a_1, X_0, X_1). This is proved in [HT15, Lemma 3.9], bearing in mind that (a_0, a_1, X_0, X_1) are denoted $(d_0, -d_1, c_0, c_1)$ in *loc. cit.*, as explained in [JNW24, Section 5.5].

Remark 3.5.35. If $\overline{\theta} = 1 + \omega^{-1}$ we will also need the following connection with the Hecke operator T_p. If $\sigma \in \{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}, \mathrm{Sym}^{p-3}\otimes\det\}$, then T_p acts on $M_{\sigma,\overline{\theta}}$ by V-functoriality. Lemma 3.5.29 provides an identification $\mathrm{End}_{\mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}})}(M_{\sigma,\overline{\theta}}) = \mathbf{F}[\![S]\!]$, and $T_p - 1$ is a uniformizer of this ring: in fact, the exactness of $\widehat{i}^*_{\overline{\theta}}$ and $\mathrm{Pro}\,V$ implies that
$$M_{\sigma,\overline{\theta}}/(T_p-1)M_{\sigma,\overline{\theta}} = V\big(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma/(T_p-1)c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma\big),$$
and hence the left-hand side is a one-dimensional $\mathbf{F}$-vector space; since $M_{\sigma,\overline{\theta}}$ is free of rank one over $\mathbf{F}[\![S]\!]$ (by Lemma 3.5.29), we conclude that indeed $T_p - 1$ is identified with a uniformizer of $\mathbf{F}[\![S]\!]$. On the other hand, the map
$$\beta : R^{\mathrm{ps}}_{\overline{\theta}} \to \mathrm{End}_{\mathrm{Mod}_c(\widetilde{R}_{\overline{\theta}})}(M_{\sigma,\overline{\theta}}) = \mathbf{F}[\![S]\!],$$
describing the restriction of the $\widetilde{R}_{\overline{\theta}}$-action on $M_{\sigma,\overline{\theta}}$ to $Z(\widetilde{R}_{\overline{\theta}}) = R^{\mathrm{ps}}_{\overline{\theta}}$ is surjective, by Lemma 3.5.29, and so it sends a_0 to a uniformizer of $\mathbf{F}[\![S]\!]$, by Remark 3.5.34. Putting the preceding observations together, we conclude that $\beta(a_0)$ is an $\mathbf{F}[\![S]\!]^{\times}$-multiple of $T_p - 1$.

Finally, we have the right $\widetilde{R}_{\overline{\theta}}$-modules $M^{\dagger}_{\sigma,\overline{\theta}} := \mathrm{Pro}\,V^{\dagger}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma)$, which are obtained from $M_{\sigma,\overline{\theta}}$ by composing with the anti-involution $\dagger$.

Lemma 3.5.36. *Let $\sigma = \sigma_{a,b}$ be a Serre weight, and let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation.*

(1) If $\overline{\theta}$ has type (ssg)*, then $M^{\dagger}_{\sigma,\overline{\theta}}$ is $R^{\mathrm{ver}}_{\overline{\theta}}$-linearly isomorphic to $(R^{\sigma}_{\overline{\theta}}/\varpi)^{\oplus 2}$.*

(2) *Assume that* $0 \le b \le p-2$*, let* $\lambda \in \mathbf{F}^\times$*, and assume that* $(b,\lambda) \neq (p-2, \pm\zeta(p)^{1/2})$*. Let* $\overline{\theta} = \mathrm{nr}_\lambda \omega^{a+b} + \mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^{a-1}$*. Then* $M^\dagger_{\sigma,\overline{\theta}} \cong \widetilde{R}_{\overline{\theta}}/\langle \ker\beta, e_{11}\rangle$*, and* $M^\dagger_{\sigma^{\mathrm{co}},\overline{\theta}} \cong \widetilde{R}_{\overline{\theta}}/\langle \ker\beta, e_{22}\rangle$ *(as right* $\widetilde{R}_{\overline{\theta}}$*-modules).*

Proof. By definition, if $g \in \widetilde{R}^\times_{\overline{\theta}}$ is the image of an element of $G_{\mathbf{Q}_p}$, then $g^\dagger = (\zeta\varepsilon^{-1})(g)g^{-1}$. Our assumptions imply that $\overline{\theta}$ does not have type (scalar) or (gen+), and so $\widetilde{R}_{\overline{\theta}}$ is an order in $M_2(R^{\mathrm{ps}}_{\overline{\theta}})$, as described in Section 3.4. It follows from this description of $\widetilde{R}_{\overline{\theta}}$ that $\dagger$ coincides, at elements of $G_{\mathbf{Q}_p}$, with the restriction to $\widetilde{R}_{\overline{\theta}}$ of the anti-involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{3.5.37}$$

of $M_2(R^{\mathrm{ps}}_{\overline{\theta}})$. Since the images of elements of $G_{\mathbf{Q}_p}$ are dense in $\widetilde{R}_{\overline{\theta}}$, we conclude that $\dagger$ coincides with (3.5.37). Thus the lemma follows from Lemma 3.5.33. $\square$

Remark 3.5.38. In case (2) of Lemma 3.5.36, it follows from (3.5.30) that the inertia group $I_{\mathbf{Q}_p}$ acts on $M^\dagger_{\sigma,\overline{\theta}}$ via the character $(\omega^{a+b})^\dagger|_{I_{\mathbf{Q}_p}} = \omega^{a-1}|_{I_{\mathbf{Q}_p}}$. Hence, if W is a left $\widetilde{R}_{\overline{\theta}}$-module, then $W \widehat{\otimes}_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}} M^\dagger_{\sigma,\overline{\theta}}$ is an $\widetilde{R}_{\overline{\theta}}$-quotient of W on which $I_{\mathbf{Q}_p}$ acts by ω^{a-1}.

Proposition 3.5.39. *Let* $\sigma = \sigma_{a,b}$ *be a Serre weight, and let* $\overline{\theta}$ *be a* 2*-dimensional* $\overline{\mathbf{F}}_p$*-valued pseudorepresentation. Let* $R^{\mathrm{ver}}_{\overline{\theta}}, R^{\sigma}_{\overline{\theta}}$ *be the versal rings defined in Definition* 3.4.4*. Assume that* $b \neq 0$*, and* $M^\dagger_{\sigma,\overline{\theta}} \neq 0$*. Then*

(1) $V^{\mathrm{ver}}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}} M^\dagger_{\sigma,\overline{\theta}}$ *has scheme-theoretic support* $\operatorname{Spec} R^\sigma_{\overline{\theta}}/\varpi$*, and*
(2) *if* $\overline{\theta}$ *does not have type* (scalar)*, then* $V^{\mathrm{ver}}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}} M^\dagger_{\sigma,\overline{\theta}} \cong R^\sigma_{\overline{\theta}}/\varpi$*.*

Remark 3.5.40. The assumption that $b \neq 0$ in Proposition 3.5.39 is due to the fact that, for any $\overline{\theta}$, we have an exact sequence

$$0 \to \widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma_{a,p-1} \to \widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}^G_{KZ}\,\sigma_{a,0} \to \widehat{i}^*_{\overline{\theta}}\big(\omega^a \mathrm{nr}_{X^2-\zeta(p)} \circ \det\big) \to 0$$

obtained by applying $\widehat{i}^*_{\overline{\theta}}$ to (2.2.37). Since $V^\dagger$ is zero on $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant representations of G, and $V^\dagger$ is exact, we deduce that $M^\dagger_{\sigma_{a,p-1},\overline{\theta}} = M^\dagger_{\sigma_{a,0},\overline{\theta}}$. Hence $V^{\mathrm{ver}}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}} V^\dagger_{\sigma_{a,0}} \cong R^{\sigma_{a,p-1}}_{\overline{\theta}}/\varpi$, which can be larger than $R^{\sigma_{a,0}}_{\overline{\theta}}/\varpi$.

Proof of Proposition 3.5.39. By definition, the Galois representation $V^{\mathrm{ver}}/\mathfrak{m}V^{\mathrm{ver}}$ is semisimple with absolutely irreducible summands. Hence, after possibly replacing $\mathbf{F}$ with a finite extension, we can assume that $\overline{\theta}$ does not have type (gen+). Furthermore, by Lemma 2.5.16 (2), $M^\dagger_{\sigma,\overline{\theta}} \neq 0$ if and only if σ is a Serre weight of $V^{\mathrm{ver}}/\mathfrak{m}V^{\mathrm{ver}}$. We now proceed by cases, according to the type of $\overline{\theta}$.

Case (ssg): Recall that $\widetilde{R}_{\overline{\theta}} = M_2(R^{\mathrm{ver}}_{\overline{\theta}})$ acting on $V^{\mathrm{ver}}_{\overline{\theta}} = (R^{\mathrm{ver}}_{\overline{\theta}})^{\oplus 2}$ in the standard representation. Hence $M^\dagger_{\sigma,\overline{\theta}}$ is $R^{\mathrm{ver}}_{\overline{\theta}}$-linearly isomorphic to the direct sum of $V^{\mathrm{ver}}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}} M^\dagger_{\sigma,\overline{\theta}}$ with itself. On the other hand, by Lemma 3.5.36 (1), $M^\dagger_{\sigma,\overline{\theta}}$ is $R^{\mathrm{ver}}_{\overline{\theta}}$-linearly isomorphic to $(R^\sigma_{\overline{\theta}}/\varpi)^{\oplus 2}$, and so $V^{\mathrm{ver}}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{R}^{\mathrm{op}}_{\overline{\theta}}} M^\dagger_{\sigma,\overline{\theta}} = R^\sigma_{\overline{\theta}}/\varpi$, as desired.

Case (gen): After a twist, we may assume that $\sigma = \mathrm{Sym}^b$ and $\overline{\theta} = \mathrm{nr}_\lambda \omega^b + \mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^{-1}$, for some $\lambda \in \mathbf{F}^\times$ and $0 \le b \le p-2$. Then Lemma 3.5.36 (2) shows that

$$V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} M_{\sigma,\overline{\theta}}^{\dagger} = V_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, e_{11}, ce_{21})V_{\overline{\theta}}^{\mathrm{ver}},$$

where we have used that $\langle \ker\beta, e_{11}\rangle = (\ker\beta, e_{11}, ce_{21})\widetilde{R}_{\overline{\theta}}$. Since $V_{\overline{\theta}}^{\mathrm{ver}}/e_{11}V_{\overline{\theta}}^{\mathrm{ver}}$ is free of rank one as an $R_{\overline{\theta}}^{\mathrm{ver}}$-module, we see that $V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} M_{\sigma,\overline{\theta}}^{\dagger}$ is a cyclic $R_{\overline{\theta}}^{\mathrm{ver}}$-module with annihilator $(\ker\beta, c)R_{\overline{\theta}}^{\mathrm{ver}}$.

The proposition will thus follow in this case once we show that $R_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c)R_{\overline{\theta}}^{\mathrm{ver}}$ coincides with $R_{\overline{\theta}}^{\sigma}/\varpi$. Since these rings are abstractly isomorphic (being power series rings over $\mathbf{F}$ in the same number of variables), it suffices to prove that the quotient map $R_{\overline{\theta}}^{\mathrm{ver}} \to R_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c)R_{\overline{\theta}}^{\mathrm{ver}}$ factors through $R_{\overline{\theta}}^{\sigma}/\varpi$. To this end, note that we have an exact sequence of $R_{\overline{\theta}}^{\mathrm{ver}}$-modules

$$0 \to V_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c, e_{22}) \to V_{\overline{\theta}}^{\mathrm{ver}} \otimes_{R_{\overline{\theta}}^{\mathrm{ver}}} R_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c)R_{\overline{\theta}}^{\mathrm{ver}} \to V_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c, e_{11}) \to 0$$

which exhibits the free rank two $R_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c)$-module $V_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c)$ as an extension of free rank one modules. The explicit form of $\widetilde{R}_{\overline{\theta}}$ given in (3.4.7) shows that this is actually a sequence of $\widetilde{R}_{\overline{\theta}}$-modules. Furthermore, since the quotient $V_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c, e_{11})$ has been proven in the previous paragraph to be isomorphic to $V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} M_{\sigma,\overline{\theta}}^{\dagger}$, Remark 3.5.38 shows that $I_{\mathbf{Q}_p}$ acts by ω^{-1} on $V_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c, e_{11})$. Since $\det V_{\overline{\theta}}^{\mathrm{ver}}$ is an unramified twist of ω^{b-1}, this shows that $V_{\overline{\theta}}^{\mathrm{ver}} \otimes_{R_{\overline{\theta}}^{\mathrm{ver}}} R_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c)R_{\overline{\theta}}^{\mathrm{ver}}$ is an ordinary deformation of $\mathrm{nr}_\lambda\omega^b \oplus \mathrm{nr}_{\lambda^{-1}\zeta(p)}\omega^{-1}$, and its classifying map $R_{\overline{\theta}}^{\mathrm{ver}} \to R_{\overline{\theta}}^{\mathrm{ver}}/(\ker\beta, c)R_{\overline{\theta}}^{\mathrm{ver}}$ factors through $R_{\overline{\theta}}^{\sigma}/\varpi$, as desired.

Case (St): This is similar to case (gen). After a twist, we may assume that $\overline{\theta} = 1 + \omega^{-1}$ and $\sigma \in \{\mathrm{Sym}^{p-1}, \mathrm{Sym}^{p-3} \otimes \det\}$. Bearing in mind the isomorphism $M_{\mathrm{Sym}^0,\overline{\theta}}^{\dagger} \cong M_{\mathrm{Sym}^{p-1},\overline{\theta}}^{\dagger}$ from Remark 3.5.40, the module $M_{\sigma,\overline{\theta}}^{\dagger}$ has thus been computed in Lemma 3.5.36. Taking into account the explicit form of $\ker\beta$ given in Remark 3.5.34, we find that

$$V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} M_{\mathrm{Sym}^{p-1},\overline{\theta}}^{\dagger} = R_{\overline{\theta}}^{\mathrm{ver}}/(\varpi, a_1, X_0, X_1, c)R_{\overline{\theta}}^{\mathrm{ver}} = \mathbf{F}[\![a_0, b_0, b_1]\!]/(a_0 b_1) \tag{3.5.41}$$

$$V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} M_{\mathrm{Sym}^{p-3}\otimes\det,\overline{\theta}}^{\dagger} = R_{\overline{\theta}}^{\mathrm{ver}}/(\varpi, a_1, X_0, X_1, b_0, b_1)R_{\overline{\theta}}^{\mathrm{ver}} = \mathbf{F}[\![a_0, c]\!] \tag{3.5.42}$$

as $R_{\overline{\theta}}^{\mathrm{ver}}$-modules. Arguing as in case (gen), and using the fact that $R^{(p,0),\mathrm{crys}}/\varpi$ is a power series ring over $\mathbf{F}[\![x, y]\!]/(xy)$, we thus see that $V \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} M_{\sigma,\overline{\theta}}^{\dagger} \cong R_{\overline{\theta}}^{\sigma}/\varpi$, as desired. (This presentation of $R^{(p,0),\mathrm{crys}}/\varpi$ can be deduced from a more careful analysis of the presentation of $\mathfrak{X}_{\overline{\theta}}$ described in Section 3.4.9, or alternatively, it can be deduced from [BCGP21, Lemma 7.3.7].)

Case (scalar): After a twist, we may assume that $\overline{\theta} = \omega^{-1} + \omega^{-1}$, and $\sigma = \mathrm{Sym}^{p-2}$. This case was excluded in Lemma 3.5.36, so we will argue in a different way, by applying the results of [San16]. In Definition 2.1.8 we have defined an irreducible $E[KZ]_\zeta$-module $\Theta(\sigma|\sigma^{\mathrm{co}})$ whose mod ϖ reduction is isomorphic to σ, and we now fix an $\mathcal{O}[KZ]_\zeta$-lattice Θ in $\Theta(\sigma|\sigma^{\mathrm{co}})$. We have isomorphisms

$$\begin{aligned} V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} M_{\sigma,\overline{\theta}}^{\dagger} &\xrightarrow{\sim} V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}} \widehat{\otimes}_{\mathcal{O}[\![G]\!]_\zeta} \widehat{i}_{\overline{\theta},*} \widehat{i}_{\overline{\theta}}^{*} c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) \\ &\xrightarrow{\sim} V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) \xrightarrow{\sim} \mathrm{Hom}_{KZ}^{\mathrm{cont}}(V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}}, \sigma^\vee)^\vee, \end{aligned}$$

where the first arrow is Lemma 2.4.22 (2), the second arrow is Lemma 2.5.26, and the third arrow is Lemma 2.2.45. Now $V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}}$ is the object denoted N in [San16] (and we employ this same notation in what follows), and so we see that $V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{R}_{\overline{\theta}}^{\mathrm{op}}} M_{\sigma,\overline{\theta}}^{\dagger}$ is the module denoted $M^{\square}(\sigma)$ in [San16, (2.12)]. Following *loc. cit.* and [Paš15, Section 2], we also consider the module

$$M(\Theta) := \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}_{KZ}^{\mathrm{cont}}(N, \Theta^d), \mathcal{O}).$$

Since $V_{\overline{\theta}}^{\mathrm{ver}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} -$ is exact, by Proposition 3.5.24, and $\widetilde{P}_{\overline{\theta}} \widehat{\otimes}_{\mathcal{O}[\![KZ]\!]_{\zeta}} -$ is exact, by Lemma 2.3.11 (4), we see that N is topologically flat in $\mathrm{Mod}_c(\mathcal{O}[\![KZ]\!]_{\zeta})$, hence projective in $\mathrm{Mod}_c(\mathcal{O}[\![KZ]\!]_{\zeta})$, by A.1.44. By [Paš15, Lemma 2.14], we conclude that $M^{\square}(\sigma)$ is isomorphic to $M(\Theta)/\varpi M(\Theta)$. We claim that $M(\Theta)$ is a faithful finitely presented $R_{\overline{\theta}}^{\sigma}$-module.

Assuming the claim, we conclude the proof of the proposition as follows. By the claim, the natural map

$$R_{\overline{\theta}}^{\sigma} \to \mathrm{End}_{R_{\overline{\theta}}^{\mathrm{ver}}}(M(\Theta))$$

is injective, and so the natural map

$$R_{\overline{\theta}}^{\sigma}/\varpi \to \mathrm{End}_{R_{\overline{\theta}}^{\mathrm{ver}}}(M(\Theta)/\varpi M(\Theta))$$

has nilpotent kernel. Since $R_{\overline{\theta}}^{\sigma}/\varpi$ is reduced, by Lemma 3.2.5, we conclude that the scheme-theoretic support of $M^{\square}(\sigma) = M(\Theta)/\varpi M(\Theta)$ is $R_{\overline{\theta}}^{\sigma}/\varpi$, as desired.

To prove the claim, observe first that $M(\Theta)$ is finitely presented over $R_{\overline{\theta}}^{\mathrm{ver}}$, by [Paš15, Proposition 2.15]. Then let $\mathfrak{a} := \mathrm{Ann}_{R_{\overline{\theta}}^{\mathrm{ver}}} M(\Theta)$. As shown during the proof of [San16, Theorem 4.10], the radical of $\mathfrak{a}$ is the kernel of $R_{\overline{\theta}}^{\mathrm{ver}} \to R_{\overline{\theta}}^{\sigma}$. However, it is also shown in *loc. cit.* that the assumptions of [San16, Theorem 2.1], or equivalently [Paš15, Theorem 2.42], are met. Hence $\mathfrak{a}$ is a radical ideal, which concludes the proof. □

Corollary 3.5.43. *Let $\overline{\theta} = 1 + \omega^{-1}$. Then $R_{\overline{\theta}}^{(p,0),\mathrm{crys}}/\varpi = \mathbf{F}[\![a_0, b_0, b_1]\!]/(a_0 b_1)$, and $R_{\overline{\theta}}^{(1,0),\mathrm{crys}}/\varpi = R_{\overline{\theta}}^{(p,0),\mathrm{crys}}/(\varpi, b_1)$.*

Proof. The claim about $R_{\overline{\theta}}^{(p,0),\mathrm{crys}}/\varpi$ is an immediate consequence of (3.5.41). We now prove the claim about $R_{\overline{\theta}}^{(1,0),\mathrm{crys}}/\varpi$. It follows from e.g. [HT15, Theorem 5.11] that the quotient map $R_{\overline{\theta}}^{\mathrm{ver}}/\varpi \to R_{\overline{\theta}}^{(1,0),\mathrm{crys}}/\varpi$ factors through a surjection $R_{\overline{\theta}}^{(p,0),\mathrm{crys}}/\varpi \to R_{\overline{\theta}}^{(1,0),\mathrm{crys}}/\varpi$ whose kernel is a minimal prime, and so it is generated by either a_0 or b_1. But it cannot be generated by a_0: if this were the case, then the map $R_{\overline{\theta}}^{\mathrm{ps}} \to R_{\overline{\theta}}^{(p,0),\mathrm{crys}}/\varpi$ would factor through $\mathbf{F}$, since the maximal ideal of $R_{\overline{\theta}}^{\mathrm{ps}}$ is generated by (a_0, a_1, X_0, X_1). This would contradict the existence of lifts of $1 \oplus \omega^{-1}$ to $\mathbf{F}[\epsilon]/\epsilon^2$ that factor through $R_{\overline{\theta}}^{(p,0),\mathrm{crys}}/\varpi$ and have nontrivial pseudocharacter (for example, the lift $\mathrm{nr}_{1-\epsilon} \oplus \mathrm{nr}_{1+\epsilon}\omega^{-1}$). □

3.6. **Complements in the Steinberg case.** We assume throughout this section that $\overline{\theta}$ is of type (St). Making a twist if necessary, we assume that in fact $\overline{\theta} = 1 + \omega^{-1}$. We will shorten notation to $R := R_{\overline{\theta}}^{\mathrm{ps}}$. Recall that in Section 3.4.9 we have given an explicit description of $R, \widetilde{R}_{\overline{\theta}}, \mathfrak{X}_{\overline{\theta}}$, and of the bundle $\mathfrak{V}_{\overline{\theta}}$ lying over $\mathfrak{X}_{\overline{\theta}}$, together with its left $\widetilde{R}_{\overline{\theta}}$-action. Using the notation of that section, we write

$$I := (X_0, X_1)R \subseteq R;$$

this is the reducibility ideal in R.

3.6.1. *The projective generator* $\mathbf{P}_{\overline{\theta}}$ *of* $\mathfrak{C}_{\overline{\theta}}$. Recall from Definition 2.4.11 and Definition 2.4.17 the projective objects $P_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}} \in \mathfrak{C}_{\overline{\theta}}$. Neither of these is a projective generator, which leads us to introduce yet another projective object.

Definition 3.6.2. Let $\overline{\theta} = 1 + \omega^{-1}$. We choose a $\mathfrak{C}_{\overline{\theta}}$-projective envelope $P_{\mathbf{1}_G^\vee}$ of the trivial character of G, we define the projective generator (with finite cosocle) $\mathbf{P}_{\overline{\theta}}$ of $\mathfrak{C}_{\overline{\theta}}$ as

$$\mathbf{P}_{\overline{\theta}} := \widetilde{P}_{\overline{\theta}} \oplus P_{\mathbf{1}_G^\vee},$$

and we write $\mathbf{E}_{\overline{\theta}} := \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}})$.

We can now apply the Morita theory of Lemma 2.3.15, taking the projective generator in the statement of that lemma to be $\mathbf{P}_{\overline{\theta}}$. Then Lemma 2.3.15 (3) shows that $\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}})$ is of finite type over $\widetilde{E}_{\overline{\theta}}$, and (2.3.16) gives an isomorphism

$$\widetilde{P}_{\overline{\theta}} \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}}) \otimes_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}}. \tag{3.6.3}$$

3.6.4. *The ideal* J *of* $\mathbf{E}_{\overline{\theta}}$. Let $\mathcal{O}_{\mathbf{1}_G^\vee}$ denote $\mathcal{O}$ regarded as an object of $\mathfrak{C}_{\overline{\theta}}$ by letting G act trivially. Since the $\mathfrak{C}_{\overline{\theta}}$-cosocle of $\mathcal{O}_{\mathbf{1}_G^\vee}$ is equal to $\mathbf{1}_G^\vee$, with multiplicity one, we find that $\mathcal{O}_{\mathbf{1}_G^\vee}$ is a quotient of $P_{\mathbf{1}_G^\vee}$, and that

$$\mathrm{Hom}_G^{\mathrm{cont}}(\mathbf{P}_{\overline{\theta}}, \mathcal{O}_{\mathbf{1}_G^\vee}) = \mathrm{Hom}_G^{\mathrm{cont}}(P_{\mathbf{1}_G^\vee}, \mathcal{O}_{\mathbf{1}_G^\vee})$$

is free of rank one over $\mathcal{O}$. Since $\mathcal{O}_{\mathbf{1}_G^\vee}$ is a quotient of $P_{\mathbf{1}_G^\vee}$, which is in turn a quotient of $\mathbf{P}_{\overline{\theta}}$, we see that $\mathrm{Hom}_G^{\mathrm{cont}}(\mathbf{P}_{\overline{\theta}}, \mathcal{O}_{\mathbf{1}_G^\vee})$ is also a quotient of $\mathbf{E}_{\overline{\theta}}$. Thus we have an isomorphism of $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-modules

$$\mathrm{Hom}_G^{\mathrm{cont}}(\mathbf{P}_{\overline{\theta}}, \mathcal{O}_{\mathbf{1}_G^\vee}) \xrightarrow{\sim} \mathbf{E}_{\overline{\theta}}^{\mathrm{op}}/J$$

for some closed right ideal J of $\mathbf{E}_{\overline{\theta}}$ with the property that $\mathcal{O} \xrightarrow{\sim} \mathbf{E}_{\overline{\theta}}^{\mathrm{op}}/J$ (as $\mathcal{O}$-modules). More precisely, J can be taken to be the annihilator of any surjection $\mathbf{P}_{\overline{\theta}} \to \mathcal{O}_{\mathbf{1}_G^\vee}$ in $\mathfrak{C}_{\overline{\theta}}$.

We claim that J is in fact a two-sided ideal. In fact, since $\mathrm{Hom}_G^{\mathrm{cont}}(\widetilde{P}_{\overline{\theta}}, \mathcal{O}_{\mathbf{1}_G^\vee}) = 0$, J contains the two-sided ideal J' generated by the idempotent of $\mathbf{E}_{\overline{\theta}}$ associated to $\widetilde{P}_{\overline{\theta}}$. Since the quotient $\mathbf{E}_{\overline{\theta}}/J'$ is commutative, we see that J is a two-sided ideal, and $\mathcal{O} \xrightarrow{\sim} \mathbf{E}_{\overline{\theta}}^{\mathrm{op}}/J$ (as $\mathcal{O}$-algebras). It also follows from Lemma 2.3.15 (1) that we have the following isomorphism in $\mathfrak{C}_{\overline{\theta}}$:

$$\mathcal{O}_{\mathbf{1}_G^\vee} \xrightarrow{\sim} (\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}/J) \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} \mathbf{P}_{\overline{\theta}}/J\mathbf{P}_{\overline{\theta}}.$$

Lemma 3.6.5. *If* M *is an object in* $\mathrm{Mod}_c(\mathbf{E}_{\overline{\theta}}^{\mathrm{op}})$, *then the natural morphism*

$$M[J] \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \to M \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}}$$

is injective, and induces an isomorphism

$$M[J] \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} (M \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)}.$$

Proof. To ease notation, write $N := M\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}\mathbf{P}_{\overline{\theta}}$; note that this is the object of $\mathfrak{C}_{\overline{\theta}}$ that corresponds to M under the Morita equivalence of Lemma 2.3.15 (1). The symbol $M[J]$ denotes the J-torsion in M; since J is a left ideal of $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$, this is also an instance of the construction in Remark A.1.50, i.e. $M[J] = \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}}(\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}/J, M)$.

By Lemmas 3.6.8 and 3.6.9 below, G acts trivially on $N^{\mathrm{SL}_2(\mathbf{Q}_p)}$, since the trivial character is the only square root of $\zeta^{-1} : \mathbf{Q}_p^{\times} \to \mathcal{O}^{\times}$ contained in $\mathfrak{C}_{\overline{\theta}}$. Hence $N^{\mathrm{SL}_2(\mathbf{Q}_p)}$ is the image of the evaluation map

$$\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathcal{O}_{\mathbf{1}_G^\vee}, N) \otimes_{\mathcal{O}} \mathcal{O}_{\mathbf{1}_G^\vee} \hookrightarrow N. \tag{3.6.6}$$

The Morita equivalence gives an isomorphism

$$\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathcal{O}_{\mathbf{1}_G^\vee}, N) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}}(\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}/J, M) = M[J],$$

and so we may rewrite (3.6.6) in the form

$$M[J] \otimes_{\mathcal{O}} \mathcal{O}_{\mathbf{1}_G^\vee} \xrightarrow{\sim} (M\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}\mathbf{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)}. \tag{3.6.7}$$

Recalling that $\mathbf{E}_{\overline{\theta}}/J = \mathcal{O}$, and that $\mathbf{P}_{\overline{\theta}}/J\mathbf{P}_{\overline{\theta}} = \mathcal{O}_{\mathbf{1}_G^\vee}$, we may rewrite the source as $M[J] \otimes_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}}$, and hence rewrite (3.6.7) as

$$M[J] \otimes_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} (M\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}\mathbf{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)}.$$

Tracing the maps through, one sees that the composition of this isomorphism with the inclusion

$$(M\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}\mathbf{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)} \hookrightarrow M \otimes_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}}$$

coincides with the natural morphism in the statement of the lemma. $\square$

Lemma 3.6.8. *Let $\mathfrak{B}$ be a block of type* (St). *If M is an object of $\mathfrak{C}_{\mathfrak{B}}$, then $M^{\mathrm{SL}_2(\mathbf{Q}_p)}$ is a subobject of M in $\mathfrak{C}_{\mathfrak{B}}$. Furthermore, we have the equality*

$$M^{\mathrm{SL}_2(\mathbf{Q}_p)} = \bigcup_{i \in \mathcal{I}} N_i,$$

where $\{N_i\}_{i\in\mathcal{I}}$ is the set of $\mathfrak{C}_{\mathfrak{B}}$-subobjects of M of finite $\mathcal{O}$-type.

Proof. The first statement is true because the condition of being $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant is a closed condition. For the second statement, observe first that if $v \in M^{\mathrm{SL}_2(\mathbf{Q}_p)}$ then $\langle G\cdot v\rangle$ is a cyclic module over $\mathcal{O}[G/\mathrm{SL}_2(\mathbf{Q}_p)]_{\zeta}$, which is finitely generated over $\mathcal{O}$. There remains to prove that if $N \subset M$ is a $\mathfrak{C}_{\mathfrak{B}}$-subobject of finite $\mathcal{O}$-type, then $N \subset M^{\mathrm{SL}_2(\mathbf{Q}_p)}$. This is a consequence of Lemma 3.6.9. $\square$

Lemma 3.6.9. *Let R be a complete Noetherian local $\mathcal{O}$-algebra with finite residue field. Let W be a finite type R-module equipped with a continuous* (*for the $\mathfrak{m}_R$-adic topology on W*) *G-action, admitting a central character, say $\alpha : \mathbf{Q}_p^{\times} \to R^{\times}$. Then α is a square and W decomposes as a direct sum of eigenspaces W_β, where β runs over the various square roots of α in the group of continuous $R^{\times}$-valued characters of $\mathbf{Q}_p^{\times}$, and G acts on W_β via $\beta\circ\det$.*

Proof. Replacing R by its Artinian quotients, we can assume that R is an Artinian ring. Then W has the discrete topology, and so the G-action is smooth. Since W has finite type, the upper unipotent subgroup U has an open subgroup that acts trivially on W, and conjugating by $\mathrm{diag}(p^{-1}, 1)$ we see that U itself also acts trivially. It follows that wUw^{-1} acts trivially, hence so does $\mathrm{SL}_2(\mathbf{Q}_p)$, which is generated by U and wUw^{-1} (see [Col10c, Proposition III.1.1] for a proof). Hence G acts on W via the determinant, and so α is a square. Twisting by $\alpha^{-1/2}$ (for some choice of this square root), we can assume that α is trivial. But then the G-action factors through $\mathbf{Q}_p^\times/(\mathbf{Q}_p^\times)^2$, and the lemma follows. $\square$

3.6.10. *The sheaves $\mathfrak{W}_{\overline{\theta}}$ and $\mathcal{W}_{\overline{\theta}}$.* In this subsection we give an alternative description of the pro-coherent sheaf

$$\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}} \in \operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}}),$$

which will be useful in Section 5.2. As usual, we have formed the completed tensor product using the canonical isomorphism $\widetilde{E}_{\overline{\theta}} \xrightarrow{\sim} \widetilde{R}_{\overline{\theta}}^{\mathrm{op}}$.

Recall from (3.6.3), and the surrounding discussion, that $\operatorname{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}})$ is an $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-module and an $\widetilde{E}_{\overline{\theta}}$-module, finitely presented over both. It is thus a complete right $\mathbf{E}_{\overline{\theta}}$-module in $\operatorname{Mod}_c(\widetilde{E}_{\overline{\theta}})$, and we obtain a right exact, cofiltered limit-preserving functor

$$\operatorname{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}}) \,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} - : \operatorname{Mod}_c(\mathbf{E}_{\overline{\theta}}) \to \operatorname{Mod}_c(\widetilde{E}_{\overline{\theta}})$$

whose composition with $\operatorname{Mod}_c(\widetilde{E}_{\overline{\theta}}) \to \operatorname{Mod}_c(\mathcal{O})$ is the usual completed tensor product. The evaluation map

$$\operatorname{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}}) \,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} \widetilde{P}_{\overline{\theta}} \tag{3.6.11}$$

is then an $\widetilde{E}_{\overline{\theta}}$-linear isomorphism. Formula (3.6.11) motivates the following definition.

Definition 3.6.12. If $\overline{\theta}$ is of type (St), we write

$$\mathcal{W}_{\overline{\theta}} := \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \operatorname{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}})$$

and

$$\mathfrak{W}_{\overline{\theta}} := \mathfrak{V}_{\overline{\theta}} \otimes_{\widetilde{E}_{\overline{\theta}}} \operatorname{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}}).$$

Then $\mathcal{W}_{\overline{\theta}}$ is a complete right $\mathbf{E}_{\overline{\theta}}$-module in $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$, $\mathfrak{W}_{\overline{\theta}}$ is a right $\mathbf{E}_{\overline{\theta}}$-module in $\operatorname{Coh}(\mathfrak{X}_{\overline{\theta}})$, and the restriction of the $\mathbf{E}_{\overline{\theta}}$-action to R coincides with the action of R through the structure map $\mathcal{X}_{\overline{\theta}} \to \operatorname{Spf} R$, resp. $\mathfrak{X}_{\overline{\theta}} \to \operatorname{Spec} R$. By (3.6.11), we have an isomorphism

$$\mathcal{W}_{\overline{\theta}} \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}}. \tag{3.6.13}$$

Since $\mathcal{V}_{\overline{\theta}}$ algebraizes to the universal rank two bundle $\mathfrak{V}_{\overline{\theta}}$ on $\mathfrak{X}_{\overline{\theta}}$, we see by Lemma 3.5.7 that $\mathcal{W}_{\overline{\theta}}$ algebraizes to $\mathfrak{W}_{\overline{\theta}}$, in the sense that $\mathcal{W}_{\overline{\theta}}$ is recovered as $\widehat{k}_{\overline{\theta}}^{*}\mathfrak{W}_{\overline{\theta}}$.

Remark 3.6.14. The sheaf $\mathfrak{W}_{\overline{\theta}}$ will often intervene in our discussion through its quotient $\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]$, where $J \subset \mathbf{E}_{\overline{\theta}}$ is the ideal defined in Section 3.6.4. This is because of Theorem 3.6.44 below (which is essentially due to Johansson–Newton–Wang-Erickson) and Proposition 5.2.18 (2).

3.6.15. *Describing* $\mathbf{E}_{\overline{\theta}}$. We next recall the structure of $\mathbf{E}_{\overline{\theta}}$. This is the subject of [Paš13, §10.5], and we refer to the discussion there for details of the computations, and for detailed explanations of the notation that we use. (See also [JNW24, §5.5].)

We first recall from Remark 2.4.16 that there is an identification of the centre of $\mathbf{E}_{\overline{\theta}}$ (equivalently, the centre of $\mathfrak{C}_{\overline{\theta}}$; or again equivalently, the centre of the category $\mathcal{A}_{\overline{\theta}}$) with the pseudodeformation ring $R = R_{\overline{\theta}}^{\mathrm{ps}}$, such that the functors $V, V^\dagger$ and $\check{\mathrm{V}}$ become R-linear.

Next, to be precise, we note that the computations of [Paš13, §10.5] concern the endomorphism ring of $P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee} \oplus P_{\mathbf{1}^\vee}$, whereas we wish to describe the endomorphism ring $\mathbf{E}_{\overline{\theta}}$ of $\mathbf{P}_{\overline{\theta}} := \widetilde{P}_{\overline{\theta}} \oplus P_{\mathbf{1}^\vee}$, where $\widetilde{P}_{\overline{\theta}}$ is the twist of $P_{\overline{\theta}} = P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee}$ described in Definition 2.4.17. (We remind the reader that where we write $P_{\pi_\alpha^\vee}$, etc., Paškūnas in [Paš13] writes $\widetilde{P}_{\pi_\alpha^\vee}$, etc; while the notation $P_{\overline{\theta}}$ and $\widetilde{P}_{\overline{\theta}}$ doesn't appear in *loc. cit.*) There are therefore three endomorphism rings to be considered in this subsection, namely $E_{\overline{\theta}} := \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P_{\overline{\theta}}), \widetilde{E}_{\overline{\theta}} := \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(\widetilde{P}_{\overline{\theta}})$ and $\mathbf{E}_{\overline{\theta}} := \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}})$. One of the ingredients in our description of $\mathbf{E}_{\overline{\theta}}$ will be a choice of basis of the free rank-one $E_{\overline{\theta}}$-module $\check{\mathrm{V}}(P_{\overline{\theta}})$, which provides an identification $\widetilde{R}_{\overline{\theta}} \xrightarrow{\sim} E_{\overline{\theta}}^{\mathrm{op}}$ as in Proposition 2.4.13 (3), and an identification $P_{\overline{\theta}} \xrightarrow{\sim} \widetilde{P}_{\overline{\theta}}$. We will describe a convenient choice of basis in Remark 3.6.26.

In preparation for this, we recall that [Paš13, (252)], and the displayed equation immediately above it on [Paš13, p. 160], yield identifications

$$E_{\overline{\theta}} := \mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P_{\overline{\theta}}) = \begin{pmatrix} Re_1 & R\varphi_{12} \\ R\varphi_{21}^0 + R\varphi_{21}^1 & Re_2 \end{pmatrix} \tag{3.6.16}$$

and

$$\mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee} \oplus P_{\mathbf{1}^\vee}) = \begin{pmatrix} Re_1 & R\varphi_{12} & R\varphi_{13}^0 + R\varphi_{13}^1 \\ R\varphi_{21}^0 + R\varphi_{21}^1 & Re_2 & R\varphi_{23}^0 + R\varphi_{23}^1 \\ R\varphi_{31} & R\beta + R\varphi_{32} & Re_3 \end{pmatrix}, \tag{3.6.17}$$

where the various φ, as well as β, denote certain homomorphisms which are constructed in *loc. cit.*, and e_i denotes the idempotent of the endomorphism ring corresponding to the i-th direct summand of $P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee} \oplus P_{\mathbf{1}^\vee}$. Of course, the isomorphism (3.6.16) is just the restriction to the upper left 2×2 block of the isomorphism (3.6.17).

Lemma 3.6.18. *There is a unique R-algebra isomorphism*

$$\begin{aligned} &\mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee} \oplus P_{\mathbf{1}^\vee}) \\ &\xrightarrow{\sim} \begin{pmatrix} R & Rc & RX_0 + RX_1 \\ Rb_0 + Rb_1 & R & Rb_0 + Rb_1 \\ R & Rb_0^{-1}a_0 + Rb_0^{-1}X_0 & R \end{pmatrix} \subset M_{3\times 3}(\mathrm{Frac}(S)) \end{aligned} \tag{3.6.19}$$

that preserves the displayed generators of (3.6.17) *and* (3.6.19).

Proof. See [JNW24, §5.5]. □

Remark 3.6.20. We can rewrite the isomorphism (3.6.19) more succinctly (but in a manner that suppresses the explicit identification of generators) as

$$\mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee} \oplus P_{\mathbf{1}^\vee}) \cong \begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & R \end{pmatrix},$$

where as usual we write $I = (X_0, X_1) \subset R$ to denote the reducibility ideal, and we write $I^\vee := \{a \in \mathrm{Frac}R : aI \subseteq R\}$ for the fractional ideal-theoretic inverse of I.

Remark 3.6.21. Taken together, (3.4.10) and (3.6.19) yield isomorphisms

$$\widetilde{R}_{\overline{\theta}} \cong \begin{pmatrix} R & Rb_0 + Rb_1 \\ Rc & R \end{pmatrix} \text{ and } E_{\overline{\theta}} \cong \begin{pmatrix} R & Rc \\ Rb_0 + Rb_1 & R \end{pmatrix}. \tag{3.6.22}$$

Analogously to the description of $\mathrm{End}_{\mathfrak{C}_{\overline{\theta}}}(P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee} \oplus P_{\mathbf{1}^\vee})$ provided by Remark 3.6.20, we can also write

$$E_{\overline{\theta}} \cong \begin{pmatrix} R & Rc \\ Ic^{-1} & R \end{pmatrix}. \tag{3.6.23}$$

Lemma 3.6.24. *We can choose a basis of* $\check{\mathrm{V}}(P_{\overline{\theta}})$ *as an* $E_{\overline{\theta}}$*-module so that the isomorphism* $R_{\overline{\theta}} \xrightarrow{\sim} E_{\overline{\theta}}^{\mathrm{op}}$ *of Proposition* 2.4.13 (3) *is given, in terms of the descriptions of its source and target provided by* (3.6.22)*, by matrix transpose.*

Proof. As noted in Proposition 2.4.13 (3), choosing a basis of $\check{\mathrm{V}}(P_{\overline{\theta}})$ gives rise to an $R_{\overline{\theta}}^{\mathrm{ps}}$-algebra isomorphism

$$\widetilde{R}_{\overline{\theta}} \xrightarrow{\sim} E_{\overline{\theta}}^{\mathrm{op}}, \tag{3.6.25}$$

and changing this choice of basis composes (3.6.25) with an inner automorphism (of either its source or target; either one amounts to the same thing).

A consideration of (3.6.22) shows that matrix transposition provides an $R_{\overline{\theta}}^{\mathrm{ps}}$-algebra isomorphism $\widetilde{R}_{\overline{\theta}} \xrightarrow{\sim} E_{\overline{\theta}}^{\mathrm{op}}$, which is necessarily the composition of (3.6.25) with an automorphism (again, of either its source or its target). It thus suffices to show that any $R_{\overline{\theta}}^{\mathrm{ps}}$-algebra automorphism of $\widetilde{R}_{\overline{\theta}}$ is inner. By Lemma 2.1.17, it in turn suffices to note that any $R_{\overline{\theta}}^{\mathrm{ps}}$-algebra automorphism of $\widetilde{R}_{\overline{\theta}}$ necessarily fixes the isomorphism classes of the two simple modules $1, \omega^{-1}$ of $\widetilde{R}_{\overline{\theta}}$, because $\mathrm{Ext}^1_{\widetilde{R}_{\overline{\theta}}}(1, \omega^{-1})$ and $\mathrm{Ext}^1_{\widetilde{R}_{\overline{\theta}}}(\omega^{-1}, 1)$ are not isomorphic. □

Remark 3.6.26. Fixing an $E_{\overline{\theta}}$-basis for $\check{\mathrm{V}}(P_{\overline{\theta}})$ as in Lemma 3.6.24 yields isomorphisms $P_{\overline{\theta}} \xrightarrow{\sim} \widetilde{P}_{\overline{\theta}}$ and $E_{\overline{\theta}} \xrightarrow{\sim} \widetilde{E}_{\overline{\theta}}$, such that the isomorphism (2.4.18) is given by matrix transpose (when its source and target are described $E_{\overline{\theta}} \xrightarrow{\sim} \widetilde{E}_{\overline{\theta}}$ and (3.6.22)). We make such a choice of basis from now on. Thus we also obtain an isomorphism $\mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee} \oplus P_{\mathbf{1}^\vee}$, and so (3.6.17) and (3.6.19) can be interpreted as an isomorphism

$$\mathbf{E}_{\overline{\theta}} \xrightarrow{\sim} \begin{pmatrix} R & Rc & RX_0 + RX_1 \\ Rb_0 + Rb_1 & R & Rb_0 + Rb_1 \\ R & Rb_0^{-1}a_0 + Rb_0^{-1}X_0 & R \end{pmatrix} = \begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & R \end{pmatrix}, \tag{3.6.27}$$

and (3.6.22) and (3.6.23) can be interpreted as an isomorphism

$$\widetilde{E}_{\overline{\theta}} \xrightarrow{\sim} \begin{pmatrix} R & Rc \\ Rb_0 + Rb_1 & R \end{pmatrix} = \begin{pmatrix} R & Rc \\ Ic^{-1} & R \end{pmatrix}. \tag{3.6.28}$$

Accordingly, from now on, we drop $E_{\overline{\theta}}$ from the notation, and work exclusively with $\widetilde{E}_{\overline{\theta}}$ and $\mathbf{E}_{\overline{\theta}}$.

Recall from Section 3.6.4 that $J \subset \mathbf{E}_{\overline{\theta}}$ denotes the two-sided ideal for which the quotient $\mathbf{E}_{\overline{\theta}}/J$ corresponds to $\mathcal{O}_{\mathbf{1}_G^\vee}$, i.e. the module $\mathcal{O}$ with its trivial G-action, under the Morita equivalence of Lemma 2.3.15.

Lemma 3.6.29. *If we write* $\mathfrak{a} := (a_0, a_1, X_0, X_1) \subset R$*, and describe* $\mathbf{E}_{\overline{\theta}}$ *via* (3.6.27)*, then we have*

$$J = \begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & \mathfrak{a} \end{pmatrix} \subset \begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & R \end{pmatrix}.$$

Proof. The surjection $\mathbf{E}_{\overline{\theta}} \to \mathbf{E}_{\overline{\theta}}/J$ is obtained by applying $\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, -)$ to the composite $\mathbf{P}_{\overline{\theta}} \to P_{\mathbf{1}_G^\vee} \to \mathcal{O}_{\mathbf{1}_G^\vee}$. Applying $\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, -)$ to the first arrow, we obtain the projection of $\begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & R \end{pmatrix}$ onto its third column. The top two entries of this third column are given by $\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\pi_\alpha^\vee}, P_{\mathbf{1}_G^\vee})$ and $\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\mathrm{St}^\vee}, P_{\mathbf{1}_G^\vee})$ respectively, each of which map to zero when we compose with the surjection $P_{\mathbf{1}_G^\vee} \to \mathcal{O}^{\mathbf{1}_G^\vee}$. Thus in fact $\mathbf{E}_{\overline{\theta}} \to \mathbf{E}_{\overline{\theta}}/J$ is the composition

$$\begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & R \end{pmatrix} \to R = \mathrm{Hom}_{\mathfrak{C}}(P_{\mathbf{1}_G^\vee}, P_{\mathbf{1}_G^\vee})$$
$$\to \mathrm{Hom}_{\mathfrak{C}}(P_{\mathbf{1}_G^\vee}, \mathcal{O}_{\mathbf{1}_G^\vee}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}}(\mathcal{O}_{\mathbf{1}_G^\vee}, \mathcal{O}_{\mathbf{1}_G^\vee}) = \mathcal{O}$$

(the first arrow being projection onto the bottom right entry; hence this factorization only occurs in the category of R-modules, and not of R-algebras). Thus the lemma will follow once we show that the kernel of the given morphism $R \to \mathcal{O}$ is indeed equal to $\mathfrak{a}$. Now [Paš13, Lem. 10.75] shows that this kernel is equal to the image of the natural morphism

$$\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee}, P_{\mathbf{1}_G^\vee}) \otimes \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\mathbf{1}_G^\vee}, P_{\pi_\alpha^\vee} \oplus P_{\mathrm{St}^\vee}) \to \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(P_{\mathbf{1}_G^\vee}, P_{\mathbf{1}_G^\vee}),$$

which we can recast more concretely (using our explicit description of $\mathbf{E}_{\overline{\theta}}$ as a matrix order) as the image of the morphism

$$(R \oplus I^\vee c) \otimes (I \oplus Ic^{-1}) \to R$$

given by performing summandwise multiplication, then adding the results. This image is equal to the ideal $I + I^\vee I$, which one computes to be $\mathfrak{a}$. ☐

3.6.30. *Describing* $\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]$. Recall that $\widetilde{R}_{\overline{\theta}}$ acts on $\mathfrak{V}_{\overline{\theta}} = S(1) \oplus S(-1)$ through its identification with a matrix order given by (3.4.10), acting via left multiplication on the elements of $\mathfrak{V}_{\overline{\theta}}$ regarded as column vectors. Since we have chosen our identification of $P_{\overline{\theta}}$ and $\widetilde{P}_{\overline{\theta}}$ to satisfy the conclusion of Lemma 3.6.24, the right action of $\widetilde{E}_{\overline{\theta}}$ on $\mathfrak{V}_{\overline{\theta}}$ is via the identification of $\widetilde{E}_{\overline{\theta}}$ with a matrix order given by (3.6.28), acting by right multiplication on elements of $\mathfrak{V}_{\overline{\theta}}$ (thought of as row vectors).

Recall also that in Section 3.6.10 we defined

$$\mathfrak{W}_{\overline{\theta}} := \mathfrak{V}_{\overline{\theta}} \otimes_{\widetilde{E}_{\overline{\theta}}} \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, \widetilde{P}_{\overline{\theta}}) \xrightarrow{\sim} \mathfrak{V}_{\overline{\theta}} \otimes_{\widetilde{E}_{\overline{\theta}}} \mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, P_{\overline{\theta}})$$

(the isomorphism being induced by our identification of $P_{\overline{\theta}}$ with $\widetilde{P}_{\overline{\theta}}$), which we can think as a coherent sheaf on $\mathfrak{X}_{\overline{\theta}}$ endowed with an $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-action, or equivalently as a graded $S \otimes_R \mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-module (regarded as a graded ring via the grading on S and the trivial grading on $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$).

Note that $\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, P_{\overline{\theta}})$ coincides, as an $(\widetilde{E}_{\overline{\theta}}, \mathbf{E}_{\overline{\theta}})$-bimodule, with the top two rows of $\mathbf{E}_{\overline{\theta}}$ when written as a matrix order as in (3.6.27); here $\mathbf{E}_{\overline{\theta}}$ acts via right multiplication, while $\widetilde{E}_{\overline{\theta}}$ acts via left multiplication, using the description of (3.6.28). In particular, reading off the first two rows of $\mathbf{E}_{\overline{\theta}}$ in this description, we find that

$$\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, P_{\overline{\theta}}) \xrightarrow{\sim} \begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \end{pmatrix}. \tag{3.6.31}$$

If we write (as usual) $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \widetilde{E}_{\overline{\theta}}$, then we see that

$$\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}, P_{\overline{\theta}}) \xrightarrow{\sim} \widetilde{E}_{\overline{\theta}} \oplus \big((\widetilde{E}_{\overline{\theta}} e_2) \otimes_R Ic^{-1}\big)$$

as an $\widetilde{E}_{\overline{\theta}}$-module. In fact, $\widetilde{E}_{\overline{\theta}} e_2$ is simply the second column of $\widetilde{E}_{\overline{\theta}}$, regarded as a left $\widetilde{E}_{\overline{\theta}}$-module. So this isomorphism simply expresses the fact that the third column of the right hand side of (3.6.31) is obtained from the second by tensoring over R with Ic^{-1}.

From this discussion, we compute that

$$\text{(3.6.32)}\quad \mathfrak{W}_{\overline{\theta}} := \big(S(1) \oplus S(-1)\big) \otimes_{\widetilde{E}_{\overline{\theta}}} \big(\widetilde{E}_{\overline{\theta}} \oplus (\widetilde{E}_{\overline{\theta}} e_2) \otimes_R Ic^{-1}\big) \xrightarrow{\sim} S(1) \oplus S(-1) \oplus \big(S(-1) \otimes_R Ic^{-1}\big)$$

as graded S-modules, where the grading on $S(-1) \otimes_R Ic^{-1}$ comes from the first factor; i.e. we take the tensor product grading with R and Ic^{-1} concentrated in degree zero. In fact, we can make $\mathbf{E}_{\overline{\theta}}$ (as described in (3.6.27)) act by right multiplication on the target of the isomorphism (3.6.32): to see that this makes sense, recall that $b_i = X_i c^{-1}$, so that Ic^{-1} naturally embeds in S, and there is a product morphism of graded S-modules

$$\text{(3.6.33)}\qquad S(-1) \otimes_R Ic^{-1} \to S(1),$$

with image equal to $(b_0, b_1)(1)$ (where (b_0, b_1) denotes the indicated homogeneous ideal of S with its natural grading). Then (3.6.32) is an isomorphism of graded $S \otimes_R \mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-modules. Finally, taking the direct sum of the identity morphisms on the first two summands of (3.6.32), and of (3.6.33) on the third summand, we obtain a surjection

$$\text{(3.6.34)}\quad \mathfrak{W}_{\overline{\theta}} \overset{(3.6.32)}{=} S(1) \oplus S(-1) \oplus \big(S(-1) \otimes_R I\big) \to S(1) \oplus S(-1) \oplus (b_0, b_1)(1).$$

Lemma 3.6.35. *The kernel of* (3.6.34) *is equal to* $\mathfrak{W}_{\overline{\theta}}[J]$. *Consequently, we have an isomorphism of graded* $S \otimes_R \mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$*-modules*

$$\text{(3.6.36)}\qquad \mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J] \xrightarrow{\sim} S(1) \oplus S(-1) \oplus (b_0, b_1)(1).$$

Proof. Since (3.6.34) is given by the identity on its first two summands, its kernel coincides with the kernel of (3.6.33). As S is an integral domain, multiplication by c^{-1} induces an isomorphism $S(-1) \xrightarrow{\sim} c^{-1}S(1)$. We may then factor (3.6.33) as the composite of this isomorphism with the morphism

$$\text{(3.6.37)}\qquad S(-1) \otimes_R I \to S(-1)$$

induced by multiplication, and so the kernel of (3.6.34) coincides with the kernel of (3.6.37).

If we consider the result of tensoring the short exact sequence of R-modules

$$0 \to I \to R \to R/I \to 0$$

with S over R, then we see that the kernel of (3.6.34) is isomorphic to $\mathrm{Tor}_1^R(S, R/I)$. Now R/I admits a free resolution with initial terms

$$R^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} a_1 & -X_1 \\ a_0 & X_0 \end{smallmatrix}\right)} R^{\oplus 2} \xrightarrow{(X_0\ X_1)} R$$

(we regard the elements of $R^{\oplus 2}$ as column vectors, and perform matrix multiplication on the left), and so $\mathrm{Tor}_1^R(S, R/I)$ can be computed as the cohomology of the three term complex

$$S^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} a_1 & -X_1 \\ a_0 & X_0 \end{smallmatrix}\right)} S^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} X_0 & X_1 \end{smallmatrix}\right)} S.$$

Now since $X_i = b_i c$, and since multiplication by c is injective, this coincides with the cohomology of the complex

$$S^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} a_1 & -b_1 c \\ a_0 & b_0 c \end{smallmatrix}\right)} S^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} b_0 & b_1 \end{smallmatrix}\right)} S.$$

One easily confirms that

$$S^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} a_1 & -b_1 \\ a_0 & b_0 \end{smallmatrix}\right)} S^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} b_0 & b_1 \end{smallmatrix}\right)} S$$

is exact, and so

(3.6.38)
$$\ker(3.6.34) \xrightarrow{\sim} \mathrm{Tor}_1^R(S, R/I) \xrightarrow{\sim} \big(S\left(\begin{smallmatrix} a_1 \\ a_0 \end{smallmatrix}\right) + S\left(\begin{smallmatrix} -b_1 \\ b_0 \end{smallmatrix}\right)\big)/\big(S\left(\begin{smallmatrix} a_1 \\ a_0 \end{smallmatrix}\right) + Sc\left(\begin{smallmatrix} -b_1 \\ b_0 \end{smallmatrix}\right)\big),$$

the sums being taken in $S^{\oplus 2}$.

With this explicit description of $\ker(3.6.34)$ in hand, it is easy to prove the lemma. Firstly, one directly checks from the description (3.6.38) that $\mathfrak{a} := (a_0, a_1, X_0, X_1) \subset R$ annihilates $\ker(3.6.34)$, and then, using the description of J given by Lemma 3.6.29, one sees that $\ker(3.6.34)$ is an $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-submodule of $\mathfrak{W}_{\overline{\theta}}$ which is annihilated by J. To complete the proof of the lemma, it remains to show that

$$\mathfrak{W}_{\overline{\theta}}/\ker(3.6.34) \xrightarrow{\sim} S(1) \oplus S(-1) \oplus (b_0, b_1)(1)$$

is J-torsion free. Again, the $\mathbf{E}_{\overline{\theta}}$-action on this module is given by right multiplication by the matrix order of (3.6.27), and so one easily checks this; indeed, extending scalars to $\mathrm{Frac}(S)$, it amounts to the fact that $M_3\big(\mathrm{Frac}(S)\big)$ acts faithfully on $\mathrm{Frac}(S)^{\oplus 3}$. $\square$

3.6.39. *Some constructions from [JNW24].* In [JNW24, Prop. 3.5.6] there is introduced a certain finitely generated graded S-module Q^*. Here $(-)^*$ denotes the internal Hom with S on the category of graded S-modules, i.e. the graded duality $\underline{\mathrm{Hom}}_{S\text{-gr}}(-, S)$; in particular, the module Q^* is constructed as $\underline{\mathrm{Hom}}_{S\text{-gr}}(Q, S)$ for some other graded S-module Q. The module Q is actually reflexive (in fact it is maximal Cohen–Macaulay over the Gorenstein ring S), and so Q can be recovered as $(Q^*)^*$. For our purposes, however, the most convenient description of Q^* is the one coming from the displayed formula above [JNW24, Prop. 3.5.6], which exhibits Q^* as the image of the morphism

$$S(-1) \oplus S(-1) \xrightarrow{\begin{pmatrix} a_1 & a_0 \\ -b_1 & b_0 \end{pmatrix}} S(-1) \oplus S(1). \tag{3.6.40}$$

Lemma 3.6.41. *If we regard Q^* as a submodule of $S(-1) \oplus S(1)$ (by regarding it as the image of the morphism (3.6.40)), then projection onto the second factor induces an isomorphism $Q^* \xrightarrow{\sim} (b_0, b_1)(1)$.*

Proof. If we regard elements of the direct sum as column vectors, then Q^* is the span of the column vectors $\left(\begin{smallmatrix} a_1 \\ -b_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right)$. Thus, the image of Q^* under the projection is equal to (b_0, b_1).

Now if s_0 and s_1 are elements of S, then

$$(s_1a_1 + s_0a_0)b_0 = a_0(s_1(-b_1) + s_0b_0).$$

Thus, since S is an integral domain, we see that kernel of the projection is trivial. This completes the proof of the lemma. $\square$

We now recall one of the main results of [JNW24, §5.5].

Definition 3.6.42. Write $X^* := S(1) \oplus S(-1) \oplus Q^*$.

Proposition 3.6.43. *The graded S-module X^* is maximal Cohen–Macaulay, and there are isomorphisms*

$$\mathbf{E}_{\overline{\theta}}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{S\text{-gr}}(X^*) \xrightarrow{\sim} \mathrm{REnd}_{S\text{-gr}}(X^*).$$

The action of $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$ is described by using the identification (3.6.27) *of $\mathbf{E}_{\overline{\theta}}$ with an order in $M_3(S)$, and having it act by matrix multiplication on the right of $X^* := S(1) \oplus S(-1) \oplus Q^*$ (regarded as a module of row vectors).*

Proof. As the notation suggests, X^* is the dual of $X := S(-1)\oplus S(1)\oplus Q$. It therefore suffices to show that $\mathrm{Ext}^i_{S\text{-gr}}(X, X) = 0$ for $i \neq 0$, and that $\mathbf{E}_{\overline{\theta}} \xrightarrow{\sim} \mathrm{End}_{S\text{-gr}}(X)$. The first statement is proved in [JNW24, Prop. 3.5.2], and the second statement is proved immediately before [JNW24, Rem. 5.5.2]. $\square$

Theorem 3.6.44. *The $S \otimes_R \mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-module $\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]$ is isomorphic to X^*, and the natural map*

$$\mathbf{E}_{\overline{\theta}}^{\mathrm{op}} \to \mathrm{REnd}_{\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}})}(\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J])$$

is an isomorphism. Consequently, the functor defined via Lemma A.10.10 (2h)*:*

$$\mathcal{W}_{\overline{\theta}}/\mathcal{W}_{\overline{\theta}}[J] \otimes^L_{\mathbf{E}_{\overline{\theta}}} - : D^b_{\mathrm{fp}}(\mathbf{E}_{\overline{\theta}}) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}),$$

is fully faithful.

Proof. Taken together, Lemmas 3.6.35 and 3.6.41 show that each of $\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]$ and X^* is isomorphic to $S(1) \oplus S(-1) \oplus (b_0, b_1)(1)$ as graded $S \otimes_R \mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-modules. This proves the first claim. The second claim follows from the first claim and Proposition 3.6.43. By Theorem C.2.15, the completion map

$$\mathrm{REnd}_{D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})}(\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]) \to \mathrm{REnd}_{\mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})}(\mathcal{W}_{\overline{\theta}}/\mathcal{W}_{\overline{\theta}}[J])$$

is an isomorphism, the final claim is a consequence of the second one and Lemma A.10.21. $\square$

3.6.45. Tor-*dimension of X^* over $\mathbf{E}_{\overline{\theta}}$.* It follows from [JNW24, Prop. 5.5.3] that $X^* \otimes^L_{\mathbf{E}_{\overline{\theta}}} -$ has amplitude $[-1, 0]$ when evaluated on objects of $\mathrm{Mod}^{\mathrm{f.l.}}(\mathbf{E}_{\overline{\theta}})$. Presumably there is a "Tor-dimension 1" variant of the local criterion for flatness which allows us to deduce that $\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J] \cong X^*$ is of Tor-dimension 1, but we give a direct proof as follows. We use the notion of Tor-dimension from Definition A.10.23.

Lemma 3.6.46. *The $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-module $\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]$ is of* Tor-*dimension* 1.

Proof. By Theorem 3.6.44, the $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-modules $\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]$ and X^* are isomorphic, so it is equivalent to show that X^* has Tor-dimension 1. The explicit computations of Example 3.6.48 below imply that the Tor-dimension of X^* is at least 1. Thus we need only show that it is also at most 1, which we now do.

To begin, recall the isomorphism (3.6.27):

$$\mathbf{E}_{\overline{\theta}} \xrightarrow{\sim} \begin{pmatrix} R & Rc & RX_0 + RX_1 \\ Rb_0 + Rb_1 & R & Rb_0 + Rb_1 \\ R & Rb_0^{-1}a_0 + Rb_0^{-1}X_0 & R \end{pmatrix} = \begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & R \end{pmatrix}$$

acting by right multiplication on $X^* = S(1) \oplus S(-1) \oplus (b_0, b_1)(1)$. Note also that

$$S = \cdots \oplus Rc^{-n} \oplus Rc^{1-n} \oplus \cdots \oplus Rc \oplus R \oplus Ic^{-1} \oplus I^2c^{-2} \oplus \cdots \oplus I^n c^{-n} \oplus \cdots,$$

while

$$(b_0, b_1) = \cdots \oplus Ic^{-n} \oplus Ic^{1-n} \oplus \cdots \oplus Ic \oplus I \oplus Ic^{-1} \oplus I^2c^{-2} \oplus \cdots \oplus I^n c^{-n} \oplus \cdots.$$

Thus (using subscripts to denote graded pieces) we have

$$(X^*)_{2n-1} = \begin{cases} Rc^{-n} \oplus Rc^{-n+1} \oplus Ic^{-n} & \text{if } n \le 0, \\ I^n c^{-n} \oplus I^{n-1}c^{-n+1} \oplus I^n c^{-n} & \text{if } n \ge 1. \end{cases}$$

In order to show that X^* has Tor-dimension ≤ 1, it suffices to show that it has a projective resolution of length two. In fact, the composite

$$\mathrm{Mod}^{\mathrm{fp}}(\mathbf{E}_{\overline{\theta}}) \xrightarrow{X^* \otimes_{\mathbf{E}_{\overline{\theta}}}} \mathrm{Coh}(\mathfrak{X}_{\overline{\theta}}) = \mathrm{GrMod}(S) \to \mathrm{Mod}(S)$$

is the usual tensor product, and so its derived functor can be computed by projective resolutions on either variable; and the forgetful functor $\mathrm{GrMod}(S) \to \mathrm{Mod}(S)$ is faithful and exact. Thus it suffices in turn to show that each $(X^*)_{2n-1}$ has such a resolution.

If $n \le 1$, then in fact $(X^*)_{2n-1}$ is isomorphic (as a right $\mathbf{E}_{\overline{\theta}}$-module) to one of the rows of $\mathbf{E}_{\overline{\theta}}$, and is thus projective over $\mathbf{E}_{\overline{\theta}}$.

If $n \ge 2$, then

$$(X^*)_{2n-1} = I^n c^{-n} \oplus I^{n-1}c^{-n+1} \oplus I^n c^{-n}.$$

This has the following two step projective resolution:

$$0 \to P_1^{\oplus n-1} c^{-n} \to P_2^{\oplus n} c^{1-n} \to (X^*)_{2n-1} \to 0$$

where $P_1 = (R, I^\vee c, R)$, $P_2 = (Ic^{-1}, R, Ic^{-1})$, the first map is given by

$$\big((-X_1, X_0, 0, \ldots, 0), (0, -X_1, X_0, \ldots, 0), \ldots, (0, \ldots, 0, -X_1, X_0)\big),$$

and the second map by $(X_0^{n-1}, X_0^{n-2}X_1, \ldots, X_1^{n-1})$. $\square$

Remark 3.6.47. Note that the projective resolution of X^* as an $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}$-module constructed in the preceding lemma is in fact a graded resolution. Furthermore, it is not difficult to describe explicit lifts of the action of each of b_0, b_1, and c to this resolution. Indeed, for $n \le 1$, the "resolution" of X^*_{2n-1} is of length one, i.e. X^*_{2n-1} is already its own projective resolution, and so nothing need be said about the action of c on these graded pieces, or the action of the b_i on X^*_{2n-1} for $n \le 0$.

If $n \geq 1$, then we have the following commutative diagram (where we regard the terms as column vectors):

$$\begin{tikzcd}[column sep=huge, row sep=huge]
0 \arrow[r] & P_1^{\oplus n-1} c^{-n} \arrow[r, "{\begin{pmatrix} -X_1 & 0 & \cdots & 0 \\ X_0 & -X_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -X_1 \\ 0 & 0 & \cdots & X_0 \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} c^{-1} & 0 & \cdots & 0 \\ 0 & c^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c^{-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}}"] & P_2^{\oplus n} c^{1-n} \arrow[r, "{\begin{pmatrix} X_0^{n-1} & X_0^{n-2}X_1 & \cdots & X_1^{n-1} \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} c^{-1} & 0 & \cdots & 0 \\ 0 & c^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c^{-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}}"] & X^*_{2n-1} \arrow[r] \arrow[d, "b_0"] & 0 \\
0 \arrow[r] & P_1^{\oplus n} c^{-n-1} \arrow[r, swap, "{\begin{pmatrix} -X_1 & 0 & \cdots & 0 \\ X_0 & -X_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -X_1 \\ 0 & 0 & \cdots & X_0 \end{pmatrix}}"] & P_2^{\oplus n+1} c^{-n} \arrow[r, swap, "{\begin{pmatrix} X_0^{n} & X_0^{n-1}X_1 & \cdots & X_1^{n} \end{pmatrix}}"] & X^*_{2n+1} \arrow[r] & 0
\end{tikzcd}$$

which gives a lift of multiplication by b_0. One easily constructs analogous diagrams which give lifts of multiplication by b_1, and of c. These then allow one to compute $\mathrm{Tor}_i^{\mathbf{E}_{\overline{\theta}}}(X^*, M)$ as a graded S-module for any $\mathbf{E}_{\overline{\theta}}$-module M.

Example 3.6.48. If π is an irreducible object of $\mathcal{A}_{\overline{\theta}}$, then there is an $\mathbf{E}_{\overline{\theta}}$-linear isomorphism

$$\mathbf{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi \xrightarrow{\sim} \mathbf{E}_{\overline{\theta}}/\mathbf{m}_\pi,$$

where $\mathbf{m}_\pi$ is a maximal left ideal in $\mathbf{E}_{\overline{\theta}}$. Concretely, writing $\mathfrak{m}$ for the maximal ideal of R, we have

$$\mathbf{m}_{\pi_\alpha} = \begin{pmatrix} \mathfrak{m} & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & R \end{pmatrix},$$

$$\mathbf{m}_{\mathrm{St}} = \begin{pmatrix} R & Rc & I \\ Ic^{-1} & \mathfrak{m} & Ic^{-1} \\ R & I^\vee c & R \end{pmatrix},$$

and

$$\mathbf{m}_{\mathbf{1}} = \begin{pmatrix} R & Rc & I \\ Ic^{-1} & R & Ic^{-1} \\ R & I^\vee c & \mathfrak{m} \end{pmatrix}.$$

Following the prescription of Remark 3.6.47, one can then compute

$$\mathrm{Tor}_i^{\mathbf{E}_{\overline{\theta}}}(X^*, \mathbf{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi) = \mathrm{Tor}_i^{\mathbf{E}_{\overline{\theta}}}(X^*, \mathbf{E}_{\overline{\theta}}/\mathbf{m}_{\pi_\alpha})$$

as a graded S-module for each irreducible π. Since these Tor-values are already computed in [JNW24, Prop. 5.5.3] by a different method (namely, by computing resolutions in $\mathfrak{C}_{\overline{\theta}}$, rather than by resolving X^* as we have done) we omit the full details of the calculation, and simply record the result for future reference. As it turns out, these Tor-values vanish for all but one value of i, and are then as follows:

$$\mathrm{Tor}_0^{\mathbf{E}_{\overline{\theta}}}(X^*, \mathbf{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi_\alpha) \cong S(1)/(a_0, a_1, b_0, b_1, \varpi),$$

$$\mathrm{Tor}_0^{\mathbf{E}_{\overline{\theta}}}(X^*, \mathbf{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \mathrm{St}) \cong S(-1)/(a_0, a_1, c, \varpi),$$

$$\mathrm{Tor}_1^{\mathbf{E}_{\overline{\theta}}}(X^*, \mathbf{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \mathbf{1}_G) \cong S(-3)/(a_0, a_1, c, \varpi).$$

3.7. **Defining an open substack of $\mathcal{X}$.** In this section, we shift our attention away from the various stacks $\mathcal{X}_{\overline{\theta}}$, and back to the stack $\mathcal{X}$. Our main goal is to describe a certain "good" open substack of $\mathcal{X}$ — to be denoted $\mathcal{U}_{\mathrm{good}}$ — and to describe its underlying reduced substack explicitly.

We begin by noting that since $\overline{\theta}$ has fixed determinant $\overline{\zeta}\omega^{-1}$, there are only finitely many $\overline{\theta}$ of types (ssg), (scalar) or (St), even if we allow ourselves to replace $\mathbf{F}$ by $\overline{\mathbf{F}}_p$. Note furthermore that $\overline{\theta}$ of type (scalar) (resp. of type (St)) exist if and only if ζ is odd (resp. even).

Definition 3.7.1.

(1) We let $\mathcal{Y}_{\mathrm{bad}}$ denote the reduced closed substack of $\mathcal{X}$ with $|\mathcal{Y}_{\mathrm{bad}}| = \pi_{\mathrm{ss}}^{-1}(|Y_{\mathrm{bad}}|)$, where Y_{bad} is defined in Definition 2.7.6. Equivalently, $\mathcal{Y}_{\mathrm{bad}}$ is the (finite) union of the $\mathcal{X}_{\overline{\theta},\mathrm{red}}$ for $\overline{\theta}$ of types (ssg), (scalar) or (St).
(2) We let $\mathcal{U}_{\mathrm{good,red}}$ be the open complement of $\mathcal{Y}_{\mathrm{bad}}$ in $\mathcal{X}_{\mathrm{red}}$, and we let $\mathcal{U}_{\mathrm{good}}$ be the open substack of $\mathcal{X}$ with underlying reduced substack $\mathcal{U}_{\mathrm{good,red}}$. Note that $|\mathcal{U}_{\mathrm{good}}| = \pi_{\mathrm{ss}}^{-1}(|U_{\mathrm{good}}|)$, where U_{good} is defined in Definition 2.7.6.

Remark 3.7.2. By the definition of $\mathcal{U}_{\mathrm{good}}$, the finite type points of $\mathcal{U}_{\mathrm{good}}$ correspond to those $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which are reducible, and furthermore have corresponding pseudorepresentation of type (gen).

Definition 3.7.3. If $\{\sigma, \sigma^{\mathrm{co}}\}$ is a pair of companion weights, we set

$$\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}} := \mathcal{U}_{\mathrm{good,red}} \cap \big(\mathcal{X}(\sigma) \cup \mathcal{X}(\sigma^{\mathrm{co}})\big),$$

$$\mathcal{U}(\sigma)_{\mathrm{red}} := \mathcal{U}_{\mathrm{good,red}} \cap \mathcal{X}(\sigma).$$

Note that $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$ is the open substack of $\mathcal{X}_{\mathrm{red}}$ whose underlying topological space is $\pi_{\mathrm{ss}}^{-1}(U(\sigma|\sigma^{\mathrm{co}}))$, where $U(\sigma|\sigma^{\mathrm{co}})$ is defined in Definition 2.7.6.

Lemma 3.7.4. *The stack $\mathcal{U}_{\mathrm{good,red}}$ decomposes as a disjoint union of connected components*

$$\mathcal{U}_{\mathrm{good,red}} = \coprod_{\sigma|\sigma^{\mathrm{co}}} \mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}},$$

where $\{\sigma, \sigma^{\mathrm{co}}\}$ runs over all companion pairs of Serre weights.

Proof. Since $\mathcal{Y}_{\mathrm{bad}}$ contains all the $\mathcal{X}_{\mathrm{red},\overline{\theta}}$ for $\overline{\theta}$ of type (St), it in particular contains all irreducible components of $\mathcal{X}_{\mathrm{red}}$ of the form $\mathcal{X}(\sigma_{a,p-1})^{\pm}$, so $\mathcal{U}_{\mathrm{good,red}}$ is the union of the $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$. Since $\mathcal{Y}_{\mathrm{bad}}$ also contains the $\mathcal{X}_{\mathrm{red},\overline{\theta}}$ for $\overline{\theta}$ of type (ssg), it follows from the description of the Serre weights associated to a representation $\overline{\rho}$ (as recalled in Section 1.3) that the $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$ are disjoint, and that each $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$ is connected, because the components $\mathcal{X}(\sigma)$ and $\mathcal{X}(\sigma^{\mathrm{co}})$ intersect along the split locus, i.e. at those $\overline{\rho}$ which are direct sums of characters. $\square$

Definition 3.7.5. We let $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})$ be the open substack of $\mathcal{X}$ with underlying reduced substack $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$.

Corollary 3.7.6. *The stack $\mathcal{U}_{\mathrm{good}}$ decomposes as a disjoint union of connected components*

$$\mathcal{U}_{\mathrm{good}} = \coprod_{\sigma|\sigma^{\mathrm{co}}} \mathcal{U}(\sigma|\sigma^{\mathrm{co}}),$$

where $\{\sigma, \sigma^{\mathrm{co}}\}$ runs over all companion pairs of Serre weights.

Proof. This is immediate from Lemma 3.7.4 and the definitions of the stacks $\mathcal{U}_{\mathrm{good}}$, $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})$. $\square$

Definition 3.7.7. Let $\{\sigma, \sigma^{\mathrm{co}}\}$ be a companion pair of Serre weights. As in Definition 2.7.6, we write

$$f(t) = \begin{cases} t & \text{if } \sigma|\sigma^{\mathrm{co}} \text{ is of type (gen)} \\ t(1-t^2) & \text{if } \sigma|\sigma^{\mathrm{co}} \text{ is of type (scalar) or (St).} \end{cases}$$

We define a group scheme $\mathbf{G}$ over $\operatorname{Spec}\mathcal{O}$ by

$$\mathbf{G} = \begin{cases} \mathbf{G}_m & \text{if } \sigma|\sigma^{\mathrm{co}} \text{ is of type (gen) or (St)} \\ C_2 \ltimes \mathbf{G}_m & \text{if } \sigma|\sigma^{\mathrm{co}} \text{ is of type (scalar),} \end{cases}$$

where C_2 denotes the cyclic group $\{1, \tau\}$ of order 2, and the semi-direct product $C_2 \ltimes \mathbf{G}_m$ is defined by $\tau \cdot u = u^{-1}$.

Proposition 3.7.8. *We have an isomorphism*

$$\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}} \cong [\operatorname{Spec}\mathbf{F}[t, f(t)^{-1}, x, y]/(xy)/\mathbf{G}], \tag{3.7.9}$$

where

- *the action of $\mathbf{G}_m \leq \mathbf{G}$ is given by*
$$u \cdot (t, x, y) = (t, u^2 x, u^{-2} y),$$
- *and if $\{\sigma, \sigma^{\mathrm{co}}\}$ is of type* (scalar)*, then the action of C_2 is given by*
$$\tau \cdot (t, x, y) = (t^{-1}, y, x).$$

If $\sigma = \sigma_{a,b}$ then the representation $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ corresponding to a finite type point of $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$ is of the form

$$\overline{\rho} \cong \begin{pmatrix} \mathrm{nr}_{t^{-1}}\overline{\zeta}\omega^{-a} & *_x \\ *_y & \mathrm{nr}_t \omega^{a-1} \end{pmatrix}. \tag{3.7.10}$$

More precisely, by (3.7.10) *we mean the following: if $y = 0$, then $\overline{\rho}$ is an extension of $\mathrm{nr}_t\omega^{a-1}$ by $\mathrm{nr}_{t^{-1}}\overline{\zeta}\omega^{-a}$, and is split if and only if $x = 0$ (note that since $t^2 \neq 1$ if $\{\sigma, \sigma^{\mathrm{co}}\}$ is of type* (scalar) *or* (St)*, there is a unique non-split extension up to isomorphism); and similarly if $x = 0$, it is an extension of $\mathrm{nr}_{t^{-1}}\overline{\zeta}\omega^{-a}$ by $\mathrm{nr}_t\omega^{a-1}$, and is split if and only if $y = 0$.*

Proof. This is a straightforward consequence of the relationship between $\mathcal{X}(\sigma|\sigma^{\mathrm{co}})$ and the special fibre of a tamely potentially crystalline stack of cuspidal type, and in particular of the explicit description of these special fibres in [LMM23]. Since [LMM23] does not fix the determinant of the stacks that it considers, we begin by explaining how to reduce to that context. To that end, we write $\mathcal{X}_2$ for the stack of projective étale (φ, Γ)-modules of rank 2 as in [EG23], and we write $\mathcal{X}_2(\sigma|\sigma^{\mathrm{co}})$ for the union of the irreducible components $\mathcal{X}_2(\sigma)$ and $\mathcal{X}_2(\sigma^{\mathrm{co}})$ of $\mathcal{X}_{2,\mathrm{red}}$.

We will construct a certain open substack $\mathcal{U}$ of $\mathcal{X}_2(\sigma|\sigma^{\mathrm{co}})$. If σ is not of type (scalar), then $\mathcal{U}$ will admit the description

$$\mathcal{U} \cong [\operatorname{Spec}\mathbf{F}[t_1^{\pm 1}, t_2^{\pm 1}, x, y]/(xy)/(\mathbf{G}_m \times \mathbf{G}_m)],$$

where the action of $\mathbf{G}_m \times \mathbf{G}_m$ is given by

$$(u_1, u_2) \cdot (t_1, t_2, x, y) = (t_1, t_2, u_1 u_2^{-1} x, u_1^{-1} u_2 y).$$

If σ is of type (scalar), then $\mathcal{U}$ will admit the description

$$\mathcal{U} \cong [\operatorname{Spec} \mathbf{F}[t_1^{\pm 1}, t_2^{\pm 1}, (t_1 - t_2)^{-1}, x, y]/(xy)/C_2 \ltimes (\mathbf{G}_m \times \mathbf{G}_m)],$$

where C_2 exchanges the two copies of $\mathbf{G}_m$, the action of $\mathbf{G}_m \times \mathbf{G}_m$ is as above, and the action of C_2 is via

$$\tau \cdot (t_1, t_2, x, y) = (t_2, t_1, y, x).$$

Furthermore, in either case, the representation $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ corresponding to a finite type point of $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$ is of the form

$$\overline{\rho} \cong \begin{pmatrix} \mathrm{nr}_{t_1}\omega^{a+b} & *_x \\ *_y & \mathrm{nr}_{t_2}\omega^{a-1} \end{pmatrix}.$$

Admitting the existence of such a $\mathcal{U}$, we deduce the proposition as follows: by the definition of $\mathcal{X}$ (i.e. Definition C.1.1), there is an open substack $\mathcal{U}'$ of $\mathcal{X}(\sigma|\sigma^{\mathrm{co}})$ determined by the pullback diagram

$$\begin{tikzcd} \mathcal{U}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U} \arrow[d, "t_1 t_2"] \\ \operatorname{Spec} \mathbf{F} \arrow[r, "(\overline{\zeta}\omega^{-2a-b})(\mathrm{Frob}_p)"] & \mathbf{G}_m \end{tikzcd}$$

It follows from the definitions that $\mathcal{U}'$ is given by the right hand side of (3.7.9), and to see that $\mathcal{U}' = \mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$, it suffices to note that they have the same $\overline{\mathbf{F}}_p$-points.

We now turn to the construction of $\mathcal{U}$. Replacing σ by σ^{co} if necessary, we may assume that $0 \le b \le (p-3)/2$ unless σ is of type (scalar). After twisting, we can and do furthermore assume that $a = 1$. We write $\Theta(\sigma|\sigma^{\mathrm{co}})$ for the cuspidal representation of $\mathrm{GL}_2(\mathbf{F}_p)$ in Definition 2.1.8 (2). By [CDT99, Lem. 3.1.1(4)], the Jordan–Hölder factors of $\Theta(\chi) \otimes_{\mathcal{O}} \mathbf{F}$ are $\{\sigma, \sigma^{\mathrm{co}}\}$. It then follows (for example[4]) from [CDT99, Lem. 4.2.4] that $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ has a potentially crystalline lift of type $\tau := \omega_2^{b+2} \oplus \omega_2^{p(b+2)}$ and Hodge–Tate weights $0, 1$ if and only if $\overline{\rho}$ corresponds to an $\overline{\mathbf{F}}_p$-point of $\mathcal{X}_2(\sigma|\sigma^{\mathrm{co}})$.

Consequently, we see that in the notation of [EG23, Defn. 4.8.8], any open substack of $(\mathcal{X}_2^{\mathrm{crys},(1,0),\tau})_{\mathbf{F}}$ which is also reduced is in fact an open substack of $\mathcal{X}_2(\sigma|\sigma^{\mathrm{co}})$. We will now exhibit $\mathcal{U}$ as such an open substack. Firstly, note that by [BBHKLLSW24, Thm. 4.5] it is equivalent to produce an open substack of the special fibre of the stack denoted $\mathcal{Z}^{\tau}$ in [LMM23], where in the notation of that paper, we take $f = 1$, so that $K = \mathbf{Q}_p$, and we have $\tau = \tau((12), (b+2, 0))$.

Suppose firstly that we are not in the (scalar) case. Write $G = \mathrm{GL}_2 / \mathbf{F}$ with its usual upper-triangular Borel subgroup $B = TU$, where T is the diagonal maximal torus, and U the subgroup of upper-triangular unipotent matrices. Write LG for its loop group in the variable v, L^+G for the positive loop group, and respectively write L^+I and L_1^+G for the Iwahori subgroup of matrices which are upper-triangular modulo v and for the principal congruence subgroup of matrices congruent to the identity modulo v.

As in [LMM23, §3.1], write $\mathcal{A}(\eta) \subset LG$ for the subfunctor with $\mathcal{A}(\eta)(R)$ given by those $A \in \mathrm{GL}_2(R((v)))$ satisfying

[4]This is a particular case (possibly a motivating one) of the Breuil–Mézard conjecture [BM02] (in a weak form in which we pay no attention to the size of the multiplicities, but only to whether or not they are non-zero), which was proved in general by Kisin [Kis09]. For a reformulation in the framework of moduli stacks of étale (φ, Γ)-modules, see [EG23, §8.5].

(1) $\det A \in v \cdot R[\![v]\!]^\times$.
(2) $A \in M_2(R[\![v]\!])$, and A is upper-triangular modulo v.
(3) vA^{-1} is upper-triangular modulo v.

Write $LG^\tau \subset LG$ for the subfunctor

$$LG^\tau(R) = \mathcal{A}(\eta)(R)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} v^{b+2} & \\ & 1 \end{pmatrix}.$$

Then by [LMM23, Lem. 3.2.1, 3.3.7] (and in the case $p=5$ and $b=1$, [LMM23, Cor. 3.3.15] and its proof) we can identify the special fibre of $\mathcal{Z}^\tau$ with the quotient stack

$$[(L_1^+G\backslash LG^\tau)/B - \mathrm{conj}]$$

By [LMM23, Lem. 3.1.2(3)], this has an open substack $\mathcal{U}$ given by

$$\left[\left(L_1^+G\backslash L^+I\begin{pmatrix} X & 1 \\ v & Y \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} v^{b+2} & \\ & 1 \end{pmatrix}\right)/B - \mathrm{conj}\right]$$

with $XY = 0$.

Note that

$$\begin{pmatrix} X & 1 \\ v & Y \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} v^{b+2} & \\ & 1 \end{pmatrix} = \begin{pmatrix} v^{b+2} & X \\ v^{b+2}Y & v \end{pmatrix}.$$

Conjugation by an element of U acts as follows:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v^{b+2} & X \\ v^{b+2}Y & v \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + xYv^{b+1} + x^2Y^2v^{b+1} & x - v^{b+1}x - v^{b+1}x^2Y \\ xY^2v^{b+1} & 1 - xYv^{b+1} \end{pmatrix}\begin{pmatrix} v^{b+2} & X \\ v^{b+2}Y & v \end{pmatrix}$$
$$\in L^+I \cdot \begin{pmatrix} v^{b+2} & X \\ v^{b+2}Y & v \end{pmatrix}$$

Since we have $B = T \ltimes U$ and $L^+G = \mathrm{GL}_2 \ltimes L_1^+G$, $L^+I = B \ltimes L_1^+G$, it follows that $\mathcal{U}$ is equal to the quotient stack

$$\left[\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} v^{b+2} & X \\ v^{b+2}Y & v \end{pmatrix}\right)/T - \mathrm{conj}\right]$$

with $XY = 0$. By construction, the étale φ-module corresponding to an $\overline{\mathbf{F}}_p$-point of this stack is given by taking φ to be the matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} v^{b+2} & X \\ v^{b+2}Y & v \end{pmatrix}$, from which we can immediately read off (3.7.10).

Suppose now that we are in the (scalar) case. We follow the analysis of this case in [LMM23, §5.5.1]. Accordingly, we set $\widetilde{w} = w_0\eta$, $\widetilde{z} = t_{(1,1)}$, and it follows from the discussion in *loc. cit.* that in this case we have an open substack $\mathcal{U}$ given by the quotient by simultaneous conjugation of the pairs of matrices (φ, N) satisfying the equation

$$N \cdot \varphi N \varphi^{-1} = 0,$$

where φ is regular semisimple and N is nilpotent (more precisely, N is in the nilpotent cone of GL_2); and the corresponding étale φ-module is given by the matrix $\varphi\,(v1_2 + N)$. Indeed in the notation of [LMM23, §5.5.1], we take

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1}, \; N = \begin{pmatrix} -D & B \\ C & D \end{pmatrix}.$$

Since the regular semisimple matrix φ is conjugate to an element of T, this stack is isomorphic to the quotient of the pairs of matrices (φ, N) as above where now $\varphi \in T$ is regular semisimple, by conjugation by the normalizer $N(T) = T \rtimes C_2$. This is the required description. $\square$

Write $j' : \mathcal{U}_{\mathrm{good}} \to \mathcal{X}$ for the open immersion.

Lemma 3.7.11. *The functor $j'_* : D^b_{\mathrm{coh}}(\mathcal{U}_{\mathrm{good}}) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})$ is t-exact.*

Proof. By Corollary 3.7.6, we can replace $\mathcal{U}_{\mathrm{good}}$ by $\mathcal{U} := \mathcal{U}(\sigma|\sigma^{\mathrm{co}})$ for a companion pair $\{\sigma, \sigma^{\mathrm{co}}\}$. Write $\mathcal{X} \xrightarrow{\sim} \mathrm{colim}\, \mathcal{X}_n$ as a colimit of algebraic stacks with the transition maps being thickenings, and define $j'_n : \mathcal{U}_n \hookrightarrow \mathcal{X}_n$ to be the base-change of the open immersion $j' : \mathcal{U} \hookrightarrow \mathcal{X}$. Then (as noted in Remark B.2.17), the functor $j'_* : D^b_{\mathrm{coh}}(\mathcal{U}) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})$ is equivalent to the colimit of functors $(j'_n)_* : D^b_{\mathrm{coh}}(\mathcal{U}_n) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}_n)$, these functors (or, more precisely, their Ind-extensions) being instances of the general construction (B.1.10) associated to an open immersion of Noetherian algebraic stacks admitting affine diagonals.

To prove the lemma it thus suffices to show that each $(j'_n)_*$ is t-exact. By Remark B.1.11 (and the discussion that precedes it), it suffices to show that each of the open immersions j'_n is cohomologically affine; this we now do. (Recall that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *cohomologically affine* ([Alp13, Defn. 3.1]) if it is quasi-compact, and if furthermore $f_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact.)

By Proposition 3.7.8, $\mathcal{U}_{\mathrm{red}}$ is the quotient of an affine scheme by a linearly reductive group (note that since we are assuming that $p > 2$, the group scheme $\mathbf{G}/\operatorname{Spec} \mathbf{F}$ is linearly reductive by [Alp13, Ex. 12.4(2)]; recall that an *fppf* group scheme G/S is *linearly reductive* if the corresponding morphism $BG \to S$ is cohomologically affine). Thus $\mathcal{U}_{\mathrm{red}}$ is cohomologically affine over $\operatorname{Spec} \mathbf{F}$ (see [Alp13, Ex. 12.9]), and consequently each $\mathcal{U}_n$ is cohomologically affine over $\operatorname{Spec} \mathcal{O}$, by [Alp13, Prop. 3.9]. Furthermore each $\mathcal{X}_n$ has affine (and hence cohomologically affine) diagonal. We now factor j'_n as

$$\mathcal{U}_n \xrightarrow{\Gamma_{j'_n}} \mathcal{U}_n \times_{\operatorname{Spec} \mathcal{O}} \mathcal{X}_n \to \mathcal{X}_n,$$

the first arrow being the graph of j'_n, and the second arrow being the second projection. Since $\Gamma_{j'_n}$ is a base-change of $\Delta_{\mathcal{X}_n}$, it is affine (and in particular cohomologically affine). Since $\mathcal{U}_n \to \operatorname{Spec} \mathcal{O}$ is cohomologically affine, so is its pullback over $\mathcal{X}_n$. Thus j'_n is the composite of cohomologically affine morphisms, and hence is itself cohomologically affine, as required. $\square$

Although we don't need it in what follows, we end this section by showing that the morphism $\pi_{\mathrm{ss}} : |\mathcal{X}| \to |X|$ defined in (3.3.9) is a quotient map.

Lemma 3.7.12. *Let $\{\sigma, \sigma^{\mathrm{co}}\}$ be a companion pair of ζ-compatible Serre weights. Then the restriction*

$$\pi_{\mathrm{ss}} : |\mathcal{U}(\sigma|\sigma^{\mathrm{co}})| = \pi_{\mathrm{ss}}^{-1}(|U(\sigma|\sigma^{\mathrm{co}})|) \to |U(\sigma|\sigma^{\mathrm{co}})|$$

is a quotient map.

Proof. Since (as noted in the proof of Lemma (3.7.11)) the group $\mathbf{G}$ in Definition 3.7.7 is linearly reductive over $\mathbf{F}$, it follows from (3.7.9) and [Alp13, Theorem 13.2] that the algebraic stack $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$ has a good moduli space $p : \mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}} \to U$, given by the spectrum U of the ring of $\mathbf{G}$-invariants of $\mathbf{F}[t, f(t)^{-1}, x, y]/(xy)$. This ring of invariants is equal to $\mathbf{F}[t, f(t)^{-1}]$, resp. $\mathbf{F}[t + t^{-1}, 1/(t^2 + t^{-2} - 2)]$, if $\{\sigma, \sigma^{\mathrm{co}}\}$ is of type (gen), resp. of type (scalar), and the induced morphism on $\overline{\mathbf{F}}_p$-valued points sends an $\overline{\mathbf{F}}_p$-valued point of $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})$ of the form (3.7.10) to the point $\overline{\mathbf{F}}_p$-valued point t, resp. $t + t^{-1}$, of U.

Now we may (uniquely) identify $\operatorname{Spec} U$ with $U(\sigma|\sigma^{\mathrm{co}})$ in such a way that an $\overline{\mathbf{F}}_p$-valued point t, resp. $t + t^{-1}$, of U is identified with the $\overline{\mathbf{F}}_p$-valued point of $U(\sigma|\sigma^{\mathrm{co}})$

corresponding to the pseudorepresentation $\mathrm{nr}_{t^{-1}}\overline{\zeta}\omega^{-a} + \mathrm{nr}_t\omega^{a-1}$. With this identification, we see (using the above description of its effect on $\overline{\mathbf{F}}_p$-valued points) that the morphism

$$|\mathcal{U}(\sigma|\sigma^{\mathrm{co}})|_{\mathrm{ft}} = |\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}|_{\mathrm{ft}} \to |U| = |U(\sigma|\sigma^{\mathrm{co}})|_{\mathrm{ft}}$$

induced by the good moduli space morphism p is precisely (the restriction to $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})$ of) the morphism (3.3.7) which maps each finite type point to its associated pseudorepresentation. Soberizing, we find that the morphism induced by p on underlying topological spaces in fact coincides with π_{ss}. By [Alp13, Theorem 4.16(v)], p induces a quotient map, so we are done. □

Lemma 3.7.13. *The morphism $\pi_{\mathrm{ss}} : |\mathcal{X}| \to |X|$ is a quotient map of topological spaces.*

Proof. Since $|\mathcal{X}|$ is Noetherian, and since $|\mathcal{X}|_{\mathrm{ft}} \to |X|_{\mathrm{ft}}$ is surjective, the morphism $|\mathcal{X}| \to |X|$ is also surjective, by Lemma 3.7.14 below. Hence it suffices to show that $\pi_{\mathrm{ss}} : \pi_{\mathrm{ss}}^{-1}(|X(\sigma|\sigma^{\mathrm{co}})|) \to |X(\sigma|\sigma^{\mathrm{co}})|$ is a quotient map for each companion pair $\{\sigma, \sigma^{\mathrm{co}}\}$. Let $S \subset |X(\sigma|\sigma^{\mathrm{co}})|$, and assume that $\pi_{\mathrm{ss}}^{-1}(S)$ is closed in $\pi_{\mathrm{ss}}^{-1}(|X(\sigma|\sigma^{\mathrm{co}})|)$, or equivalently in $|\mathcal{X}|$. We need to prove that S is closed. By Lemma 3.7.12, we see that $S \cap |U(\sigma|\sigma^{\mathrm{co}})|$ is closed in $|U(\sigma|\sigma^{\mathrm{co}})|$. Hence either $S \cap |U(\sigma|\sigma^{\mathrm{co}})|$ is a finite set of closed points, or $|U(\sigma|\sigma^{\mathrm{co}})| \subset S$. Since $X(\sigma|\sigma^{\mathrm{co}}) \setminus U(\sigma|\sigma^{\mathrm{co}})$ is a finite set of closed points, the first possibility implies that S is a finite set of closed points, which concludes the proof in this case. Similarly, the second possibility implies that $X(\sigma|\sigma^{\mathrm{co}}) \setminus S$ is a finite set of closed points, and so S is open in $X(\sigma|\sigma^{\mathrm{co}})$. Hence $\pi_{\mathrm{ss}}^{-1}(S)$ is a closed and open subset of $\pi_{\mathrm{ss}}^{-1}(|X(\sigma|\sigma^{\mathrm{co}})|)$, which is a connected topological space. This implies that $\pi_{\mathrm{ss}}^{-1}(S) = \pi_{\mathrm{ss}}^{-1}(|X(\sigma|\sigma^{\mathrm{co}})|)$ and so $S = X(\sigma|\sigma^{\mathrm{co}})$, since π_{ss} is surjective. This concludes the proof that S is closed. □

Lemma 3.7.14. *If $f : T \to S$ is a continuous surjective map of topological spaces, and T is Noetherian, then the induced map $\mathrm{sob}(T) \to \mathrm{sob}(S)$ is surjective.*

Proof. Let A be an irreducible closed subspace of S. Let $\{B_i\}$ denote the (finite, because T is Noetherian) set of irreducible components of $f^{-1}(A)$. Then, since f is surjective, we have $A = \bigcup \overline{f(B_i)}$, and so (since A is irreducible), we have $A = \overline{f(B_{i_0})}$ for one of the components B_{i_0}. Thus the induced map $\mathrm{sob}(T) \to \mathrm{sob}(S)$ is indeed surjective. □

Remark 3.7.15. In fact the morphism f has a lift to a morphism $\mathcal{X}_{\mathrm{red}} \to X$, which in turn admits a thickening to a morphism $\mathcal{X} \to \widetilde{X}$, where $\widetilde{X}$ is a certain formal scheme over $\mathrm{Spf}\,\mathcal{O}$ with underlying reduced equal to X. The formal scheme $\widetilde{X}$, and the morphism from $\mathcal{X}$ to it, are characterized as being initial in the category of morphisms from $\mathcal{X}$ to formal algebraic spaces over $\mathrm{Spf}\,\mathcal{O}$. In other words, $\widetilde{X}$ is the formal moduli space associated to $\mathcal{X}$. Since we do not need this result in this paper, we don't give a proof here (although we intend to give a construction elsewhere).

4. $D^\natural \boxtimes \mathbf{P}^1$

In this section we study analogues for general coefficients of some of the constructions of [Col10c]. Throughout the discussion, all étale (φ, Γ)-modules are automatically understood to be projective, with the exception that from Section 4.7

onward we exploit the relationship between étale (φ, Γ)-modules and Galois representations, and so allow ourselves at times to consider étale (φ, Γ)-modules (and especially so-called *formal* étale (φ, Γ)-modules) that are not necessarily projective.

Our ring of coefficients is typically taken to be a Noetherian $\mathcal{O}/\varpi^a$-algebra, for some $a \geq 1$; the main exception to this occurs in our use of formal étale (φ, Γ)-modules, where the coefficients are instead taken to be a complete Noetherian local $\mathcal{O}$-algebra with finite residue field. At certain points we also restrict our coefficients to be a finite type $\mathcal{O}/\varpi^a$-algebra; we do this sometimes for topological reasons (the various topological modules under consideration will then be Polish topological groups, which lets us apply the open mapping theorem), and sometimes because we wish to complete at maximal ideals and pass to the formal context.

4.1. **Tate modules and topologies.** The coefficient rings $\mathbf{A}_A$ and $\mathbf{A}_A^+$ introduced in Section 3.1, and many of the various modules over these rings that we study below, are examples of *Tate A-modules* in the sense of [Dri06, §3]. We now recall this notion and some of its basic properties; see also [EG21, §5] and [EG23, App. D] for some related results and discussion.

Definition 4.1.1. Let A be a commutative ring. An *elementary Tate A-module* is a topological A-module which is isomorphic to $P \oplus Q^*$, where P, Q are discrete projective A-modules, and $Q^* := \mathrm{Hom}_A(Q, A)$ equipped with its natural projective limit topology (where we write Q^* as the projective limit of the $(Q')^*$, where Q' is a finite direct summand of Q, and give each $(Q')^*$ the discrete topology). A *Tate A-module* is a direct summand of an elementary Tate A-module.

A morphism of Tate modules is a continuous morphism of the underlying A-modules.

Definition 4.1.2. A submodule L of a Tate A-module M is a *lattice* if it is open, and if furthermore for every open submodule $U \subseteq L$, the quotient L/U is a finitely generated A-module. A subset of a Tate A-module is *bounded* if it is contained in some lattice.

Remark 4.1.3. If A is Noetherian, then a submodule L of a Tate A-module M is a lattice if and only if it is open and bounded. Indeed if L is open and bounded, then by definition there is a lattice $L' \supseteq L$, and for any open submodule $U \subseteq L$, the quotient L'/U is a finitely generated A-module; so the submodule $L/U \subseteq L'/U$ is also a finitely generated A-module.

If A is an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, then $\mathbf{A}_A^+$ and $\mathbf{A}_A$ are elementary Tate modules, and $\mathbf{A}_A^+$ is a lattice in $\mathbf{A}_A$; furthermore every finitely generated projective $\mathbf{A}_A$-module has a natural topology, making it a Tate A-module. (See for example [EG21, Ex. 5.1.13].)

In fact, by [Dri06, Thm. 3.10], there is a natural bijection between finitely generated projective $\mathbf{A}_A$-modules, and pairs (M, T) consisting of a Tate A-module M and a topologically nilpotent automorphism $T : M \to M$, by giving M the $\mathbf{A}_A$-module structure determined by $Tm := T(m)$. (Here T is topologically nilpotent if and only if for each pair of lattices $L, L' \subseteq M$, we have $T^n L \subseteq L'$ for all sufficiently large n.)

Definition 4.1.4. If M is a finitely generated projective $\mathbf{A}_A$-module, then we refer to an $\mathbf{A}_A^+$-submodule $\mathfrak{M} \subset M$ which is also a lattice as an $\mathbf{A}_A^+$-*lattice* (or $\mathbf{A}^+$-*lattice* if A is clear from the context).

Remark 4.1.5. Let A be an $\mathcal{O}/\varpi^a$-algebra. If M is a finitely generated projective $\mathbf{A}_A$-module, and $\mathfrak{M} \subset M$ is a lattice, then $\mathfrak{M}$ need not be an $\mathbf{A}_A^+$-lattice. However, since multiplication by T is topologically nilpotent, for all sufficiently large n we have $T^n\mathfrak{M} \subset \mathfrak{M}$, so that $\mathfrak{M}$ is an $A[\![T^n]\!]$-module. Since M is a finite projective $A((T^n))$-module, and the rings $A[\![T^n]\!]$, $A((T^n))$ are respectively isomorphic to $\mathbf{A}_A^+$ and $\mathbf{A}_A$, this means that we can often reduce questions about lattices to the case of $\mathbf{A}^+$-lattices.

We recall the following lemma due to Drinfeld [EG21, Lem. 5.1.20].

Lemma 4.1.6. *If M is a finitely generated projective $\mathbf{A}_A$-module, and if $\mathfrak{M}$ is a lattice in M, then $M/\mathfrak{M}$ is A-flat if and only if it is A-projective. If $\mathfrak{M}$ is furthermore an $\mathbf{A}_A^+$-lattice, then these conditions are in turn equivalent to $\mathfrak{M}$ being $\mathbf{A}_A^+$-projective.*

4.1.7. *The weak topology on lattices.* If A is a complete Noetherian local ring, and L is a lattice in an A-Tate module, then in addition to its Tate-module topology, it has a *weak topology*, as we now explain. Lemma 4.1.8 below shows that we have an isomorphism (of A-modules, i.e. disregarding topologies) $L \xrightarrow{\sim} \varprojlim_k L/\mathfrak{m}_A^k L$. If we endow $L/\mathfrak{m}_A^k L$ with its quotient topology induced by the Tate module topology on L, then the right-hand side inherits an inverse limit topology, which we then can transport to L via this isomorphism; this is (by definition) the weak topology on L.

Lemma 4.1.8. *If A is a complete Noetherian local ring, and $L \subset M$ is a lattice in a Tate A-module, then L is $\mathfrak{m}_A$-adically complete. The A-module L with the weak topology is an object of* $\mathrm{Mod}_c(A)$, *hence it is profinite if A has finite residue field.*

Proof. Since M is a direct summand of an elementary Tate module, and the direct sum of lattices is a lattice, it suffices to prove the lemma under the assumption that $M = P \oplus Q^*$ is elementary.

We begin by proving the lemma when $L = Q^*$. Since A is a local ring, Q is free, and so Q^* is isomorphic to a product $\prod_{i\in I} A$ for some index set I. Then Q^* is $\mathfrak{m}_A$-adically complete because $\otimes_A A/\mathfrak{m}_A^n$ commutes with arbitrary products, since $A/\mathfrak{m}_A^n$ is finitely presented. For the second claim of the lemma, note that the Tate module topology on $\prod_{i\in I} A$ is the product topology with respect to the discrete topology on A, hence the quotient topology on $\prod_{i\in I} A/\mathfrak{m}_A^n$ is the product topology with respect to the discrete topology on $A/\mathfrak{m}_A^n$. It follows that the weak topology on $\prod_{i\in I} A$ is the product topology with respect to the $\mathfrak{m}_A$-adic topology on A, which makes $\prod_{i\in I} A$ an object of $\mathrm{Mod}_c(A)$.

Now, if $L \subset Q^*$ is any lattice, then by definition there exists an A-finite direct summand $Q' \subset Q$ such that $(Q')^\perp \subset L$. Writing Q'' for the complement of Q', so that $Q = Q' \oplus Q''$, we see that $Q^* = Q'^* \oplus Q''^*$ and $(Q')^\perp = Q''^*$. Hence there exists an A-submodule $V \subset Q'^*$ such that $L = V \oplus Q''^*$. Since Q'^* is finite, we see that V is also finite. By the results of the previous paragraph, this immediately implies that L is $\mathfrak{m}_A$-adically complete, and that the weak topology makes L an object of $\mathrm{Mod}_c(A)$.

Finally, if $L \subset M$ is any lattice, then it is contained in $P' \oplus Q^*$ for some finite A-summand $P' \subset P$. Since $M = (P/P') \oplus (P' \oplus Q^*)$, and $(P' \oplus Q^*) \cong (P'^* \oplus Q)^*$, we can conclude the proof by an application of the results in the previous paragraph (applied with Q^* replaced by $(P'^* \oplus Q)^*$). $\square$

Example 4.1.9. In the case when L is an $\mathbf{A}_A^+$-lattice in a finitely generated projective $\mathbf{A}_A$-module, the weak topology on L coincides with its $(\mathfrak{m}_A, T)$-adic topology, whereas the Tate module topology coincides with its T-adic topology.

4.1.10. *Completions and completed tensor products.* Suppose that A is a ring, and that M is a topological A-module endowed with an A-linear topology, i.e. M is an A-module with a topological abelian group structure, such that $0 \in M$ admits a neighbourhood basis consisting of A-submodules of M. Note that the product morphism $A \times M \to M$ is then jointly continuous, when A is equipped with its discrete topology. We may then consider the completion of M (either as a topological group, or as a topological A-module; the result is the same). In symbols, if we let U run over the partially ordered set of open A-submodules of M, then

$$\widehat{M} := \varprojlim_U M/U.$$

Suppose now that $A \to B$ is a ring homomorphism. If M is a topological A-module endowed with an A-linear topology, then we equip $M \otimes_A B$ with the B-linear topology defined by the images of the various $U \otimes_A B$, as U runs over the open A-submodules of M. We then define $M \widehat{\otimes}_A B$ to be the completion of $M \otimes_A B$ with respect to this topology. Thus, if we let $\overline{U \otimes_A B}$ denote the image of $U \otimes_A B$ in $M \otimes_A B$, then

$$M \widehat{\otimes}_A B := \varprojlim_U (M \otimes_A B)/(\overline{U \otimes_A B}) = \varprojlim_U (M/U) \otimes_A B \tag{4.1.11}$$

(each of the terms in each of the inverse limits being endowed with its discrete topology). The symbol $\widehat{\otimes}_A$ was also used in Section A.1.61 to denote the completed tensor product on categories of compact modules, but this should not lead to ambiguities, since in the current context, the ring A is not a profinite topological ring.

We will most often apply this notion in the case when $A \to B$ is a morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras, in situations related to the above discussion of finite projective $\mathbf{A}_A$-modules and their lattices. If M_A is a finite projective $\mathbf{A}_A$-module, then its Tate module topology is A-linear, and M_A is complete, as is any lattice in M_A. The tensor product $M_B := M_A \otimes_{\mathbf{A}_A} \mathbf{A}_B$ is a finite projective $\mathbf{A}_B$-module, and hence is again complete. There is a canonical continuous morphism $M_A \otimes_A B \to M_A \otimes_{\mathbf{A}_A} \mathbf{A}_B$. Since the target is complete, it induces a morphism from the completion of the source to the target, which is in fact an isomorphism

$$M_A \widehat{\otimes}_A B \xrightarrow{\sim} M_A \otimes_{\mathbf{A}_A} \mathbf{A}_B. \tag{4.1.12}$$

If $\mathfrak{M}_A \subseteq M_A$ is a lattice, then we may also consider the completed tensor product $\mathfrak{M}_A \widehat{\otimes}_A B$. If $\mathfrak{M}_A$ is an $\mathbf{A}_A^+$-lattice, then this completion naturally identifies with $\mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+$, in the sense that the natural map

$$\mathfrak{M}_A \widehat{\otimes}_A B \to \mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+ \tag{4.1.13}$$

is a topological isomorphism. Indeed, the natural A-linear topology on $\mathfrak{M}_A$ is the $T\mathbf{A}_A^+$-adic topology, hence the B-linear topology on

$$\mathfrak{M}_A \otimes_A B = \mathfrak{M}_A \otimes_{\mathbf{A}_A^+} (\mathbf{A}_A^+ \otimes_A B)$$

is the $T(\mathbf{A}_A^+ \otimes_A B)$-adic topology. Since $\mathbf{A}_A^+ \otimes_A B$ is a Noetherian ring with T-adic completion $\mathbf{A}_B^+$, and $\mathfrak{M}_A$ is a finite $\mathbf{A}_A^+$-module, we see that $\mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+$ is the

T-adic completion of $\mathfrak{M}_A \otimes_A B$. The inclusion $\mathfrak{M}_A \subseteq M_A$ induces a continuous morphism $\mathfrak{M}_A \widehat{\otimes}_A B \to M_B$, which is open (by Lemma 4.1.14 (4) below), but which need not be an inclusion in general.

The following lemma collects together various basic statements about lattices in finite projective $\mathbf{A}_A$-modules, and their behaviour under completed tensor product.

Lemma 4.1.14. *Let $A \to B$ be a morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras. Let M_A be a finite projective $\mathbf{A}_A$-module, and as above write $M_B = M_A \widehat{\otimes}_A B = M_A \otimes_{\mathbf{A}_A} \mathbf{A}_B$.*

(1) An $\mathbf{A}_A^+$-submodule $\mathfrak{M}_A \subset M_A$ is an $\mathbf{A}_A^+$-lattice if and only if it is finitely generated and the $\mathbf{A}_A$-span of $\mathfrak{M}_A$ is M_A.

(2) M_A contains an $\mathbf{A}_A^+$-lattice $\mathfrak{M}_A$.

(3) If L_1 is a lattice in M_A, and L_2 is an A-submodule of M_A, then L_2 is a lattice if and only if it is commensurate to L_1, in the sense that for n sufficiently large we have $T^n L_1 \subseteq L_2 \subseteq T^{-n} L_1$.

(4) If $\mathfrak{M}_A \subseteq M_A$ is a lattice then the evident sequence

$$\mathfrak{M}_A \widehat{\otimes}_A B \to M_B \to (M_A/\mathfrak{M}_A) \otimes_A B \to 0, \tag{4.1.15}$$

is exact, and the image of the morphism $\mathfrak{M}_A \widehat{\otimes}_A B \to M_A \widehat{\otimes}_A B = M_B$ is a lattice in M_B, while its kernel is finitely generated over B. In addition, this kernel is discrete (with respect to the topology induced on it by $\mathfrak{M}_A \widehat{\otimes}_A B$), and the morphism $\mathfrak{M}_A \widehat{\otimes}_A B \to M_B$ is open.

If furthermore either $A \to B$ is flat or $M_A/\mathfrak{M}_A$ is flat as an A-module then (4.1.15) *is exact on the left as well, i.e. we have a short exact sequence*

$$0 \to \mathfrak{M}_A \widehat{\otimes}_A B \to M_B \to (M_A/\mathfrak{M}_A) \otimes_A B \to 0.$$

In particular, $\mathfrak{M} \widehat{\otimes}_A B$ is then a lattice in M_B.

(5) If $A \hookrightarrow B$ is injective, and $\mathfrak{M}$ is a lattice in M_B, then $\mathfrak{M} \cap M_A$ is a lattice in M_A.

Proof. Part (1) is [EG23, Lem. D.9]. Part (2) follows from part (1), because we can choose a surjection $f : (\mathbf{A}_A)^r \to M_A$ for some sufficiently large r, and set $\mathfrak{M}_A = f((\mathbf{A}_A^+)^r)$. The "only if" direction of (3) is immediate from the topological nilpotence of T. For the converse, since L_2 contains $T^n L_1$, it is open, while if $U \subseteq L_2$ is open, then L_2/U is a submodule of the finitely generated A-module $(T^{-n}L_1)/U$, and is therefore finitely generated, so L_2 is a lattice by definition.

We turn to proving part (4). By Remark 4.1.5 we can, without loss of generality, assume that $\mathfrak{M}_A$ is an $\mathbf{A}^+$-lattice. As explained above, in this case we have that

$$\mathfrak{M}_A \widehat{\otimes}_A B = \mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+$$

and that

$$M_A \widehat{\otimes}_A B = M_A \otimes_{\mathbf{A}_A} \mathbf{A}_B = M_A \otimes_{\mathbf{A}_A^+[1/T]} \mathbf{A}_B^+[1/T] = M_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+.$$

Now $M_A/\mathfrak{M}_A = \bigcup_{n \geq 0} (T^{-n}\mathfrak{M}_A)/\mathfrak{M}_A$, so that

$$\begin{aligned}(M_A/\mathfrak{M}_A) \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+ &= \bigcup_{n \geq 0} \big((T^{-n}\mathfrak{M}_A)/\mathfrak{M}_A\big) \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+ \\ &= \bigcup_{n \geq 0} \big((T^{-n}\mathfrak{M}_A)/\mathfrak{M}_A\big) \otimes_A B = (M_A/\mathfrak{M}_A) \otimes_A B.\end{aligned}$$

Thus tensoring the short exact sequence

$$0 \to \mathfrak{M}_A \to M_A \to M_A/\mathfrak{M}_A \to 0$$

with $\mathbf{A}_B^+$ over $\mathbf{A}_A^+$, and reinterpreting the various terms according to the preceding isomorphisms, we obtain the exact sequence (4.1.15), establishing the first claim of (4). Furthermore, since the image of $\mathfrak{M}_A \widehat{\otimes}_A B = \mathfrak{M}_A \widehat{\otimes}_{\mathbf{A}_A^+} \mathbf{A}_B^+$ in M_B is finitely generated over $\mathbf{A}_B^+$ and generates M_B over $\mathbf{A}_B$, it is a lattice in M_B, by (1).

If $A \to B$ is flat, then so is $\mathbf{A}_A^+ \to \mathbf{A}_B^+$, and thus the preceding right exact sequence becomes short exact. On the other hand, if $M_A/\mathfrak{M}_A$ is A-flat, then Lemma 4.1.6 shows that $\mathfrak{M}_A$ is projective over $\mathbf{A}_A^+$. Thus $\mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+$ is projective over $\mathbf{A}_B^+$, and in particular is T-torsion free, and so again we find that it embeds into its localization $(\mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+)[T^{-1}] = M_B$.

To complete the proof of (4), we must prove our claims about the kernel of the left-most arrow in (4.1.15), and also prove that $\mathfrak{M} \otimes_A B \to M_B$ is an open mapping. Since M_A is a finite projective $\mathbf{A}_A$-module by assumption, we may find a second finite projective $\mathbf{A}_A$-module M'_A so that $M_A \oplus M'_A$ is free of finite rank over $\mathbf{A}_A$. If we choose an $\mathbf{A}_A^+$-lattice $\mathfrak{M}'_A$ in M'_A, then $\mathfrak{M}_A \oplus \mathfrak{M}'_A$ is a lattice in $M_A \oplus M'_A$. The formation of the various completed tensor products, kernels, and images in play is compatible with taking direct sums, and so replacing M_A by $M_A \oplus M'_A$, and $\mathfrak{M}_A$ by $\mathfrak{M}_A \oplus \mathfrak{M}'_A$, we may assume without loss of generality that M_A is free. In this case, we may choose a free $\mathbf{A}_A^+$-lattice $\mathfrak{M}_A \subseteq \mathfrak{M}''_A \subset M_A$. Furthermore, part (3) shows that there exists $n_0 \geq 0$ so that $T^n\mathfrak{M}''_A \subseteq \mathfrak{M}_A$ for $n \geq n_0$, and the sublattices $T^n\mathfrak{M}''_A$ $(n \geq n_0)$ then form a basis of neighbourhoods of 0 in $\mathfrak{M}_A$.

Since M_A is flat over $\mathbf{A}_A^+$, we see that the kernel of the morphism

$$\mathfrak{M}_A \widehat{\otimes}_A B = \mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+ \to M_B$$

is identified with $\mathrm{Tor}_1^{\mathbf{A}_A^+}(M_A/\mathfrak{M}_A, \mathbf{A}_B^+)$, and we begin by explaining why this is a finite B-module. From what we have already proved, we know that

$$\mathfrak{M}''_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+ \to M_B \to (M_A/\mathfrak{M}''_A) \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+ \to 0$$

is exact on the left, so that $\mathrm{Tor}_1^{\mathbf{A}_A^+}(M_A/\mathfrak{M}''_A, \mathbf{A}_B^+) = 0$. A consideration of the short exact sequence

$$0 \to \mathfrak{M}''_A/\mathfrak{M}_A \to M_A/\mathfrak{M}_A \to M_A/\mathfrak{M}''_A \to 0$$

then yields a surjection

$$\mathrm{Tor}_1^{\mathbf{A}_A^+}(\mathfrak{M}''_A/\mathfrak{M}_A, \mathbf{A}_B^+) \to \mathrm{Tor}_1^{\mathbf{A}_A^+}(M_A/\mathfrak{M}_A, \mathbf{A}_B^+).$$

Now the source of this surjection is finite over $\mathbf{A}_B^+$ (since $\mathfrak{M}''_A/\mathfrak{M}_A$ is finite over $\mathbf{A}_A^+$) and killed by some power of T (since $\mathfrak{M}''_A/\mathfrak{M}_A$ is). Thus it is finite over B, and thus so is $\mathrm{Tor}_1^{\mathbf{A}_A^+}(M_A/\mathfrak{M}_A, \mathbf{A}_B^+)$, which is what we wanted to prove.

Next, consider the neighbourhood basis $T^n\mathfrak{M}''_A \subseteq \mathfrak{M}_A$ of 0 in $\mathfrak{M}_A$. By definition, the images

$$\overline{(T^n\mathfrak{M}''_A) \widehat{\otimes}_A B} \coloneqq \mathrm{im}\big((T^n\mathfrak{M}''_A) \widehat{\otimes}_A B \to \mathfrak{M}''_A \widehat{\otimes}_A B\big)$$

form a neighbourhood basis of the topology on $\mathfrak{M}''_A \widehat{\otimes}_A B$. Again, by what we have already proved, for each n the morphism $(T^n\mathfrak{M}''_A) \widehat{\otimes}_A B \to M_B$ is injective, and so $\overline{(T^n\mathfrak{M}''_A) \widehat{\otimes}_A B)}$ has trivial intersection with $\ker(\mathfrak{M}_A \widehat{\otimes}_A B \to M_B)$. This shows that this kernel is discrete in $\mathfrak{M}_A \widehat{\otimes}_A B$. Also, the image of $(T^n\mathfrak{M}''_A) \widehat{\otimes}_A B$ is a lattice in

M_B, and so $\mathfrak{M}_A \widehat{\otimes}_A B$ has a neighbourhood basis of zero whose members have open image in M_B. Thus $\mathfrak{M}_A \widehat{\otimes}_A B \to M_B$ is open, as claimed. This completes the proof of (4).

