LOCALIZATION OF SMOOTH p-POWER TORSION REPRESENTATIONS OF $\mathrm{GL}_2(\mathbf{Q}_p)$

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ABSTRACT. We show that the category of smooth representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ on p-power torsion modules localizes over a certain projective scheme, and give some applications.

CONTENTS

1. INTRODUCTION

In this paper we establish a localization theory for the category of smooth representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $\mathbf{Z}_p$-modules on which p is locally nilpotent, for any prime $p \geq 5$.

1.1. **Initial statement of results.** In order to make a precise statement, we introduce some notation. Let $\mathcal{O}$ denote the ring of integers in a finite extension E of $\mathbf{Q}_p$ and let $\mathcal{A}$ denote the category of smooth representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $\mathcal{O}$-modules on which p acts locally nilpotently, and which have a central character equal to some fixed character $\zeta : \mathbf{Q}_p^\times \to \mathcal{O}^\times$. Let $\mathbf{F}$ denote the residue field of $\mathcal{O}$.

Let X denote a chain of projective lines over $\mathbf{F}$ with ordinary double points, of length $(p \pm 1)/2$ (where the sign is equal to $-\zeta(-1)$). Our definition of X is motivated by the mod p semisimple local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ due to Breuil [Bre03b, Defn. 1.1]. Indeed, one can think of X as a moduli space of semisimple representations $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, with the intersection points of the

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$\mathbf{P}^1$s corresponding to irreducible representations. It then follows from Paškūnas' results [Paš13] that the blocks in the subcategory $\mathcal{A}^{\mathrm{l.adm}}$ of locally admissible representations with central character ζ are in bijection with the closed points of X.

Given a closed subset Y of X with open complement $U := X \setminus Y$, we let $\mathcal{A}_Y$ denote the subcategory of $\mathcal{A}$ consisting of those representations all of whose irreducible subquotients lie in blocks corresponding to closed points of Y. The subcategory $\mathcal{A}_Y$ is localizing (in the usual sense, recalled in Appendix A.2), and is in particular a Serre subcategory of $\mathcal{A}$, and we set $\mathcal{A}_U := \mathcal{A}/\mathcal{A}_Y$.

Our main result shows that the category $\mathcal{A}$ can be *localized* over X, in the following precise sense (see Theorem 3.3.1 and Remark 3.3.2).

Theorem 1.1.1. *The collection $\{\mathcal{A}_U\}$ forms a stack (of abelian categories) over the Zariski site of X.*

1.2. **Applications.** In Section 4 we demonstrate that Theorem 1.1.1 (and the results that go into its proof) has many concrete consequences for the representation theory of $\mathrm{GL}_2(\mathbf{Q}_p)$. In particular we compute various Ext^1 groups between compact inductions of Serre weights, and between such compact inductions and irreducible representations. For example, in Proposition 4.2.2 we show that if σ is a Serre weight and π is absolutely irreducible, then

$$\dim \mathrm{Ext}^1(\pi, c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)) \leq 1;$$

in addition, we explicitly describe all of the non-split extensions that arise. Similarly, in Proposition 4.2.4 we compute the dimensions of the Ext^1 groups for extensions in the opposite direction (with a genericity assumption), and in Section 4.3 we compute the Ext^1 groups between full compact inductions of Serre weights.

In addition, in Section 4.1 we make precise the fashion in which the category of finitely generated smooth representations is built out of finite length representations (which admit finite filtrations by irreducible representations), together with compact inductions of Serre weights. In particular we prove Proposition 4.1.3, which shows that every finitely generated representation π admits a maximal subobject π_{fl} which is of finite length, and the quotient π/π_{fl} is a successive extension of submodules of representations $c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma$, for σ a Serre weight. (The structure of these submodules is not mysterious; indeed, as recalled in Remark 4.1.4, most such submodules are isomorphic to $c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma$.)

1.3. **The relationship to the Bernstein centre.** Our results can be understood by comparison with other, more classical contexts, in the representation theory of p-adic reductive groups. Note that the particular choice of coefficient ring $\mathcal{O}$ does not affect things much, as long as its residue field $\mathbf{F}$ has characteristic p. Thus it makes sense to compare our results with the classical case of smooth representations of a p-adic reductive group G over a field $\mathbf{F}$ of characteristic zero.

The first thing to note is that, in this classical case just as in our case, there are many smooth representations, indeed naturally occuring ones, that are not admissible. For example, if $G = \mathrm{GL}_2(\mathbf{Q}_p)$ and V is a finite dimensional representation of $KZ = \mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$, then $c\text{-}\mathrm{Ind}_{KZ}^G\,V$ is admissible if and only if V has vanishing "Jacquet module" (see e.g. [Bus90, Thm. 1 supp.] for a proof of a more general result), a condition which certainly need not hold in characteristic zero (e.g. if V is the trivial representation). In characteristic p, $c\text{-}\mathrm{Ind}_{KZ}^G\,V$ is *never* admissible, provided V itself is non-zero. At least morally, this is because in characteristic p, the Jacquet module of a non-zero V will never vanish.

However, if $\mathbf{F}$ is a field of characteristic zero, if G is any p-adic reductive group, and if $\mathfrak{Z}$ denotes the Bernstein centre of the category of smooth G-representations on $\mathbf{F}$-vector spaces, then Bernstein's results [Ber84] show that any finitely generated smooth G-representation is admissible *over* $\mathfrak{Z}$. (This is the p-adic analogue of a theorem of Harish-Chandra, to the effect that if G is a real reductive group with Lie algebra $\mathfrak{g}$, then any finitely generated $\big(U(\mathfrak{g}), K\big)$-module is admissible over the centre $\mathfrak{Z}$ of $U(\mathfrak{g})$.) Thus we may think of a not-necessarily-admissible smooth G-representation in characteristic zero as being a "family" of admissible representations parameterized by some subscheme of $\mathrm{Spec}(\mathfrak{Z})$. An analogous result has also recently been proved by Dat–Helm–Kurinczuk–Moss [DHKM22] for the category of smooth representations of G over a field of characteristic $\ell \neq p$, or more generally over a Noetherian $\mathbf{Z}_\ell$-algebra.

On the other hand, the analogous result is *not* true for our category $\mathcal{A}$. Indeed, independent results of A.D. and Ardakov–Schneider [AS21, Dot21] show that the Bernstein centre of $\mathcal{A}$ is trivial. Thus any finitely generated non-admissible representation (such as the $c\text{-}\mathrm{Ind}_{KZ}^G V$ introduced above) provides a counterexample to the analogue of Bernstein's result.

The localization theory of this paper was developed in part to rectify this absence of an interesting Bernstein centre for $\mathcal{A}$: rather than simply regarding objects of $\mathcal{A}$ as lying over the Spec of its Bernstein centre, we localize them over the non-affine variety X. Then, by forming the Bernstein centres of the various categories $\mathcal{A}_U$, we may endow X with the structure of a topologically ringed space, and we expect to prove (in forthcoming work) that this endows X with the structure of a formal scheme over $\mathcal{O}$ — which we denote by $\widehat{X}$ — such that X itself is the underlying reduced scheme over $\mathbf{F}$ of $\widehat{X}$. The triviality of the Bernstein centre of $\mathcal{A}$ would then correspond to the fact that the only global sections of the structure sheaf $\mathcal{O}_{\widehat{X}}$ are the constant functions. In turn, this statement itself would be an extension to the thickening $\widehat{X}$ of the fact that the only globally defined functions on the connected projective variety X are the constant functions — thus the fact that X is projective, rather than affine, is closely related to the fact that the Bernstein centre of $\mathcal{A}$ is trivial.

1.4. **The relationship with local Langlands.** Our definition of X suggests that there is a connection between our results and the mod p or p-adic local Langlands correspondence, and indeed, this is the case. To explain and motivate this, we first recall the analogous result in the $\ell \neq p$ context. Namely, if $\mathbf{F}$ is a field of characteristic zero or $\ell \neq p$, or more generally if $\mathcal{O}$ is a complete DVR whose residue field is such a field $\mathbf{F}$, then, for any p-adic field F and any $n \geq 1$, we may consider the moduli stack $\mathcal{X}$ parameterizing n-dimensional representations of the Weil–Deligne group of F over $\mathcal{O}$-algebras, and the local Langlands correspondence over $\mathcal{O}$ identifies $\mathfrak{Z}$ — the Bernstein centre of $\mathcal{O}[\mathrm{GL}_n(F)]$ — with the ring of functions on $\mathcal{X}$, by a theorem of Helm–Moss [HM18]. Since $\mathcal{X}$ is a reductive quotient of an affine scheme, we may rephrase this as saying that $\mathrm{Spec}\,\mathfrak{Z}$ is the moduli space associated to $\mathcal{X}$, or more precisely an adequate moduli space in the sense of Alper [Alp14].