Finally for (5), by parts (2) and (3) there is an $\mathbf{A}_B^+$-lattice $\mathfrak{M}' \subseteq M_B$ and an integer $n \geq 0$ such that $T^n \mathfrak{M}' \subseteq \mathfrak{M} \subseteq T^{-n}\mathfrak{M}'$. By [EG23, Lem. D.11], $\mathfrak{M}' \cap M_A$ is a lattice in M_A, and $(T^i\mathfrak{M}') \cap M_A = T^i(\mathfrak{M}' \cap M_A)$ for all integers i. In particular we have

$$T^n(\mathfrak{M}' \cap M_A) \subseteq \mathfrak{M} \cap M_A \subseteq T^{-n}(\mathfrak{M}' \cap M_A),$$

so $\mathfrak{M} \cap M_A$ is a lattice in M_A by (3). $\square$

We also note the following result, showing that certain tensor products are automatically complete.

Lemma 4.1.16. *If $A \to B$ is a finite morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras and M_A is a finite projective $\mathbf{A}_A$-module, then the natural morphism $M_A \otimes_A B \to M_A \widehat{\otimes}_A B$ is a topological isomorphism. Furthermore, if $\mathfrak{M}_A$ is a lattice in M_A, then the natural morphism $\mathfrak{M}_A \otimes_A B \to \mathfrak{M}_A \widehat{\otimes}_A B$ is a topological isomorphism.*

Proof. We begin by proving the second statement. By Remark 4.1.5, without loss of generality $\mathfrak{M}_A$ is an $\mathbf{A}_A^+$-lattice. By (4.1.13), the completed tensor product $\mathfrak{M}_A \widehat{\otimes}_A B$ identifies with $\mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+$. Since $\mathbf{A}_A^+ = A[\![T]\!]$ is Noetherian and $\mathfrak{M}_A$ is a lattice in M_A, $\mathfrak{M}_A$ is finitely presented over $\mathbf{A}_A^+$. The map

$$\mathfrak{M}_A \otimes_A B \to \mathfrak{M}_A \otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+$$

is part of a natural transformation of right-exact functors on finitely presented $\mathbf{A}_A^+$-modules. Hence, to prove that it is an isomorphism, it suffices to prove that it is an isomorphism when $\mathfrak{M}_A = \mathbf{A}_A^+$, i.e. that $\mathbf{A}_A^+ \otimes_A B = \mathbf{A}_B^+$. Since $B[T]$ is finitely presented as an $A[T]$-module, this follows because $\mathbf{A}_A^+ \otimes_A B = A[\![T]\!] \otimes_{A[T]} B[T]$ is the T-adic completion of $B[T]$.

We now prove the first statement. By (4.1.12) we have $M_A \widehat{\otimes}_A B = M_A \otimes_{\mathbf{A}_A} \mathbf{A}_B$, so we need to prove that the natural map

$$M_A \otimes_A B \to M_A \otimes_{\mathbf{A}_A} \mathbf{A}_B \tag{4.1.17}$$

is a topological isomorphism. Since it is continuous and open (by Lemma 4.1.14 4), it suffices to prove that it is bijective. Since M_A is a direct summand of a finite free $\mathbf{A}_A$-module, it suffice to prove that (4.1.17) is bijective when $M_A = \mathbf{A}_A$. This holds because $\mathbf{A}_A = \mathbf{A}_A^+[1/T]$ and $\mathbf{A}_B^+ = \mathbf{A}_A^+ \otimes_A B$, as proved in the previous paragraph. $\square$

We close this section by establishing some additional properties related to the topology of finite projective $\mathbf{A}_A$-modules.

Lemma 4.1.18. *Let A be a Noetherian ring, let M be a topological A-module endowed with an A-linear topology, and suppose furthermore that there is a sequence $(\mu_n)_{n\geq 0}$ of continuous morphisms $\mu_n : M \to A$ for which the induced morphism $M \to \prod_{n=0}^{\infty} A$ is injective. Then the induced topology on any finitely generated A-submodule of M is discrete.*

Proof. Consider a morphism $f : A^m \to M$, and write $\nu_n := \mu_n \circ f$. Let e_i ($i = 1, \ldots, m$) denote the standard basis vectors of A^m, and write

$$S_r := \{(\nu_0(e_1), \ldots, \nu_0(e_m)), \ldots, (\nu_r(e_1), \ldots, \nu_r(e_m))\} \subseteq A^m.$$

Then, if $K_r := \ker(\nu_0 \times \cdots \times \nu_r)$, we find that $K_r = S_r^{\perp}$ (the "orthogonal complement" of S_r with respect to the usual pairing $A^m \times A^m \to A$).

The S_r form an increasing sequence of subsets of A^m, and hence the $S_r^{\perp}$ form a decreasing sequence of orthogonal complements. The Artinian property of orthogonal complements[5] implies that this decreasing sequence stabilizes. If we recall that $M \to \prod_{n=0}^{\infty} A$ is injective by assumption, we then see that $K := \ker(f) = \bigcap_{r \geq 0} K_r = K_{r_0}$ for some sufficiently large r_0. Hence, if $N := \operatorname{im}(f)$ then

$$N \cap \ker\Big(\prod_{n=0}^{\infty} A \to \prod_{n=0}^{r_0} A\Big) = 0.$$

Hence N inherits the discrete topology, as claimed. □

Example 4.1.19.

(1) If A is a Noetherian $\mathcal{O}/\varpi^a$-algebra, then Lemma 4.1.18 applies to finite projective $\mathbf{A}_A$-modules, and hence also to lattices in such modules.
(2) If $(M_n)_{n \geq 0}$ is an inverse system of topological modules satisfying the hypotheses of Lemma 4.1.18, then the same is true of the inverse limit $M := \varprojlim_n M_n$.

Lemma 4.1.20. *Let A be a Noetherian $\mathcal{O}/\varpi^a$-algebra, I-adically complete with respect to some ideal $I \subseteq A$. If M_A is a finite projective $\mathbf{A}_A$-module, then any lattice in M_A is closed in the I-adic topology of M_A.*

Proof. Let $\mathfrak{M}$ be any lattice in M_A. Applying Remark 4.1.5, we see that it is no loss of generality to assume that $\mathfrak{M}$ is an $\mathbf{A}_A^+$-lattice. Choose another finite projective $\mathbf{A}_A$-module M'_A such that $M_A \oplus M'_A$ is free, and choose an $\mathbf{A}_A^+$-lattice $\mathfrak{M}'$ in M'_A. Then it suffices to prove the result for the $\mathbf{A}_A^+$-lattice $\mathfrak{M} \oplus \mathfrak{M}'$ in $M_A \oplus M'_A$, and so it is no loss of generality to further assume that M_A is free. Then we may choose a free $\mathbf{A}_A^+$-lattice L in M_A, and by part (3) of Lemma 4.1.14, we may assume (scaling L by a power of T if necessary) that L contains $\mathfrak{M}$.

Now the I-adic topology on $\mathbf{A}_A := A((T))$ evidently induces the I-adic topology on $\mathbf{A}_A^+ := A[\![T]\!]$, and so the I-adic topology on M_A induces the I-adic topology on L. Thus it suffices to prove that the I-adic topology on L induces the I-adic topology on $\mathfrak{M}$. This follows directly from Artin–Rees (applied to the ideal $I\mathbf{A}_A^+$ in the Noetherian ring $\mathbf{A}_A^+$, using the fact that $\mathbf{A}_A^+$-lattices are finitely generated over $\mathbf{A}_A^+$, by Lemma 4.1.14 (1)). □

Remark 4.1.21. Note that the statements of parts (4) and (5) of Lemma 4.1.14, and the statements of Lemmas 4.1.18 and 4.1.20, depend only on the structure of M_A as a Tate A-module. Thus if M_A is any Tate A-module which is topologically isomorphic (as a topological A-module) to a finite projective $\mathbf{A}_A$-module, these statements apply to M_A.

4.2. $D^{\natural}$ and $D^{\sharp}$.

Unless stated otherwise, in this section we assume that A is a Noetherian $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$. We prove some analogues with coefficients of some results from [Col10c; Col10a]. Our overall strategy, and many of our arguments, is based on those of Colmez, but the presence of the coefficient ring A means that we sometimes have to find new ways to argue.

---

[5] If S is a subset of A^m, then the formation of $S^{\perp}$ is order reversing, and $S \subseteq S^{\perp\perp}$. These properties then imply that $S^{\perp} = S^{\perp\perp\perp}$, and that $S \mapsto S^{\perp\perp}$ is order preserving. Thus $S_r^{\perp\perp}$ is an increasing sequence of submodules of the Noetherian module A^m, and so stabilizes. Thus the decreasing sequence $S_r^{\perp} = S_r^{\perp\perp\perp}$ also stabilizes.

Remark 4.2.1. Throughout this section we do not make any use of the action of Γ, and our results are equally valid for étale φ-modules over $\mathbf{A}_A$. We will occasionally exploit this in our proofs.

We begin by defining the ψ-operator on (φ, Γ)-modules.

Proposition 4.2.2. *Let D be an étale (φ, Γ)-module with A-coefficients. Then there is a unique A-linear morphism $\psi : D \to D$ such that*

$$\psi(\varphi(a)m) = a\psi(m),$$
$$\psi(a\varphi(m)) = \psi(a)m$$

for any $a \in \mathbf{A}_A$, $m \in D$. The morphism ψ is furthermore continuous, open, and surjective, and commutes with Γ.

Proof. Since $D = \mathbf{A}_A\varphi(D)$, we see that the required properties of ψ uniquely determine it if it exists. In fact, the same observation, expressed in the more precise form of the isomorphism $\Phi_D : \mathbf{A}_A \otimes_{\mathbf{A}_A,\varphi} D \xrightarrow{\sim} D$, allows us to construct ψ, namely as the composite

$$D \overset{\Phi_D^{-1}}{\to} \mathbf{A}_A \otimes_{\mathbf{A}_A,\varphi} D \overset{\psi\otimes 1}{\to} \mathbf{A}_A \otimes_{\mathbf{A}_A} D = D.$$

The relations $\psi(\varphi(a)m) = a\psi(m)$, $\psi(a\varphi(m)) = \psi(a)m$ then follow immediately. That ψ is continuous and open follows from the continuity of Φ_D^{-1} and the continuity and openness[6] of ψ on $\mathbf{A}_A$, the surjectivity of ψ follows similarly (and is also immediate from the relation $\psi(\varphi(m)) = m$), and the fact that ψ commutes with the Γ-action again follows by the same argument (or by the uniqueness). $\square$

Lemma 4.2.3. *Let $\mathfrak{M}$ be an $\mathbf{A}_A^+$-lattice in an étale (φ, Γ)-module D with A-coefficients. Then:*

(1) $\psi(\mathfrak{M})$ is an $\mathbf{A}_A^+$-module.
(2) If $\varphi(\mathfrak{M}) \subseteq \mathfrak{M}$ then $\mathfrak{M} \subseteq \psi(\mathfrak{M})$.
(3) If $\mathfrak{M} \subseteq \mathbf{A}_A^+ \cdot \varphi(\mathfrak{M})$, then $\psi(\mathfrak{M}) \subseteq \mathfrak{M}$.
(4) If $\psi(\mathfrak{M}) \subseteq \mathfrak{M}$, then $\psi(T^{-1}\mathfrak{M}) \subseteq T^{-1}\mathfrak{M}$, and for any $x \in D$ there exists $N \geq 0$ such that for all $n \geq N$, we have $\psi^n(x) \in T^{-1}\mathfrak{M}$.

Proof. In the case $A = \mathcal{O}/\varpi^a$ this is [Col10a, Lem. II.4.1], and the same proof works more generally, as we now briefly sketch. The first two parts follow from the identities $a\psi(x) = \psi(\varphi(a)x)$ and $\psi(a\varphi(x)) = \psi(a)x$ for $a \in \mathbf{A}_A$, $x \in D$. For the third part, note that $\psi(\sum_i a_i\varphi(x_i)) = \sum_i \psi(a_i)x_i$.

Finally for the fourth part, note firstly that

$$\psi(T^{-1}\mathfrak{M}) \subseteq \psi(\varphi(T)^{-1}\mathfrak{M}) \subseteq T^{-1}\psi(\mathfrak{M}) \subseteq T^{-1}\mathfrak{M}.$$

Furthermore, for each $k \geq 1$ we have

$$\psi(\varphi^k(T)^{-1}\mathfrak{M}) \subseteq \varphi^{k-1}(T)^{-1}\psi(\mathfrak{M}) \subseteq \varphi^{k-1}(T)^{-1}\mathfrak{M}.$$

For any $x \in D$ we have $x \in \varphi^k(T)^{-1}\mathfrak{M}$ for all k sufficiently large, and the result follows. $\square$

Lemma 4.2.4. *Let D be an étale (φ, Γ)-module with A-coefficients. Then there exists an $\mathbf{A}^+$-lattice $\mathfrak{M} \subset D$ with $\psi(\mathfrak{M}) = \mathfrak{M}$. Furthermore if $\mathfrak{M}' \subseteq D$ is another $\mathbf{A}^+$-lattice with $\psi(\mathfrak{M}') = \mathfrak{M}'$ then $T\mathfrak{M} \subseteq \mathfrak{M}' \subseteq T^{-1}\mathfrak{M}$.*

---

[6] The openness of ψ on $\mathbf{A}_A$ is clear from the fact that $\psi(\varphi(T^n)\mathbf{A}_A^+) = T^n\psi(\mathbf{A}_A^+) = T^n\mathbf{A}_A^+$ for any $n \geq 0$.

Proof. By, for example, [EG21, Lem. 5.2.7], we can choose $\mathbf{A}^+$-lattices $\mathfrak{N}_0, \mathfrak{N}_1 \subset D$ with $\varphi(\mathfrak{N}_0) \subseteq \mathfrak{N}_0 \subseteq \mathfrak{N}_1 \subseteq \mathbf{A}_A^+\varphi(\mathfrak{N}_1)$. For each $n \geq 0$, set $\mathfrak{M}_n = \psi^n(\mathfrak{N}_0)$; then by parts (1)-(3) of Lemma 4.2.3, $(\mathfrak{M}_n)_{n\geq 0}$ is an increasing sequence of $\mathbf{A}^+$-lattices, each of which contains $\mathfrak{N}_0$ and is contained in $\mathfrak{N}_1$. By Lemma 4.1.14 (3) $\mathfrak{M} := \cup_{n\geq 0}\mathfrak{M}_n$ is an $\mathbf{A}^+$-lattice in D, and by construction we have $\psi(\mathfrak{M}) = \mathfrak{M}$.

We now show that $\mathfrak{M}' \subseteq T^{-1}\mathfrak{M}$, by arguing as in the proof of Lemma 4.2.3 (4). Indeed since any two $\mathbf{A}^+$-lattices are commensurate (by Lemma 4.1.14 (3)), there exists some integer $n \geq 0$ such that $\mathfrak{M}' \subseteq \varphi^n(T)^{-1}\mathfrak{M}$, so that $\mathfrak{M}' = \psi^n(\mathfrak{M}') \subseteq T^{-1}\mathfrak{M}$.

Finally, the inclusion $T\mathfrak{M} \subseteq \mathfrak{M}'$ is equivalent to $\mathfrak{M} \subseteq T^{-1}\mathfrak{M}'$, and thus follows from the previous paragraph upon reversing the roles of $\mathfrak{M}, \mathfrak{M}'$. $\square$

Proposition 4.2.5. *Let D be an étale (φ,Γ)-module with A-coefficients.*

1. *D contains a maximal $\mathbf{A}^+$-lattice $D^\sharp \subset D$ with the property that $\psi(D^\sharp) = D^\sharp$. Furthermore, if $\mathfrak{N}$ is any bounded $\mathbf{A}^+$-submodule for which $\psi(\mathfrak{N}) = \mathfrak{N}$, then $\mathfrak{N} \subseteq D^\sharp$.*
2. *D contains a minimal ψ-stable $\mathbf{A}^+$-lattice $D^\natural$; furthermore $\psi(D^\natural) = D^\natural$.*
3. *We have $D^\natural \subseteq D^\sharp \subseteq T^{-1}D^\natural$.*

Proof. By Lemma 4.2.4, there exists an $\mathbf{A}^+$-lattice $\mathfrak{M} \subset D$ with $\psi(\mathfrak{M}) = \mathfrak{M}$. Let $D^\sharp$ denote the sum of all the $\mathbf{A}^+$-lattices $\mathfrak{M}' \subset D$ with $\psi(\mathfrak{M}') = \mathfrak{M}'$; then $\mathfrak{M} \subseteq D^\sharp \subseteq T^{-1}\mathfrak{M}$ by Lemma 4.2.4, so $D^\sharp$ is an $\mathbf{A}^+$-lattice by Lemma 4.1.14 (3). By construction $\psi(D^\sharp) = D^\sharp$, and (again by construction) $D^\sharp$ is maximal with this property. If $\mathfrak{N}$ is any bounded $\mathbf{A}^+$-submodule for which $\psi(\mathfrak{N}) = \mathfrak{N}$, then $D^\sharp + \mathfrak{N}$ is an $\mathbf{A}^+$-lattice satisfying $\psi(D^\sharp + \mathfrak{N}) = D^\sharp + \mathfrak{N}$. Thus $D^\sharp + \mathfrak{N} \subseteq D^\sharp$ by the property we've just proved, and so $\mathfrak{N} \subseteq D^\sharp$. This concludes the proof of part (1).

We claim that if $\mathfrak{M} \subseteq D^\sharp$ is an $\mathbf{A}^+$-lattice with $\psi(\mathfrak{M}) \subseteq \mathfrak{M}$, then in fact $\psi(\mathfrak{M}) = \mathfrak{M}$. Granting this, parts (2) and (3) follow easily. Indeed, we let $D^\natural$ be the intersection of all ψ-stable $\mathbf{A}^+$-lattices in D; then $D^\natural \subseteq D^\sharp$ (because $D^\sharp$ is a ψ-stable $\mathbf{A}^+$-lattice), so that (by the claim) $D^\natural$ is equal to the intersection of all the $\mathbf{A}^+$-lattices $\mathfrak{M} \subseteq D^\sharp$ with $\psi(\mathfrak{M}) = \mathfrak{M}$. It follows from Lemma 4.2.4 that for any such $\mathbf{A}^+$-lattice $\mathfrak{M}$ we have $TD^\sharp \subseteq \mathfrak{M}$, so we have $TD^\sharp \subseteq D^\natural \subseteq D^\sharp$ (which establishes part (3)), so $D^\natural$ is an $\mathbf{A}^+$-lattice by another application of Lemma 4.1.14 (3). Part (2) follows from another application of the claim, since $D^\natural$ is ψ-stable (by construction).

It remains to prove the claim. To this end, suppose that $\mathfrak{M} \subseteq D^\sharp$ and $\psi(\mathfrak{M}) \subseteq \mathfrak{M}$. Then we argue as in the proof of [Col10a, Prop. II.5.11]: since $\psi(D^\sharp) = D^\sharp$, the composite

$$D^\sharp/\mathfrak{M} \xrightarrow{\psi} D^\sharp/\psi(\mathfrak{M}) \to D^\sharp/\mathfrak{M}$$

is a surjective endomorphism of the finitely generated A-module $D^\sharp/\mathfrak{M}$, and is thus an isomorphism. In particular $D^\sharp/\psi(\mathfrak{M}) \to D^\sharp/\mathfrak{M}$ is injective, so $\psi(\mathfrak{M}) = \mathfrak{M}$, as required. $\square$

Lemma 4.2.6. *Let D be an étale (φ,Γ)-module with A-coefficients, and let $\mathfrak{M} \subset D$ be any $\mathbf{A}^+$-lattice. Then there exists an $N \geq 0$ such that for all $n \geq N$, we have $\psi^n(\mathfrak{M}) \subseteq T^{-1}D^\natural$.*

Proof. Since any two $\mathbf{A}^+$-lattices are commensurate (by Lemma 4.1.14 (3)), there exists some integer $N \geq 0$ such that $\mathfrak{M} \subset \varphi^N(T)^{-1}D^\natural$. Then for any $n \geq N$ we have $\psi^n(\mathfrak{M}) \subseteq \psi^{n-N}(T^{-1}D^\natural) \subseteq T^{-1}D^\natural$ (recalling that $T^{-1}D^\natural$ is ψ-stable, by Lemma 4.2.3). $\square$

We close this subsection by describing the behaviour of $D^\sharp$ and $D^\natural$ under morphisms of (φ,Γ)-modules, and under change of coefficients.

Lemma 4.2.7. *If $f : D_1 \to D_2$ is a morphism of étale (φ,Γ)-modules with A-coefficients, then f restricts to morphisms $f^\sharp : D_1^\sharp \to D_2^\sharp$ and $f^\natural : D_1^\natural \to D_2^\natural$. If f is furthermore injective (resp. surjective) then so are $f^\natural$ and $f^\sharp$ (resp. $f^\natural$).*

Proof. Once we know that $f^\sharp$ and $f^\natural$ exist, the injectivity statement is obvious, while the surjectivity statement for $f^\natural$ follows from the defining property of $D_2^\natural$, since $f(D_1^\natural)$ is a ψ-stable $\mathbf{A}^+$-lattice in D_2 if f is surjective.

We next note that $f(D_1^\sharp)$ is a bounded $\mathbf{A}^+$-submodule of D_2 on which ψ acts surjectively (since $D_1^\sharp$ is an $\mathbf{A}^+$-lattice in D_1 on which ψ acts surjectively), and so $f(D_1^\sharp) \subset D_2^\sharp$ by Proposition 4.2.5 (1), as claimed. Since $D_1^\natural \subseteq D_1^\sharp$, we then find that $f(D_1^\natural) \subseteq D_2^\sharp$. Let $\mathfrak{M} := f^{-1}(D_2^\natural) \cap D_1^\sharp$. Then $\mathfrak{M}$ is an $\mathbf{A}^+$-lattice, because it contains $TD_1^\sharp$ (note that $f(TD_1^\sharp) = Tf(D_1^\sharp) \subseteq TD_2^\sharp \subseteq D_2^\natural$ by Proposition 4.2.5 (3)). Furthermore $\mathfrak{M}$ is ψ-stable by definition, so it contains $D_1^\natural$, as required. $\square$

Lemma 4.2.8. *Suppose that $A \to B$ is a morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras, and that D_A is an étale (φ,Γ)-module with A-coefficients. Let $D_B := D_A \widehat{\otimes}_A B$. The following then hold:*

1. *(1) $D_B^\sharp$ contains the image of the natural map $D_A^\sharp \widehat{\otimes}_A B \to D_B$.*
2. *(2) $D_B^\natural$ is equal to the image of the natural map $D_A^\natural \widehat{\otimes}_A B \to D_B$.*
3. *(3) If $A \to B$ is surjective, then there is a canonical surjection $D_A^\natural \to D_B^\natural$.*
4. *(4) If $A \to B$ is flat, then there is a canonical isomorphism $D_A^\natural \widehat{\otimes}_A B \xrightarrow{\sim} D_B^\natural$.*

Proof. Since $D_A^\sharp$ is a ψ-stable $\mathbf{A}_A^+$-lattice satisfying $\psi(D_A^\sharp) = D_A^\sharp$, we see that the image $\mathfrak{N}$ of the map $D_A^\sharp \widehat{\otimes}_A B \to D_B$ is a ψ-stable $\mathbf{A}_B^+$-lattice for which $\psi(\mathfrak{N})$ is dense in $\mathfrak{N}$. Since ψ is open, we see that in fact $\psi(\mathfrak{N}) = \mathfrak{N}$, and so $\mathfrak{N} \subseteq D_B^\sharp$, proving (1).

Now write $\mathfrak{N}'$ for the image of the map $D_A^\natural \widehat{\otimes}_A B \to D_B$; to prove (2), we must show that $\mathfrak{N}'$ is equal to $D_B^\natural$. By Lemma 4.1.14 (4), $\mathfrak{N}'$ is a ψ-stable $\mathbf{A}_B^+$-lattice in D_B, and therefore contains $D_B^\natural$. To see that $\mathfrak{N}'$ is contained in $D_B^\natural$, we argue as in the proof of Lemma 4.2.7. Namely, let $\mathfrak{M}$ be the preimage of $D_B^\natural$ in $D_A^\natural$ under the natural morphism $D_A^\natural \subseteq D_A \to D_B$; we must show that $\mathfrak{M} = D_A^\natural$.

To see this, note that $\mathfrak{M}$ is an $\mathbf{A}_A^+$-lattice, because it contains $TD_A^\sharp$, by Proposition 4.2.5 (3) together with part (1) of the present lemma. Since $\mathfrak{M}$ is ψ-stable, we must then have $\mathfrak{M} = D_A^\natural$, as required. This concludes the proof of part (2).

As discussed prior to the statement of Lemma 4.1.14, the completed tensor product of (2) can be reinterpreted as the usual tensor product $-\otimes_{\mathbf{A}_A^+} \mathbf{A}_B^+$. Since $\mathbf{A}_A^+ \to \mathbf{A}_B^+$ is surjective when $A \to B$ is, part (3) then follows.

Finally, part (4) follows from part (2) and Lemma 4.1.14 (4). $\square$

Remark 4.2.9. Part (4) of Lemma 4.2.8 shows that the formation of $D^\natural$ satisfies flat base-change in A. Unlike $D^\natural$, $D^\sharp$ need not satisfy flat base-change in general.

4.3. **D^+, D^{++}, and D^{nr}.** We continue to assume that A is a Noetherian $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and write D to denote an étale (φ,Γ)-module with coefficients in $\mathbf{A}_A$. As in the preceding subsection, we will prove various analogues with coefficients of some results from [Col10c; Col10a]. Just as was noted in Remark 4.2.1

regarding the results of the preceding subsection, the Γ-action continues to play no role in the discussion of this subsection.

Definition 4.3.1. Let D^+ be the A-submodule of D consisting of those $z \in D$ for which the sequence $\{\varphi^n(z)\}_{n\geq 0}$ is bounded. Let D^{++} be the A-submodule of D consisting of those $z \in D$ for which the sequence $\{\varphi^n(z)\}_{n\geq 0}$ tends to zero (i.e. the submodule consisting of those z on which φ acts topologically nilpotently).

Lemma 4.3.2. *$TD^+ \subseteq D^{++} \subseteq D^+$.*

Proof. The second inclusion is obvious from the definitions, while the first follows from the fact that if (a_n) is a bounded sequence, then $(\varphi^n(T)a_n)$ is a sequence converging to 0. $\square$

Lemma 4.3.3. *$D^+ \subseteq D^\natural$.*

Proof. We follow the proof of [Col10a, Prop. II.5.14]. Fix some $z \in D^+$. Then also $\varphi(z) \in D^+$, and so Lemma 4.3.2 shows that $T\varphi(z) \in D^{++}$; i.e. that the sequence $\{\varphi^n(T\varphi(z))\}_{n\geq 0}$ tends to zero. Since $D^\natural$ is a lattice, we thus find that $\varphi^n(T\varphi(z)) \in D^\natural$ for some sufficiently large value of n. Since $D^\natural$ is ψ-stable, and since $\psi^{n+1}(\varphi^n(T\varphi(z))) = \psi(T\varphi(z)) = \psi(T)z = -z$, we thus have $z \in D^\natural$, as required. $\square$

The following corollary provides a useful characterization of D^+.

Corollary 4.3.4. *D^+ is an $\mathbf{A}_A^+$-lattice in D, and is the (unique) maximal φ-invariant bounded subset of D.*

Proof. We saw in Lemma 4.3.2 that $TD^+ \subseteq D^+$. Thus, if we show that D^+ is a lattice in D, it is in fact an $\mathbf{A}_A^+$-lattice.

By definition any φ-bounded subset of D is contained in D^+. In particular, if $\mathfrak{M}$ is any φ-stable lattice in D, then $\mathfrak{M} \subseteq D^+$. Such a lattice exists, by [EG21, Lem. 5.2.7], and thus D^+ contains a lattice, and so is open. By Remark 4.1.3, this proves the lemma, provided that we show that D^+ is also bounded; but this follows from Lemma 4.3.3. $\square$

Our goal in what follows is to show that the formation of each of D^{++} and D^+ is compatible with flat base change. In the case of D^{++} this is relatively straightforward, but the case of D^+ is more involved.

A general observation is that, when analyzing constructs such as D^+ and D^{++} (and the related construct D^{nr} of Definition 4.3.13 below) whose definitions involve dynamical aspects of the action of φ, complications can sometimes arise from the existence of a descending chain of ideals in A that do not stabilize. The following lemma (which is presumably standard) sometimes allows us to get around this problem, by reducing our analysis to the case of an Artinian coefficient ring, for which this complication cannot occur.

Lemma 4.3.5. *If A is a Noetherian ring, then there exists an embedding $A \hookrightarrow B$ with B Artinian.*

Proof. First recall that if f is any element of A, then for n large enough, the natural map $A \to A_f \times A/f^n$ is an embedding. Now, if A is non-zero, and if $\mathfrak{p}$ is a minimal prime of A, then we may choose an element $f \in A$ such that the distinguished open $D(f)$ is a neighbourhood of $\mathfrak{p}$ in $\operatorname{Spec} A$ which contains no other associated primes of A. Thus $\mathfrak{p}$ is the unique associated prime of A_f, and so the natural map $A_f \to A_{\mathfrak{p}}$ is an embedding. Thus, for n large enough, the natural map $A \to A_{\mathfrak{p}} \times A/f^n$ is an embedding. Since $\mathfrak{p}$ was chosen to be minimal, the localization $A_{\mathfrak{p}}$ is Artinian.

We continue by applying the same argument to A/f^n. By Noetherian induction (in $\operatorname{Spec} A$) the process eventually terminates, and we obtain an embedding of A into a finite product of Artinian local rings, as required. $\square$

We now return to the setting of étale (φ, Γ)-modules over Noetherian $\mathcal{O}/\varpi^a$-algebras.

Lemma 4.3.6. *The induced action of φ on each of D/D^+ and D/D^{++} is injective.*

Proof. This follows from the fact that each of D^+ and D^{++} is φ-saturated in D (i.e. if $\varphi(x)$ is in D^+, resp. D^{++}, then so is x itself), as is evident from their definitions. $\square$

The preceding lemma admits a kind of converse, characterizing D^{++}.

Lemma 4.3.7. *D^{++} is an $\mathbf{A}_A^+$-lattice in D, and is furthermore the* (*unique*) *minimal φ-invariant lattice $\mathfrak{M}$ of D for which φ acts injectively on $D/\mathfrak{M}$.*

Proof. It follows from Lemma 4.3.2 and Corollary 4.3.4 that D^{++} is a T-invariant lattice, and thus is an $\mathbf{A}_A^+$-lattice in D. The injectivity of φ on D/D^{++} was proved in Lemma 4.3.6. If $\mathfrak{M}$ is any φ-invariant lattice in D, and if $x \in D^{++}$, then by definition we see that $\varphi^n(x) \in \mathfrak{M}$ for some value of n. Thus, if φ is injective on $D/\mathfrak{M}$, then we see that $x \in \mathfrak{M}$, and hence that $D^{++} \subseteq \mathfrak{M}$. $\square$

We also note the following technical property of D^{++}.

Lemma 4.3.8. *The action of φ on D^{++} is* uniformly *topologically nilpotent, i.e. given any lattice $\mathfrak{M} \subset D$, there is some n such that $\varphi^n(D^{++}) \subset \mathfrak{M}$.*

Proof. Any lattice $\mathfrak{M}$ contains an $\mathbf{A}_A^+$-invariant sublattice, and replacing $\mathfrak{M}$ by such a sublattice if necessary, we may assume that $\mathfrak{M}$ is itself an $\mathbf{A}_A^+$-lattice. Since D^{++} is a lattice, by Lemma 4.3.7, it is finitely generated over $\mathbf{A}_A^+$, say by elements $x_1, \ldots, x_m$. By definition, we may find some n such that $\varphi^n(x_i) \in \mathfrak{M}$ for each i. Then

$$\varphi^n(D^{++}) = \varphi^n(\mathbf{A}_A^+\langle x_1, \ldots, x_m\rangle) \subseteq \mathbf{A}_A^+\langle \varphi^n(x_1), \ldots, \varphi^n(x_m)\rangle \subseteq \mathfrak{M},$$

as required. $\square$

It is now easy to establish flat base-change for D^{++}.

Lemma 4.3.9. *If $A \to B$ is a flat morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras, if D_A is an étale (φ, Γ)-module over $\mathbf{A}_A$, and if we write $D_B := D_A \widehat{\otimes}_A B$, then the resulting morphism $D_A^{++} \widehat{\otimes}_A B \to D_B$ induces an isomorphism $D_A^{++} \widehat{\otimes}_A B \xrightarrow{\sim} D_B^{++}$.*

Proof. Since $A \to B$ is flat, and since D_A^{++} is a lattice in D_A by Lemma 4.3.7, it follows from Lemma 4.1.14 (4) that $D_A^{++} \widehat{\otimes}_A B \to D_B$ identifies its source with a lattice in D_B. Using the uniform topological nilpotency of Lemma 4.3.8, we see that φ acts topologically nilpotently on the completed tensor product $D_A^{++} \widehat{\otimes}_A B$, and so this lattice is contained in D_B^{++}.

Furthermore, since B is flat over A, and since φ is injective on D_A/D_A^{++} by Lemma 4.3.6, we see that φ is injective on $(D_A/D_A^{++}) \otimes_A B = D_B/(D_A^{++} \widehat{\otimes}_A B)$.

(The indicated identification is provided by Lemma 4.1.14 (4).) Lemma 4.3.7 now shows the reverse inclusion, namely that $D_B^{++} \subseteq D_A^{++} \widehat{\otimes}_A B$. This proves the lemma. $\square$

Our next goal is to prove Lemma 4.3.12, which shows that D^+ is compatible with flat base-change for D^+.

Lemma 4.3.10. *Let $A \hookrightarrow B$ is an embedding of Noetherian $\mathcal{O}/\varpi^a$-algebras, and D_A is an étale (φ, Γ)-module with A coefficients, so that the embedding of A into B induces an embedding $D_A \hookrightarrow D_B := D_A \widehat{\otimes}_A B$, then we have $D_A^+ = D_A \cap D_B^+$. Equivalently, the natural map $D_A/D_A^+ \to D_B/D_B^+$ is injective.*

Proof. Since D_A^+ is a bounded φ-invariant subset of D_B, it is contained in D_B^+, and thus in the indicated intersection. On the other hand, this intersection is a bounded φ-invariant subset of D_A (using Lemma 4.1.14 (5) to see the boundedness), and thus is contained in D_A^+, by Corollary 4.3.4. $\square$

Lemma 4.3.11. *Let D be an étale (φ, Γ)-module over the Noetherian $\mathcal{O}/\varpi^a$-algebra A. Then for all sufficiently large n, we have $\varphi^{-n}\big((D/D^+)[T]\big) \cap \big((D/D^+)[T]\big) = 0$.*

Proof. Let M_n denote the kernel of the endomorphism $T\varphi^n$ of D/D^+; we must show that $M_n[T] = 0$ if n is sufficiently large. Applying Lemmas 4.3.5 and 4.3.10, we see that we may replace A by an Artinian overring, and thus we assume that A is Artinian for the remainder of the proof.

Since φ is injective on D/D^+, by Lemma 4.3.6, we find that

$$M_n := \ker T\varphi^n = \ker \varphi(T\varphi^n) = \ker \varphi(T)\varphi^{n+1} \supseteq \ker T\varphi^{n+1} =: M_{n+1}.$$

Thus $M_n[T]$ is a descending sequence of A-submodules of the finite type A-module $(D/D^+)[T]$, which therefore stabilizes. Suppose that the sequence stabilizes at n_0, and write $N := M_{n_0}[T] = M_n[T]$ for $n \geq n_0$. Then we see (by definition of the M_n) that $\varphi^n(N) \subset (D/D^+)[T]$ for $n \geq n_0$. Thus $\bigcup_n \varphi^n(N)$ is a bounded subset of D/D^+ (i.e. it is contained in a finite type A-module) and so $N = 0$ (by the definition of D^+). This is what we had to prove. $\square$

Lemma 4.3.12. *If $A \to B$ is a flat morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras, if D_A is an étale (φ, Γ)-module over $\mathbf{A}_A$, and if we write $D_B := D_A \widehat{\otimes}_A B$, then the resulting morphism $D_A^+ \widehat{\otimes}_A B \to D_B$ induces an isomorphism $D_A^+ \widehat{\otimes}_A B \xrightarrow{\sim} D_B^+$.*

Proof. Since $A \to B$ is flat, and since D_A^+ is a lattice in D_A by Corollary 4.3.4, it follows from Lemma 4.1.14 (4) that $D_A^+ \widehat{\otimes}_A B \to D_B$ identifies its source with a lattice in D_B. This lattice is furthermore φ-stable, and thus is contained in D_B^+ (by another application of Corollary 4.3.4). Our goal is to show that in fact $D_A^+ \widehat{\otimes}_A B = D_B^+$, or equivalently that $D_B^+/(D_A^+ \widehat{\otimes}_A B) = 0$. Since every element of $D_B/(D_A^+ \widehat{\otimes}_A B)$ is T-power torsion, it in fact suffices to show that $\big(D_B^+/(D_A^+ \widehat{\otimes}_A B)\big)[T] = 0$, and this is what we will do.

Lemma 4.3.11 shows that the morphism $(D_A/D_A^+)[T] \to D_A/D_A^+$ induced by $T\varphi^n$ is injective if n is sufficiently large. Base-changing over the flat A-algebra B, we find that the morphism $\big(D_B/(D_A^+ \widehat{\otimes}_A B)\big)[T] \to D_B/(D_A^+ \widehat{\otimes}_A B)$ induced by $T\varphi^n$ is again injective if n is sufficiently large. (Here and below we use Lemma 4.1.14 (4) to make the evident manipulations of various tensor products and completed tensor products with B.) Composing with φ^r for any r, and recalling Lemma 4.3.6, we that the same is true of the morphism $\big(D_B/(D_A^+\widehat{\otimes}_A B)\big)[T] \to D_B/(D_A^+\widehat{\otimes}_A B)$ induced by $\varphi^r(T)\varphi^n$, if n is sufficiently large (depending on r). Thus no non-zero element of $\big(D_B/(D_A^+\widehat{\otimes}_A B)\big)[T]$ has a bounded φ-orbit; in fact, since $D_A^+\widehat{\otimes}_A B$ is a lattice in D_B, the bounded subsets of $D_B^+/D_A^+\widehat{\otimes}_A B$ are precisely those contained in some B-module of finite type. So we see that indeed $\big(D_B^+/(D_A^+\widehat{\otimes}_A B)\big)[T] = 0$, as required. □

Finally, we introduce and study some basic properties of D^{nr} in our context of (φ,Γ)-modules with coefficients.

Definition 4.3.13. If D is an étale (φ,Γ)-module with coefficients in $\mathbf{A}_A$, for some Noetherian $\mathcal{O}/\varpi^a$-algebra A, then we write $D^{\mathrm{nr}} := \bigcap_n \varphi^n(D)$, equipped with its actions of φ and Γ.

Remark 4.3.14. In the case that $A = \mathcal{O}/\varpi^a$, the A-module D^{nr} is related to the maximal abelian subrepresentation of the corresponding Galois representation (see [Col10a, Rem. II.1.2], and Lemma 4.7.7 below). Since this notion does not behave well in the context of general families of étale (φ,Γ)-modules (e.g. because of the fundamental phenomenon of generically reducible families specializing to irreducible objects), it should not come as a surprise that D^{nr} does not behave well in families; for example, it is not compatible with flat base-change in general.

Lemma 4.3.15. *D^{nr} is a finite type A-submodule of D^+.*

Proof. Clearly D^{nr} is φ-stable, while since φ is injective on D we see that D^{nr} is also φ-saturated. Since $D^{\mathrm{nr}} \subseteq \varphi(D)$ by definition, we then see that $\varphi(D^{\mathrm{nr}}) = D^{\mathrm{nr}}$.

Since D^{++} is also both φ-stable and φ-saturated, we see that $D^{\mathrm{nr}} \cap D^{++}$ is again φ-stable and φ-saturated. It then follows that also $\varphi(D^{\mathrm{nr}} \cap D^{++}) = D^{\mathrm{nr}} \cap D^{++}$. The uniform topological nilpotency of Lemma 4.3.8 then shows that $D^{\mathrm{nr}} \cap D^{++} = 0$.

Now consider the image M of D^{nr} in D/D^+; evidently $\varphi(M) = M$. Suppose that $m \in M[T]$, and let $m' \in M$ be such that $\varphi(m') = m$. Then $\varphi(Tm') = \varphi(T)\varphi(m') = \varphi(T)m = 0$, and thus $Tm' = 0$ (by Lemma 4.3.6). Consequently we see that $\varphi(M[T]) \supseteq M[T]$, and hence that $\varphi^n(M[T]) \supseteq M[T]$ for all $n \geq 0$. Lemma 4.3.11 then implies that $M[T] = 0$, and so we see that in fact $M = 0$. Thus $D^{\mathrm{nr}} \subseteq D^+$. (In more detail: if $x \in M[T]$, then the inclusion $\varphi^n(M[T]) \supseteq M[T]$ shows that there exists $y \in M[T]$ such that $\varphi^n(y) = x$. Hence $y \in \varphi^{-n}\left((D/D^+)[T]\right) \cap (D/D^+)[T]$, which by Lemma 4.3.11 is zero for n large enough, and so $x = \varphi^n(y)$ is also equal to 0.)

Putting what we've shown together, we see that D^{nr} embeds into D^+/D^{++}, which is a finite type A-module by Lemma 4.3.2 together with Corollary 4.3.4, and hence D^{nr} is itself a finite type A-module. □

Corollary 4.3.16. *We have $D^{\mathrm{nr}} \subset D^\natural$.*

Proof. This is immediate from Lemmas 4.3.15 and 4.3.3. □

4.4. **From (φ,Γ)-modules to equivariant sheaves.** We now explain how some of the constructions of [Col10a, §III.1, §V] extend to our setting. We apply some results from [SVZ14, §3], in which some of the more formal aspects of Colmez's constructions are presented in a natural level of generality. With these general results in hand, the extension of Colmez's constructions to the case of étale (φ,Γ)-modules with coefficients is for the most part immediate, and we frequently refer to [Col10a] for the proofs of results when they are literally identical in our setting.

We write P^+ for the monoid $\left(\begin{smallmatrix} \mathbf{Z}_p\setminus\{0\} & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$, and P for the group $\left(\begin{smallmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & 1 \end{smallmatrix}\right)$. Then there is a natural action of P^+ on $\mathbf{Z}_p$ (respectively of P on $\mathbf{Q}_p$) via $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)x = ax+b$. The basic motivation for Colmez's constructions is as follows: we can regard an étale (φ,Γ)-module D as being the global sections of a P^+-equivariant sheaf of A-modules on $\mathbf{Z}_p$. We are then able to extend its P^+-equivariance to an equivariance under the category of all non-degenerate[7] piecewise affine-linear maps between open subsets of $\mathbf{Z}_p$, and then (by an appropriate limiting process) to the category of all *local diffeomorphisms* (in the sense of Definition 4.4.8 below) between open subsets of $\mathbf{Z}_p$.

With these constructions in place, it becomes possible to localize D in an equivariant manner over other locally analytic $\mathbf{Z}_p$-manifolds, such as P-equivariantly over $\mathbf{Q}_p$, and (our ultimate goal) $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariantly over $\mathbf{P}^1(\mathbf{Q}_p)$. In fact, the exposition becomes simpler if we construct the localization over $\mathbf{Q}_p$ en route to the construction of the equivariant structure under local diffeomorphisms, and so we do this. We discuss localization over $\mathbf{P}^1(\mathbf{Q}_p)$ in the next subsection.

Suppose now that M is any P^+-module with coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A. Write $U^+ := \left(\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$. Let φ denote the endomorphism

$$\varphi : \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & px \\ 0 & 1 \end{smallmatrix}\right)$$

of U^+, as well as the endomorphism of M induced by the action of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then if $u \in U^+$ and $m \in M$, we find that $\varphi(um) = \varphi(u)\varphi(m)$. Consequently for each $n \geq 1$ we see that $\varphi^n(M)$ is $\varphi^n(U^+)$-stable, and so we obtain an induced morphism

$$A[U^+] \otimes_{A[\varphi^n(U^+)]} \varphi^n(M) \to M. \tag{4.4.1}$$

Definition 4.4.2. We say that a P^+-module M with coefficients in A is *étale* if the morphisms (4.4.1) are isomorphisms for all $n \geq 1$.

Our interest in étale P^+-modules comes from the following lemma.

Lemma 4.4.3. *If D is an étale (φ,Γ)-module over a Noetherian $\mathcal{O}/\varpi^a$-algebra A, then D is an étale P^+-module with A-coefficients, where the action of P^+ on D is via the continuous maps*

$$\begin{pmatrix} p^k a & b \\ 0 & 1 \end{pmatrix} z = (1+T)^b \varphi^k \circ \sigma_a(z)$$

for $a \in \mathbf{Z}_p^\times$, $b \in \mathbf{Z}_p$, $k \in \mathbf{Z}_{\geq 0}$, and $z \in D$. (Recall that $\sigma_a \in \Gamma$ is the element with $\chi(\sigma_a) = a$, where χ is the cyclotomic character.)

Proof. This is immediate from the definitions. Indeed by definition, the condition that D is étale as a (φ,Γ)-module is equivalent to the morphisms (4.4.1) being isomorphisms. □

By [SVZ14, Thm. 3.32], there is an equivalence of abelian categories between the category of étale P^+-modules with A-coefficients, and the category of P-equivariant sheaves of A-modules on $\mathbf{Q}_p$; the inverse functor is given by passage to the module of sections over $\mathbf{Z}_p$. Following Colmez, for any open set $U \subset \mathbf{Q}_p$, we write $D \boxtimes U$ for the sections over U of the sheaf corresponding to D, so that in particular we have $D \boxtimes \mathbf{Z}_p = D$. As is the case for any P-equivariant sheaf of A-modules on $\mathbf{Q}_p$,

[7] In the sense of having non-zero derivative at every point.

there is a P-equivariant action on $D \boxtimes \mathbf{Q}_p$ of $\mathcal{C}(\mathbf{Q}_p, A)$, the ring of A-valued locally constant functions on $\mathbf{Q}_p$, and thus there is in particular a P-equivariant action of the ring $\mathcal{C}_c(\mathbf{Q}_p, A)$ of compactly supported A-valued locally constant functions. Similarly for each open $U \subseteq \mathbf{Q}_p$ there is a natural action of $\mathcal{C}_c(U, A)$ on $D \boxtimes U$.

If $U \subseteq \mathbf{Q}_p$ is open then we write $\mathrm{Res}_U : D \boxtimes \mathbf{Q}_p \to D \boxtimes U$ for the restriction map. If $U \subseteq V \subseteq \mathbf{Q}_p$ are open subsets then we have the restriction map $\mathrm{Res}_U^V : D \boxtimes V \to D \boxtimes U$, which we will usually denote simply by Res_U. If U is compact then the restriction map $\mathrm{Res}_U : D \boxtimes \mathbf{Q}_p \to D \boxtimes U$ admits a section given by extension by zero, and we accordingly identify the action of the locally constant function $1_U \in \mathcal{C}_c(\mathbf{Q}_p, A)$ with Res_U.

By [SVZ14, Prop. 3.26] (and its proof), the global sections $D \boxtimes \mathbf{Q}_p$ are given explicitly by

$$D \boxtimes \mathbf{Q}_p = \varprojlim_{\psi} D, \tag{4.4.4}$$

where the right hand side denotes the inverse limit of the inverse system

$$\cdots \xrightarrow{\psi} D \xrightarrow{\psi} D \xrightarrow{\psi} \cdots;$$

equivalently,

$$D \boxtimes \mathbf{Q}_p \coloneqq \{(z_n)_{n \geq N} \mid \psi(z_{n+1}) = z_n \text{ for all } n\}.$$

(Here N can be any element of $\mathbf{Z} \cup \{-\infty\}$; we take advantage of this flexibility in some of the constructions introduced below.) We can therefore endow $D \boxtimes \mathbf{Q}_p$ with the projective limit topology (where D has its canonical Tate module topology). Note that if U is compact and open, then $D \boxtimes U$ also has a canonical Tate module topology; indeed $D \boxtimes U$ is a direct summand of some $D \boxtimes \frac{1}{p^n}\mathbf{Z}_p$, which is homeomorphic to $D \boxtimes \mathbf{Z}_p = D$.

Furthermore, the action of P on $D \boxtimes \mathbf{Q}_p$ is determined by the following properties.

(1) The restriction map $\mathrm{Res}_{\mathbf{Z}_p} : D \boxtimes \mathbf{Q}_p \to D = D \boxtimes \mathbf{Z}_p$ is given by
$$(z_n)_{n \in \mathbf{Z}} \mapsto z_0 \in D = D \boxtimes \mathbf{Z}_p.$$
(2) The extension by zero map $D = D \boxtimes \mathbf{Z}_p \to D \boxtimes \mathbf{Q}_p$ is given by
$$z \mapsto \Big\{ \big(\varphi^n(z)\big)_{n \geq 0} \Big\};$$
this is continuous and P^+-equivariant.
(3) $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts via $(z_n)_{n \in \mathbf{Z}} \mapsto (z_{n+1})_{n \in \mathbf{Z}}$.
(4) the elements $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, for $b \in p^{-N}\mathbf{Z}_p$, act via the transformation
$$(z_n)_{n \geq N} \mapsto \big((1+T)^{p^n b} z_n\big)_{n \geq N}.$$

It follows from the definitions that this action of P is continuous.

Definition 4.4.5. If $\alpha \in \mathcal{C}_c(U, A)$, we write m_α for the corresponding continuous endomorphism of $D \boxtimes U$.

Lemma 4.4.6.

(1) *For any $n \geq 0$ the restriction map $\mathrm{Res}_{p^n\mathbf{Z}_p} : D \boxtimes \mathbf{Z}_p \to D \boxtimes p^n\mathbf{Z}_p$ is equal to $\varphi^n \circ \psi^n$.*

(2) Let $\alpha : U \to A$ be locally constant, and let $n \geq 0$ be large enough that for all $a \in U$, we have $a + p^n\mathbf{Z}_p \subseteq U$, and $\alpha|_{a+p^n\mathbf{Z}_p}$ is constant. Let $I_n(U)$ for a system of coset representatives for U modulo $p^n\mathbf{Z}_p$. Then

$$m_\alpha = \sum_{i \in I_n(U)} \alpha(i) \operatorname{Res}_{i+p^n\mathbf{Z}_p}.$$

Proof. The first part follows from the identity $\operatorname{Res}_{p^n\mathbf{Z}_p} = \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) \circ 1_{\mathbf{Z}_p} \circ \left(\begin{smallmatrix} p^{-n} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and the second is immediate from the identity $\alpha = \sum_{i \in I_n(U)} \alpha(i) 1_{i+p^n\mathbf{Z}_p}$ $\square$

Example 4.4.7. For future reference, note that $D \boxtimes \mathbf{Z}_p^\times = D^{\psi=0}$, with the identification being induced via the embedding $D \boxtimes \mathbf{Z}_p^\times \hookrightarrow D \boxtimes \mathbf{Z}_p \coloneqq D$ given by extension by zero. Indeed, the sections of $D = D \boxtimes \mathbf{Z}_p$ which are supported on $\mathbf{Z}_p^\times$ are precisely those in the kernel of $\operatorname{Res}_{p\mathbf{Z}_p} = \varphi \circ \psi$, and φ is injective.

Our next goal is to extend the P-equivariant localization of D over $\mathbf{Q}_p$ to an equivariant structure under the category of local diffeomorphisms. As already noted, this will then allow us to localize D over more general one-dimensional p-adic manifolds, such as $\mathbf{P}^1(\mathbf{Q}_p)$.

Definition 4.4.8. If U, V are compact open subsets of $\mathbf{Q}_p$, then we say that a map $f : U \to V$ is a local diffeomorphism if it is $\mathcal{C}^1$ (in the sense of [Col10b, Section I.5.1]), and if its derivative is non-vanishing on U.

Definition 4.4.9. For all $n \geq 0$, let $X_n \subset D$ be a subset. We say that the sequence X_n tends uniformly to 0 as $n \to \infty$ if for any A-lattice $\mathfrak{M} \subset D$ there exists $N > 0$ such that $X_n \subset \mathfrak{M}$ for all $n > N$. (This is a slight rephrasing of the definition in [Col10a, Section V.1.2].)

Fix $m > 0$, and for all $n \geq 0$, let $X_n \subset D \boxtimes \frac{1}{p^m}\mathbf{Z}_p$ be a subset. We say that the sequence X_n tends uniformly to 0 as $n \to \infty$ if its image under $\varphi^m : D \boxtimes \frac{1}{p^m}\mathbf{Z}_p \xrightarrow{\sim} D$ tends uniformly to 0.

Proposition 4.4.10. *Suppose that $f : U \to V$ is a local diffeomorphism. Then there is a continuous A-linear morphism $f_* : D \boxtimes U \to D \boxtimes V$, defined as follows: For each sufficiently large n, write $I_n(U)$ for a system of coset representatives for U modulo $p^n\mathbf{Z}_p$, and then for each $z \in D \boxtimes U$, define*

$$f_*(z) \coloneqq \lim_{n \to \infty} \sum_{i \in I_n(U)} \begin{pmatrix} f'(i) & f(i) \\ 0 & 1 \end{pmatrix} \operatorname{Res}_{p^n\mathbf{Z}_p} \left(\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} z \right);$$

this limit exists, and is independent of the choices of $I_n(U)$.

Furthermore,

$$\left\{ \operatorname{Res}_{j+p^n\mathbf{Z}_p} \left(f_*(z) - \sum_{i \in I_n(U)} \begin{pmatrix} f'(i) & f(i) \\ 0 & 1 \end{pmatrix} \operatorname{Res}_{p^n\mathbf{Z}_p} \left(\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} z \right) \right) : j \in V \right\}$$

tends uniformly to zero as $n \to \infty$.

Example 4.4.11. Let $f : \mathbf{Z}_p \to p\mathbf{Z}_p$ be multiplication by p. Then

$$f_*(z) = \lim_{n \to \infty} \sum_{i \in I_n(U)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \operatorname{Res}_{i+p^n\mathbf{Z}_p}(z) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z = \varphi(z)$$

and so $f_* = \varphi : D \to D \boxtimes p\mathbf{Z}_p = \varphi(D)$.

Similarly, let $f : p\mathbf{Z}_p \longrightarrow \mathbf{Z}_p$ be multiplication by p^{-1}. Then

$$f_* : \varphi(D) =: D \boxtimes p\mathbf{Z}_p \to D \boxtimes \mathbf{Z}_p := D$$

is given by $\varphi^{-1} : \varphi(D) \to D$. Thus the composite

$$D =: D \boxtimes \mathbf{Z}_p \xrightarrow{\mathrm{Res}_{p\mathbf{Z}_p}} D \boxtimes p\mathbf{Z}_p \xrightarrow{f_*} D \boxtimes \mathbf{Z}_p := D$$

is given by $\varphi^{-1} \circ \varphi\psi$, i.e. by the operator ψ.

Proof of Proposition 4.4.10. Note firstly that if the limit exists, it is necessarily independent of the choices of $I_n(U)$. Indeed if $I_n(U)$, $I'_n(U)$ are two such choices, this follows by considering the limit for both choices together with the third choice $I''_n(U)$ defined by $I''_n(U) = I_n(U)$ if n is even and $I'_n(U)$ if n is odd.

Exactly as in the proof of [Col10a, Prop. V.1.3], we can reduce the statement of the proposition to the case that $U = V = \mathbf{Z}_p$, and that f is *regular* in the sense that $v_p(f'(x)) = 0$ for all $x \in \mathbf{Z}_p$. If A is a finite $\mathcal{O}/\varpi^a$-algebra, this case is [Col10a, Lem. V.1.2], and essentially the same proof works in our setting, as we now explain. Write

$$u_n = \sum_{i \in I_n(U)} \begin{pmatrix} f'(i) & f(i) \\ 0 & 1 \end{pmatrix} \mathrm{Res}_{p^n\mathbf{Z}_p}\left(\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} z\right).$$

We will show next that for any $\mathbf{A}^+$-lattice $\mathfrak{M}$ containing z, $u_n - u_{n-1}$ tends to 0 uniformly in $z \in \mathfrak{M}$; it then follows that $f_*(z)$ exists, and that f_* is A-linear and continuous.

For $i \in \mathbf{Z}_p$, write $r_{n,i}(z) = \psi^n(\left(\begin{smallmatrix} 1 & -i \\ 0 & 1 \end{smallmatrix}\right)z)$. The argument of the second paragraph of the proof of [Col10a, Lem. V.1.2] goes through unchanged, and shows that we can write

$$u_n - u_{n-1} = \sum_{j \in I_n(\mathbf{Z}_p)} \begin{pmatrix} 1 & f(j) \\ 0 & 1 \end{pmatrix} \varphi^n((g_j - h_{n,j}) \cdot r_{n,j}(z)), \tag{4.4.12}$$

where $g_j, h_{n,j} \in P^+$ (and the difference $g_j - h_{n,j}$ is as elements of the group ring, rather than as elements of P^+). Furthermore, $g_j, h_{n,j}$ enjoy the following properties: we have $g_j \in \left(\begin{smallmatrix} \mathbf{Z}_p^\times & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $g_j^{-1}h_{n,j} \in \left(\begin{smallmatrix} 1+p^{a(n)}\mathbf{Z}_p & p^{a(n)}\mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ for some integer $a(n)$, and $a(n) \to \infty$ as $n \to \infty$.

Let $\mathfrak{M}$ be an $\mathbf{A}^+$-lattice containing z. Then there is a Γ-stable $\mathbf{A}^+$-lattice $\mathfrak{M}' \supseteq \mathfrak{M}$ such that $r_{n,j}(\mathfrak{M}) \subseteq \mathfrak{M}'$ for all $j \in \mathbf{Z}_p$, $n \geq 0$. Indeed $\left(\begin{smallmatrix} 1 & -j \\ 0 & 1 \end{smallmatrix}\right)\mathfrak{M} = (1+T)^j\mathfrak{M} = \mathfrak{M}$ for all $j \in \mathbf{Z}_p$, and by Lemma 4.2.6, there is an $\mathbf{A}^+$-lattice $\mathfrak{M}''$ containing all the $\psi^n(\mathfrak{M})$. Then we can take $\mathfrak{M}'$ to be the $\mathbf{A}^+$-submodule $\Gamma\mathfrak{M}''$ of D generated by γm for $\gamma \in \Gamma, m \in \mathfrak{M}''$, which is a lattice by compactness of Γ (see [EG23, Lemma 5.1.5]).

Writing

$$(g_j - h_{n,j}) \cdot r_{n,j}(z) = g_j \cdot (1 - g_j^{-1}h_{n,j}) \cdot r_{n,j}(z),$$

we see that it suffices to show that there is a nested sequence of $\mathbf{A}^+$-lattices $\mathfrak{M}_n$ with the properties that for all j we have $(1 - g_j^{-1}h_{n,j})(\mathfrak{M}') \subseteq \mathfrak{M}_n$, that the intersection of the $\mathfrak{M}_n$ is 0, and that $\mathfrak{M}_n$ is φ-stable and Γ-stable for n sufficiently large. Indeed it then follows from (4.4.12) that $u_n - u_{n-1} \in \mathfrak{M}_n$ for n sufficiently large, uniformly in $z \in \mathfrak{M}$.

It is therefore enough to show that there exists a sequence $m(n)$, tending to ∞ with n, such that if we put $\mathfrak{M}_n := T^{m(n)}\mathfrak{M}'$ then $(1 - g_j^{-1}h_{n,j})(\mathfrak{M}') \subseteq \mathfrak{M}_n$. In fact,

$\mathfrak{M}_n$ is Γ-stable for all n, since $\mathfrak{M}'$ is Γ-stable; and $\mathfrak{M}_n$ is φ-stable for all n large enough, by [EG21, Lemma 5.2.7]. Recalling that $g_j^{-1}h_{n,j} \in \left(\begin{smallmatrix} 1+p^{a(n)}\mathbf{Z}_p & p^{a(n)}\mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ and that $a(n) \to \infty$, we see in turn that it is enough to show that for each $m \geq 0$, there exists an $M \geq 0$ such that if $h \in \left(\begin{smallmatrix} 1+p^M\mathbf{Z}_p & p^M\mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$, then $(1-h)(\mathfrak{M}') \subseteq T^m(\mathfrak{M}')$. Writing $h = \left(\begin{smallmatrix} 1+p^M a & p^M b \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$(1-h)(z) = (1-(1+T)^{p^M b})z + (1+T)^{p^M b}(z - \sigma_{1+p^M a}(z)).$$

Certainly $(1-(1+T)^{p^M b}) \in T^m \mathbf{A}_A^+$ for M sufficiently large uniformly in b (because A is an $\mathcal{O}/\varpi^a$-algebra), so it suffices to show that $z - \sigma_{1+p^M a}(z) \in T^m\mathfrak{M}'$ for M sufficiently large, for all $z \in \mathfrak{M}'$ and $a \in \mathbf{Z}_p$. This follows from [EG23, Lem. D.28 (3)], since A is an $\mathcal{O}/\varpi^a$-algebra.

Finally, we note that by (4.4.12) we have for any $k \geq 1$ and $z \in \mathfrak{M}$

$$\mathrm{Res}_{i+p^n\mathbf{Z}_p}(u_{n+k}-u_{n+k-1}) = \sum_{j \in I_{n+k}(\mathbf{Z}_p), f(j) \in i+p^n\mathbf{Z}_p} \begin{pmatrix} 1 & f(j) \\ 0 & 1 \end{pmatrix} \varphi^{n+k}((g_j - h_{n+k,j}) \cdot r_{n+k,j}(z)),$$

which is contained in $\mathfrak{M}_{n+k}$ for n sufficiently large (uniformly in $z \in \mathfrak{M}$ and $i \in \mathbf{Z}_p$). Thus we have in particular that $\mathrm{Res}_{i+p^n\mathbf{Z}_p}(f_*(z) - u_n) \in \mathfrak{M}_n$, and the result follows. $\square$

The operators m_α and f_* enjoy the following properties.

Proposition 4.4.13. *Let U, V be compact open subsets of $\mathbf{Z}_p$.*

1. *For all locally constant maps $\alpha_1, \alpha_2 : U \to A$, we have $m_{\alpha_1} \circ m_{\alpha_2} = m_{\alpha_1\alpha_2}$.*
2. *If $f : U \to V$ is a local diffeomorphism and $\alpha : V \to A$ is locally constant, then*
$$f_* \circ m_{\alpha \circ f} = m_\alpha \circ f_*.$$
3. *If $f : U \to V$ and $g : V \to W$ are local diffeomorphisms, then $(g \circ f)_* = g_* \circ f_*$.*
4. *If $\alpha : U \to A$ is locally constant and $V \subseteq U$, then m_α commutes with Res_V.*
5. *If $\alpha : \mathbf{Z}_p^\times \to A$ is constant on $a + p^n\mathbf{Z}_p$ for all $a \in \mathbf{Z}_p^\times$, then*
$$m_\alpha = \sum_{i \in (\mathbf{Z}/p^n\mathbf{Z})^\times} \alpha(i)\,\mathrm{Res}_{i+p^n\mathbf{Z}_p}.$$

Proof. All but the second and third properties were already established above. In the case that A is a finite $\mathcal{O}/\varpi^a$-algebra, the second and third results are respectively [Col10a, Prop. V.2.4, V.1.6], the proofs of which go over unchanged in our setting. $\square$

In summary, the sheaf $U \mapsto D \boxtimes U$ is equivariant for the action of local diffeomorphisms.

4.4.14. *Defining $- \boxtimes \mathbf{Q}_p$ more generally.* For any ψ-stable A-submodule $M \subset D$, we let $M \boxtimes \mathbf{Z}_p = M$, and we let $M \boxtimes \mathbf{Q}_p$ denote the set of sequences $(x_n)_{n\geq 0}$ with $x_n \in M$ and $\psi(x_{n+1}) = x_n$ for all $n \geq 0$. Equivalently, we have $M \boxtimes \mathbf{Q}_p := \varprojlim_\psi M$. We endow $M \boxtimes \mathbf{Q}_p$ with its usual projective limit topology, where $M \subseteq D$ has the subspace topology, and then $M \boxtimes \mathbf{Q}_p$ is closed in the infinite product $\prod_{n\geq 0} M$, and has the subspace topology (with the product having the product topology). To see that it is closed, one can for example note that it is the intersection of the sets of sequences (x_n) with $\psi(x_{n+1}) = x_n$ for $n \leq N$, which are closed for any N because ψ

is continuous. Note that $M \boxtimes \mathbf{Q}_p$ need not be an $A[P]$-submodule of $D \boxtimes \mathbf{Q}_p$, e.g. if M is not an $\mathbf{A}_A^+$-submodule of D. However, this will be the case when $M = D^\natural, D^\sharp$: see Lemma 4.4.18.

Remark 4.4.15. As Colmez explains in [Col10a, Section V.1.2], Proposition 4.4.10 is motivated by considering the case of $\mathbf{A}_A^+ := A[\![T]\!]$ with its (φ, Γ)-structure. Indeed, we may interpret $\mathbf{A}_A^+$ as the A-module of A-valued measures on $\mathbf{Z}_p$, and Proposition 4.4.10 in the case of a local diffeomorphism $f : \mathbf{Z}_p \to \mathbf{Z}_p$ computes the pushforward of measures under f. Here we should note that $\mathbf{A}_A^+$ is also ψ-invariant, because $\mathbf{A}_A^+ = \mathbf{A}_A^\natural$, and so the preceding discussion can be applied to it: indeed, by [SVZ14, Corollary 3.30], an $A[P^+]$-submodule of an étale (φ, Γ)-module D with A-coefficients is étale in the sense of Definition 4.4.2 if and only if it is ψ-stable.

A general étale (φ, Γ)-module D need not contain an $\mathbf{A}_A^+$-lattice that is simultaneously φ and ψ-invariant — it contains such a lattice if and only if $D^+ = D^\natural$ — and so in general it is only D that localizes, rather than any of its lattices. Nevertheless, as we will see in what follows, the lattices D^+, $D^\natural$, and $D^\sharp$ do interact in important ways with the localization of D.

4.4.16. *Key examples: $D^\natural \boxtimes \mathbf{Q}_p$ and $D^\sharp \boxtimes \mathbf{Q}_p$.* Since each of $D^\natural$ and $D^\sharp$ is ψ-stable, both $D^\natural \boxtimes \mathbf{Q}_p$ and $D^\sharp \boxtimes \mathbf{Q}_p$ are defined. Furthermore, the inclusion $D^\natural \subseteq D^\sharp$ evidently induces an embedding

$$D^\natural \boxtimes \mathbf{Q}_p \hookrightarrow D^\sharp \boxtimes \mathbf{Q}_p. \tag{4.4.17}$$

Lemma 4.4.18.

(1) The embedding (4.4.17) *is open.*
(2) We have natural identifications of finite A-modules

$$(D^\sharp \boxtimes \mathbf{Q}_p)/(D^\natural \boxtimes \mathbf{Q}_p) = (D^\sharp/D^\natural) \boxtimes \mathbf{Q}_p = D^\sharp/D^\natural.$$

(3) $D^\natural \boxtimes \mathbf{Q}_p$ and $D^\sharp \boxtimes \mathbf{Q}_p$ are $B(\mathbf{Q}_p)$-stable A-submodules of $D \boxtimes \mathbf{Q}_p$, and the embedding (4.4.17) *is $A[B(\mathbf{Q}_p)]$-equivariant.*

Proof. We have a short exact sequence

$$0 \to D^\natural \to D^\sharp \to D^\sharp/D^\natural \to 0.$$

Since ψ acts surjectively on each term, the passage to $\varprojlim_\psi$ is exact, and we have a short exact sequence

$$0 \to D^\natural \boxtimes \mathbf{Q}_p \to D^\sharp \boxtimes \mathbf{Q}_p \to (D^\sharp/D^\natural) \boxtimes \mathbf{Q}_p \to 0.$$

Since $D^\sharp/D^\natural$ is a finite A-module, its surjective endomorphism ψ is bijective, so we may identify $(D^\sharp/D^\natural)\boxtimes\mathbf{Q}_p$ with $D^\sharp/D^\natural$. This proves (2), and also shows that (4.4.17) identifies $D^\natural \boxtimes \mathbf{Q}_p$ with the kernel of the composite

$$D^\sharp \boxtimes \mathbf{Q}_p \xrightarrow{\text{proj. to 1st factor}} D^\sharp \longrightarrow D^\sharp/D^\natural ,$$

proving (1). Finally, part (3) is immediate from the definitions. □

4.5. **The $\mathrm{GL}_2(\mathbf{Q}_p)$-representations $D\boxtimes\mathbf{P}^1$.** Continue to let A denote a Noetherian $\mathcal{O}/\varpi^a$-algebra, and let D be an étale (φ, Γ)-module with A-coefficients. In the preceding subsection we have explained how to localize D to a sheaf over $\mathbf{Q}_p$, equivariantly with respect to the action of local diffeomorphisms. By general principles, this structure then allows us to localize D over other 1-dimensional locally analytic p-adic manifolds, such as $\mathbf{P}^1(\mathbf{Q}_p)$. Since $\mathrm{PGL}_2(\mathbf{Q}_p)$ acts on $\mathbf{P}^1(\mathbf{Q}_p)$ via diffeomorphisms, the resulting sheaf on $\mathbf{P}^1(\mathbf{Q}_p)$ will be $\mathrm{PGL}_2(\mathbf{Q}_p)$-equivariant, and in particular its global sections will afford a $\mathrm{PGL}_2(\mathbf{Q}_p)$-representation.

In fact, we follow Colmez in making a slightly more involved construction (via explicit formulas) which depends on our choice of character ζ, and leads to a G-representation with central character $\zeta\varepsilon^{-2}$. In the case when $\zeta = \varepsilon^2$ we obtain the construction of the preceding paragraph.

Let $w := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then w acts on $\mathbf{Z}_p^\times$ via the local diffeomorphism $z \mapsto 1/z$, and we let $H_w : D \boxtimes \mathbf{Z}_p^\times \to D \boxtimes \mathbf{Z}_p^\times$ be the composite $w_* \circ m_{\zeta\varepsilon^{-2}}$ (where we think of $\zeta\varepsilon^{-2}$ as the function $\zeta\varepsilon^{-2} : \mathbf{Z}_p^\times \to \mathcal{O}^\times \to A^\times \subset A$). We set

$$D \boxtimes \mathbf{P}^1 := \{z = (z_1, z_2) \in D \times D,\ \mathrm{Res}_{\mathbf{Z}_p^\times}(z_2) = H_w(\mathrm{Res}_{\mathbf{Z}_p^\times}(z_1))\}.$$

This is a closed subspace of $D \times D$.

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, and U is a compact open subset of $\mathbf{Q}_p$ not containing $-d/c$, then we write $\alpha_g : U \to A$ for the function $\alpha_g(x) = (\zeta\varepsilon^{-2})(cx + d)$. The action of g on $\mathbf{P}^1(\mathbf{Q}_p)$ via Möbius transformations induces a diffeomorphism $g : U \to gU$, and we define $H_g : D \boxtimes U \to D \boxtimes gU$ to be $g_* \circ m_{\alpha_g}$. Note that this specialises to the definition of H_w above.

We have a restriction map $\mathrm{Res}_U : D \boxtimes \mathbf{P}^1 \to D \boxtimes U$ defined by

$$\mathrm{Res}_U(z) = \mathrm{Res}_{U\cap\mathbf{Z}_p}(z_1) + H_w(\mathrm{Res}_{wU\cap p\mathbf{Z}_p}(z_2)) = \mathrm{Res}_{U\cap p\mathbf{Z}_p}(z_1) + H_w(\mathrm{Res}_{wU\cap\mathbf{Z}_p}(z_2)).$$

Theorem 4.5.1. *There is a unique action $(g, z) \mapsto g \cdot z$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $D \boxtimes \mathbf{P}^1$ such that for all open compact subsets U of $\mathbf{Q}_p$, and all elements $z = (z_1, z_2)$ of $D \boxtimes \mathbf{P}^1$, we have*

$$\mathrm{Res}_U(g \cdot z) = H_g(\mathrm{Res}_{g^{-1}U\cap\mathbf{Z}_p}(z_1)) + H_{gw}(\mathrm{Res}_{(gw)^{-1}U\cap p\mathbf{Z}_p}(z_2)). \tag{4.5.2}$$

Proof. In the case that A is a finite $\mathcal{O}/\varpi^a$-algebra this is [Col10c, Thm. II.1.4]. The proof only makes use of the properties of f_* and m_α in Proposition 4.4.13, and therefore goes over immediately in our setting. □

Remark 4.5.3. It is immediate from the definition that $D \boxtimes \mathbf{P}^1$ has central character $\zeta\varepsilon^{-2}$. Furthermore, comparing our definitions with [Col10c, Construction II.1], we see that our $D \boxtimes \mathbf{P}^1$ coincides (in the case that A is a finite $\mathcal{O}/\varpi^a$-algebra) with the representation denoted $D \boxtimes_{\zeta\varepsilon^{-2}} \mathbf{P}^1$ in the notational scheme of *loc. cit.*

As explained in [Col10c, Rem. II.1.7], it follows from the proof of Theorem 4.5.1 that if $z = (z_1, z_2) \in D \boxtimes \mathbf{P}^1$, then $z_1 = \mathrm{Res}_{\mathbf{Z}_p} z$, and $z_2 = \mathrm{Res}_{\mathbf{Z}_p}(w \cdot z)$. These relations imply that $w(z_1, z_2) = (z_2, z_1)$ for all $(z_1, z_2) \in D \boxtimes \mathbf{P}^1$. Furthermore, [Col10c, Rem. II.1.7 (iii)] shows that if $U \subset \mathbf{Q}_p$ is compact open, then Res_U admits a continuous A-linear section ι_U, defined by

$$\begin{aligned} \iota_U : D \boxtimes U &\to D \boxtimes \mathbf{P}^1 \\ z &\mapsto \left(\mathrm{Res}_{U\cap\mathbf{Z}_p}(z), H_w(\mathrm{Res}_{U\cap w\mathbf{Z}_p}(z))\right). \end{aligned} \tag{4.5.4}$$

The A-module $D \boxtimes U$ is therefore a direct summand of $D \boxtimes \mathbf{P}^1$, and we will sometimes identify it with this direct summand, and write Res_U for $\iota_U \circ \mathrm{Res}_U$. Similarly, if $\mathfrak{M} \subseteq D$ is an A-submodule, we will write $\mathfrak{M} \boxtimes \mathbf{Z}_p$ for the image of $\mathfrak{M}$ under $\iota_{\mathbf{Z}_p}$. We single out a special case of this discussion in the next lemma.

Lemma 4.5.5. *The map*

$$D \boxtimes \mathbf{P}^1 \to (D \boxtimes \mathbf{Z}_p) \oplus (D \boxtimes p\mathbf{Z}_p) \tag{4.5.6}$$

given by $z \mapsto (\mathrm{Res}_{\mathbf{Z}_p} z, \mathrm{Res}_{p\mathbf{Z}_p}(w\cdot z))$ *(or equivalently by* $(z_1,z_2)\mapsto (z_1, \mathrm{Res}_{p\mathbf{Z}_p}(z_2))$*) is a homeomorphism, the inverse morphism being given by*

$$\iota_{\mathbf{Z}_p} + w\circ \iota_{p\mathbf{Z}_p} : (y_1,y_2)\mapsto (y_1, y_2 + H_w(\mathrm{Res}_{\mathbf{Z}_p^\times}(y_1))).$$

The induced action of w *on* $(D\boxtimes \mathbf{Z}_p)\oplus(D\boxtimes p\mathbf{Z}_p)$ *is*

$$w(y_1,y_2) = (y_2 + H_w(\mathrm{Res}_{\mathbf{Z}_p^\times}(y_1)), \mathrm{Res}_{p\mathbf{Z}_p}(y_1)).$$

Proof. The verification that the maps are mutually inverse is formal, and their continuity follows from the continuity of H_w, $\mathrm{Res}_{\mathbf{Z}_p^\times}$, and $\mathrm{Res}_{p\mathbf{Z}_p}$. The formula for w is an immediate consequence of the fact that $w(z_1,z_2) = (z_2,z_1)$ for all $(z_1,z_2)\in D\boxtimes \mathbf{P}^1$. □

Corollary 4.5.7. *For all* $g\in G$*, the map* $D\boxtimes\mathbf{P}^1\to D\boxtimes\mathbf{P}^1, z\mapsto g\cdot z$ *is continuous.*

Proof. By Lemma 4.5.5, it suffices to prove that $z\mapsto \mathrm{Res}_U(g\cdot z)$ is continuous for all open compact subsets $U\subseteq \mathbf{Q}_p$. This is true by the formula for $\mathrm{Res}_U(g\cdot z)$ given in Theorem 4.5.1, together with the fact that the operators f_* and m_α are continuous (using Proposition 4.4.10 for f_*). □

Remark 4.5.8. Since $D\boxtimes p\mathbf{Z}_p = (\varphi\circ\psi)(D) = \varphi(D)$ is a direct summand of $D=\varphi(D)\oplus D^{\psi=0}$, the subspace topology on $D\boxtimes p\mathbf{Z}_p$ makes it a Tate A-module. The map $\varphi: D\to D\boxtimes p\mathbf{Z}_p$ is a topological isomorphism with inverse ψ. Similarly, $D\boxtimes p^n\mathbf{Z}_p$ is a Tate A-module for all $n\in\mathbf{Z}$, and it is topologically isomorphic to D via the action of $\mathrm{diag}(p^{\mathbf{Z}},1)\subseteq P$. The homeomorphism (4.5.6) then shows that $D\boxtimes\mathbf{P}^1$ is a Tate A-module. In particular, it makes sense to talk about A-submodules of $D\boxtimes\mathbf{P}^1$ being lattices, and we will do so without further comment.

Lemma 4.5.9. *Let* $n\in\mathbf{Z}$*. Then* $\mathrm{Res}_{p^n\mathbf{Z}_p}: D\boxtimes\mathbf{P}^1\to D\boxtimes p^n\mathbf{Z}_p$ *is open, and sends lattices to lattices. The same is true for* $\mathrm{Res}_{p^n\mathbf{Z}_p}: D\to D\boxtimes p^n\mathbf{Z}_p$ *if* $n\geq 0$*.*

Proof. By construction, the map $\mathrm{Res}_{p^n\mathbf{Z}_p}$ is continuous, surjective and A-linear. We now show that it is open. The equivariance properties of $\mathrm{Res}_{p^n\mathbf{Z}_p}$ under the action of G show that it suffices to prove this when $n=0$, in which case $\mathrm{Res}_{\mathbf{Z}_p}$ is the projection onto a direct summand, by Lemma 4.5.5. Hence it is open.

There remains to prove that $\mathrm{Res}_{p^n\mathbf{Z}_p}$ sends lattices to lattices. It suffices to show that if $f: M\to N$ is a surjective, continuous and open morphism between A-Tate modules, and $\mathfrak{M}\subset M$ is a lattice, then $f(\mathfrak{M})$ is a lattice in N. Let $U\subset f(\mathfrak{M})$ be an open A-submodule; we need to prove that $f(\mathfrak{M})/U$ is a finite A-module. Since f induces an isomorphism

$$\mathfrak{M}/(\mathfrak{M}\cap f^{-1}(U))\cong f(\mathfrak{M})/U,$$

this is a consequence of the fact that $\mathfrak{M}$ is a lattice and $\mathfrak{M}\cap f^{-1}(U)$ is open. □

Lemma 4.5.10. *Let* $\mathfrak{M}\subset D$ *be a lattice. Then:*

1. $(\mathfrak{M}\boxtimes\mathbf{Z}_p) + w(\mathfrak{M}\boxtimes\mathbf{Z}_p)$ *is open in* $D\boxtimes\mathbf{P}^1$*, and*
2. $\{z\in D\boxtimes\mathbf{P}^1 : \mathrm{Res}_{\mathbf{Z}_p} z\in\mathfrak{M}, \mathrm{Res}_{\mathbf{Z}_p}(wz)\in\mathfrak{M}\}$ *is a lattice in* $D\boxtimes\mathbf{P}^1$*.*

Proof. For any A-submodule $L\subset D\boxtimes\mathbf{P}^1$, it follows from Lemma 4.5.5 that L is open if and only if the image of L under

$$\begin{aligned}\rho: D\boxtimes\mathbf{P}^1 &\to D\oplus\varphi(D),\\ z&\mapsto (z, \mathrm{Res}_{p\mathbf{Z}_p}(w\cdot z)),\\ (z_1,z_2)&\mapsto (z_1,\mathrm{Res}_{p\mathbf{Z}_p}(z_2))\end{aligned}$$

is open.

Let $L := (\mathfrak{M} \boxtimes \mathbf{Z}_p) + w(\mathfrak{M} \boxtimes \mathbf{Z}_p)$. Recall from (4.5.4) that

$$\mathfrak{M} \boxtimes \mathbf{Z}_p := \iota_{\mathbf{Z}_p}(\mathfrak{M}) = \{(m, H_w \operatorname{Res}_{\mathbf{Z}_p^\times} m) : m \in \mathfrak{M}\} \subset D \boxtimes \mathbf{P}^1.$$

Then

$$\rho(\mathfrak{M} \boxtimes \mathbf{Z}_p) = \{(m, 0) : m \in \mathfrak{M}\} \subset D \oplus \varphi(D)$$

and

$$\rho(w(\mathfrak{M} \boxtimes \mathbf{Z}_p)) = \{(H_w \operatorname{Res}_{\mathbf{Z}_p^\times} m, \operatorname{Res}_{p\mathbf{Z}_p} m) : m \in \mathfrak{M}\} \subset D \oplus \varphi(D).$$

To see that $\rho(L)$ is open, consider the A-linear map

$$\lambda : D \oplus D \to D \oplus \varphi(D), (x, y) \mapsto (x + H_w \operatorname{Res}_{\mathbf{Z}_p^\times} y, \operatorname{Res}_{p\mathbf{Z}_p} y).$$

Then $\rho(L) = \lambda(\mathfrak{M} \oplus \mathfrak{M})$, and since $\mathfrak{M} \oplus \mathfrak{M}$ is open in $D \oplus D$, it suffices to prove that λ is open. To do so, it suffices to find neighborhood bases $(M_n), (L_n)$ of $0 \in D$ such that $\lambda(M_n \oplus L_n)$ is open for all n. Choose M_n arbitrarily, and choose L_n such that $H_w \operatorname{Res}_{\mathbf{Z}_p^\times}(L_n) \subseteq M_n$, using the continuity of $H_w \circ \operatorname{Res}_{\mathbf{Z}_p^\times}$. Then

$$\lambda(M_n \oplus L_n) \supseteq M_n \oplus \operatorname{Res}_{p\mathbf{Z}_p}(L_n).$$

Since $\operatorname{Res}_{p\mathbf{Z}_p} : D \to \varphi(D)$ is open (because it coincides with $\varphi\psi$), this concludes the proof of part (1).

Now let $L := \{z \in D \boxtimes \mathbf{P}^1 : \operatorname{Res}_{\mathbf{Z}_p} z \in \mathfrak{M}, \operatorname{Res}_{\mathbf{Z}_p}(wz) \in \mathfrak{M}\}$. Since $\operatorname{Res}_{\mathbf{Z}_p}$ and w are continuous (using Corollary 4.5.7 for w), L is open. Hence $\rho(L) \subset D \oplus \varphi(D)$ is also open. On the other hand, $\rho(L)$ is contained in $\mathfrak{M} \oplus \operatorname{Res}_{p\mathbf{Z}_p} \mathfrak{M}$, which is a lattice in $D \oplus \varphi(D)$, by Lemma 4.5.9, so that $\rho(L)$ is also bounded, and so it is a lattice, by Remark 4.1.3. ☐

We now study the continuity of the G-action on $D \boxtimes \mathbf{P}^1$.

Lemma 4.5.11. *The action of G on $D \boxtimes \mathbf{P}^1$ that was constructed in Theorem* 4.5.1 *is continuous.*

Proof. We claim that it is enough to show that the action map

$$H \times (D \boxtimes \mathbf{P}^1) \to D \boxtimes \mathbf{P}^1 \tag{4.5.12}$$

is jointly continuous for some open subgroup H of G.

Indeed, assume that this is the case, and let $(g, x) \in G \times (D \boxtimes \mathbf{P}^1)$ be arbitrary. Any open neighbourhood of $gx \in D \boxtimes \mathbf{P}^1$ contains a neighbourhood of the form $gx + \mathfrak{M}$, where $\mathfrak{M} \subset D \boxtimes \mathbf{P}^1$ is a lattice. Since $g \in G$ acts continuously on $D \boxtimes \mathbf{P}^1$, $g^{-1}\mathfrak{M} \subset D \boxtimes \mathbf{P}^1$ is open. By the continuity of (4.5.12), there is an open subgroup H' of H and a lattice $L \subset D \boxtimes \mathbf{P}^1$ such that $H'(x + L) \subseteq x + g^{-1}\mathfrak{M}$; so $(gH')(x + L) \subseteq gx + \mathfrak{M}$, and $gH' \times (x + L) \subset G \times D \boxtimes \mathbf{P}^1$ is an open neighbourhood of (g, x).

We now turn to establishing the continuity of (4.5.12). If $H = \cup_i U_i$ and $D \boxtimes \mathbf{P}^1 = \cup_j V_j$ are open covers of H and $D \boxtimes \mathbf{P}^1$ respectively, then it suffices to check that each $U_i \times V_j \to D \boxtimes \mathbf{P}^1$ is jointly continuous. We claim that it suffices to show that $H \times D \to D \boxtimes \mathbf{P}^1$ is continuous. Indeed, since w is a continuous automorphism of $D \boxtimes \mathbf{P}^1$, if we write

$$D \boxtimes \mathbf{P}^1 = D \boxtimes \mathbf{Z}_p \cup w(D \boxtimes \mathbf{Z}_p) = D \cup wD,$$

then we deduce that $wHw \times wD \to D \boxtimes \mathbf{P}^1$ is continuous, and so the action map

$$(H \cap wHw) \times D \boxtimes \mathbf{P}^1 \to D \boxtimes \mathbf{P}^1$$

is continuous, and since $H \cap wHw$ is also an open subgroup of G, we will be done.

We now take H to be the usual Iwahori subgroup of matrices which are upper triangular modulo p, so that by the Iwahori decomposition we may write

$$H = wU(p\mathbf{Z}_p)wB(\mathbf{Z}_p)$$

where B is the (upper triangular) Borel subgroup, and $U(p\mathbf{Z}_p) := \left(\begin{smallmatrix} 1 & p\mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$. In summary, we have reduced to showing that

$$wU(p\mathbf{Z}_p)wB(\mathbf{Z}_p) \times D \to D \boxtimes \mathbf{P}^1 \tag{4.5.13}$$

is jointly continuous.

Now, the continuity of

$$B(\mathbf{Z}_p) \times D \to D = D \boxtimes \mathbf{Z}_p \subset D \boxtimes \mathbf{P}^1 \tag{4.5.14}$$

is more or less immediate from the definition of the $B(\mathbf{Z}_p)$-action on D and the continuity assumptions in the definition of a (φ, Γ)-module. More precisely, it follows from the existence of a neighborhood basis of zero in D consisting of (φ, Γ)-stable lattices, which in turn follows from the compactness of Γ (see [EG23, Lemma 5.1.5] for a proof in the case that A is a Noetherian $\mathbf{F}_p$-algebra, which goes over unchanged to the case of Noetherian $\mathcal{O}/\varpi^a$-algebras).

Accordingly, we can factor (4.5.13) as

$$wU(p\mathbf{Z}_p)wB(\mathbf{Z}_p) \times D \to wU(p\mathbf{Z}_p)w \times D \to D \boxtimes \mathbf{P}^1 \tag{4.5.15}$$

with the first arrow continuous; so we just have to show that the second arrow is continuous. Since w is a continuous automorphism, this amounts to showing that

$$U(p\mathbf{Z}_p) \times w(D \boxtimes \mathbf{Z}_p) \to D \boxtimes \mathbf{P}^1$$

is jointly continuous. Writing $D \boxtimes \mathbf{Z}_p = D \boxtimes \mathbf{Z}_p^\times \cup D \boxtimes p\mathbf{Z}_p$, and taking into account the continuity of (4.5.14), we see that it is enough to show that

$$U(p\mathbf{Z}_p) \times w(D \boxtimes p\mathbf{Z}_p) \to D \boxtimes \mathbf{P}^1$$

is jointly continuous.

It follows from the definitions (see the proof of [Col10c, Prop. II.1.8]) that $U(p\mathbf{Z}_p)$ preserves $w(D \boxtimes p\mathbf{Z}_p)$, and more precisely if $b \in p\mathbf{Z}_p$, then $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ takes $(0, z) \in w(D \boxtimes p\mathbf{Z}_p) \subset D \boxtimes \mathbf{P}^1$ to $(0, z')$ with

$$z' = (\zeta\varepsilon^{-2})^{-1}(1+b)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} (1+b)^2 & b(1+b) \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & 1/(1+b) \\ 0 & 1 \end{pmatrix} z.$$

Noting that $\left(\begin{smallmatrix} 1 & 1/(1+b) \\ 0 & 1 \end{smallmatrix}\right)$ takes $p\mathbf{Z}_p$ to $1 + p\mathbf{Z}_p$, and that w is a continuous automorphism of $D \boxtimes (1 + p\mathbf{Z}_p)$, the result follows from the continuity of (4.5.14). $\square$

In our later applications, rather than appealing to this continuity directly, we will use the following consequence of it.

Corollary 4.5.16. *The G-action on $D \boxtimes \mathbf{P}^1$ extends to an $A[\![G]\!]$-module structure on $D \boxtimes \mathbf{P}^1$, uniquely characterized by the requirement that the induced $A[\![K]\!]$-action on $D \boxtimes \mathbf{P}^1$ is continuous. Furthermore, the endomorphisms of $D \boxtimes \mathbf{P}^1$ induced by the elements of $A[\![G]\!]$ are all continuous.*

Proof. We will show that the action of K on $D \boxtimes \mathbf{P}^1$ extends to a continuous action of $A[\![K]\!]$. Such an extension is unique if it exists (by continuity, together with the density of $A[K]$ in $A[\![K]\!]$), and by the definition of $A[\![G]\!]$, this will yield the required $A[\![G]\!]$-action. The final claim of the lemma will also then follow, by combining the continuity of the G-action and the $\mathcal{O}[\![K]\!]$-action on $D \boxtimes \mathbf{P}^1$.

In order to construct the required $A[\![K]\!]$-action, choose a lattice $L \subset D \boxtimes \mathbf{P}^1$. Lemma 4.5.11 shows that the action map $K \times (D \boxtimes \mathbf{P}^1) \to D \boxtimes \mathbf{P}^1$ is jointly continuous, and so we see that for each $k \in K$, there is an open subgroup U_k of K and a lattice $L_k \subseteq L$ such that $(kU_k) \cdot L_k \subseteq L$. Since K is compact, it is covered by finitely many of the kU_k, and if we let L' denote the intersection of the corresponding lattices L_k, we see that $K \cdot L' \subseteq L$. The A-submodule generated by $K \cdot L'$ is therefore open and bounded, and so it is a lattice. Replacing $K \cdot L'$ by this lattice, and recalling that L was arbitrary, we see that $D \boxtimes \mathbf{P}^1$ contains a basis of K-stable lattices L_n. Then each $(D \boxtimes \mathbf{P}^1)/L_n$ is a discrete A-module with a continuous action of K, hence is a smooth K-representation, and thus is canonically an $A[\![K]\!]$-module (with jointly continuous action map). Consequently

$$D \boxtimes \mathbf{P}^1 = \varprojlim_n (D \boxtimes \mathbf{P}^1)/L_n$$

inherits a continuous $A[\![K]\!]$-action, as required. $\square$

We now need to consider the restriction map $\mathrm{Res}_{\mathbf{Q}_p} : D \boxtimes \mathbf{P}^1 \to D \boxtimes \mathbf{Q}_p$, which by definition is given by

$$z \mapsto \left(\mathrm{Res}_{\mathbf{Z}_p}\left(\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right) z\right)\right)_{n\geq 0}.$$

The verification that this defines a morphism to $D \boxtimes \mathbf{Q}_p$ is formal; see the proof of [Col10c, Prop. II.1.14]. The action of P on $D \boxtimes \mathbf{Q}_p$ extends to an action of the Borel subgroup $B(\mathbf{Q}_p)$ of G by letting the diagonal matrix $\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right)$ act via $(\zeta\varepsilon^{-2})(z)$.

Lemma 4.5.17. $\mathrm{Res}_{\mathbf{Q}_p} : D \boxtimes \mathbf{P}^1 \to D \boxtimes \mathbf{Q}_p$ *is continuous and* $B(\mathbf{Q}_p)$*-equivariant.*

Proof. The $B(\mathbf{Q}_p)$-equivariance is formal; see the proof of [Col10c, Prop. II.1.14]. Since $D \boxtimes \mathbf{Q}_p$ has the inverse limit topology, the continuity of $\mathrm{Res}_{\mathbf{Q}_p}$ follows from the continuity of $\mathrm{Res}_{\mathbf{Z}_p}$ and of the action of $\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$. $\square$

We then define

$$D^\natural \boxtimes \mathbf{P}^1 := \{z \in D \boxtimes \mathbf{P}^1,\ \mathrm{Res}_{\mathbf{Q}_p}(z) \in D^\natural \boxtimes \mathbf{Q}_p\},$$
$$D^\sharp \boxtimes \mathbf{P}^1 := \{z \in D \boxtimes \mathbf{P}^1,\ \mathrm{Res}_{\mathbf{Q}_p}(z) \in D^\sharp \boxtimes \mathbf{Q}_p\}.$$

By Lemma 4.5.17, $D^\natural \boxtimes \mathbf{P}^1$ and $D^\sharp \boxtimes \mathbf{P}^1$ are closed A-submodules of $D \boxtimes \mathbf{P}^1$. In the case that D has rank two and determinant $\zeta\varepsilon^{-1}$, we will show that they are in fact lattices, but this is not obvious; indeed our proof will be closely intertwined with the proof that $D^\natural \boxtimes \mathbf{P}^1$ is G-stable, and as in the original work of Colmez, this is ultimately established via p-adic interpolation from the crystalline case. We end this section by establishing two properties of these modules.

Lemma 4.5.18. $(D^\sharp \boxtimes \mathbf{P}^1)/(D^\natural \boxtimes \mathbf{P}^1)$ *is a finite* A*-module, and* $D^\natural \boxtimes \mathbf{P}^1$ *is an open submodule of* $D^\sharp \boxtimes \mathbf{P}^1$*.*

Proof. By definition, $\mathrm{Res}_{\mathbf{Q}_p}$ induces an injection

$$(D^\sharp \boxtimes \mathbf{P}^1)/(D^\natural \boxtimes \mathbf{P}^1) \hookrightarrow (D^\sharp \boxtimes \mathbf{Q}_p)/(D^\natural \boxtimes \mathbf{Q}_p),$$

and the target is a finite A-module by Lemma 4.4.18. This proves the first statement, and since $\mathrm{Res}_{\mathbf{Q}_p}$ is continuous, the second statement also follows from Lemma 4.4.18, which asserts that $D^\natural \boxtimes \mathbf{Q}_p$ is open in $D^\sharp \boxtimes \mathbf{Q}_p$. $\square$

Lemma 4.5.19. *$D^\natural \boxtimes \mathbf{P}^1$ contains $D^+ \boxtimes \mathbf{Z}_p$.*

Proof. Let $t := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By definition, we need to prove that if $z \in D^+ \boxtimes \mathbf{Z}_p$ and $n \geq 0$ then $\mathrm{Res}_{\mathbf{Z}_p}(t^n z) \in D^\natural$. Recall that $D^+ \boxtimes \mathbf{Z}_p := \iota_{\mathbf{Z}_p}(D^+)$. Applying (4.5.4), we see that there exists $z_0 \in D^+$ such that $z = (z_0, H_w \mathrm{Res}_{\mathbf{Z}_p^\times} z_0)$ as elements of $D \boxtimes \mathbf{P}^1$. Since D^+ is φ-stable, and $D^+ \subset D^\natural$ by Lemma 4.3.3, it thus suffices to prove that $\mathrm{Res}_{\mathbf{Z}_p}(t^n z) = \varphi^n(z_0)$ for all $n \geq 0$.

Applying (4.5.2) we see that

$$\mathrm{Res}_{\mathbf{Z}_p}(t^n z) = H_{t^n}(z_0) :$$

indeed, the second summand in (4.5.2) vanishes, because $\mathrm{Res}_{p\mathbf{Z}_p}(H_w \mathrm{Res}_{\mathbf{Z}_p^\times} z_0) = 0$. We now observe that $H_{t^n} = t^n_*$, where $t^n : \mathbf{Z}_p \to p^n\mathbf{Z}_p$ is the diffeomorphism of multiplication by p^n: indeed, by definition

$$H_{t^n} = t^n_* \circ m_{\alpha_{t^n}},$$

and the function α_{t^n} is identically 1 on $\mathbf{Z}_p$. Now Example 4.4.11 shows that

$$t^n_* = \varphi^n : D \to D \boxtimes p^n\mathbf{Z}_p = \varphi^n(D),$$

which concludes the proof. □

4.6. **The action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on lattices in $D\boxtimes\mathbf{P}^1$.** We continue to assume that A is a Noetherian $\mathcal{O}/\varpi^a$-algebra. Our goal in this section is to prove Proposition 4.6.7, which gives a criterion for $D^\natural \boxtimes \mathbf{P}^1$ to be G-stable, or (equivalently) $\mathcal{O}[\![G]\!]$-stable (with respect to the $\mathcal{O}[\![G]\!]$-action on $D \boxtimes \mathbf{P}^1$ given by Corollary 4.5.16). The study of this question is intertwined with the study of certain lattices in $D \boxtimes \mathbf{P}^1$.

We embed $\mathbf{Q}_p^\times$ into G via

$$\mathbf{Q}_p^\times \xrightarrow{\sim} \begin{pmatrix} \mathbf{Q}_p^\times & 0 \\ 0 & 1 \end{pmatrix}; \tag{4.6.1}$$

in particular, the composite of this embedding with $\det : G \to \mathbf{Q}_p^\times$ is the identity on $\mathbf{Q}_p^\times$.

Remark 4.6.2. Note that since $D^{\mathrm{nr}} \subseteq D$ is φ- and Γ-stable, $D^{\mathrm{nr}} \subseteq D = D \boxtimes \mathbf{Z}_p \subset D \boxtimes \mathbf{Q}_p \subset D \boxtimes \mathbf{P}^1$ is $\mathbf{Q}_p^\times$-stable.

Lemma 4.6.3. *The kernel of $\mathrm{Res}_{\mathbf{Q}_p} : D \boxtimes \mathbf{P}^1 \to D \boxtimes \mathbf{Q}_p$ is $w \cdot D^{\mathrm{nr}}$. In particular, $\ker \mathrm{Res}_{\mathbf{Q}_p}$ is a finite A-module.*

Proof. Since D^{nr} is a finite A-module by Lemma 4.3.15, it is enough to prove that $\ker \mathrm{Res}_{\mathbf{Q}_p} = w \cdot D^{\mathrm{nr}}$. This follows formally from the definitions, exactly as in the proof of [Col10c, Prop. II.1.14]. □

We have the following variant on [Col10a, Lem. III.3.6], which is proved in the same way.

Lemma 4.6.4. *If $M \subseteq D \boxtimes \mathbf{Q}_p$ is a closed $B(\mathbf{Q}_p)$-stable A-submodule, then*

$$M = M_0 \boxtimes \mathbf{Q}_p,$$

where $M_0 := \mathrm{Res}_{\mathbf{Z}_p}(M) \subseteq D$ is a Γ- and ψ-stable $\mathbf{A}_A^+$-submodule on which ψ acts surjectively.

Proof. Since M is $B(\mathbf{Q}_p)$-stable, it follows from the definition of the action of P on $D \boxtimes \mathbf{Q}_p$ that M_0 is $\mathbf{A}_A^+$-, Γ-, and ψ-stable, and that ψ acts surjectively on M_0. It also follows that, if we write $D \boxtimes \mathbf{Q}_p \coloneqq \varprojlim_\psi D$, then the image of M under the kth projection (for any $k \geq 0$) is equal to M_0. We therefore have $M \subseteq M_0 \boxtimes \mathbf{Q}_p$, and it remains to show that this inclusion is an equality. Thus, let $z = (z_n)_{n \geq 0} \in M_0 \boxtimes \mathbf{Q}_p$. The just-established equality of images shows that for each $k \geq 0$ we can find $u_k = (u_{k,n})_{n\geq 0} \in M$ with $u_{k,k} = z_k$. By the definition of the projective limit topology on $D \boxtimes \mathbf{Q}_p$, we see that $u_k \to z$ as $k \to \infty$, and we are done, since M is closed in $D \boxtimes \mathbf{Q}_p$ by hypothesis. □

Lemma 4.6.5. *If $\mathfrak{M} \subset D \boxtimes \mathbf{P}^1$ is a $B(\mathbf{Q}_p)$-stable lattice, then*

$$D^\natural \boxtimes \mathbf{Q}_p \subseteq \operatorname{Res}_{\mathbf{Q}_p}(\mathfrak{M}) \subseteq D^\sharp \boxtimes \mathbf{Q}_p.$$

Proof. Since each $\operatorname{Res}_{p^{-n}\mathbf{Z}_p} : D \boxtimes \mathbf{P}^1 \to D \boxtimes p^{-n}\mathbf{Z}_p$ takes lattices to lattices, by Lemma 4.5.9, we see that

$$\operatorname{Res}_{\mathbf{Q}_p}(\mathfrak{M}) \coloneqq \varprojlim_n \operatorname{Res}_{p^{-n}\mathbf{Z}_p} \mathfrak{M} \hookrightarrow D \boxtimes \mathbf{Q}_p \coloneqq \varprojlim_n D \boxtimes p^{-n}\mathbf{Z}_p,$$

is an inverse limit of open, and hence closed, A-submodules of the various $D \boxtimes p^{-n}\mathbf{Z}_p$, and so is a closed A-submodule of $D \boxtimes \mathbf{Q}_p$. Now Lemma 4.6.4 shows that $\operatorname{Res}_{\mathbf{Q}_p}(\mathfrak{M}) \xrightarrow{\sim} \mathfrak{M}_0 \boxtimes \mathbf{Q}_p$, where $\mathfrak{M}_0 \coloneqq \operatorname{Res}_{\mathbf{Z}_p} \mathfrak{M}$ is an $\mathbf{A}_A^+$-submodule of $D \boxtimes \mathbf{Z}_p = D$ which is ψ-stable and on which ψ acts surjectively. As already noted, $\mathfrak{M}_0$ is also a lattice in D. Thus $D^\natural \subseteq \mathfrak{M}_0 \subseteq D^\sharp$, by Proposition 4.2.5, and the lemma follows. □

Lemma 4.6.6. *If $\mathfrak{M} \subset D \boxtimes \mathbf{P}^1$ is a $B(\mathbf{Q}_p)$-stable lattice, then it is G-stable, and hence in fact $\mathcal{O}[\![G]\!]$-stable.*

Proof. Since the group G has a neighbourhood basis of the identity given by open subgroups (e.g. the congruence subgroups of $\mathrm{GL}_2(\mathbf{Z}_p)$), it follows from Lemma 4.5.11 and [EG23, Lem. D.13] that there is an open subgroup $H \subseteq G$ such that $H\mathfrak{M} = \mathfrak{M}$. Since $B(\mathbf{Q}_p)\mathfrak{M} = \mathfrak{M}$ we have $B(\mathbf{Q}_p)HB(\mathbf{Q}_p)\mathfrak{M} = \mathfrak{M}$, and since we have $B(\mathbf{Q}_p)HB(\mathbf{Q}_p) = G$ by the Bruhat decomposition, we conclude that $\mathfrak{M}$ is G-stable.

To see the final assertion, it suffices to note that since $\mathcal{O}[K]$ is dense in $\mathcal{O}[\![K]\!]$, and since $\mathfrak{M}$ is closed in $D \boxtimes \mathbf{P}^1$, the K-stability of $\mathfrak{M}$ implies the $\mathcal{O}[\![K]\!]$-stability of $\mathfrak{M}$, and G-stability and $\mathcal{O}[\![K]\!]$-stability together imply $\mathcal{O}[\![G]\!]$-stability. □

Proposition 4.6.7. *Suppose that there is a $B(\mathbf{Q}_p)$-stable lattice $\mathfrak{M} \subset D \boxtimes \mathbf{P}^1$. Then $D^\natural \boxtimes \mathbf{P}^1$ is an $\mathcal{O}[\![G]\!]$-stable lattice in $D \boxtimes \mathbf{P}^1$.*

Proof. By Lemma 4.6.5, we have

$$D^\natural \boxtimes \mathbf{P}^1 \subseteq \mathfrak{M} + \ker \operatorname{Res}_{\mathbf{Q}_p} \subseteq D^\sharp \boxtimes \mathbf{P}^1.$$

Since $\ker \operatorname{Res}_{\mathbf{Q}_p}$ is a finite A-module, by Lemma 4.6.3, it follows that $\mathfrak{M} + \ker \operatorname{Res}_{\mathbf{Q}_p}$ is a lattice. By Lemma 4.5.18, $(D^\sharp \boxtimes \mathbf{P}^1)/(D^\natural \boxtimes \mathbf{P}^1)$ is a finite A-module, hence $D^\sharp \boxtimes \mathbf{P}^1$ is a lattice, and $D^\natural \boxtimes \mathbf{P}^1$ is open in $D^\sharp \boxtimes \mathbf{P}^1$, hence $D^\natural \boxtimes \mathbf{P}^1$ is a lattice. Since $D^\natural \boxtimes \mathbf{P}^1$ is $B(\mathbf{Q}_p)$-stable, the proposition follows from Lemma 4.6.6. □

We also have the following "converse" to Proposition 4.6.7.

Lemma 4.6.8. *If $D^\natural \boxtimes \mathbf{P}^1$ is G-stable, then it is a lattice in $D \boxtimes \mathbf{P}^1$.*

Proof. Under the assumption that $D^\natural \boxtimes \mathbf{P}^1$ is G-stable, we have

$$D^\natural \boxtimes \mathbf{P}^1 \subseteq \{x \in D \boxtimes \mathbf{P}^1 \mid \mathrm{Res}_{\mathbf{Z}_p} x \in D^\natural \text{ and } \mathrm{Res}_{\mathbf{Z}_p} wx \in D^\natural\}.$$

Since $D^\natural$ is a lattice in D, it follows from Lemma 4.5.10 (2) that $D^\natural \boxtimes \mathbf{P}^1$ is a closed A-submodule of a lattice. By Remark 4.1.3, it thus suffices to show that $D^\natural \boxtimes \mathbf{P}^1$ is open. To this end, note that it follows from Lemma 4.5.19 that $D^\natural \boxtimes \mathbf{P}^1$ contains $D^+ \boxtimes \mathbf{Z}_p$, and since $D^\natural \boxtimes \mathbf{P}^1$ is assumed to be G-stable, it also contains $w \cdot (D^+ \boxtimes \mathbf{Z}_p)$. It therefore contains $D^+ \boxtimes \mathbf{Z}_p + w \cdot (D^+ \boxtimes \mathbf{Z}_p)$, which by Lemma 4.5.10 (1) is an open neighborhood of the identity, as required. □

4.7. **Formal (φ, Γ)-modules.** In this subsection, which is an interlude in our main discussion, we briefly extend some of the previous theory to the context of what we call *formal* étale (φ, Γ)-modules.

Let R be a complete local Noetherian $\mathcal{O}$-algebra with finite residue field. Following Dee [Dee01] we let $\widehat{\mathbf{A}}_R$ denote the $\mathfrak{m}_R$-adic completion of $\mathbf{A}_R$, and we define a *formal étale* (φ, Γ)*-module* with R-coefficients to be an étale (φ, Γ)-module over $\widehat{\mathbf{A}}_R$; that is, a finitely generated $\widehat{\mathbf{A}}_R$-module equipped with commuting semilinear actions of φ and Γ, with the underlying φ-module being étale in the usual sense. We do not demand that a formal étale (φ, Γ)-module is projective; if it is, we explicitly refer to it as a *projective* formal étale (φ, Γ)-module; note that it is then in fact *free*, since $\widehat{\mathbf{A}}_R$ is a local ring. Note that if R is furthermore a finite $\mathcal{O}$-module, then $\widehat{\mathbf{A}}_R = \mathbf{A}_R$, so that the categories of (projective) formal étale (φ, Γ)-modules with R-coefficients and étale (φ, Γ)-modules with R-coefficients are equivalent.

Of course, in the case that R is Artinian this agrees with the usual definition of a (not necessarily projective) étale (φ, Γ)-module, and as we recalled in Section 3.1, in this case there is an equivalence of categories between the category of étale (φ, Γ)-modules with R-coefficients and the category of finitely generated R-modules equipped with a continuous action of $G_{\mathbf{Q}_p}$. We write $V \mapsto \mathbf{D}(V)$ for the functor taking a representation of $G_{\mathbf{Q}_p}$ to the corresponding (φ, Γ)-module, and $D \mapsto \mathbf{V}(D)$ for the quasi-inverse functor.

Returning to the general case that R is a complete local Noetherian $\mathcal{O}$-algebra with finite residue field, it is shown in [Dee01] that this equivalence extends to an exact equivalence of categories between the category of formal étale (φ, Γ)-modules with R-coefficients and the category of finitely generated R-modules with a continuous action of $G_{\mathbf{Q}_p}$, where now

$$\mathbf{D}(V) \coloneqq \varprojlim_k \mathbf{D}(V/\mathfrak{m}_R^k V)$$

and similarly for $\mathbf{V}(D)$. (It is also shown in [Dee01] that this construction is compatible with extension of scalars, so we do not record R in the notation $\mathbf{D}$.)

If D is a formal étale (φ, Γ)-module with R-coefficients then we write $D_k \coloneqq D/\mathfrak{m}_R^k D$. Then D_k is a not-necessarily-projective étale (φ, Γ)-module over $R/\mathfrak{m}_R^k$, which however is projective if D is, and $D = \varprojlim_k D_k$. We extend many of the definitions that we have made in earlier parts of Section 4 to the case of formal étale (φ, Γ)-modules by passage to projective limits. Strictly speaking, in case D is not projective, so that the D_k need not be projective, the various definitions and constructions that we have made will not apply to the D_k; however, since $R/\mathfrak{m}_R^k$ is a finite $\mathcal{O}$-algebra, D_k is of finite length over $\mathbf{A}$, and we may ignore the coefficients and directly apply the definitions of [Col10a] and [Col10c]. To be precise, we make the following definition.

Definition 4.7.1. Let R be a complete Noetherian local $\mathcal{O}$-algebra with finite residue field, and let D be a formal étale (φ, Γ)-module with R-coefficients. We set:

- $D^\sharp := \varprojlim_k D_k^\sharp$ and $D^\natural := \varprojlim_k D_k^\natural$. These are ψ-stable submodules of D.
- $D \boxtimes \mathbf{P}^1 := \varprojlim_k D_k \boxtimes \mathbf{P}^1$, with its natural action of $\mathcal{O}[\![G]\!]$.
- $D^\sharp \boxtimes \mathbf{Q}_p := \varprojlim_\psi D^\sharp$ with its natural action of B. Hence $D^\sharp \boxtimes \mathbf{Q}_p \cong \varprojlim_k D_k^\sharp \boxtimes \mathbf{Q}_p$. We define $D^\natural \boxtimes \mathbf{Q}_p$ and $D \boxtimes \mathbf{Q}_p$ in a similar way as inverse limits over k, so there is a B-equivariant map
$$\mathrm{Res}_{\mathbf{Q}_p} : D \boxtimes \mathbf{P}^1 \to D \boxtimes \mathbf{Q}_p$$
defined as the inverse limit of the restriction maps for D_k.
- $D^\sharp \boxtimes \mathbf{P}^1 := \varprojlim_k D_k^\sharp \boxtimes \mathbf{P}^1$, and $D^\natural \boxtimes \mathbf{P}^1 := \varprojlim_k D_k^\natural \boxtimes \mathbf{P}^1$.
- $D^{\mathrm{nr}} := \varprojlim_k D_k^{\mathrm{nr}}$.

Each of the objects appearing in these various inverse limits has a natural topology, and so each of the objects being defined has a natural inverse limit topology. In fact, except for $D \boxtimes \mathbf{P}^1$, each of the objects appearing in the various inverse systems in these definitions is profinite, and so these inverse limit topologies are also profinite. Recall also that the formation of inverse limits of profinite R-modules is exact; we will use this fact frequently below.

Lemma 4.7.2. *Let R be a complete local Noetherian $\mathcal{O}$-algebra with finite residue field, and let D be a formal étale (φ, Γ)-module with R-coefficients. Then:*

(1) $(D^\sharp \boxtimes \mathbf{Q}_p)/(D^\natural \boxtimes \mathbf{Q}_p) = (D^\sharp/D^\natural) \boxtimes \mathbf{Q}_p = D^\sharp/D^\natural = \varprojlim_k D_k^\sharp/D_k^\natural$.

(2) $D^{\mathrm{nr}} = \bigcap_n \varphi^n(D)$.

Proof. The first claim follows from Lemma 4.4.18 and the exactness of the formation of inverse limits of profinite modules. Noting that since φ is injective we may identify $\bigcap_n \varphi^n(D)$ with $\varprojlim_\varphi D$, the second claim is clear, as limits commute with limits. $\square$

We next recall the description of D^{nr} and $D^\sharp/D^\natural$ in the context of the comparison between formal étale (φ, Γ)-modules and Galois representations. Using this, we will see in Lemma 4.7.11 below that D^{nr} and $D^\sharp/D^\natural$ are both finite R-modules.

Definition 4.7.3. We equip every finitely generated R-module with its canonical (i.e. $\mathfrak{m}_R$-adic) topology. We write $\mathrm{Rep}_R(G_{\mathbf{Q}_p})$ for the category of continuous representations of $G_{\mathbf{Q}_p}$ on finitely generated R-modules, and $\mathrm{Rep}_R^{\mathrm{ab}}(G_{\mathbf{Q}_p})$ for the subcategory of abelian representations, i.e. representations for which the action of $G_{\mathbf{Q}_p}$ factors through $G_{\mathbf{Q}_p}^{\mathrm{ab}}$. We also write $\mathrm{Rep}_R(\mathbf{Q}_p^\times)$ for the category of continuous representations of $\mathbf{Q}_p^\times$ on finitely generated R-modules.

We recall the following result, which is simply a reformulation of local class field theory for $\mathbf{Q}_p$ in our context.

Lemma 4.7.4. *Pullback along the local Artin map $\mathbf{Q}_p^\times \to G_{\mathbf{Q}_p}^{\mathrm{ab}}$ induces an equivalence of categories $\mathrm{Rep}_R^{\mathrm{ab}}(G_{\mathbf{Q}_p}) \xrightarrow{\sim} \mathrm{Rep}_R(\mathbf{Q}_p^\times)$.*

Proof. The local Artin map identifies $G_{\mathbf{Q}_p}^{\mathrm{ab}}$ with the profinite completion of $\mathbf{Q}_p^\times$. The lemma follows immediately from this, together with the fact that any continuous action of $\mathbf{Q}_p^\times$ on a finitely generated R-module extends to its profinite completion (since R is profinite). $\square$

Definition 4.7.5. If V is an object of $\mathrm{Rep}_R(G_{\mathbf{Q}_p})$, then we write V^{ab}, respectively V_{ab}, for the maximal abelian $R[G_{\mathbf{Q}_p}]$-submodule of V, respectively abelian $R[G_{\mathbf{Q}_p}]$-quotient module of V (where, as above, "abelian" has the meaning that the action of $G_{\mathbf{Q}_p}$ factors through $G_{\mathbf{Q}_p}^{\mathrm{ab}}$).

Definition 4.7.6. We let $\mathbf{D}^{\mathrm{nr}} : \mathrm{Rep}_R(G_{\mathbf{Q}_p}) \to \mathrm{Rep}_R(\mathbf{Q}_p^\times)$ be the functor given by $\mathbf{D}^{\mathrm{nr}}(V) \coloneqq \mathbf{D}(V)^{\mathrm{nr}}$ (which has an action of $\mathbf{Q}_p^\times$ by Remark 4.6.2).

Lemma 4.7.7.

(1) If $V \in \mathrm{Rep}_R(G_{\mathbf{Q}_p})$ then $\mathbf{D}^{\mathrm{nr}}(V) = \mathbf{D}^{\mathrm{nr}}(V^{\mathrm{ab}})$.

(2) The restriction of the functor $\mathbf{D}^{\mathrm{nr}}$ to $\mathrm{Rep}_R^{\mathrm{ab}}(G_{\mathbf{Q}_p})$ is naturally isomorphic to the equivalence of categories $\mathrm{Rep}_R^{\mathrm{ab}}(G_{\mathbf{Q}_p}) \xrightarrow{\sim} \mathrm{Rep}_R(\mathbf{Q}_p^\times)$ of Lemma 4.7.4.

Proof. For the first part, since $\varprojlim_k$ commutes with limits, we have $\varprojlim_k (V_k)^{\mathrm{ab}} \cong V^{\mathrm{ab}}$. Indeed, V^{ab} is naturally isomorphic to $\ker(V \xrightarrow{x_j - 1} \prod_{j \in J} V)$, where $\{x_j : j \in J\}$ is a set of generators of the commutator subgroup of $G_{\mathbf{Q}_p}$. Hence it suffices to prove that the natural map $\mathbf{D}^{\mathrm{nr}}((V_k)^{\mathrm{ab}}) \to \mathbf{D}^{\mathrm{nr}}(V_k)$ is an isomorphism. This is proved in [Col10a, Rem. II.1.2].

Turning to the second part, if we regard $\mathbf{Z}_{p^n}$ as a $\mathbf{Z}_p[\mathbf{Z}/n\mathbf{Z}]$-module (via the action of geometric Frobenius), then there is an isomorphism of $\mathbf{Z}_p[\mathbf{Z}/n\mathbf{Z}]$-modules $\mathbf{Z}_p[\mathbf{Z}/n\mathbf{Z}] \xrightarrow{\sim} \mathbf{Z}_{p^n}$. We can choose these isomorphisms compatibly in n, in the sense that, for $m \mid n$, the natural surjection $\mathbf{Z}_p[\mathbf{Z}/n\mathbf{Z}] \to \mathbf{Z}_p[\mathbf{Z}/m\mathbf{Z}]$ becomes identified with the trace map $\mathrm{Tr}^n_m : \mathbf{Z}_{p^n} \to \mathbf{Z}_{p^m}$. Equivalently, we may find elements δ_n of $\mathbf{Z}_{p^n}$ such that δ_n freely generates $\mathbf{Z}_{p^n}$ as a $\mathbf{Z}_p[\mathbf{Z}/n\mathbf{Z}]$-module, and such that $\mathrm{Tr}^n_m(\delta_n) = \delta_m$ if $m \mid n$.

Next, if V is a finite-cardinality $\mathbf{Z}_p$-module with a continuous action of $\widehat{\mathbf{Z}}$, choose n such that this action factors through $\mathbf{Z}/n\mathbf{Z}$, and define an isomorphism

$$V \xrightarrow{\sim} (V \otimes_{\mathbf{Z}_p} \mathbf{Z}_{p^n})^{\widehat{\mathbf{Z}}} = (V \otimes_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p)^{\widehat{\mathbf{Z}}} \tag{4.7.8}$$

(invariants being taken with respect to the diagonal action of $\widehat{\mathbf{Z}}$) via

$$v \mapsto \sum_{i=0}^{n-1} \langle i \rangle (v \otimes \delta_n),$$

where $\langle i \rangle$ denotes the action of the coset $i \bmod n \in \mathbf{Z}/n\mathbf{Z}$. The trace relations between the various δ_n ensure that this isomorphism is independent of the choice of n (provided that the action of $\widehat{\mathbf{Z}}$ on V factors through $\mathbf{Z}/n\mathbf{Z}$). This isomorphism intertwines the action of $\widehat{\mathbf{Z}}$ on V with the inverse of the action of $\widehat{\mathbf{Z}}$ on the second factor of $(V \otimes_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p)^{\widehat{\mathbf{Z}}}$.

Now, if V is endowed with a continuous action of $G_{\mathbf{Q}_p}^{\mathrm{ab}} \xrightarrow{\sim} \widehat{\mathbf{Q}_p^\times} \xrightarrow{\sim} p^{\widehat{\mathbf{Z}}} \times \Gamma$, where p corresponds to geometric Frobenius, and the projection to $\Gamma = \mathbf{Z}_p^\times$ is the cyclotomic character, then $\mathbf{D}^{\mathrm{nr}}(V) \xrightarrow{\sim} (V \otimes_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p)^{\mathrm{Frob}}$, with the φ-action being induced by the action of Frobenius on $\breve{\mathbf{Z}}_p$ and the Γ-action being induced by the action of Γ on V. The isomorphism (4.7.8) then induces a $\mathbf{Q}_p^\times$-equivariant isomorphism

$$V \xrightarrow{\sim} \mathbf{D}^{\mathrm{nr}}(V).$$

By construction it is natural in V, and so by passing to inverse limits, it induces a corresponding isomorphism for any finite type R-module endowed with a continuous $G_{\mathbf{Q}_p}^{\mathrm{ab}}$-representation. $\square$

Lemma 4.7.9. *There is a natural isomorphism of functors* $\mathrm{Rep}_R(G_{\mathbf{Q}_p}) \to \mathrm{Rep}_R(\mathbf{Q}_p^\times)$

$$\mathbf{D}(V)^\sharp/\mathbf{D}(V)^\natural \cong \mathbf{D}^{\mathrm{nr}}(V_{\mathrm{ab}}) \otimes \varepsilon^{-1}.$$

Proof. Since $D \mapsto D^\sharp/D^\natural$ and $D \mapsto D^{\mathrm{nr}}$ are both compatible with passage to inverse limits, it suffices to prove this when R is an Artin ring. Write $(-)^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(-, \mathbf{Q}_p/\mathbf{Z}_p)$. It is shown in [Col10a, Prop. I.2.3] that if we put $D := \mathbf{D}(V)$ and $\check{D} := \mathbf{D}(V^\vee \otimes \varepsilon)$ then there is a $\mathbf{Z}_p$-linear map

$$D \otimes_R \check{D} \to \mathbf{Q}_p/\mathbf{Z}_p$$

which identifies $\check{D}$ with the Pontrjagin dual of D (i.e. the space of continuous $\mathbf{Z}_p$-linear maps $D \to \mathbf{Q}_p/\mathbf{Z}_p$, equipped with the weak topology), and vice versa. This map is invariant under φ and Γ, and by [Col10a, Prop. II.5.19] it descends to a map

$$D^\sharp/D^\natural \otimes_R \check{D}^{\mathrm{nr}} \to \mathbf{Q}_p/\mathbf{Z}_p$$

which once again is (φ, Γ)-invariant, and puts the two factors in Pontrjagin duality (which now coincides with $\mathbf{Z}_p$-duality, since the two sides have finite $\mathbf{Z}_p$-length). By Lemma 4.7.7 we have

$$\check{D}^{\mathrm{nr}} = \mathbf{D}^{\mathrm{nr}}((V^\vee \otimes \varepsilon)^{\mathrm{ab}}) \cong \mathbf{D}^{\mathrm{nr}}((V_{\mathrm{ab}})^\vee \otimes \varepsilon),$$

so it suffices now to prove that $\mathbf{D}^{\mathrm{nr}}$ is compatible with duals and tensor products. This is standard: for example, in order to prove that the natural map

$$\mathbf{D}^{\mathrm{nr}}(V_1) \otimes_{\mathbf{Z}_p} \mathbf{D}^{\mathrm{nr}}(V_2) \to \mathbf{D}^{\mathrm{nr}}(V_1 \otimes_{\mathbf{Z}_p} V_2)$$

is an isomorphism, it suffices to extend scalars to $\breve{\mathbf{Z}}_p$, where it yields the isomorphism

$$(\breve{\mathbf{Z}}_p \otimes_{\mathbf{Z}_p} V_1) \otimes_{\breve{\mathbf{Z}}_p} (\breve{\mathbf{Z}}_p \otimes_{\mathbf{Z}_p} V_2) \xrightarrow{\sim} \breve{\mathbf{Z}}_p \otimes_{\mathbf{Z}_p} (V_1 \otimes_{\mathbf{Z}_p} V_2). \qquad \square$$

Remark 4.7.10. To illustrate the twist in Lemma 4.7.9 with an example, recall from [Col10a, Ex. II.4.5, II.5.16] that when V is the trivial character of $\mathbf{F}[G_{\mathbf{Q}_p}]$ we have

$$\mathbf{D}(V)^\sharp = T^{-1}\mathbf{F}[\![T]\!] \text{ and } \mathbf{D}(V)^\natural = \mathbf{F}[\![T]\!].$$

Then the action of Γ on T^{-1} is via

$$\gamma(T^{-1}) = ((1+T)^{\varepsilon(\gamma)} - 1)^{-1} = (\overline{\varepsilon}(\gamma)T + O(T^2))^{-1} = \overline{\varepsilon}^{-1}(\gamma)T^{-1}u(T)$$

for some $u(T) \in 1 + T\mathbf{F}[\![T]\!]$.

Lemma 4.7.11. *Let R be a complete Noetherian local $\mathcal{O}$-algebra with finite residue field, and let D be a formal étale (φ, Γ)-module with R-coefficients. Then $D^\sharp/D^\natural$ and D^{nr} are finite R-modules.*

Proof. By Lemma 4.7.9 it suffices to prove this for D^{nr}. Write $V := \mathbf{V}(D)$, so that by Lemma 4.7.7 the lemma is equivalent to the claim that $\mathbf{D}^{\mathrm{nr}}(V^{\mathrm{ab}})$ is R-finite. By definition $\mathbf{D}^{\mathrm{nr}}(V^{\mathrm{ab}})$ is $\mathfrak{m}_R$-adically complete, and since D^{nr} is exact on abelian $R[G_{\mathbf{Q}_p}]$-modules (by Lemma 4.7.7), we know that $\mathbf{D}^{\mathrm{nr}}(V^{\mathrm{ab}}) \otimes_R \mathbf{F} \cong \mathbf{D}^{\mathrm{nr}}(V^{\mathrm{ab}} \otimes_R \mathbf{F})$. By Lemma 4.3.15, the module $\mathbf{D}^{\mathrm{nr}}(V^{\mathrm{ab}} \otimes_R \mathbf{F})$ is $\mathbf{F}$-finite. This implies the required finiteness by [Stacks, Tag 031D]. $\square$

We also note the following application of the preceding results, which we will need below. If $x \in \operatorname{Spec}(R)$, we use the notation $k(x)$ to denote the residue field of the localization of R at the prime ideal corresponding to x.

Lemma 4.7.12. *Let R be a reduced $\mathcal{O}$-flat complete Noetherian local $\mathcal{O}$-algebra with finite residue field. Let V be a continuous $R[G_{\mathbf{Q}_p}]$-module which is finite free over R of rank at least 2, and such that there exists a Zariski dense set of closed points $x \in \operatorname{Spec}(R[1/p])$ such that $V \otimes_R k(x)$ is an absolutely irreducible $k(x)[G_{\mathbf{Q}_p}]$-module. Then $\mathbf{D}^{\mathrm{nr}}(V) = 0$.*

Proof. By Lemma 4.7.7 we need to prove that $V^{\mathrm{ab}} = 0$. Since V^{ab} is $\mathcal{O}$-torsion free, and $V^{\mathrm{ab}}[1/p] \subset V[1/p]^{\mathrm{ab}}$ (where $V[1/p]^{\mathrm{ab}}$ is the maximal abelian submodule of $V[1/p]$), it suffices to prove that $V[1/p]^{\mathrm{ab}} = 0$. However, $V[1/p]^{\mathrm{ab}}$ is contained in a finite free $R[1/p]$-module, hence its associated primes are minimal in $R[1/p]$, since $R[1/p]$ is reduced. To show that $V[1/p]^{\mathrm{ab}} = 0$ it thus suffices to show that it vanishes after localization at any minimal prime η of $R[1/p]$. To do so, it suffices to prove that $V[1/p] \otimes_R k(\eta)$ is an absolutely irreducible $k(\eta)[G_{\mathbf{Q}_p}]$-module. However, if this is not the case, then the properness of the Grassmannian of $V[1/p]$ over $\operatorname{Spec} R[1/p]$ shows that $V[1/p] \otimes_R k(x)$ is an absolutely reducible $k(x)[G_{\mathbf{Q}_p}]$-module for all $x \in \overline{\{\eta\}}$: in fact, the $G_{\mathbf{Q}_p}$-fixed locus is closed in the Grassmannian. This contradicts our assumption on V. $\square$

We next establish some properties of the functor $D^\sharp \boxtimes \mathbf{Q}_p$. It turns out that this functor is very well behaved in the formal context, thanks to the following result of Colmez, which shows that the formation of $D^\sharp \boxtimes \mathbf{Q}_p$ is exact on étale (φ, Γ)-modules of finite $\mathbf{A}$-length.

Theorem 4.7.13. *Let R be a complete Noetherian local $\mathcal{O}$-algebra with finite residue field. For any short exact sequence*

$$0 \to D_1 \to D_2 \to D_3 \to 0$$

of (not necessarily projective, formal) étale (φ, Γ)-modules with R-coefficients, each of which is of finite length as an $\mathbf{A}$-module, the sequence

$$0 \to D_1^\sharp \boxtimes \mathbf{Q}_p \to D_2^\sharp \boxtimes \mathbf{Q}_p \to D_3^\sharp \boxtimes \mathbf{Q}_p \to 0$$

is also short exact.

Proof. This is immediate from [Col10a, Thm. III.3.5]. $\square$

Corollary 4.7.14. *Let R be a complete Noetherian local $\mathcal{O}$-algebra with finite residue field. Then $D \mapsto D^\sharp \boxtimes \mathbf{Q}_p$ is a right exact functor on the category of formal étale (φ, Γ)-modules with R-coefficients.*

Proof. Given an exact sequence $0 \to D_1 \to D \to D_2 \to 0$ of formal étale (φ, Γ)-modules with R-coefficients, the sequence $(D_1)_k \to D_k \to (D_2)_k \to 0$ is exact for all $k \geq 1$, and its terms have finite $\mathbf{A}$-length. By Theorem 4.7.13, it stays exact after applying $(-)^\sharp \boxtimes \mathbf{Q}_p$, and then the corollary follows from the exactness of $\varprojlim_k$ on profinite R-modules. $\square$

Before formulating our next result, we note that if D is a formal étale (φ, Γ)-module over R, and if M is a finitely generated R-module, then $D \otimes_R M$ (which agrees with $D\,\widehat{\otimes}_{\widehat{\mathbf{A}}_R}(\widehat{\mathbf{A}}_R \otimes_R M)$; see Lemma A.1.41 and Remark A.1.42) is again a formal étale (φ,Γ)-module, and

$$(D \otimes_R M)_k = D_k \otimes_R M = D_k \otimes_{R/\mathfrak{m}_R^k} (M/\mathfrak{m}_R^k M)$$

is a not necessarily projective étale (φ,Γ)-module over $R/\mathfrak{m}_R^k$, for each $k \geq 1$.

Lemma 4.7.15. *Let R be a complete Noetherian local $\mathcal{O}$-algebra with finite residue field, and let D be a formal étale (φ,Γ)-module over R.*

(1) If M is a finitely generated R-module, then there is a natural isomorphism

$$(D^\sharp \boxtimes \mathbf{Q}_p) \otimes_R M \xrightarrow{\sim} (D \otimes_R M)^\sharp \boxtimes \mathbf{Q}_p.$$

(2) If D is furthermore projective, then $D^\sharp \boxtimes \mathbf{Q}_p$ is topologically flat over R (and so in particular flat over R).

Proof. For the first part, by Proposition A.1.48 (1) it suffices to show that the functor $M \mapsto (D \otimes_R M)^\sharp \boxtimes \mathbf{Q}_p$ is right exact (on the category of finitely generated R-modules). This follows from the right exactness of $M \mapsto D \otimes_R M$ and the right exactness of $D \mapsto D^\sharp \boxtimes \mathbf{Q}_p$ (Corollary 4.7.14).

Now suppose that $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of profinite R-modules. Write this as the inverse limit of a short exact sequences of finite-cardinality R-modules $0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0$. Since D is projective, it is R-flat, and so each of the sequences

$$0 \to D \otimes_R M_{1,i} \to D \otimes_R M_{2,i} \to D \otimes_R M_{3,i} \to 0$$

is again short exact. By Theorem 4.7.13, we find that

$$0 \to (D \otimes_R M_{1,i})^\sharp \boxtimes \mathbf{Q}_p \to (D \otimes_R M_{2,i})^\sharp \boxtimes \mathbf{Q}_p \to (D \otimes_R M_{3,i})^\sharp \boxtimes \mathbf{Q}_p \to 0$$

is short exact. Applying the isomorphism of (1) (noting that it does indeed apply, since the finite R-modules $M_{j,i}$ are in particular finitely generated), we find that the sequence

$$0 \to (D^\sharp \boxtimes \mathbf{Q}_p) \otimes_R M_{1,i} \to (D^\sharp \boxtimes \mathbf{Q}_p) \otimes_R M_{2,i} \to (D^\sharp \boxtimes \mathbf{Q}_p) \otimes_R M_{3,i} \to 0$$

is short exact. Passing to the inverse limit over i, and recalling that the formation of inverse limits of profinite modules is exact, we find that the sequence

$$0 \to (D^\sharp \boxtimes \mathbf{Q}_p)\,\widehat{\otimes}_R M_1 \to (D^\sharp \boxtimes \mathbf{Q}_p)\,\widehat{\otimes}_R M_2 \to (D^\sharp \boxtimes \mathbf{Q}_p)\,\widehat{\otimes}_R M_3 \to 0$$

is exact. Bearing in mind Lemma A.1.40, this proves (2). ∎

Lemma 4.7.16. *Let $R \to S$ be a morphism of complete local Noetherian $\mathcal{O}$-algebras with finite residue fields, let D be a formal étale (φ,Γ)-module over R, and write $D_S \coloneqq D\,\widehat{\otimes}_R S$. Then the natural map*

$$(D^\natural \boxtimes \mathbf{Q}_p)\,\widehat{\otimes}_R S \to D_S^\natural \boxtimes \mathbf{Q}_p$$

is surjective.

Proof. Suppose to begin with that R and S are both Artinian. It follows from Lemma 4.2.8 that the morphism $D^\natural \otimes_R S \to D_S^\natural$ is surjective. (Since S, being Artinian, is finite over R, the tensor product is automatically complete. Also, although the cited Lemma assumes projectivity of D, the claimed surjectivity follows in general from [Col10a, Prop. II.5.17].) Passing to the inverse limit over ψ, and taking into account (as always) that these inverse limits of profinite $\mathcal{O}$-modules are exact, we find that

$$\varprojlim_{\psi}(D^\natural \otimes_R S) \to D^\natural_S \boxtimes \mathbf{Q}_p$$

is surjective. Now the natural morphism

$$(D^\natural \boxtimes \mathbf{Q}_p)\,\widehat{\otimes}_R S \to \varprojlim_{\psi}(D^\natural \otimes_R S)$$

evidently has dense image, and hence is surjective (being a continuous map of profinite $\mathcal{O}$-modules). This completes the proof of the lemma in the Artinian case.

The general case follows by writing the morphism $R \to S$ as the inverse limit of the morphisms $R/\mathfrak{m}_R^k \to S/\mathfrak{m}_S^k$, and then passing to the inverse limit from the Artinian case (yet again using the fact the formation of inverse limits of profinite $\mathcal{O}$-modules is exact). $\square$

4.8. **Base-change and the action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $D^\natural \boxtimes \mathbf{P}^1$.** We now return to the setting of a morphism $A \to B$ of Noetherian $\mathcal{O}/\varpi^a$-algebras, and a projective étale (φ, Γ)-module with A-coefficients D_A. As in Section 4.1, we write $D_B \coloneqq D_A\,\widehat{\otimes}_A B$; then D_B is naturally an étale (φ, Γ)-module with B-coefficients.

Remark 4.8.1. Lemma 4.5.5 gives a topological isomorphism of Tate A-modules

$$D_A \boxtimes \mathbf{P}^1 \xrightarrow{\sim} D_A \oplus (D_A \boxtimes p\mathbf{Z}_p).$$

Now D_A is a finite projective $\mathbf{A}_A$-module, while $D_A \boxtimes p\mathbf{Z}_p = \varphi(D_A)$ is a finite projective $\varphi(\mathbf{A}_A)$-module. Since φ induces an isomorphism of $\mathbf{A}_A$ onto its image, we see that $D_A \boxtimes p\mathbf{Z}_p$ is isomorphic (as a Tate A-module) to a finite projective $\mathbf{A}_A$-module. Hence $D_A \boxtimes \mathbf{P}^1$ itself is isomorphic to a finite projective $\mathbf{A}_A$-module, and hence Example 4.1.19 and Remark 4.1.21 apply to $D_A \boxtimes \mathbf{P}^1$.

Lemma 4.8.2. *Let $A \to B$ be a morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras for some $a \geq 1$. Let D_A be an étale (φ, Γ)-module with A-coefficients. Then there is a natural isomorphism*

$$(D_A \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B \xrightarrow{\sim} D_B \boxtimes \mathbf{P}^1,$$

which induces a morphism

$$(D_A^\natural \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B \to D_B^\natural \boxtimes \mathbf{P}^1.$$

Proof. The isomorphism $D_A^{\oplus 2}\,\widehat{\otimes}_A B \xrightarrow{\sim} D_B^{\oplus 2}$ induces a morphism $(D_A \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B \to D_B \boxtimes \mathbf{P}^1$. To show that this is an isomorphism, consider the following commutative diagram, where the vertical maps are the homeomorphisms $z \mapsto (\mathrm{Res}_{\mathbf{Z}_p} z, \mathrm{Res}_{p\mathbf{Z}_p}(w \cdot z))$ of Lemma 4.5.5:

$$\begin{array}{ccc} (D_A \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B & \longrightarrow & D_B \boxtimes \mathbf{P}^1 \\ \downarrow & & \downarrow \\ (D_A \boxtimes \mathbf{Z}_p)\,\widehat{\otimes}_A B \oplus (D_A \boxtimes p\mathbf{Z}_p)\,\widehat{\otimes}_A B & \longrightarrow & (D_B \boxtimes \mathbf{Z}_p) \oplus (D_B \boxtimes p\mathbf{Z}_p) \end{array}$$

Now $(D_A \boxtimes \mathbf{Z}_p)\,\widehat{\otimes}_A B \coloneqq D_A\,\widehat{\otimes}_A B$ maps isomorphically to $D_B \eqqcolon D_B \boxtimes \mathbf{Z}_p$ by the very definition of D_B, and so it suffices to show that also

$$D_B \boxtimes p\mathbf{Z}_p = (D_A \boxtimes p\mathbf{Z}_p)\,\widehat{\otimes}_A B,$$

i.e. that

$$\varphi\psi(D_B) = (\varphi\psi(D_A))\,\widehat{\otimes}_A B.$$

Since $\varphi\psi$ is the projection onto a direct summand as a $\varphi(\mathbf{A}_A)$-module, we are done.

To see that this induces a morphism $(D_A^\natural \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B \to D_B^\natural \boxtimes \mathbf{P}^1$ as claimed, it suffices (by the definition of $D^\natural$) to consider the commutativity of the diagram

$$\begin{tikzcd}
(D_A \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B \arrow[r, "\sim"] \arrow[d] & D_B \boxtimes \mathbf{P}^1 \arrow[d] \\
(D_A \boxtimes \mathbf{Q}_p)\,\widehat{\otimes}_A B \arrow[r] & D_B \boxtimes \mathbf{Q}_p
\end{tikzcd}$$

induced by $\mathrm{Res}_{\mathbf{Q}_p}$, together with the fact that, by Lemma 4.2.8 (2), the bottom horizontal arrow induces a morphism $(D_A^\natural \boxtimes \mathbf{Q}_p)\,\widehat{\otimes}_A B \to D_B^\natural \boxtimes \mathbf{Q}_p$. $\square$

Lemma 4.8.3. *Let $A \to B$ be a morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras for some $a \geq 1$, and let D_A be an étale (φ, Γ)-module with A-coefficients.*

1. *If $\mathfrak{M}_A \subseteq D_A \boxtimes \mathbf{P}^1$ is a lattice then the image of the natural morphism $\mathfrak{M}_A\,\widehat{\otimes}_A B \to (D_A \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B = D_B \boxtimes \mathbf{P}^1$ is a lattice in $D_B \boxtimes \mathbf{P}^1$. In particular if $A \to B$ is flat then $\mathfrak{M}_A\,\widehat{\otimes}_A B$ is a lattice in $D_B \boxtimes \mathbf{P}^1$.*
2. *If $A \hookrightarrow B$ is injective, and $\mathfrak{M}$ is a lattice in $D_B \boxtimes \mathbf{P}^1$, then $\mathfrak{M} \cap (D_A \boxtimes \mathbf{P}^1)$ is a lattice in $D_A \boxtimes \mathbf{P}^1$.*

Proof. This follows from Lemma 4.8.2 together with parts (4) and (5) of Lemma 4.1.14 (taking into account Remark 4.8.1). $\square$

Lemma 4.8.4. *Let $A \to B$ be a morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras for some $a \geq 1$. Let D_A be an étale (φ, Γ)-module with A-coefficients, and let $D_B = D_A\,\widehat{\otimes}_A B$.*

1. *If $A \hookrightarrow B$ is injective, and if $D_B^\natural \boxtimes \mathbf{P}^1$ is an $\mathcal{O}[\![G]\!]$-stable lattice in $D_B \boxtimes \mathbf{P}^1$, then $D_A^\natural \boxtimes \mathbf{P}^1$ is an $\mathcal{O}[\![G]\!]$-stable lattice in $D_A \boxtimes \mathbf{P}^1$.*
2. *If $D_A^\natural \boxtimes \mathbf{P}^1$ is an $\mathcal{O}[\![G]\!]$-stable lattice in $D_A \boxtimes \mathbf{P}^1$, then $D_B^\natural \boxtimes \mathbf{P}^1$ is a $\mathcal{O}[\![G]\!]$-stable lattice in $D_B \boxtimes \mathbf{P}^1$.*
3. *In the situation of (2), if $A \to B$ is furthermore flat, then $(D_A^\natural \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B$ maps isomorphically to $D_B^\natural \boxtimes \mathbf{P}^1$.*

Proof. We begin with (1). By Proposition 4.6.7, $D_A^\natural \boxtimes \mathbf{P}^1$ is a $\mathcal{O}[\![G]\!]$-stable lattice if and only if $D_A^\natural \boxtimes \mathbf{P}^1$ is a lattice if and only if there exists some $B(\mathbf{Q}_p)$-stable lattice in $D_A \boxtimes \mathbf{P}^1$. Now, $(D_B^\natural \boxtimes \mathbf{P}^1) \cap (D_A \boxtimes \mathbf{P}^1)$ is a lattice in $D_A \boxtimes \mathbf{P}^1$ by Lemma 4.8.3 (2), and it is $B(\mathbf{Q}_p)$-stable because both $D_B^\natural \boxtimes \mathbf{P}^1$ and $D_A \boxtimes \mathbf{P}^1$ are $\mathcal{O}[\![G]\!]$-stable, so we are done.

We now turn to (2). By Lemma 4.8.3 (1), the image $\mathfrak{M}_B$ of the morphism

$$(D_A^\natural \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B \to D_B \boxtimes \mathbf{P}^1$$

is a lattice in $D_B \boxtimes \mathbf{P}^1$. It is $\mathcal{O}[\![G]\!]$-stable by construction, so it follows from Proposition 4.6.7 that $D_B^\natural \boxtimes \mathbf{P}^1$ is a $\mathcal{O}[\![G]\!]$-stable lattice.

Finally, let us put ourselves in the situation of (3), so that $A \to B$ is furthermore assumed to be flat. Then Lemma 4.8.3 (1) shows that $(D_A^\natural \boxtimes \mathbf{P}^1)\,\widehat{\otimes}_A B$ maps isomorphically onto its image in $D_B \boxtimes \mathbf{P}^1$, i.e. onto $\mathfrak{M}_B$, while Lemma 4.8.2 shows that $\mathfrak{M}_B \subseteq D_B^\natural \boxtimes \mathbf{P}^1$. Combining this last fact with the conclusion of Lemma 4.6.5, we find that $\mathrm{Res}_{\mathbf{Q}_p}(\mathfrak{M}_B) = D_B^\natural \boxtimes \mathbf{Q}_p$. Thus to show that $\mathfrak{M}_B = D_B^\natural \boxtimes \mathbf{P}^1$, it suffices by Lemma 4.6.3 to show that $\ker \mathrm{Res}_{\mathbf{Q}_p} = wD_B^{\mathrm{nr}}$ is contained in $\mathfrak{M}_B$. Since $\mathfrak{M}_B$ is G-invariant, it is equivalent to show that $D_B^{\mathrm{nr}} \subseteq \mathfrak{M}_B$. Since $D_B^{\mathrm{nr}} \subseteq D_B^+$, it suffices in turn to show that D_B^+ (regarded as a submodule of $D_B = D_B \boxtimes \mathbf{Z}_p$ and thus of $D_B \boxtimes \mathbf{P}^1$) is contained in $\mathfrak{M}_B$.

Now, by Lemma 4.5.19, we have an inclusion $D_A^+ = D_A^+ \boxtimes \mathbf{Z}_p \subseteq D_A^\natural \boxtimes \mathbf{P}^1$, which induces a morphism $(D_A^+)\widehat{\otimes}_A B \to \mathfrak{M}_B$. By Lemma 4.3.12, this morphism is an injection whose image is identified with D_B^+ when we identify $\mathfrak{M}_B$ with its image in $D_B^\natural \boxtimes \mathbf{P}^1$. This completes the proof. $\square$

Lemma 4.8.5. *Let $(A, \mathfrak{m})$ be a complete local Noetherian $\mathcal{O}/\varpi^a$-algebra, and suppose that D_A is an étale (φ, Γ)-module with A-coefficients. Suppose that for each $n \geq 1$, $(D_{A/\mathfrak{m}^n})^\natural \boxtimes \mathbf{P}^1$ is an $\mathcal{O}[\![G]\!]$-stable lattice in $D_{A/\mathfrak{m}^n} \boxtimes \mathbf{P}^1$. Then $D_A^\natural \boxtimes \mathbf{P}^1$ is an $\mathcal{O}[\![G]\!]$-stable lattice in $D_A \boxtimes \mathbf{P}^1$.*

Proof. By Lemma 4.6.8, it is enough to prove that $D_A^\natural \boxtimes \mathbf{P}^1$ is $\mathcal{O}[\![G]\!]$-stable; so by definition we need to prove that if $f \in \mathcal{O}[\![G]\!]$ and if $x \in D_A \boxtimes \mathbf{P}^1$ satisfies $\mathrm{Res}_{\mathbf{Z}_p}\left(\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)x\right) \in D_A^\natural$ for all $n \geq 0$, then $\mathrm{Res}_{\mathbf{Z}_p}\left(\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)fx\right) \in D_A^\natural$ for all $n \geq 0$.

Now, for each $m \geq 1$ the assumption that $(D_{A/\mathfrak{m}^m})^\natural \boxtimes \mathbf{P}^1$ is $\mathcal{O}[\![G]\!]$-stable means that the image of $\mathrm{Res}_{\mathbf{Z}_p}\left(\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)fx\right)$ in $D_{A/\mathfrak{m}^m}$ is contained in $D^\natural_{A/\mathfrak{m}^m}$. By Lemma 4.2.8 (3), this means that for each $n \geq 0$, $\mathrm{Res}_{\mathbf{Z}_p}\left(\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)fx\right)$ is contained in $D_A^\natural + \mathfrak{m}^m D_A$. Since this holds for all m, the lemma follows by an application of Lemma 4.1.20. $\square$

We now specialize to the case that D has rank 2.

Theorem 4.8.6. *Let A be a complete local Noetherian $\mathcal{O}/\varpi^a$-algebra with finite residue field, and suppose that D_A is an étale (φ, Γ)-module of rank 2 with A-coefficients and determinant $\zeta\varepsilon^{-1}$. Then $D_A^\natural \boxtimes \mathbf{P}^1$ is a $\mathcal{O}[\![G]\!]$-stable lattice in $D_A \boxtimes \mathbf{P}^1$.*

Proof. By Lemma 4.8.5, we may assume (by replacing A by $A/\mathfrak{m}^n$ where $\mathfrak{m}$ is the maximal ideal of A) that A is Artinian, in which case D_A corresponds to a 2-dimensional representation $\rho_A : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(A)$. By Lemma 4.8.4 (2), we can assume that A is a quotient of the universal (fixed determinant) framed deformation ring S for $\rho_A \pmod{\mathfrak{m}}$.

As explained in [Col10c, §II.3.4], there is a universal formal étale (φ, Γ) module D over S, and it is proved there that $D^\natural \boxtimes \mathbf{P}^1$ is G-stable; indeed, by [Col10c, Lem. II.3.6], this follows from the results of Berger–Breuil [BB10] together with the Zariski density of the so-called crystalline benign points in the generic fibre $\operatorname{Spec} S[1/p]$, which footnote 7 of [Col10c] explains is known for $p \geq 3$. (In fact, Zariski density statements of this kind are now known in much greater generality; see [BIP23a, Thm. 6.1, Rem. 6.2] and [BIP23b, Cor. 1.3].) The first paragraph of the proof of [Col10c, Prop. II.2.15] then shows that $D_A^\natural \boxtimes \mathbf{P}^1$ is G-stable, and hence also $\mathcal{O}[\![G]\!]$-stable, for any Artinian quotient A of S, as required. $\square$

Before stating the next main result of this section, we note the following (presumably standard) lemma.

Lemma 4.8.7. *If A is a Noetherian ring, then for some finite set $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ of maximal ideals of A, the natural map $A \to \prod_{i=1}^r \widehat{A}_{\mathfrak{m}_i}$ is injective.*

Proof. As in the proof of Lemma 4.3.5, we may assume A is non-zero, and then choose an element $f \in A$ and a positive integer n so that $A \to A_f \times A/f^n$ is injective, and so that A_f contains a unique associated prime $\mathfrak{p}$. We then see that $A_f \to A_{\mathfrak{m}_1}$, and hence also $A_f \to \widehat{A}_{\mathfrak{m}_1}$, is injective, for any maximal ideal $\mathfrak{m}_1 \supset \mathfrak{p}$. Note that maximal ideals $\mathfrak{n}$ of A/f^n correspond to maximal ideals of A containing f, and that for any such $\mathfrak{n}$, the composite $A \to A/f^n \to (A/f^n)_{\mathfrak{n}} \to \widehat{(A/f^n)}_{\mathfrak{n}}$ factors through the natural map $A \to \widehat{A}_{\mathfrak{n}}$. Thus we may continue by Noetherian induction and establish the lemma. □

Theorem 4.8.8. *Suppose that $p \geq 3$. Assume that A is a Noetherian $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and that D is a projective étale (φ, Γ)-module of rank 2 with A-coefficients and determinant $\zeta\varepsilon^{-1}$. Then $D^\natural \boxtimes \mathbf{P}^1$ is a $\mathcal{O}[\![G]\!]$-stable lattice in $D \boxtimes \mathbf{P}^1$.*

Proof. Since $\mathcal{X}$ is of finite presentation, the morphism $\operatorname{Spec} A \to \mathcal{X}$ that classifies D may be factored as $\operatorname{Spec} A \to \operatorname{Spec} A_0 \to \mathcal{X}$ for some finite type $\mathcal{O}/\varpi^a$-subalgebra A_0 of A. The morphism $\operatorname{Spec} A_0 \to \mathcal{X}$ corresponds to a rank 2 étale (φ, Γ)-module D_{A_0} with A_0-coefficients, such that $D \xrightarrow{\sim} D_{A_0} \widehat{\otimes}_{A_0} A$. By Lemma 4.8.4 (2), it then suffices to show the result for D_{A_0}; replacing A by A_0, we may thus suppose that A is in fact of finite type over $\mathcal{O}/\varpi^a$.

By Lemma 4.8.7 and Lemma 4.8.4 (1), we can further reduce to the case that A is a finite product of complete local Noetherian $\mathcal{O}/\varpi^a$-algebras with finite residue fields. This immediately reduces to the case of a single such complete local Noetherian $\mathcal{O}/\varpi^a$-algebra, which is Theorem 4.8.6. □

Remark 4.8.9. Note in particular that as a consequence of Lemmas 4.6.3 and 4.6.5, and Theorem 4.8.8, we have a short exact sequence of B-representations

$$0 \to \ker \operatorname{Res}_{\mathbf{Q}_p} = wD^{\mathrm{nr}} \to D^\natural \boxtimes \mathbf{P}^1 \to D^\natural \boxtimes \mathbf{Q}_p \to 0. \tag{4.8.10}$$

Lemma 4.1.18 (and Remark 4.8.1) show that the induced topology on wD^{nr} is discrete.

Corollary 4.8.11. *Let R be a complete local Noetherian $\mathcal{O}$-algebra with finite residue field, and let D be a free rank 2 formal étale (φ, Γ)-module with R-coefficients and determinant $\zeta\varepsilon^{-1}$. Then $D^\natural \boxtimes \mathbf{P}^1$ is $\mathcal{O}[\![G]\!]$-stable, and we have a short exact sequence of B-representations*

$$0 \to \ker \operatorname{Res}_{\mathbf{Q}_p} = wD^{\mathrm{nr}} \to D^\natural \boxtimes \mathbf{P}^1 \to D^\natural \boxtimes \mathbf{Q}_p \to 0. \tag{4.8.12}$$

Proof. Using the exactness of projective limits of profinite sets, this is immediate from Theorem 4.8.8 and Remark 4.8.9. □

Recall from Definition 2.2.5 and Lemma 2.2.7 that if A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, the Iwasawa algebra $A[\![K]\!]$ is a topological A-algebra (where A has the discrete topology), whose topology agrees with the $\mathfrak{a}$-adic topology, where $\mathfrak{a}$ denotes the ideal in $A[\![K]\!]$ generated by the augmentation ideal of $A[\![K_1]\!]$.

Corollary 4.8.13. *Suppose that $p \geq 3$. Assume that A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and that D is an étale (φ, Γ)-module of rank 2 with A-coefficients and determinant $\zeta\varepsilon^{-1}$. Then $D^\natural \boxtimes \mathbf{P}^1$ is a finitely generated $A[\![K]\!]$-module. Furthermore, the topology on $D^\natural \boxtimes \mathbf{P}^1$ coincides with its $\mathfrak{a}$-adic topology.*

Proof. Consider the $A[\![K]\!]$-submodule $A[\![K]\!]D^+ \subseteq D^\natural \boxtimes \mathbf{P}^1$. This is a lattice (since it contains D^+ and wD^+), and it is finitely generated over $A[\![K]\!]$, because D^+ is finitely generated over $\mathbf{A}_A^+ = A[\![T]\!]$ (being an $\mathbf{A}_A^+$-lattice in D) and thus over $A[\![U(\mathbf{Z}_p)]\!]$.

The quotient $(D^\natural \boxtimes \mathbf{P}^1)/(A[\![K]\!]D^+)$ is finitely generated over A and thus over $A[\![K]\!]$, so that $D^\natural \boxtimes \mathbf{P}^1$ is indeed finitely generated over $A[\![K]\!]$. Now Corollary 4.5.16 shows that $D^\natural \boxtimes \mathbf{P}^1$ is in fact a topological $A[\![K]\!]$-module, when endowed with its natural topology as a lattice in an A-Tate module. Since this topology is complete and first countable (and so completely metrizable), the general theory of finitely generated modules over Polish topological rings (cf. [EG23, Prop. C.6] as well as Remark 2.2.10) shows that the topology on $D^\natural \boxtimes \mathbf{P}^1$ is its canonical topology as a finitely generated $A[\![K]\!]$-module, which in turn is precisely its $\mathfrak{a}$-adic topology (as was noted in Remark 2.2.10). ☐

4.9. **Non-flat base-change.** Let $A \to B$ be a morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras, and suppose given a projective rank two étale (φ, Γ)-module D_A over $\mathbf{A}_A$ with determinant $\zeta\varepsilon^{-1}$. As usual, we then write

$$D_B \coloneqq D_A \otimes_{\mathbf{A}_A} \mathbf{A}_B = D_A \widehat{\otimes}_A B.$$

Then by Lemma 4.8.2 we have a natural base change morphism

$$(D_A^\natural \boxtimes \mathbf{P}^1)\, \widehat{\otimes}_A B \to D_B^\natural \boxtimes \mathbf{P}^1. \tag{4.9.1}$$

Lemma 4.9.2. *The morphism* (4.9.1) *is an open mapping, whose image is a lattice in $D_B \boxtimes \mathbf{P}^1$. Its kernel and cokernel are both finite over B, and are both discrete* (*with respect to their induced and quotient topologies, respectively*).

Proof. By Theorem 4.8.8, the source and target of (4.9.1) are lattices in $D_A \boxtimes \mathbf{P}^1$, resp. $D_B \boxtimes \mathbf{P}^1$. By Lemma 4.1.14 (4) (together with Remark 4.8.1), the morphism (4.9.1) is thus an open mapping, whose kernel is finite over B and discrete, and whose image is a lattice in $D_B \boxtimes \mathbf{P}^1$. Thus the cokernel of (4.9.1) is a quotient of lattices, and hence is discrete and B-finite. ☐

Our main focus in this section is on base-change results for non-flat morphisms; in the flat case, we have the following result.

Lemma 4.9.3. *If $A \to B$ is a flat morphism of Noetherian $\mathcal{O}/\varpi^a$-algebras, and D_A is an étale (φ, Γ)-module of rank 2 with A-coefficients and determinant $\zeta\varepsilon^{-1}$, then $(D_A^\natural \boxtimes \mathbf{P}^1)\widehat{\otimes}_A B = D_B^\natural \boxtimes \mathbf{P}^1$.*

Proof. This follows from Lemma 4.8.4 (3) and Theorem 4.8.8. ☐

4.9.4. *Base-change in the formal context.* Let $R \to S$ be a finite morphism of complete Noetherian local $\mathcal{O}$-algebras, with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively, each having finite residue field. Let D_R denote a projective rank 2 formal étale (φ, Γ)-module with R-coefficients and with determinant $\zeta\varepsilon^{-1}$, and set $D_S \coloneqq D_R \widehat{\otimes}_R S = D_R \otimes_R S$. Taking inverse limits of the morphisms (4.9.1) for $A = R/\mathfrak{m}^n$, $B = S/\mathfrak{n}^n$, and using Lemma 4.1.16 to replace a completed tensor product by a tensor product, we obtain a natural base-change morphism

$$(D_R^\natural \boxtimes \mathbf{P}^1) \otimes_R S \to D_S^\natural \boxtimes \mathbf{P}^1. \tag{4.9.5}$$

Lemma 4.9.6. *The kernel $\mathcal{K}$ and cokernel $\mathcal{C}$ of* (4.9.5) *are finite S-modules.*

Proof. Consider the short exact sequence

$$0 \to D_R^\natural \boxtimes \mathbf{Q}_p \to D_R^\sharp \boxtimes \mathbf{Q}_p \to D_R^\sharp / D_R^\natural \to 0$$

arising from Lemma 4.7.2. Taking the tensor product of this short exact sequence with S over R, and taking into account Lemma 4.7.15 (2), we obtain an exact sequence of B-representations

$$0 \to \mathrm{Tor}_1^R(D_R^\sharp/D_R^\natural, S) \to (D_R^\natural \boxtimes \mathbf{Q}_p) \otimes_R S \to (D_R^\sharp \boxtimes \mathbf{Q}_p) \otimes_R S \to (D_R^\sharp/D_R^\natural) \otimes_R S \to 0.$$

Lemma 4.7.15 (1) shows that $(D_R^\sharp \boxtimes \mathbf{Q}_p) \otimes_R S \xrightarrow{\sim} D_S^\sharp \boxtimes \mathbf{Q}_p$, while Lemma 4.7.16 shows that $(D_R^\natural \boxtimes \mathbf{Q}_p) \otimes_R S \to D_S^\natural \boxtimes \mathbf{Q}_p$ is surjective. Thus we can rewrite the preceding exact sequence in the form

$$0 \to \mathrm{Tor}_1^R(D_R^\sharp/D_R^\natural, S) \to (D_R^\natural \boxtimes \mathbf{Q}_p) \otimes_R S \to D_S^\natural \boxtimes \mathbf{Q}_p \to 0. \tag{4.9.7}$$

Next, the exact sequences of (4.8.12) give rise to a diagram of B-representations

$$\begin{tikzcd}
 & (w \cdot D_R^{\mathrm{nr}}) \otimes_R S \arrow[r] \arrow[d] & (D_R^\natural \boxtimes \mathbf{P}^1) \otimes_R S \arrow[r] \arrow[d, "(4.9.5)"] & (D_R^\natural \boxtimes \mathbf{Q}_p) \otimes_R S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & w \cdot D_S^{\mathrm{nr}} \arrow[r] & D_S^\natural \boxtimes \mathbf{P}^1 \arrow[r] & D_S^\natural \boxtimes \mathbf{Q}_p \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma, we obtain an exact sequence of B-representations

$$\begin{aligned} \ker\big((w \cdot D_R^{\mathrm{nr}}) \otimes_R S \to w \cdot D_S^{\mathrm{nr}}\big) \to \mathcal{K} \to \mathrm{Tor}_1^R(D_R^\sharp/D_R^\natural, S) \\ \to \mathrm{coker}\big((w \cdot D_R^{\mathrm{nr}}) \otimes_R S \to w \cdot D_S^{\mathrm{nr}}\big) \to \mathcal{C} \to 0. \end{aligned} \tag{4.9.8}$$

By Lemma 4.7.11, D_R^{nr} and $D_R^\sharp/D_R^\natural$ are finite R-modules, and D_S^{nr} is a finite S-module. It follows from (4.9.8) that $\mathcal{K}$ and $\mathcal{C}$ are finite S-modules, as required. ☐

We can now exploit the correspondence between formal étale (φ, Γ)-modules and Galois representations to say something more precise about $\mathcal{K}$ and $\mathcal{C}$. Write $V := \mathbf{V}(D_R)$, so that $D_R \xrightarrow{\sim} \mathbf{D}(V)$. Lemma 4.7.7 shows that

$$D_R^{\mathrm{nr}} \xrightarrow{\sim} \mathbf{D}^{\mathrm{nr}}(V) \xrightarrow{\sim} V^{\mathrm{ab}}$$

as R-linear $\mathbf{Q}_p^\times$-representations, while Lemma 4.7.9 shows that

$$D_R^\sharp/D_R^\natural \xrightarrow{\sim} \mathbf{D}^\sharp(V)/\mathbf{D}^\natural(V) \xrightarrow{\sim} V_{\mathrm{ab}} \otimes \varepsilon^{-1}$$

as R-linear $\mathbf{Q}_p^\times$-representations. Of course, there are analogous isomorphisms for $V_S := V \otimes_R S = \mathbf{V}(D_S)$.

Since $\mathcal{K}$ and $\mathcal{C}$ are finite over S, Lemma 3.6.9 and Remark 4.5.3 show that G acts on each of them through a direct sum of $\beta \circ \det$-eigenspaces, where β runs over the characters $\beta : \mathbf{Q}_p^\times \to \mathcal{O}^\times$ with $\beta^2 = \zeta\varepsilon^{-2}$. We can then read off β as the restriction of the G-action to the image of the embedding (4.6.1); in other words, the G-actions on $\mathcal{K}$ and $\mathcal{C}$ are entirely determined by their restriction to this image, i.e. by their structure as $\mathbf{Q}_p^\times$-modules.

We can then rewrite the exact sequences of B-representations (4.9.7) and (4.9.8) in the form

$$0 \to \mathrm{Tor}_1^R(V_{\mathrm{ab}} \otimes \varepsilon^{-1}, S) \to (D_R^\natural \boxtimes \mathbf{Q}_p) \otimes_R S \to D_S^\natural \boxtimes \mathbf{Q}_p \to 0 \tag{4.9.9}$$

and

$$\begin{aligned} \ker\big((w \cdot V^{\mathrm{ab}}) \otimes_R S \to w \cdot (V \otimes_R S)^{\mathrm{ab}}\big) \to \mathcal{K} \to \mathrm{Tor}_1^R(V_{\mathrm{ab}} \otimes \varepsilon^{-1}, S) \\ \to \mathrm{coker}\big((w \cdot V^{\mathrm{ab}}) \otimes_R S \to w \cdot (V \otimes_R S)^{\mathrm{ab}}\big) \to \mathcal{C} \to 0. \end{aligned} \tag{4.9.10}$$

As an example of how the preceding results can be applied, we establish the following result, which greatly constrains the circumstances in which $\mathcal{K}$ and $\mathcal{C}$ can be non-zero.

Lemma 4.9.11. *Write $\overline{V} := V/\mathfrak{m}$, a continuous two dimensional representation of $G_{\mathbf{Q}_p}$ over the residue field $k := R/\mathfrak{m}$ (which is a finite extension of $\mathbf{F}$). Then the base-change morphism* (4.9.5) *is an isomorphism unless $\overline{V}^{\mathrm{ss}} \cong \chi \oplus \chi\omega$ for some continuous character $\chi : G_{\mathbf{Q}_p} \to k^\times$.*

Proof. As already noted, each of $\mathcal{K}$ and $\mathcal{C}$ is a finite S-module equipped with a continuous G-action, and admitting $\zeta\varepsilon^{-1}$ as a central character, so that, by Lemma 3.6.9, G acts on each of them through a direct sum of $\beta \circ \det$-eigenspaces, where $\beta : \mathbf{Q}_p^\times \to \mathcal{O}^\times$ and $\beta^2 = \zeta\varepsilon^{-2}$.

Considering (4.9.9) and (4.9.10), we see that any such character β must then contribute to one of $V^{\mathrm{ab}} \otimes_R S$, $(V_S)^{\mathrm{ab}}$, or $\mathrm{Tor}_1^R(V_{\mathrm{ab}} \otimes \varepsilon^{-1}, S)$. Thus the reduction $\overline{\beta} : \mathbf{Q}_p^\times \to \mathbf{F}^\times$ must be a constituent of either $\overline{V}$ or $\overline{V} \otimes \omega^{-1}$. Since $\det \overline{V} = \overline{\zeta}\omega^{-1}$, while $\overline{\beta}^2 = \overline{\zeta}\omega^{-2}$, we find that $\overline{V}^{\mathrm{ss}} \xrightarrow{\sim} \overline{\beta} \oplus \overline{\beta}\omega$, proving the lemma. □

4.9.12. *Compatibility with completion.* If R is a complete local Noetherian $\mathcal{O}/\varpi^a$-algebra with finite residue field (for some $a \geq 1$), then in addition to the theory of formal étale (φ, Γ)-modules that we have discussed in Section 4.7, we can also consider the context of the theory of projective étale (φ, Γ)-modules with R-coefficients, as developed in the preceding sections. Given a projective étale (φ, Γ)-module D with R-coefficients, we may form $\widehat{D} := \varprojlim_k D/\mathfrak{m}_R^k D$, a projective formal étale (φ, Γ)-module with R-coefficients. We wish to compare some of the preceding constructions for $\widehat{D}$ with the analogous constructions for D itself.

Before doing this, we need a brief discussion on topologies. Recall that if L is a lattice in an R-Tate module, then in addition to its Tate-module topology, it has a *weak topology*, as described in Section 4.1.7. In particular, $D^\natural$ admits a weak topology, obtained as the inverse limit topology of the quotient topology on $D^\natural/\mathfrak{m}_R^k D^\natural$ under the isomorphism $D^\natural \xrightarrow{\sim} \varprojlim_k (D^\natural/\mathfrak{m}_R^k D^\natural)$ of Lemma 4.1.8. On the other hand, $(\widehat{D})^\natural := \varprojlim_k (D/\mathfrak{m}_R^k D)^\natural$ is endowed with the inverse limit topology with respect to the $R/\mathfrak{m}_R^k$-Tate module topology on $(D/\mathfrak{m}_R^k D)^\natural$.

Recalling that $D^\natural \boxtimes \mathbf{Q}_p := \varprojlim_\psi D^\natural$, we also define a weak topology on $D^\natural \boxtimes \mathbf{Q}_p$, as the inverse limit topology arising from the weak topology on each of the terms in the inverse limit that defines it.

Lemma 4.9.13. *Let R be a complete Noetherian local $\mathcal{O}/\varpi^a$-algebra with finite residue field, and let D be a projective étale (φ, Γ)-module with R-coefficients. Then there are natural topological isomorphisms:*

(1) $D^\natural \xrightarrow{\sim} (\widehat{D})^\natural$,
(2) $D^\natural \boxtimes \mathbf{Q}_p \xrightarrow{\sim} (\widehat{D})^\natural \boxtimes \mathbf{Q}_p$,
(3) $D^{\mathrm{nr}} \xrightarrow{\sim} (\widehat{D})^{\mathrm{nr}}$,

where $D^\natural$ and $D^\natural \boxtimes \mathbf{Q}_p$ are endowed with their weak topologies, and D^{nr} (a finite R-module, by Lemma 4.3.15*) is endowed with its $\mathfrak{m}_R$-adic topology. If D furthermore has rank two and determinant $\zeta\varepsilon^{-1}$, then there is a natural topological isomorphism:*

(4) $D^\natural \boxtimes \mathbf{P}^1 \xrightarrow{\sim} (\widehat{D})^\natural \boxtimes \mathbf{P}^1$,

where $D^\natural \boxtimes \mathbf{P}^1$ is endowed with its weak topology.

Proof. Throughout the proof we use the fact that completed tensor products $-\widehat{\otimes}_R R/\mathfrak{m}_R^k$ coincide with usual tensor products $-\otimes_R R/\mathfrak{m}_R^k = (-)/\mathfrak{m}_R^k$, since $R/\mathfrak{m}_R^k$ is finitely presented over R (see Lemma A.1.41)

Write $D_k := D/\mathfrak{m}_R^k D$ for $k \geq 1$, so that $\widehat{D} := \varprojlim_k D_k$. The natural map $D \to \widehat{D}$ is an injection (because $\mathbf{A}_R \to \widehat{\mathbf{A}}_R$ is an injection and D is projective). Lemma 4.2.8 (2) shows that for each k we have a short exact sequence

$$0 \to M_k \to D^\natural/\mathfrak{m}_R^k D^\natural \to (D_k)^\natural \to 0$$

(defining M_k), and Lemma 4.1.14 (4) shows that each M_k is a finite $R/\mathfrak{m}_R^k$-module, and so is a finite set. Note that all arrows in this exact sequence are continuous for the $R/\mathfrak{m}_R^k$-Tate module topology on $(D_k)^\natural$ and the quotient topology on $D^\natural/\mathfrak{m}_R^k D^\natural$. Passing to the inverse limit over k, we obtain a continuous surjection $D^\natural \to (\widehat{D})^\natural$. Since this map is also obtained by restricting the injection $D \hookrightarrow \widehat{D}$, it is also an injection. Since $D^\natural$ is weakly compact, by Lemma 4.1.8, and $(\widehat{D})^\natural$ is Hausdorff, this map is a topological isomorphism. This proves (1).

Part 2 follows directly from part (1). More precisely, we see that

$$D^\natural \boxtimes \mathbf{Q}_p := \varprojlim_{\psi} D^\natural \xrightarrow{\sim} \varprojlim_{\psi} (\widehat{D})^\natural =: (\widehat{D})^\natural \boxtimes \mathbf{Q}_p.$$

We now turn to part (3). Note firstly that each surjection $D \to D_k$ restricts to a morphism $D^{\mathrm{nr}} \to (D_k)^{\mathrm{nr}}$, which, taken together, yield a morphism $D^{\mathrm{nr}} \to \varprojlim_k (D_k)^{\mathrm{nr}}$. This is in fact an injection, being the restriction of the embedding $D \hookrightarrow \widehat{D}$. Since $(D_k)^{\mathrm{nr}} \subseteq (D_k)^\natural$, we furthermore find (using part (1)) that $\varprojlim_k (D_k)^{\mathrm{nr}} \subseteq D^\natural \subset D$. Now φ acts bijectively on each $(D_k)^{\mathrm{nr}}$, and thus on $\varprojlim (D_k)^{\mathrm{nr}}$. Thus in fact $(\widehat{D})^{\mathrm{nr}} := \varprojlim (D_k)^{\mathrm{nr}} \subseteq D^{\mathrm{nr}}$ (by the definition of D^{nr}). This gives the opposite inclusion to the one already proved, and so establishes part (3).

Finally, we turn to part (4). By our definition of the functors $(-)^{\mathrm{nr}}, (-)^\natural \boxtimes \mathbf{P}^1$ and $(-)^\sharp \boxtimes \mathbf{P}^1$ on formal étale (φ, Γ)-modules (see Definition 4.7.1), we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & w \cdot D^{\mathrm{nr}} & \longrightarrow & D^\natural \boxtimes \mathbf{P}^1 & \longrightarrow & D^\natural \boxtimes \mathbf{Q}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & w \cdot (\widehat{D})^{\mathrm{nr}} & \longrightarrow & (\widehat{D})^\natural \boxtimes \mathbf{P}^1 & \longrightarrow & (\widehat{D})^\natural \boxtimes \mathbf{Q}_p. & &
\end{array}$$

The first, resp. third vertical arrow is an isomorphism by part (3), resp. part (2). The first, resp. second row is exact by Remark 4.8.9, resp. Lemma 4.6.3. An application of the snake lemma then shows that the middle arrow is a bijection. Since $D^\natural \boxtimes \mathbf{P}^1$ is profinite in its weak topology, by Lemma 4.1.8, and the middle arrow is continuous for this topology, it is therefore a topological isomorphism, as desired. (To see this continuity, it suffices to check that each $D^\natural \boxtimes \mathbf{P}^1 \to D_k^\natural \boxtimes \mathbf{P}^1$ is weakly continuous; since this map factors through $(D^\natural \boxtimes \mathbf{P}^1)/\mathfrak{m}^k$, it is equivalent to check that it is strongly continuous; and this is true because it is a restriction of $D^{\oplus 2} \to D_k^{\oplus 2}$.) $\square$

4.9.14. *Base-change in the finite type context.* We now turn to the setting of a morphism of finite type $\mathcal{O}/\varpi^a$-algebras $A \to B$, and a projective rank two étale (φ, Γ)-module D_A over $\mathbf{A}_A$ with determinant $\zeta\varepsilon^{-1}$, corresponding to a morphism $\operatorname{Spec} A \to \mathcal{X}$. Recall (Definition 3.4.11) that we have defined a substack $\mathcal{X}(\mathrm{St})$ of $\mathcal{X}$.

Corollary 4.9.15. *Let $A \to B$ be a morphism of finite type $\mathcal{O}/\varpi^a$-algebras, and let D_A be a projective rank two étale (φ, Γ)-module with A-coefficients, with determinant $\zeta\varepsilon^{-1}$. Then the kernel and cokernel of the base-change morphism* (4.9.1)

$$(D_A^\natural \boxtimes \mathbf{P}^1) \widehat{\otimes}_A B \to D_B^\natural \boxtimes \mathbf{P}^1$$

are supported on $\operatorname{Spec} B \times_{\mathcal{X}} \mathcal{X}(\mathrm{St})$.

Proof. Extend (as we may) the morphism $A \to B$ to a surjection $A[x_1, \ldots, x_n] \to B$ (for some $n \geq 1$). Since $A \to A[x_1, \ldots, x_n]$ is flat, the base-change morphism

$$(D_A^\natural \boxtimes \mathbf{P}^1) \widehat{\otimes}_A A[x_1, \ldots, x_n] \to D_{A[x_1,\ldots,x_n]}^\natural \boxtimes \mathbf{P}^1$$

is an isomorphism (by Lemma 4.8.4 (3)), and so we see that, in proving the lemma, we may replace A by $A[x_1, \ldots, x_n]$ and D_A by $D_{A[x_1,\ldots,x_n]}$, and hence assume that the morphism $A \to B$ is furthermore surjective (and so, in particular, finite).

Now let $\mathcal{K}$, resp. $\mathcal{C}$, denote the kernel, resp. cokernel, of (4.9.1). By Lemma 4.9.2, (4.9.1) is open, and $\mathcal{K}$ and $\mathcal{C}$ are each finite over B.

We now study the completions of $\mathcal{K}$ and $\mathcal{C}$, in order to reduce to the results in the formal case considered above.

Let $\mathfrak{n}$ be a maximal ideal of B, with preimage $\mathfrak{m}$ in A. Since $A \to \widehat{A}_{\mathfrak{m}}$ is flat, Lemma 4.8.4 (3) shows that the base-change morphism

$$(D_A^\natural \boxtimes \mathbf{P}^1) \widehat{\otimes}_A \widehat{A}_{\mathfrak{m}} \to D_{\widehat{A}_{\mathfrak{m}}}^\natural \boxtimes \mathbf{P}^1$$

is an isomorphism. Similarly,

$$(D_B^\natural \boxtimes \mathbf{P}^1) \widehat{\otimes}_B \widehat{B}_{\mathfrak{n}} \to D_{\widehat{B}_{\mathfrak{n}}}^\natural \boxtimes \mathbf{P}^1$$

is an isomorphism. Thus, tensoring (4.9.1) with $\widehat{B}_{\mathfrak{n}}$ over B, we obtain an exact sequence that can be rewritten as

$$0 \to \widehat{\mathcal{K}}_{\mathfrak{n}} \to (D_{\widehat{A}_{\mathfrak{m}}}^\natural \boxtimes \mathbf{P}^1) \widehat{\otimes}_{\widehat{A}_{\mathfrak{m}}} \widehat{B}_{\mathfrak{n}} \to D_{\widehat{B}_{\mathfrak{n}}}^\natural \boxtimes \mathbf{P}^1 \to \widehat{\mathcal{C}}_{\mathfrak{n}} \to 0.$$

(Recall that if M is a finite B-module with the discrete topology, then $M \widehat{\otimes}_B \widehat{B}_{\mathfrak{n}} = M \otimes_B \widehat{B}_{\mathfrak{n}} \xrightarrow{\sim} \widehat{M}_{\mathfrak{n}}$.) Taking into account the isomorphism of Lemma 4.9.13 (4), we may further replace $D_{\widehat{A}_m}$ and $D_{\widehat{B}_n}$ by their formal completions, obtaining the exact sequence

$$0 \to \widehat{\mathcal{K}}_{\mathfrak{n}} \to (\widehat{D}_{\widehat{A}_{\mathfrak{m}}}^\natural \boxtimes \mathbf{P}^1) \widehat{\otimes}_{\widehat{A}_{\mathfrak{m}}} \widehat{B}_{\mathfrak{n}} \to \widehat{D}_{\widehat{B}_{\mathfrak{n}}}^\natural \boxtimes \mathbf{P}^1 \to \widehat{\mathcal{C}}_{\mathfrak{n}} \to 0.$$

By Lemma 4.1.16, we can rewrite this as

$$0 \to \widehat{\mathcal{K}}_{\mathfrak{n}} \to (\widehat{D}_{\widehat{A}_{\mathfrak{m}}}^\natural \boxtimes \mathbf{P}^1) \otimes_{\widehat{A}_{\mathfrak{m}}} \widehat{B}_{\mathfrak{n}} \to \widehat{D}_{\widehat{B}_{\mathfrak{n}}}^\natural \boxtimes \mathbf{P}^1 \to \widehat{\mathcal{C}}_{\mathfrak{n}} \to 0.$$

The result is now immediate from Lemma 4.9.11. □

4.10. **Some explicit base-change computations.** In this section we investigate two particular instances of the base-change we studied in Section 4.9.4. In light of Lemma 4.9.11, the interesting cases are those when $D \coloneqq \mathbf{D}(V)$, for V a free rank 2 module over the complete Noetherian local ring R equipped with a continuous action of $G_{\mathbf{Q}_p}$, such that furthermore $\overline{V}^{\mathrm{ss}}$ is a twist of $\omega \oplus 1$. We will be primarily focused on the case when $\overline{V}$ is in fact already semisimple, and so (up to a twist) isomorphic to $\omega \oplus 1$; thus we begin with some considerations related to the corresponding versal deformation.

Throughout this section we let $\overline{\theta} = \omega+1$, and we will denote the versal deformation ring $R^{\mathrm{ver}}_{\overline{\theta}}$ from Definition 3.4.4, resp. the versal object $V^{\mathrm{ver}}_{\overline{\theta}}$, by R, resp. V. Then V is a versal deformation of $\omega \oplus 1$ with fixed determinant ε, and R is equal to the $\mathfrak{m}$-adic completion of the ring S introduced in Section 3.4.9, where $\mathfrak{m}$ is the maximal ideal of S corresponding to the closed point of $\mathcal{X}_{\overline{\theta}}$ given by the representation $\omega \oplus 1$; namely $\mathfrak{m} = (a_0, a_1, b_0, b_1, c)$. Thus $R = \mathcal{O}[\![a_0, a_1, b_0, b_1, c]\!]/(a_0b_1 + a_1b_0)$ (and the character ζ is implicitly fixed to be ε^2, so that $\det(V) = \zeta\varepsilon^{-1}$). The natural map $R^{\mathrm{ps}}_{\overline{\theta}} \to R$ from the pseudodeformation ring is injective with image $R^{\mathrm{ps}}_{\overline{\theta}} = \mathcal{O}[\![a_0, a_1, b_0c, b_1c]\!]/(a_0b_1c + a_1b_0c)$. Furthermore, $V \cong R^{\oplus 2}$ endowed with a certain $G_{\mathbf{Q}_p}$-action, which can be written down essentially explicitly, see for example [HT15, Section 3], [JNW24, Section 3.4] or [Paš13, Appendix B], but we won't need that here. Rather, we use the fact (see (3.4.10) for a discussion) that $\mathcal{O}[\![G_{\mathbf{Q}_p}]\!]$ acts on V through its quotient $\widetilde{R}_{\overline{\theta}}$, and that the action map $\widetilde{R}_{\overline{\theta}} \to M_2(R)$ is injective with image

$$\begin{pmatrix} R^{\mathrm{ps}}_{\overline{\theta}} & R^{\mathrm{ps}}_{\overline{\theta}}b_0 + R^{\mathrm{ps}}_{\overline{\theta}}b_1 \\ R^{\mathrm{ps}}_{\overline{\theta}}c & R^{\mathrm{ps}}_{\overline{\theta}} \end{pmatrix}.$$

4.10.1. *Computing abelian subrepresentations and quotients.* Our key tool for studying base-change problems involving V and its various specializations will be the exact sequence (4.9.10). In order to use it, we will need to compute V^{ab} and V_{ab} (and the analogous objects for the specializations of V). A convenient way to do this is in terms of the "abelianization" of $\widetilde{R}_{\overline{\theta}}$, i.e. its maximal commutative quotient. One easily verifies that this quotient is equal to

$$\begin{pmatrix} R^{\mathrm{ps}}_{\overline{\theta}}/(b_0c, b_1c) & 0 \\ 0 & R^{\mathrm{ps}}_{\overline{\theta}}/(b_0c, b_1c) \end{pmatrix},$$

which is the quotient of $\widetilde{R}_{\overline{\theta}}$ by its ideal

$$J := \begin{pmatrix} (b_0c, b_1c) & R^{\mathrm{ps}}_{\overline{\theta}}b_0 + R^{\mathrm{ps}}_{\overline{\theta}}b_1 \\ R^{\mathrm{ps}}_{\overline{\theta}}c & (b_0c, b_1c) \end{pmatrix}.$$

Thus, if S is any quotient of R, and $V_S := V \otimes_R S = S^{\oplus 2}$ is the corresponding $G_{\mathbf{Q}_p}$-representation, then we see that

$$(V_S)^{\mathrm{ab}} = V_S[J] = S[c] \oplus S[(b_0, b_1)] \subseteq S \oplus S, \tag{4.10.2}$$

while

$$(V_S)_{\mathrm{ab}} = V_S/JV_S = S/(b_0, b_1)S \oplus S/cS. \tag{4.10.3}$$

Since $V/\mathfrak{m}V = \omega \oplus 1$, we see that $G_{\mathbf{Q}_p}$ acts on the first summand of $(V_S)_{\mathrm{ab}}$ through a deformation of ω, and on the second summand through a deformation of 1. Since $G_{\mathbf{Q}_p}$ acts on both summands through its abelianization, we can regard these as representations of $\mathbf{Q}_p^{\times}$.

For example, if we apply the above discussion to the case of V itself (i.e. we take $S = R$), we obtain the following result.

Lemma 4.10.4.

(1) $V^{\mathrm{ab}} = 0$.

(2) $V_{\mathrm{ab}} = R/(b_0, b_1) \oplus R/(c)$, *with* $\mathbf{Q}_p^{\times}$ *acting on the first summand via a deformation of* ω*, and on the second summand via a deformation of* 1.

Proof. The first claim follows from (4.10.2) and the fact that R is an integral domain, so that $R[c] = R[(b_0, b_1)] = 0$. (Alternatively, it also follows from Lemma 4.7.12.) The second claim follows immediately from (4.10.3). $\square$

If I denotes an ideal in the versal ring R, and if we write $D \coloneqq \mathbf{D}(V)$ and $D' \coloneqq \mathbf{D}(V/IV)$, then we can study the base-change morphism

$$(D^\natural \boxtimes \mathbf{P}^1) \otimes_R R/I \to D'^\natural \boxtimes \mathbf{P}^1. \tag{4.10.5}$$

If we let $\mathcal{K}$ and $\mathcal{C}$ denote the kernel and cokernel respectively of this morphism, then (since, as already noted in Lemma 4.10.4 above, we have $V^{\mathrm{ab}} = 0$), we deduce from (4.9.10), together with (4.10.2) and (4.10.3), that $\mathcal{K}$ and $\mathcal{C}$ admit the alternate descriptions as the kernel and cokernel of a morphism

$$\mathrm{Tor}_1^R(V/JV \otimes \varepsilon^{-1}, R/I) \to w \cdot (V/IV)[J]; \tag{4.10.6}$$

in particular they are finite R/I-modules.

More generally, choose a descending sequence of ideals $I_n \subset I$, write

$$D_n \coloneqq \mathbf{D}(V/I_nV) = \mathbf{D}(V) \otimes_R R/I_n,$$

and let $\mathcal{K}_n$ and $\mathcal{C}_n$ denote the kernel and cokernel of the base-change morphism

$$(D_n^\natural \boxtimes \mathbf{P}^1) \otimes_{R/I_n} R/I \to D'^\natural \boxtimes \mathbf{P}^1. \tag{4.10.7}$$

Then $\{\mathcal{K}_n\}_{n\geq 1}$ and $\{\mathcal{C}_n\}_{n\geq 1}$ are projective systems of finitely generated R/I-modules. There are evident morphisms $\mathcal{K} \to \varprojlim_n \mathcal{K}_n$ and $\mathcal{C} \to \varprojlim_n \mathcal{C}_n$, which are easily seen to be isomorphisms. In the following lemma we make the stronger observation that, under a natural assumption on I_n, these in fact are isomorphisms in the category $\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fg}}(R/I)$.

Lemma 4.10.8. *Maintain the notation of the previous paragraph, and assume further that the ideals I_n are cofinal with the powers of I_1. Then the natural maps induce isomorphisms $\mathcal{K} \xrightarrow{\sim} \lim_n \mathcal{K}_n$ and $\mathcal{C} \xrightarrow{\sim} \lim_n \mathcal{C}_n$ in $\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fg}}(R/I)$. In particular, both $\lim_n \mathcal{K}_n$ and $\lim_n \mathcal{C}_n$, which* a priori *are objects of $\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fg}}(R/I)$, are in fact isomorphic to objects of $\mathrm{Mod}^{\mathrm{fg}}(R/I)$.*

Proof. Since the pro-objects $\mathcal{K}_n$ and $\mathcal{C}_n$ arising from I_n and I_1^n are isomorphic, we can assume without loss of generality that $I_n = I_1^n$. Just as in the preceding discussion, we then apply (4.9.10), together with (4.10.2) and (4.10.3), to deduce that the $\mathcal{K}_n$ and $\mathcal{C}_n$ fit into the projective system of exact sequences

$$\begin{aligned} w \cdot \ker\Big(\big((V/I_1^nV)[J]\big) \otimes_{R/I_1^n} R/I \to (V/IV)[J]\Big) \to \mathcal{K}_n \\ \to \mathrm{Tor}_1^{R/I_1^n}\Big(\big((V/I_1^nV)/J(V/I_1^nV)\big) \otimes \varepsilon^{-1}, R/I\Big) \\ \to w \cdot \mathrm{coker}\Big(\big((V/I_1^nV)[J]\big) \otimes_{R/I_1^n} R/I \to (V/IV)[J]\Big) \to \mathcal{C}_n \to 0. \end{aligned}$$

If we regard this projective system of exact sequences as giving an exact sequence in the category $\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fg}}(R/I)$, then the Artin–Rees lemma shows that it is isomorphic in that category to the exact sequence

$$\begin{aligned} w \cdot \ker\big((V[J]) \otimes_R R/I \to (V/IV)[J]\big) \to \lim_n \mathcal{K}_n \to \mathrm{Tor}_1^R\big((V/JV) \otimes \varepsilon^{-1}, R/I\big) \\ \to w \cdot \mathrm{coker}\big((V[J]) \otimes_R R/I \to (V/IV)[J]\big) \to \lim_n \mathcal{C}_n \to 0. \end{aligned} \tag{4.10.9}$$

(More precisely, we are using the fact that the functor

$$\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(R), M \mapsto \lim_n M/I_1^n M$$

is fully faithful and exact, which is a standard consequence of the Artin–Rees lemma.) As already noted in Lemma 4.10.4 above, we have $V^{\mathrm{ab}} = V[J] = 0$, so that (4.10.9) simplifies to

$$0 \to \lim_n \mathcal{K}_n \to \mathrm{Tor}_1^R\big((V/JV) \otimes \varepsilon^{-1}, R/I\big) \to w \cdot (V/IV)[J] \to \lim_n \mathcal{C}_n \to 0.$$

Since the inner two terms are in fact objects of $\mathrm{Mod}^{\mathrm{fg}}(R/I)$, the same is true of the outer two terms, as claimed. Furthermore, taking into account the description of $\mathcal{K}$ and $\mathcal{C}$ as the kernel and cokernel of (4.10.6) (and the evident naturality of the formation of (4.9.10)), we obtain the claimed isomorphisms. $\square$

In the remainder of this section we consider the "extreme case" of base-change along $R \to R/\mathfrak{m}_R \xrightarrow{\sim} \mathbf{F}$. Thus, again writing $D \coloneqq \mathbf{D}(V)$, and also writing

$$\overline{D} \coloneqq \mathbf{D}(\overline{V}) = \mathbf{D}(\omega \oplus 1),$$

we will study the base-change morphism

$$(D^\natural \boxtimes \mathbf{P}^1) \otimes_R \mathbf{F} \to \overline{D}^\natural \boxtimes \mathbf{P}^1. \tag{4.10.10}$$

Lemma 4.10.11. *Let $\pi_\alpha := \mathrm{Ind}_B^G(\omega \otimes \omega^{-1})$. Then we have an isomorphism*

$$\overline{D}^\natural \boxtimes \mathbf{P}^1 \xrightarrow{\sim} \pi_\alpha^\vee \oplus (\mathbf{1}_G - \mathrm{St})^\vee,$$

where $(\mathbf{1}_G - \mathrm{St})^\vee$ is the Pontrjagin dual to the unique non-split extension of St *by $\mathbf{1}_G$.*

Proof. This follows from [Col10c, Section VII.4.7], where it is shown that

$$\mathbf{D}(1)^\natural \boxtimes \mathbf{P}^1 \xrightarrow{\sim} (\mathbf{1}_G - \mathrm{St})^\vee$$

and

$$\mathbf{D}(\omega)^\natural \boxtimes \mathbf{P}^1 \xrightarrow{\sim} \pi_\alpha^\vee.$$

(Note that, since we have $\zeta = \varepsilon^2$ in this subsection, our symbol $\boxtimes$ coincides with the $\boxtimes$ of *loc. cit.*) $\square$

As usual, we denote the kernel and cokernel of (4.10.10) respectively by $\mathcal{K}$ and $\mathcal{C}$. Taking Lemma 4.10.4 into account, we see that the exact sequence (4.9.10) of B-representations associated to the base-change morphism (4.10.10) may be written as

$$0 \to \mathcal{K} \to (1 \otimes 1)^{\oplus 2} \oplus (\omega^{-1} \otimes \omega) \to (1 \otimes 1) \oplus (\omega^{-1} \otimes \omega) \to \mathcal{C} \to 0. \tag{4.10.12}$$

(Here we have used the fact that the $\mathbf{F}$-dimension of $\mathrm{Tor}_1^R(R/I, \mathbf{F})$ is equal to the minimal number of generators of I, for any ideal $I \subseteq R$.)

Since, by Lemma 3.6.9, the G-action on each of $\mathcal{K}$ and $\mathcal{C}$ (and hence also the B-action) has to be a direct sum of representations of the form $\beta \circ \det$, we see that in fact either

$$\mathcal{K} \xrightarrow{\sim} \mathbf{1}_G^\vee \quad \text{and} \quad \mathcal{C} = 0$$

or

$$\mathcal{K} \xrightarrow{\sim} (\mathbf{1}_G^\vee)^{\oplus 2} \quad \text{and} \quad \mathcal{C} \xrightarrow{\sim} \mathbf{1}_G^\vee.$$

We write $\mathbf{1}_G^\vee$ here, rather than 1 as in (4.10.12), to emphasize the fact that here we regard $\mathcal{K}$ and $\mathcal{C}$ as G-representations, as opposed to $G_{\mathbf{Q}_p}$-representations; in other words, since $\mathcal{K}$ and $\mathcal{C}$ are the kernel and cokernel of (4.10.10), which is a morphism in $\mathfrak{C}_{\overline{\theta}}$, we consider them as objects of $\mathfrak{C}_{\overline{\theta}}$.

Since $\mathrm{Ext}^1_{\mathcal{A}}(\mathbf{1}_G, \pi_\alpha) = 0$ by [Paš13, (165)], we deduce from Lemma 4.10.11 together with the preceding computations of $\mathcal{K}$ and $\mathcal{C}$ that

$$(D^\natural \boxtimes \mathbf{P}^1) \otimes_R \mathbf{F} \xrightarrow{\sim} \pi_\alpha^\vee \oplus \mathbf{E}^\vee, \tag{4.10.13}$$

where $\mathbf{E}$ is either a three-step extension of the form $(\mathbf{1}_G - \mathrm{St} - \mathbf{1}_G)$ or else an extension of the form $(\mathrm{St} - \mathbf{1}_G^{\oplus 2})$, depending on which of the two possibilities for the structure of $\mathcal{K}$ and $\mathcal{C}$ holds.

We will show that in fact the second case is the one that holds, and, furthermore, that $\mathbf{E}$ is the universal extension $E(\mathrm{St})$ of $\mathbf{1}_G$ by St. We will do this by finding certain $\mathcal{O}$-specializations V' of V such that the induced map $(D^\natural \boxtimes \mathbf{P}^1) \otimes_R \mathbf{F} \to (D(V')^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F}$ is surjective, and such that the target is a successive extension of $\pi_\alpha^\vee$ by $\mathrm{St}^\vee$ by $\mathbf{1}_G^\vee$, the latter extension having been arbitrarily chosen in advance.

4.10.14. *Construction of specializations.* Let $\tau \neq 0 \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}_p^\times, \mathbf{F})$, and let $\tau^\perp$ be its orthogonal line in $H^1(G_{\mathbf{Q}_p}, \mathbf{F}(\omega))$ under local Tate duality. We write $\overline{\rho}_\tau$ for the non-split extension

$$0 \to \omega \to \overline{\rho}_\tau \to 1 \to 0$$

corresponding to $\tau^\perp$, and we choose a lift $\rho_\tau : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathcal{O})$ of $\overline{\rho}_\tau$ with determinant ε such that $\rho_\tau[1/p]$ is absolutely irreducible. To construct this lift, we may need to replace E with a finite extension; since we ultimately need to construct such lifts just for the two members of some chosen $\mathbf{F}$-basis of $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}_p^\times, \mathbf{F})$, we make such a replacement once and for all. The existence of the lift is then a very special case of [BIP23a, Prop. 1.12], but it is easily seen to exist by a direct argument; indeed $\overline{\rho}_\tau$ is unobstructed, so the corresponding universal deformation ring is formally smooth, and the reducible locus is formally smooth of codimension 1. We write $\overline{D}_\tau := \mathbf{D}(\overline{\rho}_\tau)$ and $D_\tau := \mathbf{D}(\rho_\tau)$.

Lemma 4.10.15. *The natural map $(D_\tau^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} \to \overline{D}_\tau^\natural \boxtimes \mathbf{P}^1$ is an isomorphism.*

Proof. This follows by a consideration of (4.9.10) (with the map of local rings taken to be $\mathcal{O} \to \mathbf{F}$). Indeed, taking into account Lemma 4.7.12, and noting that $(\rho_\tau)_{\mathrm{ab}} \otimes_{\mathcal{O}} \mathbf{F} \xrightarrow{\sim} (\overline{\rho}_\tau)_{\mathrm{ab}} \xrightarrow{\sim} 1$, we see that this sequence simplifies to

$$0 \to \mathcal{K} \to \omega^{-1} \otimes \omega \to \omega^{-1} \otimes \omega \to \mathcal{C} \to 0,$$

from which (recalling, as always, that the action of G on $\mathcal{K}$ and $\mathcal{C}$ has to be via representations of the form $\beta \circ \det$) we see that $\mathcal{K} = \mathcal{C} = 0$, as required. $\square$

Recall from [Col10c, Prop. VII.4.27] (and the discussion preceding the statement of that result) that we have a short exact sequence

$$0 \to (\mathbf{1}_G - \pi_\alpha)^\vee \to \overline{D}_\tau^\natural \boxtimes \mathbf{P}^1 \to \mathrm{St}^\vee \to 0,$$

where $(\mathbf{1}_G - \pi_\alpha)^\vee$ denotes the dual to unique non-split extension of π_α by $\mathbf{1}_G$. Indeed, in the notation of *loc. cit.* , we have $D_1 = \mathbf{D}(\omega)$ and $D_2 = \mathbf{D}(1)$, $D_1^\sharp \boxtimes \mathbf{P}^1 = (\mathbf{1}_G - \pi_\alpha)^\vee$, and $(D_2^\natural \boxtimes \mathbf{P}^1)_0 = \mathrm{St}^\vee$. Write

$$E_\tau^\vee := (\overline{D}_\tau^\natural \boxtimes \mathbf{P}^1)/\pi_\alpha^\vee, \tag{4.10.16}$$

so that $E_\tau^\vee$ is an extension of $\mathrm{St}^\vee$ by $\mathbf{1}_G^\vee$, dual to an extension E_τ of $\mathbf{1}_G$ by St. By [Col10c, Thm. VII.4.18, Prop. VII.4.27], the map $E_\tau \mapsto \tau$ is an $\mathbf{F}$-linear isomorphism

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathbf{1}_G, \mathrm{St}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}_p^\times, \mathbf{F}). \tag{4.10.17}$$

Of course, the representations ρ_τ are not specializations of our versal representation V, since they have the wrong residual representations. However, after enlarging E if necessary (e.g. replacing E by $E(\varpi^{1/2})$ suffices), we may find a $G_{\mathbf{Q}_p}$-invariant lattice $\rho_{\mathrm{ss}} \subset \rho_\tau$ with reduction equal to $\omega \oplus 1$, and such that

$$\varpi\rho_{\mathrm{ss}} \subset \varpi\rho_\tau \subset \rho_{\mathrm{ss}} \subset \rho_\tau. \tag{4.10.18}$$

Lemma 4.10.19. *There is an isomorphism*

$$(D(\rho_{\mathrm{ss}}) \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} \xrightarrow{\sim} \pi_\alpha^\vee \oplus E_\tau^\vee.$$

Proof. We apply (4.9.10) to the base-change of $D(\rho_{\mathrm{ss}}) \boxtimes \mathbf{P}^1$ along the surjection $\mathcal{O} \to \mathbf{F}$. Taking into account Lemma 4.7.12, and the fact that $(\rho_{\mathrm{ss}})_{\mathrm{ab}} \otimes_{\mathcal{O}} \mathbf{F} = \omega \oplus 1$, we see that this sequence simplifies to

$$0 \to \mathcal{K} \to (1 \otimes 1) \oplus (\omega^{-1} \otimes \omega) \to (1 \otimes 1) \oplus (\omega^{-1} \otimes \omega) \to \mathcal{C} \to 0.$$

Thus either $\mathcal{K} = \mathcal{C} = 0$, or $\mathcal{K} = \mathcal{C} = \mathbf{1}_G^\vee$. If we also take into account Lemma 4.10.11, we conclude that, correspondingly, either

$$(D(\rho_{\mathrm{ss}}) \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} \xrightarrow{\sim} \pi_\alpha^\vee \oplus (\mathbf{1}_G - \mathrm{St})^\vee,$$

or else that

$$(D(\rho_{\mathrm{ss}}) \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} \xrightarrow{\sim} \pi_\alpha^\vee \oplus (\mathrm{St} - \mathbf{1}_G)^\vee,$$

for some extension $(\mathrm{St} - \mathbf{1}_G)$ of $\mathbf{1}_G$ by St.

To conclude the proof, it therefore suffices to show that there is an embedding

$$E_\tau^\vee \hookrightarrow \big(D(\rho_{\mathrm{ss}}) \boxtimes \mathbf{P}^1\big) \otimes_{\mathcal{O}} \mathbf{F}.$$

To this end, observe first that the functor $D \mapsto D^\natural \boxtimes \mathbf{P}^1$ preserves injections: in fact, inspecting Definition 4.7.1 we see that there are natural injections $D^\natural \boxtimes \mathbf{P}^1 \to D \boxtimes \mathbf{P}^1 \to D^{\oplus 2}$, and $D \mapsto D^{\oplus 2}$ is exact.

We thus see that $D_\tau^\natural \boxtimes \mathbf{P}^1$ is ϖ-torsion free, and applying $(-)^\natural \boxtimes \mathbf{P}^1$ to (4.10.18), we obtain a chain of inclusions

$$\varpi(D_{\mathrm{ss}}^\natural \boxtimes \mathbf{P}^1) \subset \varpi(D_\tau^\natural \boxtimes \mathbf{P}^1) \subset D_{\mathrm{ss}}^\natural \boxtimes \mathbf{P}^1 \subset D_\tau^\natural \boxtimes \mathbf{P}^1,$$

from which we deduce that $(D_{\mathrm{ss}}^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F}$ has a submodule isomorphic to

$$\mathrm{coker}\big((D_{\mathrm{ss}}^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} \to (D_\tau^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F}\big).$$

It therefore suffices to prove that this cokernel is isomorphic to $E_\tau^\vee$. By Lemma 4.10.15 and (4.10.16) (i.e. by the definition of E_τ), it suffices to prove that

$$\mathrm{image}\big((D_{\mathrm{ss}}^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} \to (D_\tau^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} \xrightarrow{\sim} \overline{D}_\tau^\natural \boxtimes \mathbf{P}^1\big) = \pi_\alpha^\vee.$$

We have a commutative diagram

$$\begin{array}{ccc} (D_{\mathrm{ss}}^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} & \longrightarrow & (D_\tau^\natural \boxtimes \mathbf{P}^1) \otimes_{\mathcal{O}} \mathbf{F} \\ \downarrow & & \downarrow\wr \\ \overline{D}_{\mathrm{ss}}^\natural \boxtimes \mathbf{P}^1 & \longrightarrow & \overline{D}_\tau^\natural \boxtimes \mathbf{P}^1 \end{array}$$

where the lower horizontal arrow is induced by a non-zero map $\omega \oplus 1 \to \overline{\rho}_\tau$, and so has image $\mathbf{D}(\omega)^\natural \boxtimes \mathbf{P}^1 = \pi_\alpha^\vee$. This concludes the proof, since the cokernel $\mathcal{C}$ of the left vertical arrow is finite-dimensional. □

Definition 4.10.20. Recall that $\dim_{\mathbf{F}} \mathrm{Ext}^1_{\mathcal{A}}(\mathbf{1}_G, \mathrm{St}) = 2$ (e.g. by [Paš13, Section 10.1], or (4.10.17) above). The universal extension $E(\mathrm{St})$ is defined to be the pushout along St of any two linearly independent extensions of $\mathbf{1}_G$ by St. Alternatively, we may describe $E(\mathrm{St})$ as the preimage in $\mathrm{inj}_{\mathbf{F}[G/Z]}(\mathrm{St})$ of the $\mathbf{1}_G$-isotypic component of the first layer of the socle filtration. It sits in a short exact sequence

$$0 \to \mathrm{St} \to E(\mathrm{St}) \to \mathbf{1}_G^{\oplus 2} \to 0.$$

Thus its Pontrjagin dual sits in a short exact sequence

$$0 \to (\mathbf{1}_G^\vee)^{\oplus 2} \to E(\mathrm{St})^\vee \to \mathrm{St}^\vee \to 0.$$

Theorem 4.10.21. *We have an isomorphism*

$$(D^\natural \boxtimes \mathbf{P}^1) \otimes_R \mathbf{F} \cong \pi_\alpha^\vee \oplus E(\mathrm{St})^\vee.$$

Proof. Let ρ_{ss} be one of the lattices considered in Lemma 4.10.19. Then ρ_{ss} is obtained by base-changing V along some $\mathcal{O}$-algebra morphism $R \to \mathcal{O}$. Considering the exact sequence (4.9.10) for this base-change, and again taking into account Lemma 4.7.12 (which applies to both V and ρ_{ss}), we deduce that the induced map

$$D \boxtimes \mathbf{P}^1 \to D(\rho_{\mathrm{ss}}) \boxtimes \mathbf{P}^1$$

is surjective. Consequently

$$(D \boxtimes \mathbf{P}^1) \otimes_R \mathbf{F} \to \big(D(\rho_{\mathrm{ss}}) \boxtimes \mathbf{P}^1\big) \otimes_{\mathcal{O}} \mathbf{F}$$

is also surjective. Lemma 4.10.19 shows that the target has a direct summand isomorphic to $E_\tau^\vee$. Considering (4.10.13), we deduce that the direct summand $\mathbf{E}^\vee$ of the source surjects onto $E_\tau^\vee$. Letting τ range over a pair of basis vectors of $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}_p^\times, \mathbf{F})$, we find that in fact $\mathbf{E}^\vee$ appearing in (4.10.13) must be the dual to the universal extension $E(\mathrm{St})$, as required. □

In fact, in the sequel, our application of Theorem 4.10.21 will be via the following corollary, which does not use its full strength, but only its particular consequence

$$\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}((D^\natural \boxtimes \mathbf{P}^1) \otimes_R \mathbf{F}, \mathbf{1}_G^\vee) = 0. \tag{4.10.22}$$

Corollary 4.10.23. *Suppose that $i_S : \mathrm{Spf}\, S \to \mathcal{X}_{\overline{\theta}}$ is a versal morphism at the closed point of $\mathcal{X}_{\overline{\theta}}$, where S is a complete local Noetherian $\mathcal{O}$-algebra with finite residue field. Write D_S for the corresponding formal étale (φ, Γ)-module with S-coefficients (i.e. $D_S := \mathbf{D}(\widehat{i}_S^* \mathcal{V}_{\overline{\theta}})$). Then $D_S^\natural \boxtimes \mathbf{P}^1$ has no $\mathfrak{C}_{\overline{\theta}}$-quotients of $\mathcal{O}$-finite length.*

Proof. Without loss of generality (*cf.* [Stacks, Tag 06T5]) we can and do suppose that $S = R[\![x_1, \dots, x_r]\!]$ is a power series ring over R, where R is the versal ring considered above. If

$$\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}(D_S^\natural \boxtimes \mathbf{P}^1, \mathbf{1}_G^\vee) \neq 0,$$

then there exists $k > 0$ such that

$$\mathrm{Hom}_{\mathfrak{C}_{\overline{\theta}}}\big((D_S^\natural \boxtimes \mathbf{P}^1) \otimes_S S/\mathfrak{m}_S^k, \mathbf{1}_G^\vee\big) \neq 0.$$

(This is because $D_S^\natural \boxtimes \mathbf{P}^1$ is a profinite S-module, hence has the $\mathfrak{m}_S$-adic topology by Lemma A.1.32 (8), and we are working with continuous homomorphisms.) Filtering S by powers of the maximal ideal, we deduce that

$$\operatorname{Hom}_{\mathfrak{C}_{\overline{\theta}}}\big((D_S^\natural \boxtimes \mathbf{P}^1) \otimes_S \mathbf{F}, \mathbf{1}_G^\vee\big) \neq 0. \tag{4.10.24}$$

Writing $S_n := (R/\mathfrak{m}_R^n)[x_1, \dots, x_r]/(x_1, \dots, x_r)^n$, we have $S = \varprojlim_n S_n$, and by definition

$$D_S^\natural \boxtimes \mathbf{P}^1 = \varprojlim_n D_{S_n}^\natural \boxtimes \mathbf{P}^1.$$

Applying part (3) of Lemma 4.8.4 to the finite free morphism $R/\mathfrak{m}_R^n \to S_n$, and using Lemma 4.1.16 to replace $\widehat{\otimes}$ with $\otimes$, we find that

$$D_S^\natural \boxtimes \mathbf{P}^1 \cong \varprojlim_n (D_{R/\mathfrak{m}_R^n}^\natural \boxtimes \mathbf{P}^1) \otimes_{R/\mathfrak{m}_R^n} S_n.$$

Since $\mathbf{F}$ is finitely presented over R and S, and cofiltered limits are exact in $\mathfrak{C}_{\overline{\theta}}$, we conclude that

$$(D_S^\natural \boxtimes \mathbf{P}^1) \otimes_S \mathbf{F} \cong \varprojlim_n ((D_{R/\mathfrak{m}_R^n}^\natural \boxtimes \mathbf{P}^1) \otimes_{R/\mathfrak{m}_R^n} S_n \otimes_S \mathbf{F}) \cong (D^\natural \boxtimes \mathbf{P}^1) \otimes_R \mathbf{F}.$$

Then (4.10.24) contradicts (4.10.22), and we are done. □

5. The functor

We now construct our functor $\mathrm{F} : D^b_{\mathrm{fp}}(\mathcal{A}) \to D^b_{\mathrm{coh}}(\mathcal{X})$, and prove our main theorem, i.e. that F is fully faithful.

5.1. **The definition of the functor.**

In this section we construct the pro-coherent sheaf L_∞ of $\mathcal{O}[\![G]\!]_\zeta$-modules over $\mathcal{X}$, and use it to define our functor F.

5.1.1. *Interpreting linearly topological modules as pro-coherent sheaves.* We begin by explaining a general procedure that converts certain topological modules into pro-coherent sheaves. We have the following (slightly *ad hoc*) definition.

Definition 5.1.2. Let A be a Noetherian ring endowed with the discrete topology. We say that a linearly topological A-module M is *pro-coherent* if M is complete, and if for any open A-submodule U of M, the quotient M/U is finitely generated over A. (Of course, it suffices to check this condition for U running over a neighbourhood basis of 0.) Note that the natural morphism $M \to \varprojlim_U M/U$ (the projective limit being taken over all open submodules of M, where each quotient is endowed with its discrete topology) is then a topological isomorphism.

Remark 5.1.3. Any lattice in an A-Tate module is pro-coherent when equipped with its induced topology.

For any finitely generated A-module N, we write $\widetilde{N}$ to denote the coherent sheaf obtained by localizing N over $\operatorname{Spec} A$.

Definition 5.1.4. If M is a pro-coherent linearly topological A-module, then we write $\widetilde{M} := \lim_U \widetilde{M/U}$, regarded as a pro-coherent sheaf on $\operatorname{Spec} A$, i.e. an object of $\operatorname{Pro}\operatorname{Coh}(\operatorname{Spec} A)$.

Remark 5.1.5. Any finitely generated A-module may be regarded as a pro-coherent A-module by endowing it with the discrete topology; in this case the two possible meanings of $\widetilde{N}$ evidently coincide.

Remark 5.1.6. If A is a finite type $\mathcal{O}/\varpi^a$-algebra, and M is a finitely presented $A[\![K]\!]$-module, the canonical completely metrizable topology on M coincides with its $\mathfrak{a}$-adic topology, and so it is pro-coherent. The functor $\mathrm{Mod}^{\mathrm{fp}}(A[\![K]\!]) \to \mathrm{Pro}\,\mathrm{Coh}(\mathrm{Spec}\,A), M \mapsto \widetilde{M}$ is isomorphic to $M \mapsto \lim_n M/\mathfrak{a}^n M$, and so it is exact, by Lemma 2.2.7 (5).

Remark 5.1.7. Not every pro-coherent sheaf $\lim_i \mathcal{F}_i$ on $\mathrm{Spec}\,A$ need be of the form $\widetilde{M}$ for some pro-coherent linearly topological A-module, even if the transition morphisms $\mathcal{F}_{i'} \to \mathcal{F}_i$ are surjective. Indeed, if we let $M \coloneqq \varprojlim_i \mathcal{F}_i(A)$, then the morphisms $M \to \mathcal{F}_i(A)$ need not be surjective (see e.g. [Stacks, Tag 0ANX]).

On the other hand, if the indexing set I is countable and the transition morphisms are surjective, then M is indeed a pro-coherent module, and the natural morphism $\widetilde{M} \to \lim_i \mathcal{F}_i$ is an isomorphism.

The following lemma shows that the formation of $\widetilde{M}$ gives rise to a fully faithful functor with certain natural exactness properties.

Lemma 5.1.8. *The formation of $\widetilde{M}$ induces a fully faithful A-linear functor from the A-linear additive category of pro-coherent linearly topological A-modules* (*whose morphisms are continuous A-linear morphisms*) *to the A-linear abelian category* $\mathrm{Pro}\,\mathrm{Coh}(\mathrm{Spec}\,A)$*. Furthermore:*

(1) If $f : M \to N$ is a topological embedding of pro-coherent linearly topological A-modules, then the induced morphism $\widetilde{M} \to \widetilde{N}$ is a monomorphism.
(2) If $f : M \to N$ is a continuous, open, and surjective morphism of pro-coherent linearly topological A-modules, then the induced morphism $\widetilde{M} \to \widetilde{N}$ is an epimorphism.

Proof. Let M and N be two pro-coherent linearly topological A-modules. Then

$$\begin{aligned}\mathrm{Hom}_A^{\mathrm{cont}}(M,N) &\xrightarrow{\sim} \varprojlim_V \mathrm{Hom}_A^{\mathrm{cont}}(M,N/V)\\ &\xrightarrow{\sim} \varprojlim_V \varinjlim_U \mathrm{Hom}_A(M/U,N/V) \xrightarrow{\sim} \varprojlim_V \varinjlim_U \mathrm{Hom}_{\mathrm{Coh}(\mathrm{Spec}\,A)}(\widetilde{M/U},\widetilde{N/V})\end{aligned}$$

where U (resp. V) runs over the open submodules of M (resp, N): the first isomorphism follows from the isomorphism $N \xrightarrow{\sim} \varprojlim_V N/V$, the second follows from the fact that N/V is discrete, so that any continuous morphism $M \to N/V$ factors through M/U for some open submodule U of M, and the third follows from the fact that localization induces an equivalence of categories $\mathrm{Mod}^{\mathrm{fp}}(A) \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Spec}\,A)$. The right-most expression in this sequence of isomorphisms is precisely equal to $\mathrm{Hom}_{\mathrm{Pro}\,\mathrm{Coh}(\mathrm{Spec}\,A)}(\widetilde{M},\widetilde{N})$; thus we do indeed obtain a fully faithful functor.

If $M \hookrightarrow N$ is a topological embedding, then as V runs over the open submodules of N, the intersections $M \cap V$ run through a neighborhood basis at 0 consisting of open submodules of M. Thus the induced morphism $\widetilde{M} \to \widetilde{N}$ arises as the formal inverse limit of the inverse system of injections $M/(M\cap V) \hookrightarrow N/V$. This proves (1).

If $M \to N$ is continuous, open, and surjective, then every open submodule of N is the image $\overline{U}$ of an open submodule U of M. In this case the induced morphism $\widetilde{M} \to \widetilde{N}$ corresponds to the formal inverse limit of the inverse system of surjections $M/U \to N/\overline{U}$, proving (2). ☐

In the next lemma, if $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism of affine schemes, then we continue to write f^* for the Pro-extension of the usual pullback $f^* : \operatorname{Coh}(\operatorname{Spec} A) \to \operatorname{Coh}(\operatorname{Spec} B)$.

Lemma 5.1.9. *Suppose that M is a pro-coherent and first countable linearly topological module over the Noetherian ring A. If B is a Noetherian A-algebra, and $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is the corresponding morphism of schemes, then $f^*\widetilde{M} \xrightarrow{\sim} (M \widehat{\otimes}_A B)^\sim$.*

Proof. Let $\{U_n\}_{n\geq 0}$ be a cofinal sequence of open submodules of M, which exists by virtue of our assumption that M is first countable. Then $M \widehat{\otimes}_A B = \varprojlim_n (M/U_n) \otimes_A B$ by (4.1.11); to ease notation, let N denote this (first countable, pro-coherent) linearly topological B-module. Since the transition morphisms $(M/U_{n+1}) \otimes_A B \to (M/U_n) \otimes_A B$ are surjective, we see that each of the canonical morphisms $N \to (M/U_n) \otimes_A B$ is surjective, and that if we let V_n denote the kernel of this morphism, then V_n forms a cofinal sequence of open submodules of N.

Now $\widetilde{M} := \lim_n \widetilde{M/U_n}$, so that

$$f^*\widetilde{M} = \lim_n f^*\widetilde{M/U_n} = \lim_n \big(\widetilde{(M/U_n) \otimes_A B}\big) \xrightarrow{\sim} \lim_n \widetilde{N/V_n} =: \widetilde{N},$$

as claimed. □

If L is a lattice in a Tate module over a complete Noetherian local $\mathcal{O}$-algebra, then L is equipped not only with its Tate-module topology, but also with its weak topology, as defined in Section 4.1.7. We let L^w denote L equipped with its weak topology; this is again a pro-coherent topology on L, and the aim of the following lemma is to give a description of the associated pro-coherent sheaf.

Lemma 5.1.10. *Let S be a complete Noetherian local $\mathcal{O}$-algebra, and let L be a lattice in a Tate S-module. Then, under the identification $\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(S) \xrightarrow{\sim} \operatorname{Pro}\operatorname{Coh}(\operatorname{Spec} S)$, $\widetilde{L^w}$ is an object of $\operatorname{Pro}\operatorname{Mod}^{\mathrm{f.l.}}(S) \subset \operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(S)$, and it is isomorphic to the image of $\widetilde{L}$ under the map*

$$\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(S) \to \operatorname{Pro}\operatorname{Mod}^{\mathrm{f.l.}}(S)$$

induced by (A.1.36) *and* (A.1.37).

Proof. By Lemma 4.1.8, L^w is an object of $\operatorname{Mod}_c(S)$, and so every open S-submodule of L^w has cofinite S-length. By definition, this shows that $\widetilde{L^w} \in \operatorname{Pro}\operatorname{Mod}^{\mathrm{f.l.}}(S)$. This proves the first claim.

For the second claim, choose a neighborhood basis of the origin $U_i \subset L$ consisting of open submodules such that L/U_i is S-finite, for all i. Then, by definition, $\widetilde{L} \in \operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(S)$ is $\lim_i L/U_i$, and so its image in $\operatorname{Pro}\operatorname{Mod}^{\mathrm{f.l.}}(S)$ is $\lim_{i,n} L/(U_i, \mathfrak{m}^n L)$. Since $(U_i, \mathfrak{m}^n L)$ forms a basis of the weak topology on L, we see that $\widetilde{L^w}$ is isomorphic to the image of $\widetilde{L}$ in $\operatorname{Pro}\operatorname{Mod}^{\mathrm{f.l.}}(S)$. □

5.1.11. *Interpreting $D^\natural \boxtimes \mathbf{P}^1$ as a pro-coherent sheaf.* Suppose now that A is a Noetherian $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and that D is an étale (φ, Γ)-module of rank 2 with A-coefficients and determinant $\zeta\varepsilon^{-1}$. Then we have seen (in Theorem 4.8.8) that $D^\natural \boxtimes \mathbf{P}^1$ is an $\mathcal{O}[\![G]\!]$-stable lattice in $D \boxtimes \mathbf{P}^1$. In particular, it is pro-coherent as a topological A-module, and so we may form the associated pro-coherent sheaf $\widetilde{D^\natural \boxtimes \mathbf{P}^1}$. Since $\mathcal{O}[\![G]\!]$ acts on $D^\natural \boxtimes \mathbf{P}^1$ by continuous endomorphisms (by Corollary 4.5.16), we obtain an induced action of $\mathcal{O}[\![G]\!]$ on $\widetilde{D^\natural \boxtimes \mathbf{P}^1}$, making it a left $\mathcal{O}[\![G]\!]$-module object in $\operatorname{Pro}\operatorname{Coh}(\operatorname{Spec} A)$.

We intend to apply the construction of $\widetilde{D^\natural \boxtimes \mathbf{P}^1}$ to the universal case, i.e. over $\mathcal{X}$. Since $\mathcal{X}$ is only a formal algebraic stack, rather than a scheme, we will do this via descent. We first describe the construction over algebraic stacks, before moving to the case of the formal algebraic stack $\mathcal{X}$.

5.1.12. *Descent for algebraic stacks.* Suppose given a Noetherian algebraic $\mathcal{O}/\varpi^a$-stack $\mathcal{Z}$ (for some $a \geq 1$) with affine diagonal, equipped with a morphism $\mathcal{Z} \to \mathcal{X}$. Pulling back the universal rank 2 étale (φ, Γ)-module on $\mathcal{X}$ to $\mathcal{Z}$ gives rise to a rank 2 étale (φ, Γ)-module $D_{\mathcal{Z}}$ on $\mathcal{Z}$, and we wish to construct the associated pro-coherent sheaf of $\mathcal{O}[\![G]\!]$-modules $\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}$ on $\mathcal{Z}$.

Let $\operatorname{Spec} A \to \mathcal{Z}$ be a flat surjection of finite presentation from the spectrum of a Noetherian $\mathcal{O}/\varpi^a$-algebra. Write $R \coloneqq \operatorname{Spec} A \times_{\mathcal{Z}} \operatorname{Spec} A$. Then R is an affine scheme with the structure of an *fppf* groupoid over $\operatorname{Spec} A$. The morphism $\mathcal{Z} \to \mathcal{X}$ then corresponds to a rank 2 étale (φ, Γ)-module D with A-coefficients and determinant $\zeta\varepsilon^{-1}$, endowed with descent data with respect to R. Concretely, if we let $s, t : R \rightrightarrows \operatorname{Spec} A$ denote the two projections, and if we let s^*D, respectively t^*D, denote the base-changes of D, computed with respect to the corresponding (flat) morphisms $A \rightrightarrows B$ (where we write $R = \operatorname{Spec} B$), then we have an isomorphism $s^*D \xrightarrow{\sim} t^*D$ of étale (φ, Γ)-modules over B, satisfying an appropriate cocycle condition.

Lemma 4.8.2 then shows that $D \boxtimes \mathbf{P}^1$ is again equipped with descent data with respect to R, and Lemma 4.8.4 (3) shows that these descent data restrict to descent data on the $\mathcal{O}[\![G]\!]$-stable sublattice $D^\natural \boxtimes \mathbf{P}^1$. (We will write that $D^\natural \boxtimes \mathbf{P}^1$ is *preserved* by the descent data.) These descent data are necessarily G-equivariant, since the G-action on $D^\natural \boxtimes \mathbf{P}^1$ is computed in terms of the (φ, Γ)-module structure on D, and the descent data on $D^\natural \boxtimes \mathbf{P}^1$ are induced by descent data of (φ, Γ)-modules on D. Since $\mathcal{O}[K]$ is dense in $\mathcal{O}[\![K]\!]$, and acts continuously on $D^\natural \boxtimes \mathbf{P}^1$, we find that the descent data are furthermore $\mathcal{O}[\![K]\!]$-equivariant, and hence also $\mathcal{O}[\![G]\!]$-equivariant.

Lemma 5.1.13. *In the context of the preceding paragraph, there exists a neighborhood basis $\mathfrak{M}_n \subset D^\natural \boxtimes \mathbf{P}^1$ of the origin consisting of $A[\![K]\!]$-stable lattices that are preserved by the descent data.*

Proof. Assume first that there is a neighborhood basis $\mathfrak{N}_n$ of the origin consisting of lattices that are preserved by the descent data. Since $D^\natural \boxtimes \mathbf{P}^1$ is finitely generated over $A[\![K]\!]$, and the action of $A[\![K]\!]$ is continuous, for all n there exists i such that $\mathfrak{N}_n$ is an $A[\![K_i]\!]$-module. Hence the (finite) intersection $\mathfrak{M}_n \coloneqq \bigcap_{g \in K} g(\mathfrak{N}_n)$ is an $A[\![K]\!]$-stable lattice, and it is preserved by the descent data, since these data are $\mathcal{O}[\![K]\!]$-equivariant. Thus $\mathfrak{M}_n$ is the required neighborhood basis.

There remains to construct a neighborhood basis $\mathfrak{N}_n$ of the origin consisting of R-equivariant lattices. Since $D^\natural \boxtimes \mathbf{P}^1$ is an R-equivariant lattice in $D \boxtimes \mathbf{P}^1$, it suffices to do this for $D \boxtimes \mathbf{P}^1$. By Lemma 4.5.5 we have an isomorphism of R-equivariant Tate A-modules

$$D \boxtimes \mathbf{P}^1 \xrightarrow{\sim} D \oplus (D \boxtimes p\mathbf{Z}_p),$$

and $D \boxtimes p\mathbf{Z}_p$ is a direct summand of D. So it suffices to construct a neighborhood basis of the origin $\mathfrak{N}'_n \subset D$ such that each $\mathfrak{N}'_n$ is preserved by the descent data. We can achieve this by setting $\mathfrak{N}'_n := T^n \cdot D^+$; these are lattices in D, by Corollary 4.3.4, and they are preserved by the descent data, by Lemma 4.3.12. $\square$

Lemma 5.1.13 shows that the descent data on $D^\natural \boxtimes \mathbf{P}^1$ preserve a neighbourhood basis of the origin, and hence the associated pro-coherent sheaf $\widetilde{D^\natural \boxtimes \mathbf{P}^1}$ may again be equipped with descent data, and so descends to a pro-coherent sheaf on $\mathcal{Z}$, which we will denote by $\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}$. Concretely, if we choose a basis $\mathfrak{M}_n$ as in Lemma 5.1.13, and define $X_n := (D^\natural \boxtimes \mathbf{P}^1)/\mathfrak{M}_n$, then we have an isomorphism

$$\widetilde{D^\natural \boxtimes \mathbf{P}^1} \xrightarrow{\sim} \lim_n \widetilde{X_n}, \tag{5.1.14}$$

where each $\widetilde{X_n}$ is a coherent sheaf with $\mathcal{O}[\![K]\!]$-action (i.e. a left $\mathcal{O}[\![K]\!]$-module in $\mathrm{Coh}(\operatorname{Spec} A)$) and descent data to $\mathcal{Z}$. Note that X_n is a smooth representation of $\mathcal{O}[\![K]\!]$, and so the $\mathcal{O}[\![K]\!]$-action on X_n factors through $\mathcal{O}[\![K]\!]/\mathfrak{a}^i$ for some i, since X_n is finitely generated over A. In other words, using terminology from Remark 2.2.12, $\widetilde{X}_n$ is an $\mathfrak{a}$-power torsion $\mathcal{O}[\![K]\!]$-module in $\mathrm{Coh}(\operatorname{Spec} A)$.

The right-hand side of (5.1.14) descends to a pro-system of $\mathfrak{a}$-power torsion $\mathcal{O}[\![K]\!]$-modules $\mathcal{F}_n$ in $\mathrm{Coh}(\mathcal{Z})$, and so

$$\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1} := \lim_n \mathcal{F}_n \tag{5.1.15}$$

acquires the structure of an $\mathcal{O}[\![K]\!]$-module in $\operatorname{Pro}\mathrm{Coh}(\mathcal{Z})$. Since the descent data on $D^\natural \boxtimes \mathbf{P}^1$ are also $\mathcal{O}[\![G]\!]$-equivariant (as we noted above), this $\mathcal{O}[\![K]\!]$-module structure on $\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}$ extends to an $\mathcal{O}[\![G]\!]$-module structure.

Remark 5.1.16. One easily verifies that the pro-coherent sheaf of $\mathcal{O}[\![G]\!]$-modules $\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}$ is canonically independent of the choice of presentation of $\mathcal{Z}$ of the form $[\operatorname{Spec} A/R]$. Indeed, any two such presentations may be dominated by a third, and by Lemma 4.8.4 (3), if $\operatorname{Spec} A' \to \operatorname{Spec} A \to \mathcal{Z}$ are *fppf* morphisms, giving rise to rank 2 étale (φ, Γ)-modules D and D' over A and A' respectively, then $D^\natural \boxtimes \mathbf{P}^1$ is obtained from $(D')^\natural \boxtimes \mathbf{P}^1$ via descent along the first arrow. This implies the analogous fact for the corresponding pair of pro-coherent sheaves of $\mathcal{O}[\![G]\!]$-modules $\widetilde{(D')^\natural \boxtimes \mathbf{P}^1}$ and $\widetilde{D^\natural \boxtimes \mathbf{P}^1}$, and hence the pro-coherent sheaves of $\mathcal{O}[\![G]\!]$-modules on $\mathcal{Z}$ obtained by descending each of these are canonically isomorphic, as required.

The next lemma describes the base-change properties of $\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}$.

Lemma 5.1.17. *Let $\mathcal{Z}$ be a Noetherian algebraic stack over $\mathcal{O}/\varpi^a$, with affine diagonal, and equipped with a morphism $\mathcal{Z} \to \mathcal{X}$, inducing the pro-coherent sheaf of $\mathcal{O}[\![G]\!]$-modules $\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}$ on $\mathcal{Z}$, as above. Further, let $f : \mathcal{W} \to \mathcal{Z}$ be a morphism of Noetherian algebraic stacks, giving rise to the composite morphism $\mathcal{W} \to \mathcal{Z} \to \mathcal{X}$, and hence similarly inducing a pro-coherent sheaf of $\mathcal{O}[\![G]\!]$-modules $\widetilde{D^\natural_{\mathcal{W}} \boxtimes \mathbf{P}^1}$ on $\mathcal{W}$. Then there is a base-change morphism of pro-coherent sheaves of $\mathcal{O}[\![G]\!]$-modules*

$$f^*(\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}) \to \widetilde{D^\natural_{\mathcal{W}} \boxtimes \mathbf{P}^1},$$

whose kernel and cokernel are coherent sheaves on $\mathcal{W}$, and which is furthermore an isomorphism if f is flat.

If $\mathcal{Z}$ and $\mathcal{W}$ are furthermore of finite type over $\mathcal{O}/\varpi^a$ (for some $a \geq 1$), then the kernel and cokernel of the base-change morphism are set-theoretically supported on $\mathcal{W} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St})$.

Proof. We may find a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "g"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \mathcal{W} \arrow[r, "f"] & \mathcal{Z} \end{tikzcd}$$

in which the vertical arrows are *fppf* surjections, and then use these vertical arrows to construct each of $\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}$ and $\widetilde{D^\natural_{\mathcal{W}} \boxtimes \mathbf{P}^1}$ via descent from the corresponding pro-coherent sheaves $\widetilde{D^\natural_A \boxtimes \mathbf{P}^1}$ and $\widetilde{D^\natural_B \boxtimes \mathbf{P}^1}$ (using evident notation).

Lemmas 5.1.9 and 4.8.2 then yield a canonical morphism

$$g^*(\widetilde{D^\natural_A \boxtimes \mathbf{P}^1}) \to \widetilde{D^\natural_B \boxtimes \mathbf{P}^1},$$

which, by Lemma 4.8.4 (3), is even an isomorphism if f is flat (since then the morphism $g : \operatorname{Spec} B \to \operatorname{Spec} A$ is also flat). This morphism is compatible with the descent data, and so induces the desired morphism

$$f^*(\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}) \to \widetilde{D^\natural_{\mathcal{W}} \boxtimes \mathbf{P}^1}.$$

The claim about its kernel and cokernel can be checked after pulling back to $\operatorname{Spec} B$ (i.e. after "undoing" the descent), where it follows from Lemma 4.9.2. The final claim in the finite type case similarly follows from Corollary 4.9.15. □

5.1.18. *Descent for $\mathcal{X}$.* We now wish to define the pro-coherent sheaf of $\mathcal{O}[\![G]\!]$-modules $\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}$ associated to the universal étale (φ, Γ)-module $D_{\mathcal{X}}$ on the formal algebraic stack $\mathcal{X}$.

To this end, write $\mathcal{X} \xrightarrow{\sim} \operatorname{colim} \mathcal{X}_n$ as a colimit of algebraic stacks under thickenings; let $i_n : \mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ denote the nth transition morphism, and let $k_n : \mathcal{X}_n \hookrightarrow \mathcal{X}$ be the closed immersion. By Lemma 5.1.17, we have pro-coherent sheaves of $\mathcal{O}[\![G]\!]$-modules $\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}$, endowed with $\mathcal{O}[\![G]\!]$-equivariant morphisms

$$\widetilde{D^\natural_{\mathcal{X}_{n+1}} \boxtimes \mathbf{P}^1} \to i_{n,*}(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1})$$

(obtained by applying the adjunction between i_n^* and $i_{n,*}$ to the base-change morphisms of Lemma 5.1.17), which we may push forward to $\mathcal{X}$ so as to obtain $\mathcal{O}[\![G]\!]$-equivariant morphisms

$$k_{n+1,*}(\widetilde{D^\natural_{\mathcal{X}_{n+1}} \boxtimes \mathbf{P}^1}) \to k_{n+1,*} i_{n,*}(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}) \cong k_{n,*}(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}).$$

We thus obtain a projective system $\big(k_{n,*}(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1})\big)$ of pro-coherent sheaves of $\mathcal{O}[\![G]\!]$-modules on $\mathcal{X}$, and we define

$$\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1} := \lim_n k_{n,*}(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}), \tag{5.1.19}$$

the inverse limit being formed in the category $\operatorname{Pro}\operatorname{Coh}(\mathcal{X})$. It has the structure of a left $\mathcal{O}[\![G]\!]$-module in $\operatorname{Pro}\operatorname{Coh}(\mathcal{X})$. Since any two descriptions of $\mathcal{X}$ as a colimit of closed algebraic substacks are mutually cofinal, it is well-defined independently of the choice of such a description used in its construction.

Remark 5.1.20. If we combine (5.1.19) and (5.1.15), we find that there is a countable sequence $\mathcal{F}_n$ of $\mathfrak{a}$-power torsion $\mathcal{O}[\![K]\!]$-modules in $\mathrm{Coh}(\mathcal{X})$, and an $\mathcal{O}[\![K]\!]$-linear isomorphism

$$\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1} \xrightarrow{\sim} \lim_n \mathcal{F}_n \tag{5.1.21}$$

in $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$.

Our next results describe some of the base-change properties of $\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}$. We first consider its behaviour with respect to versal rings. In the statement of the next lemma, we make use of the completed pullback functor $\widehat{i}_x^*$ from Section B.3.16.

Lemma 5.1.22. *Let $\mathcal{Z} \to \mathcal{X}$ be a morphism of finite type with $\mathcal{Z}$ an algebraic stack with affine diagonal. Let S be a complete local Noetherian $\mathcal{O}$-algebra with finite residue field, and let $i_x : \mathrm{Spf}\, S \to \mathcal{Z}$ be a versal morphism at a finite type point x of $|\mathcal{Z}|$, corresponding to a formal étale (φ, Γ)-module $\widehat{D}_S$ with coefficients in S. Then $\widehat{i}_x^*\big(\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}\big)$, which is an object of* $\mathrm{Pro}\,\mathrm{Coh}(\mathrm{Spf}\, S) = \mathrm{Mod}_c(S)$*, is naturally isomorphic to $\widehat{D}^\natural_S \boxtimes \mathbf{P}^1$, as defined in Section* 4.7.

Proof. Since $\mathcal{Z}$ is algebraic, i_x is the $\mathfrak{m}_S$-adic completion of a morphism $i_x : \mathrm{Spec}\, S \to \mathcal{Z}$, i.e. the formal étale (φ, Γ)-module $\widehat{D}_S$ is the $\mathfrak{m}_S$-adic completion of an étale (φ, Γ)-module with coefficients in S, which we denote by D_S. By Lemma B.3.37, $\widehat{i}_x^*$ is the composite

$$\mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}) \xrightarrow{i_x^*} \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(S) \xrightarrow{(A.1.36)} \mathrm{Mod}_c(S)$$

where the first map is the Pro-extension of the usual coherent pullback $i_x^* : \mathrm{Coh}(\mathcal{Z}) \to \mathrm{Mod}^{\mathrm{fp}}(S)$. Since $\mathcal{Z}$ is of finite type over $\mathcal{O}/\varpi^a$ for some a, we see from [Stacks, Tag 0DR2] that i_x is flat. Hence Lemma 5.1.17 shows that

$$i_x^*\big(\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}\big) = \big(\widetilde{D^\natural_S \boxtimes \mathbf{P}^1}\big).$$

Applying Lemma 5.1.10, we conclude that

$$\widehat{i}_x^*\big(\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}\big) = (D^\natural_S \boxtimes \mathbf{P}^1)^w \in \mathrm{Mod}_c(S),$$

where the right-hand side is endowed with its weak topology. By Lemma 4.9.13 (4), we conclude that

$$\widehat{i}_x^*\big(\widetilde{D^\natural_{\mathcal{Z}} \boxtimes \mathbf{P}^1}\big) = \widehat{D}^\natural_S \boxtimes \mathbf{P}^1,$$

as desired. $\square$

Lemma 5.1.23. *Let S be a complete local Noetherian $\mathcal{O}$-algebra with finite residue field, and let $i_x : \mathrm{Spf}\, S \to \mathcal{X}$ be a versal morphism at a finite type point x of $|\mathcal{X}|$, corresponding to a formal étale (φ, Γ)-module $\widehat{D}_S$ with coefficients in S. Then $\widehat{i}_x^*\big(\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}\big)$, which is an object of* $\mathrm{Pro}\,\mathrm{Coh}(\mathrm{Spf}\, S) = \mathrm{Mod}_c(S)$*, is naturally isomorphic to $\widehat{D}^\natural_S \boxtimes \mathbf{P}^1$, as defined in Section* 4.7.

Proof. Fix a presentation $\mathcal{X} = \mathrm{colim}_n \mathcal{X}_n$ as a colimit of algebraic stacks under thickenings, and let $\mathrm{Spf}\, S_n \coloneqq \mathrm{Spf}\, S \times_{\mathcal{X}} \mathcal{X}_n$, so that $\mathrm{Spf}\, S = \mathrm{colim}_n \mathrm{Spf}\, S_n$. Let $i_n : \mathrm{Spf}\, S_n \to \mathcal{X}_n$ be the morphism induced by i_x, which is versal to $\mathcal{X}_n$ at x. By Lemma B.3.9 we have

$$\widehat{i}_x^*\big(\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}\big) = \lim_n \widehat{i}_n^*\big(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}\big),$$

the limit being taken in $\mathrm{Mod}_c(S)$. By Lemma 5.1.22, we have natural isomorphisms

$$\widehat{i}_n^*\big(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}\big) = \widehat{D}^\natural_{S_n} \boxtimes \mathbf{P}^1. \tag{5.1.24}$$

Applying $\lim_n$ to (5.1.24) concludes the proof. □

We now consider the base-change properties of $\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}$ with respect to finite type morphisms of algebraic stacks into $\mathcal{X}$.

Remark 5.1.25. In anticipation of the statement of Lemma 5.1.26, we recall the stack $\mathcal{X}(\mathrm{St})$ from Definition 3.4.11, and we note that if $\mathcal{Y} \to \mathcal{X}$ is a finite type morphism whose domain is an algebraic stack, then $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St})$ is a formal algebraic substack of $\mathcal{Y}$, equal to the completion of $\mathcal{Y}$ along its closed algebraic substack $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St})_{\mathrm{red}}$. Thus $\mathrm{Coh}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St}))$ embeds as a full subcategory of $\mathrm{Coh}(\mathcal{Y})$, with essential image equal to the full subcategory of $\mathrm{Coh}(\mathcal{Y})$ consisting of sheaves set-theoretically supported on $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St})_{\mathrm{red}}$. Consequently, $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St}))$ embeds as a full subcategory of $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{Y})$.

Lemma 5.1.26. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a finite type morphism whose domain is an algebraic stack with affine diagonal (so that $\mathcal{Y}$ is necessarily of finite type over $\mathcal{O}/\varpi^a$ for some $a \geq 1$), and let $D_{\mathcal{Y}}$ denote the pullback to $\mathcal{Y}$ of the universal (φ, Γ)-module over $\mathcal{X}$. Then there is a base-change morphism*

$$f^*\big(\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}\big) \to \widetilde{D^\natural_{\mathcal{Y}} \boxtimes \mathbf{P}^1}$$

(*where f^* denotes the pullback functor on pro-coherent sheaves of Remark* B.3.2) *of $\mathcal{O}[\![G]\!]$-module objects in* $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{Y})$, *whose kernel and cokernel, which* a priori *are $\mathcal{O}[\![G]\!]$-module objects of* $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{Y})$*, are in fact $\mathcal{O}[\![G]\!]$-module objects of* $\mathrm{Pro}\,\mathrm{Coh}\big(\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St})\big)$.

If f is furthermore a closed immersion, then this kernel and cokernel are in fact objects of $\mathrm{Coh}\big(\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St})\big)$.