In forthcoming work we expect to prove a p-adic analogue of this result. Namely, we will show that $\widehat{X}$ — the thickening of X induced by the localized Bernstein centre of $\mathcal{A}$ discussed above — is a formal moduli space associated to the stack $\mathcal{X}_{\mathbf{Q}_p}$

parameterizing (φ, Γ)-modules arising from continuous rank two p-adic representations of $\mathrm{Gal}_{\mathbf{Q}_p}$ of appropriately fixed determinant.

Remark 1.4.1. In the $\ell \neq p$ case, the identification of the Bernstein centre with the ring of functions on $\mathcal{X}$ is an outward manifestation of a deeper phenomenon, namely the categorical local Langlands correspondence considered in [BZCHN20, FS21, Hel20, Zhu20]. In the forthcoming paper [DEG], we establish (using the results of this paper as one our tools) an analogous categorical p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, in the form of a fully faithful functor from the derived category of $\mathcal{A}$ to an appropriate derived category of coherent sheaves on the stack of (φ, Γ)-modules $\mathcal{X}_{2,\mathbf{Q}_p}$ mentioned above.

However, while we anticipate that fully faithful functors of this kind exist in great generality (in particular, for the representations of $\mathrm{GL}_n(F)$ for any n and any p-adic local field F), we don't expect the results of the present paper to generalise in any obvious way, even to $\mathrm{GL}_2(F)$ with $F \neq \mathbf{Q}_p$.

Relatedly, on the Galois side of the Langlands correspondence, we do not expect the stacks of (φ, Γ)-modules to admit interesting associated moduli spaces beyond the cases of $\mathrm{GL}_2(\mathbf{Q}_p)$ and $\mathrm{GL}_1(F)$, F being any finite extension of $\mathbf{Q}_p$. For example, if $\mathcal{X}_{2,\mathbf{Q}_{p^2}}$ denotes the analogous stack for $\mathrm{GL}_2(\mathbf{Q}_{p^2})$, then the various specialisation relations in reducible families cause any morphism $\mathcal{X}_{2,\mathbf{Q}_{p^2}} \to Z$ with Z being a locally separated formal algebraic space to factor through the structure morphism $\mathcal{X}_{2,\mathbf{Q}_{p^2}} \to \operatorname{Spf} \mathcal{O}$.

1.5. **Methods of proof.** The subcategory $\mathcal{A}^{\mathrm{adm}}$ of $\mathcal{A}$ consisting of admissible representations is well-understood, thanks to the results of Paškūnas [Paš13]. As already mentioned above, it factors into a product of *blocks*, which are labelled by the closed points of X. In our perspective, the objects of $\mathcal{A}^{\mathrm{l.adm}}$ are supported on finite closed subsets of X, so that our localization theory, for these objects, is just a restatement of Paškūnas's results.

The novel aspects of our theory are seen when the objects under consideration are not locally admissible, since then they will localize over subsets of X with non-empty interior. A basic consequence of the localization theory is that, if two representations have disjoint support, then all Ext^i groups between them vanish. Conversely, proving such statements is the key to proving Theorem 1.1.1. The key result is Lemma 3.1.6, which (at least morally) shows that if σ is a Serre weight, and π is a supported on some closed subset Y, then the Ext^i between π and $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ are also supported on Y. It is proved by exploiting the fact that these Ext^i groups are typically not finite dimensional over $\mathbf{F}$, but they are of countable dimension. Since they are also modules over the Hecke algebra $\mathcal{H}(\sigma) \cong \mathbf{F}[T]$, this makes them amenable to an application of a well-known technique of Dixmier [Dix63], at least when $\mathbf{F}$ is uncountable; and we can always arrange that $\mathbf{F}$ is uncountable via an appropriate base-change. With this result in hand, we are able to show that for any open cover of X, the corresponding "Čech resolution" of any π is acyclic, and we deduce Theorem 1.1.1 from this acyclicity.

Remark 1.5.1. We do not explicitly use the p-adic local Langlands correspondence in our arguments. However, we do use the results of Paškūnas [Paš13], whose proofs use p-adic local Langlands. In particular we frequently use the classification of blocks in the locally admissible category $\mathcal{A}^{\mathrm{l.adm}}$, and for some of the results in

Section 3.8, we furthermore make use of Paškūnas's description of the Bernstein centres of these blocks.

Remark 1.5.2. We expect that the results of this paper extend to the cases of $p = 2$, 3, with minor modifications to the statements (e.g. to the definition of X in the case $p = 2$: in this case X should consist of a single $\mathbf{P}^1$). Our arguments would necessarily become more complicated in the cases $p = 2, 3$, so that we could no longer argue uniformly in p. Note that this already happens for the description of blocks in [Paš13], which is not carried out in loc. cit. for $p = 2, 3$, but is rather in [Paš14, PT21]. Thus we have not attempted to make the modifications of our arguments that would be required to treat these cases.

1.6. **A guide to the paper.** In Section 2 we recall some basic results on the smooth p-adic representation theory of $\mathrm{GL}_2(\mathbf{Q}_p)$, and explain our interpretation of Paškūnas' classification [Paš13] of the blocks of locally admissible representations in terms of the closed points of X. In Section 2.5 we prove some general results on Ext groups, including finiteness properties, compatibility with extension of scalars, and compatibility with colimits. Finally in Section 2.6 we prove some technical results on the Bernstein centres of these blocks.

We establish our localization theory in Section 3. We begin with the basic definitions of our localizing categories, and then prove the crucial Lemma 3.1.6 mentioned above, and (with some work) deduce Theorem 1.1.1. The rest of Section 3 is devoted to a discussion of the analogue on the representation theory side of completion along a closed subset of X. In particular, we prove Theorem 3.8.1, which is the analogue of Beauville–Laszlo gluing in this setting. As well as being of independent interest, this is used crucially in our forthcoming paper [DEG].

In Section 4 we give a variety of examples and application of our localization theory, showing in particular how it can be used to compute extensions between compact inductions $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$, and extensions between such compact inductions and admissible representations. Finally in Appendix A we recall some background material in category theory that we use in the body of the paper.

1.7. **Notation and conventions.** We fix throughout the paper a prime $p \geq 5$. Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, and write $G_{\mathbf{Q}_p}$ for the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let $\mathcal{O}$ denote the ring of integers in a fixed finite extension E of $\mathbf{Q}_p$, and let $\mathbf{F}$ be the residue field of $\mathcal{O}$; all representations considered in this paper will be on $\mathcal{O}$-modules.

We write $G = \mathrm{GL}_2(\mathbf{Q}_p)$, $K = \mathrm{GL}_2(\mathbf{Z}_p)$, and $Z = \mathbf{Q}_p^\times$ for the centre of G. We fix a continuous character $\zeta : \mathbf{Q}_p^\times \to \mathcal{O}^\times$ throughout the paper. We write $\omega : \mathbf{Q}_p^\times \to \mathbf{F}_p^\times$ for the reduction mod p of the character $\mathbf{Q}_p^\times \to \mathbf{Z}_p^\times, x \mapsto x|x|$.

We always let A denote a complete Noetherian local $\mathcal{O}$-algebra, where "local" is understood in the strong sense that A is a local ring, and the morphism $\mathcal{O} \to A$ is a local morphism; given the first assumption, this second hypothesis is equivalent to requiring that the residue field k be of characteristic p. We then let $\mathcal{A}_A$ denote the category of smooth G-representations with central character ζ on locally $\mathfrak{m}_A$-torsion A-modules. In more detail, $\mathfrak{m}_A$ denotes the maximal ideal of A, a module is *locally* $\mathfrak{m}_A$*-torsion* if each element is annihilated by some power of $\mathfrak{m}_A$, and a representation is *smooth* if each element is fixed by some open subgroup of G; so we are considering precisely those representations over A which are called smooth in [Eme10a] (and which have the required central character). In the case that $A = \mathcal{O}$

we write $\mathcal{A}$ for $\mathcal{A}_A$. If V is a finite length object of $\mathcal{A}$ we will write $\mathrm{JH}(V)$ for the multiset of Jordan–Hölder factors of V.

We will write I_ζ for the two-sided ideal of $A[G]$ or $A[KZ]$ (or other group algebras of groups containing Z) generated by $[z]-\zeta(z)$ for $z \in Z$.

We write $\mathcal{C}_A$ for the category of smooth KZ-representations with central character ζ on A-modules on which p is locally nilpotent. In the case that $A = \mathcal{O}$ we write $\mathcal{C}$ for $\mathcal{C}_A$. A *Serre weight* is an irreducible object of $\mathcal{C}_A$, or equivalently of $\mathcal{C}_k$. Since every irreducible k-representation of K is defined over $\mathbf{F}_p$, and Z is acting by ζ, the set of isomorphism classes of Serre weights is independent of A.

More explicitly, the isomorphism classes of Serre weights are represented by the representations $\mathrm{Sym}^b\,\mathbf{F}_p^2 \otimes \det^a$ of $\mathrm{GL}_2(\mathbf{F}_p)$, where $0 \le a < p-1$ and $0 \le b \le p-1$. It is sometimes convenient to view a as an element of $\mathbf{Z}/p\mathbf{Z}$, and we will do so without further comment.

We make some use of tame types, by which we mean the principal series and cuspidal representations of $\mathrm{GL}_2(\mathbf{F}_p)$. For any pair of characters $\chi_1 \ne \chi_2 : \mathbf{F}_p^\times \to \mathcal{O}^\times$, we have the principal series representation $I(\chi_1,\chi_2) = \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} \chi_1 \otimes \chi_2$, and for any character $\chi : \mathbf{F}_{p^2}^\times \to \mathcal{O}^\times$ which does not factor through the norm, the corresponding cuspidal representation $\Theta(\chi)$. (Up to sign, these are the Deligne–Lusztig inductions of regular characters of the nonsplit maximal torus in $\mathrm{GL}_2(\mathbf{F}_p)$.) These representations are irreducible of dimensions $p+1$, $p-1$ respectively, and the only isomorphisms between them are that $I(\chi_1,\chi_2) \cong I(\chi_2,\chi_1)$ and $\Theta(\chi) \cong \Theta(\chi^p)$.

The Jordan–Hölder factors of their reductions modulo p are as follows. Firstly, if we write $\overline{\chi}_i(x) = x^{n_i}$, then the Jordan–Hölder factors of the reduction of $I(\chi_1,\chi_2)$ are $\mathrm{Sym}^{[n_1-n_2]} \otimes \det^{n_2}$ and $\mathrm{Sym}^{[n_2-n_1]} \otimes \det^{n_1}$, where $[n_i - n_j]$ denotes the unique integer in $(0, p-1)$ congruent to $n_i - n_j \pmod{p-1}$.

Secondly, if $\overline{\chi}(x) = x^{i+(p+1)j}$ with $1 \le i \le p$, then if $1 < i < p$ then the reduction of $\Theta(\chi)$ has two Jordan–Hölder factors, namely $\mathrm{Sym}^{i-2} \otimes \det^{1+j}$ and $\mathrm{Sym}^{p-1-i} \otimes \det^{i+j}$; while if $i = 1$ or p, then it is irreducible and isomorphic to $\mathrm{Sym}^{p-2} \otimes \det^{1+j}$.

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2. Preliminaries on smooth $\mathrm{GL}_2(\mathbf{Q}_p)$-representations

2.1. **Irreducible representations.** If $A = k$ is an algebraically closed field of characteristic p, the irreducible objects of $\mathcal{A}_k$ have been classified in [BL94, Bre03a]. As we will often need to extend scalars to transcendental extensions of k, in this section we will extend their results to arbitrary and possibly imperfect extensions $k/\mathbf{F}_p$.

To state the classification, recall more generally that the restriction functor from $\mathcal{A}_A$ to $\mathcal{C}_A$ has an exact left adjoint given by compact induction and denoted

$$V \mapsto c\text{-}\mathrm{Ind}_{KZ}^G(V).$$

The case when $V = \sigma$ is an irreducible $k[KZ]$-module (a "Serre weight") is of particular importance. We begin by recalling the description, due to Barthel and Livné [BL94], of their endomorphism rings, which we denote by $\mathcal{H}(\sigma) := \mathrm{End}_G(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)$.

Lemma 2.1.1. *We have an isomorphism*

$$\mathcal{H}(\sigma) \cong k[T], \tag{2.1.2}$$

generated by a specific choice of Hecke operator T. Furthermore, $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ is free over $k[T]$.

In what follows we will denote T by T_p in case of a notational clash, for instance if $T \in k[T]$ is a formal variable.

We also recall (again from [BL94]) the classification of G-equivariant morphisms between the compact inductions of Serre weights.

Lemma 2.1.3. *If σ and σ' are distinct Serre weights then $\mathrm{Hom}_G(c\text{-}\mathrm{Ind}_{KZ}^G \sigma', c\text{-}\mathrm{Ind}_{KZ}^G \sigma) =$ 0 except in the case when $\{\sigma, \sigma'\} = \{\det^s, \mathrm{Sym}^{p-1} \otimes \det^s\}$ for some s, in which case we have the following:*

(1) All elements of

$$\mathrm{Hom}_G\big(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 \otimes \det^s, c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \otimes \det^s\big)$$

and of

$$\mathrm{Hom}_G\big(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \otimes \det^s, c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 \otimes \det^s\big)$$

are $k[T]$-equivariant, where $k[T]$ acts on each of source and target in the natural way, i.e. with T acting by the appropriate T_p.

(2) Each of

$$\mathrm{Hom}_G\big(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 \otimes \det^s, c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \otimes \det^s\big)$$

and

$$\mathrm{Hom}_G\big(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \otimes \det^s, c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 \otimes \det^s\big)$$

is free of rank 1 over $k[T]$.

(3) We can choose $k[T]$-generators

$$\alpha \in \mathrm{Hom}_G\big(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 \otimes \det^s, c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \otimes \det^s\big)$$

and

$$\beta \in \mathrm{Hom}_G\big(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \otimes \det^s, c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0 \otimes \det^s\big)$$

such that

$$\alpha \circ \beta = \beta \circ \alpha = T^2 - 1.$$

The cokernels of α and β have length 2.

This lemma has the following useful corollary.

Corollary 2.1.4.

(1) If σ is not a twist of Sym^0 or Sym^{p-1}, then any subobject of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ is of the form $f(T)c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ for some $f(T) \in k[T]$.

(2) If σ is a twist of Sym^0 or Sym^{p-1}, then any subobject π of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ satisfies inclusions

$$(T^2 - 1)f(T)c\text{-}\mathrm{Ind}_{KZ}^G \sigma \subseteq \pi \subseteq f(T)c\text{-}\mathrm{Ind}_{KZ}^G \sigma$$

for some $f(T) \in k[T]$. Furthermore, π is a $k[T]$-submodule of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$.

Proof. If π is non-zero, then it contains a Serre weight σ', and the inclusion $\pi \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ induces a G-equivariant morphism $c\text{-}\mathrm{Ind}_{KZ}^G \sigma' \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma$. The corollary now follows easily from Lemma 2.1.3. $\square$

This corollary is the beginning of the classification of irreducible G-representations. Namely, we see that if π is irreducible, then by choosing a Serre weight σ in π, we obtain a surjection

$$c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f(T)c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma \to \pi,$$

for some irreducible polynomial $f(T) \in k[T]$. Classifying irreducibles then amounts to analyzing the structure of such quotients $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma/f(T)c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$ (with $f(T)$ irreducible). In the case when k is algebraically closed, so that $f(T) = T - \lambda$ for some $\lambda \in k$, this was done by Barthel and Livné when $\lambda \neq 0$, and by Breuil [Bre03a] when $\lambda = 0$. Theorem 2.1.5 and Lemma 2.1.6 below summarize their results.

Theorem 2.1.5. *Let k be an algebraically closed field of characteristic p. Every irreducible object of $\mathcal{A}_k$ is isomorphic to a representation in the following list:*

(1) $\eta \circ \det$ *for some smooth character* $\eta : \mathbf{Q}_p^{\times} \to k^{\times}$*;*
(2) $(\eta \circ \det) \otimes \mathrm{St}$ *for some smooth character* $\eta : \mathbf{Q}_p^{\times} \to k^{\times}$*, where* St *is the Steinberg representation of G; or*
(3) $c\text{-}\mathrm{Ind}_{KZ}^{G}(\sigma)/(f(T))$*, where σ is an irreducible representation of* $k[KZ]$ *and* $f = T - \lambda$ *for some* $\lambda \in k$*, and*

$$(\sigma, \lambda) \notin \{(\mathrm{Sym}^0 \otimes \det^s, \pm 1), (\mathrm{Sym}^{p-1} \otimes \det^s, \pm 1)\}.$$

Conversely, all these representations are irreducible and pairwise nonisomorphic, with the following exceptions:

$$(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^0 \otimes \det^s)/(T - \lambda) \cong (c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{p-1} \otimes \det^s)/(T - \lambda) \text{ if } \lambda \neq \pm 1.$$
$$(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^r \otimes \det^s)/T \cong (c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{p-1-r} \otimes \det^{r+s})/T.$$

We recall what happens in the excluded cases of part (3) of the preceding theorem.