Proof. As in the discussion at the beginning of Section 5.1.18, write $\mathcal{X}$ as a colimit of thickenings of closed algebraic substacks $k_n : \mathcal{X}_n \hookrightarrow \mathcal{X}$, so that $\mathcal{X} \coloneqq \mathrm{colim}\, \mathcal{X}_n$. Since $\mathcal{Y}$ is an algebraic stack, and the morphism f is of finite type, we see that f factors through $\mathcal{X}_n$ for n sufficiently large, and so relabelling the $\mathcal{X}_n$ if necessary, we may assume that f factors through a morphism $f_n : \mathcal{Y} \to \mathcal{X}_n$ for each n.

From (5.1.19) we obtain isomorphisms

$$f^* \widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1} \xrightarrow{\sim} \lim_n f^* k_{n,*}(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}) \xrightarrow{\sim} \lim_n f_n^*(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}). \tag{5.1.27}$$

(To see the first isomorphism, recall that f^* is *defined* to be the pro-extension of pullback on coherent sheaves, and so commutes with arbitrary cofiltered limits; the second isomorphism holds simply because $f = k_n \circ f_n$ and $k_{n,*}$ is fully faithful.) Now Lemma 5.1.17 gives us, for each n, a base-change morphism

$$f_n^*(\widetilde{D^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1}) \to \widetilde{D^\natural_{\mathcal{Y}} \boxtimes \mathbf{P}^1}, \tag{5.1.28}$$

whose kernel and cokernel, which we denote by $\mathcal{K}_n$ and $\mathcal{C}_n$ respectively, are coherent sheaves on $\mathcal{Y}$ which are set-theoretically supported on $\mathcal{Y}(\mathrm{St}) := \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}(\mathrm{St})$. By definition, this means we may regard $\mathcal{C}_n$ and $\mathcal{K}_n$ as coherent sheaves on the formal algebraic stack $\mathcal{Y}(\mathrm{St})$. The morphisms (5.1.28) form a projective system as n varies, and thus so do the $\mathcal{C}_n$ and $\mathcal{K}_n$; we then let $\mathcal{C} := \lim_n \mathcal{C}_n$ and $\mathcal{K} := \lim_n \mathcal{K}_n$, the inverse limits being formed in $\operatorname{Pro}\operatorname{Coh}(\mathcal{Y}(\mathrm{St}))$, or equivalently in $\operatorname{Pro}\operatorname{Coh}(\mathcal{Y})$.

Passing to the inverse limit over the morphisms (5.1.28), and taking into account the isomorphism (5.1.27), we obtain the desired base-change morphism

$$f^*(\widetilde{D^{\natural}_{\mathcal{X}} \boxtimes \mathbf{P}^1}) \to \widetilde{D^{\natural}_{\mathcal{Y}} \boxtimes \mathbf{P}^1},$$

whose kernel and cokernel are isomorphic to $\mathcal{K}$ and $\mathcal{C}$ respectively.

Assume now that f is a closed immersion; it remains to prove that in this case $\mathcal{C}$ and $\mathcal{K}$ are in fact objects of $\operatorname{Coh}(\mathcal{Y}(\mathrm{St}))$ (and not merely pro-objects). Since f is a closed immersion, the stack $\mathcal{Y}(\mathrm{St})$ is a closed formal algebraic substack of the formal algebraic stack $\mathcal{X}(\mathrm{St})$, and so $\operatorname{Coh}(\mathcal{Y}(\mathrm{St}))$ fully faithfully embeds into $\operatorname{Coh}\big(\mathcal{X}(\mathrm{St})\big)$. Thus, we may regard each of $\mathcal{C}_n$ and $\mathcal{K}_n$ as objects of $\operatorname{Coh}\big(\mathcal{X}(\mathrm{St})\big)$, and hence regard $\mathcal{C}$ and $\mathcal{K}$ as objects of $\operatorname{Pro}\operatorname{Coh}\big(\mathcal{X}(\mathrm{St})\big)$. It then suffices to show that $\mathcal{C}$ and $\mathcal{K}$ are in fact objects of $\operatorname{Coh}\big(\mathcal{X}(\mathrm{St})\big)$.

If $\mathcal{X}(\mathrm{St})$ is empty, then of course this entire discussion is vacuous and there is nothing to prove. Otherwise, after making an unramified quadratic base-change in our coefficients and a twist, if necessary, we may write

$$\mathcal{X}(\mathrm{St}) = \coprod \mathcal{X}_{\overline{\theta}}$$

where $\overline{\theta}$ runs through the four quadratic twists of $1 + \omega^{-1}$. By the results of Section 3.4.9, for each $\overline{\theta}$ there exists a finite type $\mathbf{Z}$-graded $R^{\mathrm{ps}}_{\overline{\theta}}$-algebra S, and an isomorphism

$$\operatorname{colim}_n [\operatorname{Spec}(S/\mathfrak{m}^n_{R^{\mathrm{ps}}_{\overline{\theta}}} S)/\mathbf{G}_m] \xrightarrow{\sim} \mathcal{X}_{\overline{\theta}}.$$

Writing $i_x : \mathcal{X}_x \to \mathcal{X}_{\overline{\theta}}$ for the completion of $\mathcal{X}_{\overline{\theta}}$ at its unique closed point, we thus deduce from Theorem B.4.17 (3) that the completed pullback is an equivalence

$$\widehat{i}_x : \operatorname{Coh}(\mathcal{X}_{\overline{\theta}}) \xrightarrow{\sim} \operatorname{Coh}(\mathcal{O}_{\mathcal{X}_x}).$$

Thus, in order to show that $\lim_n \mathcal{K}_n$ and $\lim_n \mathcal{C}_n$ are objects of $\operatorname{Coh}(\mathcal{X}(\mathrm{St}))$, it suffices to show (for each $\overline{\theta}$) that $\lim_n \widehat{i}_x \mathcal{K}_n$ and $\lim_n \widehat{i}_x \mathcal{C}_n$ are objects of $\operatorname{Coh}(\mathcal{O}_{\mathcal{X}_x})$.

Let $v : \operatorname{Spf} R^{\mathrm{ver}} \to \mathcal{X}_{\overline{\theta}}$ be the versal ring at the closed point induced by the completion of S at its maximal homogeneous ideal $\mathfrak{m}_S$, so that there is an isomorphism

$$\operatorname{colim}_n [\operatorname{Spec}(R^{\mathrm{ver}}/\mathfrak{m}^n_{R^{\mathrm{ver}}} R^{\mathrm{ver}})/\mathbf{G}_m] \xrightarrow{\sim} \mathcal{X}_x. \tag{5.1.29}$$

To ease notation, we write $R := R^{\mathrm{ver}}$ for the remainder of the proof, and let $I \subset R$, resp. $I_n \subset R$, be the ideal of the pullback of $\mathcal{Y} \to \mathcal{X}$ to $\operatorname{Spf} R$, resp. the ideal of the pullback of $\mathcal{X}_n \to \mathcal{X}$ to $\operatorname{Spf} R$. Since $\mathcal{X}$ is Noetherian, the I_n are cofinal with the powers I_1^n.

Since $v : \operatorname{Spf} R \to \mathcal{X}_x$ is smooth and surjective, and the coherence of a pro-coherent sheaf can be tested after pulling back to a flat cover (by Remark B.3.25), it now suffices to prove that the pullbacks $v^* \lim_n \widehat{i}^*_x \mathcal{K}_n$ and $v^* \lim_n \widehat{i}^*_x \mathcal{C}_n$ are objects of $\operatorname{Coh}(\operatorname{Spf} R)$. Bearing in mind the isomorphism (5.1.29), we see that the morphism $v : \operatorname{Spf} R \to \mathcal{X}_x$ is representable by algebraic stacks, and so the pullback v^* coincides with the completed pullback $\widehat{v}^*$, by Remark B.3.15. For simplicity, in the following we will also write v for the induced morphisms $\operatorname{Spf} R/I \to \mathcal{Y}_x, \operatorname{Spf} R/I_n \to \mathcal{X}_{n,x}$.

Applying $v^*\widehat{i}_x^*$ to (5.1.28), we obtain an exact sequence

$$0 \to v^*\widehat{i}_x^*\mathcal{K}_n \to v^*\widehat{i}_x^* f_n^*(\widetilde{D}^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1) \to v^*\widehat{i}_x^* \widetilde{D^\natural_{\mathcal{Y}} \boxtimes \mathbf{P}^1} \to v^*\widehat{i}_x^*\mathcal{C}_n \to 0.$$

Lemma 5.1.22 shows that $v^*\widehat{i}_x^* \widetilde{D^\natural_{\mathcal{Y}} \boxtimes \mathbf{P}^1} = \widehat{D}^\natural_{R/I} \boxtimes \mathbf{P}^1$. Similarly, using the fact that $v^* = \widehat{v}^*$, the commutative diagram (B.3.12) implies that

$$v^*\widehat{i}_x^* f_n^*(\widetilde{D}^\natural_{\mathcal{X}_n} \boxtimes \mathbf{P}^1) = (\widehat{D}^\natural_{R/I_n} \boxtimes \mathbf{P}^1) \otimes_{R/I_n} R/I,$$

and so we have an exact sequence

$$0 \to v^*\widehat{i}_x^*\mathcal{K}_n \to (\widehat{D}^\natural_{R/I_n} \boxtimes \mathbf{P}^1) \otimes_{R/I_n} R/I \to \widehat{D}^\natural_{R/I} \boxtimes \mathbf{P}^1 \to v^*\widehat{i}_x^*\mathcal{C}_n \to 0.$$

By Lemma 4.10.8, it follows that $\lim_n v^*\widehat{i}_x^*\mathcal{K}_n$ and $\lim_n v^*\widehat{i}_x^*\mathcal{C}_n$ are objects of $\mathrm{Coh}(\mathrm{Spf}\, R)$, as desired. $\square$

5.1.30. *Defining* L_∞ *and* F. Recall from Remark 4.5.3 that the centre Z of G acts on $D^\natural \boxtimes \mathbf{P}^1$ by multiplication by $\zeta\varepsilon^{-2} : \mathbf{Q}_p^\times \to \mathcal{O}^\times$, i.e. $D^\natural \boxtimes \mathbf{P}^1$ is a left $\mathcal{O}[\![G]\!]_{\zeta\varepsilon^{-2}}$-module.

Definition 5.1.31. We let L_∞ denote the right $\mathcal{O}[\![G]\!]_\zeta$-module in $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$ obtained by twisting the left $\mathcal{O}[\![G]\!]_{\zeta\varepsilon^{-2}}$-action on $\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}$ by $\zeta^{-1}\varepsilon \circ \det$, and then turning the resulting left $\mathcal{O}[\![G]\!]_{\zeta^{-1}}$-action into a right $\mathcal{O}[\![G]\!]_\zeta$-action by precomposing with $g \mapsto g^{-1}$.

Lemma A.10.10 (2h) (and the equivalence $D^b(\mathrm{Coh}(\mathcal{X})) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{X})$ of (B.1.3)) then allows us to define a right t-exact functor

$$D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}), \quad \pi \mapsto L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} \pi. \tag{5.1.32}$$

Definition 5.1.33. We let

$$\mathrm{F} : D^b_{\mathrm{fp}}(\mathcal{A}) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}) \tag{5.1.34}$$

denote the right t-exact functor

$$\pi \mapsto L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} \pi$$

given by the composite of (5.1.32) and the t-exact fully faithful functor

$$D^b_{\mathrm{fp}}(\mathcal{A}) \hookrightarrow D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta)$$

induced by (2.2.26).

Remark 5.1.35. We will see below in Proposition 5.3.23 that F factors through a functor $D^b_{\mathrm{fp}}(\mathcal{A}) \to D^b_{\mathrm{coh}}(\mathcal{X})$, though this does not seem obvious from its definition.

The following result will allow us to get something of a handle on the pro-coherent structure of L_∞.

Lemma 5.1.36. *If $i : \mathcal{Y} \hookrightarrow \mathcal{X}$ is the inclusion of a closed algebraic substack, then:*

(1) *each $(i^*L_\infty)_{K_n}$ is a coherent sheaf on $\mathcal{Y}$, and*
(2) *$i^*L_\infty \xrightarrow{\sim} \lim_n (i^*L_\infty)_{K_n}$.*

Proof. Evidently the twists and duals in Definition 5.1.31 play no role in these assertions, and so it is enough to prove the statements of the lemma with L_∞ replaced by $\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}$. Let $p : \mathrm{Spec}\, A \to \mathcal{Y}$ be an *fppf* and surjective morphism. By Remark B.3.25, the coherence of a pro-coherent sheaf can be established after pulling back to a flat cover, so it suffices to prove that

(1) each $(\widetilde{p^*i^*D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1})_{K_n}$ is a coherent sheaf on $\operatorname{Spec} A$, and
(2) $\widetilde{p^*i^*D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1} \xrightarrow{\sim} \lim_n(\widetilde{p^*i^*D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1})_{K_n}$.

Both of these statements follow immediately from the assertion that $\widetilde{p^*i^*D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}$ is isomorphic to $\widetilde{M}$ for some finitely presented $A[\![K]\!]$-module M, which we now prove.

Consider the base-change morphism

$$p^*i^*(\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}) \to \widetilde{D^\natural_A \boxtimes \mathbf{P}^1} \tag{5.1.37}$$

of Lemma 5.1.26. That lemma shows that we have an exact sequence of right $\mathcal{O}[\![G]\!]$-modules in $\operatorname{Pro}\operatorname{Coh}(\operatorname{Spec} A)$ of the form

$$0 \to \mathcal{K} \to p^*i^*(\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1}) \to \widetilde{D^\natural_A \boxtimes \mathbf{P}^1} \to \mathcal{C} \to 0, \tag{5.1.38}$$

where $\mathcal{K}$ and $\mathcal{C}$ are furthermore coherent sheaves.

Recall from Remarks 2.2.12 and 2.2.13 that we write $A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$ for the category of $\mathfrak{a}$-power torsion $\mathcal{O}[\![K]\!]$-modules in $\operatorname{Coh}(\operatorname{Spec} A)$, and $\mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A)$ for the category of $\mathcal{O}[\![K]\!]$-modules in $\operatorname{Pro}\operatorname{Coh}(\operatorname{Spec} A)$. The forgetful functor $A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A) \to \operatorname{Coh}(\operatorname{Spec} A)$ is exact and faithful, and by Remark 2.2.13 it extends to an exact, fully faithful, cofiltered limit-preserving functor

$$\operatorname{Pro} A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A) \to \mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A). \tag{5.1.39}$$

The functor

$$\operatorname{Mod}^{\mathrm{fp}}(A[\![K]\!]) \to \mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A),\ M \mapsto \widetilde{M} := \lim_i M/\mathfrak{a}^i M \tag{5.1.40}$$

factors through (5.1.39) via the functor

$$\operatorname{Mod}^{\mathrm{fp}}(A[\![K]\!]) \to \operatorname{Pro} A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A),\ M \mapsto \lim_i M/\mathfrak{a}^i M \tag{5.1.41}$$

obtained from (2.2.8), and Corollary 4.8.13 implies that $\widetilde{D^\natural_A \boxtimes \mathbf{P}^1}$, which is *a priori* an object of $\mathcal{O}[\![K]\!]\text{-}\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(A)$, is contained in the essential image of (5.1.41). By (5.1.21), the same is true for $p^*i^*(\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1})$. The exact sequence (5.1.38) can therefore be viewed as an exact sequence in $\operatorname{Pro} A[\![K]\!]\text{-}\operatorname{Mod}^{\mathrm{fp}}(A)$.

We are now in a position to apply Lemma 2.2.7 (6) (bearing in mind Remark 2.2.12), which shows that $p^*i^*(\widetilde{D^\natural_{\mathcal{X}} \boxtimes \mathbf{P}^1})$ is contained in the essential image of (5.1.41) if this is true for $\mathcal{K}$ and $\mathcal{C}$ (recalling that we have already seen that this is true of $\widetilde{D^\natural_A \boxtimes \mathbf{P}^1}$). Finally, since $\mathcal{K}$ and $\mathcal{C}$ are objects of $\operatorname{Coh}(\operatorname{Spec} A)$ in the essential image of (5.1.39), they are the coherent sheaves underlying $\mathfrak{a}$-adically discrete $\mathcal{O}[\![K]\!]$-modules in $\operatorname{Mod}^{\mathrm{fp}}(A)$, and so (again bearing in mind Remark 2.2.12) they are contained in the essential image of (5.1.41), as desired. □

Remark 5.1.42. As already alluded to above, the significance of Lemma 5.1.36 is that it gives us a concrete description of the pro-coherent structure of L_∞. Namely, if we write $\mathcal{X} = \operatorname{colim}_m \mathcal{X}_m$ as an Ind-algebraic stack with closed transition morphisms, and write $i_m : \mathcal{X}_m \hookrightarrow \mathcal{X}$ for the corresponding closed immersion, then (bearing in mind Lemma B.3.5) we have

$$L_\infty \xrightarrow{\sim} \lim_{m,n}(i_m^* L_\infty)_{K_n}. \tag{5.1.43}$$

Actually, it follows from Proposition 5.3.20 below that that $(L_\infty)_{K_n}$ is supported on a ϖ-adic formal algebraic closed substack of $\mathcal{X}$, and so (5.1.43) does not convey the whole truth of the matter. Nevertheless, it is a key intermediate step in the ultimate analysis of L_∞.

5.2. **Completing L_∞.** A key tool in our analysis of the functor F will be completion: both in the sense of completion at a block $\mathcal{A}_{\overline{\theta}}$ of the category $\mathcal{A}$, and completion along the various substacks $\mathcal{X}_{\overline{\theta}}$ of $\mathcal{X}$. In fact, we will show that these two notions of completion are compatible, in an appropriate sense, with respect to F (see Proposition 5.3.6). Relatedly, we will introduce a "completed-at-$\overline{\theta}$" version of F, and prove that it is fully faithful (see Theorem 5.2.24). This will be a key ingredient in the proof that F itself is fully faithful.

5.2.1. *A completed version of* F. We begin by defining a completed version $\mathrm{F}_{\overline{\theta}}$ of F. It is in fact convenient to do this for any closed subspace $Y \subseteq |X|$. Recall that in (3.3.9) we have defined a continuous map $\pi_{\mathrm{ss}} : |\mathcal{X}| \to |X|$.

Definition 5.2.2.

(1) If $Y \subseteq |X|$ is a closed subset, then we write $\mathcal{X}_Y$ for the completion of $\mathcal{X}$ along $\pi_{\mathrm{ss}}^{-1}(Y)$. By Proposition B.2.8 we identify $D^b_{\mathrm{coh}}(\mathcal{X}_Y)$ with $D^b_{\mathrm{coh},Y}(\mathcal{X})$, i.e. the full subcategory of complexes whose cohomology sheaves are set-theoretically supported on Y.

(2) We write $\widehat{i}'_{Y,*} : \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_Y) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X})$, $\widehat{i}'^*_Y : \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_Y)$ for the adjoint pair of t-exact functors defined in Appendix B.3.16.

Remark 5.2.3. We remind the reader that we have already defined functors $i_{Y,*}$ and $\widehat{i}_{Y,*}$, on derived categories of G-representations, in (2.6.1) and (2.6.9), and similarly for $\widehat{i}^*_Y$. This is the reason for the notation i'_Y.

Remark 5.2.4. In the case that $Y = \{\overline{\theta}\}$ is a single closed point (identified with a conjugacy class of $\overline{\mathbf{F}}_p$-valued pseudorepresentations as in Section 2.5.1) we also have the stack $\mathcal{X}_{\overline{\theta}} \to \mathcal{X}$ introduced in (3.4.2). By Theorem 3.4.3, $\mathcal{X}_{\overline{\theta}}$ is canonically isomorphic to $\mathcal{X}_Y$, which we will therefore also denote by $\mathcal{X}_{\overline{\theta}}$. Furthermore, we will tacitly use results and notation from Section 3.4 and Section 3.5, so that, for example, $k_{\overline{\theta}} : \mathcal{X}_{\overline{\theta}} \to \mathfrak{X}_{\overline{\theta}}$ is the $\mathfrak{m}$-adic completion morphism.

Lemma 5.2.5. *For any closed subset $Y \subseteq |X|$, we have a natural isomorphism of right t-exact functors $D^b_{\mathrm{fp}}(\mathcal{O}[\![G]\!]_\zeta) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_Y)$*

$$\widehat{i}'^*_Y(L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} -) \xrightarrow{\sim} \widehat{i}'^*_Y L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} -.$$

Proof. By Lemma A.10.10 (2), it suffices to note that each functor takes $\mathcal{O}[\![G]\!]_\zeta$ to $\widehat{i}'^*_Y L_\infty$. (Note that this argument is just a particular case of an evident variant, in the derived context, of Lemma A.1.55.) ☐

Definition 5.2.6. For any closed subset $Y \subseteq |X|$, we let F_Y denote the functor

$$\mathrm{F}_Y := \widehat{i}'^*_Y \mathrm{F}\, i_{Y,*} : D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_Y).$$

In particular, for any 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation $\overline{\theta}$, we let $\mathrm{F}_{\overline{\theta}}$ denote the functor

$$\mathrm{F}_{\overline{\theta}} := \widehat{i}'^*_{\overline{\theta}} \mathrm{F}\, i_{\overline{\theta},*} : D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}). \tag{5.2.7}$$

By Lemma 5.2.5 we have a natural isomorphism of functors $D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$

$$\mathrm{F}_{\overline{\theta}}(-) := \widehat{i}'^*_{\overline{\theta}} \mathrm{F}(i_{\overline{\theta},*}(-)) = \widehat{i}'^*_{\overline{\theta}}(L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-)) \xrightarrow{\sim} \widehat{i}'^*_{\overline{\theta}} L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-). \tag{5.2.8}$$

5.2.9. *An alternative description of* $\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty$. Our next goal is to give an alternate description of the functors $\mathrm{F}_{\overline{\theta}}$, which will allow us to relate them to the functors studied in [JNW24], and thus also to the results of Colmez [Col10c] and Paškūnas [Paš13]. In view of (5.2.8), the key step in giving such a description is to describe the sheaf $\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty$, which we do in Proposition 5.2.18 below, by relating it to the sheaf $\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}}$, whose definition we now recall.

We have defined in Definition 2.4.17 a projective object $\widetilde{P}_{\overline{\theta}}$ of $\mathfrak{C}_{\overline{\theta}}$, whose endomorphism algebra $\widetilde{E}_{\overline{\theta}}$ is equipped with a canonical isomorphism $\widetilde{E}_{\overline{\theta}} \xrightarrow{\sim} \widetilde{R}^{\mathrm{op}}_{\overline{\theta}}$. If $\overline{\theta}$ is not of type (St), then $\widetilde{P}_{\overline{\theta}}$ is even a projective generator of $\mathfrak{C}_{\overline{\theta}}$. If $\overline{\theta}$ is of type (St), then $\widetilde{P}_{\overline{\theta}}$ is not a projective generator of $\mathfrak{C}_{\overline{\theta}}$; rather, we have the projective generator $\mathbf{P}_{\overline{\theta}}$, with endomorphism algebra $\mathbf{E}_{\overline{\theta}}$, introduced in Definition 3.6.2. In Section 3.6.4, we have also introduced a two-sided ideal $J \subset \mathbf{E}_{\overline{\theta}}$.

On the other hand, recall from (3.5.22) that $\mathcal{V}_{\overline{\theta}}$ denotes the universal object on $\mathcal{X}_{\overline{\theta}}$, and is a complete right $\widetilde{E}_{\overline{\theta}}$-module in $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$. When $\overline{\theta}$ has type (St), recall furthermore the sheaf $\mathcal{W}_{\overline{\theta}}$ from Definition 3.6.12: it is a complete right $\mathbf{E}_{\overline{\theta}}$-module in $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$.

We will need two preliminary results. For the first one, recall from (B.4.6) the exact and fully faithful functor

$$\begin{gathered}\operatorname{Coh}(\mathcal{O}_{\mathcal{X}_{\overline{\theta}}}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}}),\\ \mathcal{F} \mapsto \lim_n \mathcal{F}/\mathfrak{m}^{n+1}_{R^{\mathrm{ps}}_{\overline{\theta}}}\mathcal{F}.\end{gathered}$$

Lemma 5.2.10. *For every compact open subgroup $H \subset G$, the pro-coherent sheaves*

$$\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}})_H$$

and

$$(\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty)_H$$

on $\mathcal{X}_{\overline{\theta}}$ are objects of $\operatorname{Coh}(\mathcal{O}_{\mathcal{X}_{\overline{\theta}}})$. Furthermore, the natural maps

$$\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty \xrightarrow{\sim} \lim_H (\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty)_H \tag{5.2.11}$$

and

$$\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}} \xrightarrow{\sim} \lim_H \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}})_H \tag{5.2.12}$$

are isomorphisms in $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$.

Proof. Let $R := R^{\mathrm{ps}}_{\overline{\theta}}, \mathfrak{m} := \mathfrak{m}_{R^{\mathrm{ps}}_{\overline{\theta}}}$ and $\mathcal{F} := \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}})_H$. To show that $\mathcal{F} \in \operatorname{Coh}(\mathcal{O}_{\mathcal{X}_{\overline{\theta}}})$ it suffices to prove that $\mathcal{F} \xrightarrow{\sim} \lim_n \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$, and that each $\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ is an object of $\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$.

For the first statement, we begin with the isomorphism $\widetilde{P}_{\overline{\theta}} \xrightarrow{\sim} \lim_n \widetilde{P}_{\overline{\theta}}/\mathfrak{m}^{n+1}\widetilde{P}_{\overline{\theta}}$ arising from the fact that $\widetilde{P}_{\overline{\theta}}$ is a complete $\widetilde{E}_{\overline{\theta}}$-module in $\operatorname{Mod}_c(\mathcal{O})$ (by Lemma A.1.59), hence a complete R-module (by Lemma A.10.18). We then use the facts that the functor $(-)_H$ preserves cofiltered limits in $\operatorname{Mod}_c(\mathcal{O}[\![H]\!]^{\mathrm{op}})$ (by Lemma A.1.51), and the functor $\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} -$ preserves cofiltered limits in $\operatorname{Mod}_c(\widetilde{E}_{\overline{\theta}})$ (by definition). This implies that $\mathcal{F} \xrightarrow{\sim} \lim_n \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$, and also that

$$\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} = \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}}/\mathfrak{m}^{n+1}\widetilde{P}_{\overline{\theta}})_H.$$

Now $\widetilde{P}_{\overline{\theta}}/\mathfrak{m}^{n+1}\widetilde{P}_{\overline{\theta}}$ is coadmissible: in fact, it is a quotient of $\widetilde{P}_{\overline{\theta}}/\operatorname{rad}(\widetilde{E}_{\overline{\theta}})^j\widetilde{P}_{\overline{\theta}}$ for some j, since $\widetilde{E}_{\overline{\theta}}$ is a finite R-module, and $\widetilde{P}_{\overline{\theta}}/\operatorname{rad}(\widetilde{E}_{\overline{\theta}})^j\widetilde{P}_{\overline{\theta}}$ is coadmissible, by Lemma 2.3.11 (3). Hence $(\widetilde{P}_{\overline{\theta}}/\mathfrak{m}^{n+1}\widetilde{P}_{\overline{\theta}})_H$ is finitely presented over $\widetilde{E}_{\overline{\theta}}$ (in fact, even over $\mathcal{O}$). We conclude that

$$\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} = \mathcal{V}_{\overline{\theta}} \otimes_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}}/\mathfrak{m}^{n+1}\widetilde{P}_{\overline{\theta}})_H = (\mathcal{V}_{\overline{\theta}}/\mathfrak{m}^{n+1}\mathcal{V}_{\overline{\theta}}) \otimes_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}}/\mathfrak{m}^{n+1}\widetilde{P}_{\overline{\theta}})_H,$$

and since $\mathcal{V}_{\overline{\theta}}/\mathfrak{m}^{n+1}\mathcal{V}_{\overline{\theta}}$ is a coherent sheaf on $\mathcal{X}_{\overline{\theta}}$, this implies that $\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ is a coherent sheaf on $\mathcal{X}_{\overline{\theta}}$, as desired. This concludes the proof that $\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}})_H \in \operatorname{Coh}(\mathcal{O}_{\mathcal{X}_{\overline{\theta}}})$.

Similarly, let $\mathcal{G} := (\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty)_H$. By definition we have $\mathcal{X}_{\overline{\theta}} = \operatorname{colim}_n \mathcal{X}_{\overline{\theta}}/\mathfrak{m}^{n+1}$, and $\mathcal{X}_{\overline{\theta}}/\mathfrak{m}^{n+1}$ is an algebraic stack. Writing $i'_{\overline{\theta},n+1} : \mathcal{X}_{\overline{\theta}}/\mathfrak{m}^{n+1} \to \mathcal{X}$ for the restriction of $i'_{\overline{\theta}}$, Lemma B.3.5 shows that

$$\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty = \lim_n (i^{\prime *}_{\overline{\theta},n+1} L_\infty), \tag{5.2.13}$$

where $i^{\prime *}_{\overline{\theta},n+1} L_\infty = \widehat{i}^{\prime *}_{\overline{\theta}} L_\infty/\mathfrak{m}^{n+1}\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty$. Since $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$ has exact cofiltered limits, the coinvariant functor $(-)_H$ preserves cofiltered limits of right $\mathcal{O}[\![H]\!]$-modules in $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$ (by Lemma A.1.51). We thus deduce that

$$\mathcal{G} = \lim_n (i^{\prime *}_{\overline{\theta},n+1} L_\infty)_H,$$

and since $(i^{\prime *}_{\overline{\theta},n+1} L_\infty)_H = \mathcal{G}/\mathfrak{m}^{n+1}\mathcal{G}$, we conclude that $\mathcal{G} \xrightarrow{\sim} \lim_n \mathcal{G}/\mathfrak{m}^{n+1}\mathcal{G}$, and (using Lemma 5.1.36 (1)) that $\mathcal{G}/\mathfrak{m}^{n+1}\mathcal{G}$ is coherent. This concludes the proof that $(\widehat{i}^{\prime *}_{\overline{\theta}} L_\infty)_H \in \operatorname{Coh}(\mathcal{O}_{\mathcal{X}_{\overline{\theta}}})$.

The isomorphism (5.2.11) now follows from (5.2.13) and Lemma 5.1.36 (2). Finally, the isomorphism (5.2.12) follows from the isomorphism $\widetilde{P}_{\overline{\theta}} \xrightarrow{\sim} \lim_H (\widetilde{P}_{\overline{\theta}})_H$ (which is valid for every object of $\mathfrak{C}$) and the fact that $\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} -$ preserves cofiltered limits. $\square$

Before stating the next result, we recall from Remark 2.2.28 that objects of $\mathfrak{C}_{\overline{\theta}}$, such as $\widetilde{P}_{\overline{\theta}}$, can be regarded as right $\mathcal{O}[\![G]\!]_\zeta$-modules in a natural way.

Proposition 5.2.14. *Let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation of $G_{\mathbf{Q}_p}$ of determinant $\zeta\varepsilon^{-1}$.*

(1) If S is a complete Noetherian local $\mathcal{O}$-algebra with finite residue field, and $i'_S : \operatorname{Spf} S \to \mathcal{X}_{\overline{\theta}}$ is a versal morphism at the closed point of $\mathcal{X}_{\overline{\theta}}$, then there is a natural isomorphism of right $\mathcal{O}[\![G]\!]_\zeta$-modules

$$(\widehat{i}^{\prime *}_S \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})/(\widehat{i}^{\prime *}_S \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)} \xrightarrow{\sim} \widehat{i}^{\prime *}_S L_\infty, \tag{5.2.15}$$

where the left-hand side has the right $\mathcal{O}[\![G]\!]_\zeta$-action induced from that on $\widetilde{P}_{\overline{\theta}}$, and the right-hand side has the right $\mathcal{O}[\![G]\!]_\zeta$-action induced from that on L_∞. (Here, "natural" means "natural in the versal morphism i'_S".)

(2) If $\overline{\theta}$ is not of type (St)*, then* (5.2.15) *induces a natural isomorphism of right $\mathcal{O}[\![G]\!]_\zeta$-modules*

$$\widehat{i}^{\prime *}_S \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}} \xrightarrow{\sim} \widehat{i}^{\prime *}_S L_\infty. \tag{5.2.16}$$

(3) If $\overline{\theta}$ is of type (St)*, then* (5.2.15) *induces a natural isomorphism of right $\mathcal{O}[\![G]\!]_\zeta$-modules*

$$(\widehat{i}'^*_S\mathcal{W}_{\overline{\theta}}/\widehat{i}'^*_S\mathcal{W}_{\overline{\theta}}[J])\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}\mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} \widehat{i}'^*_S L_\infty, \tag{5.2.17}$$

where $J \subset \mathbf{E}_{\overline{\theta}}$ is the two-sided ideal defined in Section 3.6.4*, and $\widehat{i}'^*_S\mathcal{W}_{\overline{\theta}}[J]$ denotes the $\mathbf{E}_{\overline{\theta}}$-submodule of $\widehat{i}'^*_S\mathcal{W}_{\overline{\theta}}$ consisting of elements annihilated by J.*

Proof. The existence of these isomorphisms is a consequence of Colmez's results in [Col10c, IV.4], as we now explain. It will be convenient to follow the exposition of [CD14].

We first note that the isomorphism i'_S gives rise to a formal étale (φ,Γ)-module D_S with coefficients in S, and Lemma 5.1.23, together with Definition 5.1.31, shows that

$$\widehat{i}'^*_S L_\infty \xrightarrow{\sim} (D_S^\natural \boxtimes \mathbf{P}^1)(\zeta^{-1}\varepsilon\circ\det).$$

Secondly, we recall that the morphism $\mathcal{X}_{\overline{\theta}} \to \mathcal{X}$ is defined so the induced family of étale (φ,Γ)-modules over $\mathcal{X}_{\overline{\theta}}$ is the one that corresponds to the universal Galois representation $\mathcal{V}_{\overline{\theta}}$. Consequently, we see that $\mathbf{V}(D_S)$ (the 2-dimensional representation of $G_{\mathbf{Q}_p}$ over S determined by D_S) coincides with $\widehat{i}'^*_S\mathcal{V}_{\overline{\theta}}$. Our desired isomorphism (5.2.15) can then be rewritten in the form

$$\big(\mathbf{V}(D_S)\,\widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}}\widetilde{P}_{\overline{\theta}}\big)/\big(\mathbf{V}(D_S)\,\widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}}\widetilde{P}_{\overline{\theta}}\big)^{\mathrm{SL}_2(\mathbf{Q}_p)} \xrightarrow{\sim} (D_S^\natural \boxtimes \mathbf{P}^1)(\zeta^{-1}\varepsilon\circ\det),$$

and it is in this form that we will prove it.

Let A be an Artinian quotient of S, and set $D_A \coloneqq D_S\otimes_S A$. Then $\mathbf{V}(D_A)\otimes_{\widetilde{E}_{\overline{\theta}}}\widetilde{P}_{\overline{\theta}}$ is a finite length object of $\mathfrak{C}_{\overline{\theta}} \subset \mathfrak{C}$, so it is the Pontrjagin dual $\Pi^\vee$ of an object Π of the category $\mathrm{Rep}_{\mathrm{tors}}(\zeta)$ considered in [CD14, Section III.1].

The (φ,Γ)-module denoted $\mathbf{D}(\Pi)$ in [Col10c] is, by the definition just preceding the statement of [Col10c, Thm. IV.2.12], isomorphic to the (φ,Γ)-module of $V(\Pi)^\vee\otimes\varepsilon$, where V is normalized as in [Col10c]. This is the same as $V(\Pi)^\vee$ in our notation. Hence $\mathbf{D}(\Pi)$ is equal to (in our notation)

$$\mathbf{D}(V(\Pi)^\vee) = \mathbf{D}(\check{\mathrm{V}}(\Pi^\vee)\otimes\zeta^{-1}\varepsilon) = \mathbf{D}(\check{\mathrm{V}}(\mathbf{V}(D_A)\otimes_{\widetilde{E}_{\overline{\theta}}}\widetilde{P}_{\overline{\theta}})\otimes\zeta^{-1}\varepsilon).$$

By Lemma 2.4.22 (1) (i.e. the very definitions of $\widetilde{P}_{\overline{\theta}}$ and $E_{\overline{\theta}}$), we have

$$\check{\mathrm{V}}(\mathbf{V}(D_A)\otimes_{\widetilde{E}_{\overline{\theta}}}\widetilde{P}_{\overline{\theta}}) = \mathbf{V}(D_A),$$

and so we conclude that

$$\mathbf{D}(\Pi) = D_A\otimes\zeta^{-1}\varepsilon.$$

Now [CD14, Prop. III.44] produces a natural map of left $\mathcal{O}[\![G]\!]_{\zeta^{-1}}$-modules

$$\beta_{\mathbf{P}^1} : \Pi^\vee \to \mathbf{D}(\Pi)^\sharp \boxtimes_{\zeta^{-1}} \mathbf{P}^1$$

whose kernel is equal to $(\Pi^\vee)^{\mathrm{SL}_2(\mathbf{Q}_p)}$. Furthermore, by [Col10c, Prop. II.1.11], we have

$$\mathbf{D}(\Pi)^\sharp \boxtimes_{\zeta^{-1}} \mathbf{P}^1 = (D_A^\sharp \boxtimes \mathbf{P}^1)(\zeta^{-1}\varepsilon\circ\det),$$

since by our conventions, $\boxtimes$ is shorthand for $\boxtimes_{\zeta\varepsilon^{-2}}$ (compare Remark 4.5.3).

We thus obtain a map of left $\mathcal{O}[\![G]\!]_{\zeta^{-1}}$-modules

$$\beta_{\mathbf{P}^1} : \mathbf{V}(D_A)\otimes_{\widetilde{E}_{\overline{\theta}}}\widetilde{P}_{\overline{\theta}} \to (D_A^\sharp \boxtimes \mathbf{P}^1)(\zeta^{-1}\varepsilon\circ\det),$$

whose kernel is equal to $(\mathbf{V}(D_A) \otimes_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)}$. We claim that $\beta_{\mathbf{P}^1}$ factors through $(D_A^\natural \boxtimes \mathbf{P}^1)(\zeta^{-1}\varepsilon \circ \det)$. By [CD14, Prop. III.44(3)], this holds provided that $\Pi^{\mathrm{SL}_2(\mathbf{Q}_p)} = 0$, i.e. provided that $(\mathbf{V}(D_A) \otimes_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})_{\mathrm{SL}_2(\mathbf{Q}_p)} = 0$. By construction, $\mathbf{V}(D_A) \otimes_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}}$ is a quotient of products of copies of $\widetilde{P}_{\overline{\theta}}$. By definition such a product never contains the projective envelope of a character as a factor, and thus $\mathbf{V}(D_A) \otimes_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}}$ admits no non-zero map to a character, and hence has no non-zero $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant quotients, as required.

Thus we have obtained a natural morphism of left $\mathcal{O}[\![G]\!]_{\zeta^{-1}}$-modules

$$\beta_{\mathbf{P}^1} : \mathbf{V}(D_A) \otimes_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}} \to (D_A^\natural \boxtimes \mathbf{P}^1)(\zeta^{-1}\varepsilon \circ \det),$$

whose kernel is equal to $(\mathbf{V}(D_A) \otimes_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)}$. Furthermore, it is shown in the second paragraph of the proof of [CD14, Thm. III.45] that the cokernel of $\beta_{\mathbf{P}^1}$ is $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant and of finite $\mathcal{O}$-length.

Applying the above discussion to $A := S/\mathfrak{m}_S^k$, we have for each $k \geq 1$ a natural exact sequence

$$0 \to \ker_k \to \mathbf{V}(D_{S/\mathfrak{m}_S^k}) \otimes_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}} \overset{\beta_{\mathbf{P}^1}}{\to} (D^\natural_{S/\mathfrak{m}_S^k} \boxtimes \mathbf{P}^1)(\zeta^{-1}\varepsilon \circ \det) \to \mathrm{coker}_k \to 0$$

where $\ker_k$ and coker_k are of finite $\mathcal{O}$-length. Thus by Mittag-Leffler we can pass to the inverse limit and obtain an exact sequence

$$0 \to \varprojlim_k \ker_k \to \mathbf{V}(D_S) \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}} \to (D_S^\natural \boxtimes \mathbf{P}^1)(\zeta^{-1}\varepsilon \circ \det) \to \varprojlim_k \mathrm{coker}_k \to 0$$

whose kernel is equal to $(\mathbf{V}(D_S) \otimes_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)}$, and whose cokernel is $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant. This cokernel vanishes by Corollary 4.10.23, so after turning the left $\mathcal{O}[\![G]\!]_{\zeta^{-1}}$-action into a right $\mathcal{O}[\![G]\!]_\zeta$-action, we obtain the isomorphism (5.2.15).

If $\overline{\theta}$ is not of type (St), then $(\mathbf{V}(D_S) \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)} = 0$, because there are no $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant objects in $\mathfrak{C}_{\overline{\theta}}$, so we obtain (5.2.16).

Finally, if $\overline{\theta}$ of type (St), then by (3.6.11) and the definition of $\mathcal{W}_{\overline{\theta}}$ (i.e. Definition 3.6.12), we have a natural isomorphism of left $\mathcal{O}[\![G]\!]_{\zeta^{-1}}$-modules

$$\widehat{i}_S'^* \mathcal{W}_{\overline{\theta}} \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} \widehat{i}_S'^* \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} \mathbf{V}(D_S) \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}}.$$

Thus, to prove (5.2.17), it suffices to note the natural isomorphism

$$\widehat{i}_S'^* \mathcal{W}_{\overline{\theta}}[J] \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} (\widehat{i}_S'^* \mathcal{W}_{\overline{\theta}} \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}})^{\mathrm{SL}_2(\mathbf{Q}_p)}$$

provided by Lemma 3.6.5. □

We can now give our alternative description of $\widehat{i}_{\overline{\theta}}'^* L_\infty$.

Proposition 5.2.18.

(1) If $\overline{\theta}$ is not *of type* (St)*, then there is an $\mathcal{O}[\![G]\!]_\zeta$-equivariant isomorphism*

$$\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}} \xrightarrow{\sim} \widehat{i}_{\overline{\theta}}'^* L_\infty. \tag{5.2.19}$$

(2) If $\overline{\theta}$ is of type (St)*, then there is an $\mathcal{O}[\![G]\!]_\zeta$-equivariant isomorphism*

$$(\mathcal{W}_{\overline{\theta}}/\mathcal{W}_{\overline{\theta}}[J]) \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} \mathbf{P}_{\overline{\theta}} \xrightarrow{\sim} \widehat{i}_{\overline{\theta}}'^* L_\infty. \tag{5.2.20}$$

Proof. We give the argument in case (1); case (2) is formally identical, and we leave it to the reader. We simplify our notation by writing $R := R^{\mathrm{ps}}_{\overline{\theta}}$, with maximal ideal $\mathfrak{m}$.

Let H be a compact open subgroup of G. Recall from Lemma 5.2.10 that $(\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})_H$ and $(\widehat{i}'^{*}_{\overline{\theta}} L_\infty)_H$ are objects of $\mathrm{Coh}(\mathcal{O}_{\mathcal{X}_{\overline{\theta}}})$. As in the statement of Proposition 5.2.14, we let S be a complete Noetherian local $\mathcal{O}$-algebra with finite residue field, and $i'_S : \operatorname{Spf} S \to \mathcal{X}_{\overline{\theta}}$ be a versal morphism at the closed point of $\mathcal{X}_{\overline{\theta}}$. Since $\widehat{i}'^{*}_S$ is a right exact functor, it commutes with the formation of H-coinvariants, by Lemma A.1.55. Thus the isomorphism (5.2.16) induces an isomorphism

$$\widehat{i}'^{*}_S\big((\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})_H\big) \xrightarrow{\sim} \widehat{i}'^{*}_S\big((\widehat{i}'^{*}_{\overline{\theta}} L_\infty)_H\big), \tag{5.2.21}$$

which is natural in the morphism i'_S. By Proposition C.2.16, this collection of isomorphisms induces an isomorphism

$$(\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}})_H \xrightarrow{\sim} (\widehat{i}'^{*}_{\overline{\theta}} L_\infty)_H. \tag{5.2.22}$$

Since the isomorphisms (5.2.21) are deduced from the $\mathcal{O}[\![G]\!]_\zeta$-module isomorphisms (5.2.16), they are compatible in H as H-varies, and indeed compatible with the $\mathcal{O}[\![G]\!]_\zeta$-module structure on the projective systems formed (by allowing H to vary) by their left and right hand sides. Passing to the limit over H in the isomorphisms (5.2.22), and taking into account the isomorphisms (5.2.11) and (5.2.12), we obtain the desired isomorphism (5.2.19). $\square$

5.2.23. *An alternative description of* $\mathrm{F}_{\overline{\theta}}$. Using Proposition 5.2.18, we can now give an alternate description of $\mathrm{F}_{\overline{\theta}}$, and establish its full faithfulness.

Theorem 5.2.24. *Let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation of $G_{\mathbf{Q}_p}$.*

(1) The functor $\mathrm{F}_{\overline{\theta}} : D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$ is fully faithful, and moreover factors through a (necessarily fully faithful) functor $D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$.

(2) If $\overline{\theta}$ is not *of type* (St)*, then $\mathrm{F}_{\overline{\theta}}$ is t-exact.*

(3) If $\overline{\theta}$ is of type (St)*, then $\mathrm{F}_{\overline{\theta}}$ has amplitude $[-1, 0]$.*

Proof. We begin by supposing that $\overline{\theta}$ is not of type (St). We have natural isomorphisms of functors

$$\begin{aligned}\mathrm{F}_{\overline{\theta}}(-) &\xrightarrow[(5.2.8)]{\sim} \widehat{i}'^{*}_{\overline{\theta}} L_\infty \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-) \xrightarrow[(5.2.19)]{\sim} (\mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \widetilde{P}_{\overline{\theta}}) \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-) \\ &\xrightarrow[\text{Lem. 3.5.18}]{\sim} \mathcal{V}_{\overline{\theta}} \widehat{\otimes}^L_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-)).\end{aligned} \tag{5.2.25}$$

By Lemma 3.5.17, the functor

$$\widetilde{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-) : D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}})$$

is t-exact. By Lemma 2.3.15 (2), its restriction to the heart $\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}$ is an equivalence

$$\widetilde{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-) : \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}} \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}}).$$

Hence, by Corollary A.7.20, the functor $\widetilde{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-)$ factors through a t-exact equivalence

$$\widetilde{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-) : D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \xrightarrow{\sim} D^b_{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}}).$$

It therefore suffices to prove that

$$\mathcal{V}_{\overline{\theta}} \widehat{\otimes}^L_{\widetilde{E}_{\overline{\theta}}} - : \operatorname{Pro} D^b_{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$$

restricts to a t-exact and fully faithful functor $D^b_{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}}) \to D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$. To do so, we apply Lemma 3.5.13, which produces a commutative diagram

$$\begin{tikzcd}
D^b_{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}}) \arrow[r, hook, "(\mathrm{A}.10.6)"] \arrow[d, "\mathcal{V}_{\overline{\theta}} \otimes^L_{\widetilde{E}_{\overline{\theta}}} -"] & D^b_{\mathrm{fp}}(\widetilde{E}_{\overline{\theta}}) \arrow[r, hook, "(\mathrm{A}.10.8)"] \arrow[d, "\mathfrak{V}_{\overline{\theta}} \otimes^L_{\widetilde{E}_{\overline{\theta}}} -"] \arrow[dr, "\mathcal{V}_{\overline{\theta}} \otimes^L_{\widetilde{E}_{\overline{\theta}}} -"] & \operatorname{Pro} D^b_{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}}) \arrow[d, "\mathcal{V}_{\overline{\theta}} \widehat{\otimes}^L_{\widetilde{E}_{\overline{\theta}}} -"] \\
D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}) \arrow[r, hook, "k_{\overline{\theta},*}"] & D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}) \arrow[r, hook, "\widehat{k}^*_{\overline{\theta}}"] & \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}).
\end{tikzcd}$$

Commutativity of the outer rectangle shows that $\mathcal{V}_{\overline{\theta}} \widehat{\otimes}^L_{\widetilde{E}_{\overline{\theta}}} -$ restricts to a functor

$$\mathcal{V}_{\overline{\theta}} \otimes^L_{\widetilde{E}_{\overline{\theta}}} - : D^b_{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}}) \to D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}).$$

Bearing in mind the commutativity of the upper triangle, we now see that this restriction is fully faithful, by Theorem 3.5.25 (1), and t-exact, by Theorem 3.5.25 (2). This concludes the proof in the case that $\overline{\theta}$ does not have type (St).

We now turn to the case that $\overline{\theta}$ is of type (St). Using (5.2.20) and arguing as above, we see that there is a natural isomorphism of functors from $D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})$ to $\operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$

$$\mathrm{F}_{\overline{\theta}}(-) \xrightarrow{\sim} (\mathcal{W}_{\overline{\theta}}/\mathcal{W}_{\overline{\theta}}[J]) \widehat{\otimes}^L_{\mathbf{E}_{\overline{\theta}}} (\mathbf{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-)). \tag{5.2.26}$$

Again, the functor $\mathbf{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-)$ induces a t-exact equivalence $D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \xrightarrow{\sim} D^b_{\mathrm{f.l.}}(\mathbf{E}_{\overline{\theta}})$. We see by Lemma 3.5.13 that there is a commutative diagram

$$\begin{tikzcd}
D^b_{\mathrm{f.l.}}(\mathbf{E}_{\overline{\theta}}) \arrow[r, hook] \arrow[d, "\mathcal{W}_{\overline{\theta}}/\mathcal{W}_{\overline{\theta}}[J] \otimes^L_{\mathbf{E}_{\overline{\theta}}} -"] & D^b_{\mathrm{fp}}(\mathbf{E}_{\overline{\theta}}) \arrow[r, hook, "(\mathrm{A}.10.8)"] \arrow[d, "\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J] \otimes^L_{\mathbf{E}_{\overline{\theta}}} -"] & \operatorname{Pro} D^b_{\mathrm{f.l.}}(\mathbf{E}_{\overline{\theta}}) \arrow[d, "\mathcal{W}_{\overline{\theta}}/\mathcal{W}_{\overline{\theta}}[J] \widehat{\otimes}^L_{\mathbf{E}_{\overline{\theta}}} -"] \\
D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}) \arrow[r, hook, "k_{\overline{\theta},*}"] & D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}) \arrow[r, hook, "\widehat{k}^*_{\overline{\theta}}"] & \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}).
\end{tikzcd} \tag{5.2.27}$$

This immediately implies that

$$\mathrm{F}_{\overline{\theta}}(-) \xrightarrow{\sim} (\mathcal{W}_{\overline{\theta}}/\mathcal{W}_{\overline{\theta}}[J]) \otimes^L_{\mathbf{E}_{\overline{\theta}}} (\mathbf{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} i_{\overline{\theta},*}(-)). \tag{5.2.28}$$

factors through $D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$. Furthermore, by Lemma 3.6.46, the functor

$$(\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]) \otimes^L_{\mathbf{E}_{\overline{\theta}}} : D^b_{\mathrm{fp}}(\mathbf{E}_{\overline{\theta}}) \to D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})$$

has amplitude $[-1, 0]$, so we see that indeed $\mathrm{F}_{\overline{\theta}}$ has amplitude $[-1, 0]$. Finally, the full faithfulness of $\mathrm{F}_{\overline{\theta}}$ follows as above, using (5.2.26) and Theorem 3.6.44. $\square$

In the context of Theorem 5.2.24 (3) we have the following more precise description of the values of $\mathrm{F}_{\overline{\theta}}$ on irreducible objects of $\mathcal{A}_{\overline{\theta}}$. In its statement, we will identify coherent sheaves on $\mathfrak{X}_{\overline{\theta}}$ with graded S-modules, where S is the graded ring in Section 3.4.9, and we will continue to denote by $\widehat{k}^*_{\overline{\theta}} : D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$ the t-exact, fully faithful completion functor.

Corollary 5.2.29. *Let $\overline{\theta} = 1 + \omega^{-1}$. Then:*

$$\mathrm{F}_{\overline{\theta}}(\pi_\alpha) \cong \widehat{k}^*_{\overline{\theta}} S(1)/(a_0, a_1, b_0, b_1, \varpi)[0],$$

$$\mathrm{F}_{\overline{\theta}}(\mathrm{St}) \cong \widehat{k}^*_{\overline{\theta}} S(-1)/(a_0, a_1, c, \varpi)[0],$$

$$\mathrm{F}_{\overline{\theta}}(\mathbf{1}_G) \cong \widehat{k}^*_{\overline{\theta}} S(-3)/(a_0, a_1, c, \varpi)[1].$$

Proof. By Theorem 3.6.44, the graded S-module corresponding to $\mathfrak{W}_{\overline{\theta}}/\mathfrak{W}_{\overline{\theta}}[J]$ is the module X^* from Definition 3.6.42. Bearing in mind the isomorphism (5.2.28), and the commutativity of the leftmost square in (5.2.27), the corollary thus follows from the computations of $\mathrm{Tor}_i^{\mathbf{E}_{\overline{\theta}}}(X^*, –)$ in Example 3.6.48. ☐

We next note the evident extension of Theorem 5.2.24 to the case of a finite closed subset $Y \subset X$.

Corollary 5.2.30. *For any finite set of closed points $Y \subset X$, the functor $\mathrm{F}_Y : D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_Y)$ factors through a fully faithful functor $D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to D^b_{\mathrm{coh}}(\mathcal{X}_Y)$.*

Proof. We can write $\mathrm{F}_Y : D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_Y)$ as a product

$$\prod_{\overline{\theta}\in Y} \mathrm{F}_{\overline{\theta}} : \prod_{\overline{\theta}\in Y} D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to \prod_{\overline{\theta}\in Y} \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}),$$

so the result follows immediately from Theorem 5.2.24. ☐

In view of Theorem 5.2.24 (1) and Corollary 5.2.30, we will freely regard $\mathrm{F}_{\overline{\theta}}$ (resp. F_Y) as a fully faithful functor $D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$ (resp. a fully faithful functor $D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to D^b_{\mathrm{coh}}(\mathcal{X}_Y)$) from now on.

5.2.31. *The relationship to Colmez's functor V.* Recall from Section 2.4.1 that Colmez's functor V is exact, so in particular it induces a t-exact functor $D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to D^b_{\mathrm{f.l.}}(\widetilde{E}_{\overline{\theta}})$. If $\overline{\theta}$ is not of type (St), then by Lemma 2.4.22 we have a natural isomorphism of t-exact functors

$$\widetilde{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} (–) \xrightarrow{\sim} V^\dagger(–),$$

so we see from (5.2.25) that there is a natural isomorphism of t-exact functors from $D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})$ to $D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$

$$\mathrm{F}_{\overline{\theta}}(–) \xrightarrow{\sim} \mathcal{V}_{\overline{\theta}} \widehat{\otimes}^L_{\widetilde{E}_{\overline{\theta}}} (\widetilde{P}_{\overline{\theta}} \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} –) \xrightarrow{\sim} \mathcal{V}_{\overline{\theta}} \widehat{\otimes}^L_{\widetilde{E}_{\overline{\theta}}} V^\dagger(–). \tag{5.2.32}$$

If $\overline{\theta}$ is of type (St) then (5.2.32) does not hold in general, and $\mathrm{F}_{\overline{\theta}}$ is no longer t-exact. However, we have the following related result, which will be used in the proof of Proposition 5.3.20 below.

Corollary 5.2.33. *Suppose that $\overline{\theta}$ is of type* (St)*, and that $\pi \in \mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}$ admits no $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant quotient. Then there is a natural isomorphism*

$$H^0\mathrm{F}_{\overline{\theta}}(\pi) \xrightarrow{\sim} \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} V^\dagger(\pi).$$

Proof. By Lemma A.1.58 (noting that products are exact in the category of pro-coherent sheaves on a formal algebraic stack) and (5.2.26), we have a right exact sequence

$$\mathcal{W}_{\overline{\theta}}[J] \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} (\mathbf{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi) \to \mathcal{W}_{\overline{\theta}} \widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}} (\mathbf{P}_{\overline{\theta}} \otimes_{\mathcal{O}[\![G]\!]_\zeta} \pi) \to H^0\mathrm{F}_{\overline{\theta}}(\pi) \to 0.$$

The leftmost term vanishes, because

$$\begin{aligned}\mathcal{W}_{\overline{\theta}}[J]\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}\otimes_{\mathcal{O}[\![G]\!]_\zeta}\pi) &= (\mathcal{W}_{\overline{\theta}}[J]\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}\mathbf{P}_{\overline{\theta}})\otimes_{\mathcal{O}[\![G]\!]_\zeta}\pi\\ &= (\mathcal{W}_{\overline{\theta}}[J]\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}\mathbf{P}_{\overline{\theta}}/J)\otimes_{\mathcal{O}[\![G]\!]_\zeta}\pi\\ &= \mathcal{W}_{\overline{\theta}}[J]\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}/J\otimes_{\mathcal{O}[\![G]\!]_\zeta}\pi)\\ &= \mathcal{W}_{\overline{\theta}}[J]\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}(\mathcal{O}\otimes_{\mathcal{O}[\![G]\!]_\zeta}\pi),\end{aligned}$$

which vanishes by our assumption that π admits no $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant quotient. (The second equality holds because, by construction, the map $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}\to\mathrm{End}_{\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})}(\mathcal{W}_{\overline{\theta}}[J])$ factors through $\mathbf{E}_{\overline{\theta}}^{\mathrm{op}}/J$, hence the functor $\mathcal{W}_{\overline{\theta}}[J]\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}$ factors through $\mathrm{Mod}_c(\mathbf{E}_{\overline{\theta}})\to\mathrm{Mod}_c(\mathbf{E}_{\overline{\theta}}/J)$, $M\mapsto M/J$.)

It follows from the discussion above that $H^0\mathrm{F}_{\overline{\theta}}(\pi)$ is naturally isomorphic to $\mathcal{W}_{\overline{\theta}}\,\widehat{\otimes}_{\mathbf{E}_{\overline{\theta}}}(\mathbf{P}_{\overline{\theta}}\otimes_{\mathcal{O}[\![G]\!]_\zeta}\pi)$, which in turn is isomorphic to $\mathcal{V}_{\overline{\theta}}\,\widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}}(\widetilde{P}_{\overline{\theta}}\otimes_{\mathcal{O}[\![G]\!]_\zeta}\pi)$, by (3.6.13). The corollary now follows from Lemma 2.4.22 (2). $\square$

5.3. **Finiteness properties of** F. Our next goal is to prove Proposition 5.3.23, which shows that our functor F takes finitely presented representations to (bounded complexes of) coherent sheaves. Since one of the main steps of the argument will be to establish finiteness and concentration properties of the completions of $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)$ at closed points, we begin by establishing a compatibility between F and F_Y, where Y is a finite set of closed points of X.

5.3.1. *Compatibility between* F *and* F_Y. Let Y be a finite set of closed points of X. Then Corollary 5.2.30 shows that the functor F_Y factors through a fully faithful functor $D^b_{\mathrm{fp}}(\mathcal{A}_Y)\to D^b_{\mathrm{coh}}(\mathcal{X}_Y)$, and we write $\mathrm{Pro}\,\mathrm{F}_Y:\mathrm{Pro}\,D^b_{\mathrm{fp}}(\mathcal{A}_Y)\to\mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_Y)$ for its Pro-extension

$$\mathrm{Pro}\,\mathrm{F}_Y=\mathrm{Pro}(\widehat{i}_Y^{\prime *}\mathrm{F}\,\widehat{i}_{Y,*})=\widehat{i}_Y^{\prime *}\,\mathrm{Pro}(\mathrm{F})\widehat{i}_{Y,*},\tag{5.3.2}$$

which is also fully faithful by Lemma A.4.6. Note that since $\mathrm{F}=L_\infty\otimes^L_{\mathcal{O}[\![G]\!]_\zeta}-$, the Pro-extension of F is the functor

$$L_\infty\,\widehat{\otimes}^L_{\mathcal{O}[\![G]\!]_\zeta}-:\mathrm{Pro}\,D^b_{\mathrm{fp}}(\mathcal{A})\to\mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X})$$

from Lemma A.10.10 (2j). If we precompose (5.3.2) with $\widehat{i}_Y^*$, the unit of the adjunction gives a natural transformation of functors $D^b_{\mathrm{fp}}(\mathcal{A})\to\mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_Y)$

$$\widehat{i}_Y^{\prime *}\mathrm{F}\to(\mathrm{Pro}\,\mathrm{F}_Y)\widehat{i}_Y^*\tag{5.3.3}$$

which identifies with the natural transformation

$$\widehat{i}_Y^{\prime *}(L_\infty\otimes^L_{\mathcal{O}[\![G]\!]_\zeta}(-))\to\widehat{i}_Y^{\prime *}(L_\infty\,\widehat{\otimes}^L_{\mathcal{O}[\![G]\!]_\zeta}\widehat{i}_{Y,*}\widehat{i}_Y^*(-))$$

induced by the restriction to $D^b_{\mathrm{fp}}(\mathcal{A})$ of the unit of adjunction $\mathrm{id}_{\mathrm{Pro}\,D^b_{\mathrm{fp}}(\mathcal{A})}\to\widehat{i}_{Y,*}\widehat{i}_Y^*$. (Note that $L_\infty\,\widehat{\otimes}^L_{\mathcal{O}[\![G]\!]_\zeta}$ restricts to $L_\infty\otimes^L_{\mathcal{O}[\![G]\!]_\zeta}$ on $D^b_{\mathrm{fp}}(\mathcal{A})$, by the discussion above, or equivalently by (A.10.12).)

Applying Lemma 5.2.5, we can rewrite (5.3.3) as a natural transformation

$$\widehat{i}_Y^{\prime *}L_\infty\otimes^L_{\mathcal{O}[\![G]\!]_\zeta}(-)\to\widehat{i}_Y^{\prime *}L_\infty\,\widehat{\otimes}^L_{\mathcal{O}[\![G]\!]_\zeta}\widehat{i}_{Y,*}\widehat{i}_Y^*(-).\tag{5.3.4}$$

Assume now that $Y=\{\overline{\theta}\}$ is a singleton, and apply Proposition 5.2.18. This gives us a projective generator P of $\mathfrak{C}_{\overline{\theta}}$ with finite cosocle, with compact endomorphism ring E, and a complete right E-module V in $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_{\overline{\theta}})$, such that

$$V \mathbin{\widehat{\otimes}}^L_E P \xrightarrow{\sim} V \mathbin{\widehat{\otimes}}_E P \xrightarrow{\sim} \widehat{i}^{\prime *}_{\overline{\theta}} L_\infty$$

(where the first isomorphism follows from the fact that P is projective in $\operatorname{Mod}_c(E)$, by Lemma 2.3.11 (4)). By Lemma 3.5.18, the natural transformation (5.3.4) is thus isomorphic to

$$V \mathbin{\widehat{\otimes}}^L_E (P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} (-)) \to V \mathbin{\widehat{\otimes}}^L_E (P \mathbin{\widehat{\otimes}}^L_{\mathcal{O}[\![G]\!]_\zeta} \widehat{i}_{Y,*} \widehat{i}^*_Y(-)). \tag{5.3.5}$$

Proposition 5.3.6. *For any finite set of closed points $Y \subset X$, the natural transformations* (5.3.5) *and* (5.3.3) $: \widehat{i}^{\prime *}_Y \mathrm{F} \to (\operatorname{Pro}\mathrm{F}_Y)\widehat{i}^*_Y$ *are isomorphisms.*

Proof. As in the proof of Corollary 5.2.30, we may immediately reduce to the case that $Y = \overline{\theta}$ is a singleton. Since (5.3.3) is naturally isomorphic to (5.3.5), it suffices to prove that (5.3.5) is an isomorphism. To do so, it suffices to prove that for all projective $P \in \mathfrak{C}_{\overline{\theta}}$ with finite cosocle, the natural transformation

$$P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} (-) \to P \mathbin{\widehat{\otimes}}^L_{\mathcal{O}[\![G]\!]_\zeta} \widehat{i}_{\overline{\theta},*} \widehat{i}^*_{\overline{\theta}}(-) \tag{5.3.7}$$

is an isomorphism. We know that $P \otimes^L_{\mathcal{O}[\![G]\!]_\zeta} - : D^b_{\mathrm{fp}}(\mathcal{A}) \to \operatorname{Pro} D^b_{\mathrm{f.l.}}(E)$ is t-exact, by Lemma 3.5.17. Hence its Pro-extension $P \mathbin{\widehat{\otimes}}^L_{\mathcal{O}[\![G]\!]_\zeta} -$ is also t-exact, and so both sides of (5.3.7) are t-exact functors. By Corollary A.7.20, it thus suffices to prove that (5.3.7) becomes an isomorphism after restriction to $\mathcal{A}^{\mathrm{fp}}$. This is Lemma 2.5.26. $\square$

We can now prove that $\mathrm{F}(c\text{-}\operatorname{Ind}^G_{KZ}\sigma)$ is pure of degree zero (in the sense of Definition B.3.33) after completed pullback to $\mathcal{X}_{\overline{\theta}}$.

Lemma 5.3.8. *Let σ be a Serre weight. Let $\overline{\theta}$ be a 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentation. Then*

$$\widehat{i}^{\prime *}_{\overline{\theta}} \mathrm{F}(c\text{-}\operatorname{Ind}^G_{KZ}\sigma) \overset{\text{Prop. 5.3.6}}{=} \operatorname{Pro}\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}} c\text{-}\operatorname{Ind}^G_{KZ}\sigma)$$

is a pro-coherent sheaf, i.e. it is pure of degree zero. Furthermore, $\mathrm{F}_{\overline{\theta}}(c\text{-}\operatorname{Ind}^G_{KZ}\sigma/f_{\overline{\theta}}c\text{-}\operatorname{Ind}^G_{KZ}\sigma)$ *is pure of degree zero unless $\overline{\theta}$ is of type* (St) *and σ is Steinberg.*

Proof. This is immediate if $\overline{\theta}$ is not of type (St), since $\mathrm{F}_{\overline{\theta}}$ (and thus $\operatorname{Pro}\mathrm{F}_{\overline{\theta}}$) is t-exact by Theorem 5.2.24 (2). Note that in this case the only piece of information we need about $\widehat{i}^*_{\overline{\theta}} c\text{-}\operatorname{Ind}^G_{KZ}\sigma$ and $\widehat{i}^*_{\overline{\theta}}(c\text{-}\operatorname{Ind}^G_{KZ}\sigma/f_{\overline{\theta}}c\text{-}\operatorname{Ind}^G_{KZ}\sigma)$ is that they are objects of $\operatorname{Pro}\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}$, a consequence of the t-exactness of $\widehat{i}^*_{\overline{\theta}}$.

Suppose now that $\overline{\theta}$ has type (St). After twisting we may suppose that $\overline{\theta} = 1+\omega^{-1}$, so that by Lemma 2.5.16 (2), we only need to consider $\sigma \in \{\operatorname{Sym}^0, \det\otimes\operatorname{Sym}^{p-3}, \operatorname{Sym}^{p-1}\}$, as otherwise $\widehat{i}^*_{\overline{\theta}} c\text{-}\operatorname{Ind}^G_{KZ}\sigma = 0$. To handle the cases $\sigma = \operatorname{Sym}^0$ and $\det\otimes\operatorname{Sym}^{p-3}$, we first recall that by definition we have $f_{\overline{\theta}} = T_p - 1$, and by (2.5.17) we have

$$\widehat{i}^*_{\overline{\theta}} c\text{-}\operatorname{Ind}^G_{KZ}\sigma \xrightarrow{\sim} \lim_n (c\text{-}\operatorname{Ind}^G_{KZ}\sigma)/(T_p-1)^n.$$

Then, by an obvious dévissage, it suffices to show that $\mathrm{F}_{\overline{\theta}}\big((c\text{-}\operatorname{Ind}^G_{KZ}\sigma)/(T_p-1)\big)$ is pure of degree zero. If $\sigma = \det\otimes\operatorname{Sym}^{p-3}$, we have

$$\mathrm{F}_{\overline{\theta}}\big(c\text{-}\operatorname{Ind}^G_{KZ}\det\otimes\operatorname{Sym}^{p-3}/(T_p-1)\big) = \mathrm{F}_{\overline{\theta}}(\pi_\alpha),$$

which is pure of degree zero by Corollary 5.2.29.

In the case $\sigma = \mathrm{Sym}^0$, we have a non-split short exact sequence

$$0 \to \mathrm{St} \to c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 /(T_p - 1) \to \mathbf{1}_G \to 0, \tag{5.3.9}$$

so that $c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 /(T_p-1)$ is the cone of a non-zero map $\mathbf{1}_G[-1] \to \mathrm{St}$. It then follows from Corollary 5.2.29 and the full faithfulness of $\mathrm{F}_{\overline{\theta}}$ that $\mathrm{F}_{\overline{\theta}}(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 /(T_p - 1))$ is quasi-isomorphic to the image under the t-exact, fully faithful functor $\widehat{k}^*_{\overline{\theta}} : D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$ of a complex (in degrees -1 and 0)

$$S(-3)/(a_0, a_1, c, \varpi) \xrightarrow{\delta} S(-1)/(a_0, a_1, c, \varpi),$$

where δ is non-zero, and we regard graded S-modules as objects of $\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}})$. Since every non-zero S-linear endomorphism of $S/(a_0, a_1, c, \varpi) = \mathbf{F}[b_0, b_1]$ is injective (such an endomorphism is given by multiplication by a non-zero element of the integral domain $\mathbf{F}[b_0, b_1]$), we deduce that $\mathrm{F}_{\overline{\theta}}(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 /(T_p - 1))$ is pure of degree zero.

Finally, suppose that $\sigma = \mathrm{Sym}^{p-1}$. Since $\widehat{i}^*_{\overline{\theta}}$ is exact, applying it to (2.2.36) yields a short exact sequence

$$0 \to \widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 \to \widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \to \mathrm{St} \to 0. \tag{5.3.10}$$

Now Corollary 5.2.29 shows that $\mathrm{F}_{\overline{\theta}}(\mathrm{St})$ is pure of degree zero, and since we have already seen that $\mathrm{Pro}\, \mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0)$ is pure of degree zero, the same is true of $\mathrm{Pro}\, \mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1})$. $\square$

5.3.11. *Completed stalks of* $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$. Recall from Section 3.3.4 that the closed points of $\mathcal{X}$ are in bijection with $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy classes of 2-dimensional $\overline{\mathbf{F}}_p$-valued pseudorepresentations $\overline{\theta}$, or equivalently of semisimple Galois representations $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$. Given $\overline{\rho}$ with trace $\overline{\theta}$, we write $i'_{\overline{\rho}} : (\mathcal{X}/\varpi)^{\wedge}_{\overline{\rho}} \to \mathcal{X}/\varpi$ for the completion at the closed subset $\{\overline{\rho}\} \subset |\mathcal{X}/\varpi|$. In this subsection we study the pullback $\widehat{i}'^*_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$.

We begin by specializing the framework of Section B.3.30 to the situation at hand. In Definition 3.4.4 we have constructed a morphism

$$\mathrm{Spf}\, R^{\mathrm{ver}}_{\overline{\theta}} \to \mathcal{X}_{\overline{\theta}}$$

mapping the unique point of the source to the unique closed point $\overline{\rho}$ of the target. It thus induces a morphism

$$v : \mathrm{Spf}\, R^{\mathrm{ver}}_{\overline{\theta}}/\varpi \to (\mathcal{X}/\varpi)^{\wedge}_{\overline{\rho}}. \tag{5.3.12}$$

If σ is a Serre weight, we have defined a reduced closed algebraic substack $z : \mathcal{Z}(\sigma) \to \mathcal{X}/\varpi$ in Definition 3.2.3. We also have introduced in Definition 3.4.4 a quotient $R^{\sigma}_{\overline{\theta}}$ of $R^{\mathrm{ver}}_{\overline{\theta}}$ such that $\mathrm{Spf}\, R^{\sigma}_{\overline{\theta}} = \mathrm{Spf}\, R^{\mathrm{ver}}_{\overline{\theta}}/\varpi \times_{\mathcal{X}/\varpi} \mathcal{Z}(\sigma)$. Noting that we may factor $i'_{\overline{\rho}}$ as a composite

$$(\mathcal{X}/\varpi)^{\wedge}_{\overline{\rho}} \xrightarrow{i'_{\overline{\rho},\overline{\theta}}} \mathcal{X}_{\overline{\theta}}/\varpi \xrightarrow{i'_{\overline{\theta}}} \mathcal{X}/\varpi,$$

we see that we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spf} R^{\mathrm{ver}}_{\overline{\theta}}/\varpi \arrow[r, "v"] & (\mathcal{X}/\varpi)^{\wedge}_{\overline{\rho}} \arrow[r, "i'_{\overline{\rho},\overline{\theta}}"] \arrow[rr, bend left, "i'_{\overline{\rho}}"] & \mathcal{X}_{\overline{\theta}}/\varpi \arrow[r, "i'_{\overline{\theta}}"] & \mathcal{X}/\varpi \\ \operatorname{Spf} R^{\sigma}_{\overline{\theta}}/\varpi \arrow[u] \arrow[r, "v_\sigma"] & \mathcal{Z}(\sigma)^{\wedge}_{\overline{\rho}} \arrow[u, "z_{\overline{\rho}}"] \arrow[rr, "i'_{\overline{\rho},\sigma}"] & & \mathcal{Z}(\sigma), \arrow[u, "z"] \end{tikzcd} \tag{5.3.13}$$

with Cartesian squares, where $i'_{\overline{\rho},\sigma}$ is the completion at $\{\overline{\rho}\} \cap |\mathcal{Z}(\sigma)|$, and $z_{\overline{\rho}}$ is the completion of z at $\{\overline{\rho}\}$. (If $\overline{\rho} \notin |\mathcal{Z}(\sigma)|$, then $\mathcal{Z}(\sigma)^{\wedge}_{\overline{\rho}}$ is empty.)

We now note that $\widehat{i}'^{*}_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma)$ is in fact a pro-coherent sheaf, i.e. pure of degree zero. Indeed, Lemma 5.3.8 shows that $\widehat{i}'^{*}_{\overline{\theta}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma)$ is pure of degree zero, while the discussion of Section B.3.26 shows that $\widehat{i}'^{*}_{\overline{\rho},\overline{\theta}}$ is t-exact. By Remark B.3.14 we have $\widehat{i}'^{*}_{\overline{\rho}} = \widehat{i}'^{*}_{\overline{\rho},\overline{\theta}} \widehat{i}'^{*}_{\overline{\theta}}$, hence

$$\widehat{i}'^{*}_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma) \xrightarrow{\sim} \widehat{i}'^{*}_{\overline{\rho},\overline{\theta}} \widehat{i}'^{*}_{\overline{\theta}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma) \tag{5.3.14}$$

is again pure of degree zero.

As a special case of Lemma B.3.39, we have an exact and faithful pullback functor

$$\widehat{v}^{*} : \operatorname{Pro}\operatorname{Coh}\big((\mathcal{X}/\varpi)^{\wedge}_{\overline{\rho}}\big) \to \operatorname{Mod}_c(R^{\mathrm{ver}}_{\overline{\theta}}/\varpi).$$

We apply $\widehat{v}^{*}$ to the object $\widehat{i}'^{*}_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma)$ of $\operatorname{Pro}\operatorname{Coh}\big((\mathcal{X}/\varpi)^{\wedge}_{\overline{\rho}}\big)$, so as to obtain an object of $\operatorname{Mod}_c(R^{\mathrm{ver}}_{\overline{\theta}}/\varpi)$. The following proposition provides some quite precise information about this latter object.

Proposition 5.3.15. *Let σ be a Serre weight, and let $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be a semisimple Galois representation. Let $\overline{\theta}$ be the trace of $\overline{\rho}$.*

1. *If $\overline{\rho} \notin |\mathcal{Z}(\sigma)|$, then $\widehat{i}'^{*}_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma) = 0$.*
2. *If $\overline{\rho} \in |\mathcal{Z}(\sigma)|$, then $\widehat{v}^{*} \widehat{i}'^{*}_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma)$ is (the compact $R_{\overline{\theta}}/\varpi$-module associated to) a finitely presented $R^{\sigma}_{\overline{\theta}}/\varpi$-module, and the natural map*

$$R^{\sigma}_{\overline{\theta}}/\varpi \to \operatorname{End}_{R^{\sigma}_{\overline{\theta}}/\varpi}\big(\widehat{v}^{*} \widehat{i}'^{*}_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma)\big)$$

is injective.

3. *If $\overline{\rho} \in |\mathcal{Z}(\sigma)|$ and $\sigma \neq \det^{a} \otimes \operatorname{Sym}^{p-2}$ for any a, then*

$$\widehat{v}^{*} \widehat{i}'^{*}_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma) \cong R^{\sigma}_{\overline{\theta}}/\varpi.$$

Proof. The first statement is immediate from (5.3.14) together with Proposition 5.3.6, since if $\overline{\rho} \notin |\mathcal{Z}(\sigma)|$, then σ is not a Serre weight of $\overline{\rho}$, by Theorem 3.2.4 (5), and so $\widehat{i}^{*}_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma = 0$, by Lemma 2.5.16 (2).

We now prove the remaining two statements, initially under the additional assumption that if $\overline{\theta}$ has type (St), then σ is not a determinant twist of Sym^{0}. Under this additional assumption, we claim that

$$\operatorname{Pro}\mathrm{F}_{\overline{\theta}}(\widehat{i}^{*}_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma) \xrightarrow{\sim} \mathcal{V}_{\overline{\theta}} \widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} \operatorname{Pro} V^{\dagger}(\widehat{i}^{*}_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma). \tag{5.3.16}$$

Indeed, we first note that Lemma 5.3.8 shows that

$$\operatorname{Pro}\mathrm{F}_{\overline{\theta}}(\widehat{i}^{*}_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma) \xrightarrow{\sim} H^{0} \operatorname{Pro}\mathrm{F}_{\overline{\theta}}(\widehat{i}^{*}_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^{G} \sigma),$$

so that we may replace the left-hand side of (5.3.16) by its H^{0}.

If $\overline{\theta}$ does not have type (St), we then see that the isomorphism (5.3.16) follows by passing to H^0 in the (Pro-extension of) the isomorphism (5.2.32). On the other hand, if $\overline{\theta}$ does have type (St), then we are assuming that σ is a twist of either Sym^{p-3} or Sym^{p-1}, in which case $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/(T_p-1)^n$ does not have any $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant quotient, for any $n>0$. We may thus apply Corollary 5.2.33 to find that

$$\begin{aligned} H^0\,\mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) &\xrightarrow{\sim} \lim_n H^0\mathrm{F}_{\overline{\theta}}\big(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/(T_p-1)^n\big)\\ &\xrightarrow{\sim} \lim_n \mathcal{V}_{\overline{\theta}}\,\widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}} V^{\dagger}\big(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/(T_p-1)^n\big) \xrightarrow{\sim} \mathcal{V}_{\overline{\theta}}\,\widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}}\mathrm{Pro}\,V^{\dagger}\big(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma\big), \end{aligned}$$

again yielding the desired isomorphism (5.3.16).

We then find that

$$\begin{aligned} (5.3.17)\quad \widehat{v}^*\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) &= \widehat{v}^*\widehat{i}'^*_{\overline{\rho},\overline{\theta}}\widehat{i}'^*_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\\ &\overset{\text{Prop. 5.3.6}}{=} \widehat{v}^*\widehat{i}'^*_{\overline{\rho},\overline{\theta}}\,\mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) \overset{(5.3.16)}{=} \widehat{v}^*\widehat{i}'^*_{\overline{\rho},\overline{\theta}}\big(\mathcal{V}_{\overline{\theta}}\,\widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}}\mathrm{Pro}\,V^{\dagger}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\big)\\ &\overset{\text{Lem. A.1.55}}{=} (\widehat{v}^*\widehat{i}'^*_{\overline{\rho},\overline{\theta}}\mathcal{V}_{\overline{\theta}})\,\widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}}\mathrm{Pro}\,V^{\dagger}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) = V^{\mathrm{ver}}_{\overline{\theta}}\,\widehat{\otimes}_{\widetilde{E}_{\overline{\theta}}}\mathrm{Pro}\,V^{\dagger}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma), \end{aligned}$$

where we have followed Definition 3.4.4 in writing $V^{\mathrm{ver}}_{\overline{\theta}}$ for the versal object on $R^{\mathrm{ver}}_{\overline{\theta}}$. Under our running assumption that if $\overline{\theta}$ has type (St), then σ is not a determinant twist of Sym^0, the proposition follows from Proposition 3.5.39 (noting that $M^{\dagger}_{\sigma,\overline{\theta}}$ there is by definition equal to $\mathrm{Pro}\,V^{\dagger}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$).

It remains to treat the case when $\overline{\theta}$ is of type (St) and σ is a twist of Sym^0, where, as usual, we make a twist so as to assume that in fact $\overline{\theta}=1+\omega^{-1}$ and $\sigma=\mathrm{Sym}^0$. We need to show that

$$\widehat{v}^*\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^0)\cong R^{(1,0),\mathrm{crys}}/\varpi.$$

Recall that applying $\widehat{i}^*_{\overline{\theta}}$ to (2.2.36) and (2.2.37) gives rise to a sequence of injections

$$\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{p-1}\to \widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{0}\to \widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{p-1}$$

such that the composition is a unit multiple of T_p-1, the cokernel of the first map is $\mathbf{1}_G$, and the cokernel of the second map is St. Then Corollary 5.2.29 and Lemma 5.3.8 yield a sequence of morphisms

$$\begin{aligned} (5.3.18)\quad \mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{p-1}) &\twoheadrightarrow \mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{0})\\ &\hookrightarrow \mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{p-1}) \end{aligned}$$

whose composite is a unit multiple of T_p-1; note that the second arrow is (as indicated) an injection, since $\mathrm{F}_{\overline{\theta}}(\mathrm{St})$ is pure of degree 0, and the first arrow is (as indicated) a surjection, since $\mathrm{F}_{\overline{\theta}}(\mathbf{1}_G)$ is pure of degree -1.

Applying $\widehat{v}^*\widehat{i}'^*_{\overline{\rho},\overline{\theta}}$ to (5.3.18), and using the case $\sigma=\mathrm{Sym}^{p-1}$ of the proposition, which we have already proved, we obtain (in the notation of Section 3.4.9) a sequence of maps of $R^{\mathrm{ver}}_{\overline{\theta}}$-modules

$$R^{(p,0),\mathrm{crys}}/\varpi\cong \mathbf{F}[\![a_0,b_0,b_1]\!]/(a_0b_1)\twoheadrightarrow \widehat{v}^*\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^0)\hookrightarrow \mathbf{F}[\![a_0,b_0,b_1]\!]/(a_0b_1)$$

whose composite is a unit multiple of a_0, by Remark 3.5.35. Hence

$$\widehat{v}^*\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^0)\cong R^{(p,0),\mathrm{crys}}/(\varpi,b_1)\overset{(3.5.43)}{=} R^{(1,0),\mathrm{crys}}/\varpi. \qquad \square$$

5.3.19. F *on finitely presented representations.* If σ is a Serre weight, we continue to write $z : \mathcal{Z}(\sigma) \to \mathcal{X}/\varpi$ for the closed immersion defined in Definition 3.2.3.

Proposition 5.3.20. *Let σ be a Serre weight. Then $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is pure of degree zero, and is in fact a coherent sheaf, whose scheme-theoretic support is $\mathcal{Z}(\sigma)$. If furthermore $\sigma \neq \mathrm{Sym}^{p-2} \otimes \det^a$ for any a, then $z^*\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is an invertible sheaf on $\mathcal{Z}(\sigma)$.*

Proof. In the discussion preceding the statement and proof of Proposition 5.3.15, we noted that for every closed point $\overline{\rho}$ of $\mathcal{X}$, the pullback $\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is pure of degree zero. Since F is right t-exact, it then follows from Lemma B.3.34 (2) that the same is true of $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$, which is therefore a pro-coherent sheaf.

We will now show that $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is scheme-theoretically supported on $\mathcal{Z}(\sigma)$, in the sense of Definition B.3.31 (1). Recall the commutative diagram (5.3.13). By Lemma B.3.35, it suffices to prove that for every closed point $\overline{\rho}$ of $\mathcal{X}$, the pro-coherent sheaf $\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ on $(\mathcal{X}/\varpi)^\wedge_{\overline{\rho}}$ is scheme-theoretically supported on the completion $\mathcal{Z}(\sigma)^\wedge_{\overline{\rho}} \to \mathcal{Z}(\sigma)$ at $\{\overline{\rho}\} \cap |\mathcal{Z}(\sigma)|$. We thus need to prove that the unit of the adjunction

$$(z^*_{\overline{\rho}}, z_{\overline{\rho},*}) : \mathrm{Pro}\,\mathrm{Coh}\big((\mathcal{X}/\varpi)^\wedge_{\overline{\rho}}\big) \to \mathrm{Pro}\,\mathrm{Coh}\big(\mathcal{Z}(\sigma)^\wedge_{\overline{\rho}}\big)$$

of (B.3.13) is an isomorphism at $\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$. By Proposition 5.3.15, $\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is an object of the full subcategory $\mathrm{Coh}(\mathcal{O}_{(\mathcal{X}/\varpi)^\wedge_{\overline{\rho}}})$ of $\mathrm{Pro}\,\mathrm{Coh}\big((\mathcal{X}/\varpi)^\wedge_{\overline{\rho}}\big)$. The completed pullback functor $\widehat{v}^*$ is faithful on this subcategory, by Lemma B.5.4 (applied with $\widehat{\mathcal{X}} = (\mathcal{X}/\varpi)^\wedge_{\overline{\rho}}$, which is a completion of $(\mathfrak{X}_{\overline{\theta}}/\varpi)$, and so falls within the scope of Lemma B.5.4). It is therefore enough to show that the natural map

$$\widehat{v}^*\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma) \to \big(\widehat{v}^*\widehat{i}'^*_{\overline{\rho}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)\big) \otimes_{R^{\mathrm{ver}}_{\overline{\theta}}} R^{\sigma}_{\overline{\theta}}$$

is an isomorphism. This is a consequence of Proposition 5.3.15.

At this point in the proof, we have shown that $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is a pro-coherent sheaf, scheme-theoretically supported on $\mathcal{Z}(\sigma)$. In particular, the unit of adjunction $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma) \xrightarrow{\sim} z_*z^*\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is an isomorphism, and $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is isomorphic to $L_\infty \otimes_{\mathcal{O}[\![G]\!]_\zeta} c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ (i.e. we can replace the derived tensor product in the definition of F by a non-derived one). We then find that

$$\begin{aligned} z^*\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma) = z^*(L_\infty \otimes_{\mathcal{O}[\![G]\!]_\zeta} c\text{-}\mathrm{Ind}_{KZ}^G \sigma) &\overset{\text{Lem. A.1.55}}{=} z^*L_\infty \otimes_{\mathcal{O}[\![G]\!]_\zeta} c\text{-}\mathrm{Ind}_{KZ}^G \sigma \\ &= z^*L_\infty \otimes_{\mathcal{O}[\![K]\!]} \sigma = (z^*L_\infty)_{K_1} \otimes_{\mathcal{O}[\![K/K_1]\!]} \sigma. \end{aligned}$$

Since $(z^*L_\infty)_{K_1}$ is coherent, by Lemma 5.1.36 (1), we see that $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is indeed coherent.

In order to show that the scheme-theoretic support of $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is $\mathcal{Z}(\sigma)$, we need to prove that the natural map

$$\mathcal{O}_{\mathcal{Z}(\sigma)} \to \underline{\mathrm{End}}_{\mathcal{O}_{\mathcal{Z}(\sigma)}}\big(z^*\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)\big) \tag{5.3.21}$$

is injective. Applying Lemma B.3.34 to the kernel of (5.3.21), we see that it suffices to show that (5.3.21) becomes an injection after applying $\widehat{i}'^*_{\overline{\rho},\sigma}$, for any closed point $\overline{\rho}$

of $\mathcal{Z}(\sigma)$. By Lemma B.3.41, applying $\widehat{v}_\sigma^* \widehat{i}'^*_{\overline{\rho},\sigma}$ to (5.3.21) yields the natural map

$$(5.3.22) \quad R^\sigma_{\overline{\theta}}/\varpi \to \underline{\mathrm{End}}_{\mathrm{Mod}_c(R^\sigma_{\overline{\theta}})}(\widehat{v}_\sigma^* \widehat{i}'^*_{\overline{\rho},\sigma} z^* \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)) \\ = \underline{\mathrm{End}}_{\mathrm{Mod}_c(R^\sigma_{\overline{\theta}})}(\widehat{v}^* \widehat{i}'^*_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma));$$

here the equality is obtained by an application of (B.3.12) to the outer rectangle in the commutative diagram (5.3.13), taking into account the fact that $\widehat{v}^* \widehat{i}'^*_{\overline{\rho}} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is an $R^\sigma_{\overline{\theta}}$-module (by Proposition 5.3.15). The map (5.3.22) is injective, again by Proposition 5.3.15. Since $\widehat{i}'^*_{\overline{\rho}}$ pulls back the morphism (5.3.21) to a morphism in $\mathrm{Coh}(\mathcal{O}_{(\mathcal{X}/\varpi)^\wedge_{\overline{\rho}}})$ (by Lemma B.4.13), and $\widehat{v}^*$ is faithful on $\mathrm{Coh}(\mathcal{O}_{(\mathcal{X}/\varpi)^\wedge_{\overline{\rho}}})$ (as we have already seen earlier in this proof), this concludes the proof that (5.3.21) is injective.

Finally, assume that $\sigma \neq \mathrm{Sym}^{p-2} \otimes \det^a$ for any a. It remains to prove that $z^* \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is an invertible sheaf on $\mathcal{Z}(\sigma)$. Since $\mathcal{Z}(\sigma)$ is reduced, it suffices to show that its fibre dimension is equal to 1 at all finite type points of $\mathcal{Z}(\sigma)$. Since (5.3.21) is injective, the fibre dimension is positive at all finite type points of $\mathcal{Z}(\sigma)$, hence the loci of fibre dimension $= 1$ and ≤ 1 coincide. By semicontinuity, this locus is an open subset of $|\mathcal{Z}(\sigma)|$, and by Proposition 5.3.15 (3), it contains all closed points. It is therefore equal to $|\mathcal{Z}(\sigma)|$. □

Proposition 5.3.23. *The functor*

$$\mathrm{F} : D^b_{\mathrm{fp}}(\mathcal{A}) \to \mathrm{Pro}\, D^b_{\mathrm{coh}}(\mathcal{X})$$

has essential image contained in $D^b_{\mathrm{coh}}(\mathcal{X})$. Furthermore it is of amplitude $[-1, 0]$.

Proof. Since $D^b_{\mathrm{fp}}(\mathcal{A}) = D^b(\mathcal{A}^{\mathrm{fp}})$, it follows from Lemma B.2.5 that it suffices to show that for each $\pi \in \mathcal{A}^{\mathrm{fp}}$, $\mathrm{F}(\pi)$ has cohomology in $\mathrm{Coh}(\mathcal{X})$, which furthermore vanishes outside of degrees $-1, 0$.

We begin by showing the vanishing. By Lemma B.3.34, it suffices to prove that $\widehat{i}'^*_{\overline{\theta}} H^i \mathrm{F}(\pi) = 0$ for all closed points $\overline{\theta}$ of $\mathcal{X}$ and all $i \neq -1, 0$. Since $\widehat{i}'^*_{\overline{\theta}}$ is exact, this is equivalent to $H^i(\widehat{i}'^*_{\overline{\theta}} \mathrm{F}(\pi)) = 0$. Since (5.3.3) is an isomorphism (by Proposition 5.3.6) we reduce to showing that $H^i(\mathrm{Pro}\, \mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}} \pi))$ vanishes for $i \neq -1, 0$. This follows from Theorem 5.2.24.

We now prove that $H^0 \mathrm{F}(\pi)$ and $H^{-1} \mathrm{F}(\pi)$ are coherent, for all $\pi \in \mathcal{A}^{\mathrm{fp}}$. Recall that π admits a presentation

$$c\text{-}\mathrm{Ind}_{KZ}^G W \to c\text{-}\mathrm{Ind}_{KZ}^G V \to \pi \to 0$$

with V, W finite $\mathcal{O}$-modules. A dévissage using Proposition 5.3.20 shows that $H^{-1} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G V) = 0$, and $H^0 \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G V)$ is coherent. The right t-exactness of F then implies that $H^0 \mathrm{F}(\pi)$ is coherent for all $\pi \in \mathcal{A}^{\mathrm{fp}}$. Now, since $\mathcal{A}^{\mathrm{fp}}$ is abelian, we have a short exact sequence

$$0 \to \pi' \to c\text{-}\mathrm{Ind}_{KZ}^G V \to \pi \to 0$$

with $\pi' \in \mathcal{A}^{\mathrm{fp}}$. We thus obtain an exact sequence

$$0 \to H^{-1} \mathrm{F}(\pi) \to H^0 \mathrm{F}(\pi') \to H^0 \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G V) \to H^0 \mathrm{F}(\pi) \to 0,$$

and we conclude that $H^{-1} \mathrm{F}(\pi)$ is coherent, as required. □

5.3.24. F *preserves the recollement.* We now show that for any (not necessarily finite) closed subset $Y \subseteq X$, the functor F_Y takes $D^b_{\mathrm{fp}}(\mathcal{A}_Y)$ to $D^b_{\mathrm{coh}}(\mathcal{X}_Y)$.

Proposition 5.3.25. *Let Y be a closed subset of X with open complement $j : U \hookrightarrow X$, and write $\mathcal{U} := \pi_{\mathrm{ss}}^{-1}(U)$ and $j'^{*}_{\mathcal{U}} : D^b_{\mathrm{coh}}(\mathcal{X}) \to D^b_{\mathrm{coh}}(\mathcal{U})$ for the pullback functor.*

(1) *The essential image of the functor $\mathrm{F}_Y : D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_Y)$ is contained in $D^b_{\mathrm{coh}}(\mathcal{X}_Y)$, and there is a commutative diagram*

$$\begin{tikzcd} D^b_{\mathrm{fp}}(\mathcal{A}_Y) \arrow[r, "i_{Y,*}"] \arrow[d, "\mathrm{F}_Y"'] & D^b_{\mathrm{fp}}(\mathcal{A}) \arrow[d, "\mathrm{F}"] \\ D^b_{\mathrm{coh}}(\mathcal{X}_Y) \arrow[r, "i'_{Y,*}"] & D^b_{\mathrm{coh}}(\mathcal{X}) \end{tikzcd}$$

(2) *The composite $D^b_{\mathrm{fp}}(\mathcal{A}) \xrightarrow{\mathrm{F}} D^b_{\mathrm{coh}}(\mathcal{X}) \xrightarrow{j'^{*}_{\mathcal{U}}} D^b_{\mathrm{coh}}(\mathcal{U})$ factors through a functor*

$$\mathrm{F}_U : D^b_{\mathrm{fp}}(\mathcal{A}_U) \to D^b_{\mathrm{coh}}(\mathcal{U}).$$

Furthermore, if $\overline{\theta}$ is a closed point of U then there is a natural isomorphism

$$\widehat{i}'^{*}_{\overline{\theta}} \mathrm{F}_U \xrightarrow{\sim} (\operatorname{Pro} \mathrm{F}_{\overline{\theta}}) \widehat{i}^{*}_{\overline{\theta}} : D^b_{\mathrm{fp}}(\mathcal{A}_U) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}). \tag{5.3.26}$$

Proof. We begin by noting that, for every $\overline{\theta} \notin Y$, it follows from Proposition 5.3.6 that

$$\widehat{i}'^{*}_{\overline{\theta}} \mathrm{F} i_{Y,*} = (\operatorname{Pro} \mathrm{F}_{\overline{\theta}}) \widehat{i}^{*}_{\overline{\theta}} i_{Y,*} = 0 \tag{5.3.27}$$

(where we have used that $\widehat{i}^{*}_{\overline{\theta}} i_{Y,*} = 0$). It follows from (5.3.27) and Lemma B.3.34 that $j'^{*}_{\mathcal{U}} \mathrm{F} i_{Y,*} = 0$; equivalently, bearing in mind Proposition B.2.8, we see that the composite $\mathrm{F} i_{Y,*} : D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to D^b_{\mathrm{coh}}(\mathcal{X})$ has essential image contained in the full subcategory $D^b_{\mathrm{coh}}(\mathcal{X}_Y) = D^b_{\mathrm{coh},Y}(\mathcal{X})$ consisting of complexes whose cohomology sheaves are set-theoretically supported on Y. In particular, we may write $\mathrm{F} i_{Y,*} = \widehat{i}'_{Y,*} \mathrm{F}'_Y$ for some functor $\mathrm{F}'_Y : D^b_{\mathrm{fp}}(\mathcal{A}_Y) \to D^b_{\mathrm{coh}}(\mathcal{X}_Y)$.

We therefore have the following commutative diagram, which immediately gives (1).

$$\begin{tikzcd} D^b_{\mathrm{fp}}(\mathcal{A}_Y) \arrow[r, "i_{Y,*}"] \arrow[rrr, bend left, "\mathrm{F}_Y"] \arrow[dr, "\mathrm{F}'_Y"'] & D^b_{\mathrm{fp}}(\mathcal{A}) \arrow[r, "\mathrm{F}"] & D^b_{\mathrm{coh}}(\mathcal{X}) \arrow[r, "\widehat{i}'^{*}_{Y}"] & \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_Y) \\ & D^b_{\mathrm{coh}}(\mathcal{X}_Y) \arrow[ur, "i'_{Y,*}"] \arrow[urr, hook] & & \end{tikzcd}$$

The existence of F_U is a formal consequence of (5.3.27), Corollary 2.6.7 and the commutativity of the diagram. Finally, (5.3.26) follows from (5.3.3). □

Definition 5.3.28. By Proposition 5.3.25 and Corollary 2.6.7, the Ind-extensions of the functors $\mathrm{F}, \mathrm{F}_Y, \mathrm{F}_U$ give continuous, right t-exact functors, that we continue to denote by the same symbols

$$\mathrm{F} : D(\mathcal{A}) \to \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X}),$$

$$\mathrm{F}_Y : D(\mathcal{A}_Y) \to \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X}_Y),$$

$$\mathrm{F}_U : D(\mathcal{A}_U) \to \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{U}).$$

5.4. **Localization to $\mathcal{U}_{\mathrm{good}}$.** Recall that in Definition 2.7.6 we defined a finite subset $Y_{\mathrm{bad}} \subset |X|$ with dense open complement U_{good}, and in Definition 3.7.1 we defined $\mathcal{U}_{\mathrm{good}}$ as the open substack of $\mathcal{X}$ with underlying topological space $\pi_{\mathrm{ss}}^{-1}(U_{\mathrm{good}})$. Our goal in this section is to establish some properties of our functor F after restriction to U_{good} and $\mathcal{U}_{\mathrm{good}}$; in particular, we will show that the functor $\mathrm{F}_{\mathcal{U}_{\mathrm{good}}}$ is fully faithful (see Theorem 5.4.25).

By Lemma 3.7.4, $\mathcal{U}_{\mathrm{good}}$ decomposes as the disjoint union of open subsets $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})$, and correspondingly U_{good} decomposes as the disjoint union of open subsets $U(\sigma|\sigma^{\mathrm{co}})$. Accordingly, we will be able to work with each open subset $U(\sigma|\sigma^{\mathrm{co}})$ separately, and a key tool in our analysis will be the explicit description of the stack $\mathcal{U}(\sigma|\sigma^{\mathrm{co}})_{\mathrm{red}}$ in Proposition 3.7.8. We will need to study the coherent sheaf $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$; if $\{\sigma, \sigma^{\mathrm{co}}\}$ is not of type (scalar), then this is an invertible sheaf on $\mathcal{Z}(\sigma)$ (see Proposition 5.3.20), and our analysis could be simplified, but we give a uniform treatment of all $\sigma|\sigma^{\mathrm{co}}$ below.

We now fix a companion pair $\{\sigma, \sigma^{\mathrm{co}}\}$. To simplify notation, we make the following definitions.

Definition 5.4.1.

(1) We set $Y := Y_{\mathrm{bad}} \cap X(\sigma|\sigma^{\mathrm{co}})$, $U := U(\sigma|\sigma^{\mathrm{co}}) = X(\sigma|\sigma^{\mathrm{co}}) \setminus Y$, $\mathcal{U} := \mathcal{U}(\sigma|\sigma^{\mathrm{co}}) = \pi_{\mathrm{ss}}^{-1}(U)$. We also set $\mathcal{U}(\sigma) := \mathcal{U} \cap \mathcal{Z}(\sigma)$.

(2) We write $j^* : D(\mathcal{A}) \to D(\mathcal{A}_U)$, $j'^* : \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{U})$ for the t-exact localization functors, and j_*, j'_* for their t-exact fully faithful right adjoints (see Corollary 2.6.7 and Lemma 3.7.11).

(3) We write $f = f(t)$ for the polynomial defined in Definition 2.7.6, so that $f_\sigma^{-1}\big(U(\sigma|\sigma^{\mathrm{co}})\big) \subset \mathrm{Spec}\, \mathcal{H}_G(\sigma)$ is the non-vanishing locus of $f(T_p) \in \mathcal{H}_G(\sigma) = \mathbf{F}[T_p]$. Note that this implies
$$j_* j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma \xrightarrow{\sim} \mathrm{colim}_{\times f}\, c\text{-}\mathrm{Ind}_{KZ}^G \sigma,$$
by Lemma 2.5.16.

We begin by specializing Corollary 2.6.8 to the situation at hand.

Lemma 5.4.2. *The localizations $j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma$, $j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma^{\mathrm{co}}$ form a set of compact generators of $D(\mathcal{A}_U)$.*

Proof. By Corollary 2.6.8, the set of localizations $\{j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma'\}$, where σ' ranges over the ζ-compatible Serre weights, forms a set of generators of $D(\mathcal{A}_U)$. If $\sigma' \neq \sigma, \sigma^{\mathrm{co}}$, and σ' is not a twist of Sym^{p-1}, then $j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma' = 0$ by [DEG23, Prop. 3.1.13] (i.e. essentially by the very definition of j^*). It remains to note that by (2.2.36) and the definition of U_{good}, we have (for any r)
$$j^* c\text{-}\mathrm{Ind}_{KZ}^G \det{}^r \otimes \mathrm{Sym}^{p-1} = j^* c\text{-}\mathrm{Ind}_{KZ}^G \det{}^r \otimes \mathrm{Sym}^0 . \qquad \square$$

In preparation for the next result, we recall that $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma / f c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is pure of degree zero. (Indeed, $c\text{-}\mathrm{Ind}_{KZ}^G \sigma / f c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ is an object of $\mathcal{A}_Y$, and Proposition 5.3.25 (1) identifies $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma / f c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ with $\mathrm{F}_Y(c\text{-}\mathrm{Ind}_{KZ}^G \sigma / f c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$, which Lemma 5.3.8 shows is pure of degree zero; note that σ is not Steinberg, by the definition of a companion pair.) The following lemma describes the underlying topological space of the scheme-theoretic support of this coherent sheaf, and shows that its endomorphism $\mathrm{F}(f)$ is nilpotent. In its proof, we will make use of the fact that $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is scheme-theoretically supported on $\mathcal{Z}(\sigma)$, by Proposition 5.3.20, hence the set-theoretic support of its quotient $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/fc\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$ (in the sense of Definition B.3.31) is *a priori* contained in $|\mathcal{Z}(\sigma)|$.

Lemma 5.4.3. *Let* $i'_{\mathcal{Y}}:\mathcal{Y}\to\mathcal{X}$ *be the scheme-theoretic support of* $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/fc\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$. *Then*

(1) $|\mathcal{Y}|=\coprod_{\overline{\theta}\in Y_{\mathrm{bad}}}|\mathcal{Z}(\sigma)|\cap|\mathcal{X}_{\overline{\theta}}|=|\mathcal{Z}(\sigma)|\setminus|\mathcal{U}(\sigma)|$.

(2) The endomorphism $\mathrm{F}(f)$ *of the coherent sheaf* $i'^{*}_{\mathcal{Y}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$ *is nilpotent.*

Proof. By Definition 2.7.6, we have $f=\prod_{\overline{\theta}\in Y_{\mathrm{bad}}}f_{\overline{\theta}}$, where $f_{\overline{\theta}}\in\mathcal{H}(\sigma)$ is as in Definition 2.5.15, so that $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$ is the maximal multiplicity-free quotient of $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$ which is an object of $\mathcal{A}_{\overline{\theta}}$, and

$$c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/fc\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma=\prod_{\overline{\theta}\in Y_{\mathrm{bad}}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma.$$

Thus, if we let $i'_{\mathcal{Y}_{\overline{\theta}}}:\mathcal{Y}_{\overline{\theta}}\hookrightarrow\mathcal{X}$ denote the scheme-theoretic support of the coherent sheaf $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$, we find that $\mathcal{Y}=\coprod_{\overline{\theta}\in Y_{\mathrm{bad}}}\mathcal{Y}_{\overline{\theta}}$. It therefore suffices to prove, for each $\overline{\theta}\in Y_{\mathrm{bad}}$, that $|\mathcal{Y}_{\overline{\theta}}|=|\mathcal{Z}(\sigma)|\cap|\mathcal{X}_{\overline{\theta}}|$, and that $\mathrm{F}(f_{\overline{\theta}})$ acts nilpotently on $i'^{*}_{\mathcal{Y}_{\overline{\theta}}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$.

By Proposition 5.3.25 we have

$$\begin{aligned}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)&=i'_{\overline{\theta},*}\mathrm{F}_{\overline{\theta}}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\\&=i'_{\overline{\theta},*}\widehat{i}'^{*}_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma),\end{aligned}$$

so we see that $|\mathcal{Y}_{\overline{\theta}}|$ (which is *a priori* contained in $|\mathcal{Z}(\sigma)|$, as recalled before the statement of the lemma) is contained in $|\mathcal{Z}(\sigma)|\cap|\mathcal{X}_{\overline{\theta}}|$.

On the other hand, by Proposition 5.3.6 and Lemma 2.5.16 we have the identifications

$$\begin{aligned}\widehat{i}'^{*}_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)&=\mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}\big(\lim_n(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f_{\overline{\theta}}^{n}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma\big)\\&=\lim_n\mathrm{F}_{\overline{\theta}}\big((c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f_{\overline{\theta}}^{n}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\big).\end{aligned}$$

Applying $\widehat{i}'_{\overline{\theta},*}$, we find that

$$\widehat{i}'_{\overline{\theta},*}\widehat{i}'^{*}_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)=\lim_n i'_{\overline{\theta},*}\mathrm{F}_{\overline{\theta}}\big((c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f_{\overline{\theta}}^{n}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\big).\tag{5.4.4}$$

Since $i'_{\overline{\theta},*}\mathrm{F}_{\overline{\theta}}\big((c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f_{\overline{\theta}}^{n}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\big)$ is filtered by copies of the coherent sheaf $i'_{\overline{\theta},*}\mathrm{F}_{\overline{\theta}}\big((c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f_{\overline{\theta}}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\big)$, we see that the set-theoretic support of every coherent quotient of $\widehat{i}'_{\overline{\theta},*}\widehat{i}'^{*}_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$ is contained in that of

$$i'_{\overline{\theta},*}\mathrm{F}_{\overline{\theta}}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)=\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma),$$

which is $|\mathcal{Y}_{\overline{\theta}}|$. By Proposition 5.3.20, the scheme-theoretic support of $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$ is $\mathcal{Z}(\sigma)$, so in particular $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$ has a quotient $\mathcal{G}$ whose set-theoretic support is exactly equal to $|\mathcal{Z}(\sigma)|\cap|\mathcal{X}_{\overline{\theta}}|$. The unit $\mathcal{G}\to\widehat{i}'_{\overline{\theta},*}\widehat{i}'^{*}_{\overline{\theta}}\mathcal{G}$ is then necessarily an isomorphism, and so $\mathcal{G}$ is a coherent quotient of $\widehat{i}'_{\overline{\theta},*}\widehat{i}'^{*}_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$. It follows from the previous paragraph that $\mathcal{G}$ has set-theoretic support contained in that of $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$, which is $|\mathcal{Y}_{\overline{\theta}}|$. Thus $|\mathcal{Y}_{\overline{\theta}}|$ is equal to $|\mathcal{Z}(\sigma)|\cap|\mathcal{X}_{\overline{\theta}}|$, as claimed. This concludes the proof of the first part.

Turning to the second part, write $i'_{\mathcal{X}_{\overline{\theta},\mathrm{red}}} : \mathcal{X}_{\overline{\theta},\mathrm{red}} \hookrightarrow \mathcal{X}$ for the closed immersion. Since $\mathcal{X}_{\overline{\theta}}$ is the completion of $\mathcal{X}$ at $\mathcal{X}_{\overline{\theta},\mathrm{red}}$, we see from Remark B.3.23 that there is a natural surjection

$$\widehat{i}'_{\overline{\theta},*}\widehat{i}'^{*}_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) \twoheadrightarrow i'_{\mathcal{X}_{\overline{\theta},\mathrm{red}},*}i'^{*}_{\mathcal{X}_{\overline{\theta},\mathrm{red}}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$$

in $\operatorname{Pro}\operatorname{Coh}(\mathcal{X})$. We noted in (5.4.4) that the source of this surjection is equal to

$$\lim_n i'_{\overline{\theta},*}\mathrm{F}_{\overline{\theta}}\big((c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f_{\overline{\theta}}^{n}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\big),$$

so we see that the coherent sheaf $i'_{\mathcal{X}_{\overline{\theta},\mathrm{red}},*}i'^{*}_{\mathcal{X}_{\overline{\theta},\mathrm{red}}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$ is a quotient of

$$i'_{\overline{\theta},*}\mathrm{F}_{\overline{\theta}}\big((c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f_{\overline{\theta}}^{n}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\big) = \mathrm{F}\big((c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f_{\overline{\theta}}^{n}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)\big)$$

for n sufficiently large, whence it is killed by $\mathrm{F}(f_{\overline{\theta}})^n = \mathrm{F}(f_{\overline{\theta}}^n)$.

We have already seen that $|\mathcal{Y}_{\overline{\theta}}| \subseteq |\mathcal{X}_{\overline{\theta}}|$, and so $\mathcal{Y}_{\overline{\theta},\mathrm{red}} \subseteq \mathcal{X}_{\overline{\theta},\mathrm{red}}$. Thus, letting $i'_{\mathcal{Y}_{\overline{\theta},\mathrm{red}}}$ denote the closed immersion $\mathcal{Y}_{\overline{\theta},\mathrm{red}} \hookrightarrow \mathcal{X}$, the result of the previous paragraph shows that $\mathrm{F}(f_{\overline{\theta}})$ acts nilpotently on $i'^{*}_{\mathcal{Y}_{\overline{\theta},\mathrm{red}}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$. This immediately implies that $\mathrm{F}(f_{\overline{\theta}})$ also acts nilpotently on $i'^{*}_{\mathcal{Y}_{\overline{\theta}}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$, as required. □

We now study the interaction of our functor F with the localization functors in Definition 5.4.1. By construction, we have an isomorphism $\mathrm{F}_U j^* \xrightarrow{\sim} j'^*\mathrm{F}$. Postcomposing the inverse of this isomorphism with j'_*, the unit of adjunction $1 \to j'_* j'^*$ induces a morphism $\mathrm{F} \to j'_*\mathrm{F}_U j^*$. Precomposing with j_*, the counit $j^*j_* \to 1$ induces a natural transformation

$$\mathrm{F}j_* \to j'_*\mathrm{F}_U. \tag{5.4.5}$$

Proposition 5.4.6. *The natural transformation* (5.4.5) *is a natural isomorphism of functors from $D^b_{\mathrm{fp}}(\mathcal{A}_U)$ to $\operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$.*

Proof. By the continuity of all functors involved, it is equivalent to show that (5.4.5) is a natural isomorphism of the corresponding Ind-extended functors $D(\mathcal{A}_U) \to \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$. By Lemma 5.4.2, $j^*c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$, $j^*c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma^{\mathrm{co}}$ are compact generators of $D(\mathcal{A}_U)$, so by the symmetry between σ and σ^{co}, it suffices to show that the morphism induced by (5.4.5)

$$\mathrm{F}(j_*j^*c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) \to j'_*\mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) \tag{5.4.7}$$

is an isomorphism.

Recall (as noted in Definition 5.4.1) that the functors j_*, j^* and j'_*, j'^* are t-exact, and $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$ is a sheaf (i.e. is in the heart of $\operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$), so the same is true of $j'_*\mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) = j'_*j'^*\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$. Similarly (as also noted in Definition 5.4.1) we have $j_*j^*c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma = (c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)[1/f] = \operatorname{colim}_{\times f} c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$, and since F commutes with colimits, we have

$$\mathrm{F}(j_*j^*c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) = \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)[1/\mathrm{F}(f)] \coloneqq \operatorname{colim}_{\times \mathrm{F}(f)} \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma), \tag{5.4.8}$$

which is also a sheaf. Thus all the objects under consideration are in the hearts of the respective t-structures, and we work at the abelian level in what follows. Bearing in mind (5.4.8), we see that we need to prove that the natural map

$$\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)[1/\mathrm{F}(f)] \to j'_*j'^*\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$$

is an isomorphism.

Since $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is scheme-theoretically supported on the closed substack $\mathcal{Z}(\sigma)$ of $\mathcal{X}$, it is easy to check that we can replace $\mathcal{X}$ by $\mathcal{Z}(\sigma)$, and j' by its restriction to the open immersion $\mathcal{U}(\sigma) \hookrightarrow \mathcal{Z}(\sigma)$. We claim that the result now follows from Lemma 5.4.9 below, which we apply with $\mathcal{Z} = \mathcal{Z}(\sigma)$, $\mathcal{F} = \mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$, and f equal to our $\mathrm{F}(f)$. Indeed, it is only necessary to remark that in this case the coherent sheaf $\mathcal{F}/f\mathcal{F}$ is (by the right t-exactness of F) equal to $\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma / f c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$, so the hypotheses of Lemma 5.4.9 are supplied by Lemma 5.4.3. ☐

Lemma 5.4.9. *Let $\mathcal{F}$ be a coherent sheaf on a Noetherian algebraic stack $\mathcal{Z}$, and let f be an endomorphism of $\mathcal{F}$. Let $i : \mathcal{Y} \to \mathcal{Z}$ be the scheme-theoretic support of $\mathcal{F}/f\mathcal{F}$ (a closed algebraic substack of $\mathcal{Z}$) and let $j : \mathcal{U} \hookrightarrow \mathcal{Z}$ be the open immersion of the complement of $\mathcal{Y}$. Suppose further that f acts nilpotently on $i^*\mathcal{F}$. Then f becomes invertible on $j^*\mathcal{F}$, and the natural map $\mathcal{F}[1/f] \xrightarrow{\sim} j_*j^*\mathcal{F}$ is an isomorphism in* $\mathrm{Ind\,Coh}\,\mathcal{Z}$.

Proof. There is an exact sequence

$$0 \to \mathcal{F}[f] \to \mathcal{F} \xrightarrow{f} \mathcal{F} \to \mathcal{F}/f\mathcal{F} \to 0.$$

Note that the morphism $\mathcal{F}[f] \to \mathcal{F}/f^N$ is injective for N sufficiently large, so that there is an inclusion of set-theoretic supports

$$\mathrm{Supp}(\mathcal{F}[f]) \subseteq \mathrm{Supp}(\mathcal{F}/f^N) = \mathrm{Supp}(\mathcal{F}/f\mathcal{F}) = |\mathcal{Y}|.$$

Thus multiplication by f on $\mathcal{F}$ becomes an isomorphism after applying j^*, whence the unit of adjunction $\mathcal{F} \to j_*j^*\mathcal{F}$ induces a morphism $\mathcal{F}[1/f] \to j_*j^*\mathcal{F}$ in $\mathrm{Ind\,Coh}(\mathcal{Z})$, which we will show is an isomorphism. In order to do this we can replace $\mathcal{Z}$ by a smooth cover by a scheme Z, and $\mathcal{U}$ and $\mathcal{F}$ by their pullbacks to Z, which we denote by U and $\mathcal{F}$.

Let $\mathcal{A}$ be the coherent algebra of endomorphisms of $\mathcal{F}$ generated by $\mathcal{O}_Z$ together with f, and write $W = \underline{\mathrm{Spec}}\,\mathcal{A}$, equipped with its canonical finite affine morphism $\pi : W \to Z$. As usual, $\mathcal{F}$ gives rise to a coherent sheaf $\mathcal{G} \coloneqq \mathcal{O}_W \otimes_{\pi^{-1}\mathcal{A}} \pi^{-1}\mathcal{F}$ on W such that $\mathcal{F} \xrightarrow{\sim} \pi_*\mathcal{G}$. Since $\mathcal{A}$ acts faithfully on $\mathcal{F}$, we see that the support of $\mathcal{G}$ is all of W. Recall that this implies that for any coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_W$, we have that $\mathrm{Supp}(\mathcal{G}/\mathcal{I}\mathcal{G}) = V(\mathcal{I})$. In particular, $\mathrm{Supp}(\mathcal{G}/f\mathcal{G}) = V(f)$.

Now $\pi_*(\mathcal{G}/f\mathcal{G}) = \mathcal{F}/f\mathcal{F}$, and so $Y \coloneqq \mathrm{Supp}(\mathcal{F}/f\mathcal{F}) = \pi\big(V(f)\big)$. We claim that the inclusion $V(f) \subseteq \pi^{-1}(Y)$ is an equality. Given this, we find that $\pi^{-1}(U)$ is equal to the distinguished open subset $D(f)$ of W, and so, if $j' : D(f) \hookrightarrow W$ is the corresponding open immersion, we find (recalling that π is affine) that indeed

$$j_*j^*\mathcal{F} = \pi_*j'_*j'^*\mathcal{G} = \pi_*\mathcal{G}[1/f] = \mathcal{F}[1/f].$$

To prove the claim, write $\mathcal{I}' \coloneqq (\pi^{-1}\mathcal{I}_Y)\mathcal{O}_W$, so that $\mathcal{I}'$ cuts out a closed subscheme of W with underlying reduced equal to $\pi^{-1}(Y)$. Then $i^*\mathcal{F} = \pi_*(\mathcal{G}/\mathcal{I}'\mathcal{G})$, and our assumption that f acts nilpotently on $i^*\mathcal{F}$ implies that f acts nilpotently on $\mathcal{G}/\mathcal{I}'\mathcal{G}$. Hence (again using the fact that $\mathcal{G}$ has full support on W) we see that

$$\pi^{-1}(Y) = V(\mathcal{I}') = \mathrm{Supp}(\mathcal{G}/\mathcal{I}'\mathcal{G}) \subseteq V(f).$$

We've already noted the reverse inclusion, and the desired equality follows. ☐

We now make use of the explicit description of $\mathcal{U}_{\mathrm{red}}$ proved in Section 3.7. Write $B \coloneqq \mathbf{F}[t, f(t)^{-1}, x, y]/(xy)$, so that by Proposition 3.7.8 we may write $\mathcal{U}_{\mathrm{red}} = [\mathrm{Spec}\, B/\mathbf{G}]$, where the reductive group $\mathbf{G}$ and its action on B is specified in the statement of Proposition 3.7.8. From this explicit description we see that $B^{\mathbf{G}} =$

$\mathbf{F}[t, f(t)^{-1}]$ unless $\sigma|\sigma^{\mathrm{co}}$ is of type (scalar), in which case $B^{\mathbf{G}} = \mathbf{F}[s, (s^2-4)^{\pm 1}]$ where $s = t + t^{-1}$. In either case there is a finite morphism of $\mathbf{F}$-algebras

$$B^{\mathbf{G}} \to \mathbf{F}[t, f(t)^{-1}], \tag{5.4.10}$$

and the maximal ideals of $B^{\mathbf{G}}$ are in bijection with the closed points $\overline{\theta}$ of U. More precisely, if $\sigma = \sigma_{a,b}$, and $\overline{\theta}$ corresponds to the semisimple Galois representation

$$\overline{\rho}_{\overline{\theta}} := \mathrm{nr}_{\alpha}\overline{\zeta}\omega^{-a} \oplus \mathrm{nr}_{\alpha^{-1}}\omega^{a-1},$$

then the corresponding maximal ideal of $B^{\mathbf{G}}$ is $\mathfrak{n}_{\overline{\theta}} := (t-\alpha) \subseteq B^{\mathbf{G}}$ (respectively $\mathfrak{n}_{\overline{\theta}} := (s - (\alpha + \alpha^{-1})) \subseteq B^{\mathbf{G}}$ if $\sigma|\sigma^{\mathrm{co}}$ is of type (scalar)); and we have

$$\mathfrak{n}_{\overline{\theta}}\mathbf{F}[t, f(t)^{-1}] = (f_{\overline{\theta}}(t)). \tag{5.4.11}$$

We may now apply the results of Appendix B.4.14 to the stack $\mathcal{U}_{\mathrm{red}} = [\operatorname{Spec} B/\mathbf{G}]$, taking G there to be our $\mathbf{G}$, and R to be $\mathbf{F}$.