Lemma 2.1.6. *There are non-split short exact sequences*

$$0 \to (\mathrm{nr}_{\pm 1} \circ \det) \otimes \det{}^s \to (c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^{p-1} \otimes \det{}^s)/(T \pm 1) \to (\mathrm{nr}_{\pm 1} \circ \det) \otimes \mathrm{St}_G \otimes \det{}^s \to 0$$

and

$$0 \to (\mathrm{nr}_{\pm 1} \circ \det) \otimes \mathrm{St}_G \otimes \det{}^s \to (c\text{-}\mathrm{Ind}_{KZ}^{G}\,\mathrm{Sym}^0 \otimes \det{}^s)/(T \pm 1) \to (\mathrm{nr}_{\pm 1} \circ \det) \otimes \det{}^s \to 0.$$

These short exact sequences are induced by the morphisms of Lemma 2.1.3(3). It is a standard result that they are not split; we will give a proof in the course of proving Lemma 2.1.12 below.

The next result generalizes Theorem 2.1.5 to an arbitrary coefficient field. See [Paš13, Section 5.3] for similar results under the assumption that k is perfect.

Theorem 2.1.7. *Assume A is a complete Noetherian local $\mathcal{O}$ -algebra, and let π be an irreducible object of $\mathcal{A}_A$. Then π is isomorphic to one of the following:*

(1) $\eta \circ \det$ *for some character* $\eta : \mathbf{Q}_p^{\times} \to k^{\times}$*;*
(2) $(\eta \circ \det) \otimes \mathrm{St}_G$ *for some character* $\eta : \mathbf{Q}_p^{\times} \to k^{\times}$*; or*
(3) $\big(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma\big)/f(T)\big(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma\big)$ *for some irreducible* $k[KZ]$*-representation* σ *and some irreducible polynomial* $f(T) \in k[T]$*, with* $f(T) \neq T \pm 1$ *if* σ *is a twist of* Sym^0 *or* Sym^{p-1}*.*

Conversely, each of these representations is irreducible.

Proof. If π is an irreducible object of $\mathcal{A}_A$ then it is a $k[G]$-module. As noted above, there exists a surjection

$$\Pi := (c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/f(T) \to \pi$$

for some irreducible $k[KZ]$-representation σ and some irreducible polynomial $f(T) \in k[T]$. Bearing in mind Theorem 2.1.5, it suffices to prove that Π is irreducible whenever $\deg(f) > 1$. To do so, observe first that

$$\mathrm{Hom}_{k[G]}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma, \Pi)$$

is $\deg(f)$-dimensional over k. Indeed, since both representations are finitely generated over $k[G]$, by [Paš13, Lemma 5.1] this can be checked after extending scalars to an algebraic closure of k. Then it follows from the fact that $\Pi \otimes_k \overline{k}$ has $\deg(f)$ Jordan–Hölder factors, each of which contains σ as a KZ-subrepresentation with multiplicity one.

Now let $\Pi' \subset \Pi$ be a nonzero G-stable subspace. We claim that Π' contains the image of a nonzero element of $\mathrm{Hom}_{k[G]}(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma, \Pi)$. To see that this implies the theorem, notice that this space contains the field $k[T]/f(T)$, which has dimension $\deg(f)$ over k, and so coincides with it. It follows that every nonzero element is bijective, and so $\Pi' = \Pi$.

To prove the claim, notice that since Π' is nonzero it contains the image of a nonzero morphism

$$c\text{-}\mathrm{Ind}_{KZ}^{G}(\sigma') \to \Pi$$

for some irreducible $k[KZ]$-module σ'. Extending scalars to $\overline{k}$ and using Theorem 2.1.5 we see that $\sigma \cong \sigma'$, except possibly when σ and σ' are twists of Sym^0 or Sym^{p-1}. However, in this case we have that

$$c\text{-}\mathrm{Ind}_{KZ}^{G}(\mathrm{Sym}^{p-1})[1/(T^2-1)] \cong c\text{-}\mathrm{Ind}_{KZ}^{G}(\mathrm{Sym}^{0})[1/(T^2-1)],$$

and we are done since our assumption on $f(T)$ implies that Π has no Jordan–Hölder factors isomorphic to characters or Steinberg twists. $\square$

We note the following easy corollaries of these classification results.

Corollary 2.1.8. *Let σ_1, σ_2 be Serre weights and let $f_i \in \mathcal{H}(\sigma_i)$ be irreducible monic polynomials of degree greater than one. Then $c\text{-}\mathrm{Ind}_{KZ}^{G}(\sigma_1)/f_1 \cong c\text{-}\mathrm{Ind}_{KZ}^{G}(\sigma_2)/f_2$ if and only if $\sigma_1 = \sigma_2$ and $f_1 = f_2$.*

Proof. Passing to a finite extension l/k where both f_1 and f_2 split, this is an immediate consequence of Theorem 2.1.5, Theorem 2.1.7, and [Paš13, Lemma 5.1]. $\square$

Corollary 2.1.9. *Let σ be a Serre weight. If π' is a non-zero subobject of $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$, then $(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/\pi'$ is of finite length.*

Proof. Corollary 2.1.4 allows us to reduce to the case $\pi' = f(T)c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$ for some non-zero $f(T) \in k[T]$, and then by factoring $f(T)$, we reduce to the case that $f(T)$ is irreducible, in which case the corollary follows from Theorem 2.1.7 and Lemma 2.1.6. $\square$

We will use the following result in the proof of Lemma 3.6.6.

Lemma 2.1.10. *Let σ be a Serre weight, and g be a non-zero irreducible element of $\mathcal{H}(\sigma)$. Then $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]/(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is Artinian, and is in fact an essential extension of $\frac{1}{g}c\text{-}\mathrm{Ind}_{KZ}^G \sigma/c\text{-}\mathrm{Ind}_{KZ}^G \sigma$.*

Proof. Write $\Pi := c\text{-}\mathrm{Ind}_{KZ}^G \sigma$. It is enough to show that for each $n \geq 1$,

$$\frac{1}{g}\Pi \to \frac{1}{g^n}\Pi$$

is an essential extension. Let Θ be a sub-G-representation of $\frac{1}{g^n}\Pi/\Pi$; we can and do assume that Θ is not contained in $\frac{1}{g^{n-1}}\Pi/\Pi$. Let $\overline{\Theta}$ be the image of Θ under the isomorphism

$$\frac{1}{g^n}\Pi/\Pi \xrightarrow{\sim} \Pi/g^n\Pi$$

given by multiplication by g^n; we need to show $\overline{\Theta}$ has non-trivial intersection with $g^{n-1}\Pi/g^n\Pi$. By Corollary 2.1.4, $\overline{\Theta}$ is stable under the action of $\mathcal{H}(\sigma)$, and in particular under the action of g. We have $g^{n-1}\overline{\Theta} \neq 0$ by our assumption that Θ is not contained in $\frac{1}{g^{n-1}}\Pi$, and $g^{n-1}\overline{\Theta} \subseteq \overline{\Theta} \cap (g^{n-1}\Pi/g^n\Pi)$, as required. □

In fact, the Artinian property for locally admissible representations follows from finiteness of the socle, by Paškūnas's results on the structure of blocks. We prove this in the following proposition.

Proposition 2.1.11. *Let J be an injective object of $\mathcal{A}^{\mathrm{l.adm}}$. Then J is Artinian if and only if $\mathrm{soc}_G(J)$ has finite length.*

Proof. One direction is immediate, so we assume that $\mathrm{soc}_G(J)$ is finite-dimensional and prove that J is Artinian. After a finite extension of scalars, J is isomorphic to a direct sum of injective envelopes of absolutely irreducible objects, hence by [Paš13, Corollary 5.18] it suffices to prove that indecomposable injective objects of any absolutely irreducible block of $\mathcal{A}^{\mathrm{l.adm}}$ are Artinian. This is a consequence of one of the main results of [Paš13], which shows that the endomorphism rings of projective generators of blocks are right and left Noetherian: in fact, they are finite modules over their centres, which are Noetherian rings. □

We will also find the following refinement of Lemma 2.1.10 useful.

Lemma 2.1.12. *Let σ be a Serre weight, and $g = T - \lambda \in \mathcal{H}(\sigma)$ for some $\lambda \in k$. Then $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]/(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is uniserial.*

Proof. Except in the case when σ is a twist of Sym^0 or Sym^{p-1} and $g = T \pm 1$, we know by Theorem 2.1.7 that $c\text{-}\mathrm{Ind}_{KZ}^G \sigma/g(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is irreducible, in which the lemma is an easy consequence of Lemma 2.1.10. If we are in one of the remaining cases, we may twist so that $\sigma = \mathrm{Sym}^0$ or Sym^{p-1} and $g = T - 1$, and so we assume this from now on.

Since we are working modulo powers of $T - 1$, we localize each of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma^0$ and $c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1}$ at the prime ideal $(T - 1)$ of $k[T]$, so as to obtain a chain of inclusions

$$\begin{aligned}(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0)_{(T-1)} &\supset (c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1})_{(T-1)} \\ &\supset (T-1)(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0)_{(T-1)} \supset (T-1)(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1})_{(T-1)}\end{aligned}$$

with successive quotients equal to 1_G (the trivial representation of G), St_G, and 1_G again. Since any non-isomorphic map of $(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0)_{(T-1)}$ to itself factors through the quotient by $(T-1)(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^0)_{(T-1)}$, and similarly for Sym^{p-1}, we see that the induced extensions of 1 by St_G and of St_G by 1 are non-split. This easily proves the uniseriality in the cases of Sym^0 and Sym^{p-1} for $g = T-1$ (and thus completes the proof of the lemma). □

2.2. **Blocks, and a chain of $\mathbf{P}^1$s.** We now discuss the full subcategory $\mathcal{A}^{\mathrm{l.adm}}$ of $\mathcal{A}$ consisting of *locally admissible* representations. Recall [Eme10a, Defn. 2.2.15] that a representation $\pi \in \mathcal{A}$ is locally admissible if every vector $v \in \pi$ is smooth and generates an admissible representation, and is locally finite if for every v the representation generated by v is of finite length. In this setting locally admissible representations are locally finite by [Eme10a, Thm. 2.2.17]. Hence the category $\mathcal{A}^{\mathrm{l.adm}}$ is locally finite, unlike $\mathcal{A}$, and so admits a decomposition into blocks. These blocks are studied intensively in [Paš13].

Recall that, by definition, a block of $\mathcal{A}^{\mathrm{l.adm}}$ is an equivalence class of (isomorphism classes of) irreducible objects[1] under the equivalence relation generated by

$$\pi_1 \sim \pi_2 \text{ if } \mathrm{Ext}^1_{\mathcal{A}}(\pi_1, \pi_2) \neq 0 \text{ or } \mathrm{Ext}^1_{\mathcal{A}}(\pi_2, \pi_1) \neq 0.$$

As in the proof of [Paš13, Proposition 5.34], or equivalently using the spectral sequence (2.5.5), we get the same equivalence relation if we work instead with $\mathrm{Ext}^1_{\mathcal{A}_{\mathbf{F}}}$, which we will do in what follows. The blocks $\mathfrak{B}$ that contain absolutely irreducible representations are as follows:

(1) $\mathfrak{B} = \{\pi\}$ for an irreducible supersingular representation π,
(2) $\mathfrak{B} = \{\mathrm{Ind}_B^G(\chi_1 \otimes \omega^{-1}\chi_2), \mathrm{Ind}_B^G(\chi_2 \otimes \omega^{-1}\chi_1)\}$ for characters $\chi_1, \chi_2 : \mathbf{Q}_p^\times \to \mathbf{F}^\times$ such that $\chi_1\chi_2^{-1} \neq 1, \omega^{\pm 1}$,
(3) $\mathfrak{B} = \{\mathrm{Ind}_B^G(\chi \otimes \omega^{-1}\chi)\}$ for a character $\chi : \mathbf{Q}_p^\times \to \mathbf{F}^\times$, and
(4) $\mathfrak{B} = \{\chi, \chi \otimes \mathrm{St}_G, \mathrm{Ind}_B^G(\omega\chi \otimes \omega^{-1}\chi)\}$ for a character $\chi : \mathbf{Q}_p^\times \to \mathbf{F}^\times$.

The following Proposition completes the classification of the blocks; the proof makes use of Theorem 2.1.7 and [Paš13, Proposition 5.33], asserting that Ext^i commutes with finite extensions of $\mathbf{F}$ for representations of finite length (see Proposition 2.5.13 for a generalization to $\mathcal{A}^{\mathrm{fg}}$).

Proposition 2.2.1. *Let $\sigma = \mathrm{Sym}^r \otimes \det^s$ be a Serre weight, and let $f \in \mathcal{H}(\sigma)$ be an irreducible polynomial of degree $n > 1$. Let $\pi = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)/f(T)$, and let π' be an irreducible object of $\mathcal{A}$ not isomorphic to π. Then π and π' are in the same block of $\mathcal{A}^{\mathrm{l.adm}}$ if and only if*

$$\pi' \cong c\text{-}\mathrm{Ind}_{KZ}^G(\sigma')/f^*(T)$$

for $\sigma' = \mathrm{Sym}^{p-3-r} \otimes \det^{r+s+1}$ and $f^*(T) = T^n f(1/T)$.

Remark 2.2.2. By Lemma 2.1.3 we can assume without loss of generality that $0 \leq r \leq p-2$. If $r = p-2$, the exponent $p-3-r$ denotes $p-2$.

Proof of Proposition 2.2.1. Assume that $\pi \sim \pi'$, and let $\pi_0 = \pi, \pi_1, \dots, \pi_m = \pi'$ be irreducible objects of $\mathcal{A}$ such that $\mathrm{Ext}^1(\pi_j, \pi_{j+1}) \neq 0$ or $\mathrm{Ext}^1(\pi_{j+1}, \pi_j) \neq 0$ for all j. Since the map $(\sigma, f) \mapsto (\sigma', f^*)$ is an involution, by induction on m it suffices

[1] We note that it follows from the results of [BL94] and [Bre03a] that the irreducible objects of $\mathcal{A}$ are automatically admissible, and hence lie in $\mathcal{A}^{\mathrm{l.adm}}$. Thus we can equally well regard this as an equivalence relation on the irreducible objects of $\mathcal{A}$.

to prove that $\pi_1 \cong \pi$ or $\pi_1 \cong c\text{-Ind}_{KZ}^G(\sigma')/f^*(T)$. Passing to a finite extension of $\mathbf{F}$ and using [Paš13, Proposition 5.33] this is immediate from the classification of blocks containing principal series representations.

Conversely, assume that $\pi' \cong c\text{-Ind}_{KZ}^G(\sigma')/f^*(T)$. Again by [Paš13, Proposition 5.33] it suffices to prove that $\mathrm{Ext}^1_{\mathcal{A}_{\mathbf{F}'}}(\pi \otimes_{\mathbf{F}} \mathbf{F}', \pi' \otimes_{\mathbf{F}} \mathbf{F}') \neq 0$, where $\mathbf{F}'$ is a splitting field of f over $\mathbf{F}$. Since

$$\mathrm{Ext}^1_{\mathcal{A}_{\mathbf{F}'}}(c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda), c\text{-Ind}_{KZ}^G(\sigma')/(T-\lambda^{-1})) \neq 0$$

for all roots λ of f in $\mathbf{F}'$, this is immediate since $\mathbf{F}'/\mathbf{F}$ is separable. $\square$

Remark 2.2.3. In the more general case of $\mathcal{A}_k$ with coefficients in a field k of characteristic p, it could be the case that the splitting field l of f is inseparable over k. This situation can be treated using the fact that for all $i, j > 0$ we have

$$\mathrm{Ext}^1_{\mathcal{A}_l}(c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda)^i, c\text{-Ind}_{KZ}^G(\sigma')/(T-\lambda^{-1})^j) \neq 0,$$

which follows from the long exact sequences in Ext associated to

$$0 \to c\text{-Ind}_{KZ}^G(\sigma')/(T-\lambda^{-1})^{j-1} \to c\text{-Ind}_{KZ}^G(\sigma')/(T-\lambda^{-1})^j \to c\text{-Ind}_{KZ}^G(\sigma')/(T-\lambda^{-1}) \to 0$$

and

$$0 \to c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda) \to c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda)^j \to c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda)^{j-1} \to 0.$$

Now we let X denote a chain of $\mathbf{P}^1$'s over $\mathbf{F}$ with ordinary double points, of length $(p \pm 1)/2$, where the sign is positive if and only if ζ is odd. We choose coordinates on each irreducible component of X in such a way that each singular point corresponds to 0 on one intersecting component and ∞ on the other. We will refer to the points 0 and ∞ as *marked points*. We are going to label the components of X by cuspidal types $\Theta(\chi)$ with central character $\zeta = \chi^{p+1}$; note that since $\Theta(\chi) \cong \Theta(\chi')$ if and only if $\chi' = \chi$ or $\chi' = \chi^p$, there are as many cuspidal types with central character ζ as irreducible components of X. In order to construct the labelling, it will be useful to have the following definition.

Definition 2.2.4. We will say that two cuspidal types τ_1, τ_2 are *adjacent* if there exist $\sigma_i \in \mathrm{JH}(\overline{\tau}_i)$ such that $\{\sigma_1, \sigma_2\}$ is the set of Jordan–Hölder factors of a principal series representation of $\mathbf{F}[\mathrm{GL}_2(\mathbf{F}_p)]$.

Recall from [BP12, Corollary 5.6] that the extensions amongst irreducible $\mathbf{F}[\mathrm{GL}_2(\mathbf{F}_p)]$-representations are classified by the following proposition.

Proposition 2.2.5. *Let $\sigma = \mathrm{Sym}^b\, \mathbf{F}_p^2 \otimes \det^a$. Then one of the following is true:*

(1) there exist exactly two Serre weights σ_1, σ_2 such that $\mathrm{Ext}^1(\sigma, \sigma_i)$ is nonzero. One of these extensions is the mod p reduction of a lattice in a cuspidal type, and the other is the mod p reduction of a lattice in a principal series type.