Proposition 5.4.12. *The functor* F_U *induces an isomorphism*

$$\operatorname{End}_{\mathcal{A}_U^{\mathrm{fp}}}\big(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma\big) \xrightarrow{\sim} \operatorname{End}_{\mathrm{Coh}(\mathcal{U})}\big(\mathrm{F}_U(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)\big).$$

Proof. By Proposition 5.3.20, $\mathrm{F}_U(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) = j'^*\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)$ is a coherent sheaf on $\mathcal{U}$. Write

$$S := \operatorname{End}_{\mathrm{Coh}(\mathcal{U})}\big(\mathrm{F}_U(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)\big),$$

so that S is a (not necessarily commutative) $B^{\mathbf{G}}$-algebra which, by Corollary B.4.21, is of finite type as a $B^{\mathbf{G}}$-module. By (2.5.18), we have

$$(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/f] \xrightarrow{\sim} j_*j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma. \tag{5.4.13}$$

Consequently, bearing in mind the full faithfulness of j_*, we find that

$$\operatorname{End}_{\mathcal{A}_U^{\mathrm{fp}}}\big(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma\big) = \mathbf{F}[T_p, f(T_p)^{-1}]. \tag{5.4.14}$$

The morphism

$$\mathrm{F}_U : \operatorname{End}_{\mathcal{A}_U^{\mathrm{fp}}}\big(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma\big) \to \operatorname{End}_{\mathrm{Coh}(\mathcal{U})}\big(\mathrm{F}_U(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)\big)$$

induced by F_U is thus a morphism

$$\mathbf{F}[T_p, f(T_p)^{-1}] \to S. \tag{5.4.15}$$

Our goal is to show that this morphism is an isomorphism.

As above, we choose a closed point of U, corresponding to a pseudorepresentation $\overline{\theta}$ and a maximal ideal $\mathfrak{n}_{\overline{\theta}}$ of $B^{\mathbf{G}}$. Recall from Theorem 5.2.24 that

$$\operatorname{Pro}\mathrm{F}_{\overline{\theta}} : \operatorname{Pro} D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$$

is fully faithful, and recall from Lemma 5.3.8 that $\operatorname{Pro}\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) = \widehat{i}'^*_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) = \widehat{i}'^*_{\overline{\theta}}\mathrm{F}_U(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)$ is pure of degree zero. We thus obtain an isomorphism

$$\operatorname{Pro}\mathrm{F}_{\overline{\theta}} : \operatorname{End}_{\operatorname{Pro}\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}}\big(\widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma\big) \xrightarrow{\sim} \operatorname{End}_{\operatorname{Pro}\mathrm{Coh}\,\mathcal{X}_{\overline{\theta}}}\big(\widehat{i}'^*_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)\big). \tag{5.4.16}$$

By Corollary B.4.23, the morphism

$$\widehat{i}'^*_{\overline{\theta}} : S := \operatorname{End}_{\mathrm{Coh}\,\mathcal{U}}\big(\mathrm{F}_U(j^* c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)\big) \to \operatorname{End}_{\operatorname{Pro}\mathrm{Coh}\,\mathcal{X}_{\overline{\theta}}}\big(\widehat{i}'^*_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)\big)$$

is a completion at $\mathfrak{n}_{\overline{\theta}}$, i.e. it induces an isomorphism

$$\widehat{S}_{\mathfrak{n}_{\overline{\theta}}} \xrightarrow{\sim} \operatorname{End}_{\operatorname{Pro}\mathrm{Coh}\,\mathcal{X}_{\overline{\theta}}}\big(\widehat{i}'^*_{\overline{\theta}}\mathrm{F}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)\big).$$

Moreover, by Proposition 2.7.23, the map (2.7.22) induces the first of the identifications

$$\operatorname{End}_{\operatorname{Pro}\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\operatorname{Ind}_{KZ}^G\sigma) = \mathbf{F}[T_p, f(T_p)^{-1}]^\wedge_{f_{\overline{\theta}}(T_p)} = \mathbf{F}[T_p, f(T_p)^{-1}]^\wedge_{\mathfrak{n}_{\overline{\theta}}}, \tag{5.4.17}$$

the second being induced by (5.4.11). Finally, the natural isomorphism $\widehat{i}'^*_{\overline{\theta}}\mathrm{F}_U \xrightarrow{\sim} (\operatorname{Pro}\mathrm{F}_{\overline{\theta}})\widehat{i}^*_{\overline{\theta}}$ of Proposition 5.3.25 produces a commutative diagram
(5.4.18)

$$\begin{tikzcd}
\mathbf{F}[T_p, f(T_p)^{-1}] = \operatorname{End}_{\mathcal{A}^{\mathrm{fp}}_U}(j^*c\text{-}\operatorname{Ind}_{KZ}^G\sigma) \arrow[r, "(5.4.15)", "\mathrm{F}_U"'] \arrow[d, "\widehat{i}^*_{\overline{\theta}}"] & \operatorname{End}_{\operatorname{Coh}\mathcal{U}}(\mathrm{F}_U(j^*c\text{-}\operatorname{Ind}_{KZ}^G\sigma)) =: S \arrow[d, "\widehat{i}'^*_{\overline{\theta}}"] \\
\mathbf{F}[T_p, f(T_p)^{-1}]^\wedge_{\mathfrak{n}_{\overline{\theta}}} \overset{(5.4.17)}{=} \operatorname{End}_{\operatorname{Pro}\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\operatorname{Ind}_{KZ}^G\sigma) \arrow[r, "\sim", "\operatorname{Pro}\mathrm{F}_{\overline{\theta}}"'] & \operatorname{End}_{\operatorname{Pro}\operatorname{Coh}\mathcal{X}_{\overline{\theta}}}(\widehat{i}'^*_{\overline{\theta}}\mathrm{F}(c\text{-}\operatorname{Ind}_{KZ}^G\sigma)) = \widehat{S}_{\mathfrak{n}_{\overline{\theta}}}
\end{tikzcd}$$

whose top horizontal arrow is the morphism (5.4.15) that is the focus of our attention.

We now make some deductions. Firstly, we see that the completion of S (which we recall is of finite type as a $B^{\mathbf{G}}$-module) at each maximal ideal of $B^{\mathbf{G}}$ (which is itself a finite type $\mathbf{F}$-algebra) is commutative and reduced. Thus S itself is commutative and reduced, and is in particular a finite commutative $B^{\mathbf{G}}$-algebra. We have already observed that (5.4.10) makes $\mathbf{F}[T_p, f(T_p)^{-1}]$ a finite $B^{\mathbf{G}}$-algebra. Somewhat confusingly, we have not shown that (5.4.15) is a morphism of $B^{\mathbf{G}}$-algebras. However, we see from (5.4.18) that (5.4.15) induces isomorphisms of the completions of its source and target at each maximal ideal $\mathfrak{n}_{\overline{\theta}}$ of $B^{\mathbf{G}}$. Now the finiteness of each of $\mathbf{F}[T_p, f(T_p)^{-1}]$ and S over $B^{\mathbf{G}}$ shows that the maximal ideals of each of them are partitioned into finite sets according to the maximal $\mathfrak{n}_{\overline{\theta}}$ of $B^{\mathbf{G}}$ that they lie over, and then the isomorphisms given by the bottom arrow of (5.4.18) show that (5.4.15) respects these partitions of the maximal ideals in its source and target, and furthermore induces bijections on each of the finite sets of maximal ideals given by these partitions. Thus (5.4.15) in fact induces a bijection between the maximal ideals in its source and target, and an isomorphism between the corresponding completions of each of its source and target. It follows from Lemma 5.4.19 below that (5.4.15) is an isomorphism, as desired. $\square$

Lemma 5.4.19. *Let k be a field, and let $f : A \to B$ be a morphism between finite type k-algebras. Assume that f^{-1} induces a bijection on maximal ideals, and that for every maximal ideal $\mathfrak{m} \subset B$, the induced map $\widehat{f} : \widehat{A}_{f^{-1}(\mathfrak{m})} \to \widehat{B}_{\mathfrak{m}}$ is an isomorphism. Then f is an isomorphism.*

Proof. Since f^{-1} is bijective on maximal ideals, to prove that f an isomorphism, it suffices to prove that $\operatorname{Spec} f$ is an open immersion. By [Stacks, Tag 02LC], it in turn suffices to prove that f is étale and universally injective.

For each maximal ideal $\mathfrak{m}$ of B, it follows from our hypotheses together with [Stacks, Tag 039M] that the map $A_{f^{-1}(\mathfrak{m})} \to B_{\mathfrak{m}}$ is an étale homomorphism of local rings, and is in particular flat and unramified. Hence $f : A \to B$ is flat, by [Stacks, Tag 00HT] and furthermore it is unramified at all maximal ideals of B (i.e. $(\Omega^1_{B/A})_{\mathfrak{m}} = 0$ for all maximal $\mathfrak{m} \subset B$), by [Stacks, Tag 039G]. Thus $f : A \to B$ is étale.

It remains to show that f is universally injective. By [Stacks, Tag 01S4], it suffices to prove that $B \otimes_A B \to B$ induces a surjection on Spec. Since the image of $\operatorname{Spec}(B) \to \operatorname{Spec}(B \otimes_A B)$ is closed, it suffices to show that for each finite extension k'/k, any morphism $\phi : B \otimes_A B \to k'$ necessarily factors through B. Given such a map ϕ, we have two maps $\phi_1, \phi_2 : B \to k'$ (defined by $\phi_1(b) = \phi(b \otimes 1)$ and $\phi_2(b) = \phi(1 \otimes b)$) that agree on A, and we must show $\phi_1 = \phi_2$. Since ϕ_1, ϕ_2 agree on A, and f^{-1} is bijective on maximal ideals by hypothesis, we deduce that $\ker(\phi_1) = \ker(\phi_2) = \mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset B$. The map $\widehat{A}_{f^{-1}(\mathfrak{m})} \to \widehat{B}_{\mathfrak{m}}$ is an isomorphism by hypothesis, so it induces an isomorphism on residue fields $A/f^{-1}(\mathfrak{m}) \xrightarrow{\sim} B/\mathfrak{m}$; so $\phi_1 = \phi_2$, as required. $\square$

Remark 5.4.20. Although we won't need this in the sequel, the morphism (5.4.15) is in fact a morphism of $B^{\mathbf{G}}$-algebras. To see this, writing mSpec for the set of maximal ideals of a ring, it follows from the proof of Proposition 5.4.12 (in particular, from the commutativity of (5.4.18)) that the diagram

$$\begin{tikzcd} & \mathrm{mSpec}\, B^{\mathbf{G}} \\ \mathrm{mSpec}\, \mathbf{F}[T_p, f(T_p)^{-1}] \arrow[ur] & \mathrm{mSpec}\, S \arrow[l, "\mathrm{F}_U"'] \arrow[u] \end{tikzcd}$$

commutes. Since the formation of (5.4.18) is compatible with finite base change in $\mathbf{F}$, we see that in fact the diagram

$$\begin{tikzcd} & \mathrm{mSpec}(B^{\mathbf{G}} \otimes_{\mathbf{F}} \mathbf{F}') \\ \mathrm{mSpec}\, \mathbf{F}'[T_p, f(T_p)^{-1}] \arrow[ur] & \mathrm{mSpec}(S \otimes_{\mathbf{F}} \mathbf{F}') \arrow[l, "\mathrm{F}_U"'] \arrow[u] \end{tikzcd}$$

commutes whenever $\mathbf{F}'$ is a finite extension of $\mathbf{F}$, and so it also commutes when $\mathbf{F}' = \overline{\mathbf{F}}_p$. Since the functor $A \mapsto \mathrm{mSpec}(A \otimes_{\mathbf{F}} \overline{\mathbf{F}}_p)$ is faithful on reduced $\mathbf{F}$-algebras of finite type (with $\mathbf{F}$-linear morphisms) we conclude that the diagram

$$\begin{tikzcd} & B^{\mathbf{G}} \arrow[dl] \arrow[d] \\ \mathbf{F}[T_p, f(T_p)^{-1}] \arrow[r, "\mathrm{F}_U"] & S \end{tikzcd}$$

commutes, as claimed.

The following result is a mild reformulation of results from Section 2.7.

Proposition 5.4.21. *For any choice of $\sigma_1, \sigma_2 \in \{\sigma, \sigma^{\mathrm{co}}\}$, and for each $i \geq 0$:*

(1) the $\mathbf{F}[T_p, f(T_p)^{-1}]$-module

$$\mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}(j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2)$$

is finitely generated, where the $\mathbf{F}[T_p, f(T_p)^{-1}]$-action is defined through the action of $\mathcal{H}_G(\sigma_2) = \mathbf{F}[T_p]$ on the second factor.

(2) if $\overline{\theta}$ corresponds to a point of U, then the map

$$\widehat{i}^*_{\overline{\theta}} : \mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}(j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, j^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2) \to \mathrm{Ext}^i_{\mathrm{Pro}\, D^b(\mathcal{A}^{\mathrm{fp}}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, \widehat{i}^*_{\overline{\theta}} c\text{-}\mathrm{Ind}_{KZ}^G \sigma_2)$$

exhibits the target as the completion of the source at the principal ideal of $\mathbf{F}[T_p, f(T_p)^{-1}]$ generated by $f_{\overline{\theta}}(T_p)$.

Proof. Note that

$$\mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1, j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2) \xrightarrow[j_*]{\sim} \mathrm{Ext}^i_{D(\mathcal{A})}(j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1, j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2)$$

(because j_* is fully faithful on the level of derived categories), and that $\mathrm{Ext}^i_{\mathcal{A}} = \mathrm{Ext}^i_{D(\mathcal{A})}$ by definition. Part (1) is then part of Proposition 2.7.9, while part (2) is Proposition 2.7.23. ☐

We can now deduce the following key statement.

Proposition 5.4.22. *For any choice of $\sigma_1, \sigma_2 \in \{\sigma, \sigma^{\mathrm{co}}\}$, and any $i \geq 0$, the functor F_U induces an isomorphism*

$$\mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}\big(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1, j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2\big) \xrightarrow{\sim} \mathrm{Ext}^i_{D^b_{\mathrm{coh}}(\mathcal{U})}\big(\mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1), \mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2)\big).$$

Proof. The case $i = 0$ and $\sigma_1 = \sigma_2$ is immediate from Proposition 5.4.12 (bearing in mind the symmetry between σ and σ^{co}), which shows that

$$(5.4.23)\quad \mathbf{F}[T_p, f(T_p)^{-1}] \overset{(5.4.14)}{=} \mathrm{End}_{\mathcal{A}_U^{\mathrm{fp}}}(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2) \xrightarrow{\sim} \mathrm{End}_{\mathrm{Coh}(\mathcal{U})}\big(\mathrm{F}_U(j^*(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2)\big).$$

By Proposition 5.4.21 and Corollary B.4.22, it follows that for each $i \geq 0$ and pair σ_1, σ_2, both

$$\mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}\big(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1, j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2\big)$$

and

$$\mathrm{Ext}^i_{D^b_{\mathrm{coh}}(\mathcal{U})}\big(\mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1), \mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2)\big)$$

are finitely generated over the finite type $\mathbf{F}$-algebra $\mathbf{F}[T_p, f(T_p)^{-1}]$ (acting through the second factor).

Since (5.4.23) is induced by F_U, the morphism

$$\mathrm{F}_U : \mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}\big(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1, j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2\big) \to \mathrm{Ext}^i_{D^b_{\mathrm{coh}}(\mathcal{U})}\big(\mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1), \mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2)\big)$$

is $\mathbf{F}[T_p, f(T_p)^{-1}]$-linear, and so (since we have now seen that it is an $\mathbf{F}[T_p, f(T_p)^{-1}]$-linear morphism between finitely generated $\mathbf{F}[T_p, f(T_p)^{-1}]$-modules) to show that it is an isomorphism, it suffices to prove that it becomes one after completing at each of the principal ideals $f_{\overline{\theta}}(T_p)$ of $\mathbf{F}[T_p, f(T_p)^{-1}]$. By (5.3.26) there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^i_{D^b_{\mathrm{fp}}(\mathcal{A}_U)}(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1, j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2) \arrow[r, "\mathrm{F}_U"] \arrow[d, "\widehat{i}^*_{\overline{\theta}}"] & \mathrm{Ext}^i_{D^b_{\mathrm{coh}}(\mathcal{U})}\big(\mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1), \mathrm{F}_U(j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2)\big) \arrow[d, "\widehat{i}'^*_{\overline{\theta}}"] \\
\mathrm{Ext}^i_{\mathrm{Pro}\,D^b_{\mathrm{fp}}(\mathcal{A}_{\overline{\theta}})}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1, \widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2) \arrow[r, "\mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}"] & \mathrm{Ext}^i_{\mathrm{Pro}\,D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})}\big(\mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_1), \mathrm{Pro}\,\mathrm{F}_{\overline{\theta}}(\widehat{i}^*_{\overline{\theta}}c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma_2)\big)
\end{tikzcd}$$

Proposition 5.4.21 shows that the left vertical arrow is a completion at the principal ideal generated by $f_{\overline{\theta}}$. Corollary B.4.23 shows the same is true of the right vertical arrow. It only remains to show that the lower horizontal arrow is an isomorphism, and this is immediate from Theorem 5.2.24. ☐

Proposition 5.4.24. *For each companion pair $\{\sigma, \sigma^{\mathrm{co}}\}$, the functor*

$$\mathrm{F}_{U(\sigma|\sigma^{\mathrm{co}})} : D^b_{\mathrm{fp}}(\mathcal{A}_{U(\sigma|\sigma^{\mathrm{co}})}) \to D^b_{\mathrm{coh}}\big(\mathcal{U}(\sigma|\sigma^{\mathrm{co}})\big)$$

is fully faithful.

Proof. By Lemma A.4.6, it suffices to show that

$$\mathrm{F}_{U(\sigma|\sigma^{\mathrm{co}})} : D(\mathcal{A}_{U(\sigma|\sigma^{\mathrm{co}})}) \to \operatorname{Ind} D^b_{\mathrm{coh}}\big(\mathcal{U}(\sigma|\sigma^{\mathrm{co}})\big)$$

is fully faithful. Using Proposition 5.4.22, this is immediate from an application of the implication "(4) $\implies$ (1)" of Proposition A.2.25, using the compact generators $j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma$, $j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma^{\mathrm{co}}$ of $D(\mathcal{A}_{U(\sigma|\sigma^{\mathrm{co}})})$. □

Finally, we deduce the main results of this section.

Theorem 5.4.25.

(1) The functor $\mathrm{F}_{U_{\mathrm{good}}} : D^b_{\mathrm{fp}}(\mathcal{A}_{U_{\mathrm{good}}}) \to D^b_{\mathrm{coh}}(\mathcal{U}_{\mathrm{good}})$ *is fully faithful.*

(2) The natural transformation

$$\mathrm{F}j_{U_{\mathrm{good}},*} \to j'_{\mathcal{U}_{\mathrm{good}},*}\mathrm{F}_{U_{\mathrm{good}}}$$

of functors from $D^b_{\mathrm{fp}}(\mathcal{A}_{U_{\mathrm{good}}})$ *to* $\operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$ *is a natural isomorphism.*

Proof. Recalling the decomposition $U_{\mathrm{good}} = \coprod U(\sigma|\sigma^{\mathrm{co}})$, this is immediate from Propositions 5.4.24 and 5.4.6. □

5.5. **The main theorem.** We are now in a position to deduce our main theorem, showing that if $p \geq 5$, then there is a fully faithful functor $D(\mathcal{A}) = \operatorname{Ind} D^b_{\mathrm{fp}}(\mathcal{A}) \to \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$.

Theorem 5.5.1. *The functors*

$$\mathrm{F} : D^b_{\mathrm{fp}}(\mathcal{A}) \to D^b_{\mathrm{coh}}(\mathcal{X}) \tag{5.5.2}$$

and

$$\mathrm{F} : D(\mathcal{A}) = \operatorname{Ind} D^b_{\mathrm{fp}}(\mathcal{A}) \to \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X}) \tag{5.5.3}$$

are fully faithful.

Proof. We put ourselves in the situation of Section A.5, taking the category $\mathcal{A}^c$ there to be our $D^b_{\mathrm{fp}}(\mathcal{A})$; so (in a somewhat unfortunate clash of notation) it follows from Corollary 2.6.7 that $\mathcal{A}$ there is our $D(\mathcal{A})$. We take $(\mathcal{A}')^c$ there to be our $D^b_{\mathrm{coh}}(\mathcal{X})$. We let $\mathcal{A}^c_Z$ be $D^b_{\mathrm{fp}}(\mathcal{A}_{Y_{\mathrm{bad}}})$, and $(\mathcal{A}'_Z)^c$ be $D^b_{\mathrm{coh},Y_{\mathrm{bad}}}(\mathcal{X}) = D^b_{\mathrm{coh}}(\mathcal{X}_{Y_{\mathrm{bad}}})$.

The theorem will be an immediate consequence of Proposition A.5.3, once we check that the functor (5.5.2) satisfies the hypotheses of that result. Note firstly that by Corollary 2.6.7 and (B.2.12), we have equivalences

$$D^b_{\mathrm{fp}}(\mathcal{A})/D^b_{\mathrm{fp}}(\mathcal{A}_{Y_{\mathrm{bad}}}) \to D^b_{\mathrm{fp}}(\mathcal{A}_{U_{\mathrm{good}}})$$

and

$$D^b_{\mathrm{coh}}(\mathcal{X})/D^b_{\mathrm{coh}}(\mathcal{X}_{Y_{\mathrm{bad}}}) \to D^b_{\mathrm{coh}}(\mathcal{U}_{\mathrm{good}}).$$

Then Hypothesis (1) is the claim that the functors

$$\mathrm{F}_{U_{\mathrm{good}}} : D^b_{\mathrm{fp}}(\mathcal{A}_{U_{\mathrm{good}}}) \to D^b_{\mathrm{coh}}(\mathcal{U}_{\mathrm{good}})$$

and

$$\mathrm{F}_{Y_{\mathrm{bad}}} : D^b_{\mathrm{fp}}(\mathcal{A}_{Y_{\mathrm{bad}}}) \to D^b_{\mathrm{coh}}(\mathcal{X}_{Y_{\mathrm{bad}}})$$

are fully faithful, which is Theorem 5.4.25 (1) and Corollary 5.2.30. Hypothesis (2) is precisely Theorem 5.4.25 (2), and finally Hypothesis (3) is Proposition 5.3.6 (with $Y = Y_{\mathrm{bad}}$). □

Appendix A. Category theory

We begin by briefly collecting some standard results in category theory. In addition to the material recalled here, we refer the reader to [DEG23, App. A.3] for some recollections on Ind and Pro categories.

A.1. **1-categories.**

As we discuss in more detail at the beginning of Section A.2, we typically fix a Grothendieck universe, and sets are called *small* if they belong to this fixed universe.

A.1.1. *Subcategories of abelian categories.* Let $\mathcal{A}$ be an abelian category. By definition, a *Serre subcategory* of $\mathcal{A}$ is a full subcategory $\mathcal{B}$ of $\mathcal{A}$ such that, for any exact sequence $X \to Y \to Z$ for which X and Z are objects of $\mathcal{B}$, we also have that Y is an object of $\mathcal{B}$; equivalently, $\mathcal{B}$ is closed under passing to subquotients and extensions in $\mathcal{A}$ (cf. [Stacks, Tag 02MP]). A *localizing subcategory* of $\mathcal{A}$ is a Serre subcategory $\mathcal{B}$ of $\mathcal{A}$ for which the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ admits a right adjoint, which is then necessarily fully faithful, by [Gab62, Prop. III.2.3(a)].

Following [Stacks, Tag 02MN], we define a *weak Serre subcategory* of $\mathcal{A}$ to be a full subcategory $\mathcal{B}$ of $\mathcal{A}$ with the additional property that, whenever the four outermost terms of a five term exact sequence in $\mathcal{A}$ are objects of $\mathcal{B}$, so is the middle term. Any weak Serre subcategory $\mathcal{B}$ (and thus any Serre subcategory) of $\mathcal{A}$ is an *exact abelian subcategory* of $\mathcal{A}$; that is, it is a full subcategory of $\mathcal{A}$ which is an abelian category in its own right, and for which the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is exact. (Indeed, a weak Serre subcategory of $\mathcal{A}$ is precisely an exact abelian subcategory of $\mathcal{A}$ which is also closed under the formation of extensions in $\mathcal{A}$; c.f. [Stacks, Tag 0754].)

A.1.2. *Grothendieck categories.* Recall that a Grothendieck category $\mathcal{A}$ is an abelian category which is cocomplete, admits a set of generators, and for which the formation of filtered colimits in $\mathcal{A}$ is exact. It then follows that $\mathcal{A}$ is also complete [KS06, Prop. 8.3.27].

We say that an object X of a Grothendieck category $\mathcal{A}$ is compact if $\mathrm{Hom}_{\mathcal{A}}(X, -)$ commutes with filtered colimits, and we write $\mathcal{A}^c$ for the full subcategory of compact objects. We say that the category $\mathcal{A}$ is compactly generated if it admits a set of compact generators, in which case the natural functor $\mathrm{Ind}(\mathcal{A}^c) \to \mathcal{A}$ is an equivalence. (In the literature, compact objects are often called "finitely presented", and a compactly generated category is "locally finitely presented".)

Recall that an abelian category is called *locally Noetherian* if it is a Grothendieck category, and furthermore admits a set of Noetherian generators. A locally Noetherian abelian category is compactly generated, and its compact objects are precisely the Noetherian objects (see e.g. [DEG23, Prop. A.1.1]).

An abelian category $\mathcal{A}$ is *locally coherent* if it is a compactly generated Grothendieck category and if furthermore $\mathcal{A}^c$ is abelian, in which case $\mathcal{A}^c$ is a weak Serre subcategory, hence an exact abelian subcategory, of $\mathcal{A}$ (for example by Lemma A.1.3 below). Any locally Noetherian category is locally coherent.

A Serre subcategory $\mathcal{B}$ of a Grothendieck category $\mathcal{A}$ is localizing if and only if it is closed under arbitrary direct sums in $\mathcal{A}$. Suppose furthermore that $\mathcal{A}$ is locally Noetherian; then $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are also locally Noetherian, and the right adjoint $\mathcal{A}/\mathcal{B} \to \mathcal{A}$ preserves filtered colimits. Furthermore the compact objects of $\mathcal{B}$ are those objects of $\mathcal{B}$ which are compact in $\mathcal{A}$, and the compact objects of $\mathcal{A}/\mathcal{B}$ are the images of the compact objects in $\mathcal{A}$, so that the sequence of functors

$$\mathcal{B} \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

is the Ind-extension of the induced sequence

$$\mathcal{B}^c \to \mathcal{A}^c \to (\mathcal{A}/\mathcal{B})^c = \mathcal{A}^c/\mathcal{B}^c.$$

(See for example [DEG23, Lem. A.2.7] for these facts.)

Lemma A.1.3. *If $\mathcal{A}$ is a locally coherent abelian category, then $\mathcal{A}^c$ is a weak Serre subcategory of $\mathcal{A}$. If $\mathcal{A}$ is furthermore locally Noetherian, then $\mathcal{A}^c$ is a Serre subcategory of $\mathcal{A}$.*

Proof. For the first statement, we have to check that $\mathcal{A}^c$ is an exact abelian subcategory of $\mathcal{A}$ which is closed under extensions. This is consequence of the equivalence of (1) and (2) in [Her97, Thm. 1.6], together with [Her97, Prop. 1.5].

For the second statement, we have to check that $\mathcal{A}^c$ is closed under subquotients and extensions, which is clear, since in this case (i.e. when $\mathcal{A}$ is locally Noetherian) $\mathcal{A}^c$ coincides with the Noetherian objects in $\mathcal{A}$. ☐

A.1.4. *Ind and Pro categories.* If $\mathcal{C}$ is a small category, then, as usual, we write $\mathrm{Ind}(\mathcal{C})$ for the category of small filtered diagrams of objects of $\mathcal{C}$. We typically consider directed systems $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$, indexed by a directed set I, and write $\mathrm{colim}_{i\in I} X_i$ to denote the associated object of $\mathrm{Ind}(\mathcal{C})$. (Of course, one can equally well consider systems of objects indexed by a filtered category, and we will do so on occasion, using the same notation.) Regarding objects of $\mathcal{C}$ as constant directed systems, we obtain a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$. The object $\mathrm{colim}_{i\in I} X_i$ is then the colimit of the X_i in $\mathrm{Ind}\,\mathcal{C}$.

Recall that morphisms in $\mathrm{Ind}\,\mathcal{C}$ are computed via the formula

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_{i\in I} X_i, \mathrm{colim}_{j\in J} Y_j) = \lim_{i\in I} \mathrm{colim}_{j\in J} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

(This simply encodes the fact, already noted, that the natural functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is fully faithful, together with the additional fact that the objects of $\mathcal{C}$ are compact in $\mathrm{Ind}\,\mathcal{C}$.)

We define $\mathrm{Pro}(\mathcal{C})$ in a dual fashion, so that objects of $\mathrm{Pro}(\mathcal{C})$ can be written as cofiltered limits $\lim_{i\in I} X_i$ of objects of $\mathcal{C}$. We will usually apply these constructions when $\mathcal{C}$ is a small abelian category, in which case $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ are also abelian (but no longer small).

We have the following standard lemmas.

Lemma A.1.5. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small abelian categories, and let $f : \mathcal{C} \to \mathcal{C}'$ be an additive functor. Then:*

(1) f is left exact (resp. right exact, resp. exact) if and only if its cofiltered limit-preserving extension $\mathrm{Pro}(f) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}')$ is left exact (resp. right exact, resp. exact).

(2) If $\mathcal{C}'$ has cofiltered limits, then f is left exact if and only if its cofiltered limit-preserving extension $\mathrm{Pro}(f) : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}'$ is left exact;

(3) If $\mathcal{C}'$ has exact cofiltered limits, then f is right exact if and only if $\mathrm{Pro}(f) : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}'$ is right exact.

Proof. By for example [KS06, Prop. 8.6.6(a)] any exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Pro}(\mathcal{C})$ can be written as a cofiltered limit of exact sequences in $\mathcal{C}$. By [KS06, Theorem 8.6.5(ii)], the functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ is fully faithful and exact.

Part (1) follows, and then part (2), resp. part (3), follows from part (1) and the fact that the inverse limit functor $\mathrm{Pro}(\mathcal{C}') \to \mathcal{C}'$ is left exact, resp. exact under the assumptions of part (3). $\square$

Lemma A.1.6. *Suppose that $g : \mathcal{C}' \to \mathcal{C}$ is an exact functor between small abelian categories, and continue to write $g : \mathrm{Pro}\,\mathcal{C}' \to \mathrm{Pro}\,\mathcal{C}$ for its* Pro*-extension. Then:*

(1) $g : \mathrm{Pro}\,\mathcal{C}' \to \mathrm{Pro}\,\mathcal{C}$ admits a left adjoint $f : \mathrm{Pro}\,\mathcal{C} \to \mathrm{Pro}\,\mathcal{C}'$, which is right exact and cofiltered limit-preserving.

(2) f is exact if and only if the restriction $f|_{\mathcal{C}} : \mathcal{C} \to \mathrm{Pro}\,\mathcal{C}'$ is exact.

Proof. By assumption, $g : \mathrm{Pro}\,\mathcal{C}' \to \mathrm{Pro}\,\mathcal{C}$ is exact and cofiltered limit-preserving, so it is limit-preserving, and thus admits a left adjoint $f : \mathrm{Pro}\,\mathcal{C} \to \mathrm{Pro}\,\mathcal{C}'$ by the adjoint functor theorem. Being a left adjoint, f is right exact. We now prove that it preserves cofiltered limits. Letting $X = \lim_{i \in I} X_i$ be a cofiltered limit of objects of $\mathrm{Pro}\,\mathcal{C}$, and Y be an object of $\mathcal{C}'$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}'}(f(\lim_i X_i), Y) &= \mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(\lim_i X_i, g(Y)) = \mathrm{colim}_i\, \mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(X_i, g(Y)) \\ &= \mathrm{colim}_i\, \mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}'}(f(X_i), Y) = \mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}'}(\lim_i f(X_i), Y),\end{aligned}$$

where the second equality is because $g(Y) \in \mathcal{C}$. Since every object of $\mathrm{Pro}\,\mathcal{C}'$ is a limit of objects of $\mathcal{C}'$, this equality remains true for arbitrary objects $Y \in \mathrm{Pro}\,\mathcal{C}'$. Hence $f(\lim_i X_i) = \lim_i f(X_i)$, as required. The second part follows from parts (2) and (3) of Lemma A.1.5, since $\mathrm{Pro}\,\mathcal{C}'$ has exact cofiltered limits. $\square$

Remark A.1.7. In the setting of Lemma A.1.6, we may form the Ind-extensions of f and g, which are a pair of adjoint functors $\mathrm{Ind}\, f : \mathrm{Ind}\,\mathrm{Pro}\,\mathcal{C} \to \mathrm{Ind}\,\mathrm{Pro}\,\mathcal{C}'$, $\mathrm{Ind}\, g : \mathrm{Ind}\,\mathrm{Pro}\,\mathcal{C}' \to \mathrm{Ind}\,\mathrm{Pro}\,\mathcal{C}$ (as can be checked directly from the definition of Hom-spaces in Ind-categories). Since Ind-completion preserves exactness, we see that $\mathrm{Ind}\, g$ is always exact, and if $f|_{\mathcal{C}} : \mathcal{C} \to \mathrm{Pro}\,\mathcal{C}'$ is exact, then so is $\mathrm{Ind}\, f$.

We now recall a standard result about projective objects of $\mathrm{Pro}\,\mathcal{C}$. (The dual statement then gives a result about injective objects in $\mathrm{Ind}\,\mathcal{C}$.) We begin with the following technical lemma, which is Exercise 6.11 in [KS06].

Lemma A.1.8. *Let $\mathcal{C}'$ be a full subcategory of a small category $\mathcal{C}$. Then:*

(1) if $A = \mathrm{colim}_i X_i \in \mathrm{Ind}\,\mathcal{C}$ is such that any $X \to A$ with $X \in \mathcal{C}$ factors through some $Y \to A$ with $Y \in \mathcal{C}'$, then $A \in \mathrm{Ind}\,\mathcal{C}'$.

(2) If $A = \lim_i X_i \in \mathrm{Pro}\,\mathcal{C}$ is such that any $A \to X$ with $X \in \mathcal{C}$ factors through some $A \to Y$ with $Y \in \mathcal{C}'$, then $A \in \mathrm{Pro}\,\mathcal{C}'$.

Proof. The second statement follows from the first by passage to opposite categories, so it suffices to prove the first. To this end, write $\mathcal{D}$ to denote the category of morphisms (in $\mathrm{Ind}\,\mathcal{C}$) $Y \to A$, with Y an object of $\mathcal{C}'$. It follows from the hypothesis (by a very similar argument to that in the last paragraph of this proof) that the category $\mathcal{D}$ is filtered. Accordingly, the functor $\alpha : \mathcal{D} \to \mathcal{C}'$ mapping the morphism $Y \to A$ to Y gives a filtered diagram in $\mathcal{C}'$, so it has colimit $\mathrm{colim}_{\mathcal{D}}\, Y$ in $\mathrm{Ind}\,\mathcal{C}'$.

The morphisms $Y \to A$ that define the objects of $\mathcal{D}$ give rise to a tautological morphism $\mathrm{colim}_{\mathcal{D}}\, Y \to A$, which we will show is an isomorphism. By Yoneda's lemma, it suffices to show that for any $X \in \mathcal{C}$, the induced morphism

$$\mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(X, \mathrm{colim}_{\mathcal{D}}\, Y) \to \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(X, A) \tag{A.1.9}$$

is an isomorphism.

Since $X \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(X, \mathrm{colim}_{\mathcal{D}}\, Y) = \mathrm{colim}_{\mathcal{D}}\, \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

and the surjectivity of (A.1.9) follows from the hypothesis on A. For the injectivity, suppose that $X \rightrightarrows \mathrm{colim}_{\mathcal{D}}\, Y$ are two morphisms that induce the same morphism to A. These morphisms then factor through some common Y, and we have that the two composites $X \rightrightarrows Y \to A$ coincide. Now the morphism $Y \to A$ factors through some X_i, and if we choose i large enough, the composites $X \rightrightarrows Y \to X_i$ will coincide. Finally, the map $X_i \to A$ factors through some $Z \to A$ with Z an object of $\mathcal{C}'$, by hypothesis. So our morphisms $X \rightrightarrows \mathrm{colim}_{\mathcal{D}}\, Y$ factor through the tautological map $Z \to \mathrm{colim}_{\mathcal{D}}\, Y$, via $X \rightrightarrows Y \to X_i \to Z$, and already coincide as maps to Z. Hence they coincide, and we've proved injectivity. $\square$

Lemma A.1.10. *Let $\mathcal{C}$ be a small abelian category. Then $\mathrm{Pro}\,\mathcal{C}$ has enough projectives. If $\mathcal{C}$ has enough projectives, then every projective object in $\mathrm{Pro}\,\mathcal{C}$ is of the form $\lim_i P_i$ with the P_i projective objects of $\mathcal{C}$.*

Proof. The first claim is dual to [KS06, Corollary 9.6.5]. We now prove the second claim. The analogous result for injective objects of Ind-categories is immediate from [KS06, Prop. 15.2.3], and we follow the proof of that result. Let P be a projective object of $\mathrm{Pro}\,\mathcal{C}$, and let $P \to X$ be a morphism in $\mathrm{Pro}\,\mathcal{C}$, with X an object of $\mathcal{C}$. Since $\mathcal{C}$ has enough projectives, there exists a surjection $Q \to X$, with Q a projective object of $\mathcal{C}$. Since P is projective, the morphism $P \to X$ factors as $P \to Q \to X$. The result follows from Lemma A.1.8 (with $\mathcal{C}'$ there being the full subcategory of projective objects of $\mathcal{C}$). $\square$

We finish up this discussion of Pro-categories by recording a technical result related to countably indexed pro-objects.

Definition A.1.11. If $\mathcal{C}$ is a small category, we say that an object X of $\mathrm{Pro}\,\mathcal{C}$ is countably indexed if we can write $X = \lim_n X_n$, for some sequence of morphisms $\cdots X_{n+1} \to X_n \to \cdots \to X_0$ in $\mathcal{C}$.

Lemma A.1.12. *Let $\mathcal{C}$ be a small abelian category, and $0 \to A \to B \to C \to 0$ a short exact sequence in $\mathrm{Pro}\,\mathcal{C}$. If each of A and C is countably indexed, then the same is true of B. Furthermore, if $A = \lim_n A_n$ for some sequence of morphisms $\cdots \to A_{n+1} \to A_n \to \cdots \to A_0$ in $\mathcal{C}$, then we can write $0 \to A \to B \to C \to 0$ as a projective limit $0 \to A_n \to B_n \to C_n \to 0$ of short exact sequences in $\mathcal{C}$, with the same transition maps $A_{n+1} \to A_n$ as before.*

Proof. If we push out the given exact sequence along the various morphisms $A \to A_n$, we obtain exact sequences

$$0 \to A_n \to B_n \to C \to 0 \tag{A.1.13}$$

which are compatible as n varies. Since the formation of cofiltered limits is exact in $\mathrm{Pro}\,\mathcal{C}$, we find that the limit of these sequences recovers the original short exact sequence; in particular, $B \xrightarrow{\sim} \lim_n B_n$.

The class of B_n gives an element in $\mathrm{Ext}^1_{\mathrm{Pro}\,\mathcal{C}}(C, A_n)$. If we write $C = \lim_m C_m$, then [DEG23, Lems. A.3.13, A.3.14(2)] give an isomorphism

$$\mathrm{colim}_m\, \mathrm{Ext}^1_{\mathcal{C}}(C_m, A_n) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Pro}\,\mathcal{C}}(C, A_n).$$

In particular, for some m_n there is an extension

$$0 \to A_n \to B_{n,m_n} \to C_{m_n} \to 0 \tag{A.1.14}$$

from which (A.1.13) is obtained by pulling back along the morphism $C \to C_{m_n}$. If

$$0 \to A_n \to B_{n,m} \to C_m \to 0 \tag{A.1.15}$$

denotes the short exact sequence in $\mathcal{C}$, obtained by pulling back (A.1.15) along the transition morphism $C_m \to C_{m_n}$ (for $m \geq m_n$), then we find that (A.1.13) maps isomorphically to the limit of these short exact sequences, and in particular, that $B_n \xrightarrow{\sim} \lim_m B_{n,m}$. Consequently, we see that B itself is countably indexed.

Choosing an appropriate strictly increasing function $m = m(n)$, and considering the corresponding diagonal system of short exact sequences

$$0 \to A_n \to B_{n,m(n)} \to C_{m(n)} \to 0$$

we obtain an inverse system of short exact sequences in $\mathcal{C}$ whose limit over n coincides with the given short exact sequence $0 \to A \to B \to C \to 0$. (More precisely, suppose we have chosen $m(n)$ suitably. The composite of the morphism $B_{n+1} \to B_n$ (that arises via their construction as pushouts) with the morphism $B_n \to B_{n,m(n)}$ factors through some $B_{n+1,m(n+1)}$, inducing a corresponding morphism of short exact sequences (A.1.15) from level $\big(n+1, m(n+1)\big)$ to $\big(n, m(n)\big)$.) This concludes the proof. $\square$

A.1.16. *Socle and radical filtrations.* We briefly recall some facts about socle and radical filtrations in an abelian category $\mathcal{C}$. We say that an object $M \in \mathcal{C}$ is *simple* if its only subobjects (equivalently, quotients) are 0 and M. We say that M is *semisimple* if it is a direct sum of simple objects, and that M has *finite length* if it has a finite composition series, i.e. a finite filtration whose graded pieces are simple. The length of a composition series is then independent of the choice of filtration, and defined to be the length of M. We introduce the following notation.

Definition A.1.17. Let $\mathcal{C}$ be an abelian category. We write $\mathcal{C}^{\text{f.l.}}$ for the full subcategory of $\mathcal{C}$ of objects of finite length.

When $\mathcal{C} = \operatorname{Mod}(R)$ for a ring R, we will also write $\operatorname{Mod}^{\text{f.l.}}(R)$ for $\mathcal{C}^{\text{f.l.}}$. Assume now that $\mathcal{C}$ is complete and cocomplete (e.g. a Grothendieck category). If $M \in \mathcal{C}$, we let

$$\operatorname{rad}(M) \coloneqq \ker\left(\prod_{q: M \to N : N \text{ is simple}} q\right) = \bigcap_{q: M \to N : N \text{ is simple}} \ker(q)$$

and

$$\operatorname{soc}(M) \coloneqq \operatorname{im}\left(\coprod_{q: N \to M : N \text{ is simple}} q\right).$$

Then we define

$$\operatorname{rad}^0(M) \coloneqq M, \operatorname{rad}^{i+1}(M) \coloneqq \operatorname{rad}(\operatorname{rad}^i(M))$$

and

$$\operatorname{soc}_{-1}(M) \coloneqq 0, \operatorname{soc}_{i+1}(M) \coloneqq \text{preimage in } M \text{ of } \operatorname{soc}(M/\operatorname{soc}_i(M)).$$

Finally, we set

$$\operatorname{cosoc}(M) \coloneqq M/\operatorname{rad}(M).$$

If we write $\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ for the natural anti-equivalence, an induction on i then shows that

$$(M/\operatorname{rad}^i M)^\vee = \operatorname{soc}^i(M^\vee).$$

As the following lemmas show, these notions behave mostly as expected in Grothendieck categories: for example, since coproducts are then exact, we find that

$$\operatorname{soc}(M) = \sum_{q:N\to M:N \text{ is simple}} q(N), \tag{A.1.18}$$

hence $\operatorname{soc}(M)$ is the sum of all simple subobjects of M. Lemma A.1.19 (2) then shows that $\operatorname{soc}(M)$ is a semisimple subobject of M (indeed, the maximal semisimple subobject of M). However, $\operatorname{cosoc}(M)$ need not be a semisimple quotient of M: see Remark A.1.22 for a counterexample, which also shows that the next lemma need not hold in an arbitrary abelian category.

Lemma A.1.19. *Let $\mathcal{C}$ be an abelian category, and let M be an object of $\mathcal{C}$. Assume furthermore either that $\mathcal{C}$ is a Grothendieck category, or that M is of finite length.*

1. *If M is semisimple, then every subobject, resp. quotient, of M is semisimple, and every exact sequence $0 \to M' \to M \to M'' \to 0$ splits.*
2. *If $M = \sum_{i\in I} M_i$ for a set I of simple subobjects, then M is semisimple.*

Proof. Assume that M is semisimple, and consider an exact sequence as in the statement of (1). By assumption, $M = \bigoplus_{i\in I} M_i$ for a set I of simple objects of $\mathcal{C}$. Furthermore, either $\mathcal{C}$ is a Grothendieck category, or I is finite. In either case, there is a (possibly empty) subset $J \subseteq I$ of I maximal with respect to the property that

$$M' \cap \bigoplus_{i\in J} M_i = 0.$$

If I is finite, this is obvious. Otherwise $\mathcal{C}$ is a Grothendieck category by assumption, so the formation of filtered colimits is exact in $\mathcal{C}$. Thus, for any $J \subseteq I$, we have that

$$M' \cap \bigoplus_{i\in J} M_i = \varinjlim_{K\subset J \text{ finite}} \left(M' \cap \bigoplus_{i\in K} M_i \right), \tag{A.1.20}$$

and so the existence of a maximal J follows from Zorn's lemma.

Having chosen such a subset J, we define $N := M' + \bigoplus_{i\in J} M_i$. Note that this sum is direct. Now if $N \neq M$ then there exists $i \in I$ such that $M_i \not\subset N$, and so $N + M_i$ is a direct sum, which contradicts the maximality of J. We have thus proved that the exact sequence in (1) splits, and that M'' is semisimple.

As an aside, note that this already implies (2), since (momentarily adopting the notation of that statement) we have a surjection $\bigoplus M_i \to M$ with semisimple source.

Returning to the proof of (1), it remains to show that M' is semisimple. We first note that if $M' \neq 0$ then it contains a simple subobject. Indeed, if $M' \neq 0$, then (A.1.20) shows that there exists a finite $K \subset I$ such that $M' \cap \bigoplus_{i\in K} M_i \neq 0$. Now $M' \cap \bigoplus_{i\in K} M_i$ has finite length (since K is finite) and so it necessarily contains a simple subobject, which is then a simple subobject of M'.

We now prove that M' is semisimple. By (2) (which we've already proved) it suffices to show that M' is equal to the sum of its simple subobjects. To this end, then, let M'_0 be the sum of the simple subobjects of M'. By what we've already shown, we know that the inclusion $M'_0 \hookrightarrow M$ is split, and so the inclusion $M'_0 \hookrightarrow M'$ is also split (by the restriction to M' of a retraction of $M'_0 \hookrightarrow M$). Thus we can write $M' = M'_0 \oplus M'_1$ for some $M'_1 \subset M'$. By the construction of M'_0, we see that M'_1 contains no simple subobjects. By what we've already proved, we conclude that $M'_1 = 0$, so that $M'_0 = M'$, as required. □

Corollary A.1.21. *Let $\mathcal{C}$ be a complete and cocomplete abelian category, and let $M \in \mathcal{C}$.*

(1) *If $\mathcal{C}$ is a Grothendieck category, then $\operatorname{soc}_i M/\operatorname{soc}_{i-1} M$ is semisimple for all $i \geq 0$.*
(2) *If M has finite length, then $\operatorname{soc}_i M/\operatorname{soc}_{i-1} M$ and $\operatorname{rad}^i M/\operatorname{rad}^{i+1} M$ are semisimple for all $i \geq 0$.*

Proof. The statements about soc_i are immediate by Lemma A.1.19 (2), which shows that $\operatorname{soc} M$ is semisimple, together with the fact that $\operatorname{soc}_i M/\operatorname{soc}_{i-1} M \xrightarrow{\sim} \operatorname{soc}(M/\operatorname{soc}_{i-1} M)$ for all i, by definition. Assume therefore that M has finite length. Arguing as in the previous case, it suffices to prove that $M/\operatorname{rad} M$ is semisimple. By construction, $M/\operatorname{rad} M$ is a finite length subobject of a product $\prod_{i \in I} M_i$ of simple objects. Since

$$\prod_{i \in I} M_i = \varprojlim_{K \subset I \text{ finite}} \prod_{i \in K} M_i$$

and $M/\operatorname{rad} M$ is Artinian, there exists a finite $K \subset I$ such that

$$M/\operatorname{rad} M \to \prod_{i \in K} M_i = \bigoplus_{i \in K} M_i$$

is a monomorphism. Since K is finite, Lemma A.1.19 (1) now implies that $M/\operatorname{rad} M$ is semisimple, as desired. □

Remark A.1.22. Lemma A.1.19 need not hold true if $\mathcal{C}$ is not a Grothendieck category. For example, let $\mathcal{C} = \operatorname{Mod}(\mathbf{Z})^{\mathrm{op}}$ be the opposite of the category of abelian groups. The simple objects in $\mathcal{C}$ are the groups $\mathbf{Z}/q$ for q a prime number. Then $G := \prod_q \mathbf{Z}/q$ is a semisimple object in $\mathcal{C}$, but the subgroup generated by 1 in all coordinates is a quotient of G in $\mathcal{C}$ which is isomorphic to $\mathbf{Z}$, which is not a semisimple object of $\mathcal{C}$.

Similarly, now viewing G as an object of $\mathcal{C}^{\mathrm{op}}$ (i.e. of $\operatorname{Mod}(\mathbf{Z})$), we see that $\operatorname{rad}_{\operatorname{Mod}(\mathbf{Z})}(G) = 0$. However, G is not semisimple, since it contains non-torsion elements; hence the cosocle of an object need not be semisimple (even in a Grothendieck category).

Lemma A.1.23. *Let $\mathcal{C}$ be an abelian category, and let M, N be objects of $\mathcal{C}$.*

(1) *If $\mathcal{C}$ is a Grothendieck category, and N is a subobject of M, then $\operatorname{soc}_i N = N \cap \operatorname{soc}_i M$ for all $i \geq 0$.*
(2) *If $\mathcal{C}^{\mathrm{op}}$ is a Grothendieck category, and $q : M \to N$ is an epimorphism, then $\operatorname{rad}^i N = q(\operatorname{rad}^i M)$ for all i.*

Proof. We begin by proving part (1), by induction on i. For the base case $i = 0$, the inclusion $\operatorname{soc} N \subset N \cap \operatorname{soc} M$ is true by (A.1.18). On the other hand, the inclusion $N \cap \operatorname{soc} M \subset \operatorname{soc} N$ holds because $N \cap \operatorname{soc} M$ is semisimple, by Lemma A.1.19 (1), and so it is the sum of its simple subobjects; and by (A.1.18), these are also subobjects of $\operatorname{soc}(N)$. By inductive assumption, we now have an inclusion $N/\operatorname{soc}_i N \to M/\operatorname{soc}_i M$, and the case $i = 0$ implies that

$$\operatorname{soc}(N/\operatorname{soc}_i N) = (N/\operatorname{soc}_i N) \cap \operatorname{soc}(M/\operatorname{soc}_i M).$$

Hence the preimage in N of $\mathrm{soc}(N/\mathrm{soc}_i N)$, which is $\mathrm{soc}_{i+1} N$, coincides with the preimage in N of $\mathrm{soc}(M/\mathrm{soc}_i M)$, which is $N \cap \mathrm{soc}_{i+1} M$. This concludes the proof of part (1).

Part (2) now follows by duality, since it is equivalent to the statement that $q|_{\mathrm{rad}^i M}$ is an epimorphism onto $\mathrm{rad}^i N$, and we already know that $N^\vee \to M^\vee \to M^\vee/\mathrm{soc}_i M^\vee$ factors through a monomorphism $N^\vee/\mathrm{soc}_i N^\vee \to M^\vee/\mathrm{soc}_i M^\vee$. □

We conclude this subsection with the following well-known fact about socle and radical filtrations of objects of finite length.

Lemma A.1.24. *Let $\mathcal{C}$ be a complete and cocomplete abelian category, and let $M \in \mathcal{C}$ be an object of finite length. Then the socle and radical filtration of M have the same length (i.e. number of nonzero graded pieces), called the* Loewy length *of M.*

Proof. Let m be the length of the radical filtration, and n the length of the socle filtration. Then m is minimal such that $\mathrm{rad}^m(M) = 0$, and n is minimal such that $\mathrm{soc}_{n-1}(M) = M$. Since, by Corollary A.1.21, $\mathrm{soc}_i M/\mathrm{soc}_{i-1} M$ is semisimple for all i, we see by induction on i that $\mathrm{rad}^i(M) \subseteq \mathrm{soc}_{n-1-i}(M)$. This implies that $\mathrm{rad}^n(M) = 0$, and so $m \leq n$. Similarly, Corollary A.1.21 and induction on j implies that $\mathrm{rad}^{m-1-j}(M) \subseteq \mathrm{soc}_j(M)$ for all j. Hence $\mathrm{soc}_{m-1}(M) = M$, and so $n \leq m$. □

A.1.25. *Locally finite categories.* A *locally finite category* is a Grothendieck category admitting a set of generators of finite length.

Lemma A.1.26.

(1) *If $\mathcal{A}$ is a locally finite category, and $M \in \mathcal{A}$, then M is compact if and only if it is Noetherian, if and only if it has finite length.*
(2) *If $\mathcal{A}$ is a locally finite category, and $\mathcal{A}^{\mathrm{f.l.}}$ is the full subcategory of objects of finite length, then the inclusion $\mathcal{A}^{\mathrm{f.l.}} \to \mathcal{A}$ induces an equivalence $\mathrm{Ind}\,\mathcal{A}^{\mathrm{f.l.}} \to \mathcal{A}$.*
(3) *If $\mathcal{A}^{\mathrm{op}}$ is a locally finite category, and $\mathcal{A}^{\mathrm{f.l.}}$ is the full subcategory of $\mathcal{A}$ whose objects are of finite length, then the inclusion $\mathcal{A}^{\mathrm{f.l.}} \to \mathcal{A}$ induces an equivalence $\mathrm{Pro}\,\mathcal{A}^{\mathrm{f.l.}} \to \mathcal{A}$.*

Proof. For parts (1) and (2), observe that the locally finite category $\mathcal{A}$ is locally Noetherian, hence its compact objects coincide with its Noetherian objects by [DEG23, Proposition A.1.1], and (writing $\mathcal{A}^{\mathrm{Noeth}}$ for the subcategory of Noetherian objects) we have $\mathrm{Ind}\,\mathcal{A}^{\mathrm{Noeth}} \xrightarrow{\sim} \mathcal{A}$ by [KS06, Proposition 6.3.4]. So it suffices to prove that $\mathcal{A}^{\mathrm{Noeth}} = \mathcal{A}^{\mathrm{f.l.}}$. Since it is immediate that $\mathcal{A}^{\mathrm{f.l.}} \subseteq \mathcal{A}^{\mathrm{Noeth}}$, it suffices to prove that every compact object $X \in \mathcal{A}$ has finite length, which is immediate since X is a filtered colimit of subobjects of finite length. Part (3) is dual to part (2). □

If $\mathcal{A}$ is a locally finite category, then a *block* of $\mathcal{A}$ is an equivalence class of isomorphism classes of simple objects under the equivalence relation generated by

$$S_1 \sim S_2 \text{ if } \mathrm{Ext}^1_{\mathcal{A}}(S_1, S_2) \neq 0 \text{ or } \mathrm{Ext}^1_{\mathcal{A}}(S_2, S_1) \neq 0.$$

If $\mathfrak{B}$ is a block of $\mathcal{A}$, then we let $\mathcal{A}_{\mathfrak{B}}$ denote the full subcategory of $\mathcal{A}$ containing precisely those objects all of whose irreducible subquotients lie in $\mathfrak{B}$.

We let $\mathcal{A}^{\mathrm{f.l.}}_{\mathfrak{B}}$ denote the full subcategory of objects in $\mathcal{A}_{\mathfrak{B}}$ that are of finite length in $\mathcal{A}_{\mathfrak{B}}$, and note the following evident result.

Lemma A.1.27. *For any block $\mathfrak{B}$, the category $\mathcal{A}_{\mathfrak{B}}^{\mathrm{f.l.}}$ coincides with the subcategory of objects of $\mathcal{A}_{\mathfrak{B}}$ that are finite length, equivalently compact, equivalently Noetherian, in $\mathcal{A}$.*

Proof. Since $\mathcal{A}_{\mathfrak{B}}$ is a localizing subcategory of $\mathcal{A}$, an object of $\mathcal{A}_{\mathfrak{B}}$ is of finite length in $\mathcal{A}$ if and only it is of finite length when regarded as an object of $\mathcal{A}_{\mathfrak{B}}$. The present lemma then follows from Lemma A.1.26. □

We have the following general structure theorem for locally finite categories.

Proposition A.1.28. *Let $\mathcal{A}$ be a locally finite category. Then there is a canonical direct product decomposition*

$$\mathcal{A} \cong \prod_{\overline{\theta}} \mathcal{A}_{\mathfrak{B}}, \tag{A.1.29}$$

where the product is over all the blocks $\mathfrak{B}$ of $\mathcal{A}$. For each block $\mathfrak{B}$, there exists a pseudocompact topological ring $E_{\mathfrak{B}}$ and an equivalence between $\mathcal{A}_{\mathfrak{B}}^{\mathrm{op}}$ and the category of right-pseudocompact modules over $E_{\mathfrak{B}}$.

Proof. This is a consequence of the results of [Gab62, Section IV.2]. □

The rings $E_{\mathfrak{B}}$ appearing in Proposition A.1.28 are only well-defined up to their categories of pseudocompact modules — that is, up to some form of Morita equivalence. In our applications, we will in fact be able to choose the rings $E_{\mathfrak{B}}$ to be profinite, so that the categories of right-pseudocompact modules will simply be the categories $\mathrm{Mod}_c(E_{\mathfrak{B}}^{\mathrm{op}})$ of profinite right $E_{\mathfrak{B}}$-modules, as we now describe.

A.1.30. *Profinite modules over profinite rings.* Recall that if R is a topological ring whose underlying space is profinite, and which is equipped with an $\mathcal{O}$-algebra structure as an abstract ring, then R is in fact a profinite $\mathcal{O}$-algebra in the strongest possible sense, namely we may write $R \xrightarrow{\sim} \varprojlim_i R_i$, where the R_i are $\mathcal{O}$-algebras of finite cardinality, the transition maps are $\mathcal{O}$-algebra homomorphisms, and R is endowed with the inverse limit topology. This is a consequence of the equivalence of [RZ10, Prop. 5.1.2 (b), (e)]. We will refer to such R as *profinite topological $\mathcal{O}$-algebras* from now on. (Note that by *loc. cit.*, any compact (Hausdorff) ring is necessarily profinite, and so we could equally well speak of *compact topological $\mathcal{O}$-algebras*; but *profinite* seems clearer.) The examples we care about in the main body of the paper include the completed group ring $\mathcal{O}[\![\Gamma]\!]$ of a profinite group Γ, the Cayley–Hamilton algebras of Section 2.1.9, and the endomorphisms algebras of various profinite $\mathcal{O}[\![G]\!]_{\zeta}$-modules.

Any profinite topological $\mathcal{O}$-algebra R satisfies the requirements of [Gab62, § IV.3]; i.e. R is a *left- and right-pseudocompact* ring. We can thus consider the associated category of *left-pseudocompact R-modules*, as introduced there, and recalled in the next definition.

Definition A.1.31. We define $\mathrm{Mod}_c(R)$ to be the category whose objects are separated and complete topological left R-modules which admit a neighborhood basis at zero consisting of open R-submodules of finite R-colength. The morphisms are the continuous R-linear maps. (These are precisely the left-pseudocompact R-modules in the sense of [Gab62, Section IV.3].)

The category $\mathrm{Mod}_c(R)$ has the following standard properties. Note that, as a consequence of Lemma A.1.32 (3)(4), every object of $\mathrm{Mod}_c(R)$ is topologically isomorphic to an inverse limit of R-modules of finite cardinality, and so it has a profinite topology.

Lemma A.1.32. *Let R be a profinite topological $\mathcal{O}$-algebra.*

(1) The category $\mathrm{Mod}_c(R)$ is abelian, and the forgetful functor

$$\mathrm{Mod}_c(R) \to \mathrm{Mod}(R) \tag{A.1.33}$$

is faithful and exact.

(2) $\mathrm{Mod}_c(R)^{\mathrm{op}}$ is a locally finite category.

(3) There is an equivalence

$$\mathrm{Pro}\,\mathrm{Mod}_c(R)^{\mathrm{f.l.}} \xrightarrow{\sim} \mathrm{Mod}_c(R) \tag{A.1.34}$$

through the functor that sends a cofiltered diagram of objects of $\mathrm{Mod}_c(R)^{\mathrm{f.l.}}$ to its limit in $\mathrm{Mod}_c(R)$.

(4) The forgetful functor (A.1.33) induces a functor

$$\mathrm{Mod}_c(R)^{\mathrm{f.l.}} \to \mathrm{Mod}(R)^{\mathrm{f.l.}}. \tag{A.1.35}$$

The objects of $\mathrm{Mod}_c(R)^{\mathrm{f.l.}}$ carry the discrete topology, and they are finite sets. The functor (A.1.35) is fully faithful.

(5) Any finitely presented R-module has a unique topology making it an object of $\mathrm{Mod}_c(R)$, referred to as its canonical topology*. Passing to the canonical topology yields a fully faithful, exact embedding*

$$\mathrm{Mod}^{\mathrm{fp}}(R) \hookrightarrow \mathrm{Mod}_c(R). \tag{A.1.36}$$

(6) The Jacobson radical $\mathrm{rad}(R)$ of R is the intersection of all maximal closed left ideals of R, and so it is closed.

(7) If $\mathrm{rad}(R) = 0$, then every $M \in \mathrm{Mod}_c(R)$ is projective, and can be written as a direct product

$$M = \prod_{\mathfrak{m} \subset R} (R/\mathfrak{m})^{I_{\mathfrak{m}}}$$

over maximal closed left ideals $\mathfrak{m} \subset R$ (with each $I_{\mathfrak{m}}$ being an appropriately chosen indexing set).

(8) If R is Noetherian, then $\mathrm{rad}(R)^n$ is open for all $n \geq 0$, the canonical topology on any $M \in \mathrm{Mod}^{\mathrm{fp}}(R)$ is the $\mathrm{rad}(R)$-adic topology, and the forgetful functor (A.1.35) is an equivalence. The equivalence (A.1.34) may thus be rewritten as an equivalence

$$\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(R) \xrightarrow{\sim} \mathrm{Mod}_c(R). \tag{A.1.37}$$

Proof. Part (1) is proved in [Gab62, Section IV.3]. Part (2) is [Gab62, Section IV.3, Th. 3], and it implies part (3) by Lemma A.1.26.

We now prove part (4). The existence of (A.1.35) follows from [VV97, Lemma 3.4(1)]. Furthermore, any $M \in \mathrm{Mod}_c(R)^{\mathrm{f.l.}}$ is discrete: this is because M is Artinian in $\mathrm{Mod}_c(R)$, and so it has a minimal open submodule M', which must be unique (as can be seen by considering the intersection of two minimal open submodules); and since M is separated, necessarily $M' = 0$, which shows that M is discrete. Hence (A.1.35) is fully faithful. Finally, to see that M is a finite set, we can assume without loss of generality that M is simple, and then it suffices to note that there exists a continuous surjection $R \to M$, hence M is both compact and discrete. This concludes the proof of part (4).

We now prove part (5). To prove existence of the canonical topology, note that $\mathrm{End}_{\mathrm{Mod}_c(R)}(R) = \mathrm{End}_{\mathrm{Mod}(R)}(R) = R^{\mathrm{op}}$. By (for example) Proposition A.1.48, we thus obtain a functor $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}_c(R)$ whose composite with the forgetful functor (A.1.33) is the embedding $\mathrm{Mod}^{\mathrm{fp}}(R) \hookrightarrow \mathrm{Mod}(R)$. This functor is fully faithful, by the "automatic continuity" result in [VV97, Proposition 3.5]. The same result implies the uniqueness of the canonical topology. This concludes the proof of part (5).

Part 6 follows from [Gab62, Section IV, Proposition 12], or can be proved directly. We now turn to part 7, and so assume that $\mathrm{rad}(R) = 0$, so that $R \to \prod_{\mathfrak{m}} R/\mathfrak{m}$ is a monomorphism (the product ranging over closed maximal left ideals of R). Since $\mathrm{Mod}_c(R)^{\mathrm{op}}$ is locally finite (hence a Grothendieck category) the dual of Lemma A.1.19 shows that R is a product of simple objects of $\mathrm{Mod}_c(R)$. Now any $M \in \mathrm{Mod}_c(R)$ admits a surjection $\prod_{s\in S} R \to M$ for some index set S: to see this, it suffices to use part (4) to write $M = \lim_{i\in I} M_i$ as a limit of finite-cardinality objects of $\mathrm{Mod}_c(R)$, and then take the limit of the system of surjections

$$\prod_{m\in M_i} R = \bigoplus_{m\in M_i} R \to M_i.$$

This produces a surjection $\prod_{m\in M} R \to M$. Applying Lemma A.1.19 to this surjection, we conclude that M is a direct product of simple objects (because so is R), and a direct factor of $\prod_{m\in M} R$. Since $\prod_{s\in S} R$ is indeed projective in $\mathrm{Mod}_c(R)$ for any S, by e.g. [Bru66, Cor. 1.3], this concludes the proof of (7).

The first two statements of part (8) are [VV97, Corollary 3.14] (note that M is a Noetherian object of $\mathrm{Mod}(R)$, hence of $\mathrm{Mod}_c(R)$). Finally, to see that (A.1.35) is an equivalence, it suffices to note that, when R is Noetherian, the forgetful functor (A.1.33) actually induces an equivalence on the full subcategories of Noetherian objects, by [VV97, Proposition 3.21]. $\square$

There are the usual tensor product functors

$$- \otimes_R - : \mathrm{Mod}(R^{\mathrm{op}}) \times \mathrm{Mod}(R) \to \mathrm{Mod}(\mathcal{O})$$
$$-\widehat{\otimes}_R- : \mathrm{Mod}_c(R^{\mathrm{op}}) \times \mathrm{Mod}_c(R) \to \mathrm{Mod}_c(\mathcal{O}).$$

From the perspective of the equivalence (A.1.34), the completed tensor product $-\widehat{\otimes}_R-$ can be interpreted as the Pro-extension of the usual tensor product, when we restrict it to a functor $-\otimes_R - : \mathrm{Mod}_c(R^{\mathrm{op}})^{\mathrm{f.l.}} \times \mathrm{Mod}_c(R)^{\mathrm{f.l.}} \to \mathrm{Mod}^{\mathrm{f.l.}}(\mathcal{O})$. If M and N are objects of $\mathrm{Mod}_c(R^{\mathrm{op}})$ and $\mathrm{Mod}_c(R)$ respectively, then there is a natural map of $\mathcal{O}$-modules

$$M \otimes_R N \to M \widehat{\otimes}_R N, \tag{A.1.38}$$

where the source is the usual tensor product, formed without regard to the topologies on M and N, and the target is the completed tensor product, regarded as an $\mathcal{O}$-module by forgetting its topology. The image of (A.1.38) is dense in the defining topology on $M \widehat{\otimes}_R N$. Looking ahead, from the perspective of Lemma A.1.53, if R is furthermore Noetherian, the functor $M \widehat{\otimes}_R -$ is associated to the complete right R-module M in $\mathrm{Mod}_c(\mathcal{O})$, by Lemma A.1.59.

If R is a profinite $\mathcal{O}$-algebra, the equivalence (A.1.34) shows that $\mathrm{Mod}_c(R)$ is the Pro-category of an abelian category, and so it has enough projectives (by Lemma A.1.10; alternatively, this fact was established in the proof of Lemma A.1.32 (7)). We write $\widehat{\mathrm{Tor}}_i^R$ for the left derived functors of $-\widehat{\otimes}_R-$.

Definition A.1.39. We say that a module $M \in \mathrm{Mod}_c(R^{\mathrm{op}})$, respectively $M \in \mathrm{Mod}_c(R)$, is *topologically flat* if $M \widehat{\otimes}_R -$ (resp. $-\widehat{\otimes}_R M$) is an exact functor.

Lemma A.1.40. *Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $M \in \mathrm{Mod}_c(R^{\mathrm{op}})$. Then M is topologically flat if and only if M is a flat R-module (in the usual sense that $M \otimes_R (-)$ is exact on $\mathrm{Mod}(R)$).*

Proof. Since Definition A.1.39 is a specialization of Definition A.10.23, this lemma is a consequence of Lemma A.10.24. ☐

Since a right exact functor on an abelian category with enough projectives is exact if and only if its left derived functors vanish, we see that an object M of $\mathrm{Mod}_c(R^{\mathrm{op}})$ (resp. of $\mathrm{Mod}_c(R)$) is topologically flat if and only if $\widehat{\mathrm{Tor}}_i^R(M,-) = 0$ for all $i > 0$, respectively $\widehat{\mathrm{Tor}}_i^R(-,M) = 0$ for all $i > 0$. From the very construction of left derived functors, we see that projective objects of $\mathrm{Mod}_c(R^{\mathrm{op}})$ and $\mathrm{Mod}_c(R)$ are topologically flat; we will see in Lemma A.1.44 below that the converse also holds. The following lemma shows that the (derived) tensor product with finitely generated R-modules coincides with the (derived) completed tensor product.

Lemma A.1.41. *Let R be a Noetherian profinite $\mathcal{O}$-algebra. If M is an object of $\mathrm{Mod}_c(R^{\mathrm{op}})$ and N is an object of $\mathrm{Mod}^{\mathrm{fp}}(R)$, then the natural map* (A.1.38) *induces isomorphisms $\mathrm{Tor}_i^R(M,N) \to \widehat{\mathrm{Tor}}_i^R(M,N)$.*

Proof. The statement for $i = 0$ is immediate from the fact that the restriction of $M \widehat{\otimes}_R -$ to $\mathrm{Mod}^{\mathrm{fp}}(R)$ is naturally isomorphic to $M \otimes_R -$ (see Corollary A.1.54 for a generalization). In turn, this implies the statement for all i, since we can compute the left derived functors via a resolution of N by finite free R-modules. ☐

Remark A.1.42. In the context of Lemma A.1.41, we will often regard $M \otimes_R N$ as a compact R-module, by identifying it with $M \widehat{\otimes}_R N$. Note that the case $i = 0$ of Lemma A.1.41 is true without the Noetherian assumption on R, and in fact applies in the more general context of pseudocompact rings and modules (compare [Bru66, Lemma 2.1]).

Remark A.1.43. Let R be a profinite $\mathcal{O}$-algebra. Since $\mathrm{rad}(R)$ is closed in R (by Lemma A.1.32 (6)), it inherits a profinite topology making it a subobject of R in $\mathrm{Mod}_c(R)$. Thus we may consider the quotient $R/\mathrm{rad}(R)$, and the associated completed tensor product functor $(R/\mathrm{rad}(R)) \widehat{\otimes}_R -$ on $\mathrm{Mod}_c(R)$. Evidently, $R/\mathrm{rad}(R)$ is again a profinite $\mathcal{O}$-algebra, and this functor takes values in $\mathrm{Mod}_c\big(R/\mathrm{rad}(R)\big)$. Lemma A.1.32 (7) shows that if M is an object of $\mathrm{Mod}_c(R)$, then $(R/\mathrm{rad}(R)) \widehat{\otimes}_R M$ is a product of simple discrete R-modules. In fact, $(R/\mathrm{rad}(R)) \widehat{\otimes}_R M$ may be characterized as the maximal quotient of M in $\mathrm{Mod}_c(R)$ which is a product of simple discrete R-modules. To see this, note that if N is a product of simple discrete R-modules, and $M \to N$ is a surjection in $\mathrm{Mod}_c(R)$, then $M \to N$ factors through $M/\mathrm{rad}(R)M$. Since $N \in \mathrm{Mod}_c(R)$, it actually factors through the quotient of M by the closure of $\mathrm{rad}(R)M$ in M. Using the fact that (A.1.38) has dense image (applied to $\mathrm{rad}(R) \otimes_R M \to \mathrm{rad}(R) \widehat{\otimes}_R M$) we see that this quotient is $(R/\mathrm{rad}(R)) \widehat{\otimes}_R M$.

We next show that topological flatness is equivalent to being projective in $\mathrm{Mod}_c(R)$. (This is standard, see for example [PT21, Lemma 2.4], but we provide a proof for convenience.)

Lemma A.1.44. *Let R be a profinite $\mathcal{O}$-algebra, and let $M \in \mathrm{Mod}_c(R)$. Then M is topologically flat over R if and only if it is projective in $\mathrm{Mod}_c(R)$.*

Proof. If M is projective, then it is topologically flat, since $\widehat{\mathrm{Tor}}_i^R(-, M) = 0$ for all $i > 0$. We now prove the converse. By Remark A.1.43, we know that $(R/\operatorname{rad}(R)) \widehat{\otimes}_R M$ can be written as a direct product

$$(R/\operatorname{rad}(R)) \widehat{\otimes}_R M = \prod_{\mathfrak{m}} (R/\mathfrak{m})^{I_{\mathfrak{m}}}$$

over maximal closed left ideals $\mathfrak{m} \subset R$, for some appropriately chosen indexing sets $I_{\mathfrak{m}}$. Now choose a projective envelope $R_{\mathfrak{m}} \to R/\mathfrak{m}$ of $R/\mathfrak{m}$ in $\mathrm{Mod}_c(R)$ (whose existence is guaranteed by the fact that $\mathrm{Mod}_c(R)^{\mathrm{op}}$ is a Grothendieck category, which has injective envelopes, by [Gab62, Thm. II.6.2]). By definition, $R_{\mathfrak{m}}$ has a unique discrete simple quotient (namely $R/\mathfrak{m}$), and so Remark A.1.43 shows that $(R/\operatorname{rad}(R)) \widehat{\otimes}_R R_{\mathfrak{m}} \xrightarrow{\sim} R/\mathfrak{m}$. So we obtain a morphism

$$\prod_{\mathfrak{m}} R_{\mathfrak{m}}^{I_{\mathfrak{m}}} \to M \tag{A.1.45}$$

which induces an isomorphism after applying $R/\operatorname{rad} R \widehat{\otimes}_R (-)$. We now apply Nakayama's lemma for $\mathrm{Mod}_c(R)$, which asserts that if N is an object of $\mathrm{Mod}_c(R)$ for which $\big(R/\operatorname{rad}(R)\big) \widehat{\otimes}_R N = 0$, then $N = 0$. (See for example [VV97, Lemma 3.23].) Together with the assumption that M is topologically flat, this implies that (A.1.45) is an isomorphism, and so M is projective, as desired. $\square$

A.1.46. *Tensoring objects of abelian categories by modules.* In preparation for the statements of Morita theory, we now discuss a general formalism for tensor products in abelian categories.

Definition A.1.47. Let R be a ring, and let $\mathcal{A}$ be an abelian category. Then a *right R-module in $\mathcal{A}$* is a pair consisting of an object M of $\mathcal{A}$, and a ring homomorphism $R^{\mathrm{op}} \to \mathrm{End}_{\mathcal{A}}(M)$. The category of right R-modules in $\mathcal{A}$ is the category whose objects are right R-modules in $\mathcal{A}$, and whose morphisms are the morphisms of underlying of objects of $\mathcal{A}$ which are compatible with the homomorphisms from R^{op}.

Given a functor $F : \mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$, we see that $F(R)$ is naturally a right R-module in $\mathcal{A}$. We have the following standard result.

Proposition A.1.48. *Let R be a ring, and let $\mathcal{A}$ be an abelian category.*

(1) *The functor $F \mapsto F(R)$ is an equivalence of categories between the category of right exact functors $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$ and the category of right R-modules M in $\mathcal{A}$; we denote the functor F associated to M by $M \otimes_R -$.*
(2) *If $\mathcal{A}$ is cocomplete, then the functor $F \mapsto F(R)$ is an equivalence of categories between the category of colimit-preserving functors $\mathrm{Mod}(R) \to \mathcal{A}$ and the category of right R-modules M in $\mathcal{A}$; we denote the functor F associated to M by $M \otimes_R -$. Furthermore, the functor F is then left adjoint to the functor $\mathrm{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \mathrm{Mod}(R)$.*
(3) *If $\mathcal{A}$ is complete, then the functor $F \mapsto F(R)$ is an equivalence of categories between the category of right exact, cofiltered limit-preserving functors $\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$ and the category of right R-modules in $\mathcal{A}$; we denote the functor F associated to M by $M \widehat{\otimes}_R -$.*

(4) *The functor $F \mapsto F(R)$ is an equivalence of categories between the category of right exact, cofiltered limit-preserving functors* $\operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(R) \to \operatorname{Pro}\mathcal{A}$ *sending R to an object of $\mathcal{A}$, and the category of right R-modules in $\mathcal{A}$; we denote the functor F associated to M by $M \mathbin{\widehat{\otimes}}_R -$.*

Proof. If R is commutative, then (1) is [Stacks, Tag 0GNN], and the same proof works for general R. Part (2) then follows by passage to filtered colimits, see [Stacks, Tag 0GNQ] and [Stacks, Tag 0GNS].

Alternatively, (2) is the special case $k = \mathbf{Z}$ of [NS16, Thm. 3.1]. Part (1) can then be deduced by applying (2) with $\mathcal{A}$ replaced by $\operatorname{Ind}\mathcal{A}$, noting that because M is compact in $\operatorname{Ind}\mathcal{A}$ (being an object of $\mathcal{A}$), the functor $\operatorname{Hom}_{\operatorname{Ind}\mathcal{A}}(M, -) : \operatorname{Ind}\mathcal{A} \to \operatorname{Mod}(R)$ is filtered colimit-preserving, from which one deduces that its left adjoint F preserves compact objects. (Cf. Lemma A.2.32 below for the analogous statement in the context of functors between stable ∞-categories.)

Finally (3) follows from (1) by passing to cofiltered limits, and part (4) follows from parts (3) and (1). □

Remark A.1.49. The equivalences of categories in Proposition A.1.48 give the usual bifunctoriality of the tensor product, and even the trifunctoriality of the tensor-Hom adjunction (see for example [Mit65, Chapter VI, Thm. 3.1] for this last point). In addition, if S is a commutative ring, and R is an S-algebra, then there is an obvious S-linear variant of Proposition A.1.48; see [NS16, Cor. 2.3].

Remark A.1.50. Proposition A.1.48 also applies to left R-modules M in $\mathcal{A}$, which (by definition) are right R^{op}-modules. As a consequence, if M is a left R-module in $\mathcal{A}$, we will sometimes write $- \otimes_R M : \operatorname{Mod}^{\mathrm{fp}}(R^{\mathrm{op}}) \to \mathcal{A}$ for the right exact functor previously denoted $M \otimes_{R^{\mathrm{op}}} -$. If $\mathcal{A}$ is cocomplete, it extends to a functor $\operatorname{Mod}(R^{\mathrm{op}}) \to \mathcal{A}$ which is left adjoint to $\operatorname{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \operatorname{Mod}(R^{\mathrm{op}})$.

On the other hand, a left R-module M in $\mathcal{A}$ can also be regarded as a right R-module in $\mathcal{A}^{\mathrm{op}}$, and so it defines a (covariant) right exact functor $\operatorname{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}^{\mathrm{op}}$. We will write $\operatorname{Hom}_R(-, M) : \operatorname{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$ for the corresponding *contravariant* right exact functor. If $\mathcal{A}$ is complete, so that Proposition A.1.48 (2) applies to $\mathcal{A}^{\mathrm{op}}$, then the functor $\operatorname{Hom}_R(-, M)$ extends to a contravariant functor $\operatorname{Mod}(R) \to \mathcal{A}$ (sending colimits to limits) and there is a bifunctorial isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(X, \operatorname{Hom}_R(N, M)) \cong \operatorname{Hom}_R(N, \operatorname{Hom}_{\mathcal{A}}(X, M))$$

for all $X \in \mathcal{A}$ and all $N \in \operatorname{Mod}(R)$.

In applications, we will have a finitely generated two-sided ideal J of R, and a left R-module M in $\mathcal{A}$, and we will write $M[J] \coloneqq \operatorname{Hom}_R(R/J, M)$. Then $M[J]$ is naturally a subobject of M in the category of left R-modules in $\mathcal{A}$, and the R-action on $M[J]$ factors through R/J. The right exact functor $\otimes_R M[J]$ is therefore naturally isomorphic to the composition

$$\operatorname{Mod}^{\mathrm{fp}}(R^{\mathrm{op}}) \xrightarrow{\otimes_R R/J} \operatorname{Mod}^{\mathrm{fp}}((R/J)^{\mathrm{op}}) \xrightarrow{\otimes_{R/J} M[J]} \mathcal{A}.$$

Lemma A.1.51. *Let R be a ring, and let $\mathcal{A}$ be a complete abelian category with exact cofiltered limits. Let M_i be a cofiltered system of right R-modules in $\mathcal{A}$, and let $M \coloneqq \varprojlim_i M_i$. Then the natural map $(\varprojlim_i M_i \otimes_R -) \to \varprojlim_i (M_i \otimes_R -)$ is an isomorphism of functors* $\operatorname{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$.

Proof. Since both sides send R to $\varprojlim_i M_i$, it suffices (by Proposition A.1.48) to prove that $\varprojlim_i (M_i \otimes_R -)$ is a right-exact functor on $\mathrm{Mod}^{\mathrm{fp}}(R)$, which is a consequence of the exactness of cofiltered limits in $\mathcal{A}$. $\square$

We will also need the following variant of the above constructions, which produces a formalism of "completed tensor products" in complete abelian categories.

Definition A.1.52. Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be an abelian category. Then we say that a right R-module M in $\mathcal{A}$ is *complete* if the natural morphisms $M \to M \otimes_R \big(R/\operatorname{rad}(R)^n\big)$ give rise to an isomorphism $M \to \varprojlim_n M \otimes_R \big(R/\operatorname{rad}(R)^n\big)$ (i.e. this limit exists in $\mathcal{A}$, and is given by M).

Lemma A.1.53. *Let R be a Noetherian profinite topological $\mathcal{O}$-algebra, and let $\mathcal{A}$ be a complete abelian category. Then the assignment $F \mapsto F(R)$ is an equivalence between the category of right exact and cofiltered limit-preserving functors $\mathrm{Mod}_c(R) \to \mathcal{A}$, and the category of complete right R-modules M in $\mathcal{A}$; we denote the functor F associated to M by $M \widehat{\otimes}_R -$.*

Proof. Suppose that $F : \mathrm{Mod}_c(R) \to \mathcal{A}$ is right exact and cofiltered limit-preserving, and write $M := F(R)$. Then the restriction of F to $\mathrm{Mod}^{\mathrm{fp}}(R)$ is given by $M \otimes_R -$, and since by Lemma A.1.32 (8) we have $R \xrightarrow{\sim} \varprojlim_n \big(R/\operatorname{rad}(R)^n\big)$ in $\mathrm{Mod}_c(R)$, we see that M is necessarily a complete right R-module. This shows that $F \mapsto F(R)$ defines a functor between the categories in the statement of the lemma.

We now construct a quasi-inverse to $F \mapsto F(M)$. If $M \in \mathcal{A}$ is a complete right R-module, then we have the right exact and cofiltered limit-preserving functor $M \widehat{\otimes}_R - : \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$ of Proposition A.1.48 (3). Restricting this functor to $\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(R)$, and recalling (A.1.37), we obtain a right exact and cofiltered limit-preserving functor $\mathrm{Mod}_c(R) \xrightarrow{\sim} \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(R) \to \mathcal{A}$.

Given M, the functor $M \widehat{\otimes}_R -$ takes R to $\varprojlim_n M \otimes_R \big(R/\operatorname{rad}(R)^n\big)$, which coincides with M, since M is assumed to be complete. On the other hand, given F, the functors F and $F(M) \widehat{\otimes}_R - : \mathrm{Mod}_c(R) \to \mathcal{A}$ preserve cofiltered limits, and are isomorphic to $F(M) \otimes_R -$ after restriction to $\mathrm{Mod}^{\mathrm{f.l.}}(R)$; they are therefore naturally isomorphic. This concludes the proof that $F \mapsto F(M)$ is an equivalence, with quasi-inverse $M \mapsto M \widehat{\otimes}_R -$. $\square$

Corollary A.1.54. *Let R be a Noetherian profinite $\mathcal{O}$-algebra. Let $\mathcal{A}$ be a complete abelian category, and let $M \in \mathcal{A}$ be a complete right R-module. Then the restriction of $M \widehat{\otimes}_R - : \mathrm{Mod}_c(R) \to \mathcal{A}$ to $\mathrm{Mod}^{\mathrm{fp}}(R)$ is $M \otimes_R -$.*

Proof. This is immediate by construction (or by the defining properties of the tensor products). $\square$

Next, we record a compatibility between our tensor product functors, and right exact functors between abelian categories.

Lemma A.1.55.

(1) *If $\mathcal{A}, \mathcal{B}$ are abelian categories, R is a ring, $F : \mathcal{A} \to \mathcal{B}$ is right exact, and M is an R-module in $\mathcal{A}$, then there is a natural isomorphism of functors $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{B}$*

$$(F(M) \otimes_R -) \to F(M \otimes_R -). \tag{A.1.56}$$

(2) *If $\mathcal{A}, \mathcal{B}$ are complete abelian categories, R is a Noetherian profinite topological $\mathcal{O}$-algebra, $F : \mathcal{A} \to \mathcal{B}$ is right exact and cofiltered limit-preserving, and M is a complete R-module in $\mathcal{A}$, then there is a natural isomorphism of functors* $\mathrm{Mod}_c(R) \to \mathcal{B}$

$$(F(M) \widehat{\otimes}_R -) \to F(M \widehat{\otimes}_R -). \tag{A.1.57}$$

Proof. The existence of the natural morphism of functors (A.1.56) follows from adjunction (i.e. from the deduction of part (1) of Proposition A.1.48 from part (2)) Since the functor $F(M \otimes_R -)$ is right exact and takes R to $F(M)$, this morphism is an isomorphism by Proposition A.1.48 (1).

For part (2), since $\mathrm{Mod}_c(R) \xrightarrow{\sim} \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(R)$ (by Lemma A.1.32 (8)), and both sides of (A.1.57) preserve cofiltered limits, it suffices to construct the isomorphism (A.1.57) after restricting to $\mathrm{Mod}^{\mathrm{f.l.}}(R)$, which is a consequence of part (1) and Corollary A.1.54. □

Now let $\mathcal{A}$ be a complete abelian category, and let R be a Noetherian profinite $\mathcal{O}$-algebra. If $X \in \mathrm{Mod}_c(R)$, then Lemma A.1.53 shows that the completed tensor product with X gives rise to a functor

$$-\widehat{\otimes}_R X : (\text{complete right } R\text{-modules in } \mathcal{A}) \to \mathcal{A}.$$

More precisely, given a morphism $M_1 \to M_2$ of complete right R-modules in $\mathcal{A}$, the morphism $M_1 \widehat{\otimes}_R X \to M_2 \widehat{\otimes}_R X$ is the evaluation at X of the natural transformation $M_1 \widehat{\otimes}_R - \to M_2 \widehat{\otimes}_R -$ arising from Lemma A.1.53.

Lemma A.1.58. *Let $\mathcal{A}$ be a complete abelian category in which products are exact, and let R be a Noetherian profinite $\mathcal{O}$-algebra. Then for any object X of $\mathrm{Mod}_c(R)$, and for any sequence of morphisms $M_1 \to M_2 \to M_3 \to 0$ of complete right R-modules in $\mathcal{A}$ which is exact when viewed as a sequence in $\mathcal{A}$, the resulting sequence $M_1 \widehat{\otimes}_R X \to M_2 \widehat{\otimes}_R X \to M_3 \widehat{\otimes}_R X \to 0$ is exact; or, more succinctly, for any $X \in \mathrm{Mod}_c(R)$, the functor $-\widehat{\otimes}_R X$ (regarded as a functor from the category of complete right R-modules in $\mathcal{A}$ to $\mathcal{A}$) is right exact.*

Proof. Note firstly that the claim is immediate if $X = R$. For general X, we may choose a presentation $\prod_{i\in I} R \to \prod_{j \in J} R \to X \to 0$, so that for any right exact sequence $M_1 \to M_2 \to M_3 \to 0$ of R-modules, we have a commutative diagram

$$\begin{array}{ccccccc}
\prod_{i\in I} M_1 & \longrightarrow & \prod_{i\in I} M_2 & \longrightarrow & \prod_{i\in I} M_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod_{j\in J} M_1 & \longrightarrow & \prod_{j\in J} M_2 & \longrightarrow & \prod_{j\in J} M_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
M_1 \widehat{\otimes}_R X & \longrightarrow & M_2 \widehat{\otimes}_R X & \longrightarrow & M_3 \widehat{\otimes}_R X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

with all rows and columns exact, except possibly the last row, which is therefore also exact. □

Lemma A.1.59. *Let R be a Noetherian profinite $\mathcal{O}$-algebra. Let M be an object of $\mathrm{Mod}_c(R^{\mathrm{op}})$. Then M is a complete right R-module in $\mathrm{Mod}_c(\mathcal{O})$. The functor*

$M \widehat{\otimes}_R - : \mathrm{Mod}_c(R) \to \mathrm{Mod}_c(\mathcal{O})$ *obtained from Lemma* A.1.53 *coincides with the usual completed tensor product with* M.

Proof. As a consequence of Lemma A.1.32 (3)(4), the canonical topology on M is profinite, and so M is a right R-module in $\mathrm{Mod}_c(\mathcal{O})$. We now show that M is complete, i.e. that the natural map $M \to \varprojlim_n M \otimes_R R/\operatorname{rad}(R)^n$ is an isomorphism. This map is part of a natural transformation of functors $\mathrm{Mod}_c(R^{\mathrm{op}}) \to \mathrm{Mod}_c(\mathcal{O})$, which induces the identity after evaluating at $M = R$, by Lemma A.1.32 (8). So it suffices to prove that both functors preserve cofiltered limits, and to do so, it suffices to prove that $M \mapsto M \otimes_R R/\operatorname{rad}(R)^n$ preserves cofiltered limits. Since $R/\operatorname{rad}(R)^n$ is finitely presented, this is a consequence of Lemma A.1.51. This concludes the proof that M is complete. The last statement of the corollary now follows, because the functor $M \widehat{\otimes}_R -$ and the usual completed tensor product are right exact, cofiltered limit-preserving functors mapping R to M. ☐

Remark A.1.60. Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be a complete abelian category. If M is a right R-module in $\mathcal{A}$, we have associated to M a pro-extended functor $M \widehat{\otimes}_R - : \operatorname{Pro}\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$ (Proposition A.1.48 (3)). It is right exact, and cofiltered limit-preserving. Since $\mathrm{Mod}_c(R) = \operatorname{Pro}\mathrm{Mod}^{\mathrm{f.l.}}(R)$, there is an exact and cofiltered limit-preserving inclusion functor

$$g : \mathrm{Mod}_c(R) \to \operatorname{Pro}\mathrm{Mod}^{\mathrm{fp}}(R),$$

which sends R to $\lim_n R/\operatorname{rad}(R)^n$. Restricting $M \widehat{\otimes}_R -$ through g gives rise to a right exact, cofiltered-limit preserving functor $\mathrm{Mod}_c(R) \to \mathcal{A}$. It sends R to the right R-module $\varprojlim_n (M \otimes_R R/\operatorname{rad}(R)^n)$. If M is complete, we thus obtain the functor associated to M in Lemma A.1.53. This explains why we have denoted both functors by $M \widehat{\otimes}_R -$.

The above discussion can also be understood in terms of the left adjoint f to g, which is pro-extended from the inclusion $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}_c(R)$ described in Lemma A.1.32 (5): in fact, if $\lim_{i\in I} M_i$ is a cofiltered system of finitely presented R-modules, and $N = \varprojlim_{j\in J} N_j$ is a compact R-module written as a cofiltered limit of finite length R-modules, then

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}_c(R)}(\varprojlim_i M_i, N) &= \varprojlim_j \mathrm{Hom}_{\mathrm{Mod}_c(R)}(\varprojlim_i M_i, N_j) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathrm{Mod}_c(R)}(M_i, N_j) \\
&= \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathrm{Mod}^{\mathrm{fp}}(R)}(M_i, N_j) = \mathrm{Hom}_{\operatorname{Pro}\mathrm{Mod}^{\mathrm{fp}}(R)}(\lim_i M_i, \lim_j N_j) \\
&= \mathrm{Hom}_{\operatorname{Pro}\mathrm{Mod}^{\mathrm{fp}}(R)}(\lim_i M_i, g(N)).
\end{aligned}$$

In more detail, the second equality is because objects of $\mathrm{Mod}^{\mathrm{f.l.}}(R)$ are compact in $\mathrm{Mod}_c(R)^{\mathrm{op}} = \mathrm{Ind}(\mathrm{Mod}^{\mathrm{f.l.}}(R)^{\mathrm{op}})$, and the third equality is because finite length objects are finitely presented, and $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}_c(R)$ is fully faithful. It follows immediately that the counit $fg \to \mathrm{id}_{\mathrm{Mod}_c(R)}$ is an isomorphism. Furthermore, by Lemma A.1.6, the functor f is exact. We are therefore in the situation of [Gab62, Proposition 5, §III.2], and we conclude that f is a Serre quotient functor.

The condition that $M \in \mathcal{A}$ be "complete" can therefore be understood as an explicit characterization of the condition that the functor $M \widehat{\otimes}_R - : \operatorname{Pro}\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$ factors through f (and, indeed, this characterization then shows that f is the quotient of $\operatorname{Pro}\mathrm{Mod}^{\mathrm{fp}}(R)$ by the Serre subcategory generated by the kernel of the canonical morphism $R \to \lim_n R/\operatorname{rad}(R)^n$). If this condition holds, then we can compute the resulting functor $\mathrm{Mod}_c(R) \to \mathcal{A}$ by restricting through g, since $fg \xrightarrow{\sim} \mathrm{id}_{\mathrm{Mod}_c(R)}$.

A.1.61. *Morita theory for abelian categories.* Now let $\mathcal{C}$ be a cocomplete abelian category, and let P be an object of $\mathcal{C}$. Let $E \coloneqq \mathrm{End}_{\mathcal{C}}(P)$. Then E is a right E^{op}-module in $\mathcal{A}$, and so the functor

$$\mathrm{Hom}_{\mathcal{C}}(P, –) : \mathcal{C} \to \mathrm{Mod}(E^{\mathrm{op}}) \tag{A.1.62}$$

admits a left adjoint $– \otimes_E P$, by Proposition A.1.48 (2).

Assume now that P is projective. Recall that the right orthogonal to P, which we denote by $\mathcal{T}$, is defined to be the full subcategory of $\mathcal{C}$ consisting of objects X for which $\mathrm{Hom}_{\mathcal{C}}(P, X) = 0$. The object P is then a generator of $\mathcal{C}$ if and only if $\mathcal{T} = 0$. We have the following result.

Lemma A.1.63. *Let $\mathcal{C}$ be a cocomplete abelian category, let $P \in \mathcal{C}$ be projective, and let $\mathcal{T}$ be the right orthogonal to P. Then:*

1. *$\mathcal{T}$ is a Serre subcategory of $\mathcal{C}$.*
2. *If, for any object X of $\mathcal{C}$, we let $\overline{X}$ denote the image of X in $\mathcal{Q} \coloneqq \mathcal{C}/\mathcal{T}$, then $\mathrm{Hom}_{\mathcal{C}}(P, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Q}}(\overline{P}, \overline{X})$.*
3. *The image $\overline{P}$ of P in $\mathcal{C}/\mathcal{T}$ is a projective generator of $\mathcal{Q}$.*

Proof. Part (1) is because $\mathcal{T}$ is the kernel of the exact functor $\mathrm{Hom}_{\mathcal{C}}(P, –)$. Part (2) can be proved in the same way as [Gab62, Lem. III.2.1(c)]. We now prove part (3), starting with the claim that $\overline{P}$ is projective in $\mathcal{Q}$. If $\overline{X} \to \overline{Y}$ is a surjection in $\mathcal{Q}$, then by [Gab62, Cor. III.1.1] we can replace it by an isomorphic surjection, and assume that it is represented by a surjection $X \to Y$ in $\mathcal{C}$. Since P is projective in $\mathcal{C}$, this shows that $\mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y)$ is surjective. Now (2) shows that $\mathrm{Hom}_{\mathcal{Q}}(\overline{P}, \overline{X}) \to \mathrm{Hom}_{\mathcal{Q}}(\overline{P}, \overline{Y})$ is surjective. Hence $\overline{P}$ is projective in $\mathcal{Q}$. Now it remains to prove that the orthogonal to $\overline{P}$ in $\mathcal{Q}$ is zero, which is a direct consequence of part (2). ☐

In the context of Lemma A.1.63, assume furthermore that $\mathcal{C}$ is a Grothendieck category, and that P is a compact projective object of $\mathcal{C}$ (or equivalently, $\mathrm{Hom}_{\mathcal{C}}(P, –)$ is exact and commutes with direct sums). If $E \coloneqq \mathrm{End}_{\mathcal{C}}(P)$, then from Lemma A.1.63 (2) we see that also $E \coloneqq \mathrm{End}_{\mathcal{Q}}(\overline{P})$, and then Lemma A.1.63 (3) together with [Gab62, Cor. V.1.1] implies that

$$\mathrm{Hom}_{\mathcal{Q}}(\overline{P}, –) : \mathcal{Q} \to \mathrm{Mod}(E^{\mathrm{op}})$$

is an equivalence. Its left adjoint $– \otimes_E \overline{P}$, as constructed in Proposition A.1.48 (2), is then a quasi-inverse to this equivalence. In particular, if P is a compact projective generator of $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(P, –)$ induces an equivalence $\mathcal{C} \xrightarrow{\sim} \mathrm{Mod}(E^{\mathrm{op}})$ with quasi-inverse $– \otimes_E P$.

Some of the Grothendieck categories of interest to us do not admit projective generators (compact or not), and so the preceding discussion will not apply. However, Lemma A.1.26 shows that, if $\mathcal{A}$ is locally finite, then its opposite category is naturally the Pro-category of an abelian category, and so in particular does contain projective objects. We now describe a variant of the preceding results which applies in this context.

Let $\mathcal{C}$ be the opposite category to a locally finite category, and let P be a projective generator of $\mathcal{C}$. Then, following [Gab62, Section IV.4], the ring $E \coloneqq \mathrm{End}_{\mathcal{C}}(P)$ can be endowed with the structure of a right-pseudocompact topological ring, and the functor $\mathrm{Hom}_{\mathcal{C}}(P,-)$ factors through $\mathrm{Mod}_c(E^{\mathrm{op}})$. We make the additional assumption that P has finite cosocle, and that E is a Noetherian profinite $\mathcal{O}$-algebra; this will be true in all cases of interest in this paper. By [Gab62, Thm. IV.4.4], the functor

$$\mathrm{Hom}_{\mathcal{C}}(P,-) : \mathcal{C} \to \mathrm{Mod}_c(E^{\mathrm{op}}) \tag{A.1.64}$$

is then an equivalence: this is in fact how Proposition A.1.28 is proved.

Proposition A.1.65. *Under the assumptions in the previous paragraph, P is a complete left E-module in $\mathcal{C}$, and the functor $-\widehat{\otimes}_E P : \mathrm{Mod}_c(E^{\mathrm{op}}) \to \mathcal{C}$ (as defined in Lemma* A.1.53) *is a quasi-inverse to* (A.1.64).

Proof. We begin by proving that P is complete, i.e. $P \xrightarrow{\sim} \varprojlim_n P/\mathrm{rad}(E)^n P$. Write $\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ for the natural anti-equivalence. Let $I := P^\vee \in \mathcal{C}^{\mathrm{op}}$, and let $I_n := (P/\mathrm{rad}(E)^n P)^\vee$. Choose generators $x_1, \ldots, x_t$ of $\mathrm{rad}(E)^n$ as a left ideal. Then

$$I_n = \bigcap_{i=1}^{t} \ker(x_i : I \to I),$$

and we need to prove that $I = \varinjlim_n I_n$. By assumption, $\mathcal{C}^{\mathrm{op}}$ is a locally finite category, so it suffices to prove that for each finite length subobjects $M \subset I$, there exists n such that

$$\mathrm{rad}(E)^n \subset \mathrm{Ann}_{E^{\mathrm{op}}}(M) := \{x \in E^{\mathrm{op}} : x|_M = 0\}.$$

The right-pseudocompact topology on $E = \mathrm{End}_{\mathcal{C}}(P)$, which coincides with the left-pseudocompact topology on $E^{\mathrm{op}} = \mathrm{End}_{\mathcal{C}^{\mathrm{op}}}(I)$, is the $\mathrm{rad}(E)$-adic topology, by Lemma A.1.32 (8). So we need to prove that $\mathrm{Ann}_{E^{\mathrm{op}}}(M)$ is open in the left-pseudocompact topology on E^{op}. This is true by definition of this topology, see [Gab62, Section IV.4, end of page 396], where $\mathrm{Ann}_{E^{\mathrm{op}}}(M)$ is denoted $\mathfrak{l}(M)$. This concludes the proof that P is a complete left E-module in $\mathcal{C}$.

It now suffices to prove that $-\widehat{\otimes}_E P$ is left adjoint to the equivalence $\mathrm{Hom}_{\mathcal{C}}(P,-)$. Recall that $\mathcal{C} = \mathrm{Pro}\,\mathcal{C}^{\mathrm{f.l.}}$ and $\mathrm{Mod}_c(E^{\mathrm{op}}) = \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(E^{\mathrm{op}})$. We claim that $\mathrm{Hom}_{\mathcal{C}}(P,-)$ and $-\widehat{\otimes}_E P$ are Pro-extended from functors between $\mathcal{C}^{\mathrm{f.l.}}$ and $\mathrm{Mod}^{\mathrm{f.l.}}(E^{\mathrm{op}})$.

Assuming the claim, it suffices to prove that these restricted functors are adjoint. By Corollary A.1.54, $-\widehat{\otimes}_E P$ and $-\otimes_E P : \mathrm{Mod}(E^{\mathrm{op}}) \to \mathrm{Ind}(\mathcal{C})$ have the same restriction to $\mathrm{Mod}^{\mathrm{f.l.}}(E^{\mathrm{op}})$. By Proposition A.1.48, $-\otimes_E P$ is left adjoint to $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(P,-)$, whose restriction to $\mathcal{C}^{\mathrm{f.l.}}$ is $\mathrm{Hom}_{\mathcal{C}}(P,-)$. Putting these together, we see that indeed $-\widehat{\otimes}_E P$ and $\mathrm{Hom}_{\mathcal{C}}(P,-)$ restrict to adjoint functors between $\mathrm{Mod}^{\mathrm{f.l.}}(E^{\mathrm{op}})$ and $\mathcal{C}^{\mathrm{f.l.}}$. This concludes the proof of the proposition.

There remains to prove the claim. Since $\mathrm{Hom}_{\mathcal{C}}(P,-)$ and $-\widehat{\otimes}_E P$ commute with cofiltered limits, it suffices to prove that they preserve objects of finite length. This is true for $\mathrm{Hom}_{\mathcal{C}}(P,-)$ because it is an equivalence. By dévissage, it now suffices to prove that $M \widehat{\otimes}_E P$ has finite length if M is a simple E^{op}-module. Now $M \widehat{\otimes}_E P$ is a quotient of $(E/\mathrm{rad}(E)) \widehat{\otimes}_E P$, which equals $P/\mathrm{rad}(E)P$, by Corollary A.1.54. Since P has cosocle of finite length, it suffices to prove that $\mathrm{rad}(E)P = \mathrm{rad}(P)$. By [Gab62, Prop. IV.4.12], we have

$$\mathrm{rad}(E) = \{\varphi \in E : \varphi(P) \subseteq \mathrm{rad}(P)\},$$

and so $\mathrm{rad}(E)P \subseteq \mathrm{rad}(P)$. If $\mathrm{rad}(P)/\mathrm{rad}(E)P \neq 0$, then there is a non-zero map $\overline{\varphi} : P \to \mathrm{rad}(P)/\mathrm{rad}(E)P$, because P is a generator of $\mathcal{C}$; and $\overline{\varphi}$ lifts to a map $\varphi : P \to \mathrm{rad}(P)$, because P is a projective object of $\mathcal{C}$. But then $\varphi \in \mathrm{rad}(E)$, which contradicts the fact that $\overline{\varphi} \neq 0$. $\square$

A.2. **∞-categories.** We now recall some basic results about ∞-categories, and especially stable ∞-categories, that we will use below. Our main references are ultimately [Lur09; Lur17; Kerodon], but we find it convenient to refer to [BGT13; NS18; Cis19] for some of the results we use. In order to deal with set-theoretic issues, we follow the approach of [Lur09]. In particular, we typically fix a Grothendieck universe, and sets are called *small* if they belong to this fixed universe. Furthermore, all limits and colimits are assumed to be small. At one point in the sequel we employ the technical device of enlarging the universe (allowing us to regard an *a priori* large ∞-category as in fact being small), and so in Subsection A.4 below we briefly recall how the process of enlarging the universe interacts with the ∞-categorical constructions in which we're interested. However, throughout the present subsection, the universe remains fixed.

It will sometimes be useful to refer to the more traditional literature of triangulated categories. To this end, we recall that the homotopy category of a stable ∞-category is a triangulated category [Lur17, Rem. 1.1.2.15], and that a functor $\mathcal{A} \to \mathcal{B}$ between stable ∞-categories is an equivalence (resp. is fully faithful) if and only if the induced functor on homotopy categories is an equivalence (resp. is fully faithful).

The ∞-categories that we consider typically come in two sizes: *small*, and *presentable*. More precisely, we say that an ∞-category $\mathcal{C}$ is *small* if it is equivalent to an ∞-category whose underlying simplicial set is small, i.e. if it is essentially small in the sense of [Kerodon, Tag 03SM]. Following [Kerodon, Tag 06K6, Tag 06NF], for every small regular cardinal κ, we say that $\mathcal{C}$ is *κ-accessible* if there exists a small ∞-category $\mathcal{C}_0$ such that $\mathcal{C}$ is equivalent to the κ-completion $\mathrm{Ind}_\kappa(\mathcal{C}_0)$. We then say that $\mathcal{C}$ is *κ-presentable* if it is κ-accessible and cocomplete, and we say that $\mathcal{C}$ is *presentable* if it is κ-presentable for some small regular cardinal κ. By [Kerodon, Tag 06PU], presentable ∞-categories are complete. In applications, we will usually be able to take $\kappa = \omega$, and we refer to Section A.2.14 for a review of the Ind_ω-construction (which we will simply denote by Ind).

If $\mathcal{C}$ is a stable ∞-category, then it is ω-presentable if and only if it is ω-accessible: in fact, if it is ω-accessible, then its cocompleteness follows from the fact that it has filtered colimits and finite colimits. Hence, if $\mathcal{C}_0$ is a small stable ∞-category, then $\mathrm{Ind}(\mathcal{C}_0)$ is ω-presentable, because it is stable, as explained in Remark A.2.16, and ω-accessible, by definition.

We will refer to colimit-preserving functors between cocomplete stable ∞-categories as *continuous*. In many texts, such as [Lur09, Def. 5.3.4.5], *continuous* signifies the preservation of filtered colimits, and so the functors between cocomplete stable ∞-categories that we call continuous would there be called exact and continuous. Since we have no occasion to consider non-exact functors, we have opted to incorporate the exactness condition into our definition of continuous functors.

Finally, we will say that an arrow $f : x \to y$ in an ∞-category $\mathcal{C}$ is an *isomorphism* if it represents an isomorphism in the homotopy category $h\mathcal{C}$. In the literature, this is often referred to as f being an *equivalence*, and we will sometimes use this terminology interchangeably.

A.2.1. *Anima, and morphisms in ∞-categories.* In the theory of ∞-categories, the ∞-category of *anima*[8] (referred to as the ∞-category of *spaces* in [Lur09; Lur17], and as the ∞-category of *∞-groupoids* in other references), which we denote Ani, plays

[8]This is the terminology of [ČS24], and is short for *animated set*.

the role that the category of sets plays in ordinary category theory. In particular, if x and y are objects of an ∞-category $\mathcal{C}$, then we may form $\mathrm{Maps}_{\mathcal{C}}(x, y)$, the anima of morphisms from x to y. The basic constructions related to this notion are described e.g. in [Lur09, §1.2.2] and [Cis19, §3.7].

The formation of $\mathrm{Maps}_{\mathcal{C}}(-,-)$ is functorial in each of its arguments. One way to formulate this statement is via the existence of the Yoneda embedding (see [Lur09, Prop. 5.1.3.1] or [Cis19, Thm. 5.8.13]), which is a fully faithful functor

$$\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani}); \tag{A.2.2}$$

here the target denotes the ∞-category of functors from $\mathcal{C}^{\mathrm{op}}$ to Ani.

A.2.3. *Spectra.* Since the theory of stable ∞-categories inevitably bumps up against the theory of *spectra* (in the algebraic topological sense), we give some brief reminders regarding this latter theory for the non-expert. In fact, in the present paper, all the stable ∞-categories that we consider will be R-linear (in a suitable sense) for some commutative ring R, and hence, as we explain below, all the spectra that we encounter will actually be objects of $D(R)$ (the derived category of R-modules). Nevertheless, the references we give for ∞-categorical results typically work in a framework that involves the consideration of general spectra; thus we hope the following recollections will be helpful.

Intuitively, the ∞-category Sp of spectra is obtained from the ∞-category of pointed anima by making the suspension functor Σ become invertible. In particular, to each pointed anima x, there is associated a "suspension spectrum" $\Sigma^\infty x$, which is the image of x in the ∞-category of spectra with respect to the canonical functor from pointed anima to spectra.

The category Sp carries a canonical right complete t-structure[9], whose connective part $\mathrm{Sp}^{\leq 0}$ contains the full subcategory of suspension spectra, and is obtained from this by taking the closure under colimits and extensions [Lur17, Rem. 1.4.3.5]. Right completeness means that any spectrum s can be written as a colimit

$$\mathrm{colim}\, s_n[-n] \xrightarrow{\sim} s,$$

where the s_n are connective spectra. In fact, we may even arrange things so that the s_n are suspension spectra, say $s_n = \Sigma^\infty x_n$. Eliding the difference between a pointed anima and its associated suspension spectrum, we may then write this as

$$\mathrm{colim}\, \Sigma^{-n} x_n \text{``}\xrightarrow{\sim}\text{''} s,$$

which in turn prompts the most naive notion of spectrum: namely, a spectrum consists of a sequence (x_n) of pointed anima equipped with transition morphisms $\Sigma x_n \to x_{n+1}$. (From this perspective, the suspension spectrum of a pointed anima x is given by the particular sequence $(\Sigma^n x)$.) These transition morphisms may be expressed in adjoint form as maps $x_n \to \Omega x_{n+1}$, and we say that the sequence (x_n) is an Ω-*spectrum* if these maps are homotopy equivalences. It turns out that any "naive" spectrum is equivalent to an Ω-spectrum, and so one may restrict attention to Ω-spectra when developing the theory.

Another way of thinking of an Ω-spectrum is that we start with a pointed anima x_0, which we then enhance to a sequence (x_n) equipped with equivalences $x_n \xrightarrow{\sim} \Omega x_{n+1}$. In other words, (x_n) is a sequence of successive deloopings of x_0,

[9]We recall the notion of t-structure in Section A.6 below, and the notion of right completeness in Section A.6.2.

endowing x_0 with the structure of an infinite loop space. Motivated by this, we write $x_0 := \Omega^\infty s$; this construction then yields a functor $\Omega^\infty : \mathrm{Sp} \to \mathrm{Ani}_*$ (the codomain denoting the ∞-category of pointed anima) which is right adjoint to Σ^∞.

The preceding discussion is of course rather informal, but one rigorous version of it can be found in [Lur17, §1.4]. The discussion of [Lur17, §1.4.2] presents a general explanation of how to "stabilize" an ∞-category $\mathcal{C}$ admitting finite limits, yielding a stable ∞-category $\mathrm{Sp}(\mathcal{C})$. If $\mathcal{C}$ is furthermore pointed, then [Lur17, Prop. 1.4.2.24] shows that $\mathrm{Sp}(\mathcal{C})$ may be described as the ∞-category of "Ω-spectrum objects" in $\mathcal{C}$. (If $\mathcal{C}$ admits finite limits, and so in particular a final object, then we may consider the ∞-category $\mathcal{C}_*$ of pointed objects of $\mathcal{C}$. The forgetful functor $\mathcal{C}_* \to \mathcal{C}$ then induces an equivalence $\mathrm{Sp}(\mathcal{C}_*) \to \mathrm{Sp}(\mathcal{C})$ [Lur17, Rem. 1.4.2.18], and so it is in fact no loss of generality to assume that $\mathcal{C}$ is pointed; cf. [Lur17, Rem. 1.4.2.25].) Another discussion of the description of $\mathrm{Sp}(\mathcal{C})$ in terms of spectrum objects, for a pointed ∞-category $\mathcal{C}$ admitting finite limits, can be found in [BGT13, §2.3].

In particular, taking $\mathcal{C}$ to be the ∞-category Ani_* of pointed anima, we obtain the stable ∞-category Sp as its stabilization in the sense just discussed. (The functor Ω^∞ is described in [Lur17, 1.4.2.20]. The construction of the t-structure on Sp is given by [Lur17, Prop. 1.4.3.4], and the left adjoint Σ^∞ to Ω^∞ is introduced in the course of proving that result (and again in [Lur17, Prop. 1.4.4.4]). The description of $\mathrm{Sp}^{\leq 0}$ is given in [Lur17, Rem. 1.4.3.5].)

The heart of the t-structure on Sp is the category of abelian groups. If A is an abelian group, then one typically denotes the corresponding spectrum by HA; it is the so-called *Eilenberg–MacLane spectrum* of A. We may furthermore describe the homotopy groups of spectra in terms of the t-structure on Sp. Indeed, if s is a spectrum, then we write $\pi_0(s) := \tau^{\geq 0}\tau^{\leq 0}s$, and more generally $\pi_{-n}(s) := \pi_0(s[n]) = (\tau^{\geq n}\tau^{\leq n}s)[n]$. Of course, the homotopy groups of spectra, defined in this manner, also admit an interpretation as stable homotopy groups.[10]

If R is a commutative ring, then its associated Eilenberg–MacLane spectrum HR has a natural E_∞-structure. By [Lur17, Thm. 7.1.2.13], which is an unbounded version of the Dold–Kan correspondence, we may identify the stable ∞-category of HR-module spectra with the ∞-categorical version of the derived category $D(R)$. As indicated at the beginning of this section, these sorts of spectra are the only ones that we will actually have occasion to consider in the present paper.

A.2.4. RHom *in stable ∞-categories.* If $\mathcal{C}$ is a *stable* ∞-category and x and y are two objects of $\mathcal{C}$, then in addition to forming the anima of morphisms from x to y, we may also form the *mapping spectrum* from x to y, denoted by $\mathrm{RHom}_{\mathcal{C}}(x,y)$. One way to express this is by saying that $\mathcal{C}$ is canonically enriched over spectra. (The reference [HM24, App. C] gives a rather general discussion of enriched categories, which applies in particular in our present context.) In the following discussion we state, and sketch the proofs, of the basic results that we need related to this enriched structure.

[10]For example, if we describe s as an Ω-spectrum (x_m), then $\pi_{-n}(s) = \pi_{m-n}(x_m)$ (for any $m \geq n$), where $\pi_\bullet$ on the right hand side denotes the usual homotopy groups, because the equivalence $x_m \xrightarrow{\sim} \Omega x_{m+1}$ induces an isomorphism $\pi_{m-n}(x_m) \xrightarrow{\sim} \pi_{m-n+1}(x_{m+1})$. More generally, if s is the spectrum associated to a sequence (x_m) of pointed anima equipped with morphisms $\Sigma x_m \to x_{m+1}$, then $\pi_{-n}(s) = \mathrm{colim}_m\, \pi_{m-n}(x_m)$, the transition morphisms being induced by the morphisms $\pi_{m-n}(x_m) \to \pi_{m-n+1}(\Sigma x_m) \to \pi_{m-n+1}(x_{m+1})$. So, if x is a pointed anima, then $\pi_{-n}\Sigma^\infty x$ is the $(-n)$th stable homotopy group of x.

For a terse explanation of this enriched structure, see e.g. [Lur17, Rem. 7.1.2.2]. More details can be found in the discussion of [BGT13, §2.3]; see also the discussion following [BGT13, Def. 2.15] (noting that those authors write maps where we write Maps, and write Maps where we write RHom). Concretely, the discussion following [Lur17, Rem. 1.1.2.8] shows that if x and y are objects of the stable ∞-category $\mathcal{C}$, then we have $\mathrm{Maps}_{\mathcal{C}}(\Sigma^{-n}x, y) \xrightarrow{\sim} \Omega \,\mathrm{Maps}_{\mathcal{C}}(\Sigma^{-n-1}x, y)$, and so the sequence $\big(\mathrm{Maps}_{\mathcal{C}}(\Sigma^{-n}x, y)\big)_{n\in\mathbf{Z}}$ forms a spectrum, or more specifically, an Ω-spectrum. This Ω-spectrum is the mapping spectrum from x to y, which (as already indicated) we denote by $\mathrm{RHom}_{\mathcal{C}}(x, y)$. (Note that when $\mathcal{C}$ is a stable ∞-category, the mapping anima $\mathrm{Maps}_{\mathcal{C}}(-,-)$ are canonically pointed, by the zero morphism.)

We may recover Maps from RHom by applying the functor Ω^∞:

$$\mathrm{Maps}_{\mathcal{C}}(x, y) \xrightarrow{\sim} \Omega^\infty \,\mathrm{RHom}_{\mathcal{C}}(x, y).$$

Also, we find that

$$\pi_{-n}\,\mathrm{RHom}_{\mathcal{C}}(x, y) = \pi_0 \,\mathrm{Maps}_{\mathcal{C}}(x[-n], y) = \mathrm{Hom}_{h\mathcal{C}}(x[-n], y), \tag{A.2.5}$$

where $h\mathcal{C}$ denotes the underlying homotopy category of $\mathcal{C}$.

In the examples we care about, our stable ∞-categories will (in a suitable sense) be R-linear for some commutative ring R; indeed, they will be derived categories of R-linear abelian categories, or closely related to such. The corresponding mapping spectra $\mathrm{RHom}_{\mathcal{C}}(x, y)$ will thus be module spectra over the Eilenberg–MacLane spectrum HR of R. As already noted above, [Lur17, Thm. 7.1.2.13] identifies the stable ∞-category of HR-module spectra with the ∞-categorical version of the derived category $D(R)$. Accordingly, we regard $\mathrm{RHom}_{\mathcal{C}}(x, y)$ as taking values in $D(R)$.

In traditional homological algebra, one often defines RHom on the derived category of an R-linear abelian category as a derived functor (again with values in $D(R)$), and so *a priori* RHom may be ambiguously defined. However, the exactness properties that characterize derived functors in the ∞-categorical context (see e.g. the various results discussed in Section A.7.14 below) will show that RHom as defined above will coincide with any definition of RHom as a derived functor that comes up in practice.

Relatedly, if $\mathcal{A}$ is an abelian category and $\mathcal{C} \coloneqq D(\mathcal{A})$ (in the sense of Definition A.7.1 below), then for any two objects x, y of $\mathcal{A}$ (thought of as a full subcategory of $\mathcal{C}$ in the canonical way), we deduce from (A.2.5) and standard homological algebra (see e.g. [Stacks, Tag 06XU]) that $\pi_{-n}\,\mathrm{RHom}_{\mathcal{C}}(x, y)$ agrees with the Yoneda Ext-group $\mathrm{Ext}^n_{\mathcal{A}}(x, y)$. In keeping with this observation, and with [Lur17, Notation 1.1.2.17], we employ the usual notation

$$\mathrm{Ext}^n_{\mathcal{C}}(x, y) \coloneqq \pi_{-n}\,\mathrm{RHom}_{\mathcal{C}}(x, y) \tag{A.2.6}$$

for any objects x, y of an arbitrary stable ∞-category $\mathcal{C}$.[11]

[11]As indicated by our notation $\mathrm{Sp}^{\leq 0}$, we have employed cohomological conventions for the t-structure on Sp. This contrasts with [Lur17], which uses homological conventions, and defines $\pi_n(x) := \tau_{\leq 0}\tau_{\geq 0}x[-n]$ [Lur17, Def. 1.2.1.11]. However, the underlying stable ∞-category of spectra is the same, hence so is the suspension functor on it; and since we have defined $\pi_{-n}(x)$ to be $\tau^{\geq 0}\tau^{\leq 0}x[n]$, which equals $\tau_{\leq 0}\tau_{\geq 0}x[n]$, our notion of $\pi_{-n}(x)$ coincides with the one in [Lur17]. Note also that, because of this, our $\mathrm{Sp}^{\leq 0}$ coincides with the category of spectra whose stable homotopy is concentrated in *nonnegative* degrees.