(2) $b = p-2$, there exists a cuspidal type τ with $\overline{\tau} = \{\sigma\}$, and there exists a unique Serre weight σ_1 such that $\mathrm{Ext}^1(\sigma, \sigma_1) \neq 0$. This nonsplit extension is the mod p reduction of a lattice in a principal series type.

(3) $b = 0$, and there exists a unique Serre weight σ_1 such that $\mathrm{Ext}^1(\sigma, \sigma_1) \neq 0$. This nonsplit extension is the mod p reduction of a lattice in a cuspidal type.

(4) $b = p-1$, and σ is a projective $\mathbf{F}[\mathrm{GL}_2(\mathbf{F}_p)]$-module.

Corollary 2.2.6. *Let τ be a cuspidal type. Then one of the following is true:*

(1) there exist exactly two cuspidal types adjacent to τ, or
(2) $\overline{\tau}$ contains a twist of Sym^0 *or* Sym^{p-2}*, and there exists precisely one cuspidal type adjacent to τ.*

Proof. This is immediate from Proposition 2.2.5. □

Proposition 2.2.7. *There exist exactly two bijections $\tau \mapsto X(\tau)$, from the set of cuspidal types with central character ζ to the set of irreducible components of X, with the following property: $X(\tau_1) \cap X(\tau_2)$ is not empty if and only if τ_1 and τ_2 are adjacent.*

Proof. Assume first that ζ is odd. Twisting by the determinant, we may assume that $\zeta|_{\mu_{p-1}(\mathbf{Q}_p)} = \omega^{-1}$. Then there exists a cuspidal type τ with central character ζ such that $\overline{\tau} \cong \mathrm{Sym}^{p-2}$, which therefore needs to be sent to one of the components with only one singular point. By Corollary 2.2.6, the bijection is determined by choosing which one.

Similarly, if ζ is even we can twist and assume that $\zeta|_{\mu_{p-1}(\mathbf{Q}_p)} = 1$. In this case, the bijection is determined by the image of the cuspidal type whose reduction contains the trivial $\mathbf{F}[\mathrm{GL}_2(\mathbf{F}_p)]$-representation. □

In what follows we will make an arbitrary choice amongst the two bijections constructed in Proposition 2.2.7.

Remark 2.2.8. This ambiguity is related to the fact that the group $\mathbf{Z}/2 \times \mathbf{Z}/2$ acts on the category $\mathcal{A}$ by autoequivalences arising from twisting by the quadratic characters $\mathbf{Q}_p^\times \to \mathbf{F}^\times$, since twisting by ramified characters changes the K-socle of an irreducible object of $\mathcal{A}$. On the other hand, twisting by the unramified quadratic character gives rise to an isomorphism

$$(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \otimes \det{}^s)/(T+\lambda) \cong (\mathrm{nr}_{-1} \circ \det) \otimes (c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \otimes \det{}^s)/(T-\lambda).$$

We record an additional property of the map $\tau \mapsto X(\tau)$.

Lemma 2.2.9. *If τ_1, τ_2 are adjacent cuspidal types, the pair (σ_1, σ_2) in Definition 2.2.4 is unique.*

Proof. Let σ_1', σ_2' be the other Jordan–Hölder factors of $\overline{\tau}_i$, if any exist. Let $x = \dim \sigma_1$. Then $\dim \sigma_2 = p + 1 - x$, $\dim \sigma_1' = p - 1 - x$, and $\dim \sigma_2' = x - 2$. Since σ_2 is not isomorphic to σ_2', and there is a principal series extension between σ_1 and σ_2, there is no principal series extension between σ_1 and σ_2'. Similarly, there is no principal series extension between σ_1' and σ_2. Finally, $\dim \sigma_1' + \dim \sigma_2' = p - 3$, so there is no principal series extension between σ_1' and σ_2'. □

The following corollary is immediate (recalling that by definition, a principal series type is irreducible).

Corollary 2.2.10. *The map $\tau \mapsto X(\tau)$ induces a bijection from singular points of X to isomorphism classes of principal series types.*

2.3. **Coordinates on X.** Choose a Serre weight σ (with compatible central character). Recall that we have chosen coordinates on each irreducible component of X in such a way that each singular point corresponds to 0 on one intersecting component and ∞ on the other. We will use the isomorphism (2.1.2) and the map $\tau \mapsto X(\tau)$ to construct morphisms from the spectra of various Hecke algebras $\mathcal{H}(\sigma)$ to X. Thus we will be able to regard $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ as "lying over" a copy of

$\mathbf{A}^1 = \operatorname{Spec} \mathbf{F}[T] = \operatorname{Spec} \mathcal{H}(\sigma)$, which will form the basis of our localization theory for $\mathcal{A}_A$.

Definition 2.3.1. Let σ be a Serre weight, and assume σ is not a twist of Sym^{p-1}. Then σ is contained in a unique cuspidal type τ. Define a map $f_\sigma : \operatorname{Spec} \mathcal{H}(\sigma) \to X$ in the following way.

(1) If σ is not a twist of $\operatorname{Sym}^0, \operatorname{Sym}^{p-2}$, then it is contained in a unique principal series type τ'. Then f_σ is the inclusion of $\operatorname{Spec} \mathcal{H}(\sigma) \cong \mathbf{A}^1$ in $X(\tau)$ preserving the given coordinates and sending 0 to the point of $X(\tau)$ corresponding to τ' under Corollary 2.2.10.
(2) If $\sigma \cong (\chi \circ \det) \otimes \operatorname{Sym}^0$, then f_σ is the inclusion of $\operatorname{Spec} \mathcal{H}(\sigma) \cong \mathbf{A}^1$ in $X(\tau)$ preserving the given coordinate and sending 0 to the nonsingular marked point.
(3) If $\sigma \cong (\chi \circ \det) \otimes \operatorname{Sym}^{p-2}$, then f_σ is the degree-two map $\mathbf{A}^1 \to X(\tau), (x \mapsto x + x^{-1})^{\pm 1}$. (The sign is chosen so that 0 is sent to the singular point.)

We extend this definition to the case $\sigma \cong (\chi \circ \det) \otimes \operatorname{Sym}^{p-1}$ by letting f_σ be the same map as in Case (2) for $\sigma \cong (\chi \circ \det) \otimes \operatorname{Sym}^0$.

Remark 2.3.2. The definition in the case of twists of Sym^{p-2} is motivated by the existence of extensions between nonisomorphic irreducible quotients of $c\text{-}\operatorname{Ind}_{KZ}^G(\sigma)/(T-\lambda)$, which does not happen for any other weights.

The definition for twists of Sym^{p-1} and Sym^0 is motivated by the isomorphism

$$c\text{-}\operatorname{Ind}_{KZ}^G(\operatorname{Sym}^{p-1}) \left[\frac{1}{T^2-1}\right] \cong c\text{-}\operatorname{Ind}_{KZ}^G(\operatorname{Sym}^0) \left[\frac{1}{T^2-1}\right]$$

constructed in Lemma 2.1.3. Furthermore, by Lemma 2.1.6 if $T = \pm 1$ we still have an isomorphism after semisimplification: more precisely,

$$\mathrm{JH}\left(c\text{-}\operatorname{Ind}_{KZ}^G(\operatorname{Sym}^{p-1})/(T \pm 1)\right) = \mathrm{JH}\left(c\text{-}\operatorname{Ind}_{KZ}^G(\operatorname{Sym}^0)/(T \pm 1)\right).$$

The key property of the maps f_σ is given in the following proposition.

Proposition 2.3.3. *Let x be a closed point of X. Define*

$$\mathfrak{B}_x = \bigcup_{\sigma \text{ s.t. } x \in f_\sigma(\mathbf{A}^1)} \mathrm{JH}\left(c\text{-}\operatorname{Ind}_K^G(\sigma) \otimes_{\mathcal{H}(\sigma)} x\right).$$

Then $\mathfrak{B}_x$ is a block of $\mathcal{A}_{\mathbf{F}}$, and the map $x \mapsto \mathfrak{B}_x$ is a bijection from the set of closed points of X to the set of blocks of $\mathcal{A}_{\mathbf{F}}$.

Proof. We begin with the case of $\mathbf{F}$-rational points $x \in X(\mathbf{F})$. Assume first that ζ is even and x is not a marked point. There are two possibilities for the set of weights σ such that $f_\sigma(\mathbf{A}^1)$ contains x: up to twist, it has the form

$$\{\operatorname{Sym}^i, \operatorname{Sym}^{p-3-i} \otimes \det^{i+1}\}$$

for $i \neq 0$ (which are the factors of a cuspidal type), or

$$\{\operatorname{Sym}^0, \operatorname{Sym}^{p-1}, \operatorname{Sym}^{p-3} \otimes \det\}$$

(which are the factors of a cuspidal type, together with a Steinberg weight). By Remark 2.3.2 it follows that in either case there exists $\lambda \in k^\times$ such that

$$\mathfrak{B}_x = \mathrm{JH}\left(c\text{-}\operatorname{Ind}_{KZ}^G(\operatorname{Sym}^i)/(T-\lambda)\right) \cup \mathrm{JH}\left(c\text{-}\operatorname{Ind}_{KZ}^G(\operatorname{Sym}^{p-3-i} \otimes \det^{i+1})/(T-\lambda^{-1}\right),$$

which in the terminology of Section 2.2 is a block of type (2) if $i \neq 0$ and a block of type (4) if $i = 0$.

Now assume that x is a marked point. Up to twist, the set of σ such that $f_\sigma(\mathbf{A}^1)$ contains x has the form
$$\{\mathrm{Sym}^i, \mathrm{Sym}^{p-1-i} \otimes \det{}^i\}$$
which are the factors of a principal series type. We have
$$\mathfrak{B}_x = \mathrm{JH}\left(c\text{-}\mathrm{Ind}_{KZ}^G(\mathrm{Sym}^i)/T\right) \cup \mathrm{JH}\left(c\text{-}\mathrm{Ind}_{KZ}^G(\mathrm{Sym}^{p-1-i} \otimes \det{}^i)/T\right),$$
but these are two isomorphic supersingular irreducible representations, so we get a block of type (1).

The analysis in the case that ζ is odd is similar. The only new case is when x is a point of a component indexed by a cuspidal type whose reduction is isomorphic to a twist of $\sigma = \mathrm{Sym}^{p-1}$ In this case, if x is a singular point of X then $f_\sigma(x)^{-1} = 0$, so again $\mathfrak{B}_x$ is a block of type (1). On the other hand, if x is a regular point of X then there exists $\lambda \in \mathbf{F}^\times$ such that $f_\sigma^{-1}(x) = \{\lambda, \lambda^{-1}\}$. The corresponding set
$$\mathfrak{B}_x = \left\{c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)/(T-\lambda), c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)/(T-\lambda^{-1})\right\}$$
is a block of type (2) if $\lambda \notin \{\pm 1\}$ and a block of type (3) if $\lambda \in \{\pm 1\}$.

This defines a map from $X(\mathbf{F})$ to the set of blocks of $\mathcal{A}^{\mathrm{l.adm}}$ containing absolutely irreducible representations, and the classification in Section 2.2 implies that it is a bijection. There remains to treat the case where x is not $\mathbf{F}$-rational. Then x corresponds under f_σ to a maximal ideal of $\mathcal{H}(\sigma)$ generated by an irreducible polynomial f of degree $n > 1$, and
$$c\text{-}\mathrm{Ind}_{KZ}^G(\sigma) \otimes_{\mathcal{H}(\sigma)} x = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)/(f)$$
is irreducible. Proposition 2.2.1 shows that $\mathfrak{B}_x$ is a block consisting of two non-isomorphic irreducible representations, neither of which is absolutely irreducible. Again by Proposition 2.2.1, the map $x \mapsto \mathfrak{B}_x$ defines a bijection from the set of closed points of X not defined over $\mathbf{F}$ to the set of blocks of $\mathcal{A}_{\mathbf{F}}$ not containing absolutely irreducible representations. This completes the proof. □

2.4. **Categories of representations.** Recall that we assume throughout the paper that A is a complete local Noetherian $\mathcal{O}$-algebra with residue field of characteristic p. We say that a representation of G on an A-module is finitely generated if it is finitely generated as an $A[G]$-module. We note that a smooth representation of G on an A-module, admitting a central character, is finitely generated if and only if it is a quotient of a compactly induced representation $c\text{-}\mathrm{Ind}_{KZ}^G V$, for some finite length A-module V endowed with a smooth action of KZ. Any object of $\mathcal{A}_A$ is thus a colimit of objects of the form $c\text{-}\mathrm{Ind}_{KZ}^G V$, where V is a finite length A-module endowed with a smooth KZ-action.

We frequently use the following result:

Theorem 2.4.1. *If A is a complete Noetherian local $\mathcal{O}$-algebra, then any subrepresentation of a finitely generated smooth representation of G over A with fixed central character $\zeta : Z \to A^\times$ is again finitely generated. Equivalently, any finitely generated smooth representation of G over A with central character ζ is Noetherian.*

Proof. Let π be a finitely generated smooth $A[G]$-representation. By definition, π is a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G(V)$ for some smooth $A[KZ]$-module V of finite length over $A[KZ]$. Since quotients, subobjects and extensions of Noetherian objects are

Noetherian, it suffices to prove the theorem when $\pi = c\text{-}\mathrm{Ind}_{KZ}^G(V)$, and it further suffices to prove it in the case that V has length one, hence is an irreducible $k[KZ]$-module. In this case the result follows from Corollary 2.1.9. □

Remark 2.4.2. Theorem 2.4.1 does not hold in general for non-abelian p-adic reductive groups. For example, it fails for $\mathrm{GL}_2(\mathbf{Q}_{p^2})$, and presumably any time we're outside the case of $\mathrm{GL}_2(\mathbf{Q}_p)$, or some essentially equivalent context.

The following Lemma (and its proof) goes through unchanged for general p-adic reductive groups.

Lemma 2.4.3. *If A is a complete Noetherian local $\mathcal{O}$-algebra, then $\mathcal{A}_A$ is a Grothendieck category.*

Proof. Since the category of A-modules is a Grothendieck category, it is enough to show that $\mathcal{A}_A$ has a generator. To form a generator of $\mathcal{A}_A$, let W run over a set of isomorphism class representatives of smooth finite length $A[KZ]$-modules. Since K has a cofinal countable system of open normal subgroups with finite quotient, these representatives form a countable set. We can take the generator to be $\bigoplus_W c\text{-}\mathrm{Ind}_{KZ}^G W$. □

Corollary 2.4.4. *If A is a complete Noetherian local $\mathcal{O}$-algebra, the abelian category $\mathcal{A}_A$ is locally Noetherian. The Noetherian objects of $\mathcal{A}_A$ are precisely the finitely generated objects.*

Proof. By Lemma 2.4.3, filtered colimits are exact in $\mathcal{A}_A$; so in order to see that $\mathcal{A}_A$ is locally Noetherian, we only need to exhibit a set of Noetherian generators of $\mathcal{A}_A$. By Theorem 2.4.1, the compact inductions $c\text{-}\mathrm{Ind}_{KZ}^G(V)$ are Noetherian whenever V is a finite length smooth $A[KZ]$-module, since the finite length quotients of A are Artinian rings. To see that these are generators, let $f : \pi_1 \to \pi_2$ be a nonzero morphism between objects of $\mathcal{A}_A$. We need to find a morphism $c\text{-}\mathrm{Ind}_{KZ}^G(V) \to \pi_1$ whose composition with f is nonzero. To do so, it suffices to choose $v \in \pi_1$ such that $f(v) \neq 0$, and then notice that it generates a finite length $A[KZ]$-module, since π_1 is a smooth representation. This concludes the proof of the first assertion.

By Theorem 2.4.1, the finitely generated objects of $\mathcal{A}_A$ are Noetherian, and the converse is immediate from the definition. □

2.5. **Generalities about** Ext**'s in** $\mathcal{A}_A$**.** The inclusion of $\mathcal{A}_A$ into the category of all $A[G]$-modules with central character ζ is exact and fully faithful, and admits a right adjoint, given by passing to the submodule consisting of smooth vectors annihilated by some power of $\mathfrak{m}_A$. This right adjoint preserves injectives, and so since the latter category admits enough injectives (being the category of modules over a ring), so does the category $\mathcal{A}_A$. Of course, since $\mathcal{A}_A$ is a Grothendieck category, it also admits enough injectives for abstract reasons.

Recall that the functor $c\text{-}\mathrm{Ind}_{KZ}^G$ is an exact functor from $\mathcal{C}_A$ to $\mathcal{A}_A$. It is left adjoint to the forgetful functor (i.e. restriction) from $\mathcal{A}_A$ to $\mathcal{C}_A$, and so this latter functor preserves injectives. Thus we have natural isomorphisms

$$\mathrm{Ext}^i_{\mathcal{A}_A}(c\text{-}\mathrm{Ind}_{KZ}^G V, \pi) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{C}_A}(V, \pi) \tag{2.5.1}$$

whenever V is a object of $\mathcal{C}_A$ and π is an object of $\mathcal{A}_A$. We will use (2.5.1) to reduce some questions about Ext groups in $\mathcal{A}_A$ to Ext groups in $\mathcal{C}_A$. We begin with the following basic statement about these Ext groups, whose statement and proof extend in an obvious way to arbitrary compact p-adic analytic groups.