A.2.7. *Spectral variant of the Yoneda embedding.* If $\mathcal{C}$ is a stable ∞-category, then we may use $\mathrm{RHom}_{\mathcal{C}}$ to construct a variant of the usual Yoneda embedding (A.2.2). Namely, $y \mapsto \mathrm{RHom}_{\mathcal{C}}(-, y)$ gives a fully faithful exact functor

$$\mathcal{C} \hookrightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \tag{A.2.8}$$

where Sp denotes the stable ∞-category of spectra, and $\mathrm{Fun}^{\mathrm{ex}}$ denotes the stable ∞-category of exact functors between stable ∞-categories. One construction of this functor is given in [BGT13, Def. 2.15], where it is obtained as the stabilization (in the sense of [Lur17, §1.4.2]) of the usual Yoneda embedding (A.2.2), bearing in mind that $\mathrm{Sp}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$ when $\mathcal{C}$ is itself a stable ∞-category.

A.2.9. RHom *as a right adjoint.* If $\mathcal{C}$ is a presentable stable ∞-category, then we have an action of spectra on $\mathcal{C}$ [Lur17, Rem. 7.1.2.2, Prop. 4.8.2.18, Rem. 4.8.2.20]. More precisely, the ∞-category $\mathcal{P}r^L$ whose objects are presentable ∞-categories and whose morphisms are left adjoint functors admits a symmetric monoidal structure $(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \otimes \mathcal{D}$, where $\mathcal{C} \otimes \mathcal{D}$ is the presentable ∞-category characterized by the property that it receives a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which is colimit preserving in each variable separately, and which is universal for this property. Furthermore, the stable ∞-category Sp of spectra is idempotent with respect to this structure, and the Sp-local objects of $\mathcal{P}r^L$, i.e. the presentable ∞-categories for which $\mathcal{C} \xrightarrow{\sim} \mathcal{C} \otimes \mathrm{Sp}$, are precisely the presentable stable ∞-categories [Lur17, Prop. 4.8.2.18]. The inverse equivalence $\mathcal{C} \otimes \mathrm{Sp} \to \mathcal{C}$ then gives the action of Sp on $\mathcal{C}$; in particular, this action is colimit preserving in either variable.

The $\mathrm{RHom}_{\mathcal{C}}$ construction may also be interpreted as right adjoint to the action of Sp on $\mathcal{C}$. In a little more detail, for any object x of $\mathcal{C}$, the functor $- \otimes x : \mathrm{Sp} \to \mathcal{C}$ is colimit preserving, and so admits a right adjoint, which we (temporarily) denote by G. If $\mathbf{S}$ denotes the sphere spectrum (i.e. $\Sigma^\infty S^0$, where S^0 denotes the pointed 0-sphere), then $\mathbf{S}$ is a unit object for $- \otimes x$, and so, for any object y of $\mathcal{C}$, we find that

$$\mathrm{Maps}_{\mathcal{C}}(x, y) \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{C}}(\mathbf{S} \otimes x, y) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Sp}}\big(\mathbf{S}, G(y)\big).$$

Now for any object t of Sp, we have that

$$\mathrm{Maps}_{\mathrm{Sp}}(\mathbf{S}, t) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Sp}}(\Sigma^\infty S^0, t) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Ani}_*}(S^0, \Omega^\infty t).$$

Thus

$$\mathrm{Maps}_{\mathcal{C}}(x, y) \xrightarrow{\sim} \Omega^\infty G(y),$$

and replacing y by $\Sigma^n y$, we find that

$$\mathrm{Maps}_{\mathcal{C}}(\Sigma^{-n} x, y) \xrightarrow{\sim} \Omega^\infty \Sigma^n G(y).$$

Now any spectrum t is identified with the Ω-spectrum given by the sequence $(\Omega^\infty \Sigma^n t)_{n \geq 0}$. Thus we see that $G(y)$ is identified with the Ω-spectrum $\mathrm{Maps}_{\mathcal{C}}(\Sigma^{-n} x, y)$, which is to say with $\mathrm{RHom}_{\mathcal{C}}(x, y)$. To summarize: if s is a spectrum and $x, y \in \mathcal{C}$, we have a natural isomorphism

$$\mathrm{Maps}_{\mathcal{C}}(s \otimes x, y) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Sp}}\big(s, \mathrm{RHom}_{\mathcal{C}}(x, y)\big). \tag{A.2.10}$$

A.2.11. *Formal properties of* RHom. We recall (and verify) some properties of $\mathrm{RHom}_{\mathcal{C}}$, which are analogues of the usual formal properties for $\mathrm{Maps}_{\mathcal{C}}$. (The notion of a stable ∞-category being compactly generated is recalled in some detail below; in particular, it implies presentability.)

Lemma A.2.12. *Let $\mathcal{C}$ be a compactly generated stable ∞-category, and let $x \in \mathcal{C}$ be an object of $\mathcal{C}$. Then:*

(1) The functor $\mathrm{RHom}_{\mathcal{C}}(x, –)$ commutes with small limits.
(2) If $x = \mathrm{colim}_{i\in I}\, x_i$, then $\mathrm{RHom}_{\mathcal{C}}(x, –) = \lim_{i\in I} \mathrm{RHom}_{\mathcal{C}}(x_i, –)$.
(3) If x is compact in $\mathcal{C}$, then $\mathrm{RHom}_{\mathcal{C}}(x, –)$ commutes with small colimits.

Proof. The description (A.2.10) of $\mathrm{RHom}_{\mathcal{C}}(x,–)$ as a right adjoint immediately implies part (1).

We now prove part (2). Since $s \otimes -$ preserves colimits for any $s \in \mathrm{Sp}$, we obtain a natural isomorphism

$$\mathrm{Maps}_{\mathrm{Sp}}(s, \mathrm{RHom}_{\mathcal{C}}(\mathrm{colim}\, x_i, –)) \xrightarrow{\sim} \lim_i \mathrm{Maps}_{\mathrm{Sp}}(s, \mathrm{RHom}_{\mathcal{C}}(x_i, –))$$

and so part (2) follows from the Yoneda lemma in Sp.

Finally, we prove part (3). By Lemma A.2.32 below, applied to $F \coloneqq - \otimes x$ and $G \coloneqq \mathrm{RHom}_{\mathcal{C}}(x, –)$, it suffices to prove that $- \otimes x$ preserves compact objects. The compact objects in Sp are precisely the colimits of finite diagrams, all of whose objects are shifts of the sphere spectrum $\mathbf{S}$, or zero. Since $\mathbf{S} \otimes x = x$, and $- \otimes x$ preserves colimits, we see that $- \otimes x$ sends compact objects to colimits of finite diagrams, all of whose objects are shifts of x, or zero. Since such colimits are compact if x is compact, this concludes the proof. □

A.2.13. RHom *and adjoints.* If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are (left, resp. right) adjoint functors between ∞-categories, then they are necessarily exact (being adjoints) and so are compatible with suspension. One then immediately verifies that $\mathrm{RHom}_{\mathcal{D}}\big(F(x), y\big) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{C}}\big(x, G(y)\big)$ for any object x of $\mathcal{C}$ and y of $\mathcal{D}$.

If $F : \mathcal{C} \to \mathcal{D}$ is a continuous functor between presentable stable ∞-categories, then from the canonical nature of the Sp-action on each of $\mathcal{C}$ and $\mathcal{D}$, one deduces that F is compatible with these actions. Since F admits a right adjoint G, one can also deduce this from the compatibility of adjunctions with RHom noted in the preceding paragraph, together with the adjunction (A.2.10).

A.2.14. *Ind-categories.* We recall the notion of Ind-completion of ∞-categories, following the exposition in [Lur09] and [BGT13, §2.4]. If $\mathcal{C}$ is a small ∞-category, the Ind-category $\mathrm{Ind}(\mathcal{C})$ is defined to be the formal closure of $\mathcal{C}$ under filtered colimits. Dually, we have the Pro-category $\mathrm{Pro}(\mathcal{C})$, which may be defined as $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$, and enjoys analogous properties. The ∞-category $\mathrm{Ind}(\mathcal{C})$ is characterised by the property that it admits filtered colimits, and admits a fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ which induces (via restriction) an equivalence of ∞-categories of functors

$$\mathrm{Fun}'\big(\mathrm{Ind}(\mathcal{C}), \mathcal{D}\big) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D}), \tag{A.2.15}$$

for any $\mathcal{D}$ admitting filtered colimits, where the domain denotes the full sub-∞-category of $\mathrm{Fun}\big(\mathrm{Ind}(\mathcal{C}), \mathcal{D}\big)$ consisting of those functors that preserve filtered colimits. See [Lur09, Prop. 5.3.5.10]. The embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ preserves all limits, and if $\mathcal{C}$ admits finite limits then so does $\mathrm{Ind}(\mathcal{C})$.

If we take the target ∞-category $\mathcal{D}$ in (A.2.15) to be $\mathrm{Ani}^{\mathrm{op}}$, and compose the resulting equivalence with the Yoneda embedding (A.2.2) for $\mathrm{Ind}(\mathcal{C})$, we obtain a functor

$$\mathrm{Ind}(\mathcal{C}) \hookrightarrow \mathrm{Fun}'\big(\mathrm{Ind}(\mathcal{C}), \mathrm{Ani}^{\mathrm{op}}\big)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathrm{Ani}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani}),$$

which is again fully faithful, and preserves filtered colimits (since the objects of $\mathcal{C}$ are compact in $\mathrm{Ind}(\mathcal{C})$). This functor identifies $\mathrm{Ind}(\mathcal{C})$ with the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani})$ generated by $\mathcal{C}$ under the formation of filtered colimits [Lur09, Cor. 5.3.5.4].

Remark A.2.16. If $\mathcal{C}$ is furthermore a stable ∞-category, then so is $\mathrm{Ind}(\mathcal{C})$ [Lur17, Prop. 1.1.3.6], and the canonical fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}\,\mathcal{C}$ is exact (see [Lur09, Prop. 5.1.3.2, 5.3.5.14]). Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories, then (A.2.15) restricts to an equivalence

$$\mathrm{Fun}^{\mathrm{ex}\prime}(\mathrm{Ind}\,\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \tag{A.2.17}$$

(the source denoting exact and filtered colimit preserving — or equivalently, continuous — functors, and the target denoting exact functors). Indeed, since the inclusion $\mathcal{C} \hookrightarrow \mathrm{Ind}\,\mathcal{C}$ is exact, the restriction of an exact functor $\mathrm{Ind}\,\mathcal{C} \to \mathcal{D}$ is again exact; while conversely, if $F : \mathcal{C} \to \mathcal{D}$ is exact, then $\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ preserves cofibre sequences, and so is also exact. (To see this claim regarding preservation of cofibre sequences, note that by the proof of [Lur17, Prop. 1.1.3.6], every cofibre sequence in $\mathrm{Ind}(\mathcal{C})$ is a filtered colimit of cofibre sequences in $\mathcal{C}$. Since F preserves filtered colimits, and a colimit of cofibre sequences in $\mathcal{D}$ is a cofibre sequence, this implies that $\mathrm{Ind}(F)$ preserves cofibre sequences, as claimed.)

In particular, taking $\mathcal{D}$ to be $\mathrm{Sp}^{\mathrm{op}}$ in (A.2.17) and composing with the spectral Yoneda embedding (A.2.8) for $\mathrm{Ind}(\mathcal{C})$, we obtain an exact and fully faithful functor $\mathrm{Ind}\,\mathcal{C} \hookrightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Sp}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$. In this way we may alternatively regard $\mathrm{Ind}\,\mathcal{C}$ as the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ generated by $\mathcal{C}$ under filtered colimits.

Remark A.2.18. In fact, the equivalence (A.2.15) is part of an adjunction, as we now briefly explain. Namely, we may consider the functor

$$\mathrm{Fun}\big(\mathrm{Ind}(\mathcal{C}), \mathcal{D}\big) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \tag{A.2.19}$$

given by restriction. Then "Indization", i.e. the inverse to the equivalence (A.2.15), is a left adjoint to this functor. This is a consequence of [Lur09, Prop. 4.3.3.7], because the indization of a functor $F : \mathcal{C} \to \mathcal{D}$ is its left Kan extension through the canonical embedding $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$, by [Lur09, Lem. 5.3.5.8]. The left adjoint to (A.2.19) is furthermore fully faithful, since it induces an equivalence onto its essential image $\mathrm{Fun}'\big(\mathrm{Ind}(\mathcal{C}), \mathcal{D}\big)$.

Remark A.2.20. If $\mathcal{C}$ is a 1-category, then [Kerodon, Tag 065F], shows that the formation of $\mathrm{Ind}\,\mathcal{C}$ is independent (up to canonical equivalence) of whether we regard $\mathcal{C}$ as a 1-category and form its Ind-completion in the sense of Section A.1.4, or whether we regard it as an ∞-category and form its Ind-completion in the sense of the present discussion.

If $\mathcal{C}$ admits filtered colimits then we let $\mathcal{C}^c$ denote the full sub-∞-category of $\mathcal{C}$ consisting of compact objects.

Lemma A.2.21. *Suppose that $\mathcal{C}$ is an ∞-category admitting filtered colimits for which $\mathcal{C}^c$ is small. If $\mathcal{C}'$ is any sub-∞-category of $\mathcal{C}^c$, then the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ extends* (*essentially*) *uniquely to a filtered colimit preserving functor* $\mathrm{Ind}\,\mathcal{C}' \hookrightarrow \mathcal{C}$, *which is again fully faithful.*

Proof. The defining property (A.2.15) (taking $\mathcal{C}$ there to be $\mathcal{C}'$ and $\mathcal{D}$ to be $\mathcal{C}$) shows the existence and essential uniqueness of the extension. Its full faithfulness is then an immediate application of [Lur09, Prop. 5.3.5.11(1)] (taking the κ there to be ω), and is in any case easily verified directly. (It is here that we use the compactness in $\mathcal{C}$ of the objects of $\mathcal{C}'$.) $\square$

We say that $\mathcal{C}$ is *compactly generated* if it is ω-accessible, or equivalently if it admits filtered colimits, if $\mathcal{C}^c$ is small, and if the natural functor $\mathrm{Ind}(\mathcal{C}^c) \to \mathcal{C}$ of Lemma A.2.21 (sending a filtered diagram in $\mathcal{C}^c$ to its colimit in $\mathcal{C}$) is an equivalence.[12] If $\mathcal{A}$ is a small ∞-category, then $\mathrm{Ind}(\mathcal{A})$ is idempotent-complete [Lur09, Cor. 4.4.5.16], and the compact objects of $\mathrm{Ind}(\mathcal{A})$ are the idempotent completion of $\mathcal{A}$ [Lur09, Lem. 5.4.2.4]. We thus obtain a correspondence [Lur09, Prop. 5.4.2.17] between small idempotent-complete ∞-categories and compactly generated ∞-categories. [13] If $\mathcal{C}$ is compactly generated and furthermore stable, then it is in fact cocomplete (since by the definitions of compact generation and stability respectively, it admits both filtered and finite colimits). Hence, if $\mathcal{C}$ is a stable ∞-category, then it is compactly generated if and only if it is ω-presentable.

We say that a sub-∞-category $\mathcal{A}$ of a stable ∞-category $\mathcal{B}$ is a *stable sub-∞-category* if it is full as a sub-∞-category, stable as an ∞-category, and if the inclusion $\mathcal{A} \subseteq \mathcal{B}$ is exact. (This is what in [NS18] is called a stable subcategory.) If $\mathcal{C}$ is stable and admits filtered colimits, then $\mathcal{C}^c$ is a stable sub-∞-category of $\mathcal{C}$, for example by [Lur17, Lem. 1.1.3.3]. On the other hand, as we already noted, the Ind-completion of a small stable ∞-category is stable. In particular, we see that a compactly generated ∞-category $\mathcal{C}$ is stable if and only if $\mathcal{C}^c$ is stable.

Any functor $F : \mathcal{C} \to \mathcal{D}$ between small ∞-categories induces a filtered colimit-preserving functor $F_{\mathrm{Ind}(\mathcal{C})} : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$, corresponding under (A.2.15) to the composite functor $\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathrm{Ind}(\mathcal{D})$; we will sometimes denote $F_{\mathrm{Ind}(\mathcal{C})}$ simply by F if this will not cause confusion. If F is fully faithful, then so is $F_{\mathrm{Ind}(\mathcal{C})}$ [Lur09, Prop. 5.3.5.11(1)]. If $F : \mathcal{C} \to \mathcal{D}$ is an exact functor between small stable ∞-categories then $F_{\mathrm{Ind}(\mathcal{C})} : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ preserves all colimits, because it preserves filtered colimits and finite colimits, and thus it is continuous.

Remark A.2.22. Recall that there is a general notion of what it means for a set of objects X belonging to a cocomplete stable ∞-category $\mathcal{C}$ to *generate* $\mathcal{C}$; see e.g. [EGH25, Def. A.8.5]. Briefly: we define $X^\perp$ to be the full sub-∞-category of $\mathcal{C}$ whose objects are those y for which $\mathrm{RHom}_{\mathcal{C}}(x, y) = 0$ for all $x \in X$, and we say that X *generates* $\mathcal{C}$ if $X^\perp = 0$. Equivalently, by e.g. [EGH25, Cor. A.8.7], X generates $\mathcal{C}$ if and only if the only cocomplete stable subcategory of $\mathcal{C}$ containing all objects of X is $\mathcal{C}$ itself. We say that X is a *set of compact generators* of $\mathcal{C}$ if every object of X is compact, and if X generates $\mathcal{C}$.

[12]Note that in [Lur09, Def. 5.5.7.1] and [BGT13, §2.4], compactly generated is taken to mean ω-presentable (rather than merely ω-accessible), while [Kerodon, Tag 0673] gives essentially the same definition as the one we give here, but omits the requirement that $\mathcal{C}^c$ be small. In the stable case, ω-accessible and ω-presentable coincide, so in this case (which is our primary focus) our definition coincides with those of [Lur09] and [BGT13].

[13]By [Lur09, Prop. 5.4.2.17], this correspondence can be upgraded to an equivalence of ∞-categories, where the functors between compactly generated ∞-categories are those which preserve filtered colimits and compact objects, but we will not need this result; and indeed we will consider functors between compactly generated categories which at least *a priori* do not preserve compact objects.

Lemma A.2.23 below shows that the notion of compact generation introduced above is compatible with this more general notion of generation. In particular, it follows immediately from Lemma A.2.23 (2) that if $\mathcal{C}$ has a set of compact generators, then $\mathcal{C}$ is compactly generated (because the existence of a set of compact generators implies that $(\mathcal{C}^c)^\perp = 0$).

Lemma A.2.23. *Let $\mathcal{C}$ be a cocomplete stable ∞-category.*

(1) If $\mathcal{C}$ is compactly generated, then $(\mathcal{C}^c)^\perp = 0$.
(2) Suppose that $\mathcal{C}'$ is a small stable sub-∞-category of $\mathcal{C}^c$, with the property that $(\mathcal{C}')^\perp = 0$. Then $\mathcal{C}^c$ is equal to the idempotent completion of $\mathcal{C}'$, and $\mathcal{C}$ is compactly generated. In particular, if $(\mathcal{C}^c)^\perp = 0$ then $\mathcal{C}$ is compactly generated.

Proof. Since $\operatorname{Ind}\mathcal{C}^c$ is defined to be a subcategory of the presheaf category of $\mathcal{C}^c$, we see that the right perpendicular of $\mathcal{C}^c$ in $\operatorname{Ind}\mathcal{C}^c$ is trivial; this proves (1).

We now prove (2). Since $\mathcal{C}$ is cocomplete, Lemma A.2.21 shows that the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ Ind-extends to a filtered colimit preserving fully faithful functor

$$\operatorname{Ind}\mathcal{C}' \hookrightarrow \mathcal{C}. \tag{A.2.24}$$

Part (2) will follow if we further show that (A.2.24) is an equivalence, since we have already observed that the analogue of (2) holds for $\operatorname{Ind}\mathcal{C}'$.

Since the inclusion of $\mathcal{C}'$ into $\mathcal{C}$ is an exact functor between stable ∞-categories, the same is true of its Ind-extension (A.2.24), by Remark A.2.16. Thus this Ind-extension is continuous. The source is furthermore cocomplete (being compactly generated and stable), and hence its essential image is a cocomplete and stable sub-∞-category of $\mathcal{C}$, containing $\mathcal{C}'$. Our assumption that $(\mathcal{C}')^\perp = 0$, interpreted in light of Remark A.2.22, shows that this essential image coincides with $\mathcal{C}$, as required. $\square$

The following standard result gives a convenient criterion to check the full faithfulness of a functor by checking it on compact generators.

Proposition A.2.25. *Let $F : \mathcal{C} \to \mathcal{D}$ be a continuous exact functor between compactly generated stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, and let X be a set of compact generators of $\mathcal{C}$. Suppose that F preserves compact objects. Then the following conditions are equivalent:*

(1) F is fully faithful, i.e. for all objects $x, y \in \mathcal{C}$, the induced map

$$\mathrm{Maps}_{\mathcal{C}}(x, y) \to \mathrm{Maps}_{\mathcal{D}}(F(x), F(y))$$

is an isomorphism.

(2) For all objects $x, y \in \mathcal{C}$, the induced map

$$\mathrm{RHom}_{\mathcal{C}}(x, y) \to \mathrm{RHom}_{\mathcal{D}}(F(x), F(y)) \tag{A.2.26}$$

is an isomorphism.

(3) (A.2.26) *is an isomorphism for all $x, y \in X$.*

(4) For all objects $x, y \in X$ and each $n \in \mathbf{Z}$, the induced map

$$\mathrm{Ext}^n_{\mathcal{C}}(x, y) \to \mathrm{Ext}^n_{\mathcal{D}}(F(x), F(y))$$

is an isomorphism.

Proof. The first and second conditions are equivalent by the discussion in Section A.2.11. More precisely, RHom is constructed as an Ω-spectrum from Maps, while Maps is recovered from RHom by applying Ω^∞.

The third and fourth conditions are equivalent by [Lur17, Remark 1.4.3.8]. The second condition trivially implies the third, and it remains to prove the converse.

Assume then that hypothesis (3) holds, and consider the (full, stable) subcategory Y of $\mathcal{C}$ consisting of the $y \in \mathcal{C}$ with the property that (A.2.26) is an isomorphism for all $x \in X$. By assumption, Y contains every $y \in X$. In order to show that $Y = \mathcal{C}$, it therefore suffices to show that Y is closed under colimits in $\mathcal{C}$, which follows by noting that for any $x \in X$ and any colimit $\operatorname{colim}_{i \in I} y_i$, we have (by the compactness of x and of $F(x)$, the continuity of F, and Lemma A.2.12 (3))

$$\begin{aligned}
\mathrm{RHom}_{\mathcal{C}}(x, \operatorname{colim}_{i\in I} y_i) &= \operatorname{colim}_{i\in I} \mathrm{RHom}_{\mathcal{C}}(x, y_i) \\
&= \operatorname{colim}_{i\in I} \mathrm{RHom}_{\mathcal{D}}(F(x), F(y_i)) \\
&= \mathrm{RHom}_{\mathcal{D}}(F(x), \operatorname{colim}_{i\in I} F(y_i)) \\
&= \mathrm{RHom}_{\mathcal{D}}(F(x), F(\operatorname{colim}_{i\in I} y_i)).
\end{aligned}$$

We now consider the (full, stable) subcategory Z of $\mathcal{C}$ consisting of the $x \in \mathcal{C}$ with the property that (A.2.26) is an isomorphism for all $y \in \mathcal{C}$. We have just seen that Z contains X, so to show that $X = \mathcal{C}$, it again suffices to show that Z is closed under colimits in $\mathcal{C}$.

To see this, we have (using Lemma A.2.12 (2) and the continuity of F)

$$\begin{aligned}
\mathrm{RHom}_{\mathcal{C}}(\operatorname{colim}_{i\in I} x_i, y) &= \lim_{i\in I} \mathrm{RHom}_{\mathcal{C}}(x_i, y) \\
&= \lim_{i\in I} \mathrm{RHom}_{\mathcal{D}}(F(x_i), F(y)) \\
&= \mathrm{RHom}_{\mathcal{D}}(\operatorname{colim}_{i\in I} F(x_i), F(y)) \\
&= \mathrm{RHom}_{\mathcal{D}}(F(\operatorname{colim}_{i\in I} x_i), F(y)).
\end{aligned}$$

Thus (A.2.26) is an isomorphism for all $x, y \in \mathcal{C}$, as required. $\square$

A.2.27. *Quotients.* An important role in our arguments will be played by *Verdier quotients* of stable ∞-categories. Given a stable sub-∞-category $\mathcal{A}$ of a stable ∞-category $\mathcal{B}$ (or, what is essentially the same data, an exact fully faithful functor $\mathcal{A} \hookrightarrow \mathcal{B}$ between stable ∞-categories), the Verdier quotient $\mathcal{B}/\mathcal{A}$ is a stable ∞-category receiving a functor $\mathcal{B} \to \mathcal{B}/\mathcal{A}$ whose composite with $\mathcal{A} \hookrightarrow \mathcal{B}$ is the zero functor, and which is universal (i.e. initial) for these properties. In what follows, we recall the construction of the Verdier quotient, and then discuss its interaction with the formation of Ind-categories.

The key point in the construction of $\mathcal{B}/\mathcal{A}$ comes by noting that, given an exact functor of stable ∞-categories $\mathcal{B} \to \mathcal{C}$, the objects in (the essential image of) $\mathcal{A}$ all map to zero precisely if all the arrows in $\mathcal{B}$ with fibres lying in (the essential image of) $\mathcal{A}$ become invertible in $\mathcal{C}$. Thus the construction of Verdier quotients becomes a special case of the construction of ∞-categorical localizations, a construction that we now briefly recall.

If $\mathcal{C}$ is an ∞-category, and W is a collection of arrows of $\mathcal{C}$, then the *localization* $\mathcal{C}[W^{-1}]$ is an ∞-category receiving a functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ under which all the elements of W become invertible, and which is universal (i.e. initial) for these properties. This construction (under the slightly more circumlocutious name of "the ∞-category obtained from $\mathcal{C}$ by inverting the set of morphisms W") is the subject of [Lur17, Def. 1.3.4.1, Rem. 1.3.4.2], and in the context of small ∞-categories it is also discussed in [NS18, §I.3], where it is called "Dwyer–Kan localization". It is also discussed in [Kerodon, Tag [01M4], where it is simply called "localization" (and it this last terminological choice that we follow).

In [NS18, Thm. I.3.3] it is shown that if $\mathcal{A}$ is a stable sub-∞-category of the stable ∞-category $\mathcal{B}$, and if W denotes the set of arrows in $\mathcal{B}$ whose fibre lies in $\mathcal{A}$, then the localization $\mathcal{B}[W^{-1}]$ is again stable, and hence (for the reason already explained above) satisfies the universal property of the Verdier quotient.

Remark A.2.28. As already mentioned, in [NS18, § I.3], a smallness hypothesis is imposed. In our setting, in which we use universes, this is not essential, since we can always enlarge the universe to make any given ∞-category small. One could worry that the universal property that $\mathcal{C}[W^{-1}]$ satisfies is universe dependent (i.e. that it ceases to hold if one enlarges the universe), but in fact this does not happen (i.e. the localization of $\mathcal{C}$ at W remains such after any enlargement of the universe). To see this, one notes [Kerodon, Tag [05ZN] that the size of any localization of $\mathcal{C}$ is bounded in terms of the size of $\mathcal{C}$ itself.

Among all localizations of ∞-categories, a particularly important class is given by the so-called *Bousfield localizations*. These are localizations for which the canonical functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ admits a fully faithful right adjoint. This right adjoint then realizes $\mathcal{C}[W^{-1}]$ as a full subcategory of $\mathcal{C}$ itself. In [Lur09] and [Lur17], these are referred to simply as "localizations", and are characterized in terms of the resulting full subcategories of $\mathcal{C}$; in [Kerodon] they are called "reflective localizations". See [Lur17, Ex. I.3.4.3] for one explanation of why these "localizations" (defined as they are in [Lur09], in terms of certain full subcategories of $\mathcal{C}$) are localizations in the sense that we are using here.

In practice, the right adjoint required for a Bousfield localization will be constructed via an application of the adjoint functor theorem (see the discussion below for more on this), and so Bousfield localizations typically arise in the context of large, rather than small, ∞-categories. We note, though, that [NS18, Lem. I.3.4] shows that one can use Yoneda embeddings to map (the opposite of) any localization functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ into an associated Bousfield localization of $\mathrm{Fun}(\mathcal{C}, \mathrm{Ani}) \to \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{Ani})$; thus Bousfield localizations can be used as a tool to study arbitrary localizations (such as localizations of small categories, which is the actual context under consideration in [NS18]).

Since Ind-categories especially lend themselves to the construction of right adjoints (again, see the discussion below), for our purposes, the role of Bousfield localizations will be to aid in the analysis of Verdier quotients of Ind-categories of stable ∞-categories. For example, if $\mathcal{A} \hookrightarrow \mathcal{B}$ is a fully faithful continuous functor of presentable stable ∞-categories, then it follows from [BGT13, Prop. 5.6] that the Verdier quotient $\mathcal{B}/\mathcal{A}$ is a Bousfield localization of $\mathcal{B}$.

The following proposition describes how the formation of Ind-categories and the formation of Verdier quotients interact.

Proposition A.2.29. *Let $\mathcal{A}$ be an idempotent complete stable sub-∞-category of the small idempotent complete stable ∞-category $\mathcal{B}$. Then the* Ind*-extension* $\mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{B}/\mathcal{A})$ *of the quotient functor $\mathcal{B} \to \mathcal{B}/\mathcal{A}$ (of small stable ∞-categories) is a Bousfield localization at the collection of arrows of* $\mathrm{Ind}(\mathcal{B})$ *whose cofibre lies in* $\mathrm{Ind}(\mathcal{A})$*. Furthermore, $\mathcal{A} = (\mathrm{Ind}\,\mathcal{A}) \cap \mathcal{B}$ (the intersection taking place in* $\mathrm{Ind}\,\mathcal{B}$*).*

Proof. The claim that $\mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{B}/\mathcal{A})$ is a Bousfield localization at the collection of arrows with cofibre in $\mathrm{Ind}(\mathcal{A})$ is part of [NS18, Prop. I.3.5]. It thus follows that the objects of $\mathrm{Ind}(\mathcal{A}) \cap \mathcal{B}$ are precisely the objects of $\mathcal{B}$ that become equivalent to 0 in $\mathcal{B}/\mathcal{A}$. Since $h(\mathcal{B}) \to h(\mathcal{B}/\mathcal{A})$ is by construction a Verdier quotient of triangulated categories, the equality $\mathrm{Ind}(\mathcal{A}) \cap \mathcal{B} = \mathcal{A}$ thus follows from [Stacks, Tag 05RK], since $\mathcal{A}$ is idempotent complete. □

Remark A.2.30. Suppose now that $\mathcal{C} \hookrightarrow \mathcal{D}$ is a fully faithful continuous functor between compactly generated stable ∞-categories, which furthermore preserves compact objects, so that it can be identified as the Ind-ification of its restriction $\mathcal{C}^c \hookrightarrow \mathcal{D}^c$ to the sub-∞-categories of compact objects. Then Proposition A.2.29 can be rephrased as the statement that the induced functor

$$\mathcal{D}^c/\mathcal{C}^c \to \mathcal{D}/\mathcal{C} = \mathrm{Ind}\,\mathcal{D}^c/\,\mathrm{Ind}\,\mathcal{C}^c$$

induces an equivalence

$$\mathrm{Ind}(\mathcal{D}^c/\mathcal{C}^c) \xrightarrow{\sim} \mathcal{D}/\mathcal{C}.$$

It thus follows that, in this case, the Verdier quotient $\mathcal{D}/\mathcal{C}$ is compactly generated.

A.2.31. *Adjoint functors.* By the adjoint functor theorem [Lur09, Cor. 5.5.2.9(1)], a functor $F : \mathcal{C} \to \mathcal{D}$ between cocomplete and compactly generated stable ∞-categories is continuous if and only if it admits a right adjoint G. In particular, if $F : \mathcal{C} \to \mathcal{D}$ is an exact functor between small stable ∞-categories then $F : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ always admits a right adjoint; and passing to opposite categories, we see that the induced functor $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$ always admits a left adjoint. In addition, if $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$, then the induced functors $F : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ and $G : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{C})$ are an adjoint pair. This can be seen either by the defining properties of the Ind extended functors, or by [Lur09, Prop. 5.3.5.13].

The following is a special case of [Lur09, Prop. 5.5.7.2].

Lemma A.2.32. *If $F : \mathcal{C} \to \mathcal{D}$ is a continuous functor between compactly generated stable ∞-categories, then F preserves compact objects if and only if its right adjoint G is continuous.*

A.3. **Recollements.** We will use the notion of recollements, which in the ∞-category setting were defined by Lurie in [Lur17, §A.8]. We recall the definition here, in the context of recollements of stable ∞-categories.

Definition A.3.1. Let $\mathcal{A}_U$, $\mathcal{A}_Z$, and $\mathcal{A}$ be stable ∞-categories, and let $i_* : \mathcal{A}_Z \to \mathcal{A}$ and $j_* : \mathcal{A}_U \to \mathcal{A}$ be fully faithful exact functors. We will often regard $\mathcal{A}_Z$ and $\mathcal{A}_U$ as full subcategories of $\mathcal{A}$ by identifying them with their essential images. We say that $\mathcal{A}$ is the *recollement* of $\mathcal{A}_Z$ and $\mathcal{A}_U$ if the following hypotheses hold:

(1) i_* and j_* admit (necessarily exact) left adjoints i^* and j^*.
(2) $j^*i_* = 0$.
(3) i^* and j^* are jointly conservative, in the sense that if $i^*X = 0$ and $j^*X = 0$ for some object X of $\mathcal{A}$, then $X = 0$.

We now describe a method for recognizing recollements. We will usually assume the following hypothesis.

Hypothesis A.3.2. Let $i_* : \mathcal{A}_Z \hookrightarrow \mathcal{A}$ be a fully faithful functor between compactly generated stable ∞-categories, which is continuous and preserves compact objects. (Equivalently, i_* is the Ind-extension of a fully faithful exact functor $i_* : \mathcal{A}_Z^c \hookrightarrow \mathcal{A}^c$.)

We will freely regard $\mathcal{A}_Z$ as a full subcategory of $\mathcal{A}$, by identifying it with the essential image of i_*. Set $\mathcal{A}_U \coloneqq \mathcal{A}/\mathcal{A}_Z$, and write $j^* : \mathcal{A} \to \mathcal{A}_U$ for the quotient functor.

The following lemma contains the specializations of some results from Section A.2.27 to the present context.

Lemma A.3.3. *Assume Hypothesis* A.3.2. *Then:*

(1) $\mathcal{A}_Z \cap \mathcal{A}^c = \mathcal{A}_Z^c$.

(2) $\mathcal{A}_U$ *is compactly generated, and the functor* j^* *is continuous and preserves compact objects. The kernel of* j^* *is* $\mathcal{A}_Z$.

(3) j^* *has a fully faithful continuous right adjoint* $j_* : \mathcal{A}_U \to \mathcal{A}$.

In particular, the functors i_*, j^*, j_* *are the Ind-extensions of their restrictions* $i_* : \mathcal{A}_Z^c \hookrightarrow \mathcal{A}^c$, $j^* : \mathcal{A}^c \to \mathcal{A}_U^c$, $j_* : \mathcal{A}_U^c \hookrightarrow \mathcal{A}$.

Proof. Applying Proposition A.2.29 to the inclusion $\mathcal{A}_Z^c \subseteq \mathcal{A}^c$, we see that $\mathcal{A}_Z \cap \mathcal{A}^c = \mathcal{A}_Z^c$, that j^* is the Ind-extension of the quotient functor $\mathcal{A}^c \to \mathcal{A}^c/\mathcal{A}_Z^c$ (hence it preserves compact objects), and that the kernel of j^* is $\mathcal{A}_Z$. Furthermore, j^* has a fully faithful right adjoint j_* (and so in particular is continuous), because it is a Bousfield localization, and $\mathcal{A}_U$ is compactly generated, because it is equivalent to $\mathrm{Ind}(\mathcal{A}^c/\mathcal{A}_Z^c)$. □

We can now state our criterion for recognizing recollements.

Lemma A.3.4. *Assume Hypothesis* A.3.2. *Then the following conditions are equivalent:*

(1) $\mathcal{A}$ *is the recollement of* $\mathcal{A}_Z$ *and* $\mathcal{A}_U$.

(2) i_* *admits a left adjoint* i^*.

(3) $i_*|_{\mathcal{A}_Z^c} : \mathcal{A}_Z^c \to \mathcal{A}^c$ *admits a left adjoint.*

Proof. By definition, (1) implies (2). For the converse implication, by Lemma A.3.3, the only point to verify is that if X is an object of $\mathcal{A}$ with $i^*X = 0$ and $j^*X = 0$ then $X = 0$. But if $j^*X = 0$, then X is isomorphic to i_*Y for some $Y \in \mathcal{A}_Z$. Now i_* is fully faithful, so the counit $i^*i_*Y \to Y$ is an isomorphism. If furthermore $i^*X = 0$, this implies $Y = 0$, hence $X = 0$, as desired.

We now turn to the equivalence of (2) and (3). If (3) holds, then passing to Ind-categories gives (2), by the discussion of Section A.2.31. Conversely, by Lemma A.2.32 since i_* is continuous, any left adjoint i^* preserves compact objects, and so restricts to a left adjoint of $i_*|_{\mathcal{A}_Z^c}$. □

Consider now a continuous functor $F : \mathcal{A} \to \mathcal{A}'$ of compactly generated stable ∞-categories, each of which satisfies Hypothesis A.3.2; that is, we have fully faithful, continuous, and compact object preserving functors of compactly generated stable ∞-categories $i_* : \mathcal{A}_Z \hookrightarrow \mathcal{A}$ and $i'_* : \mathcal{A}'_Z \hookrightarrow \mathcal{A}'$. We say that $F(\mathcal{A}_Z) \subseteq \mathcal{A}'_Z$ if the essential image of Fi_* is a subcategory of the essential image of i'_*.

Lemma A.3.5. *The following conditions are equivalent:*

(1) $F(\mathcal{A}_Z) \subseteq \mathcal{A}'_Z$.

(2) There exists a continuous functor $F_{\mathcal{A}_Z} : \mathcal{A}_Z \to \mathcal{A}'_Z$, *characterized by the existence of a natural isomorphism*

$$Fi_* \xrightarrow{\sim} i'_* F_{\mathcal{A}_Z}. \tag{A.3.6}$$

(3) *There exists a continuous functor* $F_{\mathcal{A}_U} : \mathcal{A}_U := \mathcal{A}/\mathcal{A}_Z \to \mathcal{A}'_U := \mathcal{A}'/\mathcal{A}'_Z$, *characterized by the existence of a natural isomorphism*

$$j'^*F \xrightarrow{\sim} F_{\mathcal{A}_U} j^*. \tag{A.3.7}$$

(4) $j'^*Fi_* = 0$.

Proof. The first, second and fourth conditions are equivalent by definition. The third implies the fourth, because $j^*i_* = 0$. Conversely, if the fourth condition holds, then $j'^*F : \mathcal{A} \to \mathcal{A}'_U$ is the zero functor on $\mathcal{A}_Z$, so induces a functor $F_{\mathcal{A}_U} : \mathcal{A}_U \to \mathcal{A}'_U$ and a natural isomorphism as in (3), by the universal property of the quotient. □

Suppose now given $\mathcal{A}$, $\mathcal{A}'$, and $F : \mathcal{A} \to \mathcal{A}'$ as above, and suppose that $F(\mathcal{A}_Z) \subseteq \mathcal{A}'_Z$, so that the equivalent conditions of Lemma A.3.5 hold. We will write η, η' (resp. ϵ, ϵ') for the units (resp. the counits) of the adjunctions $(j^*, j_*), (j'^*, j'_*)$. Precomposing the unit $\eta' : \mathrm{id}_{\mathcal{A}'} \to j'_*j'^*$ with Fj_*, and taking into account (A.3.7), we obtain a natural transformation

$$Fj_* \xrightarrow{\eta' Fj_*} j'_*j'^*Fj_* \xrightarrow{j'_*(\text{A.3.7})j_*} j'_*F_{\mathcal{A}_U}j^*j_* \xrightarrow{j'_*F_{\mathcal{A}_U}\epsilon} j'_*F_{\mathcal{A}_U}, \tag{A.3.8}$$

which need not be an isomorphism in general.

Now suppose further that that each of i_* and i'_* admits a left adjoint i^* and i'^*, so that by Lemma A.3.4, each of $\mathcal{A}$ and $\mathcal{A}'$ is a recollement. Postcomposing (A.3.6) with i'^*, and recalling that i'_* is fully faithful (and so the counit is an isomorphism), we obtain an isomorphism

$$i'^*Fi_* \xrightarrow{\sim} i'^*i'_*F_{\mathcal{A}_Z} \xrightarrow{\sim} F_{\mathcal{A}_Z}.$$

It follows that if we postcompose the unit of adjunction $1_{\mathcal{A}} \to i_*i^*$ with i'^*F, then we obtain a natural transformation

$$i'^*F \to i'^*Fi_*i^* \xrightarrow{\sim} F_{\mathcal{A}_Z}i^*, \tag{A.3.9}$$

which again need not be an isomorphism in general.

Remark A.3.10. Using terminology from [Sha22, Def. 2.3, Def. 2.6], if each of $\mathcal{A}$, $\mathcal{A}'$ is a recollement, then a functor F that satisfies the conditions of Lemma A.3.5 defines a *morphism of recollements* if and only if (A.3.9) is an isomorphism, and a *strict morphism of recollements* if and only if both (A.3.8) and (A.3.9) are isomorphisms.

In the framework just described, we have the following criterion for F to be fully faithful.

Proposition A.3.11. *Suppose that* $F : \mathcal{A} \to \mathcal{A}'$ *is a continuous functor between stable* ∞*-categories each satisfying Hypothesis* A.3.2*, and suppose that* $F(\mathcal{A}_Z) \subseteq \mathcal{A}'_Z$*. Suppose furthermore that*

(1) *each of* $i_* : \mathcal{A}_Z \to \mathcal{A}$ *and* $i'_* : \mathcal{A}'_Z \to \mathcal{A}'$ *admits a left adjoint* i^* *and* i'^* *respectively;*
(2) *the natural transformations* (A.3.8) *and* (A.3.9) *are natural isomorphisms.*

Then if each of $F_{\mathcal{A}_U}$ *and* $F_{\mathcal{A}_Z}$ *are fully faithful, so is* F*.*

Proof. Write $F(\mathcal{A}), F_{\mathcal{A}_Z}(\mathcal{A}_Z), F_{\mathcal{A}_U}(\mathcal{A}_U)$ for the essential images of $F, F_{\mathcal{A}_Z}, F_{\mathcal{A}_U}$. The natural isomorphisms (A.3.6) and (A.3.7) imply that we have a commutative diagram

$$
\begin{tikzcd}
\mathcal{A}_Z \arrow[r, "i_*"] \arrow[d, "F_{\mathcal{A}_Z}"] & \mathcal{A} \arrow[r, "j^*"] \arrow[d, "F"] & \mathcal{A}_U \arrow[d, "F_{\mathcal{A}_U}"] \\
F_{\mathcal{A}_Z}(\mathcal{A}_Z) \arrow[r, "i'_*"] & F(\mathcal{A}) \arrow[r, "j'^*"] & F_{\mathcal{A}_U}(\mathcal{A}_U).
\end{tikzcd}
\tag{A.3.12}
$$

Assumption (1) and Lemma A.3.4 show that $\mathcal{A}$ is the recollement of $\mathcal{A}_Z$ and $\mathcal{A}_U$ via i_* and j_*. Assumption (2) implies in particular that j'_* and i'^* restrict to functors $j'_* : F_{\mathcal{A}_U}(\mathcal{A}_U) \to F(\mathcal{A})$ and $F(\mathcal{A}) \to F_{\mathcal{A}_Z}(\mathcal{A}_Z)$, which are then adjoint to (the appropriate restrictions of) i'_* and j'^*. We therefore see that all the properties in Definition A.3.1 are satisfied, and so $F(\mathcal{A})$ is the recollement of $F_{\mathcal{A}_Z}(\mathcal{A}_Z)$ and $F_{\mathcal{A}_U}(\mathcal{A}_U)$ via i'_* and j'_*. Furthermore, assumption (2) also shows that (A.3.12) is a "strict morphism of recollements" in the sense of [Sha22, Def. 2.6] (compare Remark A.3.10). We can therefore replace $\mathcal{A}'$, $\mathcal{A}'_Z$, $\mathcal{A}'_U$ by $F(\mathcal{A})$, $F_{\mathcal{A}_Z}(\mathcal{A}_Z)$, $F_{\mathcal{A}_U}(\mathcal{A}_U)$ respectively, thus reducing ourselves to the case that $F_{\mathcal{A}_Z}$ and $F_{\mathcal{A}_U}$ are equivalences of categories. We may then apply [Sha22, Rem. 2.7], or equivalently [Lur17, Prop. A.8.14], to conclude that F is also an equivalence, as required. □

A.4. **Enlarging the universe, and the Ind Pro construction.** In the sequel we will use Ind Pro categories. In order to deal with the size issues that this entails, we make a digression on change of universe.

A.4.1. *Change of universe.* Writing U for the universe (implicitly) considered in Section A.2, we now choose another universe V with $U \in V$. We wish to compare size-related notions, such as smallness, small colimits, and compact objects, with respect to the two universes U and V.

To this end, we now say that an ∞-category $\mathcal{C}$ is U-small if it is small with respect to U. Given a U-small ∞-category $\mathcal{C}$, we let $\mathrm{Ind}^U \mathcal{C}$ denote the Ind category of $\mathcal{C}$ built by formally adjoining U-small filtered colimits. Given an ∞-category $\mathcal{C}$, we say that an object X in $\mathcal{C}$ is U-compact if $\mathrm{Maps}_{\mathcal{C}}(X, -)$ commutes with the formation of U-small filtered colimits, and write $\mathcal{C}^{U-c}$ to denote the full sub-∞-category of $\mathcal{C}$ consisting of U-compact objects. We say that $\mathcal{C}$ is U-compactly generated if it admits U-small filtered colimits, if $\mathcal{C}^{U-c}$ is U-small, and if $\mathrm{Ind}^U \mathcal{C}^{U-c} \to \mathcal{C}$ is an equivalence.

All these notions have evident V-analogues, given by replacing the universe U by the universe V in the definitions. It is evident that, on a literal level, the notions of U-compactly generated and V-compactly generated objects are distinct. However, there is in fact a sense in which compactly generated categories are insensitive to the choice of universe.[14] Namely, if $\mathcal{C}$ is U-compactly generated, so that

$$\mathrm{Ind}^U \mathcal{C}^{U-c} \xrightarrow{\sim} \mathcal{C},$$

then we define

$$\mathcal{C}^V := \mathrm{Ind}^V \mathcal{C}^{U-c}.$$

This definition makes sense, because the U-small category $\mathcal{C}^{U-c}$ is in particular V-small. By construction, $\mathcal{C}^V$ is V-compactly generated, and $(\mathcal{C}^V)^{V-c} = \mathcal{C}^{U-c}$. Thus $\mathcal{C}^V$ "promotes" $\mathcal{C}$ to a V-compactly generated ∞-category whose V-compact

[14]We could phrase this in a precise mathematical manner by describing the construction $\mathcal{C} \mapsto \mathcal{C}^V$ that we define below as inducing an equivalence of certain ∞-categories of ∞-categories, although we don't do that here.

objects coincide with the U-compact objects in the original ∞-category $\mathcal{C}$. One can think of $\mathcal{C}^V$ as a "version" of $\mathcal{C}$ which is adapted to the larger universe V.

Similarly, any U-continuous functor (i.e. a functor compatible with U-small filtered colimits) $F : \mathcal{C} \to \mathcal{D}$ between U-compactly generated ∞-categories extends canonically to a V-continuous functor $F^V : \mathcal{C}^V \to \mathcal{D}^V$. If F preserves U-compact objects, then F^V preserves V-compact objects.

A.4.2. *Ind Pro categories.* As we already explained, our motivation for introducing change-of-universe considerations is so that we may consider Ind Pro categories. More precisely, if $\mathcal{C}$ is a U-small ∞-category, then we may define $\operatorname{Pro}^U \mathcal{C}$ in the obvious manner. This is no longer a U-small category, but it *is* V-small (e.g. by [Hen17, Thm. 1(i)]), and so we may then define the Ind completion

$$\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C},$$

a V-compactly generated ∞-category.

We note the following lemma related to computing morphisms in $\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}$.

Lemma A.4.3. *Let $\mathcal{C}$ be a U-small ∞-category, let $X = \lim_{i\in I} X_i$ be an object of $\operatorname{Pro}^U \mathcal{C}$, written as a U-small cofiltered limit of objects X_i of $\mathcal{C}$, and let Y be an object of $\operatorname{Ind}^V \mathcal{C}$. Then the fully faithful embeddings $\operatorname{Pro}^U \mathcal{C} \hookrightarrow \operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}$ and $\operatorname{Ind}^V \mathcal{C} \hookrightarrow \operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}$ allow us to regard both these objects as belonging to $\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}$, and we have*

$$\operatorname{Maps}_{\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}}(X, Y) \xrightarrow{\sim} \operatorname{colim}_i \operatorname{Maps}_{\operatorname{Ind}^V \mathcal{C}}(X_i, Y).$$

Proof. We may write $Y = \operatorname{colim}_{j\in J} Y_j$, a V-small filtered colimit of objects Y_j of $\mathcal{C}$. Furthermore, the embedding $\operatorname{Ind}^V \mathcal{C} \hookrightarrow \operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}$ preserves V-small filtered colimits. Then

$$\begin{aligned}\operatorname{Maps}_{\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}}(X, \operatorname{colim}_j Y_j) &\xrightarrow{\sim} \operatorname{colim}_j \operatorname{Maps}_{\operatorname{Pro}^U \mathcal{C}}(X, Y_j)\\ &\xrightarrow{\sim} \operatorname{colim}_j \operatorname{colim}_i \operatorname{Maps}_{\mathcal{C}}(X_i, Y_j) \xrightarrow{\sim} \operatorname{colim}_i \operatorname{colim}_j \operatorname{Maps}_{\mathcal{C}}(X_i, Y_j)\\ &\xrightarrow{\sim} \operatorname{colim}_i \operatorname{Maps}_{\operatorname{Ind}^V \mathcal{C}}(X_i, \operatorname{colim}_j Y_j),\end{aligned}$$

as claimed. $\square$

Suppose now that $F : \mathcal{C} \to \mathcal{D}$ is an exact functor between U-small (and hence V-small) stable ∞-categories, so that the induced functor $F_{\operatorname{Ind}^V \mathcal{C}} : \operatorname{Ind}^V \mathcal{C} \to \operatorname{Ind}^V \mathcal{D}$ admits a right adjoint $G : \operatorname{Ind}^V \mathcal{D} \to \operatorname{Ind}^V \mathcal{C}$. The functor F also induces an exact functor between V-small stable ∞-categories $F_{\operatorname{Pro}^U \mathcal{C}} : \operatorname{Pro}^U \mathcal{C} \to \operatorname{Pro}^U \mathcal{D}$, whose Ind-ification

$$F_{\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}} : \operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C} \to \operatorname{Ind}^V \operatorname{Pro}^U \mathcal{D}$$

similarly admits a right adjoint

$$G_{\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{D}} : \operatorname{Ind}^V \operatorname{Pro}^U \mathcal{D} \to \operatorname{Ind}^V \operatorname{Pro}^U \mathcal{C}.$$

The following lemma shows that $G_{\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{D}}$ and G are compatible in the evident way.

Lemma A.4.4. *The functor $G_{\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{D}}$ is continuous, and so is determined by its restriction to $\operatorname{Pro}^U \mathcal{D}$. Furthermore, if $Y := \lim_{j\in J} Y_j$ is an object of $\operatorname{Pro}^U \mathcal{D}$, written as a cofiltered limit of objects Y_j of $\mathcal{D}$, then*

$$G_{\operatorname{Ind}^V \operatorname{Pro}^U \mathcal{D}}(Y) \xrightarrow{\sim} \lim_{j\in J} G(Y_j).$$

(The limit exists in $\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}$.*) In particular, the restrictions of* $G_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{D}}$ *and of* G *to* $\mathcal{D}$ *coincide.*

Proof. Since $F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}$ is defined as the Ind-ification of $F_{\mathrm{Pro}^U \mathcal{C}}$, it preserves V-compact objects, and so its right adjoint $G_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{D}}$ is continuous by Lemma A.2.32. Furthermore, right adjoints necessarily commute with limits (which also implies the claimed existence of $\lim_{j\in J} G(Y_j)$ in $\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}$). Thus it remains to prove the final claim of the lemma, namely that the restrictions to $\mathcal{D}$ of $G_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{D}}$ and of G coincide.

Let Y be an object of $\mathcal{D}$. Then $G(Y)$ is an object of $\mathrm{Ind}^V \mathcal{C}$, and we must exhibit a canonical equivalence

$$\mathrm{Maps}_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}\big(X, G(Y)\big) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{D}}\big(F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}(X), Y\big)$$

for any object X of $\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}$. Since $F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}$ is continuous, the formation of both mapping spaces is compatible with the formation of colimits in X, and so it suffices to consider the case when $X = \lim_{i\in I} X_i$ is an object of $\mathrm{Pro}^U \mathcal{C}$, written as a cofiltered limit of objects X_i of $\mathcal{C}$. We then find that

$$\begin{aligned}
&\mathrm{Maps}_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}\big(X, G(Y)\big) \xrightarrow{\sim} \mathrm{colim}_i \, \mathrm{Maps}_{\mathrm{Ind}^V \mathcal{C}}\big(X_i, G(Y)\big) \\
&\quad \xrightarrow{\sim} \mathrm{colim}_i \, \mathrm{Maps}_{\mathrm{Ind}^V \mathcal{D}}\big(F(X_i), Y\big) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{D}}\big(\lim_{i\in I} F(X_i), Y\big) \\
&\qquad \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{D}}\big(F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}(\lim_{i\in I} X_i), Y\big),
\end{aligned}$$

the first and third equivalences being provided by Lemma A.4.3, the second equivalence being provided by the adjunction between $F_{\mathrm{Ind}^V \mathcal{C}}$ and G (and recalling that $F_{\mathrm{Ind}^V \mathcal{C}}$ is the Ind-ification of F, and so coincides with F on objects of $\mathcal{C}$), and the final equivalence being provided by the definition of $F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}$ as the Ind Pro-ification of F. $\square$

We also have the following useful lemmas.

Lemma A.4.5. $F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}$ *commutes with all limits and colimits in* $\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}$.

Proof. We already observed that $F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}$ admits a right adjoint $G_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{D}}$, so it commutes with colimits. Similarly, the functor $F_{\mathrm{Pro}^U \mathcal{C}} : \mathrm{Pro}^U \mathcal{C} \to \mathrm{Pro}^U \mathcal{D}$ admits a left adjoint, and thus so does its Ind-extension $F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}$; so $F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}$ also commutes with limits. $\square$

Lemma A.4.6. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be an exact functor between* U*-small stable* ∞*-categories. Then the following conditions are equivalent.*

(1) F *is fully faithful.*
(2) $F_{\mathrm{Ind}^V \mathcal{C}}$ *is fully faithful.*
(3) $F_{\mathrm{Pro}^U \mathcal{C}}$ *is fully faithful.*
(4) $F_{\mathrm{Ind}^V \mathrm{Pro}^U \mathcal{C}}$ *is fully faithful.*

Proof. Since the restriction of $F_{\mathrm{Ind}^V \mathcal{C}}$ to $\mathcal{C}$ agrees with F by definition, the second condition implies the first. Conversely, suppose that F is fully faithful, and let X, Y be arbitrary objects of $\mathrm{Ind}^V \mathcal{C}$, which we write as filtered colimits $X = \mathrm{colim}_{i\in I} X_i$, $Y = \mathrm{colim}_{j\in J} Y_j$ of objects X_i, Y_j of $\mathcal{C}$. Then $F_{\mathrm{Ind}^V \mathcal{C}}(X) = \mathrm{colim}_{i\in I} F(X_i)$,

$F_{\mathrm{Ind}^V \mathcal{C}}(Y) = \mathrm{colim}_{j\in J} F(Y_j)$, so we have

$$\begin{aligned}\mathrm{Maps}_{\mathrm{Ind}^V \mathcal{D}}(F_{\mathrm{Ind}^V \mathcal{C}}(X), F_{\mathrm{Ind}^V \mathcal{C}}(Y)) &= \lim_{i\in I} \mathrm{colim}_{j\in J} \mathrm{Maps}_{\mathcal{D}}(F(X_i), F(Y_j)) \\ &= \lim_{i\in I} \mathrm{colim}_{j\in J} \mathrm{Maps}_{\mathcal{C}}(X_i, Y_j) = \mathrm{Maps}_{\mathrm{Ind}^V \mathcal{C}}(X, Y).\end{aligned}$$

This shows that the first two conditions are equivalent; the remaining equivalences follow by analogous arguments which we leave to the reader. □

A.4.7. *Notational conventions.* When we apply the preceding material, we will again suppress the choice of V (as well as of the original universe U), and will simply write $\mathrm{Ind}\,\mathrm{Pro}\,\mathcal{C}$. When we implement this construction, it will be furthermore understood that all U-compactly generated categories $\mathcal{D}$ under consideration, and any U-continuous functors F between them, are replaced by their "V-versions" $\mathcal{D}^V$ and F^V. This stipulation is required because, for example, we will want to view a U-compactly generated category $\mathcal{A}$ as a full subcategory of $\mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}^c)$, and we will do this by means of the canonical embedding

$$\mathcal{A}^V = \mathrm{Ind}^V(\mathcal{A}^c) \to \mathrm{Ind}^V \mathrm{Pro}^U(\mathcal{A}^c).$$

A.5. **An application of Ind Pro categories to full faithfulness.** We prove a criterion for full faithfulness (Proposition A.5.3 below) which will in turn be used in the proof of our main theorem.

A.5.1. *Constructing a recollement.* We return to the setting of Hypothesis A.3.2, so that we assume given a fully faithful, continuous, compact object preserving functor $i_* : \mathcal{A}_Z \hookrightarrow \mathcal{A}$ of compactly generated stable ∞-categories, with a corresponding Verdier quotient functor $j^* : \mathcal{A} \to \mathcal{A}_Z$ which (by Lemma A.3.3) is also continuous and compact object preserving. In fact, Lemma A.3.3 shows that this data is obtained by Ind-extending the corresponding functors $i_* : \mathcal{A}_Z^c \hookrightarrow \mathcal{A}^c$ and $j^* : \mathcal{A}^c \to \mathcal{A}_U^c$ obtained by restriction to the various full sub-stable-∞-categories of compact objects.

Starting with these functors on the categories of compact objects, we now take Pro-extensions, to obtain functors that we denote $\widehat{i}_* : \mathrm{Pro}\,\mathcal{A}_Z^c \hookrightarrow \mathrm{Pro}\,\mathcal{A}^c$ and $\widehat{j}^* : \mathrm{Pro}\,\mathcal{A}^c \to \mathrm{Pro}\,\mathcal{A}_U^c$. The evident Pro-analogue of Proposition A.2.29 shows that $\widehat{j}^*$ realizes $\mathrm{Pro}\,\mathcal{A}_U^c$ as the Verdier quotient $\mathrm{Pro}\,\mathcal{A}^c/\mathrm{Pro}\,\mathcal{A}_Z^c$.

Next we Ind-extend these Pro-extended functors (using a change of universe, as discussed in the preceding section, in order to regard these Pro-categories as being small) to obtain functors (notated in the same manner)

$$\widehat{i}_* : \mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}_Z^c \hookrightarrow \mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}^c$$

and

$$\widehat{j}^* : \mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}^c \to \mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}_U^c.$$

Again, Proposition A.2.29 shows that $\widehat{j}^*$ realizes $\mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}_U^c$ as the Verdier quotient $\mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}^c/\mathrm{Ind}\,\mathrm{Pro}\,\mathcal{A}_Z^c$. Furthermore, we are now again in the context of Hypothesis A.3.2, with the the role of $\mathcal{A}_Z^c$, $\mathcal{A}^c$, and $\mathcal{A}_U^c$ now being played by $\mathrm{Pro}\,\mathcal{A}_Z^c$, $\mathrm{Pro}\,\mathcal{A}^c$, and $\mathrm{Pro}\,\mathcal{A}_U^c$.

We can now apply Lemma A.3.3 twice. Firstly, returing to the context of $i_* : \mathcal{A}_Z \hookrightarrow \mathcal{A}$ and its quotient $\mathcal{A}_U$, we obtain the fully faithful right adjoint to j^*, which (following the notation of that Lemma) we denote by $j_* : \mathcal{A}_U \hookrightarrow \mathcal{A}$. But secondly, considering the context of $\widehat{i}_* : \operatorname{Ind}\operatorname{Pro}\mathcal{A}_Z^c \hookrightarrow \operatorname{Ind}\operatorname{Pro}\mathcal{A}^c$ and its quotient $\operatorname{Ind}\operatorname{Pro}\mathcal{A}_U^c$, we obtain the fully faithful right adjoint to $\widehat{j}^*$, which we denote by

$$\widehat{j}_* : \operatorname{Ind}\operatorname{Pro}\mathcal{A}_U^c \hookrightarrow \operatorname{Ind}\operatorname{Pro}\mathcal{A}^c.$$

We can compare these two constructions. Indeed, the canonical fully faithful embeddings $\mathcal{A}_U^c \hookrightarrow \operatorname{Pro}\mathcal{A}_U^c$ and $\mathcal{A}^c \hookrightarrow \operatorname{Pro}\mathcal{A}^c$ induce fully faithful embeddings

$$\mathcal{A}_U \xrightarrow{\sim} \operatorname{Ind}\mathcal{A}_U^c \hookrightarrow \operatorname{Ind}\operatorname{Pro}\mathcal{A}_U^c$$

and

$$\mathcal{A} \xrightarrow{\sim} \operatorname{Ind}\mathcal{A}^c \hookrightarrow \operatorname{Ind}\operatorname{Pro}\mathcal{A}^c,$$

and we see that we are in a particular instance of the general setting of Lemma A.4.4. That lemma then shows that $\widehat{j}_*$ restricts to j_*.

In the present situation, where we have made a preliminary Pro-extension before performing an Ind-extension, there is one more functor that arises (that is not part of the general package coming from Hypothesis A.3.2). Namely, the Pro-extended functor $\widehat{i}_*$ has a left adjoint

$$\widehat{i}^* : \operatorname{Pro}\mathcal{A}^c \to \operatorname{Pro}\mathcal{A}_Z^c.$$

Then, since Ind-extension preserves adjunctions, we see that its Ind-extension

$$\widehat{i}^* : \operatorname{Ind}\operatorname{Pro}\mathcal{A}^c \to \operatorname{Ind}\operatorname{Pro}\mathcal{A}_Z^c$$

(which we denote by the same symbol) is left adjoint to (the Ind-extended version of) $\widehat{i}_*$. Lemma A.3.4 then shows that the category $\operatorname{Ind}\operatorname{Pro}\mathcal{A}^c$ is the recollement of $\operatorname{Ind}\operatorname{Pro}\mathcal{A}_Z^c$ and $\operatorname{Ind}\operatorname{Pro}\mathcal{A}_U^c$, via $\widehat{i}_*$ and $\widehat{j}_*$.

A.5.2. *A criterion for full faithfulness.* Suppose now that we have a functor $F_{\mathcal{A}^c} : \mathcal{A}^c \to (\mathcal{A}')^c$ with Ind-extension $F : \mathcal{A} \to \mathcal{A}'$, Pro extension $F_{\operatorname{Pro}\mathcal{A}^c} : \operatorname{Pro}\mathcal{A}^c \to \operatorname{Pro}(\mathcal{A}')^c$, and Ind Pro extension $F_{\operatorname{Ind}\operatorname{Pro}\mathcal{A}^c} : \operatorname{Ind}\operatorname{Pro}\mathcal{A}^c \to \operatorname{Ind}\operatorname{Pro}(\mathcal{A}')^c$. Suppose also that we have a stable sub-∞-category $i'_* : (\mathcal{A}'_Z)^c \hookrightarrow (\mathcal{A}')^c$, and write $\mathcal{A}'_U$, $\widehat{i}'_*$ etc. for the corresponding constructions; and suppose further that $F_{\mathcal{A}^c}(\mathcal{A}_Z^c) \subseteq (\mathcal{A}'_Z)^c$, so that $F_{\mathcal{A}^c}$ restricts to a functor $F_{\mathcal{A}_Z^c} : \mathcal{A}_Z^c \to (\mathcal{A}'_Z)^c$. Then $F_{\mathcal{A}_Z^c}$ induces functors $F_{\mathcal{A}_Z} : \mathcal{A}_Z \to \mathcal{A}'_Z$, $F_{\operatorname{Ind}\operatorname{Pro}\mathcal{A}_Z^c} : \operatorname{Ind}\operatorname{Pro}\mathcal{A}_Z^c \to \operatorname{Ind}\operatorname{Pro}(\mathcal{A}'_Z)^c$, and functors $F_{\mathcal{A}_U^c} : \mathcal{A}_U^c \to (\mathcal{A}'_U)^c$, $F_{\mathcal{A}_U} : \mathcal{A}_U \to \mathcal{A}'_U$ and $F_{\operatorname{Ind}\operatorname{Pro}\mathcal{A}_U^c} : \operatorname{Ind}\operatorname{Pro}\mathcal{A}_U^c \to \operatorname{Ind}\operatorname{Pro}(\mathcal{A}'_U)^c$.

The following criterion for F to be fully faithful is used to prove the main theorem of the paper (see Theorem 5.5.1).

Proposition A.5.3. *Suppose, with notation as above, that the functor $F_{\mathcal{A}^c} : \mathcal{A}^c \to (\mathcal{A}')^c$ satisfies:*

(1) $F_{\mathcal{A}_U^c}$ and $F_{\mathcal{A}_Z^c}$ are fully faithful.

(2) The natural transformation of functors $\mathcal{A}_U^c \to \mathcal{A}'$

$$Fj_* \to j'_* F_{\mathcal{A}_U}$$

(arising from precomposing the unit of adjunction $\mathrm{id}_{\mathcal{A}'} \to j'_* j'^*$ with Fj_*) is a natural isomorphism.

(3) The natural transformation of functors $\mathcal{A}^c \to \operatorname{Pro}(\mathcal{A}'_Z)^c$

$$\widehat{i}'^* F_{\mathcal{A}^c} \to F_{\operatorname{Pro}\mathcal{A}_Z^c}\widehat{i}^*$$

(arising from postcomposing $\widehat{i}'^* F_{\operatorname{Pro}\mathcal{A}^c}$ with the unit of adjunction $1_{\mathcal{A}} \to \widehat{i}_*\widehat{i}^*$) is a natural isomorphism.

Then F and $F_{\mathcal{A}^c}$ are fully faithful.

Proof. By Lemma A.4.6, assumption (1) implies that $F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c_U}$ and $F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c_Z}$ are fully faithful, and that the proposition is equivalent to proving that $F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c}$ is fully faithful. We will deduce this from Proposition A.3.11, applied with the functor $F : \mathcal{A} \to \mathcal{A}'$ in that proposition being our functor $F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c} : \mathrm{Ind\,Pro}\,\mathcal{A}^c \to \mathrm{Ind\,Pro}(\mathcal{A}')^c$. In order to apply this result, we need to verify that Hypothesis A.3.2 holds (which we have already done, in the paragraph preceding the statement of this proposition) and to show that the natural transformations

$$F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c}\widehat{j}_* \to \widehat{j}'_* F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c_U} \tag{A.5.4}$$

and

$$\widehat{i}'^* F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c} \to F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c_Z}\widehat{i}^* \tag{A.5.5}$$

are natural isomorphisms. (These are the natural transformations (A.3.8) and (A.3.9), with $\mathcal{A}$ there being our $\mathrm{Ind\,Pro}\,\mathcal{A}^c$.)

These statements follow easily from our assumptions (2) and (3), using Lemma A.4.5 (for $F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c_U}$ and $F_{\mathrm{Ind\,Pro}\,\mathcal{A}^c_Z}$), Lemma A.4.4 (for $\widehat{j}_*$ and $\widehat{j}'_*$), and the compatibility of $\widehat{i}^*, \widehat{i}'^*$ with cofiltered limits. □

A.6. ***t*-structures.** We recall some of the key facts about t-structures that we will use. We refer to [EGH25, App. A] for a more complete recollection, as well as further references.

The data of a t-structure on the stable ∞-category $\mathcal{D}$ is determined by the specification of a full sub-∞-category $\mathcal{D}^{\le 0} \hookrightarrow \mathcal{D}$. The full sub-$\infty$-category $\mathcal{D}^{\ge 1} \hookrightarrow \mathcal{D}$ is then defined to be the right orthogonal to $\mathcal{D}^{\le 0}$. Of course $\mathcal{D}^{\le 0}$ should satisfy some conditions (e.g. $\mathcal{D}^{\le 0}[1] \subseteq \mathcal{D}^{\le 0}$), which amount to the requirement that the image of $\mathcal{D}^{\le 0}$ and $\mathcal{D}^{\ge 1}$ in the homotopy category of $\mathcal{D}$ define a t-structure on this homotopy category in the usual sense.

For any integer n, we define $\mathcal{D}^{\le n} := \mathcal{D}^{\le 0}[-n]$, and $\mathcal{D}^{\ge n} := \mathcal{D}^{\ge 1}[1-n]$. For any pair of integers $a \le b$, we write $\mathcal{D}^{[a,b]} := \mathcal{D}^{\ge a} \cap \mathcal{D}^{\le b}$. We say that an object X of $\mathcal{D}$ is *bounded above* (resp. *below*) if it lies in $\mathcal{D}^{\le n}$ (resp. $\mathcal{D}^{\ge n}$) for some n.

We say that the t-structure on $\mathcal{D}$ is *bounded* (resp. *bounded above*, resp. *bounded below*) if every object of $\mathcal{D}$ is bounded (resp. bounded above, resp. bounded below).

The heart of the t-structure, often denoted $\mathcal{D}^{\heartsuit}$, is defined to be $\mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0} =: \mathcal{D}^{[0,0]}$. It is an abelian category.

Each inclusion $\mathcal{D}^{\le n} \hookrightarrow \mathcal{D}$ admits a right adjoint $\tau^{\le n} : \mathcal{D} \to \mathcal{D}^{\le n}$, while each inclusion $\mathcal{D}^{\ge n} \hookrightarrow \mathcal{D}$ admits a left adjoint $\tau^{\ge n} : \mathcal{D} \to \mathcal{D}^{\ge n}$. These satisfy various relations; for example, if $a \le b$ then there is a natural isomorphism $\tau^{\ge a}\tau^{\le b} \xrightarrow{\sim} \tau^{\le b}\tau^{\ge a}$ of functors $\mathcal{D} \to \mathcal{D}^{[a,b]}$. For any integer n and any object X of $\mathcal{D}$, $\tau^{\ge n}\tau^{\le n}X$ is an object of $\mathcal{D}^{[n,n]} = \mathcal{D}^{\heartsuit}[-n]$. We then define the functor[15] $H^n : \mathcal{D} \to \mathcal{D}^{\heartsuit}$ via $H^n(X) := (\tau^{\ge n}\tau^{\le n}X)[n]$. Recall also that for each $X \in \mathcal{D}$ and $n \in \mathbf{Z}$ there is a fibre sequence

$$\tau^{\le n-1}X \to X \to \tau^{\ge n}X. \tag{A.6.1}$$

[15]We follow traditional homological algebra notation (with *cohomological* conventions) by writing H^n. This suits our purposes, since the stable ∞-categories that we consider in the body of the paper will typically be derived categories of various abelian categories of modules, or variants thereof. In the case of the stable ∞-category of spectra, and other stable ∞-categories of a more homotopical nature, it is more usual to write π_{-n}. We have applied this convention in the discussion of Section A.2.3 above, but in all other cases that we consider, we use the cohomological notation established here.

A.6.2. *Completeness.* If $\mathcal{D}$ is a stable ∞-category endowed with a t-structure, then we have a functor

$$\mathcal{D} \to \lim \mathcal{D}^{\leq n}$$

defined by $X \mapsto (\tau^{\leq n} X)$ (the transition morphisms being given by the obvious truncations). We say that $\mathcal{D}$ is *right complete* if this functor is an equivalence. Unwinding this definition, the full faithfulness of this morphism is equivalent to the requirement that for any object X of $\mathcal{D}$, the canonical morphisms $\tau^{\leq n} X \to X$ induce an isomorphism $\operatorname{colim} \tau^{\leq n} X \xrightarrow{\sim} X$. The essential surjectivity is equivalent to the requirement that if $\{X_n\}$ is any sequence of objects endowed with identifications $X_n \xrightarrow{\sim} \tau^{\leq n} X_{n+1}$, then $\operatorname{colim}_n X_n$ exists in $\mathcal{D}$.

There is a dual notion of *left complete*; namely, this is the condition that the canonical functor

$$\mathcal{D} \to \lim \mathcal{D}^{\geq n},$$

defined by $X \mapsto (\tau^{\geq n} X)$, should be an equivalence. (Note that this is a limit with $n \to -\infty$.) The full faithfulness now amounts to the requirement that for any object X of $\mathcal{D}$, the canonical morphisms $X \to \tau^{\geq n} X$ induce an isomorphism

$$X \xrightarrow{\sim} \lim_n \tau^{\geq n} X.$$

The essential surjectivity is equivalent to the requirement that if $\{X_n\}$ is any sequence of objects endowed with identifications $\tau^{\geq n} X_{n-1} \xrightarrow{\sim} X_n$, then $\lim_n X_n$ exists in $\mathcal{D}$.

A.6.3. *t-exactness of functors.* If $\mathcal{D}$ and $\mathcal{D}'$ are two stable ∞-categories endowed with t-structures, then a functor $F : \mathcal{D} \to \mathcal{D}'$ is *right t-exact* (resp. *left t-exact*) if $F(\mathcal{D}^{\leq 0}) \subseteq (\mathcal{D}')^{\leq 0}$ (resp. $F(\mathcal{D}^{\geq 0}) \subseteq (\mathcal{D}')^{\geq 0}$), and *$t$-exact* if it is both right and left t-exact.

A.6.4. *Some cautions.* If $\mathcal{D}$ is a stable ∞-category endowed with a t-structure, it is natural to ask whether if X is an object of $\mathcal{D}$ for which $H^i(X) = 0$ for all i, is it necessarily the case that $X = 0$? While we recall some positive results in this direction, Example A.6.11 below shows that it need not hold in general.

Lemma A.6.5. *If X is an object of $\mathcal{D}$ which is bounded, and $H^i(X) = 0$ for all i, then $X = 0$.*

Proof. By assumption, $X \in \mathcal{D}^{[a,b]}$ for some $a \leq b$. We have the fibre sequence

$$\tau^{\leq b-1} X \to X \to H^b(X)[-b] = 0,$$

so that $\tau^{\leq b-1} X \xrightarrow{\sim} X$. Now $\tau^{\leq b-1} X$ is an object of $\mathcal{D}^{[a,b-1]}$, and the lemma follows by descending induction on the amplitude $b - a \geq 0$. □

Lemma A.6.6. *If the t-structure on $\mathcal{D}$ is left complete, and X is an object of $\mathcal{D}$ which is bounded above and for which $H^i(X) = 0$ for all i, then $X = 0$.*

Proof. Since $H^i(X) = 0$ for all i, we have $H^i(\tau^{\geq n} X) = 0$ for all i, n, so Lemma A.6.5 shows that $\tau^{\geq n} X = 0$ for all n. Since $\mathcal{D}$ is left complete, we deduce that $X = 0$, as required. □

Remark A.6.7. There is an obvious analogue of Lemma A.6.6 for right complete t-structures and objects that are bounded below.

A.6.8. *t-structures and Ind/Pro constructions.* For later reference, we recall (from [AGH19, Prop. 2.13] and [Lur18, Lem. C.2.4.3]) the following general facts about t-structures and Ind-categories, as well as the dual version for Pro-categories.

Proposition A.6.9. *Let $\mathcal{D}$ be a stable ∞-category endowed with a t-structure.*

(1) $\operatorname{Ind}\mathcal{D}$ *inherits a t-structure, characterized by the requirements that $\mathcal{D} \hookrightarrow \operatorname{Ind}\mathcal{D}$ is t-exact, and that the inclusions $\mathcal{D}^{\leq 0} \hookrightarrow \mathcal{D}$, $\mathcal{D}^{\geq 0} \hookrightarrow \mathcal{D}$, and $\mathcal{D}^{\heartsuit} \hookrightarrow \mathcal{D}$ induce equivalences* $\operatorname{Ind}(\mathcal{D}^{\leq 0}) \xrightarrow{\sim} \operatorname{Ind}(\mathcal{D})^{\leq 0}$, $\operatorname{Ind}(\mathcal{D}^{\geq 0}) \xrightarrow{\sim} \operatorname{Ind}(\mathcal{D})^{\geq 0}$, $\operatorname{Ind}(\mathcal{D}^{\heartsuit}) \xrightarrow{\sim} \operatorname{Ind}(\mathcal{D})^{\heartsuit}$, *respectively. If the t-structure on $\mathcal{D}$ is furthermore bounded above, then the t-structure on* $\operatorname{Ind}\mathcal{D}$ *is right complete.*

(2) $\operatorname{Pro}\mathcal{D}$ *inherits a t-structure, characterized by the requirements that $\mathcal{D} \hookrightarrow \operatorname{Pro}\mathcal{D}$ is t-exact, and that the inclusions $\mathcal{D}^{\leq 0} \hookrightarrow \mathcal{D}$, $\mathcal{D}^{\geq 0} \hookrightarrow \mathcal{D}$, and $\mathcal{D}^{\heartsuit} \hookrightarrow \mathcal{D}$ induce equivalences* $\operatorname{Pro}(\mathcal{D}^{\leq 0}) \xrightarrow{\sim} \operatorname{Pro}(\mathcal{D})^{\leq 0}$, $\operatorname{Pro}(\mathcal{D}^{\geq 0}) \xrightarrow{\sim} \operatorname{Pro}(\mathcal{D})^{\geq 0}$, $\operatorname{Pro}(\mathcal{D}^{\heartsuit}) \xrightarrow{\sim} \operatorname{Pro}(\mathcal{D})^{\heartsuit}$, *respectively. If the t-structure on $\mathcal{D}$ is furthermore bounded below, then the t-structure on* $\operatorname{Pro}\mathcal{D}$ *is left complete.*

Remark A.6.10. Since neither of the references cited above explicitly state the result about the heart, we recall a proof here in the case of Ind categories; the Pro category case is formally dual.

On the one hand, the inclusion $\mathcal{D}^{\heartsuit} \hookrightarrow \mathcal{D}$ induces a corresponding inclusion $\operatorname{Ind}\mathcal{D}^{\heartsuit} \hookrightarrow \operatorname{Ind}\mathcal{D}$, which lies in $(\operatorname{Ind}\mathcal{D})^{\heartsuit}$, since it lies in both $(\operatorname{Ind}\mathcal{D})^{\leq 0} \xrightarrow{\sim} \operatorname{Ind}(\mathcal{D}^{\leq 0}) \hookrightarrow \operatorname{Ind}\mathcal{D}$ and $(\operatorname{Ind}\mathcal{D})^{\geq 0} \xrightarrow{\sim} \operatorname{Ind}(\mathcal{D}^{\geq 0}) \hookrightarrow \operatorname{Ind}\mathcal{D}$.

On the other hand, the functors $\tau^{\leq 0} : \operatorname{Ind}\mathcal{D} \to (\operatorname{Ind}\mathcal{D})^{\leq 0} \xrightarrow{\sim} \operatorname{Ind}(\mathcal{D}^{\leq 0})$ and $\tau^{\geq 0} : \operatorname{Ind}\mathcal{D} \to (\operatorname{Ind}\mathcal{D})^{\geq 0} \xrightarrow{\sim} \operatorname{Ind}(\mathcal{D}^{\geq 0})$ are the Ind-extensions of the corresponding functors for $\mathcal{D}$ (as can be seen from the second displayed equation in the proof of [AGH19, Prop. 2.13]) and in particular commute with filtered colimits. Thus, if X is any object of $(\operatorname{Ind}\mathcal{D})^{\heartsuit}$, so that $\tau^{\geq 0}\tau^{\leq 0}X \xrightarrow{\sim} X$, then, if we write $X = \operatorname{colim}_i X_i$ for some objects X_i of $\mathcal{D}$, we find that $X \xrightarrow{\sim} \tau^{\geq 0}\tau^{\leq 0}X \xrightarrow{\sim} \operatorname{colim}_i \tau^{\geq 0}\tau^{\leq 0}X_i$, which describes X as an object of $\operatorname{Ind}\mathcal{D}^{\heartsuit}$, so that $(\operatorname{Ind}\mathcal{D})^{\heartsuit} \subseteq \operatorname{Ind}\mathcal{D}^{\heartsuit}$.

Example A.6.11. Let $\mathcal{C}$ be the category of finitely generated modules over $k[\epsilon]$ (the dual numbers over a field k), and let $D^b(\mathcal{C})$ be the bounded derived category of the abelian category $\mathcal{C}$, in the sense of Definition A.7.1 and Remark A.7.5. As usual, regard k as a $k[\epsilon]$-module by having ϵ act by zero. Recall then that $\operatorname{Ext}^1_{k[\epsilon]}(k,k)$ is one-dimensional over k, and that if we let x denote a basis vector, then Yoneda product of Exts induces an isomorphism $k[x] \xrightarrow{\sim} \operatorname{Ext}^{\bullet}_{k[\epsilon]}(k,k)$. Now consider the sequence

$$k \xrightarrow{x} k[1] \xrightarrow{x} k[2] \xrightarrow{x} \cdots \xrightarrow{x} k[n] \xrightarrow{x} \ldots$$

of morphisms in $D^b(\mathcal{C})$. By the facts just recalled, the composite of any n successive morphisms in this sequence corresponds to the (non-zero!) element $x^n \in \operatorname{Ext}^n_{k[\epsilon]}(k,k)$. Thus none of the transition morphisms in this sequence are zero, and it induces a non-zero object of $\operatorname{Ind} D^b(\mathcal{C})$. This is then an example (probably the most standard example) of an object of $\operatorname{Ind} D^b(\mathcal{C})$ which is non-zero (as one can see by computing its endomorphisms), but has all cohomologies equal to zero. Note this object is bounded above, but not bounded below.

Similarly, if we instead consider the sequence

$$\cdots \xrightarrow{x} k[-n] \xrightarrow{x} \cdots \xrightarrow{x} k[-2] \xrightarrow{x} k[-1] \xrightarrow{x} k$$

of morphisms in $D^b(\mathcal{C})$, it gives rise to an object of $\operatorname{Pro} D^b(\mathcal{C})$ which is non-zero, but has all cohomologies equal to zero; it is bounded below, but not bounded above.

A.7. **Derived categories.** We recall a range of notation and results related to derived categories of abelian categories, working in the framework of stable ∞-categories.

Definition A.7.1. If $\mathcal{A}$ is an abelian category, then we define the *derived category* $D(\mathcal{A})$ of $\mathcal{A}$ to be the localization of the category $\mathrm{CoCh}(\mathcal{A})$ of cochain complexes valued in $\mathcal{A}$ with respect to the class of quasi-isomorphisms. In symbols, if we let W denote the collection of quasi-isomorphisms in $\mathrm{CoCh}(\mathcal{A})$, then $D(\mathcal{A}) \coloneqq \mathrm{CoCh}(\mathcal{A})[W^{-1}]$.

Remark A.7.2. If we continue to let W denote the collection of quasi-isomorphisms in $\mathrm{CoCh}(\mathcal{A})$, and let W_0 denote the collection of cochain homotopy equivalences, then [Lur17, Prop. 1.3.4.5] shows that $\mathrm{CoCh}(\mathcal{A})[W_0^{-1}] \xrightarrow{\sim} N_{\mathrm{dg}}\big(\mathrm{CoCh}(\mathcal{A})\big)$ (the dg-nerve of the dg-category $\mathrm{CoCh}(\mathcal{A})$). Then

$$\mathrm{CoCh}(\mathcal{A})[W^{-1}] \xrightarrow{\sim} \mathrm{CoCh}(\mathcal{A})[W_0^{-1}][W^{-1}] \xrightarrow{\sim} N_{\mathrm{dg}}\big(\mathrm{CoCh}(\mathcal{A})\big)[W^{-1}]. \tag{A.7.3}$$

Remark A.7.4. Definition A.7.1 is the most general applicable definition of the ∞-categorical version of the derived category of an abelian category of which we are aware. It is a stable ∞-category, by (A.7.3) and [NS18, Theorem I.3.3], and the homotopy category of $D(\mathcal{A})$ is the unbounded triangulated derived category of $\mathcal{A}$, as defined e.g. in [Stacks, Tag 05RU].

If $\mathcal{A}$ is a Grothendieck category, then an alternative definition of $D(\mathcal{A})$ is given in [Lur17, Def. 1.3.5.8]. The equivalence of that definition with Definition A.7.1 is provided by [Lur17, Prop. 1.3.5.15]. By [Lur17, Prop. 1.3.5.21], we thus see that $D(\mathcal{A})$ is presentable (hence cocomplete) whenever $\mathcal{A}$ is a Grothendieck category.

Remark A.7.5. If $\mathcal{A}$ is an abelian category, then the usual t-structure on the unbounded triangulated derived category of $\mathcal{A}$, i.e. the one given by the vanishing of cohomology groups (see e.g. [GM03, §IV.4]), induces a t-structure on $D(\mathcal{A})$. We can then define $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ in the usual manner. Their homotopy categories are the corresponding triangulated subcategories of the unbounded triangulated derived category of $\mathcal{A}$. Since ∞-categorical enhancements of these triangulated categories are unique up to equivalence [Ant21, Corollary 6.5, Corollary 6.10], our definitions will coincide with most variants appearing in the literature, such as [Lur17, Def. 1.3.2.7], which gives an alternative definition of $D^-(\mathcal{A})$ under the assumption that $\mathcal{A}$ admits enough projectives.

The following definition uses the notion of a weak Serre subcategory, which was recalled in Section A.1.1.

Definition A.7.6. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{B}$ be a weak Serre subcategory of $\mathcal{A}$. Then we write $D_{\mathcal{B}}(\mathcal{A})$ for the full subcategory of $D(\mathcal{A})$ consisting of those objects x all of whose cohomologies $H^n(x)$ lie in $\mathcal{B}$. We define $D^*_{\mathcal{B}}(\mathcal{A})$ analogously, when $*$ is any of the boundedness conditions $b, +$, or $-$.

Remark A.7.7. Recall that, by definition [Lur09, § 1.2.11], (full) subcategories of an ∞-category correspond to (full) subcategories of its homotopy category. Under this correspondence, the subcategory $D_{\mathcal{B}}(\mathcal{A})$ of Definition A.7.6 corresponds to the subcategory of the triangulated derived category of $\mathcal{A}$ given by [Stacks, Tag 06UP].

Lemma A.7.8. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{B}$ be a Serre subcategory. Then the natural functor $D^b(\mathcal{A})/D^b_{\mathcal{B}}(\mathcal{A}) \to D^b(\mathcal{A}/\mathcal{B})$ is a t-exact equivalence.*

Proof. This is [Miy91, Thm. 3.2]. ☐

Recall that a Serre subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is *localizing* if the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ admits a right adjoint. This right adjoint functor is necessarily fully faithful [Gab62, §III.2 Prop. 3].

Lemma A.7.9. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{B}$ be a localizing subcategory. Then the natural functor $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is a t-exact equivalence.*

Proof. This is [Kra15, Lem. 5.9]. ☐

A.7.10. *A criterion for compact generation.* By Lemma A.1.3, if $\mathcal{A}$ is a locally coherent abelian category, then $\mathcal{A}^c$ is a weak Serre subcategory of $\mathcal{A}$, so that $D^b_{\mathcal{A}^c}(\mathcal{A})$ is defined. Then we have the following lemma.

Lemma A.7.11. *If $\mathcal{A}$ is a locally coherent abelian category, then the natural functors $D^b(\mathcal{A}^c) \to D^b_{\mathcal{A}^c}(\mathcal{A})$ and $D^-(\mathcal{A}^c) \to D^-_{\mathcal{A}^c}(\mathcal{A})$ are t-exact equivalences.*

Proof. Bearing in mind that $\mathcal{A}$ is (by definition) compactly generated, and so $\mathrm{Ind}(\mathcal{A}^c) \xrightarrow{\sim} \mathcal{A}$, the lemma is an immediate consequence of [KS06, Thm. 15.3.1]. ☐

If R is a coherent ring, the category $\mathrm{Mod}(R)$ is locally coherent, and its compact objects coincide with the finitely presented modules. We introduce the notation

$$D^b_{\mathrm{fp}}(R) \coloneqq D^b_{\mathrm{Mod}^{\mathrm{fp}}(R)}(\mathrm{Mod}(R))$$

and

$$D^-_{\mathrm{fp}}(R) \coloneqq D^-_{\mathrm{Mod}^{\mathrm{fp}}(R)}(\mathrm{Mod}(R)),$$

and note the following immediate corollary of Lemma A.7.11.

Corollary A.7.12. *Let R be a coherent ring. Then the natural functors $D^b(\mathrm{Mod}^{\mathrm{fp}}(R)) \to D^b_{\mathrm{fp}}(R)$ and $D^-(\mathrm{Mod}^{\mathrm{fp}}(R)) \to D^-_{\mathrm{fp}}(R)$ are t-exact equivalences.*

The following proposition is closely related to [Kra05, Lem. 4.5]. By "a set of weak generators" of an abelian category $\mathcal{A}$, we mean a set X of objects of $\mathcal{A}$ having the property that for each $a \in \mathcal{A}$, there exists an $x \in X$ and a non-zero morphism $x \to a$.

Proposition A.7.13. *Let $\mathcal{A}$ be a locally coherent Grothendieck category, and suppose furthermore that $\mathcal{A}^c \subseteq D(\mathcal{A})^c$. Then:*

(1) $D(\mathcal{A})$ is compactly generated, and $D^b_{\mathcal{A}^c}(\mathcal{A})$ coincides with $D(\mathcal{A})^c$.
(2) Any subset $X \subseteq \mathcal{A}^c$ which is a set of weak generators of $\mathcal{A}$ is a set of compact generators of $D(\mathcal{A})$.
(3) The standard t-structure on $D(\mathcal{A})$ is the one induced by the standard t-structure on $D^b_{\mathcal{A}^c}(\mathcal{A})$ via Proposition A.6.9 and the equivalence $\mathrm{Ind}\, D^b_{\mathcal{A}^c}(\mathcal{A}) \xrightarrow{\sim} D(\mathcal{A})$ induced by part (1).

Proof. By Lemma A.7.11, the full sub-∞-category $D^b_{\mathcal{A}^c}(\mathcal{A})$ of $D(\mathcal{A})$ is well-defined, and equivalent to $D^b(\mathcal{A}^c)$. Furthermore, the assumption that $\mathcal{A}^c \subset D(\mathcal{A})^c$, together with an induction on amplitude, shows that $D^b_{\mathcal{A}^c}(\mathcal{A}) \subset D(\mathcal{A})^c$. Since $D^b_{\mathcal{A}^c}(\mathcal{A})$ is evidently idempotent complete, we see that part (1) of the proposition will follow from Lemma A.2.23 (2), applied to $\mathcal{C}' := D^b_{\mathcal{A}^c}(\mathcal{A})$ and $\mathcal{C} := D(\mathcal{A})$, provided that we prove that $\left(D^b_{\mathcal{A}^c}(\mathcal{A})\right)^\perp = 0$. To prove this, it suffices to show that if $a \in D(\mathcal{A})$ is not zero, and $X \subseteq \mathcal{A}^c$ is a set of weak generators of $\mathcal{A}$, then there exists $x \in X$ such that $\mathrm{RHom}_{D(\mathcal{A})}(x, a) \neq 0$; note that this will also establish part (2).

Suppose then that a is a non-zero object of $D(\mathcal{A})$. Since the homotopy category of $D(\mathcal{A})$ is the unbounded derived category of $\mathcal{A}$ with the usual t-structure, there exists some n such that $H^n(a) \neq 0$. We may represent the truncation $\tau^{\leq n}a$ by a complex $\cdots \to a_{n-1} \to a_n \to 0 \to 0 \to \cdots$ in $\mathrm{CoCh}(\mathcal{A})$. Since $\mathrm{coker}(a_{n-1} \to a_n) =: H^n(\tau^{\leq n}a) = H^n(a) \neq 0$, and since X is a set of weak generators of $\mathcal{A}$ by assumption, we may find an object x of X and a morphism $x \to a_n$ such that the composite $x \to H^n(\tau^{\leq n}a)$ is non-zero. Then the composite $x[-n] \to \tau^{\leq n}a \to a$ (the first arrow being induced by the chosen morphism $x \to a_n$) is non-zero (since it induces a non-zero morphism on H^n), and so in particular $\mathrm{RHom}_{D(\mathcal{A})}(x, a) \neq 0$, as required.

Finally, the claim in part (3) about t-structures is equivalent to the t-structure on $D(\mathcal{A})$ being compatible with filtered colimits, which is [Lur17, Prop. 1.3.5.21]. $\square$

A.7.14. *Derived functors.* A very general mechanism for constructing left derived functors is given by the following result [Lur17, Thm. 1.3.3.2], which characterizes $D^-(\mathcal{C})$ (for abelian categories with enough projectives) by a mapping property.

Theorem A.7.15. *If $\mathcal{C}$ is an abelian category with enough projectives, and if $\mathcal{D}$ is a stable ∞-category equipped with a left complete t-structure, then $F \mapsto \tau^{\geq 0}F_{|\mathcal{C}}$ (the restriction being taken by identifying $\mathcal{C}$ with the heart of $D^-(\mathcal{C})$) is an equivalence between the ∞-category of right t-exact functors $D^-(\mathcal{C}) \to \mathcal{D}$ which carry projective objects of $\mathcal{C}$ into $\mathcal{D}^{\heartsuit}$, and the ordinary category of right exact functors $\mathcal{C} \to \mathcal{D}^{\heartsuit}$.*

Furthermore, by restriction from $D^-(\mathcal{C})$ to $D^b(\mathcal{C})$, these categories are equivalent to the ∞-category of right t-exact functors $D^b(\mathcal{C}) \to \mathcal{D}$ which carry projective objects of $\mathcal{C}$ into $\mathcal{D}^{\heartsuit}$.

Proof. As already noted, the first claim is simply a restatement of [Lur17, Thm. 1.3.3.2]. The second claim is a consequence of the proof of that result, and is a manifestation of the role that left completeness plays in that result, as we now explain.

Namely, suppose that $F : D^-(\mathcal{C}) \to \mathcal{D}$ is a right t-exact functor. Write $D^{-,\geq n}(\mathcal{C})$ to denote the full subcategory of $D^-(\mathcal{C})$ consisting of objects whose cohomology in degrees $< n$ vanishes (i.e. the "$\geq n$" part of the t-structure on $D^-(\mathcal{C})$), and write

$$F_n := \tau^{\geq n}F_{|D^{-,\geq n}(\mathcal{C})} : D^{-,\geq n}(\mathcal{C}) \to \mathcal{D}^{\geq n}.$$

The fact that F is right t-exact shows that the canonical functor

$$\tau^{\geq n}F \to \tau^{\geq n}F\tau^{\geq n} \tag{A.7.16}$$

is an isomorphism, from which one easily verifies that the diagram of functors

$$\begin{tikzcd} D^{-,\geq n-1}(\mathcal{C}) \arrow[r, "F_{n-1}"] \arrow[d, "\tau^{\geq n}"] & \mathcal{D}^{\geq n-1} \arrow[d, "\tau^{\geq n}"] \\ D^{-,\geq n}(\mathcal{C}) \arrow[r, "F_n"] & \mathcal{D}^{\geq n} \end{tikzcd}$$

commutes. Thus, taking into account the left completeness of each of $D^-(\mathcal{C})$ (by [Lur17, Prop. 1.3.3.16]) and $\mathcal{D}$ (by hypothesis), we may form the functor

$$\lim_n F_n : D^-(\mathcal{C}) \to \mathcal{D}.$$

Again, the fact that (A.7.16) is an isomorphism shows that we have natural transformations (for X an object of $D^-(\mathcal{C})$)

$$F(X) \to \tau^{\geq n}F(X) \xrightarrow{\sim} \tau^{\geq n}F(\tau^{\geq n}X) = F_n(\tau^{\geq n}X),$$

evidently compatible as n varies, and hence a natural transformation

$$F(X) \to \lim_n \big(F_n(\tau^{\geq n} X)\big) = (\lim_n F_n)(X). \tag{A.7.17}$$

If F furthermore takes projective objects of $\mathcal{C}$ to objects of $\mathcal{D}^\heartsuit$, then (A.7.17) is a natural isomorphism. (Indeed, both source and target are right t-exact functors from $D^-(\mathcal{C})$ to $\mathcal{D}$ which coincide on $\mathcal{C}$, and take projective objects of $\mathcal{C}$ to $\mathcal{D}^\heartsuit$.) Thus F can be recovered (and constructed) from the various F_n, and thus from its restriction to $D^b(\mathcal{C})$. □

Corollary A.7.18. *If $\mathcal{C}$ is an abelian category with enough projectives, and if $\mathcal{D}$ is a stable ∞-category equipped with a left complete t-structure, then $F \mapsto F_{|\mathcal{C}}$ induces an equivalence between the ∞-category of t-exact functors $D^-(\mathcal{C}) \to \mathcal{D}$ and the ordinary category of exact functors $\mathcal{C} \to \mathcal{D}^\heartsuit$.*

Proof. This follows from Theorem A.7.15, noting that the t-exactness assumption implies that *all* objects of $\mathcal{C}$, projective or not, are carried into objects of $\mathcal{D}^\heartsuit$. See also [Lur17, Rem. 1.3.3.6]. □

In the case of t-exact functors, we can in fact state results valid for any abelian category (related, morally, to the fact that we don't need projective resolutions to compute the derived functors of exact functors). This is because of the following very general mapping property satisfied by bounded derived categories [BCKW25, Cor. 7.4.12].

Theorem A.7.19. *If $\mathcal{C}$ is an abelian category, and $\mathcal{D}$ is a stable ∞-category, then $F \mapsto F_{|\mathcal{C}}$ induces an equivalence between the ∞-category of exact functors $D^b(\mathcal{C}) \to \mathcal{D}$ and the ∞-category of finite coproduct-preserving functors $\mathcal{C} \to \mathcal{D}$ which take exact sequences in $\mathcal{C}$ to fibre sequences in $\mathcal{D}$.*

This has the following immediate corollary:

Corollary A.7.20. *If $\mathcal{C}$ is an abelian category, and if $\mathcal{D}$ is a stable ∞-category equipped with a t-structure, then there is an equivalence between the ∞-category of t-exact functors $D^b(\mathcal{C}) \to \mathcal{D}$ and the ordinary category of exact functors $\mathcal{C} \to \mathcal{D}^\heartsuit$, given by $F \mapsto F_{|\mathcal{C}}$ (the restriction being taken by identifying $\mathcal{C}$ with the heart of $D^b(\mathcal{C})$).*

We also note the following variant of the preceding corollary.

Corollary A.7.21. *If $\mathcal{C}$ is an abelian category, if $\mathcal{D}$ is a stable ∞-category equipped with a t-structure, and if $F : D^b(\mathcal{C}) \to \mathcal{D}$ is an exact functor for which $F_{|\mathcal{C}}$ takes values in $\mathcal{D}^\heartsuit$, then F is t-exact.*

Proof. The hypotheses imply that $F_{|\mathcal{C}}$ is an exact functor $\mathcal{C} \to \mathcal{D}^\heartsuit$, which by Corollary A.7.20 arises by restriction from a t-exact functor $G : D^b(\mathcal{C}) \to \mathcal{D}$. Taking into account the equivalence of categories described in the statement of Theorem A.7.19, the equality $F_{|\mathcal{C}} = G_{|\mathcal{C}}$ is induced by an isomorphism $F \xrightarrow{\sim} G$. Thus F is t-exact, as claimed. □

We now consider right derived functors. We begin with the following simple lemma.

Lemma A.7.22. *If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories, then F induces a t-exact functor $D(\mathcal{A}) \to D(\mathcal{B})$, whose restriction to the heart of $D(\mathcal{A})$ is F.*

Proof. We obtain a functor $\mathrm{CoCh}(\mathcal{A}) \to \mathrm{CoCh}(\mathcal{B})$ by applying F degreewise. Since F is exact, it preserves quasi-isomorphisms, so it induces the required functor $D(\mathcal{A}) \to D(\mathcal{B})$ by the universal property of localization. $\square$

Suppose now that $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between Grothendieck categories. Then F induces a functor $\mathrm{CoCh}(\mathcal{A}) \to \mathrm{CoCh}(\mathcal{B})$, and thus a functor $\overline{F} : \mathrm{CoCh}(\mathcal{A}) \to D(\mathcal{B})$. Write $Q : \mathrm{CoCh}(\mathcal{A}) \to D(\mathcal{A})$ for the localization functor. Following [Cis19, 7.5.23] we make the following definition, which is a "lifting" to the ∞-categorical setting of a definition on the level of homotopy categories that goes back to Quillen (see [Qui67, Def. 1.4.1], or [Cis19, Def. 2.3.1] for a more recent treatment).

Definition A.7.23. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between Grothendieck categories. A *right derived functor* RF of F is a functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ which is equipped with a natural transformation $\overline{F} \to RF \circ Q$, and is such that for any other functor $G : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation $\overline{F} \to G \circ Q$, there is a unique (up to a contractible space of choices) natural transformation $RF \to G$ giving rise to the given $\overline{F} \to G \circ Q$.

Lemma A.7.24. *If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between Grothendieck categories, then the right derived functor RF exists. Furthermore RF is t-exact, and agrees with the extension of F in Lemma* A.7.22*.*

Proof. This is immediate from the definitions. $\square$

In fact, the right derived functor RF exists for any additive functor F, although it is only well-behaved if F is left exact, in a sense made precise by Lemma A.7.25 below. Right derived functors can be computed using resolutions by K-injective complexes of injective objects: see [Stacks, Tag 079P] and [Stacks, Tag 070K] in the context of triangulated categories. Alternatively, in the context of ∞-categories and Definition A.7.23, we can give $\mathrm{CoCh}(\mathcal{A})$ the model category structure of [Lur17, Proposition 1.3.5.3]. The fibrant objects are then precisely the K-injective complexes of injective objects: compare [Hov99, Remark 2.3.18] for this fact, bearing in mind that the "dg-injective complexes" in *loc. cit.* are precisely the K-injective complexes of injective objects, by [AF91, 1.2.I] (where K-injective complexes are called π-injective). Then, as explained in [Cis19, Section 7.5.25], the existence of RF, and the fact that it can be computed by K-injective complexes of injective objects, both follow from the fact that F sends weak equivalences between fibrant objects of $\mathrm{CoCh}(\mathcal{A})$ to weak equivalences in $\mathrm{CoCh}(\mathcal{B})$, which in turn follows from [Lur17, Proposition 1.3.5.14], asserting that every weak equivalence between fibrant objects of $\mathrm{CoCh}(\mathcal{A})$ is a chain homotopy equivalence.

Lemma A.7.25. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between Grothendieck categories. Then the right derived functor RF is left t-exact. Furthermore, the natural map $F \to H^0 RF$ (induced by the natural transformation $\overline{F} \to RF \circ Q$) is an isomorphism if and only if F is left exact.*

Proof. Both claims follow easily from the existence of injective resolutions, and the fact that bounded below complexes of injective objects are K-injective, see e.g. [Stacks, Tag 070J] and [Stacks, Tag 05TD]. $\square$

We next quote two results from [EGH25] concerning the interplay between derived functors and adjunctions.

Lemma A.7.26. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact, colimit-preserving functor between Grothendieck categories, and write $G : \mathcal{B} \to \mathcal{A}$ for its right adjoint. Then the right-derived functor $RG : D(\mathcal{B}) \to D(\mathcal{A})$ is right adjoint to $F : D(\mathcal{A}) \to D(\mathcal{B})$. In particular, $F : D(\mathcal{A}) \to D(\mathcal{B})$ is continuous.*

Proof. This is [EGH25, Prop. A.7.1]. □

Theorem A.7.27. *Let $F : \mathcal{A} \hookrightarrow \mathcal{B}$ be the inclusion of a localizing subcategory into a Grothendieck category, with right adjoint $G : \mathcal{B} \to \mathcal{A}$. Suppose further that:*

(1) For any objects X and Y of $\mathcal{A}$ with Y injective, there is an epimorphism $Z \to X$ such that $\operatorname{Ext}^i_{\mathcal{B}}(F(Z), F(Y)) = 0$ for $i > 0$.
(2) The formation of products in $\mathcal{B}$ is exact.
(3) The derived right adjoint RG has finite cohomological dimension.

Then $F : D(\mathcal{A}) \to D(\mathcal{B})$ is fully faithful, with essential image equal to $D_{\mathcal{A}}(\mathcal{B})$.

Remark A.7.28. Note in particular that condition (1) is satisfied if F preserves injectives.

Proof of Theorem A.7.27. This is a combination of parts (1) and (2) of [EGH25, Prop. A.7.3]. □

We end this section with a result on the compatibility of localization of locally Noetherian categories with passage to derived categories. Rather than consider the general case, we will assume various additional hypotheses that will hold in our application of this material in Section 2.6.

Suppose that $\mathcal{B}$ is a locally Noetherian abelian category and that $i_* : \mathcal{A} \to \mathcal{B}$ is an inclusion of a localizing subcategory; so i_* is exact and is compatible with colimits, and by Lemma A.7.26, we have a continuous t-exact extension $i_* : D(\mathcal{A}) \to D(\mathcal{B})$. Write $j^* : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ for the quotient functor, which is also exact and compatible with colimits, and $j_* : \mathcal{B}/\mathcal{A} \to \mathcal{B}$ for its fully faithful right adjoint, which (as recalled in Appendix A.1) preserves filtered colimits. If we suppose further that j_* is exact, it is furthermore compatible with all colimits, and so we have continuous t-exact extensions $j^* : D(\mathcal{B}) \to D(\mathcal{B}/\mathcal{A})$, $j_* : D(\mathcal{B}/\mathcal{A}) \to D(\mathcal{B})$.

Proposition A.7.29. *Suppose as above that $\mathcal{B}$ is a locally Noetherian abelian category and that $i_* : \mathcal{A} \to \mathcal{B}$ is an inclusion of a localizing subcategory. Write $j^* : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ for the quotient functor, and $j_* : \mathcal{B}/\mathcal{A} \to \mathcal{B}$ for its right adjoint.*

Suppose that:

(a) $\mathcal{B}^c \subseteq D(\mathcal{B})^c$.
(b) $j_* : \mathcal{B}/\mathcal{A} \to \mathcal{B}$ is exact.
(c) $i_* : D(\mathcal{A}) \to D(\mathcal{B})$ is fully faithful, with essential image $D_{\mathcal{A}}(\mathcal{B})$.

Then

(1) j^* induces equivalences $D(\mathcal{B})/D(\mathcal{A}) \xrightarrow{\sim} D(\mathcal{B}/\mathcal{A})$ and $D^b(\mathcal{B}^c)/D^b(\mathcal{A}^c) \xrightarrow{\sim} D^b(\mathcal{B}^c/\mathcal{A}^c)$.
(2) $j_* : D(\mathcal{B}/\mathcal{A}) \to D(\mathcal{B})$ is fully faithful, and is right adjoint to $j^* : D(\mathcal{B}) \to D(\mathcal{B}/\mathcal{A})$.
(3) $D(\mathcal{B})$, $D(\mathcal{A})$ and $D(\mathcal{B}/\mathcal{A})$ are compactly generated.
(4) $D(\mathcal{B})^c$ coincides with $D^b_{\mathcal{B}^c}(\mathcal{B})$, which in turn is equivalent to $D^b(\mathcal{B}^c)$. The analogous statements hold for $D(\mathcal{A})^c$ and $D(\mathcal{B}/\mathcal{A})^c$.

Proof. By assumption we have $\mathcal{B}^c \subseteq D(\mathcal{B})^c$. Since $i_* : \mathcal{A} \to \mathcal{B}$ preserves colimits and compact objects, and (by assumption) $i_* : D(\mathcal{A}) \to D(\mathcal{B})$ is continuous and fully faithful, we see also that $\mathcal{A}^c \subseteq D(\mathcal{A})^c$. Then the statements of (3) and (4) for $D(\mathcal{B})$ and $D(\mathcal{A})$ follow from Proposition A.7.13 and Lemma A.7.11; furthermore $i_* : D(\mathcal{A}) \to D(\mathcal{B})$ preserves compact objects.

We now prove part (1). By assumption, i_* induces an equivalence between $D(\mathcal{A})$ and $D_{\mathcal{A}}(\mathcal{B})$, hence the first statement of part (1) follows from Lemma A.7.9; note that this implies that Hypothesis A.3.2 is satisfied by i_* and j^*. The second statement of part (1) follows similarly from Lemma A.7.8, provided we show that i_* induces an equivalence between $D^b(\mathcal{A}^c)$ and $D^b_{\mathcal{A}^c}(\mathcal{B}^c)$. Certainly i_* induces a fully faithful functor $i_* : D^b(\mathcal{A}^c) \to D^b_{\mathcal{A}^c}(\mathcal{B}^c)$. It thus suffices to note that any object of $D^b_{\mathcal{A}^c}(\mathcal{B}^c)$ is compact in $D(\mathcal{B})$ (by the already proved statement of part (4) for $D(\mathcal{B})$) and contained in $D_{\mathcal{A}}(\mathcal{B})$, and thus in the essential image of $D(\mathcal{A})$; and $D(\mathcal{B})^c \cap D(\mathcal{A}) = D(\mathcal{A})^c = D^b(\mathcal{A}^c)$ by Lemma A.3.3 (and the already proved statement of part (4) for $D(\mathcal{A})$). This concludes the proof of part (1).

Finally, by Lemmas A.7.26 and A.7.24, $j_* : D(\mathcal{B}/\mathcal{A}) \to D(\mathcal{B})$ is right adjoint to $j^* : D(\mathcal{B}) \to D(\mathcal{B}/\mathcal{A})$. Hence the statements of part (2), as well as part (3) for $D(\mathcal{B}/\mathcal{A})$, are immediate consequences of Lemma A.3.3, which also shows that j^* preserves compact objects. Thus the objects of $(\mathcal{B}/\mathcal{A})^c = j^*(\mathcal{B}^c)$ are compact in $D(\mathcal{B}/\mathcal{A})^c$, and so the statement of part (4) for $D(\mathcal{B}/\mathcal{A})$ follows from Proposition A.7.13 and Lemma A.7.11. ☐

A.7.30. *Colimits of abelian and derived categories.* If $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ is a filtered system of abelian categories indexed by $\mathcal{I}$, with exact transition functors, then we can form $\mathcal{A} := \operatorname{colim}_i \mathcal{A}_i$, which is again an abelian category. By Lemma A.7.22 we get an induced system $\{D^b(\mathcal{A}_i)\}_{i\in\mathcal{I}}$ with t-exact transition functors, and the universal property of Corollary A.7.20 shows that

$$\operatorname{colim}_i D^b(\mathcal{A}_i) \xrightarrow{\sim} D^b(\mathcal{A}),$$

the colimit being formed in the ∞-category of small ∞-categories.

Similarly, a consideration of the universal property of the Ind construction shows that we obtain an equivalence

$$\operatorname{colim}_i \operatorname{Ind} D^b(\mathcal{A}_i) \xrightarrow{\sim} \operatorname{Ind} D^b(\mathcal{A}),$$

the colimit now being formed in the ∞-category of compactly generated stable ∞-categories (whose morphisms are the continuous functors).

A.8. **Left derived functors via Pro-categories.** The goal of this section is to study how certain constructions of derived functors interact with the formation of Pro-categories. The reason for these considerations is as follows: if $\mathcal{C}$ is any abelian category, then $\operatorname{Pro}\mathcal{C}$ has enough projectives, and so passing to $\operatorname{Pro}\mathcal{C}$ can be a convenient first step in the construction of left derived functors. On the other hand, these derived functors are then *a priori* defined on $D^-(\operatorname{Pro}\mathcal{C})$, which is an inconvenient source category for the applications we have in mind; we would prefer to work with $\operatorname{Pro} D^b(\mathcal{C})$. More precisely, then, the following discussion will explain how we can form derived functors whose source is this latter category.

Another motivation for passing to $\operatorname{Pro}\mathcal{C}$ is to construct left adjoints which might otherwise not exist. With this in mind, we also show that our construction of derived functors is compatible with the formation of such adjoints.

Finally, we note that our discussion is closely related to the discussion of [KS06, Ch. 15], which considers the analogous constructions for Ind (rather than Pro) categories, in the setting of triangulated (rather than stable ∞-) categories. For a more precise explanation of the connection, see Remark A.8.12 below.

A.8.1. *The Pro-categorical context.* Throughout this discussion, we let $\mathcal{C}$ and $\mathcal{C}'$ denote two small abelian categories. We begin with the following consequence of Theorem A.7.19.

Corollary A.8.2. *If $\mathcal{C}$ and $\mathcal{C}'$ are two small abelian categories, then an exact functor $D^b(\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$ is t-exact if and only if its limit-preserving extension $\operatorname{Pro} D^b(\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$ is t-exact. Furthermore, the functors $F \mapsto F_{|\operatorname{Pro}\mathcal{C}}$ and $F \mapsto F_{|\mathcal{C}}$ induce equivalences between the following categories:*

1. *The ∞-category of t-exact limit-preserving functors $\operatorname{Pro} D^b(\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$.*
2. *The ∞-category of t-exact functors $D^b(\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$.*
3. *The ordinary category of exact limit-preserving functors from $\operatorname{Pro}\mathcal{C}$ to $\operatorname{Pro}\mathcal{C}'$.*
4. *The ordinary category of exact functors $\mathcal{C} \to \operatorname{Pro}\mathcal{C}'$.*

Proof. The first statement of the corollary is immediate from Proposition A.6.9, which also implies that the ∞-category of t-exact limit-preserving functors $\operatorname{Pro} D^b(\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$ is equivalent to the category of t-exact functors $D^b(\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$. The result is then immediate from Corollary A.7.20. $\square$

The category $\operatorname{Pro}\mathcal{C}$ has enough projectives, by Lemma A.1.10, while $\operatorname{Pro} D^b(\mathcal{C}')$ is left complete with respect to its natural t-structure, by Proposition A.6.9. Thus we have the following particular case of Theorem A.7.15 and Corollary A.7.18.

Proposition A.8.3.

1. *$F \mapsto \tau^{\geq 0} F_{|\operatorname{Pro}\mathcal{C}}$ is an equivalence between the ∞-category of right t-exact functors $D^-(\operatorname{Pro}\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$ which carry projective objects of $\operatorname{Pro}\mathcal{C}$ into $\operatorname{Pro}\mathcal{C}'$, and the ordinary category of right exact functors $\operatorname{Pro}\mathcal{C} \to \operatorname{Pro}\mathcal{C}'$.*
2. *$F \mapsto F_{|\operatorname{Pro}\mathcal{C}}$ is an equivalence between the ∞-category of t-exact functors $D^-(\operatorname{Pro}\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$ and the ordinary category of exact functors $\operatorname{Pro}\mathcal{C} \to \operatorname{Pro}\mathcal{C}'$.*

By Proposition A.8.3 (2), the natural identification $\operatorname{Pro}\mathcal{C} \xrightarrow{\sim} \big(\operatorname{Pro} D^b(\mathcal{C})\big)^{\heartsuit}$ induces a t-exact functor $p : D^-(\operatorname{Pro}\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C})$. Of course, the canonical inclusion $\mathcal{C} \hookrightarrow \operatorname{Pro}\mathcal{C}$ (which is exact) induces t-exact functors $i : D^b(\mathcal{C}) \to D^-(\operatorname{Pro}\mathcal{C})$ and $j : D^-(\mathcal{C}) \to D^-(\operatorname{Pro}\mathcal{C})$. There are also the canonical inclusions $D^b(\mathcal{C}) \hookrightarrow D^-(\mathcal{C})$ and $D^b(\mathcal{C}) \hookrightarrow \operatorname{Pro} D^b(\mathcal{C})$.

Lemma A.8.4. *We have a commutative diagram of t-exact functors:*

(A.8.5)
$$\begin{tikzcd}
 & D^-(\mathcal{C}) \arrow[d, "j"] \\
D^b(\mathcal{C}) \arrow[ur] \arrow[r, "i"] \arrow[dr] & D^-(\operatorname{Pro}\mathcal{C}) \arrow[d, "p"] \\
 & \operatorname{Pro} D^b(\mathcal{C})
\end{tikzcd}$$

The horizontal and diagonal arrows in (A.8.5) *are fully faithful.*

Proof. The commutativity follows directly from Corollary A.7.20, since all the functors are t-exact, and their restrictions to the hearts are either the identity functors or the natural embedding $\mathcal{C} \hookrightarrow \operatorname{Pro}\mathcal{C}$. The full faithfulness of the diagonal arrows holds by definition. The full faithfulness of i is a consequence of [KS06, Thm. 15.3.1]. □

As was already remarked on above, in applications, it is much more convenient to have the domain of our left-derived functors be equal to $\operatorname{Pro} D^b(\mathcal{C})$, rather than the ∞-category $D^-(\operatorname{Pro}\mathcal{C})$, which is rather hard to work with. In other words, under appropriate hypotheses, we would like to canonically factor the left derived functors constructed by Proposition A.8.3 through the functor p of (A.8.5). It turns out that this is possible whenever the functor we are deriving is cofiltered limit-preserving, i.e. it is Pro-extended from $\mathcal{C}$.

To this end, suppose that $f : \mathcal{C} \to \operatorname{Pro}(\mathcal{C}')$ is right exact, so that $\operatorname{Pro}(f) : \operatorname{Pro}(\mathcal{C}) \to \operatorname{Pro}(\mathcal{C}')$ is right exact and cofiltered limit-preserving. Write

$$F : D^-(\operatorname{Pro}\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$$

for the right t-exact functor corresponding to $\operatorname{Pro}(f)$ via Proposition A.8.3 (1). We will also consider the composite

$$(F \circ i)_{|\mathcal{C}} : \mathcal{C} \subset D^b(\mathcal{C}) \xrightarrow{i} D^-(\operatorname{Pro}\mathcal{C}) \xrightarrow{F} \operatorname{Pro} D^b(\mathcal{C}');$$

recalling from Remark A.2.20 that $\operatorname{Pro}\mathcal{C}$ is naturally identified with the ∞-categorical Pro-completion of $\mathcal{C}$, this functor can be Pro-extended to a functor

$$\operatorname{Pro}((F \circ i)_{|\mathcal{C}}) : \operatorname{Pro}\mathcal{C} \to \operatorname{Pro} D^b(\mathcal{C}').$$

Lemma A.8.6. *With the notation of the preceding paragraph, the functor $F_{|\operatorname{Pro}\mathcal{C}}$: $\operatorname{Pro}\mathcal{C} \to \operatorname{Pro} D^b(\mathcal{C}')$ preserves cofiltered limits; equivalently, it is naturally equivalent to* $\operatorname{Pro}((F \circ i)_{|\mathcal{C}})$.

Proof. Set

$$G \coloneqq \operatorname{Pro}\big((F \circ i)_{|\mathcal{C}}\big) : \operatorname{Pro}\mathcal{C} \longrightarrow \operatorname{Pro} D^b(\mathcal{C}').$$

By the Pro-version of the adjunction remarked upon in Remark A.2.18, there is a canonical natural transformation

$$\eta : F_{|\operatorname{Pro}\mathcal{C}} \longrightarrow G,$$

which we need to prove is an isomorphism. Concretely, if $X = \lim_i X_i$ with the X_i objects of $\mathcal{C}$, then $G(X) \coloneqq \lim_i F(X_i)$; and so the morphisms $F(X) \to F(X_i)$ induce a morphism $\eta_X : F(X) \to G(X)$. From this description, we see that η_X is an isomorphism whenever $X \in \mathcal{C}$.

Since the t-structure on $\operatorname{Pro} D^b(\mathcal{C}')$ is left complete, it suffices to prove for every $n \leq 0$, the truncation

$$\tau^{\geq n}\eta : \tau^{\geq n} F_{|\operatorname{Pro}\mathcal{C}} \longrightarrow \tau^{\geq n} G \tag{A.8.7}$$

is an equivalence. When $n = 0$, we have $\operatorname{Pro}(f) = \tau^{\geq 0} F_{|\operatorname{Pro}\mathcal{C}} \xrightarrow{\sim} \tau^{\geq 0} G$, so by induction it suffices to show that if (A.8.7) is an isomorphism for some $n \leq 0$, then it is also an isomorphism for $n-1$. Considering the fibre sequences

$$H^{n-1}(X)[1-n] \longrightarrow \tau^{\geq n-1} X \longrightarrow \tau^{\geq n} X,$$

we see that it suffices in turn to show that the induced morphism

$$H^{n-1}(F(X)) \to H^{n-1}(G(X)) \tag{A.8.8}$$

is an isomorphism for all objects X of $\operatorname{Pro}\mathcal{C}$. We already know that it is an isomorphism when $X \in \mathcal{C}$.

Suppose that we have a short exact sequence

$$0 \to Y \to P \to X \to 0$$

in $\operatorname{Pro}\mathcal{C}$, where P is projective. Bearing in mind that F sends projective objects of $\operatorname{Pro}\mathcal{C}$ into $\operatorname{Pro}\mathcal{C}'$, so that in particular $H^{n-1}(F(P)) = 0$, we obtain a diagram of exact sequences in $\operatorname{Pro}(\mathcal{C}')$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H^{n-1}\big(F(X)\big) \arrow[r] \arrow[d] & H^n\big(F(Y)\big) \arrow[d, "\sim"] \\
H^{n-1}\big(G(P)\big) \arrow[r] & H^{n-1}\big(G(X)\big) \arrow[r] & H^n\big(G(Y)\big)
\end{tikzcd}$$

in which the last vertical map is an isomorphism by inductive assumption. Since $\operatorname{Pro}\mathcal{C}$ has enough projectives, it follows that in order to show that (A.8.8) is an isomorphism for all X, it is enough to show that $H^{n-1}\big(G(P)\big) = 0$ for all projective objects P of $\operatorname{Pro}\mathcal{C}$.

Suppose furthermore that X is an object of $\mathcal{C}$; then the morphism $H^{n-1}\big(G(P)\big) \to H^{n-1}\big(G(X)\big)$ vanishes (because the natural morphism $H^{n-1}\big(F(X)\big) \to H^{n-1}\big(G(X)\big)$ is an isomorphism). If now $Q \to X$ is another (not necessarily epi) morphism with Q a projective object of $\operatorname{Pro}\mathcal{C}$, then by lifting to a morphism $Q \to P$, we see that the morphism $H^{n-1}\big(G(Q)\big) \to H^{n-1}\big(G(X)\big)$ also vanishes.

In particular, if we write $P = \lim_i P_i$ for objects P_i of $\mathcal{C}$ (which is possible by Lemma A.1.10), and recall that the truncation functors on $\operatorname{Pro} D^b(\mathcal{C}')$ are compatible with cofiltered limits, we have

$$H^{n-1}\big(G(P)\big) \xrightarrow{\sim} \lim_i H^{n-1}\big(G(P_i)\big).$$

Since (as shown in the previous paragraph, taking Q to be P and X to be P_i) the morphisms $H^{n-1}\big(G(P)\big) \to H^{n-1}\big(G(P_i)\big)$ are all zero, we deduce that $H^{n-1}\big(G(P)\big) = 0$, as required. □

Lemma A.8.9. *Let $f : \mathcal{C} \to \operatorname{Pro}(\mathcal{C}')$ be right exact, and let $F : D^-(\operatorname{Pro}\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$ be the right t-exact functor corresponding to* $\operatorname{Pro}(f)$ *via Proposition* A.8.3 (1). *Then we can extend the commutative diagram* (A.8.5) *to a commutative diagram*

$$\begin{tikzcd}
 & D^-(\mathcal{C}) \arrow[d, "j"] \arrow[dr, "F\circ j"] & \\
D^b(\mathcal{C}) \arrow[ur] \arrow[r, "i"] \arrow[dr, hook] & D^-(\operatorname{Pro}\mathcal{C}) \arrow[r, "F"] \arrow[d, "p"] & \operatorname{Pro} D^b(\mathcal{C}') \\
 & \operatorname{Pro} D^b(\mathcal{C}) \arrow[ur, "\operatorname{Pro}(F\circ i)"'] &
\end{tikzcd} \tag{A.8.10}$$

Proof. We only have to prove that the bottom right triangle commutes. Since p restricts to the identity on $\operatorname{Pro}\mathcal{C}$, it follows from Lemma A.8.6 that

$$\big(\operatorname{Pro}(F\circ i)\circ p\big)_{|\operatorname{Pro}\mathcal{C}} = \operatorname{Pro}(F\circ i)_{|\operatorname{Pro}\mathcal{C}} = \operatorname{Pro}\big((F\circ i)_{|\mathcal{C}}\big) = F_{|\operatorname{Pro}\mathcal{C}}.$$

Proposition A.8.3 now shows that $\operatorname{Pro}(F\circ i)\circ p$ and F are naturally isomorphic, as required. □

Theorem A.8.11. *If $\mathcal{C}$ and $\mathcal{C}'$ are two small abelian categories, then the above constructions induce equivalences between the following categories.*

(1) The ordinary category of right exact functors $f:\mathcal{C}\to\operatorname{Pro}\mathcal{C}'$.
(2) The ordinary category of right exact cofiltered limit-preserving functors $\operatorname{Pro}(f):\operatorname{Pro}\mathcal{C}\to\operatorname{Pro}\mathcal{C}'$.
(3) The ∞-category of right t-exact functors $F: D^-(\operatorname{Pro}\mathcal{C})\to\operatorname{Pro}D^b(\mathcal{C}')$ which carry projective objects of $\operatorname{Pro}\mathcal{C}$ into $\operatorname{Pro}\mathcal{C}'$ and for which $\tau^{\geq 0}F_{|\operatorname{Pro}\mathcal{C}}$ is cofiltered limit-preserving.
(4) The ∞-category of right t-exact limit-preserving functors $F':\operatorname{Pro}D^b(\mathcal{C})\to\operatorname{Pro}D^b(\mathcal{C}')$ which carry projective objects of $\operatorname{Pro}\mathcal{C}$ into $\operatorname{Pro}\mathcal{C}'$.

Under these equivalences, we have $F'=\operatorname{Pro}(F\circ i)$, and $F=F'\circ p$.

Proof. The equivalences between the first three categories are immediate from Proposition A.8.3 (1), so it remains to see that the functors $F\mapsto\operatorname{Pro}(F\circ i)$ and $F'\mapsto F'\circ p$ are inverse equivalences of categories between (3) and (4). This follows from Lemma A.8.9, which shows that the composites in each direction of the purported equivalences are naturally equivalent to the identity functors. Indeed, to show that $F'\mapsto F'\circ p\mapsto\operatorname{Pro}(F'\circ p\circ i)$ is isomorphic to the identity, we need to check that $F'=\operatorname{Pro}(F'\circ p\circ i)=\operatorname{Pro}F'_{|D^b(\mathcal{C})}$. This is precisely the statement that F' is cofiltered limit-preserving (or equivalently, limit-preserving, since F' is exact). On the other hand, the statement that $F\mapsto\operatorname{Pro}(F\circ i)\mapsto\operatorname{Pro}(F\circ i)\circ p$ is isomorphic to the identity follows from the commutativity of the bottom right triangle in (A.8.10). □

Remark A.8.12. The functor $p: D^-(\operatorname{Pro}\mathcal{C})\to\operatorname{Pro}D^b(\mathcal{C})$ (or, more precisely, its restriction to $D^b(\mathcal{C})$) constructed above is a Pro-analogue, in the ∞-categorical context, of the morphism $J: D^b(\operatorname{Ind}\mathcal{C})\to\operatorname{Ind}D^b(\mathcal{C})$ constructed in [KS06, Thm. 15.4.3]: see Section A.9 for an elaboration of this point. Our Theorem A.8.11 is then a Pro- (and ∞-categorical) analogue of [KS06, Prop. 15.4.7].

A.8.13. *A compatibility.* Suppose now that $\mathcal{C}$ itself has enough projectives, so that we can apply Theorem A.7.15 to construct derived functors whose source is $D^-(\mathcal{C})$. We will show that this is compatible with the equivalences of Theorem A.8.11. Note firstly that by Corollary A.7.18, the t-exact functor

$$j: D^-(\mathcal{C})\to D^-(\operatorname{Pro}\mathcal{C})$$

appearing in (A.8.5), which is the functor given by the functoriality of the formation of the derived category D^-, can also be thought of as being the derived functor arising from the exact functor $\mathcal{C}\to\operatorname{Pro}\mathcal{C}$.

As in Lemma A.8.9, we suppose that $f:\mathcal{C}\to\operatorname{Pro}(\mathcal{C}')$ is a right exact functor, and we write $F: D^-(\operatorname{Pro}\mathcal{C})\to\operatorname{Pro}D^b(\mathcal{C}')$ for the right t-exact functor corresponding to $\operatorname{Pro}(f)$ via Proposition A.8.3 (1). We also write $F_{\mathcal{C}}: D^-(\mathcal{C})\to\operatorname{Pro}D^b(\mathcal{C}')$,

$F^-_{\mathcal{C}} : D^-(\mathcal{C}) \to D^-(\operatorname{Pro}\mathcal{C}')$ for the right t-exact functors corresponding to f via Theorem A.7.15.

We can then consider the following variant on the commutative diagram (A.8.10):

$$\begin{tikzcd}
 & D^-(\mathcal{C}) \arrow[r, "F^-_{\mathcal{C}}"] \arrow[d, "j"] \arrow[dr, "F_{\mathcal{C}}"] & D^-(\operatorname{Pro}\mathcal{C}') \arrow[d, "p'"] \\
D^b(\mathcal{C}) \arrow[ur, "a"] \arrow[r, "i"] \arrow[dr, "b"'] & D^-(\operatorname{Pro}\mathcal{C}) \arrow[r, "F"] \arrow[d, "p"] & \operatorname{Pro} D^b(\mathcal{C}') \\
 & \operatorname{Pro} D^b(\mathcal{C}) \arrow[ur, "\operatorname{Pro}(F\circ i)"'] &
\end{tikzcd} \tag{A.8.14}$$

Lemma A.8.15. *Suppose as above that $\mathcal{C}$ has enough projectives, and that $f : \mathcal{C} \to \operatorname{Pro}(\mathcal{C}')$ is right exact. Then the diagram* (A.8.14) *commutes; in particular,* $\operatorname{Pro}(F \circ i) = \operatorname{Pro}(F_{\mathcal{C}} \circ a)$.

Proof. Note firstly that $F_{\mathcal{C}}$ is naturally isomorphic to $p' \circ F^-_{\mathcal{C}}$, by Theorem A.7.15 and the t-exactness of p'. We next show that $\operatorname{Pro}(F \circ i) = \operatorname{Pro}(F_{\mathcal{C}} \circ a)$. Since $F_{\mathcal{C}} \circ a$ is right t-exact, its Pro-extension is a right t-exact functor

$$\operatorname{Pro}(F_{\mathcal{C}} \circ a) : \operatorname{Pro} D^b(\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}')$$

satisfying

$$(\tau^{\geq 0} \operatorname{Pro}(F_{\mathcal{C}} \circ a))_{|\operatorname{Pro}\mathcal{C}} \xrightarrow{\sim} \operatorname{Pro}(\tau^{\geq 0} F_{\mathcal{C}} \circ a)_{|\operatorname{Pro}\mathcal{C}} \xrightarrow{\sim} \operatorname{Pro} f.$$

Thus, to show that $\operatorname{Pro}(F_{\mathcal{C}} \circ a)$ coincides with $\operatorname{Pro}(F \circ i)$, we have (according to Theorem A.8.11) to show that $\operatorname{Pro}(F_{\mathcal{C}} \circ a)$ takes projective objects of $\operatorname{Pro}\mathcal{C}$ to the heart $\operatorname{Pro}\mathcal{C}'$ of $\operatorname{Pro} D^b(\mathcal{C}')$.

By Lemma A.1.10, any projective object in $\operatorname{Pro}\mathcal{C}$ is of the form $\lim_i X_i$, with the X_i being projective objects of $\mathcal{C}$. Thus, for such an object,

$$\operatorname{Pro}(F_{\mathcal{C}} \circ a)(\lim_i X_i) := \lim_i F_{\mathcal{C}}(X_i)$$

indeed lies in $\operatorname{Pro}\mathcal{C}'$, since each $F_{\mathcal{C}}(X_i)$ lies in $\operatorname{Pro}\mathcal{C}'$ (as each X_i is projective in $\mathcal{C}$, and $F_{\mathcal{C}}$ is associated to f via the construction of Theorem A.7.15). This concludes the proof that $\operatorname{Pro}(F_{\mathcal{C}} \circ a) = \operatorname{Pro}(F \circ i)$.

At this point, bearing in mind Lemma A.8.9, in order to conclude the proof that (A.8.14) commutes, there remains to check that in fact $F_{\mathcal{C}} = F \circ j$, which we will do by applying Theorem A.7.15. Note first that the t-exactness of j implies that

$$\tau^{\geq 0}(F \circ j)|_{\mathcal{C}} = \tau^{\geq 0} F|_{\operatorname{Pro}\mathcal{C}} \circ j|_{\mathcal{C}} = f;$$

so by Theorem A.7.15, it suffices to show that $F \circ j$ takes projective objects of $\mathcal{C}$ to $\operatorname{Pro}\mathcal{C}'$. To this end, note that

$$(F \circ j) \circ a = F \circ i = \operatorname{Pro}(F \circ i) \circ b = \operatorname{Pro}(F_{\mathcal{C}} \circ a) \circ b = F_{\mathcal{C}} \circ a.$$

Since the restriction of a to $\mathcal{C}$ is the identity functor, this implies that the restriction of $F \circ j$ to $\mathcal{C}$ agrees with the restriction of $F_{\mathcal{C}}$ to $\mathcal{C}$, and since $F_{\mathcal{C}}$ takes projective objects of $\mathcal{C}$ to $\operatorname{Pro}\mathcal{C}'$ by construction, we are done. $\square$

A.8.16. *Deriving adjoint functors.* Let $g : \mathcal{C}' \to \mathcal{C}$ be an exact functor between small abelian categories. We continue to write g for its Pro-extension $g : \mathrm{Pro}(\mathcal{C}') \to \mathrm{Pro}(\mathcal{C})$, which by Lemma A.1.6 has a left adjoint $f : \mathrm{Pro}\,\mathcal{C} \to \mathrm{Pro}\,\mathcal{C}'$, which is right exact and cofiltered limit-preserving.

Proposition A.8.17. *As above, let $g : \mathcal{C}' \to \mathcal{C}$ be an exact functor between small abelian categories, let $g : \mathrm{Pro}\,\mathcal{C}' \to \mathrm{Pro}\,\mathcal{C}$ be its* Pro*-extension, and let $f : \mathrm{Pro}\,\mathcal{C} \to \mathrm{Pro}\,\mathcal{C}'$ be the left adjoint of g. Let $F : \mathrm{Pro}\,D^b(\mathcal{C}) \to \mathrm{Pro}\,D^b(\mathcal{C}')$ and $G : \mathrm{Pro}\,D^b(\mathcal{C}') \to \mathrm{Pro}\,D^b(\mathcal{C})$ be the limit-preserving functors determined by f, g respectively by Theorem* A.8.11 *and Corollary* A.8.2*, which are respectively right t-exact and t-exact.*

Then:

1. *(1) F is left adjoint to G.*
2. *(2) If furthermore f is exact, then F is t-exact, and is the Pro-extension of the t-exact functor $D^b(\mathcal{C}) \to \mathrm{Pro}\,D^b(\mathcal{C}')$ determined by f.*
3. *(3) Continuing to assume that f is exact, the* Ind*-extensions of F and G are adjoint t-exact functors.*

Proof. The composite $G \circ F : \mathrm{Pro}\,D^b(\mathcal{C}) \to \mathrm{Pro}\,D^b(\mathcal{C})$ is again right t-exact and limit-preserving, and takes projective objects of $\mathrm{Pro}\,\mathcal{C}$ into $\mathrm{Pro}\,\mathcal{C}$. Theorem A.8.11 (in particular, the equivalence of ∞-categories that it provides) thus implies that the unit of adjunction $\mathrm{id}_{\mathrm{Pro}\,\mathcal{C}} \to g \circ f$ induces a morphism $\mathrm{id}_{\mathrm{Pro}\,D^b(\mathcal{C})} \to G \circ F$. For any objects X of $\mathrm{Pro}\,D^b(\mathcal{C})$ and Y of $\mathrm{Pro}\,D^b(\mathcal{C}')$, this morphism induces functorial maps

$$\begin{aligned}&\mathrm{Maps}_{\mathrm{Pro}\,D^b(\mathcal{C}')}\big(F(X), Y\big)\\ &\qquad\xrightarrow{\text{apply } G} \mathrm{Maps}_{\mathrm{Pro}\,D^b(\mathcal{C})}\big(GF(X), G(Y)\big)\\ &\qquad\qquad\longrightarrow \mathrm{Maps}_{\mathrm{Pro}\,D^b(\mathcal{C})}\big(X, G(Y)\big).\end{aligned}$$

We claim that this composite is an isomorphism, thus showing that F and G are indeed adjoints.

Writing each of X and Y as the limit of objects in $D^b(\mathcal{C})$ and $D^b(\mathcal{C}')$ respectively, and using the fact that F and G are limit-preserving, we reduce to the case when X and Y are in fact objects of $D^b(\mathcal{C})$ and $D^b(\mathcal{C}')$. An obvious dévissage using the fibre sequences (A.6.1) for X and Y (and the exactness of F and G) reduces us the case that X is an object of $\mathcal{C}$ and Y is an object of $\mathcal{C}'$.

We are thus reduced to a consideration of the morphism

$$\mathrm{Maps}_{\mathrm{Pro}\,D^b(\mathcal{C}')}\big(F(X), Y\big) \to \mathrm{Maps}_{\mathrm{Pro}\,D^b(\mathcal{C})}\big(X, G(Y)\big)$$

with X and Y objects of $\mathcal{C}$ and $\mathcal{C}'$ respectively. The adjunction property of $\tau^{\geq 0}$ lets us rewrite this as

$$\mathrm{Maps}_{\mathrm{Pro}\,D^b(\mathcal{C}')}\big(\tau^{\geq 0}F(X), Y\big) \to \mathrm{Maps}_{\mathrm{Pro}\,D^b(\mathcal{C})}\big(X, G(Y)\big),$$

and then (using the t-exactness of G) as

$$\mathrm{Maps}_{\mathrm{Pro}\,\mathcal{C}'}\big(f(X), Y\big) \to \mathrm{Maps}_{\mathrm{Pro}\,\mathcal{C}}\big(X, g(Y)\big).$$

This is precisely the map induced by the unit of adjunction for f and g, and so is indeed an isomorphism, as required. This concludes the proof of the first point. The second point then follows from Corollary A.8.2, and the third point is clear, since Ind-extension preserves t-exactness and adjoints. ∎

A.9. **Pro-categories and the co-Yoneda embedding.** Let $\mathcal{C}$ be a small abelian category. We may restrict the t-exact functor p of (A.8.5) to a t-exact functor

$$p : D^b(\operatorname{Pro}\mathcal{C}) \to \operatorname{Pro} D^b(\mathcal{C}). \tag{A.9.1}$$

If $\mathcal{B}$ is an exact full abelian subcategory of $\operatorname{Pro}\mathcal{C}$, containing the image of the canonical embedding $\mathcal{C} \to \operatorname{Pro}\mathcal{C}$, then the composite of (A.9.1) with the canonical t-exact functor $D^b(\mathcal{B}) \to D^b(\operatorname{Pro}\mathcal{C})$ induces a t-exact functor

$$q : D^b(\mathcal{B}) \to \operatorname{Pro} D^b(\mathcal{C}). \tag{A.9.2}$$

The functor (A.9.1) need not be fully faithful in general, by e.g. [KS06, Rem. 15.4.5, Ex. 15.2], and so the functor q certainly need not be fully faithful in general. Nevertheless, it is sometimes the case that q is fully faithful: for example, by Lemma A.8.9, this is true when $\mathcal{B} = \mathcal{C}$. Our goal in this section is to give a useful sufficient condition for full faithfulness of q (see Lemma A.9.9).

Before doing so, it will be useful to give another description of (A.9.1). In order to do this, we first explain the "dual" set-up, where we consider Ind-categories rather than Pro-categories. This will let us speak of presheaves, Yoneda embeddings, and so on, rather than their "co" variants. Passing to opposite categories at the end of the discussion will then give the description we need.

If $\mathcal{C}$ is a small abelian category, the functor $p : D^-(\operatorname{Pro}\mathcal{C}^{\mathrm{op}}) \to \operatorname{Pro} D^b(\mathcal{C}^{\mathrm{op}})$ induces, by passage to opposite categories, a t-exact functor $\check{p} : D^+(\operatorname{Ind}\mathcal{C}) \to \operatorname{Ind} D^b(\mathcal{C})$, which we may restrict to a functor

$$\check{p} : D^b(\operatorname{Ind}\mathcal{C}) \to \operatorname{Ind} D^b(\mathcal{C}).$$

By Corollary A.7.20, $\check{p}$ is the unique (up to equivalence) t-exact functor $D^b(\operatorname{Ind}\mathcal{C}) \to \operatorname{Ind} D^b(\mathcal{C})$ inducing the identity on hearts. Using the Yoneda embedding, we can also give the following alternative description of $\check{p}$: the canonical functor $D^b(\mathcal{C}) \to D^b(\operatorname{Ind}\mathcal{C})$ gives rise to an exact functor

$$D^b(\operatorname{Ind}\mathcal{C}) \to \operatorname{Fun}^{\mathrm{ex}}\big(D^b(\mathcal{C})^{\mathrm{op}}, \mathrm{Sp}\big), \quad X \mapsto \operatorname{RHom}_{D^b(\operatorname{Ind}\mathcal{C})}(-, X). \tag{A.9.3}$$

If we regard $\operatorname{Ind} D^b(\mathcal{C})$ as a stable sub-∞-category of $\operatorname{Fun}^{\mathrm{ex}}\big(D^b(\mathcal{C})^{\mathrm{op}}, \mathrm{Sp}\big)$ via the discussion of Remark A.2.16, then we see that $\check{p}$ and (A.9.3) are exact functors

$$D^b(\operatorname{Ind}\mathcal{C}) \to \operatorname{Fun}^{\mathrm{ex}}\big(D^b(\mathcal{C})^{\mathrm{op}}, \mathrm{Sp}\big),$$

both of which induce the canonical equivalence $\operatorname{Ind}\mathcal{C} \xrightarrow{\sim} \big(\operatorname{Ind} D^b(\mathcal{C})\big)^{\heartsuit}$. By Theorem A.7.19, they must coincide.

Concretely, this means that if X is an object of $D^b(\operatorname{Ind}\mathcal{C})$ and Y is an object of $D^b(\mathcal{C})$, then

$$\operatorname{RHom}_{D^b(\operatorname{Ind}\mathcal{C})}(Y, X) \xrightarrow{\sim} \operatorname{RHom}_{\operatorname{Ind} D^b(\mathcal{C})}\big(Y, \check{p}(X)\big). \tag{A.9.4}$$

Returning now to the "un-dualized" setting of Pro-categories, we find that the restriction (A.9.1) of p admits a co-Yoneda description. Concretely, if X is an object of $D^b(\operatorname{Pro}\mathcal{C})$ and Y is an object of $D^b(\mathcal{C})$, then the canonical morphism

$$\begin{aligned}\operatorname{RHom}_{D^b(\operatorname{Pro}\mathcal{C})}(X, Y) &= \operatorname{RHom}_{D^-(\operatorname{Pro}\mathcal{C})}(X, Y) \\ &\to \operatorname{RHom}_{\operatorname{Pro} D^b(\mathcal{C})}\big(p(X), Y\big)\end{aligned} \tag{A.9.5}$$

induced by p is an isomorphism. (Here we have used the commutativity of (A.8.5) to identify the restriction of p to $D^b(\mathcal{C})$ with the canonical embedding of $D^b(\mathcal{C})$ into $\operatorname{Pro} D^b(\mathcal{C})$, and so to identify Y with $p(Y)$.)

We then have the following lemma, which specializes, in the case when $\mathcal{B}$ equals $\mathrm{Pro}\,\mathcal{C}$ and $p = q$, to the fact that (A.9.5) is an isomorphism.

Lemma A.9.6. *Let $\mathcal{C}$ be a small abelian category, and let $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathrm{Pro}\,\mathcal{C}$ be an exact full abelian subcategory of $\mathrm{Pro}\,\mathcal{C}$. If X and Y are objects of $D^b(\mathcal{B})$ and $D^b(\mathcal{C})$ respectively, then q induces a natural isomorphism*

$$\mathrm{RHom}_{D^b(\mathcal{B})}(X, Y) \xrightarrow{\sim} \mathrm{RHom}_{\mathrm{Pro}\,D^b(\mathcal{C})}\big(q(X), Y\big),$$

where we used the commutativity of (A.8.5) *to identify the restriction of p to $D^b(\mathcal{C})$ with the canonical embedding of $D^b(\mathcal{C})$ into $\mathrm{Pro}\,D^b(\mathcal{C})$, and so to identify Y with $q(Y)$.*

Proof. A standard truncation argument reduces to the case when X is an object of $\mathcal{B}$ and Y is an object of $\mathcal{C}$. We may and do write $X = \lim_i X_i$ for some objects X_i of $\mathcal{C}$. We then consider the sequence of morphisms

(A.9.7)
$$\begin{aligned} &\mathrm{colim}_i\, \mathrm{RHom}_{D^b(\mathcal{C})}(X_i, Y) \to \mathrm{colim}_i\, \mathrm{RHom}_{D^b(\mathcal{B})}(X_i, Y) \to \mathrm{RHom}_{D^b(\mathcal{B})}(X, Y) \\ &\to \mathrm{RHom}_{D^b(\mathrm{Pro}\,\mathcal{C})}(X, Y) \xrightarrow{\sim} \mathrm{RHom}_{\mathrm{Pro}\,D^b(\mathcal{C})}\big(q(X), Y\big) \xrightarrow{\sim} \mathrm{colim}_i\, \mathrm{RHom}_{D^b(\mathcal{C})}(X_i, Y) \end{aligned}$$

(with the first three morphisms being the evident ones, the first isomorphism being that of (A.9.5), and the second isomorphism arising from the definition of morphisms in the Pro category). Evidently the composite of this sequence of morphisms is the identity.

Passing to cohomology, we see that the natural morphism

$$\mathrm{Ext}^i_{D^b(\mathcal{B})}(X, Y) \to \mathrm{Ext}^i_{D^b(\mathrm{Pro}\,\mathcal{C})}(X, Y)$$

is surjective for all $i \geq 0$, for any objects X and Y as above. Since this morphism is an isomorphism when $i = 0$, a standard argument then shows that it is in fact an isomorphism for all $i \geq 0$. (For the sake of completeness, we have recalled this argument in Lemma A.9.8 below; note that Yoneda Ext functors computed in any abelian category are element-wise effaceable in the sense of that lemma.) Thus in fact $\mathrm{RHom}_{D^b(\mathcal{B})}(X, Y) \xrightarrow{\sim} \mathrm{RHom}_{D^b(\mathrm{Pro}\,\mathcal{C})}(X, Y)$, and the lemma follows by another consideration of (A.9.7). ∎

Lemma A.9.8. *Let $\eta^\bullet : F^\bullet \to G^\bullet$ be a natural transformation of abelian group-valued δ-functors on an abelian category $\mathcal{A}$. Suppose that η^0 is an isomorphism, and that η^n is an epimorphism for each n. Suppose also that F^n is* element-wise effaceable *for each $n > 0$ (in the sense that if $\alpha \in F^n(X)$ for some object X of $\mathcal{A}$, then there exists a monomorphism $X \hookrightarrow Y$ in $\mathcal{A}$ with respect to which the image of α in $F^n(Y)$ vanishes). Then $\eta^\bullet$ is in fact a natural isomorphism.*

Proof. We prove that η^n is an isomorphism for each $n \geq 0$ via induction on n, the case $n = 0$ holding by hypothesis. Thus, we assume that η^i is an isomorphism for all $0 \leq i < n$, and prove that η^n is an isomorphism. In fact, we need only show that η^n is a monomorphism, since it is an epimorphism by assumption. Choose an object X in $\mathcal{A}$, and an element $\alpha \in \ker\big(\eta^n(X) : F^n(X) \to G^n(X)\big)$.

By hypothesis we may find a monomorphism $X \hookrightarrow Y$ such that the image of α in $F^n(Y)$ vanishes. Let Z denote the cokernel of this embedding; we then have a commutative diagram with exact rows

$$\begin{tikzcd}
F^{n-1}(Y) \arrow[r] \arrow[d] & F^{n-1}(Z) \arrow[r] \arrow[d] & F^n(X) \arrow[r] \arrow[d] & F^n(Y) \arrow[d] \\
G^{n-1}(Y) \arrow[r] & G^{n-1}(Z) \arrow[r] & G^n(X) \arrow[r] & G^n(Y)
\end{tikzcd}$$

A straightforward diagram chase (using that the left two vertical arrows are isomorphisms) shows that α itself vanishes. Thus we see that η^n is indeed a monomorphism, as required. □

The next lemma provides the promised criterion for the functor q to be fully faithful.

Lemma A.9.9. *Suppose that $\mathcal{C} \subseteq \mathcal{B} \subseteq \operatorname{Pro}\mathcal{C}$ is an exact full abelian subcategory of $\operatorname{Pro}\mathcal{C}$. Suppose further that for any objects X, Y of $\mathcal{B}$, writing $Y = \lim_i Y_i$ as a cofiltered limit of objects of $\mathcal{C}$, the canonical morphism*

$$\operatorname{RHom}_{D^b(\mathcal{B})}(X, Y) \to \lim_i \operatorname{RHom}_{D^b(\mathcal{B})}(X, Y_i) \tag{A.9.10}$$

is an isomorphism. Then $q : D^b(\mathcal{B}) \to \operatorname{Pro} D^b(\mathcal{C})$ is fully faithful.

Proof. An easy argument with truncations (taking into account the exactness of q) reduces the proposition to verifying that the natural morphism

$$\operatorname{RHom}_{D^b(\mathcal{B})}(X, Y) \to \operatorname{RHom}_{\operatorname{Pro} D^b(\mathcal{C})}\big(q(X), q(Y)\big) \tag{A.9.11}$$

is an isomorphism whenever X is an object of $D^b(\mathcal{B})$ and Y is an object of $\mathcal{B}$. Similarly, our assumption that (A.9.10) is an isomorphism when $X \in \mathcal{B}$ implies that (A.9.10) is also an isomorphism when $X \in D^b(\mathcal{B})$.

Write $Y = \lim_i Y_i$. Since (A.9.10) is an isomorphism, and $q(Y)$ coincides with Y (viewed as an object of $\operatorname{Pro}\mathcal{C}$), we see that it furthermore suffices to prove that (A.9.11) is an isomorphism when Y is replaced by each of the Y_i; that is, we reduce to the case when Y is an object of $\mathcal{C}$. The result in this case follows from Lemma A.9.6. □

A.10. **Derived tensor products.** We develop derived analogues of the various constructions considered in Section A.1.46.

A.10.1. *The derived category of finite length R-modules.* Let R be a Noetherian profinite $\mathcal{O}$-algebra. By (A.1.37), we have $\operatorname{Mod}_c(R) \xrightarrow{\sim} \operatorname{Pro}\operatorname{Mod}^{\mathrm{f.l.}}(R)$.

Definition A.10.2. Write $D^b_{\mathrm{f.l.}}(R)$ to denote the full subcategory of $D^b\big(\operatorname{Mod}_c(R)\big)$ consisting of objects whose cohomologies are of finite length, i.e. lie in $\operatorname{Mod}^{\mathrm{f.l.}}(R)$.

Remark A.10.3. Since $\operatorname{Mod}^{\mathrm{f.l.}}(R)$ is a Serre subcategory of $\operatorname{Mod}_c(R)$, Definition A.10.2 makes sense, as a particular instance of Definition A.7.6. Despite what the notation may suggest, $D^b_{\mathrm{f.l.}}(R)$ *a priori* depends on the structure of R as a topological ring, since it is defined as a full subcategory of $D^b\big(\operatorname{Mod}_c(R)\big)$. Part (1) of Lemma A.10.4 shows in particular that it admits another description depending only on the structure of R as an abstract ring.

Recall that by Lemma A.1.32 we have natural fully faithful exact embeddings $\operatorname{Mod}^{\mathrm{f.l.}}(R) \hookrightarrow \operatorname{Mod}_c(R)$, $\operatorname{Mod}^{\mathrm{fp}}(R) \hookrightarrow \operatorname{Mod}_c(R)$, and $\operatorname{Mod}^{\mathrm{f.l.}}(R) \hookrightarrow \operatorname{Mod}^{\mathrm{fp}}(R)$. The following lemma shows that the derived extensions of these functors remain fully faithful, and, in the case of the first embedding, identifies the essential image with the stable ∞-category $D^b_{\mathrm{f.l.}}(R)$ introduced in Definition A.10.2.

Lemma A.10.4.

(1) The natural t-exact functor $D^b\big(\mathrm{Mod}^{\mathrm{f.l.}}(R)\big) \to D^b\big(\mathrm{Mod}_c(R)\big)$ is fully faithful, and induces an equivalence $D^b(\mathrm{Mod}^{\mathrm{f.l.}}(R)) \xrightarrow{\sim} D^b_{\mathrm{f.l.}}(R)$.

(2) The natural t-exact functor $D^b\big(\mathrm{Mod}^{\mathrm{fp}}(R)\big) \to D^b\big(\mathrm{Mod}_c(R)\big)$ is fully faithful.

(3) The natural t-exact functor $D^b\big(\mathrm{Mod}^{\mathrm{f.l.}}(R)\big) \to D^b\big(\mathrm{Mod}^{\mathrm{fp}}(R)\big)$ is fully faithful.

Proof. By Lemma A.1.32 (2), $\mathrm{Mod}_c(R)^{\mathrm{op}}$ is a locally finite category, in the sense recalled in A.1.25, i.e. it is a Grothendieck category with compact objects $\mathrm{Mod}^{\mathrm{f.l.}}(R)^{\mathrm{op}}$ (see Lemma A.1.26 (1) for this assertion regarding compact objects). Since $\mathrm{Mod}^{\mathrm{f.l.}}(R)^{\mathrm{op}}$ is abelian, $\mathrm{Mod}_c(R)^{\mathrm{op}}$ is a locally coherent abelian category, and part (1) of the lemma is therefore immediate from Lemma A.7.11 by passing to opposite categories.

To prove (2), it follows from Proposition A.2.25 (2) that it suffices to show that the embedding $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}_c(R)$ induces an isomorphism on all Ext^i. This follows from [VV97, Prop. 3.16]. (Note that Corollary 3.12 of the same reference shows that the objects of $\mathrm{Mod}^{\mathrm{fp}}(R)$ are Noetherian when regarded as objects of $\mathrm{Mod}_c(R)$, so that the cited Proposition does indeed apply.)

Part (3) follows immediately from parts (1) and (2). $\square$

Remark A.10.5. Lemma A.10.4 gives rise to the following convenient t-exact fully faithful embedding:

$$D^b_{\mathrm{f.l.}}(R) \xrightarrow{\sim} D^b\big(\mathrm{Mod}^{\mathrm{f.l.}}(R)\big) \hookrightarrow D^b\big(\mathrm{Mod}^{\mathrm{fp}}(R)\big) \xrightarrow{\sim} D^b_{\mathrm{fp}}(R), \tag{A.10.6}$$

with the first equivalence being the inverse of the equivalence of part (1) of the lemma, the fully faithful embedding being that of part (3) of the lemma, and the final equivalence being that of Corollary A.7.12. The utility of (A.10.6) is that while the source is defined with reference to the topology of R, the target depends only on R as an abstract ring.

Remark A.10.7. By the discussion in Section A.2.31, the Pro-extension of (A.10.6) has a left adjoint

$$\mathrm{Pro}\, D^b_{\mathrm{fp}}(R) \to \mathrm{Pro}\, D^b_{\mathrm{f.l.}}(R). \tag{A.10.8}$$

This left adjoint is t-exact, and maps the constant object $R \in \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(R)$ to $\lim_n R/\mathrm{rad}(R)^n \in \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(R)$. To see this, we apply Proposition A.8.17 with $g : \mathrm{Mod}^{\mathrm{f.l.}}(R) \to \mathrm{Mod}^{\mathrm{fp}}(R)$ taken to be the inclusion, noting that the left adjoint $\mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(R)$ to $\mathrm{Pro}(g)$ is exact and sends R to $\lim_n R/\mathrm{rad}(R)^n$, by Remark A.1.60.

Note furthermore that the composite

$$D^b_{\mathrm{f.l.}}(R) \xrightarrow{\text{(A.10.6)}} D^b_{\mathrm{fp}}(R) \subset \mathrm{Pro}\, D^b_{\mathrm{fp}}(R) \xrightarrow{\text{(A.10.8)}} \mathrm{Pro}\, D^b_{\mathrm{f.l.}}(R)$$

is the natural embedding of $D^b_{\mathrm{f.l.}}(R)$ in $\mathrm{Pro}\, D^b_{\mathrm{f.l.}}(R)$. (These functors being t-exact, this statement can be checked on restriction to the hearts.)

A.10.9. *Derived tensor products.* We now return to the setting of Definition A.1.47. We assume in addition that the ring R is coherent, so that $\mathrm{Mod}(R)$ is locally coherent, with compact objects given by the abelian category $\mathrm{Mod}^{\mathrm{fp}}(R)$. By Corollary A.7.12, the natural functor

$$D^b(\mathrm{Mod}^{\mathrm{fp}}(R)) \to D^b_{\mathrm{fp}}(R) := D^b_{\mathrm{Mod}^{\mathrm{fp}}(R)}(\mathrm{Mod}(R))$$

is an equivalence. Similarly, bearing in mind the fact that $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}(R)$ preserves projectives, it follows from [Lur17, Prop. 1.3.3.7] (see also [EGH25, Prop. A.5.7]) that the natural functor

$$D^-(\mathrm{Mod}^{\mathrm{fp}}(R)) \to D^-_{\mathrm{fp}}(R) := D^-_{\mathrm{Mod}^{\mathrm{fp}}(R)}(\mathrm{Mod}(R))$$

is an equivalence.

We now apply the machinery of Appendix A.8 to construct various derived versions of the tensor products provided by Proposition A.1.48 and Lemma A.1.53.

Lemma A.10.10. *Let R be a coherent ring, and let $\mathcal{A}$ be an abelian category.*

(1) *The functor $F \mapsto F(R)$ gives equivalences of categories between the category of right R-modules M in $\mathcal{A}$ and the following categories of functors.*
   (a) *The category of right exact functors*
   $$M \otimes_R - : \mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}.$$
   (b) *The category of right exact cofiltered limit-preserving functors*
   $$M \mathbin{\widehat{\otimes}}_R - : \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Pro}(\mathcal{A})$$
   taking R to an object of $\mathcal{A}$.
   (c) *The category of right t-exact functors*
   $$M \otimes^L_R - : D^b_{\mathrm{fp}}(R) \to D^-(\mathcal{A})$$
   taking R to an object of $\mathcal{A}$.
   (d) *The category of right t-exact functors*
   $$M \otimes^L_R - : D^-_{\mathrm{fp}}(R) \to D^-(\mathcal{A})$$
   taking R to an object of $\mathcal{A}$.
   (e) *The category of right t-exact limit-preserving functors*
   $$M \mathbin{\widehat{\otimes}}^L_R - : \mathrm{Pro}\, D^b_{\mathrm{fp}}(R) \to \mathrm{Pro}\, D^b(\mathcal{A})$$
   taking R to an object of $\mathcal{A}$.

   Furthermore, we have a commutative diagram

(A.10.11)
$$\begin{tikzcd} D^b_{\mathrm{fp}}(R) \arrow[r] \arrow[rr, bend left, "(1c)"] \arrow[dr] & D^-_{\mathrm{fp}}(R) \arrow[r, "(1d)"] \arrow[d] & D^-(\mathcal{A}) \arrow[r] & \mathrm{Pro}\, D^b(\mathcal{A}) \\ & \mathrm{Pro}\, D^b_{\mathrm{fp}}(R) \arrow[urr, "(1e)"'] & & \end{tikzcd}$$

(2) *The functor $F \mapsto F(R)$ gives equivalences of categories between the category of right R-modules M in $\mathrm{Pro}(\mathcal{A})$ and the following categories of functors.*
   (f) *The category of right exact functors*
   $$M \otimes_R - : \mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Pro}(\mathcal{A}).$$

(g) *The category of right exact cofiltered limit-preserving functors*

$$M \,\widehat{\otimes}_R - \colon \operatorname{Pro}\operatorname{Mod}^{\mathrm{fp}}(R) \to \operatorname{Pro}(\mathcal{A}).$$

(h) *The category of right t-exact functors*

$$M \otimes^L_R - \colon D^b_{\mathrm{fp}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$$

taking R to an object of $\operatorname{Pro}(\mathcal{A})$.

(i) *The category of right t-exact functors*

$$M \otimes^L_R - \colon D^-_{\mathrm{fp}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$$

taking R to an object of $\operatorname{Pro}(\mathcal{A})$.

(j) *The category of right t-exact limit-preserving functors*

$$M \,\widehat{\otimes}^L_R - \colon \operatorname{Pro} D^b_{\mathrm{fp}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$$

taking R to an object of $\operatorname{Pro}(\mathcal{A})$.

Furthermore, we have a commutative diagram

$$\begin{tikzcd}
D^b_{\mathrm{fp}}(R) \arrow[d] \arrow[dr, "(2h)"] & \\
D^-_{\mathrm{fp}}(R) \arrow[r, "(2i)"] \arrow[d] & \operatorname{Pro} D^b(\mathcal{A}) \\
\operatorname{Pro} D^b_{\mathrm{fp}}(R) \arrow[ur, "(2j)"'] &
\end{tikzcd} \tag{A.10.12}$$

(3) *Suppose furthermore that R is in fact a Noetherian profinite topological $\mathcal{O}$-algebra. Then the functors $F \mapsto F(R)$ and $F \mapsto \varprojlim_n F(R/\operatorname{rad}(R)^n)$ give equivalences of categories between the category of complete right R-modules in* $\operatorname{Pro}(\mathcal{A})$ (*in the sense of Definition* A.1.52) *and the following categories of functors.*

(k) *The category of right exact functors*

$$M \otimes_R - \colon \operatorname{Mod}^{\mathrm{f.l.}}(R) \to \operatorname{Pro}(\mathcal{A}).$$

(l) *The category of right exact cofiltered limit-preserving functors*

$$M \,\widehat{\otimes}_R - \colon \operatorname{Mod}_c(R) \to \operatorname{Pro}(\mathcal{A}).$$

(m) *The category of right t-exact functors*

$$M \otimes^L_R - \colon D^b_{\mathrm{f.l.}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$$

for which $\lim_n \Big(M \otimes^L_R \big(R/\operatorname{rad}(R)^n\big)\Big)$ (*computed in* $\operatorname{Pro} D^b(\mathcal{A})$) *lies in* $\operatorname{Pro}(\mathcal{A})$ (*the latter being regarded as the heart of* $\operatorname{Pro} D^b(\mathcal{A})$).

(n) *The category of right t-exact limit-preserving functors*

$$M \,\widehat{\otimes}^L_R - \colon \operatorname{Pro} D^b_{\mathrm{f.l.}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$$

taking R to an object of $\operatorname{Pro}(\mathcal{A})$.

Proof. We begin with part (1). The equivalences between the category of right R-modules in $\mathcal{A}$ and the categories of functors in (1a) and (1b) are established in Proposition A.1.48, and the existence of the derived functors in (1c)–(1e) follows from Theorem A.7.15 and Proposition A.8.3. By Theorem A.8.11 and Lemma A.8.15, in order to see that these constructions induce the claimed equivalences of categories (and to see the commutativity of (A.10.11)), we must show that in each of (1c)–(1e), the assumption that R is taken to an object of $\mathcal{A}$ implies that all projective objects in the heart of the source are taken to objects in the heart of the target. In (1c) and (1d) this is clear, since the projective objects of $\mathrm{Mod}^{\mathrm{fp}}(R)$ are direct summands of finite free R-modules. The case of (1e) then follows from Lemma A.1.10.

The proof of part (2) is almost identical. Turning to part (3), the equivalence between the category of complete right R-modules in $\mathrm{Pro}(\mathcal{A})$ and the category of functors in (3l) is established in Lemma A.1.53. The equivalence between (3k) and (3l) is immediate from the definition of the Pro-category (recalling also the equivalence $\mathrm{Mod}_c(R) \xrightarrow{\sim} \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(R)$ of (A.1.37)), as is the equivalence between (3m) and (3n). By Theorem A.8.11, we are reduced to showing that any functor as in (3n) necessarily takes projective objects of $\mathrm{Mod}_c(R)$ to objects of $\mathrm{Pro}(\mathcal{A})$. This follows from the fact that every projective object of $\mathrm{Mod}_c(R)$ is a product of finite projective R-modules (see [Gab62, §IV.3, Cor. 1]), and the assumption that the functor in (3n) is limit-preserving. □

We now describe the cohomology of the derived tensor product with a right module in $\mathrm{Pro}(\mathcal{A})$.

Lemma A.10.13. *Let R be a coherent ring, let $\mathcal{A}$ be an abelian category, and let $\{M_i : i \in I\}$ be a cofiltered system of right R-modules in $\mathcal{A}$. Let $M \coloneqq \lim_i M_i$, a right R-module in* $\mathrm{Pro}(\mathcal{A})$. *Then for all $X \in D^-_{\mathrm{fp}}(R)$ and $p \in \mathbf{Z}$, we have*

$$H^p(M \otimes^L_R X) = \lim\nolimits_i H^p(M_i \otimes^L_R X)$$

as objects of $\mathrm{Pro}\, D^b(\mathcal{A})^\heartsuit = \mathrm{Pro}(\mathcal{A})$, *where $M \otimes^L_R -$ and $M_i \otimes^L_R -$ are defined as in Lemma* A.10.10 (2i).

Proof. Since truncation functors commute with cofiltered limits in $\mathrm{Pro}\, D^b(\mathcal{A})$, the functor

$$\lim\nolimits_i (M_i \otimes^L_R -) : D^-_{\mathrm{fp}}(R) \to \mathrm{Pro}\, D^b(\mathcal{A})$$

is right t-exact. It takes R to $\lim_i M_i$, computed in $\mathrm{Pro}\, D^b(\mathcal{A})$, which coincides with the limit computed in $\mathrm{Pro}(\mathcal{A})$, which is M. (This is because $\mathrm{Pro}(\mathcal{A}) \to \mathrm{Pro}\, D^b(\mathcal{A})$ is Pro-extended, and so preserves cofiltered limits.) By the uniqueness assertion in Lemma A.10.10 (2i), $\lim_i (M_i \otimes^L_R -)$ is isomorphic to $M \otimes^L_R -$. Hence $H^p(M \otimes^L_R X) = \lim_i H^p(M_i \otimes^L_R X)$, as desired. □

Remark A.10.14. In the context of Lemma A.10.13, the cohomology $H^p(M_i \otimes^L_R X)$ does not change depending on whether we regard $M_i \otimes^L_R$ as a functor $F : D^-_{\mathrm{fp}}(R) \to D^-(\mathcal{A})$ as in Lemma A.10.10 (1d) or as a functor $F' : D^-_{\mathrm{fp}}(R) \to \mathrm{Pro}\, D^b(\mathcal{A})$ as in Lemma A.10.10 (2i). In fact, in (A.8.10) we have described a t-exact functor $pj : D^-(\mathcal{A}) \to \mathrm{Pro}\, D^b(\mathcal{A})$ inducing the natural inclusion on hearts. Now pjF is right t-exact and sends R to M_i, and so it is isomorphic to F'.

There are several compatibilities between the functors in Lemma A.10.10. Most of these follow straightforwardly from our constructions, in the manner of the following lemma.

Lemma A.10.15. *Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be an abelian category. Let M be a complete right R-module in* $\mathrm{Pro}(\mathcal{A})$*. Then the composite*

$$D^b_{\mathrm{fp}}(R) \xrightarrow{\text{(A.10.8)}} \mathrm{Pro}\, D^b_{\mathrm{f.l.}}(R) \xrightarrow{\text{(3n)}} \mathrm{Pro}\, D^b(\mathcal{A})$$

is naturally isomorphic to (2h)*.*

Proof. As explained in Remark A.10.7, the first arrow is t-exact and maps R to $\varprojlim_n R/\operatorname{rad}(R)^n$. By construction, the second arrow is right t-exact and sends $\varprojlim_n R/\operatorname{rad}(R)^n$ to M. So the composite is right t-exact and sends R to M, as desired. □

There is however one subtlety, relating to the compatibility of the functors of parts (2h) and (3m) of Lemma A.10.10. Considering this compatibility leads to the following definition and lemmas.

Definition A.10.16. Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be an abelian category. Let M be a right R-module in $\mathrm{Pro}(\mathcal{A})$. Then we say that M is *derived complete* if the natural morphisms $M \to M \otimes^L_R \big(R/\operatorname{rad}(R)^n\big)$ (where the derived tensor product is computed by the functor in part (2h) of Lemma A.10.10) give rise to an isomorphism $M \xrightarrow{\sim} \lim_n M \otimes^L_R \big(R/\operatorname{rad}(R)^n\big)$ in $\mathrm{Pro}\, D^b(\mathcal{A})$.

Lemma A.10.17. *Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be an abelian category. Let M be a derived complete right R-module in* $\mathrm{Pro}(\mathcal{A})$*. Then M is complete.*

Proof. By the definition of the t-structure on $\mathrm{Pro}(\mathcal{A})$, the functor $\tau^{\geq 0}$ preserves cofiltered limits, and so we have

$$\begin{aligned} M = \tau^{\geq 0} M &\xrightarrow{\sim} \tau^{\geq 0}\big(\lim_n M \otimes^L_R \big(R/\operatorname{rad}(R)^n\big)\big) \\ &\xrightarrow{\sim} \lim_n \tau^{\geq 0}\big(M \otimes^L_R \big(R/\operatorname{rad}(R)^n\big) \\ &\xrightarrow{\sim} \lim_n M \otimes_R \big(R/\operatorname{rad}(R)^n\big), \end{aligned}$$

as required. □

We now consider the converse to Lemma A.10.17. We begin with the following lemma.

Lemma A.10.18. *Let R be a commutative Noetherian profinite $\mathcal{O}$-algebra, and let E be a finite R-algebra (i.e. E is finite as an R-module). Let $\mathcal{A}$ be an abelian category, and let M be a right E-module in* $\mathrm{Pro}(\mathcal{A})$*.*

1. *M is complete if and only if it is complete as an R-module.*
2. *Suppose furthermore that E is flat over R. Then M is derived complete if and only if it is derived complete as an R-module.*

Proof. Writing

$$M \otimes_R R/\operatorname{rad}(R)^n = M \otimes_E (E \otimes_R R/\operatorname{rad}(R)^n) = M \otimes_E E/\operatorname{rad}(R)^n E,$$

and bearing in mind Lemma A.1.32 (8), the first part follows from the cofinality of $(\operatorname{rad}(R)^n E)_{n\geq 0}$ and $(\operatorname{rad}(E)^n)_{n\geq 0}$ (which in turn follows from the finiteness of E over R).

The second part follows in the same way, noting that if E is flat over R then we have

$$M \otimes^L_R R/\operatorname{rad}(R)^n = M \otimes^L_E (E \otimes^L_R R/\operatorname{rad}(R)^n) = M \otimes^L_E E/\operatorname{rad}(R)^n E. \qquad \square$$

The following lemma presumably holds in greater generality, but for simplicity, we will only prove it in the case of profinite $\mathcal{O}$-algebras that are finite flat over a commutative ring.

Lemma A.10.19. *Let R be a commutative Noetherian profinite $\mathcal{O}$-algebra, and let E be a finite flat R-algebra. Let $\mathcal{A}$ be an abelian category, and let M be a complete right E-module in* $\mathrm{Pro}(\mathcal{A})$*. Then M is derived complete.*

Proof. By Lemma A.10.18, we can assume without loss of generality that $E = R$ is commutative. Let $\mathfrak{m} := \mathrm{rad}(R)$. We need to prove that $\lim_n M \otimes^L_R R/\mathfrak{m}^n$ is concentrated in degree zero. Note that this coincides with $M \,\widehat{\otimes}^L_R \lim_n R/\mathfrak{m}^n$ (where $M \,\widehat{\otimes}^L_R$ is the functor in Lemma A.10.10 (2j)). Choose generators of $\mathfrak{m}$, say $\mathfrak{m} = (f_1, \dots, f_r)$, and write $\mathfrak{m}_n := (f_1^n, \dots, f_r^n)$. Then the pro-objects $\lim_n R/\mathfrak{m}^n$ and $\lim_n R/\mathfrak{m}_n$ are isomorphic. So it suffices to prove that $M \,\widehat{\otimes}^L_R \lim_n R/\mathfrak{m}_n = \lim_n M \otimes^L_R R/\mathfrak{m}_n$ is concentrated in degree zero, i.e. the pro-object $\lim_n H^p(M \otimes^L_R R/\mathfrak{m}_n)$ is zero for all $p \neq 0$. Since M is complete, Lemma A.10.13 allows us to replace M by $M/\mathfrak{m}^N M$, and so to assume that M is an $R/\mathfrak{m}^N R$-module, for some $N > 0$.

Let K_n be the Koszul complex on $(f_1^n, \dots, f_r^n)$, normalized as a cochain complex in degrees $-r, \dots, -1, 0$ with $\wedge^t R^{\oplus r}$ in degree $-t$. There is a map of complexes $K_{r+i} \to K_r$, given by $\wedge^t(\mathrm{diag}(f_1^i, \dots, f_r^i))$ in degree $-t$, and inducing the natural map $R/\mathfrak{m}_{r+i} \to R/\mathfrak{m}_r$ on H^0. Consider now the inverse system of complexes in $\mathrm{Pro}(\mathcal{A})$ given by $M \otimes_R K_r$, Since M is an $R/\mathfrak{m}^N R$-module, the transition map $M \otimes_R K_{r+N} \to M \otimes_R K_r$ is zero for any r in all degrees $t \neq 0$, because it is given by a matrix with entries in $\mathfrak{m}^N$. Hence the pro-object $\lim_r H^t(M \otimes_R K_r) \in \mathrm{Pro}(\mathcal{A})$ is zero for all $t \neq 0$. Since K_r is a complex of finite projective R-modules, the complex $M \otimes_R K_r$ represents the image of K_r under the derived functor $D^-(\mathrm{Mod}^{\mathrm{fp}}(R)) \to D^-(\mathrm{Pro}(\mathcal{A}))$ of $M \otimes_R -$. Since $M \otimes^L_R$ is the composite of this derived functor with $D^-(\mathrm{Pro}(\mathcal{A})) \to \mathrm{Pro}\, D^b(\mathcal{A})$, which is t-exact and induces an equivalence on hearts, we conclude that $\lim_r H^t(M \otimes^L_R K_r) = 0$ for all $t \neq 0$.

Of course, K_r need not be a projective resolution of $R/\mathfrak{m}_r$. However, we can define an inverse system of fibre sequences $K'_r \to K_r \to R/\mathfrak{m}_r[0]$ in $D^b_{\mathrm{fp}}(R)$, and it is proved in [Stacks, Tag 0921] that for all r there exists i such that $K'_{r+i} \to K'_r$ is zero. This shows that $\lim_r K'_r = 0$, and so $\lim_r K_r \xrightarrow{\sim} \lim_r R/\mathfrak{m}_r[0]$ in $\mathrm{Pro}\, D^b_{\mathrm{fp}}(R)$. Hence

$$\lim_r (M \otimes^L_R R/\mathfrak{m}_r[0]) = M \,\widehat{\otimes}^L_R \lim_r R/\mathfrak{m}_r[0] = M \,\widehat{\otimes}^L_R \lim_r K_r = \lim_r M \otimes^L_R K_r.$$

The previous paragraph shows that $\lim_r M \otimes^L_R K_r$ is concentrated in degree zero, hence so is $\lim_r (M \otimes^L_R R/\mathfrak{m}_r[0])$, as desired. $\square$

The following result is the reason for introducing Definition A.10.16.

Lemma A.10.20. *Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be an abelian category. Let M be a derived complete right R-module in* $\mathrm{Pro}(\mathcal{A})$*. Then we* *have a commutative diagram*

$$\begin{tikzcd}
D^b_{\text{f.l.}}(R) \arrow[r, hook, "(\text{A.10.6})"'] \arrow[d, hook] \arrow[rrd, bend left, "(3\text{m})"] & D^b_{\text{fp}}(R) \arrow[d, hook] \arrow[rd, "(2\text{h})"'] & \\
& & \operatorname{Pro} D^b(\mathcal{A}) \\
\operatorname{Pro} D^b_{\text{f.l.}}(R) \arrow[r, hook, "\operatorname{Pro}(\text{A.10.6})"] \arrow[rru, bend right, "(3\text{n})"'] & \operatorname{Pro} D^b_{\text{fp}}(R) \arrow[ru, "(2\text{j})"] &
\end{tikzcd}$$

Proof. Note firstly that by Lemma A.10.17, all of the functors in the diagram are defined. It evidently suffices to check that the functor

$$M \,\widehat{\otimes}^L_R - : \operatorname{Pro} D^b_{\text{f.l.}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$$

of (3n) is given by the composite of the embedding

$$\operatorname{Pro} D^b_{\text{f.l.}}(R) \hookrightarrow \operatorname{Pro} D^b_{\text{fp}}(R)$$

deduced from (A.10.6) and the functor

$$M \,\widehat{\otimes}^L_R - : \operatorname{Pro} D^b_{\text{fp}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$$

of (2j). Since each of these functors $\operatorname{Pro} D^b_{\text{f.l.}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$ is right t-exact and limit-preserving, it suffices by Lemma A.10.10 (3) to show that they agree on the object R of $\operatorname{Mod}_c(R) = (\operatorname{Pro} D^b_{\text{f.l.}}(R))^{\heartsuit}$. By their definitions, they respectively take R to M and to $\lim_n M \otimes^L_R \big(R/\operatorname{rad}(R)^n\big)$, and since M is derived complete by assumption, we are done. $\square$

We have the following criterion for the full faithfulness of a derived tensor product.

Lemma A.10.21. *Let R be a coherent ring, and let $\mathcal{A}$ be an abelian category. Suppose that M is a right R-module in* $\operatorname{Pro}(\mathcal{A})$*, and that the natural morphism*

$$R \to \operatorname{REnd}_{\operatorname{Pro} D^b(\mathcal{A})}(M)$$

is an isomorphism. Then the functor $M \otimes^L_R - : D^b_{\text{fp}}(R) \to \operatorname{Pro} D^b(\mathcal{A})$ defined in Lemma A.10.10 (2h) *is fully faithful.*

Proof. By Lemma A.4.6, it is equivalent to prove that the induced continuous functor

$$D(R) = \operatorname{Ind} D^b_{\text{fp}}(R) \to \operatorname{Ind} \operatorname{Pro} D^b(\mathcal{A})$$

is fully faithful. Since $D(R)$ is compactly generated by R, the result is immediate from Proposition A.2.25, together with the full faithfulness of $\operatorname{Pro} D^b(\mathcal{A}) \to \operatorname{Ind} \operatorname{Pro} D^b(\mathcal{A})$. $\square$

A.10.22. *Topological flatness.* We now introduce various notions of flatness, in the context of Lemma A.10.10.

Definition A.10.23. Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be an abelian category.

(1) Let M be a right R-module in $\mathcal{A}$. The Tor-*dimension of* M is the amplitude of the right t-exact functor $M \otimes_R^L - : D^-_{\mathrm{fp}}(R) \to D^-(\mathcal{A})$, i.e. the smallest $n \geq 0$ such that the restriction of $M \otimes_R^L -$ to $D^-_{\mathrm{fp}}(R)^\heartsuit$ (i.e. the abelian category $\mathrm{Mod}^{\mathrm{fp}}(R)$) factors through $D^-(\mathcal{A})^{\geq -n}$.
(2) Let M be a right R-module in $\mathcal{A}$. We say that M is *flat* if $M \otimes_R - : \mathrm{Mod}^{\mathrm{fp}}(R) \to \mathcal{A}$ is exact, or equivalently, if M has Tor-dimension zero.
(3) Let M be a complete right R-module in $\mathrm{Pro}(\mathcal{A})$. We say that M is *topologically flat* if $M \widehat{\otimes}_R - : \mathrm{Mod}_c(R) \to \mathrm{Pro}(\mathcal{A})$ in Lemma A.1.53 is exact.

Lemma A.10.24. *Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be an abelian category. Let M be a complete right R-module in* $\mathrm{Pro}(\mathcal{A})$. *Then M is flat if and only if M is topologically flat.*

Proof. By Corollary A.1.54, the restriction of $M \widehat{\otimes}_R -$ to $\mathrm{Mod}^{\mathrm{fp}}(R)$ is $M \otimes_R -$. Hence, if M is topologically flat, then M is flat. On the other hand, the inclusion $\mathrm{Mod}^{\mathrm{f.l.}}(R) \to \mathrm{Mod}_c(R)$ factors through $\mathrm{Mod}^{\mathrm{fp}}(R)$. Hence, if M is flat, then $M \widehat{\otimes}_R -$ restricts to an exact functor on $\mathrm{Mod}^{\mathrm{f.l.}}(R)$, and since $\mathrm{Mod}_c(R) = \mathrm{Pro}\,\mathrm{Mod}^{\mathrm{f.l.}}(R)$, we see that $M \widehat{\otimes}_R -$ is exact, by Lemma A.1.5 (2) and (3). Thus M is topologically flat, as desired. $\square$

Lemma A.10.25. *Let R be a Noetherian profinite $\mathcal{O}$-algebra, and let $\mathcal{A}$ be an abelian category. Let M be a complete right R-module in* $\mathrm{Pro}(\mathcal{A})$. *If M is topologically flat, then the functor*

$$M \widehat{\otimes}_R^L - : \mathrm{Pro}\, D^b_{\mathrm{f.l.}}(R) \to \mathrm{Pro}\, D^b(\mathcal{A}),$$

defined in Lemma A.10.10 (3n)*, is t-exact.*

Proof. By definition, $M \widehat{\otimes}_R^L -$ is the Pro-extension of the composite

$$D^b_{\mathrm{f.l.}}(R) = D^b(\mathrm{Mod}^{\mathrm{f.l.}}(R)) \xrightarrow{i} D^-(\mathrm{Mod}_c(R)) \to \mathrm{Pro}\, D^b(\mathcal{A}),$$

where the first arrow is induced by the inclusion $\mathrm{Mod}^{\mathrm{f.l.}}(R) \subset \mathrm{Mod}_c(R)$ (and is t-exact) and the second arrow is the derived functor of $M \widehat{\otimes}_R -$. Since M is topologically flat, $M \widehat{\otimes}_R -$ is exact, and so both arrows are t-exact. The lemma thus follows from Corollary A.8.2. $\square$

Lemma A.10.26. *Let R be a Noetherian profinite $\mathcal{O}$-algebra, and assume that R has finite global dimension (i.e. there exists n such that every left R-module has a projective resolution of length $\leq n$). Let $\mathcal{A}$ be an abelian category, and let M be a right R-module in $\mathcal{A}$. Then the functor $M \otimes_R^L - : D^b_{\mathrm{fp}}(R) \to D^-(\mathcal{A})$ defined in* (1c) *factors through $D^b(\mathcal{A})$.*

Proof. An induction on the amplitude of $X \in D^b_{\mathrm{fp}}(R)$ shows that it suffices to check that $M \otimes_R^L X \in D^b(\mathcal{A})$ whenever $X \in \mathrm{Mod}^{\mathrm{fp}}(R)$. However, by assumption, $M \otimes_R^L X$ is represented by a complex concentrated in degrees $[-n, 0]$. $\square$

Appendix B. Coherent sheaves on formal algebraic stacks

In this appendix, after recalling some of the basic framework and results related to the theory of coherent sheaves on algebraic stacks, we develop various constructions and results on coherent and pro-coherent sheaves on formal algebraic stacks. These include assorted pullback and pushforward functors, as well as results related to the theorem on formal functions.

B.1. **Coherent sheaf theory on algebraic stacks.** There are various competing approaches to defining both the abelian and the derived category of quasi-coherent sheaves on an algebraic stack; see [EGH25, App. B.2] for a discussion of some of them. In the context of a Noetherian algebraic stack $\mathcal{X}$ with affine diagonal, it follows from [HR17, Prop. 1.3, Rem. 1.5] that these various approaches lead to an unambiguous definition of $D^b_{\mathrm{coh}}(\mathcal{X})$, the stable ∞-category of bounded complexes of $\mathcal{O}_{\mathcal{X}}$-modules having coherent cohomology. A similar discussion applies to quasi-coherent cohomology, although the coherent case will be the most important one for us in the present paper.

For definiteness, we will use sheaves on the lisse-étale site of $\mathcal{X}$ as our model for $D^b_{\mathrm{coh}}(\mathcal{X})$, and the various related categories of sheaves that we will have to consider. We make this choice primarily to make contact with various pieces of literature (e.g. [Con05] and [AHL23]) whose results we are going to cite.

As in the preceding discussion, we assume that $\mathcal{X}$ is a Noetherian algebraic stack with affine diagonal, and we let $\mathcal{O}_{\mathcal{X}}$ denote the usual structure sheaf on the lisse-étale site of $\mathcal{X}$. We let $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ denote the abelian category of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules on the lisse-étale site of $\mathcal{X}$, and we let $D(\mathcal{O}_{\mathcal{X}})$ denote the derived ∞-category of $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$.

We let $\mathrm{Coh}(\mathcal{X})$, resp. $\mathrm{QCoh}(\mathcal{X})$, denote the abelian category of coherent, resp. quasi-coherent, sheaves of $\mathcal{O}_{\mathcal{X}}$-modules (see e.g. [Ols07, §6]). These are both weak Serre subcategories of $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$, and $\mathrm{Coh}(\mathcal{X})$ is a Serre subcategory of $\mathrm{QCoh}(\mathcal{X})$ (see e.g. [Stacks, Tag 07B4], [Stacks, Tag 0GRB]). We have the following standard lemma about the structure of $\mathrm{QCoh}(\mathcal{X})$.

Lemma B.1.1. *Let $\mathcal{X}$ be a Noetherian algebraic stack. Then $\mathrm{QCoh}(\mathcal{X})$ is a locally coherent Grothendieck category, and $\mathrm{QCoh}(\mathcal{X})^c = \mathrm{Coh}(\mathcal{X})$.*

Proof. $\mathrm{QCoh}(\mathcal{X})$ is a Grothendieck category by [Stacks, Tag 0781]. To see that coherent sheaves are compact in $\mathrm{QCoh}(\mathcal{X})$, choose a smooth surjective morphism $p : X \to \mathcal{X}$ with X a Noetherian scheme. Then p^* preserves colimits, and sends coherent sheaves on $\mathcal{X}$ to coherent sheaves on X. It then follows from [Stacks, Tag 06WT] that every $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$ is compact in $\mathrm{QCoh}(\mathcal{X})$, since $p^*\mathcal{F}$ is compact in $\mathrm{QCoh}(X)$. Conversely, compact objects of $\mathrm{QCoh}(\mathcal{X})$ are coherent, because every object of $\mathrm{QCoh}(\mathcal{X})$ is the filtered colimit of its coherent subsheaves ([LM00, Proposition 15.4] or [Stacks, Tag 0GRF]). $\square$

In particular, the embeddings $\mathrm{QCoh}(\mathcal{X}) \hookrightarrow \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ and $\mathrm{Coh}(\mathcal{X}) \hookrightarrow \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ are both exact, and so induce t-exact functors of associated derived ∞-categories $D\big(\mathrm{QCoh}(\mathcal{X})\big) \to D_{\mathrm{qc}}(\mathcal{X})$ and $D\big(\mathrm{Coh}(\mathcal{X})\big) \to D_{\mathrm{coh}}(\mathcal{X})$, where the targets denote the full subcategories of $D(\mathcal{O}_{\mathcal{X}})$ consisting of objects whose cohomology sheaves lie in $\mathrm{QCoh}(\mathcal{X})$, resp. $\mathrm{Coh}(\mathcal{X})$. The first of these functors restricts to an equivalence

$$D^+\big(\mathrm{QCoh}(\mathcal{X})\big) \xrightarrow{\sim} D^+_{\mathrm{qc}}(\mathcal{X}), \tag{B.1.2}$$

by [AJPV18, Prop. 1.6] or [HNR19, Thm. C.1]; note however that the first reference uses a slightly different framework to the one we are using here, which is explained and compared with our framework of lisse-étale sites in [AJPV15]. By Lemma A.7.11 (whose hypotheses hold because of Lemma B.1.1) the equivalence (B.1.2) restricts to a t-exact equivalence

$$D^b\big(\mathrm{Coh}(\mathcal{X})\big) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{X}). \tag{B.1.3}$$

In the body of the text, we will not work directly with quasi-coherent sheaves that are not coherent. Rather, we will work with Ind-coherent sheaves; that is, we will consider the stable ∞-category $\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})$. Proposition A.6.9 shows that the t-structure on $D^b_{\mathrm{coh}}(\mathcal{X})$ extends canonically to a t-structure on $\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})$. Since $D_{\mathrm{qc}}(\mathcal{X})$ is cocomplete, there is a t-exact functor $\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}) \to D_{\mathrm{qc}}(\mathcal{X})$, which restricts to an equivalence

$$\big(\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})\big)^+ \xrightarrow{\sim} D^+_{\mathrm{qc}}(\mathcal{X}) \tag{B.1.4}$$

(see Remark B.1.5 below). Then, when it comes to a consideration of functors such as Rj_* for an open immersion $j : \mathcal{Y} \to \mathcal{X}$, which are usually defined on the categories D^+_{qc}, we will extend them to the categories $\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})$; furthermore, we will often construct such functors using this Ind-coherent perspective, as in the discussion of Section B.1.8 below.

Remark B.1.5. In the literature one finds a definition of a stable infinity category $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{X})$ which is not necessarily compactly generated, but is rather defined by a universal mapping property, which incorporates (B.1.4); see for example [CW24, Defn. 5.10]. However, in our setting this more sophisticated definition agrees with our $\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})$; for example, all of the stacks that we consider in this paper are "coherent ind-geometric stacks" in the sense of [CW24], so this agreement follows from [CW24, Prop. 5.30].

B.1.6. *Operations on coherent and quasi-coherent sheaves.* Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks, both assumed to be Noetherian and having affine diagonal, and let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism. In this context one can define pushforward and pullback functors on both the abelian and derived categories of quasi-coherent sheaves, as well on various related categories. We now recall some details of these constructions.

Remark B.1.7. The hypotheses we have placed on $\mathcal{X}$ and $\mathcal{Y}$ are certainly unnecessarily restrictive; our goal here is simply to give some details regarding, and references for, the construction of these functors in the specific framework that we have adopted. One motivation for taking care on this point is that we wish to combine citations to works on sheaf theory on algebraic stacks, such as [AHR23; AHL23] with the ∞-categorical framework, and so we want to ensure that the functors we compare in these various different contexts are indeed the ones we imagine they are. Another is that we will extend some of these constructions to the context of formal algebraic stacks, where there are fewer existing references to rely on, and we want to be sure that our constructions in this context are well-founded (for *some* choice of foundations).

The construction of pullbacks and pushforwards is simplest in the case when f is representable by schemes. (One simple but important case when this holds is when f is an immersion.) In this case $U \mapsto U \times_{\mathcal{X}} \mathcal{Y}$ gives a continuous functor u (in the sense discussed in [Stacks, Tag 00WU]) from the lisse-étale site of $\mathcal{X}$ to the lisse-étale site of $\mathcal{Y}$. Adopting notation from *loc. cit.*, we thus obtain a functor u^s from the category of sheaves on the lisse-étale site of $\mathcal{Y}$ to the category of sheaves on the lisse-étale site of $\mathcal{X}$.

This functor u is the restriction to lisse-étale sites of the analogously defined functor on big flat sites. This latter functor induces a morphism of (ringed) topoi [Stacks, Tag 06W8]. Unfortunately, the functor u on lisse-étale sites typically *does not* define a morphism of topoi: the left adjoint u_s to u^s is typically not exact, see e.g. [Stacks, Tag 07BF].

Nevertheless, the morphism f induces functors $f_* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ and $f^* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$. In the Stacks Project, these are defined using big sites: the pullback f^* is induced by the left-adjoint of the "big site version" of the functor u^s considered above, and the push-forward f_* being defined via slightly elaborate manipulations, for which see [Stacks, Tag 070A]. In the context of small lisse-étale sites, however, we can describe f_* rather directly. Namely, there is a canonical morphism $\mathcal{O}_{\mathcal{X}} \to u^s(\mathcal{O}_{\mathcal{Y}})$, so that u^s induces a functor $f_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$, which by [Ols07, Lem. 6.5] restricts to a functor $\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$. (Note that our assumption that $\mathcal{Y}$ and $\mathcal{X}$ are Noetherian implies that f is necessarily quasi-compact.) If f is furthermore proper, then f_* restricts to a functor $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$: see [Ols07, Thm. 10.13] for a proof of the (stronger) derived statement, although note that (as the proof of this result indicates) under the assumption that f is representable by schemes, this result follows directly from the corresponding result for schemes.

We may then construct the left adjoint $f^* : \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}})$ of f_*, via composing the left adjoint u_s to u^s with the extension of scalars from $u_s\mathcal{O}_{\mathcal{X}}$ to $\mathcal{O}_{\mathcal{Y}}$. The functor f^* then restricts to functors $\mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(\mathcal{Y})$. (For this material, see again [Ols07, Lem. 6.5].)

We may right derive the left exact functor f_* to obtain a functor $Rf_* : D^+(\mathcal{Y}) \to D^+(\mathcal{X})$, which restricts to a functor $Rf_* : D^+_{\mathrm{qc}}(\mathcal{Y}) \to D^+_{\mathrm{qc}}(\mathcal{X})$, and even to a functor $Rf_* : D^+_{\mathrm{coh}}(\mathcal{Y}) \to D^+_{\mathrm{coh}}(\mathcal{X})$ if f is proper. (See [Ols07, Lem. 6.20, Thm. 10.13].) Recall that in general Rf_* need not have bounded amplitude, and consequently need not take $D_{\mathrm{qc}}(\mathcal{Y})$ to $D_{\mathrm{qc}}(\mathcal{X})$; see e.g. [Stacks, Tag 07DC].

One may also consider the left-derived functor Lf^*. The treatment of this functor in the lisse-étale formalism of [Ols07] is complicated by the failure of u to induce a morphism of topoi. However, in the very particular case when f is a smooth morphism (e.g. an open immersion, which is the case used in Section 5.4), u *does* induce a morphism of topoi; indeed, the functor u^s is then simply given by restriction along f, and so *is* exact (see the remark at the bottom of [Ols07, p. 60]). In this case f^* is exact, as it coincides with the exact functor u_s, and by Lemma A.7.22 it induces a t-exact functor $D(\mathcal{O}_{\mathcal{X}}) \to D(\mathcal{O}_{\mathcal{Y}})$, which restricts to t-exact functors $D_{\mathrm{qc}}(\mathcal{X}) \to D_{\mathrm{qc}}(\mathcal{Y})$ and $D_{\mathrm{coh}}(\mathcal{X}) \to D_{\mathrm{coh}}(\mathcal{Y})$.

If $f : \mathcal{X} \to \mathcal{Y}$ is not representable by schemes, then the construction of pullbacks and pushforwards is more involved; cf. [AJPV15, Rem., p. 488]. The reference [AJPV15] gives a construction in this case, using a similar framework to the lisse-étale site (namely, a version of the flat site), and building on the methods of [Ols07].

B.1.8. *Pushforward along an immersion.* In the main body of the paper, we will frequently need to consider the derived functors of f^* or f_* in the case when f is a closed or open immersion, which we will typically denote by i or j respectively —

possibly with additional decorations — rather than f. If i is a closed immersion, then i_* is exact: this is easily checked directly, by working locally on the target and so reducing to the case of schemes (see also [Ols07, Cor. 6.6]). Thus, by Lemma A.7.24, Ri_* is simply the t-exact functor induced by i_* following the prescription of Lemma A.7.22. We will thus write simply i_* rather than Ri_*.

If $j : \mathcal{U} \to \mathcal{X}$ is an open immersion, we can give an alternative construction of Rj_* directly as a functor on Ind-coherent complexes. To begin with, note that $j_* : \mathrm{QCoh}(\mathcal{U}) \to \mathrm{QCoh}(\mathcal{X})$ is fully faithful, hence the counit $j^*j_* \to \mathrm{id}_{\mathrm{QCoh}(\mathcal{U})}$ is an isomorphism. Since $j^* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{U})$ is exact, it follows from [Gab62, Proposition 5, §III.2] that j^* is a Serre quotient by its kernel, which is a localizing subcategory of $\mathrm{QCoh}(\mathcal{X})$. Since (by Lemma B.1.1) the categories $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{QCoh}(\mathcal{U})$ are locally coherent with compact objects $\mathrm{Coh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{U})$ respectively, [Kra97, Theorem 2.6] shows that j^* realizes $\mathrm{Coh}(\mathcal{U})$ as the quotient of $\mathrm{Coh}(\mathcal{X})$ by its kernel, which is the Serre subcategory $\mathrm{Coh}_Z(\mathcal{X})$, whose objects are those coherent sheaves on $\mathcal{X}$ with set-theoretic support contained in $Z \coloneqq |\mathcal{X}| \setminus |\mathcal{U}|$. Lemma A.7.8 then provides a t-exact equivalence

$$D^b\big(\mathrm{Coh}(\mathcal{X})\big)/D^b_{\mathrm{Coh}_Z(\mathcal{X})}\big(\mathrm{Coh}(\mathcal{X})\big) \xrightarrow{\sim} D^b\big(\mathrm{Coh}(\mathcal{U})\big),$$

which (B.1.3) allows us to interpret as a t-exact equivalence

$$D^b_{\mathrm{coh}}(\mathcal{X})/D^b_{\mathrm{coh},Z}(\mathcal{X}) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{U}), \tag{B.1.9}$$

where $D^b_{\mathrm{coh},Z}(\mathcal{X})$ is the full subcategory of complexes whose cohomology is set-theoretically supported on Z.

The functor $D^b_{\mathrm{coh}}(\mathcal{X}) \to D^b_{\mathrm{coh}}(\mathcal{U})$ implicit in (B.1.9) is of course simply the t-exact extension of j^*. The Ind-extension of this functor is then a continuous t-exact functor $j^* : \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{U})$ which restricts (via the equivalence (B.1.4) for each of $\mathcal{X}$ and $\mathcal{U}$) to the functor $j^* : D^+_{\mathrm{qc}}(\mathcal{X}) \to D^+_{\mathrm{qc}}(\mathcal{U})$ considered above. The right adjoint of j^* is then a functor

$$\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{U}) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}) \tag{B.1.10}$$

which restricts (again via the equivalence (B.1.4) for each of $\mathcal{X}$ and $\mathcal{U}$) to the functor Rj_* considered above. Since j^* preserves compact objects (equivalently, is constructed via Ind extension from the corresponding functor on compact objects), its right adjoint (B.1.10) is also continuous, by Lemma A.2.32. Thus (B.1.10) is the Ind-extension of its restriction $D^b_{\mathrm{coh}}(\mathcal{U}) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})$, which in turn may be regarded as the composite

$$D^b_{\mathrm{coh}}(\mathcal{U}) \xrightarrow{Rj_*} D^+_{\mathrm{qc}}(\mathcal{X}) \xrightarrow{\sim} \big(\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})\big)^+ \hookrightarrow \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}).$$

Consequently, the functors Rj_* and (B.1.10) canonically determine one another.

Remark B.1.11. In fact, in the cases of interest to us in the main body of the paper, the relevant open immersions j will be furthermore cohomologically affine. Hence Rj_* will be t-exact, and so, just as in the case of closed immersions, will be obtained from j_* by an appropriate application of Lemma A.7.22. From the preceding explanation of how to recover (B.1.10) from Rj_*, we conclude that (B.1.10) is also t-exact in this case. Consequently, in this case we will denote both Rj_* and its continuous extension (B.1.10) simply by j_*.

B.2. **Coherent sheaf theory on formal algebraic stacks.** Let $\mathcal{X}$ be a Noetherian formal algebraic stack with affine diagonal, written as a colimit

$$\mathcal{X} \xrightarrow{\sim} \operatorname{colim}_n \mathcal{X}_n \tag{B.2.1}$$

with the $\mathcal{X}_n$ being algebraic stacks and the transition maps being thickenings. Note that the $\mathcal{X}_n$ are then closed substacks of $\mathcal{X}$, and are thus also Noetherian with affine diagonal. We then write

$$D^b_{\mathrm{coh}}(\mathcal{X}) := \operatorname{colim}_n D^b_{\mathrm{coh}}(\mathcal{X}_n), \tag{B.2.2}$$

the transition maps being given by pushforward of sheaves as described in Section B.1.8 above, and the colimit being formed in the ∞-category of stable ∞-categories. Since any two presentations of $\mathcal{X}$ of the form (B.2.1) are mutually cofinal, we see that the resulting colimit is (up to canonical equivalence) independent of the choice of presentation (B.2.1). Since the transition maps are t-exact, the colimit $D^b_{\mathrm{coh}}(\mathcal{X})$ is endowed with a canonical t-structure. Furthermore, each $D^b_{\mathrm{coh}}(\mathcal{X}_n)$ is idempotent complete, hence the same is true of $D^b_{\mathrm{coh}}(\mathcal{X})$, by [Kerodon, Tag 042P]: to see the idempotent-completeness of $D^b_{\mathrm{coh}}(\mathcal{X}_n)$, by [Lur17, Lemma 1.2.4.6] it suffices to prove that the homotopy category of each $D^b_{\mathrm{coh}}(\mathcal{X}_n)$ is idempotent complete, which is a consequence of the equivalence (B.1.3) and the fact that the bounded derived category of an abelian category is idempotent complete [BS01, Corollary 2.10].

We let $\mathrm{Coh}(\mathcal{X})$ denote the heart of the t-structure on $D^b_{\mathrm{coh}}(\mathcal{X})$. The discussion of Section A.7.30 shows that there are canonical equivalences

$$\operatorname{colim}_n \mathrm{Coh}(\mathcal{X}_n) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{X}) \quad \text{and} \quad D^b\big(\mathrm{Coh}(\mathcal{X})\big) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{X}). \tag{B.2.3}$$

The presentation (B.2.1) allows us to define the topological space $|\mathcal{X}| := \operatorname{colim}_n |\mathcal{X}_n|$, and then $|\mathcal{X}_n| \xrightarrow{\sim} |\mathcal{X}|$ for any choice of n. If $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, then it can be represented by an object $\mathcal{F}_n \in \mathrm{Coh}(\mathcal{X}_n)$ for some n, and the set-theoretic support $\operatorname{supp} \mathcal{F}_n \subset |\mathcal{X}_n|$ is independent of the choice of representative. Using the t-structure on $D^b_{\mathrm{coh}}(\mathcal{X})$ we can thus associate to each closed subset $Z \subset |\mathcal{X}|$ of the underlying topological space of $\mathcal{X}$ (or equivalently, of $|\mathcal{X}_n|$, for any n) a full sub-∞-category $D^b_{\mathrm{coh},Z}(\mathcal{X})$ consisting of objects whose cohomology sheaves are set-theoretically supported on Z. By construction, (B.2.2) induces an equivalence

$$\operatorname{colim}_n D^b_{\mathrm{coh},Z}(\mathcal{X}_n) \xrightarrow{\sim} D^b_{\mathrm{coh},Z}(\mathcal{X}). \tag{B.2.4}$$

As well as considering the bounded derived category $D^b_{\mathrm{coh}}(\mathcal{X})$ of a Noetherian formal algebraic stack $\mathcal{X}$ with affine diagonal, we will also have occasion to consider the associated Ind and Pro categories (and even the associated Ind Pro category). If we present $\mathcal{X}$ as in (B.2.1), then (as noted in Section A.7.30) we have an equivalence

$$\operatorname{colim}_n \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X}_n) \xrightarrow{\sim} \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X}),$$

the colimit now being formed in the ∞-category of compactly generated stable ∞-categories (whose morphisms are the continuous functors). Since $D^b_{\mathrm{coh}}(\mathcal{X})$ is idempotent complete, it is the subcategory of compact objects in $\operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$. We also note the following result.

Lemma B.2.5. *Let $\mathcal{X}$ be a Noetherian formal algebraic stack with affine diagonal. Then $D^b_{\mathrm{coh}}(\mathcal{X})$ is the full subcategory of $\operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$ (resp. $\operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X})$) given by those objects whose cohomology is coherent and concentrated in finitely many degrees.*

Proof. We give the argument in the Ind case, the argument in the Pro case being dual to it. It suffices to prove that if $x \in \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})^{[a,b]}$ is a bounded object with coherent cohomology, then x is compact in $\operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$. When $a = b$ we have an isomorphism $x \xrightarrow{\sim} H^b(x)[-b]$, and $H^b(x) \in \operatorname{Ind}\operatorname{Coh}(\mathcal{X})$ is assumed to be contained in $\operatorname{Coh}(\mathcal{X}) \subset D^b_{\mathrm{coh}}(\mathcal{X})$. Hence $H^b(x)$ is compact, and so is x. In general, by induction on $b-a$, we see that x is part of a cofibre sequence $\tau^{\leq(b-1)}x \to x \to H^b(x)[-b]$ where the first and third term are compact, so x is also compact, as required. ☐

B.2.6. *Completion along a closed substack.* Let $\mathcal{X}$ be a Noetherian formal algebraic stack with affine diagonal, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack of $\mathcal{X}$. We define the completion of $\mathcal{X}$ along $\mathcal{Z}$ to be the substack $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}_{\mathcal{Z}}$ of $\mathcal{X}$ that classifies morphisms $T \to \mathcal{X}$ for which the induced map on underlying topological spaces $|T| \to |\mathcal{X}|$ has image landing in $|\mathcal{Z}|$. By definition, then, $\widehat{\mathcal{X}}$ depends only on the closed subset $Z := |\mathcal{Z}|$ of $|\mathcal{X}|$, not on its particular realization as a closed substack $\mathcal{Z}$ of $\mathcal{X}$, and we will sometimes write $\widehat{\mathcal{X}}_Z$ rather than $\widehat{\mathcal{X}}_{\mathcal{Z}}$. Note that $\widehat{\mathcal{X}}$ is again a Noetherian formal algebraic stack. To see this, note for example that [Eme, Lem 8.13] implies that $\widehat{\mathcal{X}}$ is locally Noetherian; since $\mathcal{X}$ is furthermore assumed Noetherian, i.e. to be quasi-compact and quasi-separated in addition to being locally Noetherian, it is straightforward to deduce that $\widehat{\mathcal{X}}$ is also Noetherian.

If we choose a presentation $\widehat{\mathcal{X}} \xrightarrow{\sim} \operatorname{colim}_n \mathcal{Z}_n$ for $\widehat{\mathcal{X}}$ as a colimit with respect to thickenings of Noetherian algebraic stacks $\mathcal{Z}_n$, then we may choose our presentation (B.2.1) for $\mathcal{X}$ so that each composite

$$\mathcal{Z}_n \hookrightarrow \widehat{\mathcal{X}} \to \mathcal{X}$$

factors through $\mathcal{X}_n$, inducing a closed immersion

$$i_n : \mathcal{Z}_n \hookrightarrow \mathcal{X}_n.$$

The canonical monomorphism $i : \widehat{\mathcal{X}} \hookrightarrow \mathcal{X}$ may then be written as the colimit of the closed immersions i_n. Passing to the colimit, the (t-exact) pushforward functors $i_{n,*} : D^b_{\mathrm{coh}}(\mathcal{Z}_n) \to D^b_{\mathrm{coh}}(\mathcal{X}_n)$ then induce a t-exact functor

$$i_* : D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \to D^b_{\mathrm{coh}}(\mathcal{X}). \tag{B.2.7}$$

Proposition B.2.8. *Let $\mathcal{X}$ be a Noetherian formal algebraic stack with affine diagonal, let $\mathcal{Z}$ be a closed substack of $\mathcal{X}$, and let $\widehat{\mathcal{X}}$ denote the completion of $\mathcal{X}$ along $\mathcal{Z}$. Then* (B.2.7) *is a t-exact fully faithful functor $D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \hookrightarrow D^b_{\mathrm{coh}}(\mathcal{X})$, whose essential image coincides with $D^b_{\mathrm{coh},Z}(\mathcal{X})$, i.e. the full sub-$\infty$-category of $D^b_{\mathrm{coh}}(\mathcal{X})$ consisting of objects whose cohomology sheaves are set-theoretically supported on $Z := |\mathcal{Z}|$.*

Proof. We may replace $\mathcal{Z}$ by $\mathcal{Z}_{\mathrm{red}}$, which is a closed algebraic substack of $\mathcal{X}$. Then, if we choose a presentation $\mathcal{X} \xrightarrow{\sim} \operatorname{colim}_n \mathcal{X}_n$ as in the discussion above, we see that $\mathcal{Z}$ is in fact a closed substack of each $\mathcal{X}_n$, that

$$\widehat{\mathcal{X}} \xrightarrow{\sim} \operatorname{colim}_n \widehat{\mathcal{X}_n},$$

where all the indicated completions are taken along $\mathcal{Z}$, and that

$$D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \xrightarrow{\sim} \operatorname{colim}_n D^b_{\mathrm{coh}}(\widehat{\mathcal{X}_n}),$$

where the colimit is formed in the ∞-category of stable ∞-categories.

Since a filtered colimit of fully faithful t-exact morphisms is fully faithful and t-exact, and

$$D^b_{\mathrm{coh},Z}(\mathcal{X}) = \mathrm{colim}_n\, D^b_{\mathrm{coh},Z}(\mathcal{X}_n)$$

by (B.2.4), to prove that (B.2.7) is fully faithful with essential image $D^b_{\mathrm{coh},Z}(\mathcal{X})$, it then suffices to show that pushforward induces a t-exact equivalence

$$D^b_{\mathrm{coh}}(\widehat{\mathcal{X}_n}) \xrightarrow{\sim} D^b_{\mathrm{coh},Z}(\mathcal{X}_n) \tag{B.2.9}$$

for each n. This allows us to assume that $\mathcal{X} = \mathcal{X}_n$ for some n, i.e. that $\mathcal{X}$ is algebraic.

In this case, let $\mathcal{I}$ denote the coherent ideal sheaf cutting out $\mathcal{Z}$ in $\mathcal{X}$, and let $\mathcal{Z}^{(n)}$ denote the closed substack of $\mathcal{X}$ cut out by $\mathcal{I}^n$. Then $\widehat{\mathcal{X}} \xrightarrow{\sim} \mathrm{colim}_n\, \mathcal{Z}^{(n)}$, and so we have to show that pushforward induces a fully faithful t-exact functor

$$\mathrm{colim}_n\, D^b_{\mathrm{coh}}(\mathcal{Z}^{(n)}) \hookrightarrow D^b_{\mathrm{coh}}(\mathcal{X}),$$

with essential image equal to $D^b_{\mathrm{coh},Z}(\mathcal{X})$. If $\mathcal{X}$ is an affine scheme, this is proved in [EGH25, Proposition B.1.9, Proposition B.1.17]: it amounts to the statement that the first two arrows of [EGH25, (B.1.8)] are equivalences. The general case can be proved in the same way. Alternatively, it can be deduced as a special case of [HP23, Theorem 2.2.3]. ☐

We use the same notation (namely i_*) to denote the extension of (B.2.7) to the corresponding Ind categories; this is a fully faithful embedding

$$i_* : \mathrm{Ind}\, D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \xrightarrow{\sim} \mathrm{Ind}\, D^b_{\mathrm{coh},Z}(\mathcal{X}) \hookrightarrow \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}). \tag{B.2.10}$$

Lemma B.2.11. *The functor* (B.2.10) *satisfies the conditions of Hypothesis A.3.2, taking* $\mathcal{A}^c_Z := D^b_{\mathrm{coh}}(\widehat{\mathcal{X}})$ *and* $\mathcal{A}^c := D^b_{\mathrm{coh}}(\mathcal{X})$.

Proof. This is an immediate consequence of the construction of (B.2.10) as an Ind-extension of a fully faithful functor. ☐

Write $\mathcal{U} := \mathcal{X} \setminus \mathcal{Z}$ (an open substack of $\mathcal{X}$), and let $j : \mathcal{U} \hookrightarrow \mathcal{X}$ denote the canonical open immersion. If we again choose a presentation (B.2.1) for $\mathcal{X}$, and write $\mathcal{U}_n := \mathcal{U} \times_{\mathcal{X}} \mathcal{X}_n$, then we obtain a compatible presentation $\mathrm{colim}_n\, \mathcal{U}_n \xrightarrow{\sim} \mathcal{U}$ of $\mathcal{U}$. As we noted in (B.1.9), restriction induces equivalences

$$D^b_{\mathrm{coh}}(\mathcal{X}_n)/D^b_{\mathrm{coh},Z}(\mathcal{X}_n) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{U}_n).$$

Passing to the colimit over n, and recalling the statement of Proposition B.2.8, we obtain an equivalence

$$D^b_{\mathrm{coh}}(\mathcal{X})/D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{U}). \tag{B.2.12}$$

Passing to Ind-categories, and taking into account Proposition A.2.29, we obtain an equivalence

$$\mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})/\, \mathrm{Ind}\, D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \xrightarrow{\sim} \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{U}), \tag{B.2.13}$$

which we use to identify its left and right hand sides from now on. Taking into account this identification, as well as the prescription of Hypothesis A.3.2, we then write j^* to denote the composite

$$j^* : D^b_{\mathrm{coh}}(\mathcal{X}) \to D^b_{\mathrm{coh}}(\mathcal{X})/D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{U}), \tag{B.2.14}$$

as well as its Ind-extension

$$j^* : \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X}) \to \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{X})/\, \mathrm{Ind}\, D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \xrightarrow{\sim} \mathrm{Ind}\, D^b_{\mathrm{coh}}(\mathcal{U}). \tag{B.2.15}$$

By Lemma A.3.3, the functor (B.2.15) admits a right adjoint $j_* : \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{U}) \to \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$.

Remark B.2.16. Although we don't need it in the present paper, we remark that the Pro-extension of j^* also admits a left adjoint $j_!$, which is the Pro version of "extension by zero from $\mathcal{U}$". As far as we understand, defining $j_!$ in this way was the original motivation for introducing pro-coherent sheaves (by Deligne in [Har66, Appendix]).

Remark B.2.17. The functors j^* in (B.2.14) and (B.2.15) are induced by the functors j_n^* arising from the open immersions $j_n : \mathcal{U}_n \hookrightarrow \mathcal{X}_n$, in the sense that if we write each of $D^b_{\mathrm{coh}}(\mathcal{X})$ and $D^b_{\mathrm{coh}}(\mathcal{U})$ as colimits over n, following the prescription of (B.2.2), then j^* is obtained as colimit of the functors j_n^*. Thus the notation j^* is compatible with our notation for operations on derived categories of coherent sheaves on algebraic stacks.

Each of the functors j_n^* has a corresponding right adjoint as in (B.1.10), which we denote by $j_{n,*} : D^b_{\mathrm{coh}}(\mathcal{U}_n) \to \operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X}_n)$. Since the functor j_* is compatible with the formation of (filtered, and hence all) colimits (since it is right adjoint to the compact object-preserving functor j^*), one finds that j_* is similarly the colimit of the $j_{n,*}$.

Although it might be better to write Rj_* rather than j_*, to indicate the derived nature of this functor, in the discussion of Subsection B.1.8 we have reserved the former notation to denote the derived functor of j_* in the context of derived categories of complexes of quasi-coherent sheaves, rather than in the context of Ind coherent complexes. Thus we content ourselves with writing simply j_* in the Ind coherent case, trusting that context will resolve any possible ambiguity. This also allows us to maintain notational consistency with the general discussion of Appendix A related to Hypothesis A.3.2. In any event, in our eventual applications, the relevant open immersions j will be cohomologically affine, and so the functors j_* that we have to consider will in fact be t-exact; thus no confusion should arise from our choice of notation.

Remark B.2.18. Our definition of the categories $\mathrm{Coh}(\mathcal{X})$ and $D^b_{\mathrm{coh}}(\mathcal{X})$ is less expansive than some others in the literature, since, by definition, any object of either category is supported on a closed algebraic substack of $\mathcal{X}$. Thus, for example, the structure sheaf $\mathcal{O}_{\mathcal{X}}$ will not be an object of $\mathrm{Coh}(\mathcal{X})$ or $D^b_{\mathrm{coh}}(\mathcal{X})$, unless $\mathcal{X}$ happens to be an algebraic stack, rather than merely formally algebraic. Rather, in our setting, the structure sheaf will be realized as an object of $\operatorname{Pro}\mathrm{Coh}(\mathcal{X})$, or $\operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X})$, namely the pro-object $\lim_n \mathcal{O}_{\mathcal{X}_n}$.. In other papers, the ∞-category that we denote by $D^b_{\mathrm{coh}}(\mathcal{X})$ might instead be denoted $D^b_{\mathrm{coh},\mathcal{X}_{\mathrm{red}}}(\mathcal{X})$: the category of coherent complexes whose cohomologies are set-theoretically supported on the closed substack $\mathcal{X}_{\mathrm{red}}$ of $\mathcal{X}$.

We use the definition that we do because it fits well with the general definition of Ind-coherent sheaves on Ind-schemes, and also because it ensures that there are no hidden issues related to completion or adic topologies lurking in the background; objects of $D^b_{\mathrm{coh}}(\mathcal{X})$ and $\operatorname{Ind} D^b_{\mathrm{coh}}(\mathcal{X})$ are, by definition, adically discrete. Issues of adic topologies and completions are then handled by the use of Pro categories.

We also note that we are, in fact, unaware of a general treatment of stable ∞-categories of coherent sheaves on formal algebraic stacks in the more expansive sense that includes the structure sheaf $\mathcal{O}_{\mathcal{X}}$ as one its objects, although presumably it can be realized as a special case of Clausen–Scholze's theory of stable ∞-categories of solid modules on analytic stacks. The one exception is the case of formal algebraic stacks obtained as completions of algebraic stacks; in this case, [Con05] introduces the corresponding triangulated category. We refer to Proposition B.4.10 below for a discussion of how to relate this more standard approach to our approach via Pro categories.

B.3. **Pullback of pro-coherent sheaves.** In this section we consider various pullback and pushforward functors defined on abelian categories of coherent and pro-coherent sheaves.

B.3.1. *Representable morphisms.* Assume that $\mathcal{Z}, \mathcal{X}$ are Noetherian formal algebraic stacks with affine diagonal, and assume that $f : \mathcal{Z} \to \mathcal{X}$ is representable by algebraic stacks. Choose a presentation $\mathcal{X} = \operatorname{colim}_n \mathcal{X}_n$ as in (B.2.1), and let $\mathcal{Z}_n \coloneqq \mathcal{Z} \times_{\mathcal{X}} \mathcal{X}_n$, which is a closed algebraic substack of $\mathcal{Z}$. By [Eme, Lemma 4.8], the natural map $\operatorname{colim}_n \mathcal{Z}_n \to \mathcal{Z}$ is an isomorphism, hence it is a presentation of $\mathcal{Z}$ as in (B.2.1). Write $x_{mn} : \mathcal{X}_m \to \mathcal{X}_n$ for the transition map (which is a closed immersion), and similarly for $z_{mn} : \mathcal{Z}_m \to \mathcal{Z}_n$. The map f is then the colimit over n of morphisms $f_n : \mathcal{Z}_n \to \mathcal{X}_n$ of algebraic stacks. Since the transition maps x_{mn}, z_{mn} are closed immersions, the diagrams

$$\begin{tikzcd} \operatorname{Coh}(\mathcal{X}_n) \arrow[r, "f_n^*"] & \operatorname{Coh}(\mathcal{Z}_n) \\ \operatorname{Coh}(\mathcal{X}_m) \arrow[u, "x_{mn,*}"] \arrow[r, "f_m^*"] & \operatorname{Coh}(\mathcal{Z}_m) \arrow[u, "z_{mn,*}"'] \end{tikzcd}$$

commute. Since $\operatorname{Coh}(\mathcal{X}) = \operatorname{colim}_n \operatorname{Coh}(\mathcal{X}_n)$ and $\operatorname{Coh}(\mathcal{Z}) = \operatorname{colim}_n \operatorname{Coh}(\mathcal{Z}_n)$, we can thus define

$$f^* = \operatorname{colim}_n f_n^* : \operatorname{Coh}(\mathcal{X}) \to \operatorname{Coh}(\mathcal{Z}).$$

If f is furthermore proper (in the sense of [Eme, Definition 3.11, Remark 3.13], i.e. its pullback to any algebraic stack is proper) the pushforwards f_{n*} preserve coherence, as recalled in Section B.1.6. We can thus define

$$f_* = \operatorname{colim}_n f_{n*} : \operatorname{Coh}(\mathcal{Z}) \to \operatorname{Coh}(\mathcal{X}).$$

Up to natural equivalence, the functors f^*, f_* are independent of the choice of presentation of $\mathcal{X}$. They form an adjoint pair.

Remark B.3.2. We will denote the Pro-extensions of f^* and f_* by the same symbols, to avoid confusion with the completed pullbacks to be defined in Section B.3.6.

Remark B.3.3. If $i : \mathcal{Z} \to \mathcal{X}$ is a closed immersion, then it is representable by algebraic stacks and proper, and by the discussion above we have an adjunction

$$(i^*, i_*) : \operatorname{Pro}\operatorname{Coh}(\mathcal{X}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{Z}).$$

The right adjoint i_* is exact and fully faithful, and the unit $1 \to i_* i^*$ is surjective.

If $\mathcal{Z}$ is furthermore algebraic, and we let $\widehat{\mathcal{X}}$ denote the completion of $\mathcal{X}$ along $\mathcal{Z}$, then $\operatorname{Coh}(\mathcal{Z})$ is a full subcategory of $\operatorname{Coh}(\widehat{\mathcal{X}})$, and i_* is the restriction to $\operatorname{Coh}(\mathcal{Z})$ of the functor $\operatorname{Coh}(\widehat{\mathcal{X}}) \to \operatorname{Coh}(\mathcal{X})$ obtained by restricting the t-exact functor (B.2.7) to the heart of its domain. This can be checked by computing i_* with respect to a presentation $\mathcal{X} = \operatorname{colim}_n \mathcal{X}_n$ such that $\mathcal{Z}$ is a closed substack of $\mathcal{X}_n$ for all n.

B.3.4. *Presentation of pro-coherent sheaves.* The functors i^* and i_* can be used to give presentations for pro-coherent sheaves, as in the following lemma.

Lemma B.3.5. *Let $\mathcal{X}$ be a Noetherian formal algebraic stack with affine diagonal, and choose a presentation $\mathcal{X} \xrightarrow{\sim} \operatorname{colim}_n \mathcal{X}_n$ as in* (B.2.1)*, Write $i_n : \mathcal{X}_n \hookrightarrow \mathcal{X}$ to denote the canonical closed immersion. Then, for any pro-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, we have $\mathcal{F} \xrightarrow{\sim} \lim_n i_{n,*} i_n^* \mathcal{F}$ in* $\operatorname{Pro}\operatorname{Coh}(\mathcal{X})$.

Proof. We first consider the case when $\mathcal{F}$ is actually coherent. Then by definition $\mathcal{F} = (i_m)_* \mathcal{F}_m$ for some m and some object $\mathcal{F}_m$ of $\operatorname{Coh}(\mathcal{X}_m)$. Thus

$$\lim_n i_{n,*} i_n^* \mathcal{F} = \lim_{n \geq m} i_{n,*} i_n^* (i_m)_* \mathcal{F}_m \xrightarrow{\sim} \lim_{n \geq m} (i_m)_* \mathcal{F}_m \xrightarrow{\sim} (i_m)_* \mathcal{F}_m = \mathcal{F}.$$

In the general case that $\mathcal{F}$ is pro-coherent, write $\mathcal{F} \xrightarrow{\sim} \lim_{i \in I} \mathcal{F}_i$ as a cofiltered limit of coherent sheaves $\mathcal{F}_i$. Then

$$\lim_n i_{n,*} i_n^* \mathcal{F} \xrightarrow{\sim} \lim_n \lim_{i \in I} i_{n,*} i_n^* \mathcal{F}_i \xrightarrow{\sim} \lim_{i \in I} \lim_n i_{n,*} i_n^* \mathcal{F}_i = \lim_{i \in I} \mathcal{F}_i,$$

as required. In more detail, the first isomorphism holds because $i_{n,*}$ and i_n^* are pro-extended, and so compatible with the formation of cofiltered limits; the second isomorphism is an interchange of cofiltered limits; and the third isomorphism follows from the case of coherent sheaves that we've already proved. □

B.3.6. *Completed pullbacks.* We continue to assume that $f : \mathcal{Z} \to \mathcal{X}$ is a morphism of Noetherian formal algebraic stacks with affine diagonal, but we no longer assume that f is representable by algebraic stacks.

Suppose firstly that $\mathcal{X}$ is algebraic, and choose a presentation $\mathcal{Z} = \operatorname{colim}_n \mathcal{Z}_n$ as in (B.2.1). Let $f_n : \mathcal{Z}_n \to \mathcal{X}$ be the restriction of f to $\mathcal{Z}_n$. We can then define a functor

$$\begin{aligned} \widehat{f}^* : \operatorname{Coh}(\mathcal{X}) &\to \operatorname{Pro}\operatorname{Coh}(\mathcal{Z}) \\ \mathcal{F} &\mapsto \lim_n f_n^* \mathcal{F}. \end{aligned}$$

Up to natural equivalence, this functor is independent of the choice of presentation of $\mathcal{Z}$.

Lemma B.3.7. *Let $f : \mathcal{Z} \to \mathcal{X}$ be a morphism of Noetherian formal algebraic stacks with affine diagonal. Suppose that $\mathcal{X}$ is in fact an algebraic stack, let $i : \mathcal{Y} \to \mathcal{X}$ be a closed immersion, and let $f' : \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ and $i' : \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Z}$ denote the base-changes of f and i respectively. Then the natural diagram* (*where the right hand vertical arrow is defined following Remark* B.3.2)

$$\begin{tikzcd} \operatorname{Coh}(\mathcal{X}) \arrow[r, "\widehat{f}^*"] & \operatorname{Pro}\operatorname{Coh}(\mathcal{Z}) \\ \operatorname{Coh}(\mathcal{Y}) \arrow[u, "i_*"] \arrow[r, "\widehat{f'}^*"] & \operatorname{Pro}\operatorname{Coh}(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}) \arrow[u, "i'_*"'] \end{tikzcd}$$

commutes.

Proof. By the definition of $\widehat{f}^*$, we are reduced to noting that the pushforward functor for a closed immersion of algebraic stacks commutes with arbitrary base change. □

In the more general case that $\mathcal{X}$ is not necessarily algebraic, we as usual choose a presentation $\mathcal{X} = \operatorname{colim}_n \mathcal{X}_n$. Then we can form the fibre product $\mathcal{Z}_n := \mathcal{Z} \times_{\mathcal{X}} \mathcal{X}_n$, a closed formal algebraic substack of $\mathcal{Z}$ with a map $f_n : \mathcal{Z}_n \to \mathcal{X}_n$. By Lemma B.3.7, the diagrams

$$\begin{tikzcd} \mathrm{Coh}(\mathcal{X}_n) \arrow[r, "\widehat{f}_n^*"] & \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}_n) \\ \mathrm{Coh}(\mathcal{X}_m) \arrow[u, "x_{mn,*}"] \arrow[r, "\widehat{f}_m^*"] & \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}_m) \arrow[u, "z_{mn,*}"'] \end{tikzcd}$$

commute, where the vertical arrows are induced by the transition maps $x_{mn} : \mathcal{X}_m \to \mathcal{X}_n$. We can thus pass to the colimit and obtain a functor

$$\widehat{f}^* : \mathrm{Coh}(\mathcal{X}) \to \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}), \tag{B.3.8}$$

which is independent (up to natural isomorphism) of the choice of presentation of $\mathcal{X}$. We continue to denote by $\widehat{f}^*$ the Pro-extension

$$\widehat{f}^* : \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}) \to \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}).$$

We can also consider the composite functors

$$\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}) \xrightarrow{x_n^*} \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}_n) \xrightarrow{\widehat{f}_n^*} \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}_n) \xrightarrow{z_{n*}} \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}).$$

where $x_n : \mathcal{X}_n \to \mathcal{X}$ and $z_n : \mathcal{Z}_n \to \mathcal{Z}$ denote the closed immersions. (Noting that the target of $f_n : \mathcal{Z}_n \to \mathcal{X}_n$ is algebraic, the functor $\widehat{f}_n^*$ is defined by the discussion above, while x_n^* and z_{n*} are defined in Remark B.3.2.)

Lemma B.3.9. *With notation as above, the functor* $\widehat{f}^* : \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}) \to \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z})$ *is naturally isomorphic to* $\lim_n z_{n,*}\widehat{f}_n^* x_n^*$.

Proof. Lemma B.3.5 gives a natural isomorphism of functors

$$\mathrm{id} \xrightarrow{\sim} \lim_n x_{n,*}x_n^*$$

on $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$. Applying $\widehat{f}^*$, we find that

$$\widehat{f}^* \xrightarrow{\sim} \lim_n \widehat{f}^* x_{n,*}x_n^* \xrightarrow{\sim} \lim_n z_{n,*}\widehat{f}_n^* x_n^*$$

(the first isomorphism following from the fact that $\widehat{f}$ commutes with cofiltered limits by its construction as a Pro-extension, and the second claim following from the definition of (B.3.8)), as claimed. ☐

Remark B.3.10. Let $f : \mathcal{Z} \to \mathcal{X}$ be a morphism of Noetherian formal algebraic stacks with affine diagonal, and let $i : \mathcal{Y} \to \mathcal{X}$ be a closed immersion. As in Lemma B.3.7, we can form the Cartesian diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "f"] & \mathcal{X} \\ \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \arrow[u, "i'"] \arrow[r, "f'"] & \mathcal{Y}. \arrow[u, "i"'] \end{tikzcd} \tag{B.3.11}$$

Analyzing the definitions, we see that we have commutative diagrams
(B.3.12)

$$\begin{tikzcd} \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}) \arrow[r, "\widehat{f}^*"] & \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}) \\ \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Y}) \arrow[u, "i_*"] \arrow[r, "\widehat{f}'^*"] & \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}) \arrow[u, "i'_*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}) \arrow[r, "\widehat{f}^*"] \arrow[d, "i^*"] & \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z}) \arrow[d, "i'^*"] \\ \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Y}) \arrow[r, "\widehat{f}'^*"] & \mathrm{Pro}\,\mathrm{Coh}(\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}) \end{tikzcd}$$

and furthermore, for all $\mathcal{F} \in \operatorname{Pro}\operatorname{Coh}(\mathcal{X})$, the image under $\widehat{f}^*$ of the unit $\mathcal{F} \to i_* i^* \mathcal{F}$ is naturally isomorphic to the unit of (i'^*, i'_*) evaluated at $\widehat{f}^*\mathcal{F}$. Note that commutativity of the leftmost square in (B.3.12) is a generalization of Lemma B.3.7 to the case where $\mathcal{X}$ is not necessarily algebraic.

We will often apply this construction in a context where $\mathcal{Y}$ is furthermore algebraic, and $f : \widehat{\mathcal{X}}_Z \to \mathcal{X}$ is the completion of $\mathcal{X}$ at a closed subset $Z \subset |\mathcal{X}|$. Then (B.3.11) becomes a Cartesian diagram

$$\begin{tikzcd} \widehat{\mathcal{X}}_Z \arrow[r, "f"] & \mathcal{X} \\ \widehat{\mathcal{Y}}_{|\mathcal{Y}|\cap Z} \arrow[u, "i'"] \arrow[r, "f'"] & \mathcal{Y} \arrow[u, "i"'] \end{tikzcd}$$

since the pullback $\widehat{\mathcal{X}}_Z \times_{\mathcal{X}} \mathcal{Y}$ is the completion of $\mathcal{Y}$ at $|\mathcal{Y}| \cap Z$. We will sometimes write i_Z for the closed immersion i', so that we have an adjunction

$$(i_Z^*, i_{Z,*}) : \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}}_Z) \to \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{Y}}_{|\mathcal{Y}|\cap Z}), \tag{B.3.13}$$

as in Remark B.3.3, and the image under $\widehat{f}^*$ of the unit of (i^*, i_*) is the unit of $(i_Z^*, i_{Z,*})$.

Remark B.3.14. If $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are morphisms of Noetherian formal algebraic stacks with affine diagonal, then $\widehat{gf}^*$ is naturally isomorphic to $\widehat{f}^*\widehat{g}^*$. To see this, we can assume without loss of generality that $\mathcal{Z}$ is algebraic. Choose a presentation $\mathcal{Y} = \operatorname{colim}_n \mathcal{Y}_n$, and let $\mathcal{X}_n \coloneqq \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_n$. Then $\mathcal{X}_n$ is a closed formal algebraic substack of $\mathcal{X}$, and $\mathcal{X} = \operatorname{colim}_n \mathcal{X}_n$ (note that any map $\operatorname{Spec} A \to \mathcal{X}$ factors through some $\mathcal{X}_n$, because the composite $\operatorname{Spec} A \to \mathcal{X} \to \mathcal{Y}$ factors through some $\mathcal{Y}_n$).

Choose also a presentation $\mathcal{X} = \operatorname{colim}_m \mathcal{X}'_m$, and define $\mathcal{X}_{n,m} = \mathcal{X}_n \times_{\mathcal{X}} \mathcal{X}'_m$. Then $\mathcal{X}_n = \operatorname{colim}_m \mathcal{X}_{n,m}$ for all $n \geq 0$, by [Eme, Lemma 4.8], so that $\mathcal{X} = \operatorname{colim}_{\mathbf{Z}^2_{\geq 0}} \mathcal{X}_{n,m}$, where the colimit is taken with respect to the product order on $\mathbf{Z}^2_{\geq 0}$.

Now, if $\mathcal{F} \in \operatorname{Coh}(\mathcal{Z})$, then by construction we have

$$\widehat{f}^*\widehat{g}^*(\mathcal{F}) = \widehat{f}^*(\lim_n g_n^*\mathcal{F}) = \lim_{n,m} f_{n,m}^* g_n^* \mathcal{F}$$

where $g_n : \mathcal{Y}_n \to \mathcal{Z}$, $f_{n,m} : \mathcal{X}_{n,m} \to \mathcal{Y}_n$ are the natural maps. This also coincides with $\widehat{gf}^*\mathcal{F}$ computed with respect to the presentation $\mathcal{X} = \operatorname{colim}_{\mathbf{Z}^2_{\geq 0}} \mathcal{X}_{n,m}$, as desired.

Remark B.3.15. Let $\mathcal{Z}, \mathcal{X}$ be Noetherian formal algebraic stacks with affine diagonal, and assume that $f : \mathcal{Z} \to \mathcal{X}$ is representable by algebraic stacks (which is always the case if $\mathcal{Z}$ is algebraic, by our assumption on the diagonal of $\mathcal{X}$). It then follows from the definitions that $\widehat{f}^* : \operatorname{Coh}(\mathcal{X}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{Z})$ takes values in $\operatorname{Coh}(\mathcal{Z})$, and is naturally isomorphic to f^*. It follows that their Pro-extensions are also naturally isomorphic, as functors $f^*, \widehat{f}^* : \operatorname{Pro}\operatorname{Coh}(\mathcal{X}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{Z})$.

B.3.16. *Completed pullbacks to completed stacks.* As in Section B.2.6, we let $\mathcal{X}$ be a Noetherian formal algebraic stack with affine diagonal, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack of $\mathcal{X}$. We write $i : \widehat{\mathcal{X}} \hookrightarrow \mathcal{X}$ for the completion of $\mathcal{X}$ along $\mathcal{Z}$. Following as above the notational conventions of Appendix A.5, we write

$$\widehat{i}_* : \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{X}), \tag{B.3.17}$$

$$\widehat{i}^* : \operatorname{Pro}\operatorname{Coh}(\mathcal{X}) \to \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}}) \tag{B.3.18}$$

for the Pro-extensions of $i_* : \operatorname{Coh}(\widehat{\mathcal{X}}) \to \operatorname{Coh}(\mathcal{X})$ and $\widehat{i}^* : \operatorname{Coh}(\mathcal{X}) \to \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})$.

Lemma B.3.19. *With notation as above, write $\widehat{\mathcal{X}} = \operatorname{colim}_n \mathcal{Z}_n$ as a colimit of thickenings of closed algebraic substacks of $\mathcal{X}$, and let $k_n : \mathcal{Z}_n \hookrightarrow \widehat{\mathcal{X}}$ and $i_n : \mathcal{Z}_n \hookrightarrow \mathcal{X}$ denote the corresponding closed immersions. Then the functor $\widehat{i}^* : \operatorname{Pro}\operatorname{Coh}(\mathcal{X}) \to \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})$ agrees with the functor $\lim_n k_{n,*} i_n^*$.*

Proof. Since $\mathcal{Z}_n$ is algebraic, the functor $\widehat{k}_n^*$, resp. $\widehat{i}_n^*$, coincides with k_n^*, resp. i_n^*, by Remark B.3.15. Lemma B.3.5 (applied to $\widehat{\mathcal{X}}$) shows that

$$\widehat{i}^* \xrightarrow{\sim} \lim_n k_{n,*} k_n^* \widehat{i}^* \xrightarrow{\sim} \lim_n k_{n,*} i_n^*,$$

where we have applied the natural isomorphisms $k_n^* \widehat{i}^* = \widehat{k}_n^* \widehat{i}^* \xrightarrow{\sim} \widehat{i}_n^* = i_n^*$ of Remark B.3.14, taking into account that $i_n = i k_n$ for each n. ☐

Remark B.3.20. If $\mathcal{X}$ is furthermore algebraic, the preceding lemma recovers the definition of the completed pullback $\widehat{i}^*$. On the other hand, if we take $\mathcal{X} = \widehat{\mathcal{X}}$, $i_n = k_n$, and $\widehat{i}^*$ to be the identity of $\operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})$, then we recover Lemma B.3.5.

Proposition B.3.21. *Write $i : \widehat{\mathcal{X}} \hookrightarrow \mathcal{X}$ for the inclusion of the completion $\widehat{\mathcal{X}}$ of a Noetherian formal algebraic stack $\mathcal{X}$, with affine diagonal, along a closed algebraic substack $\mathcal{Z}$. Then the functor $\widehat{i}^* : \operatorname{Pro}\operatorname{Coh}(\mathcal{X}) \to \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})$ is exact, and is left adjoint to $\widehat{i}_* : \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{X})$.*

Proof. Since the completion $\widehat{\mathcal{X}}$ only depends on the underlying topological space of $\mathcal{Z}$, if $\mathcal{X} = \operatorname{colim}_n \mathcal{X}_n$ is a presentation as in (B.2.1), then we have $\widehat{\mathcal{X}} = \operatorname{colim}_n \widehat{\mathcal{X}}_n$. It is therefore easy (and formal) to reduce to the case that $\mathcal{X}$ is algebraic, which we assume from now on.

We begin by showing exactness, for which it suffices to show that the restriction $\widehat{i}^* : \operatorname{Coh}(\mathcal{X}) \to \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})$ is exact. Write $\mathcal{I}$ for the coherent ideal sheaf of $\mathcal{Z}$ in $\mathcal{X}$, so that for any coherent sheaf $\mathcal{F}$ we have

$$\widehat{i}^* \mathcal{F} = \lim_n \mathcal{F}/\mathcal{I}^n. \tag{B.3.22}$$

The exactness of $\widehat{i}^*$ then follows from a standard Artin–Rees argument, *cf.* [Stacks, Tag 0G9M].

We now turn to the adjunction. Since both $\widehat{i}^*$ and $\widehat{i}_*$ are Pro-extended, it suffices to show that if $\mathcal{F}$ is an object of $\operatorname{Coh}(\mathcal{X})$ and $\mathcal{G}$ is an object of $\operatorname{Coh}(\widehat{\mathcal{X}})$ then

$$\operatorname{Hom}_{\operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})}(\widehat{i}^* \mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{\operatorname{Pro}\operatorname{Coh}(\mathcal{X})}(\mathcal{F}, \widehat{i}_* \mathcal{G}),$$

or equivalently that

$$\operatorname{Hom}_{\operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})}(\widehat{i}^* \mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{\operatorname{Coh}(\mathcal{X})}(\mathcal{F}, i_* \mathcal{G}).$$

As usual, we write $\widehat{\mathcal{X}} = \operatorname{colim}_n \mathcal{X}_n$, so that $i_n : \mathcal{X}_n \hookrightarrow \mathcal{X}$ is a closed immersion of an algebraic substack. Relabeling if necessary, we can assume that $\mathcal{G}$ is supported on $\mathcal{X}_1$, so that we can (slightly abusively) write $i_* \mathcal{G} = i_{n,*} \mathcal{G}$ for each n. Then we compute that

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Pro}\,\mathrm{Coh}(\widehat{\mathcal{X}})}(\widehat{i}^*\mathcal{F},\mathcal{G}) = \\ &\qquad \mathrm{Hom}_{\mathrm{Pro}\,\mathrm{Coh}(\widehat{\mathcal{X}})}(\lim_n i_n^*\mathcal{F},\mathcal{G}) = \mathrm{colim}_n \mathrm{Hom}_{\mathrm{Coh}(\widehat{\mathcal{X}})}(i_n^*\mathcal{F},\mathcal{G}) = \\ &\qquad \mathrm{colim}_n \mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_n)}(i_n^*\mathcal{F},\mathcal{G}) = \mathrm{colim}_n \mathrm{Hom}_{\mathrm{Coh}(\mathcal{X})}(\mathcal{F}, i_{n,*}\mathcal{G}) = \\ &\qquad\qquad \mathrm{Hom}_{\mathrm{Coh}(\mathcal{X})}(\mathcal{F}, i_*\mathcal{G}),\end{aligned}$$

as required. (In more detail, the first isomorphism is the definition of $\widehat{i}^*$, the second holds by virtue of how morphisms are computed in the Pro-category, the third is by the definition of $\mathrm{Coh}(\widehat{\mathcal{X}})$ and our assumption that $\mathcal{G}$ is supported on $\mathcal{X}_1$, the fourth is the adjunction between i_n^* and $i_{n,*}$, and the fifth is again by our assumption that $\mathcal{G}$ is supported on $\mathcal{X}_1$.) □

Remark B.3.23. We put ourselves in the context of Lemma B.3.19, so that $i_1 : \mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion of an algebraic stack $\mathcal{Z}$ into a Noetherian formal algebraic stack $\mathcal{X}$ with affine diagonal, $i : \widehat{\mathcal{X}} \to \mathcal{X}$ is the completion of $\mathcal{X}$ along $\mathcal{Z}$, and $k_1 : \mathcal{Z} \to \widehat{\mathcal{X}}$ is the defining closed immersion. Then for any object $\mathcal{F}$ of $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$, there is a canonical surjection

$$\widehat{i}^*\mathcal{F} \to k_{1,*}i_1^*\mathcal{F} \tag{B.3.24}$$

in $\mathrm{Pro}\,\mathrm{Coh}(\widehat{\mathcal{X}})$. In fact, as noted in Remark B.3.3, the unit of adjunction $\mathcal{F} \to i_{1,*}i_1^*\mathcal{F}$ is surjective, and so it remains surjective after applying the functor $\widehat{i}^*$, which is exact by Proposition B.3.21. It now suffices to note that $\widehat{i}^* i_{1,*}i_1^*\mathcal{F} = k_{1,*}i_1^*\mathcal{F}$, because the counit of adjunction $\widehat{i}^*\widehat{i}_* \to \mathrm{id}$ is an isomorphism, and $i_{1,*} = \widehat{i}_* k_{1,*}$.

Applying the exact functor $\widehat{i}_*$ to (B.3.24) we obtain a surjection

$$\widehat{i}_*\widehat{i}^*\mathcal{F} \to i_{1,*}i_1^*\mathcal{F}$$

of objects in $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$.

Remark B.3.25. If $\mathcal{X}$ is a Noetherian algebraic stack with affine diagonal, and $p : \mathrm{Spec}\, A \to \mathcal{X}$ is a flat and surjective morphism, then the pullback $p^* : \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}) \to \mathrm{Pro}\,\mathrm{Coh}(\mathrm{Spec}\, A)$ reflects coherence, in the sense that if $\mathcal{F} = \lim_{i\in I} \mathcal{F}_i \in \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$, and $p^*\mathcal{F}$ is an object of $\mathrm{Coh}(\mathrm{Spec}\, A)$, then $\mathcal{F}$ is an object of $\mathrm{Coh}(\mathcal{X})$. To see this, we first remark that p^* is exact and faithful (and thus conservative) on $\mathrm{Coh}(\mathcal{X})$, and so the same is true of its Pro-extension, by e.g. [KS06, Prop. 6.1.10, Cor. 8.6.8]. Now, if $\mathcal{G} := p^*\mathcal{F}$ is coherent, then $\mathcal{G}$ has descent data to $\mathcal{X}$, since it's pulled back from $\mathcal{X}$ when thought of as a pro-coherent sheaf. It thus descends to a coherent sheaf $\mathcal{G}'$ on $\mathcal{X}$. The isomorphism $p^*\mathcal{G}' \xrightarrow{\sim} p^*\mathcal{F}$ induces morphisms $p^*\mathcal{G}' \to p^*\mathcal{F}_i$ for all i, which are compatible with the transition maps in $\mathcal{F}_i$, and with the descent data on source and target. These morphisms therefore descend to a morphism $\mathcal{G}' \to \mathcal{F}$ in $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$, which becomes an isomorphism after applying p^*. Since, as we already recalled, p^* is conservative, we conclude that $\mathcal{G}' \xrightarrow{\sim} \mathcal{F}$ and so $\mathcal{F}$ is coherent, as desired.

More generally, if $\mathcal{X} = \mathrm{colim}_n \mathcal{X}_n$ is a Noetherian formal algebraic stack with affine diagonal, presented as in (B.2.1), and $\mathrm{Spf}\, A \to \mathcal{X}$ is a smooth surjective morphism which is representable by algebraic stacks, then the pullback $p^* : \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X}) \to \mathrm{Pro}\,\mathrm{Coh}(\mathrm{Spf}\, A)$ reflects coherence. To see this, let $\mathcal{F} \in \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$, and assume that $p^*\mathcal{F}$ is coherent. Then, if we define $\mathrm{Spec}\, A_n := \mathrm{Spf}\, A \times_{\mathcal{X}} \mathcal{X}_n$, and write $k_n : \mathcal{X}_n \to \mathcal{X}$ for the closed immersion and for its pullback to $\mathrm{Spf}\, A$, there exists n such that

$p^*\mathcal{F} \to k_{n,*}k_n^*p^*\mathcal{F}$ is an isomorphism. Hence $\mathcal{F} \to k_{n,*}k_n^*\mathcal{F}$ is also an isomorphism. Since $\mathcal{X}_n$ is an algebraic stack, we have thus reduced to the result of the previous paragraph.

B.3.26. *Passage to derived categories.* By Proposition B.3.21 and Corollary A.8.2, the functors (B.3.17) and (B.3.18) induce t-exact adjoint functors

$$\widehat{i}_* : \operatorname{Pro} D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \hookrightarrow \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}) \tag{B.3.27}$$

and

$$\widehat{i}^* : \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}). \tag{B.3.28}$$

Remark B.3.29. Since $\widehat{i}_*$ is fully faithful (as a consequence of Proposition B.2.8), the counit of adjunction $\widehat{i}^*\widehat{i}_* \to \mathrm{id}$ is a natural isomorphism. Thus the restriction of $\widehat{i}^*$ to $D^b_{\mathrm{coh},Z}(\mathcal{X})$ induces an inverse to the equivalence $D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \xrightarrow{\sim} D^b_{\mathrm{coh},Z}(\mathcal{X})$ of Proposition B.2.8.

B.3.30. *Scheme-theoretic support.* It is unclear to us in what generality it is sensible to define the notion of "scheme-theoretic" support for pro-coherent sheaves, but the following definitions will suffice for us.

Definition B.3.31. Let $z : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed immersion of Noetherian formal algebraic stacks with affine diagonal.

(1) Let $\mathcal{F}$ be a pro-coherent sheaf on $\mathcal{X}$. Then we say that $\mathcal{F}$ is *scheme-theoretically supported on* $\mathcal{Z}$ if $\mathcal{F}$ is in the essential image of $z_* : \operatorname{Pro}\operatorname{Coh}(\mathcal{Z}) \to \operatorname{Pro}\operatorname{Coh}(\mathcal{X})$, or equivalently, if the unit of adjunction $\mathcal{F} \to z_*z^*\mathcal{F}$ is an isomorphism.

(2) Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, and assume that $\mathcal{Z}$ is algebraic. Then we say that $\mathcal{Z}$ is the *scheme-theoretic support* of $\mathcal{F}$ if $\mathcal{F}$ is scheme-theoretically supported on $\mathcal{Z}$ and the natural map $\mathcal{O}_{\mathcal{Z}} \to \underline{\mathrm{End}}_{\mathrm{Coh}(\mathcal{Z})}(z^*\mathcal{F})$ is injective, where $\underline{\mathrm{End}}$ denotes the sheaf-End. This being the case, we say that $|\mathcal{Z}|$ is the *set-theoretic support* of $\mathcal{F}$.

Remark B.3.32. If $\mathcal{X}$ is a scheme, and $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$, then the scheme-theoretic support of $\mathcal{F}$ is the closed subscheme associated to the kernel of $\mathcal{O}_{\mathcal{X}} \to \underline{\mathrm{End}}_{\mathrm{Coh}(\mathcal{X})}(\mathcal{F})$, which is a quasicoherent ideal sheaf. Hence Definition B.3.31 (2) agrees with the usual definition of the scheme-theoretic support of $\mathcal{F}$ in this case.

Definition B.3.33. We say that an object $\mathcal{F}$ of $\operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X})$ is *pure of degree zero* if it is a pro-coherent sheaf: that is, it is contained in the heart $\operatorname{Pro}\operatorname{Coh}(\mathcal{X})$ of $\operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X})$. (Note that this is *a priori* a stronger condition than asking that the cohomology sheaves of $\mathcal{F}$ vanish outside of degree zero.)

Lemma B.3.34. *Let $\mathcal{X}$ be a Noetherian formal algebraic stack with affine diagonal.*

(1) Suppose that $\mathcal{F}$ is an object of $\operatorname{Pro}\operatorname{Coh}(\mathcal{X})$. Then $\mathcal{F} = 0$ if and only if, for every closed point $x \in |\mathcal{X}|$ with completion-at-$\{x\}$ morphism $i_x : \widehat{\mathcal{X}}_x \to \mathcal{X}$, we have $\widehat{i}_x^*\mathcal{F} = 0$.

(2) Suppose that $\mathcal{G}$ is an object of $\operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X})$ which is bounded above. Then $\mathcal{G}$ is pure of degree zero, resp. equal to zero, if and only if for every closed point $x \in |\mathcal{X}|$, $\widehat{i}_x^*\mathcal{G}$ is pure of degree zero, resp. equal to zero.

Proof. Lemma A.6.6 shows that the second statement follows from the first by considering cohomology sheaves.

For the first statement, assume that $\widehat{i}_x^*\mathcal{F} = 0$ for every closed point x of $\mathcal{X}$. Write $\mathcal{F} = \lim_j \mathcal{F}_j$ as a cofiltered limit of $\mathcal{F}_j \in \mathrm{Coh}(\mathcal{X})$. Now fix an index j. For each point x, there is (by assumption) an index j'_x such that $\mathcal{F}_{j'_x} \to \mathcal{F}_j$ becomes zero after applying $\widehat{i}_x^*$. We claim that x has an open neighbourhood $\mathcal{U}_x$ over which the restriction of the transition morphism $\mathcal{F}_{j'_x} \to \mathcal{F}_j$ vanishes. Granting this, we find that the union of these $\mathcal{U}_x$ is equal to all of $\mathcal{X}$ (since it is an open substack of $\mathcal{X}$ that contains every closed point). Since $\mathcal{X}$ is quasi-compact, we may therefore cover $\mathcal{X}$ by finitely many of these $\mathcal{U}_x$, and thus find a single index j' (some index lying above each of the finitely many corresponding j'_x) such that $\mathcal{F}_{j'} \to \mathcal{F}_j$ vanishes. This implies that $\mathcal{F}$ itself is the zero object in $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$.

It remains to prove the claim. Since $\widehat{i}_x^*$ is exact, by a consideration of the image sheaf, it suffices to prove that, if $\mathcal{G}$ is a coherent sheaf on $\mathcal{X}$, and $\widehat{i}_x^*\mathcal{G} = 0$, then there exists an open neighborhood $\mathcal{U}_x$ of x such that $\mathcal{G}|_{\mathcal{U}_x} = 0$. By definition, the coherent sheaf $\mathcal{G}$ is supported on a closed algebraic substack of $\mathcal{X}$, and replacing $\mathcal{X}$ by this closed algebraic substack, we may furthermore assume that $\mathcal{X}$ is itself algebraic. Now $\widehat{i}_x^*$ can be thought of as the functor of $\mathcal{I}$-adic completion (with $\mathcal{I}$ denoting the coherent ideal sheaf on $\mathcal{X}$ cutting out the underlying reduced substack structure on the closed subset $\{x\}$), and so this amounts to the standard fact that if the $\mathcal{I}$-adic completion of a coherent sheaf vanishes, then the coherent sheaf itself vanishes in a neighbourhood of x. For the sake of completeness (and since our completions are computed in the Pro-sense, rather than literally, and since we are working on an algebraic stack rather than a scheme), we recall the proof. To this end, we choose a smooth surjection $p : X = \mathrm{Spec}\, A \to \mathcal{X}$ from a Noetherian affine scheme onto some open neighbourhood of x; since p is open [Stacks, Tag 04XL], it suffices to prove that $p^*\mathcal{G}$ vanishes after restriction to some open neighbourhood of the closed subset $p^{-1}(x)$ of X. Let I denote the ideal in A cutting out the underlying reduced subscheme structure on $p^{-1}(x)$, and let M denote the finitely generated A-module corresponding to the coherent sheaf $p^*\mathcal{G}$. Our assumption that $\widehat{i}_x^*\mathcal{G} = 0$ implies that the morphism $M/I^nM \to M/IM$ coincides with the zero morphism for sufficiently large n; since this morphism is surjective, we see that its target vanishes, i.e. that $M = IM$. A standard application of Nakayama's lemma now implies that indeed the coherent sheaf attached to M vanishes in some neighbourhood of $p^{-1}(x)$, as required. □

Lemma B.3.35. *Let $z : \mathcal{Z} \to \mathcal{X}$ be a closed immersion of Noetherian formal algebraic stacks with affine diagonal. Then a pro-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ is scheme-theoretically supported on $\mathcal{Z}$ if and only if, for every closed point $x \in |\mathcal{X}|$ with completion-at-$\{x\}$ morphism $i_x : \widehat{\mathcal{X}}_x \to \mathcal{X}$, the pro-coherent sheaf $\widehat{i}_x^*\mathcal{F}$ is scheme-theoretically supported on $\widehat{\mathcal{Z}}_{x\cap|\mathcal{Z}|}$.*

Proof. If $x \in |\mathcal{X}|$ is a closed point, then the image under $\widehat{i}_x^*$ of the unit of adjunction $\mathcal{F} \to z_*z^*\mathcal{F}$ is isomorphic to the unit $\widehat{i}_x^*\mathcal{F} \to z_{x,*}z_x^*(\widehat{i}_x^*\mathcal{F})$, by Remark B.3.10. Since $\widehat{i}_x^*$ is exact (by Proposition B.3.21), the kernel and cokernel of $\mathcal{F} \to z_*z^*\mathcal{F}$ vanish under $\widehat{i}_x^*$ if and only if $\widehat{i}_x^*\mathcal{F}$ is scheme-theoretically supported on $\widehat{\mathcal{Z}}_{x\cap|\mathcal{Z}|}$. Thus the lemma follows from Lemma B.3.34 (1). □

B.3.36. *Pullbacks to versal rings.*

Lemma B.3.37. *Let $\mathcal{X}$ be a Noetherian algebraic stack with affine diagonal, let $x \in |\mathcal{X}|$ be a finite type point, and let $i_x : \mathrm{Spec}\, R \to \mathcal{X}$ be a morphism from the* *spectrum of a complete Noetherian local ring R with finite residue field, sending the maximal ideal $\mathfrak{m}$ to x. Let $\operatorname{Spf} R$ be the formal $\mathfrak{m}$-adic spectrum of R. Then the composite*

(B.3.38)
$$\mathrm{Coh}(\mathcal{X}) \xrightarrow{i_x^*} \mathrm{Mod}^{\mathrm{fp}}(R) \xrightarrow{\text{(A.1.36)}} \mathrm{Mod}_c(R) = \operatorname{Pro}\mathrm{Mod}^{\mathrm{f.l.}}(R) = \operatorname{Pro}\mathrm{Coh}(\operatorname{Spf} R)$$

is naturally isomorphic to the completed pullback for the composite $\operatorname{Spf} R \to \operatorname{Spec} R \xrightarrow{i_x} \mathcal{X}$*, where the first arrow is the completion at $\{\mathfrak{m}\}$.*

Proof. Let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, and let $M := i_x^*(\mathcal{F})$, a finitely presented R-module. Then the composite of (B.3.38) sends $\mathcal{F}$ to the object $\lim_n M/\mathfrak{m}^n M$ of $\operatorname{Pro}\mathrm{Coh}(\operatorname{Spf} R)$. On the other hand, we can compute the completed pullback to $\operatorname{Spf} R$ using the presentation
$$\operatorname{Spf} R = \operatorname{colim}_n \operatorname{Spec} R/\mathfrak{m}^n.$$
Then, by definition, we have $\widehat{i_x^*}\mathcal{F} = \lim_n i_x^*(\mathcal{F})/\mathfrak{m}^n i_x^*(\mathcal{F})$, as desired. □

Lemma B.3.39. *Let $\widehat{\mathcal{X}}$ be a Noetherian formal algebraic stack with affine diagonal. Assume that $|\widehat{\mathcal{X}}| = \{x\}$ is a singleton, and that $\widehat{\mathcal{X}}_{\mathrm{red}}$ is of finite type over* $\operatorname{Spec}\mathbf{F}$*. Let $v_x : \operatorname{Spf} R \to \widehat{\mathcal{X}}$ be a morphism from the formal $\mathfrak{m}$-adic spectrum of a complete Noetherian local ring R with residue field k, and assume that v_x is versal to the induced morphism* $\operatorname{Spec} k \to \mathcal{X}$*. Then*
$$\widehat{v}_x^* : \mathrm{Coh}(\widehat{\mathcal{X}}) \to \operatorname{Pro}\mathrm{Coh}(\operatorname{Spf} R) = \mathrm{Mod}_c(R)$$
is exact and faithful.

Proof. Since the formation of versal rings and of the completed pullbacks $\widehat{v}_x^*$ are both compatible with base change to closed immersions (using the commutativity of the leftmost square in (B.3.12) for the latter), we reduce to the case that $\widehat{\mathcal{X}}$ is algebraic and locally of finite type over $\operatorname{Spec}\mathcal{O}$. In this case v_x is effective, by [Stacks, Tag 0DR1], so by Lemma B.3.37 that it suffices to prove that the usual coherent pullback for the map
$$v_x : \operatorname{Spec} R \to \widehat{\mathcal{X}}$$
is exact and faithful. This follows from [Stacks, Tag 0DR2], which shows that v_x is flat and surjective. □

B.3.40. *Completed pullback and internal* Hom. If $\mathcal{X}$ is a Noetherian algebraic stack, then by [Stacks, Tag 0GQN], $\mathrm{Coh}(\mathcal{X})$ has an internal Hom functor $\underline{\mathrm{Hom}}_{\mathrm{Coh}(\mathcal{X})}(-,-)$.

Lemma B.3.41. *Let $\mathcal{X}$ be an algebraic stack with affine diagonal, which is locally of finite type over* $\operatorname{Spec}\mathcal{O}$*. Let $i_x : \operatorname{Spf} R \to \mathcal{X}$ be a versal morphism at a closed point $x \in |\mathcal{X}|$. Then we have a natural R-linear isomorphism*
$$\widehat{i}_x^*\underline{\mathrm{Hom}}_{\mathrm{Coh}(\mathcal{X})}(-,-) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}_c(R)}(\widehat{i}_x^*-, \widehat{i}_x^*-).$$

Proof. By [Stacks, Tag 0DR1], the map i_x is effective. By Lemma B.3.37, it suffices to show that we have a natural R-linear isomorphism
$$i_x^*\underline{\mathrm{Hom}}_{\mathrm{Coh}(\mathcal{X})}(-,-) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}^{\mathrm{fp}}(R)}(i_x^*-, i_x^*-),$$
where i_x^* is the coherent pullback for the map $v_x : \operatorname{Spec} R \to \mathcal{X}$. Since i_x is flat, by [Stacks, Tag 0DR2], this is a consequence of [Stacks, Tag 0GQP]. □

B.4. **Coherent sheaves and formal functions in the case of "affine mod reductive" quotient stacks.** In this section we prove some results about stacks of the form $[\operatorname{Spec} B/G]$, where B is a finite type algebra over a Noetherian ring R and G is a reductive group scheme over R. Some of our results are deduced from those of [AHL23]; since our setup is slightly different to theirs (see Remark B.2.18), we begin by relating our approach to completions of algebraic stacks to that taken in [Con05] and [AHL23].

B.4.1. *$\mathcal{O}_{\widehat{\mathcal{X}}}$-modules.* Suppose that $\mathcal{X}$ is a Noetherian algebraic stack with affine diagonal, and that $\mathcal{Z}$ is a closed substack with coherent ideal sheaf $\mathcal{I}$. We write $\widehat{\mathcal{X}}$ for the formal completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ along $\mathcal{Z}$. In this context, [Con05] defines a sheaf of rings

$$\mathcal{O}_{\widehat{\mathcal{X}}} := \lim_n \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1} \tag{B.4.2}$$

on the lisse-étale site of $\mathcal{X}$, which is morally the structure sheaf of the completion $\widehat{\mathcal{X}}$, as well as an associated abelian category $\operatorname{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$ of coherent sheaves of $\mathcal{O}_{\widehat{\mathcal{X}}}$-modules. This allows one to also introduce the stable ∞-category $D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{X}}})$. Furthermore, writing $\mathcal{Z}^{[n]}$ for the closed substack of $\mathcal{X}$ with ideal sheaf $\mathcal{I}^n$, the functors $\mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_{\widehat{\mathcal{X}}}} -$ induce an equivalence of categories (see [Con05, Theorem 2.3])

$$\operatorname{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}}) \xrightarrow{\sim} \lim_n \operatorname{Coh}(\mathcal{Z}^{[n]}). \tag{B.4.3}$$

Remark B.4.4. We note that the abelian category $\operatorname{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$ is denoted simply by $\operatorname{Coh}(\widehat{\mathcal{X}})$ in [Con05]. Similarly, the stable ∞-category $D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{X}}})$ — or, rather, its underlying triangulated category — is denoted[16] $D^b_{\mathrm{coh}}(\widehat{\mathcal{X}})$ in [Con05]. However, as indicated in Remark B.2.18, the categories defined in [Con05] are larger than the category $D^b_{\mathrm{coh}}(\widehat{\mathcal{X}})$ that we consider, and so to distinguish the two we always denote the categories of [Con05] by $\operatorname{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$ and $D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{X}}})$.

Rather than forming the actual projective limit in (B.4.2) to obtain a sheaf of rings, one can consider the formal projective limit, so as to obtain an object of $\operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})$. More precisely, since every $\mathcal{O}_{\mathcal{X}}$-module that is annihilated by a power of $\mathcal{I}$ is also an $\mathcal{O}_{\widehat{\mathcal{X}}}$-module, we have an exact and fully faithful functor

$$\operatorname{Coh}(\widehat{\mathcal{X}}) \to \operatorname{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}}), \tag{B.4.5}$$

and we can also define an exact and fully faithful functor

$$\operatorname{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}}) \hookrightarrow \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}}), \quad \mathcal{F} \mapsto \lim_n \mathcal{F}/\mathcal{I}^{n+1}. \tag{B.4.6}$$

(The claimed properties of this functor follow again from [Con05, Thm. 2.3], which implies full faithfulness, together with a standard Artin–Rees argument to deduce exactness, as in the proof of Proposition B.3.21.) The composition of (B.4.6) and (B.4.5) is the natural inclusion of $\operatorname{Coh}(\widehat{\mathcal{X}})$ in its Pro-completion $\operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{X}})$.

As remarked on [Con05, p. 1], the sheaf $\mathcal{O}_{\widehat{\mathcal{X}}}$ is a flat sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras. Thus $\mathcal{O}_{\widehat{\mathcal{X}}} \otimes_{\mathcal{O}_{\mathcal{X}}} -$ defines an exact functor

$$\operatorname{Coh}(\mathcal{X}) \to \operatorname{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}}). \tag{B.4.7}$$

[16]Strictly speaking, [Con05] only considers the "+" version of the derived category, which it denotes by $D^+_{\mathrm{coh}}(\widehat{\mathcal{X}})$.

By definition, the composite of (B.4.7) and (B.4.6) is the functor $\widehat{i}^* : \mathrm{Coh}(\mathcal{X}) \to \mathrm{Pro}\,\mathrm{Coh}(\widehat{\mathcal{X}})$ defined in Section B.3.6.

It follows from Corollary A.8.2 (2), applied to $\mathcal{C} = \mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$ and $\mathcal{C}' = \mathrm{Coh}(\widehat{\mathcal{X}})$, that (B.4.6) induces a t-exact functor

$$D^b(\mathrm{Coh}\,\mathcal{O}_{\widehat{\mathcal{X}}}) \to \mathrm{Pro}\,D^b(\mathrm{Coh}\,\widehat{\mathcal{X}}).$$

Remembering that $D^b\big(\mathrm{Coh}\,\mathcal{O}_{\widehat{\mathcal{X}}}\big) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{X}}})$ and $D^b\big(\mathrm{Coh}\,\widehat{\mathcal{X}}\big) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\widehat{\mathcal{X}})$, we obtain a t-exact functor

$$D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{X}}}) \to \mathrm{Pro}\,D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}), \tag{B.4.8}$$

whose composition with the t-exact functor

$$D^b_{\mathrm{coh}}(\widehat{\mathcal{X}}) \to D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{X}}}) \tag{B.4.9}$$

associated to (B.4.5) is again the natural inclusion of $D^b_{\mathrm{coh}}(\widehat{\mathcal{X}})$ to its Pro-completion.

Proposition B.4.10. *The functor* (B.4.8) *is fully faithful.*

Proof. Either by inspection, or by an application of Corollary A.7.20, one sees that (B.4.8) must coincide with the functor q of (A.9.2) (taking $\mathcal{C}$ to be $\mathrm{Coh}(\widehat{\mathcal{X}})$ and $\mathcal{B}$ to be $\mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$, viewed as an exact abelian subcategory of $\mathrm{Pro}\,\mathcal{C}$ via (B.4.6)). We will prove its full faithfulness by applying the criterion of Lemma A.9.9. Let $\mathcal{G}$ be an object of $\mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$. Then $\mathcal{G}$ maps to $\lim_n \mathcal{G}/\mathcal{I}^{n+1}$ under (B.4.6). It follows from [Stacks, Tag 0BKY] and the Artin–Rees lemma that in fact

$$\mathcal{G} \xrightarrow{\sim} R\lim_n \mathcal{G}/\mathcal{I}^{n+1}$$

in the unbounded derived category $D(\mathcal{O}_{\widehat{\mathcal{X}}})$ of $\mathcal{O}_{\widehat{\mathcal{X}}}$-modules, which is to say, the corresponding $R^1\lim_n$ vanishes. Thus the natural morphism

$$\mathrm{RHom}_{D(\mathcal{O}_{\widehat{\mathcal{X}}})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Rlim}_n\, \mathrm{RHom}_{D(\mathcal{O}_{\widehat{\mathcal{X}}})}(\mathcal{F}, \mathcal{G}/\mathcal{I}^{n+1})$$

is an isomorphism, for any object $\mathcal{F}$ of $D(\mathcal{O}_{\widehat{\mathcal{X}}})$. Restricting to objects $\mathcal{F}$ of the full subcategory $D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{X}}})$ of $D(\mathcal{O}_{\widehat{\mathcal{X}}})$, we see that the hypothesis of Lemma A.9.9 is satisfied, as required. $\square$

It will be useful to give an alternative definition of the abelian category $\mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$ (as a full subcategory of $\mathrm{Pro}\,\mathrm{Coh}(\widehat{\mathcal{X}})$); e.g. it allows us to give a simple proof of Lemma B.4.13 below. This definition also has the merit of applying to a more general class of formal algebraic stacks (although we don't have cause to use it in that greater level of generality).

Definition B.4.11. Let $\mathcal{Y}$ be a Noetherian formal algebraic stack with affine diagonal. We define $\mathrm{Coh}(\mathcal{O}_{\mathcal{Y}})$ as the full subcategory of $\mathrm{Pro}\,\mathrm{Coh}(\mathcal{Y})$ whose objects are the pro-coherent sheaves $\mathcal{F}$ such that, for all morphisms $i : \mathcal{Z} \to \mathcal{Y}$, where $\mathcal{Z}$ is a quasicompact algebraic stack with affine diagonal, the pullback $i^*\mathcal{F}$ is coherent.

Remark B.4.12. As the preceding discussion already implied, in the case when $\widehat{\mathcal{X}}$ is the completion of a Noetherian algebraic stack $\mathcal{X}$ at a closed substack $\mathcal{Z}^{[0]}$ with ideal sheaf $\mathcal{I}$, the category $\mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$, as defined in Definition B.4.11, coincides with the essential image of (B.4.6).

In fact, we have $\mathcal{X} = \mathrm{colim}_n\, \mathcal{Z}^{[n]}$, where $i_n : \mathcal{Z}^{[n]} \to \mathcal{X}$ is the closed substack with ideal sheaf $\mathcal{I}^{n+1}$. Every morphism $\mathcal{Z} \to \mathcal{X}$ from a quasicompact algebraic stack factors through some $\mathcal{Z}^{[n]}$ (this is a consequence of [Eme, Lem. 4.4], after passing to a cover by a quasicompact scheme), and so the essential image of (B.4.6) is contained in $\mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$. Conversely, if $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$ (as defined in Definition B.4.11), then the sequence $(i_n^*\mathcal{F})_n$ defines an object of the right-hand side of (B.4.19), and so of its left-hand side. The isomorphism $\mathcal{F} \xrightarrow{\sim} \lim_n i_{n,*}i_n^*\mathcal{F}$ from Lemma B.3.5 implies that $\mathcal{F}$ is in the essential image of (B.4.6).

Lemma B.4.13. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of Noetherian formal algebraic stacks with affine diagonal. Then the completed pullback functor $\widehat{f}$ restricts to a functor*
$$\widehat{f} : \mathrm{Coh}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Coh}(\mathcal{O}_{\mathcal{Y}}).$$

Proof. Let $i : \mathcal{Z} \to \mathcal{Y}$ be a morphism from a quasicompact algebraic stack with affine diagonal $\mathcal{Z}$, and let $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_{\mathcal{X}})$. We need to prove that $i^*\widehat{f}^*\mathcal{F}$ is coherent. By Remark B.3.15, it is isomorphic to $\widehat{i}^*\widehat{f}^*\mathcal{F}$, which is the same as $\widehat{(f \circ i)}^*\mathcal{F}$, by Remark B.3.14. Again by Remark B.3.15, we conclude that $i^*\widehat{f}^*\mathcal{F} \cong (f \circ i)^*\mathcal{F}$, which is coherent, because $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_{\mathcal{X}})$. $\square$

B.4.14. *Reductive groups.* We will require some coherent sheaf theory on stacks of the form $[\operatorname{Spec} B/G]$, where B is a finite type algebra over a Noetherian ring R and G is a reductive group scheme over R. The definition of "reductive group scheme" in the literature is not always consistent, due to different conventions about disconnected groups. However, the notion of a connected reductive group is well-defined; indeed, over an an arbitrary base scheme S, a connected reductive group G/S is a smooth S-affine group scheme all of whose geometric fibres are connected reductive group.

Turning to the not-necessarily-connected case, if G/S is a finitely presented smooth group scheme over an arbitrary base scheme S, then there is a unique open subgroup scheme $G^\circ \subseteq G$ with the property that for all points $s \in S$, the fibre $(G^\circ)_s$ is equal to the identity component $(G_s)^\circ$ of G_s. (See for example [Con14, §3.1].) The formation of G° commutes with arbitrary base change. By [Con14, Prop. 3.1.3], if G/S is separated, smooth, and of finite presentation, and each fibre G_s° is reductive, then G° is open and closed in G, and the quotient G/G° is a separated étale group scheme of finite presentation.

For us, a "reductive group scheme" will always mean the following.

Definition B.4.15. A group scheme G/S is reductive if G is smooth and affine, G° is connected reductive, and G/G° is finite étale.

The following result of Alper gives an alternative characterisation of reductive group schemes, which we use in the proof of Theorem B.4.17 below. Recall that by definition [Alp14, Defn. 9.1.1], a group scheme G/S is *geometrically reductive* if and only if it is finitely presented, flat and separated over S and the morphism $BG \to S$ is adequately affine in the sense of [Alp14, Defn. 4.1.1].

Proposition B.4.16. *Suppose that G/S is a smooth affine group scheme, and that G° is* (*connected*) *reductive. Then G is reductive* (*in the sense of Definition* B.4.15) *if and only if it is geometrically reductive.*

Proof. As noted above, since G° is connected reductive, the component group scheme G/G° is separated. The result is then immediate from [Alp14, Thm. 9.7.6]. $\square$

Now let B be a finite type R-algebra for a Noetherian ring R, let G be a reductive group scheme over R, and suppose we are given an action of G on $\operatorname{Spec} B$ (in the category of R-schemes); that is, a morphism

$$G \times_{\operatorname{Spec} R} \operatorname{Spec} B \to \operatorname{Spec} B$$

of R-schemes satisfying the usual axioms for an action. We then write $[\operatorname{Spec} B/G]$ for the usual stack quotient. In more detail, the pair of morphisms $G \times_{\operatorname{Spec} R} \operatorname{Spec} B \rightrightarrows \operatorname{Spec} B$ — the first being the obvious projection, and the second being the action morphism — are the source and target map for a smooth groupoid over $\operatorname{Spec} B$, and $[\operatorname{Spec} B/G]$ is the quotient stack of $\operatorname{Spec} B$ with respect to this groupoid.

The following theorem summarizes results of Alper and of Alper–Hall–Lim, but since the paper [AHL23] only states them in the case that G is connected reductive, we explain the straightforward deduction of the non-necessarily-connected reductive case from their results. We remind the reader that a finite type morphism $f : \mathcal{Y} \to \mathcal{X}$ of Noetherian algebraic stacks is *cohomologically proper* [AHL23, Def. 2.1(2)] if for every coherent sheaf $\mathcal{F}$ on $\mathcal{Y}$, the derived pushforward $f_*\mathcal{F}$ lies in $D^+_{\mathrm{coh}}(\mathcal{X})$.

Theorem B.4.17. *Let G be a reductive group scheme over a Noetherian ring R, and let B be a finite type R-algebra with an action of G. Then:*

(1) B^G is of finite type over R (and is in particular Noetherian).
(2) $[\operatorname{Spec} B/G] \to \operatorname{Spec} B^G$ is cohomologically proper.
(3) Suppose that $I \subseteq B$ is a G-equivariant ideal, and that B^G is I^G-adically complete. Write $\mathcal{Y} = [\operatorname{Spec} B/G]$, $\mathcal{Y}^{[n]} = [\operatorname{Spec}(B/I^{n+1})/G]$, and let $\widehat{\mathcal{Y}} = \operatorname{colim}_n \mathcal{Y}^{[n]}$ be the completion of $\mathcal{Y}$ along $i : \mathcal{Y}^{[0]} \hookrightarrow \mathcal{Y}$. Then the functor

$$\widehat{i}^* : D^b_{\mathrm{coh}}(\mathcal{Y}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\widehat{\mathcal{Y}}) \tag{B.4.18}$$

is t-exact and fully faithful, and the functor

$$\mathcal{O}_{\widehat{\mathcal{Y}}} \otimes_{\mathcal{O}_{\mathcal{Y}}} - : \operatorname{Coh}(\mathcal{Y}) \to \operatorname{Coh}(\mathcal{O}_{\widehat{\mathcal{Y}}}) \tag{B.4.19}$$

is fully faithful. If furthermore G is a linearly reductive closed subgroup of $\mathrm{GL}_{n,R}$, then the essential image of (B.4.18) is $D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{Y}}})$.

Proof. Part 1 is immediate from [Alp14, Rem. 9.1.5, Thm. 6.3.3].

We now turn to part 2. By[AHL23, Ex. 2.7], the morphism $[\operatorname{Spec} B/G^\circ] \to \operatorname{Spec} B^{G^\circ}$ is cohomologically proper. Furthermore, the morphism $\operatorname{Spec} B^{G^\circ} \to \operatorname{Spec} B^G$ is finite and therefore cohomologically proper, so that the composite $[\operatorname{Spec} B/G^\circ] \to \operatorname{Spec} B^G$ is cohomologically proper. Since the map

$$[\operatorname{Spec} B/G^\circ] \to [\operatorname{Spec} B/G]$$

is finite étale and surjective, and $[\operatorname{Spec} B/G] \to \operatorname{Spec} B^G$ is of finite type, the claim follows from [AHL23, Thm. 6.1(1)].

Finally, we consider part 3. The t-exactness of $\widehat{i}^*$ is Proposition B.3.21. As noted after (B.4.7), it follows immediately from the definitions that we may factor $\widehat{i}^*$ as the composite

$$D^b_{\mathrm{coh}}(\mathcal{Y}) \xrightarrow{\mathcal{O}_{\widehat{\mathcal{Y}}} \otimes^L_{\mathcal{O}_{\mathcal{Y}}} -} D^b_{\mathrm{coh}}(\mathcal{O}_{\widehat{\mathcal{Y}}}) \xrightarrow{\text{(B.4.8)}} \operatorname{Pro} D^b_{\mathrm{coh}}(\widehat{\mathcal{Y}}).$$

Both arrows are t-exact, and the second is fully faithful, by Proposition B.4.10. So, to prove that (B.4.18) is fully faithful (resp. an equivalence if G is linearly reductive and closed in $\mathrm{GL}_{n,R}$), it suffices to show that $\mathcal{O}_{\widehat{\mathcal{Y}}} \otimes^L_{\mathcal{O}_{\mathcal{Y}}} -$ is fully faithful (resp. an equivalence if G is linearly reductive and closed in $\mathrm{GL}_{n,R}$). This will also imply that (B.4.19) is fully faithful, and complete the proof of the proposition.

It suffices therefore to show that the pair $(\mathcal{Y}, \mathcal{Y}^{[0]})$ satisfies *derived formal functions* (resp. is *derived coherently complete*) in the sense of [AHL23, Defn. 3.2]. By [AHL23, Prop. 3.5], it is in turn enough to show the pair $(\mathcal{Y}, \mathcal{Y}^{[0]})$ satisfies *formal functions* (resp. satisfies formal functions and is *coherently complete*) in the sense of [AHL23, Defn. 3.2].

We begin by showing that $(\mathcal{Y}, \mathcal{Y}^{[0]})$ satisfies formal functions. Since the preimage of $[\mathrm{Spec}(B/I)/G]$ in $[\mathrm{Spec}\, B/G^\circ]$ is $[\mathrm{Spec}(B/I)/G^\circ]$, [AHL23, Thm. 6.1(2)] reduces us to showing that the pair $([\mathrm{Spec}\, B/G^\circ], [\mathrm{Spec}(B/I)/G^\circ])$ satisfies formal functions. Noting that B^{G° is I^{G°-adically complete by Lemma B.4.20 below, this is part of [AHL23, Cor. 4.9].

Finally, under the additional assumption that G is linearly reductive and closed in $\mathrm{GL}_{n,R}$, the fact that $(\mathcal{Y}, \mathcal{Y}^{[0]})$ is coherently complete follows from [AHR23, Thm. 1.6], bearing in mind that $[\mathrm{Spec}\, B/G] \to \mathrm{Spec}\, B^G$ is a good moduli space, by [Alp13, Thm. 13.2], and that $\mathcal{Y}^{[0]} = [\mathrm{Spec}(B/I)/G]$ has the resolution property, by [AHR23, Rem. 2.5], since $G \to \mathrm{Spec}(R)$ is assumed to be embeddable. $\square$

The following lemma and its proof are presumably standard.

Lemma B.4.20. *Let G be a finite group acting on a ring A, and let $I \subseteq A$ be a G-equivariant ideal. If A^G is I^G-adically complete, then A is I-adically complete.*

Proof. Since A is finite over A^G (being both finite type and integral), A is $I^G A$-adically complete. It therefore suffices to show that $\mathrm{rad}(I^G A) = \mathrm{rad}(I)$, i.e. that $I \subseteq \mathrm{rad}(I^G A)$. To this end, let $x \in I$ be arbitrary. Then x is a root of the monic polynomial $\prod_{g \in G}(X - g(x))$, all of whose non-leading coefficients are contained in I^G. Thus $x^{|G|} \in I^G A$, as required. $\square$

We now record some consequences of Theorem B.4.17.

Corollary B.4.21. *Let G be a reductive group scheme over a Noetherian ring R, let B be a finite type R-algebra with an action of G, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on $\mathcal{Y} = [\mathrm{Spec}\, B/G]$. Then for each $n \geq 0$, $\mathrm{Ext}^n_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G})$ is of finite type as a B^G-module.*

Proof. Since the morphism $f : \mathcal{Y} \to \mathrm{Spec}\, B^G$ is cohomologically proper (by Theorem B.4.17 2), we see that each cohomology group $H^i(\mathcal{Y}, \mathcal{H})$ is of finite type over B^G, for any coherent sheaf $\mathcal{H}$ on $\mathcal{Y}$. In particular, we can apply this observation to the Ext sheaves $\underline{\mathrm{Ext}}^\bullet_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G})$ (which are coherent, since $\mathcal{F}$ and $\mathcal{G}$ are). The result then follows from the sheaf-Ext-to-Ext spectral sequence

$$E_2^{p,q} := H^p\big(\mathcal{Y}, \underline{\mathrm{Ext}}^q_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G})\big) \implies \mathrm{Ext}^{p+q}_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G}). \qquad \square$$

Corollary B.4.22. *Let G be a reductive group scheme over a Noetherian ring R, let B be a finite type R-algebra with an action of G, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on $\mathcal{Y} = [\mathrm{Spec}\, B/G]$. Then for each $n \geq 0$, $\mathrm{Ext}^n_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G})$ is of finite type as an $\mathrm{End}_{\mathcal{O}_\mathcal{Y}}(\mathcal{F})$-module and as an $\mathrm{End}_{\mathcal{O}_\mathcal{Y}}(\mathcal{G})$-module.*

Proof. The action of B^G on $\mathcal{G}$ makes $\mathrm{End}_{\mathcal{O}_\mathcal{Y}}(\mathcal{G})$ a B^G-algebra, and the action of $\mathrm{End}_{\mathcal{O}_\mathcal{Y}}(\mathcal{G})$ on $\mathrm{Ext}^n_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G})$ is compatible with this algebra structure and the *a priori* action of B^G on the Ext^n. The corollary thus follows from Corollary B.4.22. The proof for $\mathrm{End}_{\mathcal{O}_\mathcal{Y}}(\mathcal{F})$ is the same. $\square$

Corollary B.4.23. *As above, let R be a Noetherian ring, and let B be a finitely generated R-algebra endowed with an action of a reductive group scheme G over* $\operatorname{Spec} R$. *Furthermore, let $I \subseteq B$ be a G-equivariant ideal, let $\widehat{B^G}$ denote the I^G-adic completion of B^G, and let $\widehat{\mathcal{Y}}$ denote the completion of $\mathcal{Y} := [\operatorname{Spec} B/G]$ along the zero locus $i : [\operatorname{Spec}(B/I)/G] \hookrightarrow \mathcal{Y}$ of I. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on $\mathcal{Y}$. Then for each $n \geq 0$, the functor $\widehat{i}^*$ induces an isomorphism*

$$\operatorname{Ext}^n_{D^b_{\mathrm{coh}}(\mathcal{Y})}(\mathcal{F}, \mathcal{G})^\wedge \xrightarrow{\sim} \operatorname{Ext}^n_{\operatorname{Pro} D^b_{\mathrm{coh}}(\widehat{\mathcal{Y}})}(\widehat{i}^*\mathcal{F}, \widehat{i}^*\mathcal{G}),$$

where the $(-)^\wedge$ in the source indicates the I^G-adic completion.

Proof. The morphism $R \to B$ factors through B^G, and so we may rewrite the G-action on $\operatorname{Spec} B$ in the form

$$(G \times_{\operatorname{Spec} R} \operatorname{Spec} B^G) \times_{\operatorname{Spec} B^G} \operatorname{Spec} B \to \operatorname{Spec} B.$$

Since B^G is Noetherian (by Theorem B.4.17 (1)), we may replace R by B^G, and G by $G \times_{\operatorname{Spec} R} \operatorname{Spec} B^G$, and so we assume from now on that $R = B^G$. The I^G-adic completion $\widehat{B^G}$ of B^G is flat over B^G (since B^G is Noetherian). We then set $B' = \widehat{B^G} \otimes_{B^G} B$, $G' = \operatorname{Spec} \widehat{B^G} \times_{\operatorname{Spec} B^G} G$, and $\mathcal{Y}' := \operatorname{Spec} \widehat{B^G} \otimes_{\operatorname{Spec} B^G} \mathcal{Y} = [\operatorname{Spec} B'/G']$. From the flatness of $\widehat{B^G}$ over B^G, we see that $(B')^{G'} = \widehat{B^G} \otimes_{B^G} B^G = \widehat{B^G}$. We write $i' : [\operatorname{Spec}(B'/IB')/G'] \hookrightarrow \mathcal{Y}'$ for the base-change of i.

Let $f : \mathcal{Y}' \to \mathcal{Y}$ denote the canonical morphism, and write f^* for the induced pullback morphism on coherent sheaves. Since f is flat, the derived pullback f^* is a t-exact functor

$$f^* : D^b_{\mathrm{coh}}(\mathcal{Y}) \to D^b_{\mathrm{coh}}(\mathcal{Y}').$$

For any $\mathcal{F}, \mathcal{G} \in \operatorname{Coh}(\mathcal{Y})$, flat base-change shows that then f^* induces isomorphisms

$$\widehat{B^G} \otimes_{B^G} \operatorname{Ext}^n_{D^b_{\mathrm{coh}}(\mathcal{Y})}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \operatorname{Ext}^n_{D^b_{\mathrm{coh}}(\mathcal{Y}')}(f^*\mathcal{F}, f^*\mathcal{G}). \tag{B.4.24}$$

If we let $\widehat{\mathcal{Y}'}$ denote the completion of $\mathcal{Y}'$ along the zero locus of IB', then f induces an isomorphism $\widehat{\mathcal{Y}'} \xrightarrow{\sim} \widehat{\mathcal{Y}}$. Considering the commutative diagram

$$\begin{tikzcd} \operatorname{Coh}(\mathcal{Y}) \arrow[r, "f^*"] \arrow[d, "\widehat{i}^*"] & \operatorname{Coh}(\mathcal{Y}') \arrow[d, "\widehat{i}'^*"] \\ \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{Y}}) \arrow[r, "\widehat{f}^*", "\sim"'] & \operatorname{Pro}\operatorname{Coh}(\widehat{\mathcal{Y}'}) \end{tikzcd}$$

and noting that $\widehat{i}'^*$ is fully faithful by Theorem B.4.17 (3), we deduce from (B.4.24) that $\widehat{i}^*$ induces isomorphisms

$$\begin{aligned} \widehat{B^G} \otimes_{B^G} \operatorname{Ext}^n_{D^b_{\mathrm{coh}}(\mathcal{Y})}(\mathcal{F}, \mathcal{G}) &\xrightarrow{\sim} \operatorname{Ext}^n_{\operatorname{Pro} D^b_{\mathrm{coh}}(\widehat{\mathcal{Y}'})}(\widehat{i}'^* f^*\mathcal{F}, \widehat{i}'^* f^*\mathcal{G}) \\ &\xrightarrow{\sim} \operatorname{Ext}^n_{\operatorname{Pro} D^b_{\mathrm{coh}}(\widehat{\mathcal{Y}'})}(\widehat{f}^*\widehat{i}^*\mathcal{F}, \widehat{f}^*\widehat{i}^*\mathcal{G}) \xrightarrow{\sim} \operatorname{Ext}^n_{\operatorname{Pro} D^b_{\mathrm{coh}}(\widehat{\mathcal{Y}})}(\widehat{i}^*\mathcal{F}, \widehat{i}^*\mathcal{G}). \end{aligned}$$

Corollary B.4.22 shows that each $\operatorname{Ext}^n_{D^b_{\mathrm{coh}}(\mathcal{Y})}(\mathcal{F}, \mathcal{G})$ is of finite type as a B^G-module. Thus tensoring it with $\widehat{B^G}$ over B^G coincides with I^G-adically completing it. This proves the corollary. $\square$

B.5. **Another version of formal functions.** Let R be a complete Noetherian local $\mathcal{O}$-algebra with maximal ideal $\mathfrak{m}$ and finite residue field, and let S be a finite type R-algebra equipped with an action of a reductive group scheme G over R (where "reductive" is as in Definition B.4.15). We may then form the algebraic stack $\mathfrak{X} := [\operatorname{Spec} S/G]$ over R. We furthermore assume that S^G is finite over R and local (so that S^G is again a complete Noetherian local ring). This assumption implies that $\mathfrak{X}$ has a unique closed point z_0, corresponding to a G-equivariant maximal ideal $\mathfrak{m}_S$. This is because $|\mathcal{Z}| \to |\operatorname{Spec} S^G|$ induces a bijection on closed points, by [Alp14, Thm. 5.3.1 (5)]; recall from [Alp14, Rem. 9.1.5] that the map $\mathcal{Z} \to \operatorname{Spec} S^G$ is an adequate moduli space (using that $G \to \operatorname{Spec} R$ is geometrically reductive in the sense of *loc. cit.*, by Proposition B.4.16).

Then:

(1) We write $\mathcal{X}$ to denote the $\mathfrak{m}_R$-adic completion of $\mathfrak{X}$, and $\widehat{\mathcal{X}}$ to denote the $\mathfrak{m}_S$-adic completion of $\mathfrak{X}$. Note that $|\widehat{\mathcal{X}}|$ consists of a single point (corresponding to the unique closed point z_0 of $\mathfrak{X}$).

(2) We say that a morphism $\operatorname{Spf} A \to \widehat{\mathcal{X}}$, for A a complete Noetherian local $\mathcal{O}$-algebra with finite residue field, equipped with its $\mathfrak{m}_A$-adic topology, is *versal* if it is versal to the induced morphism $\operatorname{Spec} A/\mathfrak{m}_A \to \widehat{\mathcal{X}}$, in the sense of e.g. [EG21, Defn. 2.2.9]. Precisely, this means that given a commutative square consisting of the solid arrows in the following diagram

(B.5.1)
$$\begin{tikzcd} \operatorname{Spec} B_0 \arrow[r] \arrow[d] & \operatorname{Spec} B_1 \arrow[d] \arrow[dl, dashed] \\ \operatorname{Spf} A \arrow[r] & \widehat{\mathcal{X}} \end{tikzcd}$$

where $B_1 \to B_0$ is a surjection of finite type Artinian local R-algebras, and where the corresponding homomorphisms $A \to B_0$ and $B_1 \to B_0$ induce isomorphisms on residue fields, then the dotted arrow can be filled in.

(3) We write $(\mathrm{Ver}/\widehat{\mathcal{X}})$ for the category of versal morphisms with target $\widehat{\mathcal{X}}$, where a morphism

$$(g : \operatorname{Spf} B \to \widehat{\mathcal{X}}) \to (f : \operatorname{Spf} A \to \widehat{\mathcal{X}})$$

is given by a commutative diagram of morphisms of formal algebraic stacks

$$\begin{tikzcd} \operatorname{Spf} B \arrow[r] \arrow[dr, "g"'] & \operatorname{Spf} A \arrow[d, "f"] \\ & \widehat{\mathcal{X}} \end{tikzcd},$$

or, equivalently, a local morphism $\alpha : A \to B$ of complete Noetherian local $\mathcal{O}$-algebras together with an isomorphism $f \circ (\operatorname{Spf} \alpha) \xrightarrow{\sim} g$.

Remark B.5.2. It will sometimes be useful to consider a versal morphism $\operatorname{Spf} A \to \widehat{\mathcal{X}}$ as a versal morphism to the unique closed point of $\mathcal{X}$, by implicitly composing it with the completion morphism $\widehat{\mathcal{X}} \to \mathcal{X}$, Accordingly, we sometimes refer to the objects of $(\mathrm{Ver}/\widehat{\mathcal{X}})$ as *versal morphisms to the closed point of* $\mathcal{X}$.

In this subsection we give a description of $\mathrm{Coh}(\mathcal{X})$ in terms of a category of coherent sheaves on $(\mathrm{Ver}/\widehat{\mathcal{X}})$, whose definition we turn to next. We begin by noting that if $\operatorname{Spf} A \to \widehat{\mathcal{X}}$ is an object of $(\mathrm{Ver}/\widehat{\mathcal{X}})$, then we have equivalences

$$\mathrm{Coh}(\mathcal{O}_{\operatorname{Spf} A}) \xrightarrow{\sim} \lim_n \mathrm{Coh}(\operatorname{Spec} A/\mathfrak{m}_A^n) \xleftarrow{\sim} \mathrm{Coh}(\operatorname{Spec} A).$$

Furthermore, if $\varphi : \operatorname{Spf} B \to \operatorname{Spf} A$ is a morphism in $(\mathrm{Ver}/\widehat{\mathcal{X}})$, then the completed pullback

$$\widehat{\varphi}^* : \mathrm{Pro}\,\mathrm{Coh}(\operatorname{Spf} A) \to \mathrm{Pro}\,\mathrm{Coh}(\operatorname{Spf} B)$$

restricts to a functor

$$\widehat{\varphi}^* : \mathrm{Coh}(\mathcal{O}_{\operatorname{Spf} A}) \to \mathrm{Coh}(\mathcal{O}_{\operatorname{Spf} B})$$

by Lemma B.4.13.

Definition B.5.3. A coherent sheaf $\mathcal{F}$ on $(\mathrm{Ver}/\widehat{\mathcal{X}})$ consists of the following data:

- for every object $f : \operatorname{Spf} A \to \widehat{\mathcal{X}}$ of $(\mathrm{Ver}/\widehat{\mathcal{X}})$, we have a finitely presented A-module $\mathcal{F}(x)$ (the sections of $\mathcal{F}$ over $\operatorname{Spf} A$);
- given a morphism $\varphi : g \to f$ of $(\mathrm{Ver}/\widehat{\mathcal{X}})$, lying over a morphism $\operatorname{Spf} B \to \operatorname{Spf} A$, or in more usual terms, a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spf} B \arrow[r, "\varphi"] \arrow[dr, "g"'] & \operatorname{Spf} A \arrow[d, "f"] \\ & \widehat{\mathcal{X}} \end{tikzcd},$$

  we have a pullback isomorphism $c_\varphi : \widehat{\varphi}^*\mathcal{F}(g) = B \otimes_A \mathcal{F}(g) \xrightarrow{\sim} \mathcal{F}(g)$; these pullback isomorphisms satisfy the following cocycle condition:
- given morphisms $\psi : h \to g$ and $\varphi : g \to x$, lying over $\operatorname{Spf} C \to \operatorname{Spf} B \to \operatorname{Spf} A$, the isomorphism $c_{\varphi\circ\psi}$ coincides with the composite $c_\psi \circ (\mathrm{id}_C \otimes_B c_\varphi)$, after we identify $C \otimes_A \mathcal{F}(f)$ and $C \otimes_B (B \otimes_A \mathcal{F}(f))$ in the natural manner.

If $\mathcal{F}$ and $\mathcal{G}$ are two coherent sheaves on $(\mathrm{Ver}/\widehat{\mathcal{X}})$, a morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is defined to be a collection of morphisms $\alpha_f : \mathcal{F}(f) \to \mathcal{G}(f)$, compatible in an evident manner with the pullback isomorphisms c_φ. We write $\mathrm{Coh}(\mathrm{Ver}/\widehat{\mathcal{X}})$ for the resulting category of coherent sheaves on $(\mathrm{Ver}/\widehat{\mathcal{X}})$.

Lemma B.5.4. *Let $f : \operatorname{Spf} A \to \widehat{\mathcal{X}}$ be an object of $(\mathrm{Ver}/\widehat{\mathcal{X}})$, and let $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$. Then $\widehat{f}^*\mathcal{F}$ is an object of $\mathrm{Coh}(\mathcal{O}_{\operatorname{Spf} A}) = \mathrm{Coh}(\operatorname{Spec} A)$, and the resulting functor*

$$\widehat{f}^* : \mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}}) \to \mathrm{Coh}(\operatorname{Spec} A) \tag{B.5.5}$$

is exact and faithful.

Proof. The coherence of $\widehat{f}^*\mathcal{F}$ is a consequence of Lemma B.4.13. The exactness of $\widehat{f}^*$ is immediate from Lemma B.3.39, which implies that $\widehat{f}^*$ is exact on the larger category $\mathrm{Pro}\,\mathrm{Coh}(\widehat{\mathcal{X}})$. There remains to prove that $\widehat{f}^*$ is faithful, or equivalently (because of the exactness) conservative. The isomorphism $\mathcal{F} \xrightarrow{\sim} \lim_i \mathcal{F}/\mathfrak{m}_S^{i+1}\mathcal{F}$ arising from Lemma B.3.5 exhibits any $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$ as a cofiltered limit of objects of $\mathrm{Coh}(\widehat{\mathcal{X}})$, with surjective structure maps. If $\mathcal{F} \neq 0$, then $\mathcal{F}/\mathfrak{m}_S^{i+1}\mathcal{F} \neq 0$ for some i, and then $\widehat{f}^*(\mathcal{F}/\mathfrak{m}_S^{i+1}\mathcal{F}) \neq 0$, by Lemma B.3.39. Hence $\widehat{f}^*\mathcal{F} \neq 0$, since it surjects onto $\widehat{f}^*(\mathcal{F}/\mathfrak{m}_S^{i+1}\mathcal{F})$. This concludes the proof. $\square$

By Lemma B.5.4, there is a faithful restriction functor

$$\mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}}) \to \mathrm{Coh}(\mathrm{Ver}/\widehat{\mathcal{X}}) \tag{B.5.6}$$

sending a coherent $\mathcal{O}_{\widehat{\mathcal{X}}}$-module $\mathcal{F}$ to the coherent sheaf on $(\mathrm{Ver}/\widehat{\mathcal{X}})$ whose sections on $f : \mathrm{Spf}\, A \to \widehat{\mathcal{X}}$ are given by $\widehat{f}^*\mathcal{F}(\mathrm{Spf}\, A)$. Writing $i : \widehat{\mathcal{X}} \to \mathcal{X}$ for the completion map, we also have the composite

$$\mathrm{Coh}(\mathcal{O}_{\mathcal{X}}) \xrightarrow{\widehat{i}^*} \mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}}) \xrightarrow{\text{(B.5.6)}} \mathrm{Coh}(\mathrm{Ver}/\widehat{\mathcal{X}}). \tag{B.5.7}$$

Proposition B.5.8. *With the notation and assumptions introduced above, the functor* (B.5.7) *is fully faithful.*

Proof. We will show that both arrows of (B.5.7) are fully faithful. For the first arrow, taking into account the isomorphism $\mathcal{F} \xrightarrow{\sim} \lim_n \mathcal{F}/\mathfrak{m}_R^{n+1}\mathcal{F}$ arising from Lemma B.3.5, it suffices to prove that $\widehat{i}^*$ is fully faithful on the full subcategory of $\mathfrak{m}_R$-power torsion objects, i.e. that the restriction

$$\mathrm{Coh}(\mathcal{X}) \xrightarrow{\text{(B.4.5)}} \mathrm{Coh}(\mathcal{O}_{\mathcal{X}}) \xrightarrow{\widehat{i}^*} \mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$$

is fully faithful. This restriction can be written as the composite

$$\mathrm{Coh}(\mathcal{X}) \xrightarrow{i_{\mathfrak{m}_R,*}} \mathrm{Coh}(\mathfrak{X}) \xrightarrow{\mathcal{O}_{\widehat{\mathcal{X}}}\otimes_{\mathcal{O}_{\mathfrak{X}}} -} \mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}}),$$

where $i_{\mathfrak{m}_R} : \mathcal{X} \to \mathfrak{X}$ is the $\mathfrak{m}_R$-adic completion morphism, and so it is fully faithful, by Proposition B.2.8 and Theorem B.4.17 (3). This concludes the proof that the first arrow of (B.5.7) is fully faithful.

We now prove that the second arrow, i.e. the functor (B.5.6), is fully faithful. We have already seen that it is faithful, as a consequence of Lemma B.5.4. We now have to show that it is also full.

To this end, let $v : \mathrm{Spf}\, \widehat{S} \to \widehat{\mathcal{X}}$ be the versal morphism given by the colimit of the flat covers $\mathrm{Spec}\, S/\mathfrak{m}_S^i \to [(\mathrm{Spec}\, S/\mathfrak{m}_S^i)/G_i]$.

Suppose that $\mathcal{F}$ and $\mathcal{G}$ are objects of $\mathrm{Coh}(\mathcal{O}_{\widehat{\mathcal{X}}})$, and that we are given a morphism α between their restrictions to $\mathrm{Coh}(\mathrm{Ver}/\widehat{\mathcal{X}})$; that is, for each $f : \mathrm{Spf}\, A \to \widehat{\mathcal{X}}$ in $(\mathrm{Ver}/\widehat{\mathcal{X}})$ we are given a morphism $\alpha_f : \mathcal{F}(f) \to \mathcal{G}(f)$, compatible with all base changes in $(\mathrm{Ver}/\widehat{\mathcal{X}})$. If we write $\widehat{G} \coloneqq G \times_{\mathrm{Spec}\, R} \mathrm{Spf}\, \widehat{S} = \mathrm{colim}_n\, G_n$ to denote the $\mathfrak{m}_S$-adic completion of $G_S \coloneqq G \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, S$ (compare [Eme, Lem. 4.8]), then since $\mathcal{F}(v)$ and $\mathcal{G}(v)$ are pulled back from $\mathcal{F}$ and $\mathcal{G}$, they are naturally $\widehat{G}$-equivariant modules over $\widehat{S}$, and $\mathcal{F}$ and $\mathcal{G}$ are recovered by descending $\mathcal{F}(v)$ and $\mathcal{G}(v)$ with respect to this equivariant structure. We will prove that the morphism α_v is also equivariant for the action of $\widehat{G}$; it then descends to the desired morphism from $\mathcal{F}$ to $\mathcal{G}$.

The equivariance of each of $\mathcal{F}(v)$ and $\mathcal{G}(v)$ is encoded by isomorphisms $\imath_{\mathcal{F}(v)} : \widehat{p}^*\mathcal{F}(v) \xrightarrow{\sim} \widehat{a}^*\mathcal{F}(v)$ and $\imath_{\mathcal{G}(v)} : \widehat{p}^*\mathcal{G}(v) \xrightarrow{\sim} \widehat{a}^*\mathcal{G}(v)$. Our goal then is to show that

$$\imath_{\mathcal{G}(v)}\widehat{p}^*(\alpha_v) = \widehat{a}^*(\alpha_v)\imath_{\mathcal{F}(v)} \tag{B.5.9}$$

(as morphisms $\widehat{p}^*\mathcal{F}(v) \to \widehat{a}^*\mathcal{G}(v)$ of $\mathcal{O}_{\widehat{G}}$-modules).

Since $\widehat{G}$ is the $\mathfrak{m}_S$-adic completion of G_S, which has finite type over $\mathrm{Spec}\, S$, it suffices to check that (B.5.9) holds after pulling back along every morphism

$f : \operatorname{Spf} A \to \widehat{G}$ arising as the completion of G_S at a closed point lying above $\mathfrak{m}_S$. Fixing such an f, the compositions

$$\operatorname{Spf} A \xrightarrow{f} \widehat{G} \rightrightarrows \operatorname{Spf} \widehat{S} \xrightarrow{v} \widehat{\mathcal{X}}$$

coincide, and the resulting morphism $u : \operatorname{Spf} A \to \widehat{\mathcal{X}}$ is an object of $(\mathrm{Ver}/\widehat{\mathcal{X}})$: this is a consequence of the fact that the projection map $p : \widehat{G} \to \operatorname{Spf} \widehat{S}$ is a completion of the smooth morphism $G_S \to \operatorname{Spec} S$.

The definition of morphisms in $\mathrm{Coh}(\mathrm{Ver}/\widehat{\mathcal{X}})$ gives the commutativity of the two squares in the following diagram, where the horizontal morphisms are induced by the sheaf property of $\mathcal{F}$ and the fact that $vp = va$ as morphisms $\widehat{G} \to \widehat{\mathcal{X}}$,

$$\begin{tikzcd}
\widehat{f}^*\widehat{p}^*\mathcal{F}(v) \arrow[r, "\sim"'] \arrow[rr, bend left, "\widehat{f}^*\imath_{\mathcal{F}(v)}"] \arrow[d, "\widehat{f}^*\widehat{p}^*(\alpha_v)"] & \mathcal{F}(u) \arrow[r, "\sim"'] \arrow[d, "\alpha_u"] & \widehat{f}^*\widehat{a}^*\mathcal{F}(v) \arrow[d, "\widehat{f}^*\widehat{a}^*(\alpha_v)"'] \\
\widehat{f}^*\widehat{p}^*\mathcal{G}(v) \arrow[r, "\sim"'] \arrow[rr, bend right, "\widehat{f}^*\imath_{\mathcal{G}(v)}"'] & \mathcal{G}(u) \arrow[r, "\sim"'] & \widehat{f}^*\widehat{a}^*\mathcal{G}(v)
\end{tikzcd}$$

That (B.5.9) holds after pulling back along f is now immediate from the commutativity of the diagram, and we are done. $\square$

Appendix C. Stacks of (φ, Γ)-modules

In this appendix we prove some minor variations on results of [EG23]. We freely use the notation and conventions of [EG23]. In particular, we fix a finite extension $K/\mathbf{Q}_p$, and for each integer $d \geq 1$ we write $\mathcal{X}_d$ for the corresponding moduli stack of rank d projective étale (φ, Γ)-modules.

C.1. **Fixed determinant stacks.** We begin by extending some of the results of [EG23] to the case of moduli stacks of étale (φ, Γ)-modules with fixed determinant. There is a natural morphism $\wedge^d : \mathcal{X}_d \to \mathcal{X}_1$, sending a rank d projective étale (φ, Γ)-module D to $\wedge^d D$. Let $\chi : G_K \to \mathcal{O}^\times$ be a character. Write $\overline{\chi} : G_K \to \mathbf{F}^\times$ for the reduction of χ modulo ϖ.

Definition C.1.1. *We define $\mathcal{X}_d^\chi$ as the pullback*

$$\begin{tikzcd}
\mathcal{X}_d^\chi \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{X}_d \arrow[d, "\wedge^d"] \\
\operatorname{Spf} \mathcal{O} \arrow[r, "\chi"] & \mathcal{X}_1
\end{tikzcd}$$

More explicitly, $\mathcal{X}_d^\chi(A)$ is the groupoid of pairs (D, θ) where D is a rank d projective étale (φ, Γ)-module with A-coefficients, and θ is an identification of $\wedge^d D$ with χ.

If χ is a de Rham character, $\underline{\lambda}$ is a Hodge type, and τ is an inertial type, we define

$$\mathcal{X}_d^{\mathrm{crys}, \underline{\lambda}, \tau, \chi} := \mathcal{X}_d^{\mathrm{crys}, \underline{\lambda}, \tau} \times_{\mathcal{X}_d} \mathcal{X}_d^\chi. \tag{C.1.2}$$

When τ is trivial, we will often omit it from the notation. By construction, $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau,\chi}$ is a closed substack of $\mathcal{X}_d^{\chi}$, and by [EG23, Thm. 4.8.12], it is a p-adic formal algebraic stack. It is non-zero if and only if $(\underline{\lambda},\tau)$ is compatible with χ in the following sense.

Definition C.1.3. *We say that* $(\underline{\lambda},\tau)$ *is* compatible with χ *if* $\chi\det(\tau)^{-1}$ *is crystalline, and for each* $\sigma : K \hookrightarrow \overline{\mathbf{Q}}_p$ *we have* $\mathrm{HT}_\sigma(\chi\det(\tau)^{-1}) = \sum_{i=1}^d \lambda_{\sigma,i}$.

Corollary C.1.4. *If* $(\underline{\lambda},\tau)$ *is compatible with* χ*, and* $\underline{\lambda}$ *is regular, then* $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\chi} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathbf{F}$ *is equidimensional of dimension* $[K:\mathbf{Q}_p]d(d-1)/2$.

Proof. The proof of [EG23, Thm. 4.8.14] goes over immediately, replacing the crystalline deformation rings with their fixed determinant variants (whose dimensions are known by the proof of [Kis08, Thm. 3.3.8]; for a precise statement see for example [BG19, Thm. A]), and the reductive group GL_d by SL_d. □

Definition C.1.5. *We say that a Serre weight* $\underline{k}$ is compatible with χ *(or* is compatible with $\overline{\chi}$*) if*

$$\overline{\chi}|_{I_K} = \overline{\varepsilon}^{-d(d-1)/2} \prod_{\overline{\sigma}:k\hookrightarrow\overline{\mathbf{F}}_p} \omega_{\overline{\sigma}}^{-\sum_{i=1}^d k_{\overline{\sigma},i}}.$$

Remark C.1.6. The conventions of this paper for Serre weights are opposite to those of [EG23]. This has the slightly unfortunate consequence that in the particular case that $K = \mathbf{Q}_p$ and $d = 2$, Definition C.1.5 compares with Definition 2.1.6 as follows: a Serre weight $\sigma_{a,b}$ is compatible with $\overline{\zeta}$ in the sense of Definition 2.1.6 if and only if it is compatible with $\overline{\zeta}^{-1}\overline{\varepsilon}^{-1}$ in the sense of Definition C.1.5. (To see this, take take $k_1 = a + b$, $k_2 = a$ (where we have suppressed the unique choice of $\overline{\sigma}$ from the notation), and note that $\omega = \overline{\varepsilon}$.)

Definition C.1.7. *We say that a Serre weight* $\underline{k}$ *is* Steinberg *if for all* $\overline{\sigma} : k \hookrightarrow \overline{\mathbf{F}}_p$ *and all* $1 \le i \le d-1$ *we have* $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = p - 1$.

The following is the analogue for $\mathcal{X}_d^{\chi}$ of some of the results of [EG23] for $\mathcal{X}_d$.

Theorem C.1.8. $\mathcal{X}_d^{\chi}$ *is a Noetherian formal algebraic stack. Its underlying reduced substack* $\mathcal{X}_{d,\mathrm{red}}^{\chi}$ *is of finite type over* $\mathbf{F}_p$*, and is equidimensional of dimension* $[K:\mathbf{Q}_p]d(d-1)/2$*. The irreducible components of* $\mathcal{X}_{d,\mathrm{red}}^{\chi}$ *admit a natural bijection with the Serre weights which are compatible with* χ*, except that each Steinberg weight which is compatible with* χ *corresponds to multiple irreducible components, indexed by the dth roots of unity in* $\overline{\mathbf{F}}_p$*. In each case, a component corresponding to a Serre weight* $\underline{k}$ *has a dense open substack which is maximally non-split of niveau one and weight* $\underline{k}$*.*

Proof. The argument of the proof of [EG23, Thm. 6.5.1] goes through almost unchanged to show that every irreducible component of $\mathcal{X}_{d,\mathrm{red}}^{\chi}$ has dimension at least $[K:\mathbf{Q}_p]d(d-1)/2$, replacing the appeal to [EG23, Thm. 4.8.14] with one to Corollary C.1.4. (The one possibly subtle point is that $\mathcal{X}_{d,\mathrm{red}}^{\chi}$ only depends on $\overline{\chi}$, so in making the comparison to $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\chi}$ we are free to replace χ by a crystalline character which is compatible with $\underline{\lambda}$.)

We now consider the morphism

$$f : \mathcal{X}_{d,\mathrm{red}}^{\chi} \times \mathbf{G}_m \to \mathcal{X}_{d,\mathrm{red}} \tag{C.1.9}$$

given by forgetting θ and taking unramified twists, as in [EG23, §5.3]. Note that any family of (φ,Γ)-modules with determinant χ is essentially twistable in the sense of [EG23, Defn. 5.3.1] (because any twist preserving the determinant would have to be by a dth root of unity).

Let $\mathcal{Z}$ be an irreducible component of $\mathcal{X}^{\chi}_{d,\mathrm{red}}$, and let $f(\mathcal{Z})$ be the scheme-theoretic image of $\mathcal{Z}\times\mathbf{G}_m$ in $\mathcal{X}_{d,\mathrm{red}}$. Since we have already shown that $\mathcal{Z}$ has dimension at least $[K:\mathbf{Q}_p]d(d-1)/2$, it follows from [EG23, Lem. 5.3.2], that $f(\mathcal{Z})$ has dimension at least $[K:\mathbf{Q}_p]d(d-1)/2$. (Use [Stacks, Tag 0DS4], together with the fact that the forgetful morphism $\mathcal{X}^{\chi}_{d,\mathrm{red}}\to\mathcal{X}_{d,\mathrm{red}}$ has fibres of dimension (at most) 1.) Since $\mathcal{X}_{d,\mathrm{red}}$ is equidimensional of this dimension, we see that $\mathcal{Z}$ must have dimension exactly $[K:\mathbf{Q}_p]d(d-1)/2$, and that $f(\mathcal{Z})$ is an irreducible component of $\mathcal{X}_{d,\mathrm{red}}$, so that $f(\mathcal{Z})=\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ for some Serre weight $\underline{k}$. Since $\mathcal{Z}$ was arbitrary, we have in particular shown that $\mathcal{X}^{\chi}_{d}$ is equidimensional of dimension $[K:\mathbf{Q}_p]d(d-1)/2$, as claimed.

It follows from Definition C.1.5 and [EG23, Defn. 5.5.1] (together with the defining property of $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$) that $\underline{k}$ is compatible with χ. It therefore remains to show that the association of $\underline{k}$ to $\mathcal{Z}$ is surjective, and that it is injective if $\underline{k}$ is not Steinberg, and to determine the Steinberg components.

Let $\mathcal{X}^{\underline{k},\chi}_{d}$ be the closed substack of $\mathcal{X}^{\chi}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ given by the fibre product

$$\begin{tikzcd} \mathcal{X}^{\underline{k},\chi}_{d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p} \arrow[d, "\wedge^d"] \\ \operatorname{Spec}\overline{\mathbf{F}}_p \arrow[r, "\chi"] & \mathcal{X}_1 \end{tikzcd}$$

Then $\mathcal{X}^{\underline{k},\chi}_{d}$ is nonempty and has dimension $[K:\mathbf{Q}_p]d(d-1)/2$ (for example by another appeal to [Stacks, Tag 0DS4]), so there is at least one irreducible component of $\mathcal{X}^{\chi}_{d,\mathrm{red}}$ corresponding to $\underline{k}$. Note also that if $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is generically maximally non-split of weight $\underline{k}$, so is $\mathcal{X}^{\underline{k},\chi}_{d}$.

If $\underline{k}$ is Steinberg then the classification of the irreducible components follows easily from [EG23, Thm. 5.5.12] and the definition of being maximally non-split of weight $\underline{k}$, so we assume from now on that $\underline{k}$ is not Steinberg. We need to show that $\mathcal{X}^{\underline{k},\chi}_{d}$ is irreducible. We prove this by induction on d. The inductive argument is similar to the proof of [EG23, Thm. 5.5.12] given in [EG22], but much simpler, as we are already using the conclusions of [EG23, Thm. 5.5.12]. We content ourselves with explaining the key differences in the argument. The case $d=1$ is trivial, and the case $d=2$ is easy and is left to the reader. If $d\geq 3$ then after possibly replacing all representations with their duals (which has the effect of reversing the order of the k_i) we can and do assume that $\underline{k}_{d-1}$ is also not Steinberg.

Let $\mathcal{U}^{\underline{k},\chi}$ be a dense open substack of $\mathcal{X}^{\underline{k},\chi}_{d}$ which is maximally non-split of weight $\underline{k}$, and let $\nu_1:\mathcal{U}^{\underline{k},\chi}\to\mathbf{G}_m$ be the character given by [EG23, (5.5.10)] (which by definition determines the unique quotient character of each $\overline{\mathbf{F}}_p$-point of $\mathcal{U}^{\underline{k},\chi}$; this character is the unramified twist of $\overline{\alpha}:=\overline{\varepsilon}^{1-d}\omega_{\underline{k},1}$ by ν_1). Let $\mathcal{Z}$ be an irreducible component of $\mathcal{X}^{\underline{k},\chi}_{d}$, and let $\mathcal{U}$ be its intersection with $\mathcal{U}^{\underline{k},\chi}$. Since $\mathcal{U}$ has dimension $[K:\mathbf{Q}_p]d(d-1)/2$, the locus in $\mathbf{G}_m$ where the fibres of ν_1 have dimension at least $[K : \mathbf{Q}_p]d(d-1)/2 - 1$ is nonempty, and since it is Zariski closed it is either finite or all of $\mathbf{G}_m$.

If this locus were finite, then since the dimension of $\mathcal{U}^{\underline{k},\chi}$ is $[K : \mathbf{Q}_p]d(d-1)/2$, there would be some fibre of dimension $[K : \mathbf{Q}_p]d(d-1)/2$. By the usual computations of the dimensions of extension groups, this would necessitate some $\mathcal{X}_{d-1}^{\chi'}$ having dimension at least $[K : \mathbf{Q}_p](d-1)(d-2)/2+1$, a contradiction. It follows that for each $\overline{\mathbf{F}}_p$-point of $\mathbf{G}_m$, the fibre of ν_1 has dimension $[K : \mathbf{Q}_p]d(d-1)/2 - 1$. Furthermore by the inductive hypothesis, we see that such a fibre has precisely one irreducible component of dimension $[K : \mathbf{Q}_p]d(d-1)/2 - 1$, and this irreducible component contains a dense open substack of the universal extension of the fixed twist of $\overline{\alpha}$ by a dense open substack of the (irreducible, by the inductive hypothesis) $\mathcal{X}_{d-1}^{\underline{k}_{d-1},\chi'}$ (where χ' is determined by χ and the twist of $\overline{\alpha}$ determined by the point of $\mathbf{G}_m$). Since this analysis was independent of the choice of irreducible component $\mathcal{Z}$, we see that $\mathcal{Z}$ is unique, as required. $\square$

C.2. **Stacks of Galois representations.** As in [EG23, §6.7], we write $\mathcal{X}_d^{\mathrm{Gal}}$ for Wang-Erickson's formal algebraic stack (see [Wan18, Thm. 3.8, Rem. 3.9]), which is characterised by the following property: if A is an $\mathcal{O}$-algebra in which p is nilpotent, then $\mathcal{X}_d^{\mathrm{Gal}}(A)$ is the groupoid of rank d projective A-modules T_A with a continuous action of G_K (where T_A has the discrete topology). The universal object $\mathcal{V}_d$ on $\mathcal{X}_d^{\mathrm{Gal}}$ is an object of $\operatorname{Pro}\operatorname{Coh}(\mathcal{X}_d^{\mathrm{Gal}})$. By [EG23, Thm. 6.7.2], there is a monomorphism (i.e. a fully faithful functor, compatible with the structure of fibred categories)

$$\mathcal{X}_d^{\mathrm{Gal}} \to \mathcal{X}_d. \tag{C.2.1}$$

In order to describe this monomorphism, it is convenient to recall that $\mathcal{X}_d$ can be regarded as classifying rank d projective étale (φ, G_K)-modules with A-coefficients, in the sense of [EG23, Defn. 2.7.7]; these objects are equivalent to rank d projective étale (φ, Γ)-modules with A-coefficients, by [EG23, Prop. 2.7.8]. We adopt this perspective for the duration of this discussion.

The monomorphism (C.2.1) is then given by the following construction: if $T_A \in \mathcal{X}_d^{\mathrm{Gal}}(A)$, where A is a finite type $\mathcal{O}/\varpi^a$-algebra then

$$T_A \mapsto \mathbf{D}_A(T_A) := T_A \otimes_A W(\mathbf{C}^\flat)_A = T_A \otimes_A W_a(\mathbf{C}^\flat)_A \tag{C.2.2}$$

(so that $\mathbf{D}_A(T_A)$ becomes a rank d projective étale (φ, G_K)-module with A-coefficients when equipped with the φ-action induced by the φ-action on $W(\mathbf{C}^\flat)_A$, together with the diagonal G_K-action). Recall that, by definition,

$$W(\mathbf{C}^\flat)_A := \varprojlim_i W_i(\mathbf{C}^\flat)_A,$$
$$W_i(\mathbf{C}^\flat)_A := W_i(\mathcal{O}_{\mathbf{C}}^\flat)_A[1/v],$$
$$W_i(\mathcal{O}_{\mathbf{C}}^\flat)_A = W_i(\mathcal{O}_{\mathbf{C}}^\flat)\,\widehat{\otimes}_{\mathbf{Z}_p} A := (W_i(\mathcal{O}_{\mathbf{C}}^\flat) \otimes_{\mathbf{Z}_p} A)^\wedge_{v\text{-adic}},$$

where $v \in W_i(\mathcal{O}_{\mathbf{C}}^\flat)$ is any element of the maximal ideal of $W_i(\mathcal{O}_{\mathbf{C}}^\flat)$ whose image in $\mathcal{O}_{\mathbf{C}}^\flat$ is not zero. Furthermore, the Frobenius $\varphi : W_i(\mathcal{O}_{\mathbf{C}}^\flat) \to W_i(\mathcal{O}_{\mathbf{C}}^\flat)$ is v-adically continuous, and so extends uniquely to an A-linear continuous endomorphism of $W_i(\mathbf{C}^\flat)_A$, still denoted φ.

Lemma C.2.3. *If A is a finite type $\mathcal{O}/\varpi^a$-algebra, then the sequence*

$$0 \to A \to W_a(\mathbf{C}^\flat)_A \xrightarrow{\varphi - 1} W_a(\mathbf{C}^\flat)_A \to 0$$

is exact.

Proof. The claim that $W_a(\mathbf{C}^\flat)_A^{\varphi=1} = A$ is [EG23, Lem 2.2.19]. There remains to prove that $\varphi - 1$ is surjective on $W_a(\mathbf{C}^\flat)_A$. Since p is nilpotent on $W_a(\mathbf{C}^\flat)_A$, it suffices to prove that $\varphi - 1$ is surjective on $\mathbf{C}^\flat_A = W_a(\mathbf{C}^\flat)_A/p$.

The action of $\varphi - 1$ on $\mathbf{C}^\flat$ is surjective, because $\mathbf{C}^\flat$ is algebraically closed. Thus it also acts surjectively on $\mathbf{C}^\flat \otimes_{\mathbf{Z}_p} A$. Let $z \in \mathbf{C}^\flat_A$ be arbitrary. Since $\mathbf{C}^\flat_A = (\mathcal{O}_{\mathbf{C}^\flat} \widehat{\otimes}_{\mathbf{Z}_p} A)[1/v]$, there exists $z' \in \mathbf{C}^\flat \otimes_{\mathbf{Z}_p} A$ with $y := z - z' \in v(\mathcal{O}_{\mathbf{C}^\flat} \widehat{\otimes}_{\mathbf{Z}_p} A)$. Then $y = (\varphi - 1)(-\sum_{n\geq 0} \varphi^n(y))$ is in the image of $\varphi - 1$, as is z' (since it is an element of $\mathbf{C}^\flat \otimes_{\mathbf{Z}_p} A$), and hence so is $z = y + z'$, as required. $\square$

Lemma C.2.4. *If A is a finite type $\mathcal{O}/\varpi^a$-algebra, and $M_A \in \mathcal{X}_d(A)$ is an étale (φ, G_K)-module, then M_A is in the essential image of* (C.2.1) *if and only if the natural map*

$$M_A^{\varphi=1} \otimes_A W(\mathbf{C}^\flat)_A \to M_A \tag{C.2.5}$$

is an isomorphism.

Proof. If M_A is in the essential image of (C.2.1), then (C.2.5) is an isomorphism, by a consideration of (C.2.2) and the fact that $W(\mathbf{C}^\flat)_A^{\varphi=1} = A$ (see Lemma C.2.3). Conversely, if (C.2.5) is an isomorphism, it suffices to show that the G_K-module $M_A^{\varphi=1}$ is a projective A-module of rank d. By [Stacks, Tag 058S], it suffices to show that $W(\mathbf{C}^\flat)_A$ is a faithfully flat A-algebra. By [EG23, Prop. 2.2.12], it suffices in turn to show that the Laurent series ring $A((T))$ is a faithfully flat A-algebra. Since A is Noetherian, and $A((T))$ is a localization of a completion of the flat A-algebra $A[T]$, we see that $A((T))$ is a flat A-algebra. To see that it is faithfully flat, we can use the morphism of A-modules $A((T)) \to A$ given by $\sum a_i T^i \mapsto a_0$. $\square$

Lemma C.2.6. *Let* $\operatorname{Spec} A \to \mathcal{X}_d$ *be a morphism whose source is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let I be a nilpotent ideal in A. Suppose that the composite* $\operatorname{Spec} A/I \to \operatorname{Spec} A \to \mathcal{X}_d$ *factors through $\mathcal{X}_d^{\mathrm{Gal}}$. Then the morphism* $\operatorname{Spec} A \to \mathcal{X}_d$ *itself factors through $\mathcal{X}_d^{\mathrm{Gal}}$.*

Proof. Since A is Noetherian, we immediately reduce to the case where I is generated by a single element ε such that $\varepsilon^n = 0$ for some $n \geq 1$. We prove the lemma by induction on n (the case $n = 1$ being trivial).

Let M_A be the finite projective étale (φ, G_K)-module over $W(\mathbf{C}^\flat)_A$ corresponding as above to the given morphism $\operatorname{Spec} A \to \mathcal{X}_d$. By Lemma C.2.4, we need to show that the natural map (C.2.5) is an isomorphism.

We now consider the exact sequence

$$0 \to \varepsilon M_A \to M_A \to M_A/\varepsilon M_A \to 0. \tag{C.2.7}$$

Let $J := \mathrm{Ann}_A(\varepsilon)$, so that $\varepsilon^{n-1} \in J$. Since $W(\mathbf{C}^\flat)_A$ is flat over A (as we noted in the proof of Lemma C.2.4), and M_A is finite projective over $W(\mathbf{C}^\flat)_A$, we see that the natural map $M_A/J \to \varepsilon M_A$ is an isomorphism. By our inductive assumption, the morphism $\operatorname{Spec} A/\varepsilon^{n-1}A \to \mathcal{X}_d$ classifying $M_A/\varepsilon^{n-1}M_A$ factors through $\mathcal{X}_d^{\mathrm{Gal}}$, hence the same is true for the morphism $\operatorname{Spec} A/JA \to \operatorname{Spec} A/\varepsilon^{n-1}A \to \mathcal{X}_d$ classifying $M_A/J \xrightarrow{\sim} \varepsilon M_A$.

It follows from Lemma C.2.4 that the natural map

$$(\varepsilon M_A)^{\varphi=1} \otimes_A W(\mathbf{C}^\flat)_A \to \varepsilon M_A$$

is an isomorphism. In particular, we see that $\varphi - 1$ is surjective on εM_A, since it is surjective on $W(\mathbf{C}^\flat)_A$, by Lemma C.2.3. Applying the snake lemma to the endomorphism $\varphi - 1$ of (C.2.7), we deduce that

$$0 \to (\varepsilon M_A)^{\varphi=1} \to (M_A)^{\varphi=1} \to (M_A/\varepsilon M_A)^{\varphi=1} \to 0$$

is exact. Since $W(\mathbf{C}^\flat)_A$ is flat over A, we obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & (\varepsilon M_A)^{\varphi=1} \otimes_A W(\mathbf{C}^\flat)_A \arrow[r] \arrow[d] & (M_A)^{\varphi=1} \otimes_A W(\mathbf{C}^\flat)_A \arrow[r] \arrow[d] & (M_A/\varepsilon M_A)^{\varphi=1} \otimes_A W(\mathbf{C}^\flat)_A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \varepsilon M_A \arrow[r] & M_A \arrow[r] & M_A/\varepsilon M_A \arrow[r] & 0 \end{tikzcd}$$

Since the left and right hand vertical arrows are isomorphisms (by the inductive hypothesis), the five lemma implies that so is the middle vertical arrow, as required. □

If L/K is a finite extension, then we have maps (with obvious notation) $\mathcal{X}_{K,d}^{\mathrm{Gal}} \to \mathcal{X}_{L,d}^{\mathrm{Gal}}$ and $\mathcal{X}_{K,d} \to \mathcal{X}_{L,d}$, in each case given by restricting the action of G_K to its subgroup G_L. The second morphism is analyzed in detail in [EG23, Lem. 3.7.5], where it is shown to be representable by algebraic spaces, affine, and of finite presentation.

Lemma C.2.8. *If L/K is a finite extension, then the following diagram is Cartesian.*

$$\begin{tikzcd} \mathcal{X}_{K,d}^{\mathrm{Gal}} \arrow[r] \arrow[d] & \mathcal{X}_{K,d} \arrow[d] \\ \mathcal{X}_{L,d}^{\mathrm{Gal}} \arrow[r] & \mathcal{X}_{L,d} \end{tikzcd}$$

Proof. This is immediate from Lemma C.2.4 (since the criterion of that lemma does not involve the G_K-action or G_L-action at all!). □

C.2.9. *Stacks with fixed pseudorepresentation.* Let A be an $\mathcal{O}$-algebra in which p is nilpotent, and assume $\operatorname{Spec} A$ is connected. If $T_A \in \mathcal{X}_d^{\mathrm{Gal}}(A)$, then the $\mathcal{O}$-subalgebra $B \subseteq A$ generated by the values of the A-valued pseudorepresentation θ_A associated to T_A is Artinian and local with finite residue field, by [Che14, Lem. 3.10, Defn. 3.12]. If $\overline{\theta}$ is a d-dimensional $\overline{\mathbf{F}}_p$-pseudorepresentation of G_K, we define $\mathcal{X}_{\overline{\theta}}(A) \subset \mathcal{X}_d^{\mathrm{Gal}}(A)$ to be the full subgroupoid of objects T_A such that $\theta_A \otimes_B \overline{\mathbf{F}}_p$ is $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugate to $\overline{\theta}$. It follows from [Che14, Cor. 3.14] (see also [Wan18, Thm. 3.5] for a restatement in terms of the stacks $\mathcal{X}_d^{\mathrm{Gal}}$) that

$$\mathcal{X}_d^{\mathrm{Gal}} = \coprod_{\overline{\theta}} \mathcal{X}_{\overline{\theta}}, \tag{C.2.10}$$

where $\overline{\theta}$ runs over representatives of the $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy classes of d-dimensional $\overline{\mathbf{F}}_p$-pseudorepresentations of G_K.

We write $\mathbf{F}_{\overline{\theta}}$ for the field extension of $\mathbf{F}$ generated by the values of $\overline{\theta}$. Then $\overline{\theta}$ can be regarded as a $\mathbf{F}_{\overline{\theta}}$-valued pseudorepresentation. The universal (pseudo)deformation ring of $\overline{\theta}$, to complete Noetherian local $W(\mathbf{F}_{\overline{\theta}}) \otimes_{W(\mathbf{F})} \mathcal{O}$-algebras, will be denoted $R_{\overline{\theta}}^{\mathrm{ps}}$. Note that there is a natural morphism

$$\mathcal{X}_{\overline{\theta}} \to \operatorname{Spf} R_{\overline{\theta}}^{\mathrm{ps}}, \tag{C.2.11}$$

since for any test morphism $\operatorname{Spf} A \to \mathcal{X}_{\overline{\theta}}$ (with A, as above, an $\mathcal{O}$-algebra in which p is nilpotent), the inclusion $B \subseteq A$ (where, as above, B denotes the sub-$\mathcal{O}$-algebra of traces) induces the composite morphism $R^{\mathrm{ps}}_{\overline{\theta}} \to B \subset A$, giving rise to a morphism $\operatorname{Spf} A \to \operatorname{Spf} R^{\mathrm{ps}}_{\overline{\theta}}$. Up to natural isomorphism, the ring $R^{\mathrm{ps}}_{\overline{\theta}}$ and the map (C.2.11) only depend on the $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-conjugacy class of $\overline{\theta}$.

The following theorem confirms the expectation of [EG23, Rem. 6.7.4].

Theorem C.2.12. *The morphism*

$$\mathcal{X}_{\overline{\theta}} \to \mathcal{X}_d$$

induced by (C.2.10) *and* (C.2.1) *is a monomorphism which induces a closed immersion on underlying reduced substacks, and exhibits $\mathcal{X}_{\overline{\theta}}$ as the completion of $\mathcal{X}_d$ along its closed substack $\mathcal{X}_{\overline{\theta},\mathrm{red}}$.*

Proof. That the morphism is a monomorphism is immediate from [EG23, Thm. 6.7.2]. To see that it induces a closed immersion on underlying reduced substacks, we begin by noting that there is a finite extension L/K such that every representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ corresponding to an $\overline{\mathbf{F}}_p$-point of $\mathcal{X}_{\overline{\theta}}$ factors through $\mathrm{Gal}(L/K)$. Indeed, this is a basic fact that underlies the entire framework of [Wan18]. To see it directly, note that we may firstly replace K by the fixed field of $\ker \overline{\theta}$, and thus assume that $\overline{\theta}$ is trivial, so that $\overline{\rho}(G_K)$ is unipotent. We can then repeatedly (a total of $(d-1)$ times) replace K by its maximal elementary abelian p-extension.

By [EG23, Thm. 6.6.3(2)], the trivial G_L-representation gives rise to a closed residual gerbe $Z \hookrightarrow \mathcal{X}_{L,d}$, and so also of $\mathcal{X}^{\mathrm{Gal}}_{L,d}$. As recalled above, the right hand vertical arrow in the Cartesian diagram of Lemma C.2.8 is representable by algebraic spaces, and so the fibre product $Z \times_{\mathcal{X}_{L,d}} \mathcal{X}_{K,d}$ is a closed algebraic substack of $\mathcal{X}_{K,d}$ contained in $\mathcal{X}^{\mathrm{Gal}}_{K,d}$ which (by virtue of our choice of L) contains $\mathcal{X}_{\overline{\theta},\mathrm{red}}$ as a closed substack (indeed, as a connected component). Thus the induced morphism on underlying reduced substacks is indeed a closed immersion.

Finally, write $\widehat{\mathcal{X}}_d \to \mathcal{X}_d$ for the completion of $\mathcal{X}_d$ along $\mathcal{X}_{\overline{\theta},\mathrm{red}}$. Then the morphism $\mathcal{X}_{\overline{\theta}} \to \mathcal{X}_d$ factors through a monomorphism $\mathcal{X}_{\overline{\theta}} \to \widehat{\mathcal{X}}_d$, which we need to prove is an equivalence. Since it is a monomorphism, it is fully faithful, and the essential surjectivity is a consequence of Lemma C.2.6. $\square$

We will write $\mathfrak{X}_{\overline{\theta}}$ for the algebraic stack denoted $\mathrm{Rep}_{\overline{D}}$ in [Wan18] (with $\overline{D} := \overline{\theta}$); this is of finite type over $\operatorname{Spec} R^{\mathrm{ps}}_{\overline{\theta}}$, and is in fact (by definition) a quotient stack

$$\mathrm{Rep}_{\overline{D}} = [\mathrm{Rep}^{\square}_{\overline{D}} / \mathrm{GL}_d],$$

where $\mathrm{Rep}^{\square}_{\overline{D}}$ is a finite type scheme over $\operatorname{Spec} R^{\mathrm{ps}}_{\overline{\theta}}$.

Theorem C.2.13. *The morphism* (C.2.11) *is representable by algebraic stacks, and it arises as the $\mathfrak{m}_{R^{\mathrm{ps}}_{\overline{\theta}}}$-adic completion of $\mathfrak{X}_{\overline{\theta}} \to \operatorname{Spec} R^{\mathrm{ps}}_{\overline{\theta}}$.*

Proof. This is immediate from [Wan18, Thm. 3.8]. $\square$

Corollary C.2.14. *Let $S := \Gamma(\mathrm{Rep}^{\square}_{\overline{D}}, \mathcal{O}_{\mathrm{Rep}^{\square}_{\overline{D}}})$ and $S_i := S \otimes_{R^{\mathrm{ps}}_{\overline{\theta}}} R^{\mathrm{ps}}_{\overline{\theta}}/(\mathfrak{m}_{R^{\mathrm{ps}}_{\overline{\theta}}})^i$. Then S^{GL_d} is a finite local $R^{\mathrm{ps}}_{\overline{\theta}}$-algebra (so in particular a complete local Noetherian ring), and each $S_i^{\mathrm{GL}_d}$ is a finite local $R^{\mathrm{ps}}_{\overline{\theta}}/(\mathfrak{m}_{R^{\mathrm{ps}}_{\overline{\theta}}})^i$-algebra.*

Proof. By [Wan18, Thm. 3.8], the map $R^{\mathrm{ps}}_{\overline{\theta}} \to S^{\mathrm{GL}_d}$ is an adequate homeomorphism (i.e. it is an integral universal homeomorphism, which is a local isomorphism at all points of residue characteristic zero; see [Alp14, Defn. 3.3.1]). By [Alp14, Rem. 5.2.2], the map

$$S^{\mathrm{GL}_d} \otimes_{R^{\mathrm{ps}}_{\overline{\theta}}} R^{\mathrm{ps}}_{\overline{\theta}}/(\mathfrak{m}_{R^{\mathrm{ps}}_{\overline{\theta}}})^i \to (S \otimes_{R^{\mathrm{ps}}_{\overline{\theta}}} R^{\mathrm{ps}}_{\overline{\theta}}/(\mathfrak{m}_{R^{\mathrm{ps}}_{\overline{\theta}}})^i)^{\mathrm{GL}_d} = S_i^{\mathrm{GL}_d}$$

is also an adequate homeomorphism. The property of being an adequate homeomorphism is stable under base change and composition, so that $R^{\mathrm{ps}}_{\overline{\theta}}/(\mathfrak{m}_{R^{\mathrm{ps}}_{\overline{\theta}}})^i \to S_i^{\mathrm{GL}_d}$ is an adequate homeomorphism, and by Theorem B.4.17 (1), it is furthermore of finite type.

It thus suffices to prove that if $f : R \to R'$ is an adequate homeomorphism of finite type between $\mathcal{O}$-algebras, and R is local, then R' is local and f is finite (and therefore local). That R' is local is immediate from $R \to R'$ being a homeomorphism, and that f is finite is immediate from it being integral and of finite type, as desired. $\square$

We now specialize some of the framework of Appendix B to the context at hand. In all of the following discussion, we freely use the equivalence

$$D^b\big(\mathrm{Coh}(\mathcal{Z})\big) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{Z})$$

for any Noetherian algebraic, or formal algebraic, stack $\mathcal{Z}$ having an affine diagonal (see (B.1.3)).

We write $k_{\overline{\theta}} : \mathcal{X}_{\overline{\theta}} \hookrightarrow \mathfrak{X}_{\overline{\theta}}$ for the completion map, and we write $\mathfrak{X}_0$ for the underlying reduced substack of $\mathcal{X}_{\overline{\theta}}$. Recall from Appendix B.3.16 (in particular Proposition B.3.21) that we have a t-exact fully faithful functor

$$k_{\overline{\theta},*} : D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}) \to D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}),$$

whose Pro-extension

$$\widehat{k}_{\overline{\theta},*} : \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}),$$

which is again t-exact, admits a t-exact left adjoint of "$\mathfrak{m}$-adic completion"

$$\widehat{k}^*_{\overline{\theta}} : \operatorname{Pro} D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}).$$

We use the same notation to denote the various functors on hearts that are induced by these t-exact functors; Corollary A.7.20 shows that in turn these various t-exact functors are determined by the induced functors on hearts.

The canonical morphism $\mathfrak{X}_{\overline{\theta}} \to \operatorname{Spec} R^{\mathrm{ps}}_{\overline{\theta}}$ endows any object $\mathfrak{F}$ of $\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}})$ with an $R^{\mathrm{ps}}_{\overline{\theta}}$-module structure. This structure is natural in $\mathfrak{F}$, i.e. $\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}})$ is enriched over $R^{\mathrm{ps}}_{\overline{\theta}}$-modules, and so any object of $\operatorname{Pro}\mathrm{Coh}(\mathfrak{X}_{\overline{\theta}})$ is again endowed with an $R^{\mathrm{ps}}_{\overline{\theta}}$-module structure. Similarly, the adic morphism $\mathcal{X}_{\overline{\theta}} \to \operatorname{Spf} R^{\mathrm{ps}}_{\overline{\theta}}$ induces a natural $R^{\mathrm{ps}}_{\overline{\theta}}$-module structure on any object $\mathcal{F}$ of $\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})$, or, more generally, on any object of $\operatorname{Pro}\mathrm{Coh}(\mathcal{X}_{\overline{\theta}})$. The functors $k_{\overline{\theta},*}$, $\widehat{k}_{\overline{\theta},*}$, and $\widehat{k}^*_{\overline{\theta}}$ are all compatible with these $R^{\mathrm{ps}}_{\overline{\theta}}$-module structures, i.e. they are functors of categories enriched over $R^{\mathrm{ps}}_{\overline{\theta}}$-modules.

We summarize the situation in the following result.

Theorem C.2.15. *The functor*

$$k_{\overline{\theta},*} : D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}) \hookrightarrow D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}),$$

is t-exact and fully faithful, with essential image equal to $D^b_{\mathrm{coh},\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}})$, i.e. the full sub-$\infty$-category of $D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})$ consisting of objects whose cohomology sheaves are set-theoretically supported on $\mathfrak{X}_0$.

The Pro-extended functor

$$\widehat{k}_{\overline{\theta},*} : \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}) \hookrightarrow \operatorname{Pro} D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}})$$

is again t-exact and fully faithful, as is the restriction of its left adjoint

$$\widehat{k}^*_{\overline{\theta}} : D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}).$$

The restriction of $\widehat{k}^*_{\overline{\theta}}$ to $D^b_{\mathrm{coh},\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}})$ is an inverse equivalence to the equivalence $D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}}) \xrightarrow{\sim} D^b_{\mathrm{coh},\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}})$ induced by $k_{\overline{\theta},*}$.

Proof. The claim of the first paragraph is a special case of Proposition B.2.8. That $\widehat{k}_{\overline{\theta},*}$ is t-exact and fully faithful follows immediately, as the Pro-extension of a t-exact and fully faithful functor is again t-exact and fully faithful. Remark B.3.29 shows that $\widehat{k}^*_{\overline{\theta}}$ restricts to an equivalence $D^b_{\mathrm{coh},\mathfrak{X}_0}(\mathfrak{X}_{\overline{\theta}}) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$.

It remains to show that $\widehat{k}^*_{\overline{\theta}} : D^b_{\mathrm{coh}}(\mathfrak{X}_{\overline{\theta}}) \to \operatorname{Pro} D^b_{\mathrm{coh}}(\mathcal{X}_{\overline{\theta}})$ is fully faithful. To this end, as in Corollary C.2.14 we write $\mathrm{Rep}^{\square}_{\overline{D}} = \operatorname{Spec} S$, and then apply Theorem B.4.17 (3), taking $R = R^{\mathrm{ps}}_{\overline{\theta}}$, $B = S$, $I = \mathfrak{m}_{R^{\mathrm{ps}}_{\overline{\theta}}} B$, and $G = \mathrm{GL}_d$; note that by Corollary C.2.14, B^G is a complete local ring, and is in particular I^G-adically complete. $\square$

Finally, since the assumptions of Definition B.5.3 are satisfied by the algebraic stack $\mathfrak{X}_{\overline{\theta}}$, we have the following specialization of Proposition B.5.8.

Proposition C.2.16. *Let $\widehat{\mathcal{X}}_{\overline{\theta}}$ be the completion of $\mathfrak{X}_{\overline{\theta}}$ (or equivalently, $\mathcal{X}_{\overline{\theta}}$) at its unique closed point. Then the functor*

$$\mathrm{Coh}(\mathcal{O}_{\mathcal{X}_{\overline{\theta}}}) \to \mathrm{Coh}(\mathrm{Ver}/\widehat{\mathcal{X}}_{\overline{\theta}}),$$

sending a coherent sheaf $\mathcal{F}$ to the coherent sheaf $(f : \operatorname{Spf} A \to \widehat{\mathcal{X}}_{\overline{\theta}}) \mapsto \widehat{f}^*\mathcal{F}(\operatorname{Spf} A)$, is fully faithful.

Remark C.2.17. There are obvious analogues of these results in the fixed determinant case, which are proved in exactly the same way (replacing the pseudodeformation ring with the fixed determinant version, and GL_d with SL_d; see also [JNW24, §2.1].) We will use these analogues in the main body of the paper.

References

[AF91] Luchezar L. Avramov and Hans-Bjørn Foxby. "Homological dimensions of unbounded complexes". *J. Pure Appl. Algebra* 71.2-3 (1991), 129–155.

[AGH19] Benjamin Antieau, David Gepner, and Jeremiah Heller. "K-theoretic obstructions to bounded t-structures". *Invent. Math.* 216.1 (2019), 241–300.

[AHL23] Jarod Alper, Jack Hall, and David Benjamin Lim. *Coherently complete algebraic stacks in positive characteristic*. 2023. arXiv: `2309.01388 [math.AG]`.

[AHR23] Jarod Alper, Jack Hall, and David Rydh. *The étale local structure of algebraic stacks*. 2023. arXiv: 1912.06162.

[AJPV15] Leovigildo Alonso Tarrío, Ana Jeremías López, Marta Pérez Rodríguez, and María J. Vale Gonsalves. "A functorial formalism for quasi-coherent sheaves on a geometric stack". *Expo. Math.* 33.4 (2015), 452–501.

[AJPV18] Leovigildo Alonso Tarrío, Ana Jeremías López, Marta Pérez Rodríguez, and María J. Vale Gonsalves. "On the derived category of quasi-coherent sheaves on an Adams geometric stack". *J. Pure Appl. Algebra* 222.4 (2018), 828–845.

[Alp13] Jarod Alper. "Good moduli spaces for Artin stacks". *Ann. Inst. Fourier (Grenoble)* 63.6 (2013), 2349–2402.

[Alp14] Jarod Alper. "Adequate moduli spaces and geometrically reductive group schemes". *Algebr. Geom.* 1.4 (2014), 489–531.

[Ant21] Benjamin Antieau. *On the uniqueness of ∞-categorical enhancements of triangulated categories.* https://arxiv.org/abs/1812.01526v3. 2021. arXiv: 1812.01526v3 [math.AG].

[BB10] Laurent Berger and Christophe Breuil. "Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbf{Q}_p)$". *Astérisque* 330 (2010), 155–211.

[BBHKLLSW24] Rebecca Bellovin, Neelima Borade, Anton Hilado, Kalyani Kansal, Heejong Lee, Brandon Levin, David Savitt, and Hanneke Wiersema. "Irregular loci in the Emerton-Gee stack for GL_2". *J. Reine Angew. Math.* 814 (2024), 9–46.

[BC09] Joël Bellaïche and Gaëtan Chenevier. "Families of Galois representations and Selmer groups". *Astérisque* 324 (2009), xii+314.

[BCGP21] George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni. "Abelian surfaces over totally real fields are potentially modular". *Publ. Math. Inst. Hautes Études Sci.* 134 (2021), 153–501.

[BCKW25] Ulrich Bunke, Denis-Charles Cisinski, Daniel Kasprowski, and Christoph Winges. "Controlled objects in left-exact ∞-categories and the Novikov conjecture". *Bull. Soc. Math. France* 153.2 (2025), 295–458.

[BG19] Rebecca Bellovin and Toby Gee. "G-valued local deformation rings and global lifts". *Algebra Number Theory* 13.2 (2019), 333–378.

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. "A universal characterization of higher algebraic K-theory". *Geom. Topol.* 17.2 (2013), 733–838.

[BIP23a] Gebhard Böckle, Ashwin Iyengar, and Vytautas Paškūnas. "On local Galois deformation rings". *Forum Math. Pi* 11 (2023), Paper No. e30, 54.

[BIP23b] Gebhard Böckle, Ashwin Iyengar, and Vytautas Paškūnas. "Zariski density of crystalline points". *Proc. Natl. Acad. Sci. USA* 120.13 (2023), Paper No. e2221042120, 7.

[BL94] L. Barthel and R. Livné. "Irreducible modular representations of GL_2 of a local field". *Duke Math. J.* 75.2 (1994), 261–292.

[BM02] Christophe Breuil and Ariane Mézard. “Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbf{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ en $l = p$”. *Duke Math. J.* 115.2 (2002). With an appendix by Guy Henniart, 205–310.

[BN93] Marcel Bökstedt and Amnon Neeman. “Homotopy limits in triangulated categories”. *Compositio Math.* 86.2 (1993), 209–234.

[Bre03] Christophe Breuil. “Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbf{Q}_p)$. I”. *Compositio Math.* 138.2 (2003), 165–188.

[Bru66] Armand Brumer. “Pseudocompact algebras, profinite groups and class formations”. *J. Algebra* 4 (1966), 442–470.

[BS01] Paul Balmer and Marco Schlichting. “Idempotent completion of triangulated categories”. *J. Algebra* 236.2 (2001), 819–834.

[CD14] Pierre Colmez and Gabriel Dospinescu. “Complétés universels de représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$”. *Algebra Number Theory* 8.6 (2014), 1447–1519.

[CDT99] Brian Conrad, Fred Diamond, and Richard Taylor. “Modularity of certain potentially Barsotti-Tate Galois representations”. *J. Amer. Math. Soc.* 12.2 (1999), 521–567.

[CEGS25] Ana Caraiani, Matthew Emerton, Toby Gee, and David Savitt. “The geometric Breuil-Mézard conjecture for two-dimensional potentially Barsotti-Tate Galois representations”. *Algebra Number Theory* 19.2 (2025), 287–312.

[Che14] Gaëtan Chenevier. “The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings”. In: *Automorphic forms and Galois representations. Vol. 1.* Vol. 414. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2014, 221–285.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*. Vol. 180. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2019, xviii+430.

[Col10a] Pierre Colmez. “(ϕ, Γ)-modules et représentations du mirabolique de $\mathrm{GL}_2(\mathbf{Q}_p)$”. *Astérisque* 330 (2010), 61–153.

[Col10b] Pierre Colmez. “Fonctions d’une variable p-adique”. *Astérisque* 330 (2010), 13–59.

[Col10c] Pierre Colmez. “Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (ϕ, Γ)-modules”. *Astérisque* 330 (2010), 281–509.

[Con05] Brian Conrad. “Formal GAGA for Artin stacks” (2005).

[Con14] Brian Conrad. “Reductive group schemes”. In: *Autour des schémas en groupes. Vol. I.* Vol. 42/43. Panor. Synthèses. Soc. Math. France, Paris, 2014, 93–444.

[ČS24] Kęstutis Česnavičius and Peter Scholze. “Purity for flat cohomology”. *Ann. of Math. (2)* 199.1 (2024), 51–180.

[CW24] Sabin Cautis and Harold Williams. *Ind-geometric stacks*. 2024. arXiv: `2306.03043 [math.AG]`.

[Dee01] Jonathan Dee. “Φ-Γ modules for families of Galois representations”. *J. Algebra* 235.2 (2001), 636–664.

[DEG23] Andrea Dotto, Matthew Emerton, and Toby Gee. *Localization of smooth p-power torsion representations of* $GL_2(\mathbf{Q}_p)$. 2023. arXiv: `2207.04671 [math.NT]`.

[Dot25] Andrea Dotto. "Restriction of p-adic representations of $\mathrm{GL}_2(\mathrm{Q}_p)$ to parahoric subgroups". *Compos. Math.* 161.1 (2025), 74–119.

[Dri06] Vladimir Drinfeld. "Infinite-dimensional vector bundles in algebraic geometry: an introduction". In: *The unity of mathematics*. Vol. 244. Progr. Math. Birkhäuser Boston, Boston, MA, 2006, 263–304.

[EG21] Matthew Emerton and Toby Gee. "'Scheme-theoretic images' of morphisms of stacks". *Algebr. Geom.* 8.1 (2021), 1–132.

[EG22] Matthew Emerton and Toby Gee. "Moduli stacks of étale (φ, Γ)-modules: errata" (2022).

[EG23] Matthew Emerton and Toby Gee. *Moduli stacks of étale (φ, Γ)-modules and the existence of crystalline lifts*. Vol. 215. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2023, ix+298.

[EGH25] Matthew Emerton, Toby Gee, and Eugen Hellmann. "An introduction to the categorical p-adic Langlands program". In: *The Langlands Program*. Vol. 112. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2025, 167–419.

[Eme] Matthew Emerton. "Formal algebraic stacks". Available at `http://www.math.uchicago.edu/~emerton/pdffiles/formal-stacks.pdf`.

[Eme10] Matthew Emerton. "Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties". *Astérisque* 331 (2010), 355–402.

[EP10] Matthew Emerton and Vytautas Paškūnas. "On the effaceability of certain δ-functors". *Astérisque* 331 (2010), 461–469.

[Fon90] Jean-Marc Fontaine. "Représentations p-adiques des corps locaux. I". In: *The Grothendieck Festschrift, Vol. II*. Vol. 87. Progr. Math. Boston, MA: Birkhäuser Boston, 1990, 249–309.

[Gab62] Pierre Gabriel. "Des catégories abéliennes". *Bull. Soc. Math. France* 90 (1962), 323–448.

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Second. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003, xx+372.

[Gro90] Benedict H. Gross. "A tameness criterion for Galois representations associated to modular forms (mod p)". *Duke Math. J.* 61.2 (1990), 445–517.

[Har66] Robin Hartshorne. *Residues and duality*. Vol. No. 20. Lecture Notes in Mathematics. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne. Springer-Verlag, Berlin-New York, 1966, vii+423.

[Hen17] Benjamin Hennion. "Tate objects in stable $(\infty, 1)$-categories". *Homology Homotopy Appl.* 19.2 (2017), 373–395.

[Her11] Florian Herzig. "A Satake isomorphism in characteristic p". *Compos. Math.* 147.1 (2011), 263–283.

[Her97] Ivo Herzog. "The Ziegler spectrum of a locally coherent Grothendieck category". *Proc. London Math. Soc. (3)* 74.3 (1997), 503–558.

[Hey21] Claudius Heyer. *On the smooth part functor*. https://arxiv.org/abs/2108.05262. 2021. arXiv: 2108.05262 [math.NT].

[Hey23] Claudius Heyer. "The left adjoint of derived parabolic induction". *Math. Z.* 305.3 (2023), Paper No. 46, 60.

[Hey24] Claudius Heyer. *The geometrical lemma for smooth representations in natural characteristic*. https://doi.org/10.48550/arXiv.2303.14721. 2024.

[HM24] Claudius Heyer and Lucas Mann. *6-Functor Formalisms and Smooth Representations*. 2024. arXiv: 2410.13038 [math.CT].

[HNR19] Jack Hall, Amnon Neeman, and David Rydh. "One positive and two negative results for derived categories of algebraic stacks". *J. Inst. Math. Jussieu* 18.5 (2019), 1087–1111.

[Hov99] Mark Hovey. *Model categories*. Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999, xii+209.

[HP23] Daniel Halpern-Leistner and Anatoly Preygel. "Mapping stacks and categorical notions of properness". *Compos. Math.* 159.3 (2023), 530–589.

[HR17] Jack Hall and David Rydh. "Perfect complexes on algebraic stacks". *Compos. Math.* 153.11 (2017), 2318–2367.

[HT15] Yongquan Hu and Fucheng Tan. "The Breuil-Mézard conjecture for non-scalar split residual representations". *Ann. Sci. Éc. Norm. Supér. (4)* 48.6 (2015), 1383–1421.

[JNW24] Christian Johansson, James Newton, and Carl Wang-Erickson. *Moduli stacks of Galois representations and the p-adic local Langlands correspondence for* $\mathrm{GL}_2(\mathbf{Q}_p)$. https://arxiv.org/abs/2403.19565. 2024.

[Kerodon] Jacob Lurie. *Kerodon*. https://kerodon.net.

[Kis08] Mark Kisin. "Potentially semi-stable deformation rings". *J. Amer. Math. Soc.* 21.2 (2008), 513–546.

[Kis09] Mark Kisin. "The Fontaine-Mazur conjecture for GL_2". *J. Amer. Math. Soc.* 22.3 (2009), 641–690.

[Kra05] Henning Krause. "The stable derived category of a Noetherian scheme". *Compos. Math.* 141.5 (2005), 1128–1162.

[Kra15] Henning Krause. "Deriving Auslander's formula". *Doc. Math.* 20 (2015), 669–688.

[Kra97] Henning Krause. "The spectrum of a locally coherent category". *J. Pure Appl. Algebra* 114.3 (1997), 259–271.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*. Vol. 332. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, x+497.

[Laz65] Michel Lazard. "Groupes analytiques p-adiques". *Inst. Hautes Études Sci. Publ. Math.* 26 (1965), 389–603.

[LM00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Berlin: Springer-Verlag, 2000, xii+208.

[LMM23] Bao Viet Le Hung, Ariane Mézard, and Stefano Morra. *Local model theory for non-generic tame potentially Barsotti–Tate deformation rings*. 2023. arXiv: `2311.16617 [math.NT]`.

[LO96] Huishi Li and Freddy van Oystaeyen. *Zariskian filtrations*. Vol. 2. K-Monographs in Mathematics. Kluwer Academic Publishers, Dordrecht, 1996, x+252.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, xviii+925.

[Lur17] Jacob Lurie. *Higher Algebra (version of September 18, 2017)*. 2017.

[Lur18] Jacob Lurie. *Spectral Algebraic Geometry (version of February 3, 2018)*. 2018.

[Mit65] Barry Mitchell. *Theory of categories*. Vol. Vol. XVII. Pure and Applied Mathematics. Academic Press, New York-London, 1965, xi+273.

[Miy91] Jun-ichi Miyachi. "Localization of triangulated categories and derived categories". *J. Algebra* 141.2 (1991), 463–483.

[NS16] A. Nyman and S. P. Smith. "A generalization of Watts's theorem: right exact functors on module categories". *Comm. Algebra* 44.7 (2016), 3160–3170.

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". *Acta Math.* 221.2 (2018), 203–409.

[Ols07] Martin Olsson. "Sheaves on Artin stacks". *J. Reine Angew. Math.* 603 (2007), 55–112.

[Paš13] Vytautas Paškūnas. "The image of Colmez's Montreal functor". *Publ. Math. Inst. Hautes Études Sci.* 118 (2013), 1–191.

[Paš15] Vytautas Paškūnas. "On the Breuil–Mézard conjecture". *Duke Math. J.* 164.2 (2015), 297–359.

[PT21] Vytautas Paškūnas and Shen-Ning Tung. "Finiteness properties of the category of mod p representations of $\mathrm{GL}_2(\mathbf{Q}_p)$". *Forum Math. Sigma* 9 (2021), Paper No. e80, 39.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Vol. No. 43. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1967, iv+156 pp. (not consecutively paged).

[RZ10] Luis Ribes and Pavel Zalesskii. *Profinite groups*. Second. Vol. 40. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2010, xvi+464.

[San16] Fabian Sander. "A local proof of the Breuil-Mézard conjecture in the scalar semi-simplification case". *J. Lond. Math. Soc. (2)* 94.2 (2016), 447–461.

[Ser87] Jean-Pierre Serre. "Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$". *Duke Math. J.* 54.1 (1987), 179–230.

[Sha22] Jay Shah. *Recollements and stratification*. 2022. arXiv: `2110.06567 [math.AT]`.

[Sho20] Jack Shotton. "The category of finitely presented smooth mod p representations of $GL_2(F)$". *Doc. Math.* 25 (2020), 143–157.

[Stacks] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`.

[SVZ14] Peter Schneider, Marie-France Vigneras, and Gergely Zabradi. "From étale P_+-representations to G-equivariant sheaves on G/P". In: *Automorphic forms and Galois representations. Vol. 2*. Vol. 415. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2014, 248–366.

[Swa19] Richard G. Swan. "K-theory of coherent rings". *J. Algebra Appl.* 18.9 (2019), 1950161, 16.

[Tim23] James Timmins. "Coherence of augmented Iwasawa algebras". *Adv. Math.* 417 (2023), Paper No. 108916, 86.

[Ven02] Otmar Venjakob. "On the structure theory of the Iwasawa algebra of a p-adic Lie group". *J. Eur. Math. Soc. (JEMS)* 4.3 (2002), 271–311.

[VV97] Martine Van Gastel and Michel Van den Bergh. "Graded modules of Gelfand-Kirillov dimension one over three-dimensional Artin-Schelter regular algebras". *J. Algebra* 196.1 (1997), 251–282.

[Wan18] Carl Wang-Erickson. "Algebraic families of Galois representations and potentially semi-stable pseudodeformation rings". *Math. Ann.* 371.3-4 (2018), 1615–1681.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, xiv+450.

Email address: `andrea.dotto@kcl.ac.uk`

Department of Mathematics, King's College London, London WC2R 2LS, UK

Email address: `emerton@math.uchicago.edu`

Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637, USA

Email address: `toby.gee@imperial.ac.uk`

Department of Mathematics, Imperial College London, London SW7 2AZ, UK