Lemma 2.5.2. *Assume that $(A, \mathfrak{m}, k)$ is a complete Noetherian local $\mathcal{O}$-algebra and that V, W are objects of $\mathcal{C}_A$ which are finitely generated over A. Then*

$$\mathrm{Ext}^i_{\mathcal{C}_A}(V, W)$$

is finitely generated over A.

Proof. By induction on $\mathrm{length}_K(V)$, it suffices to prove the lemma when V is irreducible. We begin by proving that $\mathrm{Ext}^i_{\mathcal{C}_k}(V, W)$ is finite-dimensional over k whenever W is an object of $\mathcal{C}_k$ of finite dimension over k. By induction on $\mathrm{length}_K(W)$, it suffices to prove this claim when W is irreducible. Let K_1 be the first congruence subgroup of K, which acts trivially on V and W. Since

$$\mathrm{Hom}_{\mathcal{C}_k}(V, -) = \mathrm{Hom}_{k[KZ/K_1]/I_\zeta}(V, (-)^{K_1})$$

there is a spectral sequence

$$\mathrm{Ext}^i_{k[KZ/K_1]/I_\zeta}(V, H^j(K_1, W)) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{C}_k}(V, W). \tag{2.5.3}$$

The group K_1 acts trivially on W, which is a finite-dimensional k-vector space, and so $H^j(K_1, W)$ is also finite-dimensional for all j: see for example [SW00, Cor. 4.2.5, Thm. 5.1.2] for a discussion of this fact, which follows from Lazard's results on the structure of $k[[K]]$. Finally, since $k[KZ/K_1]/I_\zeta$ is a finite k-algebra, we see that the groups on the E_2-page are also finite-dimensional, completing the proof of the claim.

Now let W be an A-finite object of $\mathcal{C}_A$. Since A is Noetherian we know that $\mathrm{Ext}^j_A(k, W)$ is a finite k-vector space for all j, hence the lemma follows from the claim proved before and the spectral sequence (2.5.5) to follow. $\square$

Lemma 2.5.4. *Let $(A, \mathfrak{m}, k)$ be a complete Noetherian local $\mathcal{O}$-algebra, and let V, W be objects of $\mathcal{C}_k$. Then there is a spectral sequence*

$$\mathrm{Ext}^i_{\mathcal{C}_k}(V, \mathrm{Ext}^j_A(k, W)) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{C}_A}(V, W). \tag{2.5.5}$$

The same is true with $\mathcal{A}_k$ replacing $\mathcal{C}_k$ and $\mathcal{A}_A$ replacing $\mathcal{C}_A$.

Proof. Notice that

$$\mathrm{Hom}_{\mathcal{C}_A}(V, -) = \mathrm{Hom}_{\mathcal{C}_k}(V, (-)[\mathfrak{m}])$$

as functors from $\mathcal{C}_A$ to k-vector spaces.

If I is an injective object of $\mathcal{C}_A$, then $I[\mathfrak{m}]$ is injective in $\mathcal{C}_k$, since $(-)[\mathfrak{m}]$ is right adjoint to the inclusion of $\mathcal{C}_k$ into $\mathcal{C}_A$, which is exact. Furthermore, I is also injective as an A-module, so that the right derived functors of $(-)[\mathfrak{m}]$ on $\mathcal{C}_A$ coincide with (the composite of the forgetful functor to A-modules and) the right derived functors of $(-)[\mathfrak{m}]$ on the category of A-modules. To see this A-module injectivity of I, notice that if $\mathfrak{a} \to A$ is the inclusion of an ideal and $\varphi : \mathfrak{a} \to I$ is A-linear then $\varphi(\mathfrak{a})$ is contained in I^{K_0} for some open normal subgroup $K_0 \subset K$, since A is Noetherian. Viewing $\mathfrak{a}$ and A as trivial K_0-modules, it follows that φ extends to a map $A \to I$, since I is also injective as a K_0-representation.

Putting together the observations of the preceding paragraphs gives a Grothendieck spectral sequence as in the statement of the lemma. The same proof works in the case of $\mathcal{A}_A$. (To see that the forgetful functor from $\mathcal{A}_A$ to A-modules preserves injectives, we can factor this functor through the forgetful functor to $\mathcal{C}_A$, and then combine what was proved above with the fact that restriction to a compact open subgroup preserves injectives.) $\square$

Lemma 2.5.6. *Assume A is a complete Noetherian local $\mathcal{O}$-algebra. If π is a finitely generated object of $\mathcal{A}_A$, then each $\mathrm{Ext}^i_{\mathcal{A}_A}(\pi, –)$ commutes with filtered colimits.*

Proof. This follows from Corollary 2.4.4 and Proposition A.1.1 (3). □

Lemma 2.5.7. *Assume that A is a complete Noetherian local $\mathcal{O}$-algebra. If π and π' are objects of $\mathcal{A}_A$, with π being finitely generated and π' being countably generated, then each $\mathrm{Ext}^i_{\mathcal{A}_A}(\pi, \pi')$ is a countably generated A-module.*

Proof. Since A is Noetherian, every submodule of a countably generated A-module is countably generated. Since π is finitely generated, and thus finitely presented by Theorem 2.4.1, a standard dimension-shifting argument reduces us to checking the claim in the case when $\pi = c\text{-}\mathrm{Ind}_{KZ}^G V$, for some finitely generated A-module V endowed with a smooth KZ-action. We then consider the isomorphisms

$$\mathrm{Ext}^i_{\mathcal{A}_A}(c\text{-}\mathrm{Ind}_{KZ}^G V, \pi') \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{C}_A}(V, \pi') \xrightarrow{\sim} \varinjlim_W \mathrm{Ext}^i_{\mathcal{C}_A}(V, W),$$

where W runs over the finitely generated $A[KZ]$-submodules of π'. (The first isomorphism is by (2.5.1), and the last isomorphism is proved in the same way as Lemma 2.5.6.) The directed set of such W contains a countable cofinal subset, since π' is countably generated, and thus the lemma follows from the fact that each $\mathrm{Ext}^i_{\mathcal{C}_A}(V, W)$ is finitely generated, by Lemma 2.5.2. □

We will sometimes make use of the following comparison (due to Paškūnas) between Ext groups in the locally admissible and smooth categories.

Lemma 2.5.8. *If π, π' are objects of $\mathcal{A}^{\mathrm{l.adm}}$, then $\mathrm{Ext}^i_{\mathcal{A}^{\mathrm{l.adm}}}(\pi, \pi') = \mathrm{Ext}^i_{\mathcal{A}}(\pi, \pi')$.*

Proof. This is immediate from [Paš13, Cor. 5.18]. □

A different (more elementary) comparison of Ext groups occurs if we consider $\mathcal{A}^{\mathrm{fg}}$ inside $\mathcal{A}$. The former category does not have enough injectives, but (as noted in A.1.2) we can define Ext^i in $\mathcal{A}^{\mathrm{fg}}$ via Yoneda extensions. We then have the following result.

Lemma 2.5.9. *If π and π' are objects of $\mathcal{A}^{\mathrm{fg}}$, then $\mathrm{Ext}^i_{\mathcal{A}^{\mathrm{fg}}}(\pi, \pi') \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{A}}(\pi, \pi')$.*

Proof. This follows from Corollary 2.4.4 and Lemma A.1.3. □

We next establish some base-change results about Ext groups.

Lemma 2.5.10. *If A is a finite-dimensional associative algebra over a field k, if I is an injective A-module, and if l is any extension of k, then $l \otimes_k I$ is an injective $l \otimes_k A$-module.*

Proof. We begin with the case that I is finite-dimensional over k. It follows from Baer's criterion that I is injective if and only if $I^\vee$ is projective over A^{op}. Similarly $l \otimes_k I$ is injective over $l \otimes_k A$ if and only if $(l \otimes_k I)^\vee \cong l \otimes_k I^\vee$ is projective over $l \otimes_k A^{\mathrm{op}}$. Thus it suffices to prove the analogue of the lemma for projective modules, and thus for free modules (using the characterization of projective modules as direct summands of free modules). The case of free modules is clear, and thus the lemma is proved in this case.

Since A and $l \otimes_k A$ are Noetherian, the property of being injective over these rings is preserved under the formation of filtered colimits. Since A^{op} is finite dimensional, we know that every simple A^{op}-module has a finite-dimensional projective envelope. Thus any simple A-module has a finite-dimensional injective envelope. It follows that any injective A-module is a filtered colimit of finite dimensional injective A-modules, since it contains an injective envelope of any finite-dimensional submodule. Hence the lemma follows from the finite-dimensional case. $\square$

Corollary 2.5.11. *If $k \subseteq l$ is an extension of fields, and if V is a finite dimensional object of $\mathcal{C}_k$, then for any object W of $\mathcal{C}_k$, the base-change map*

$$l \otimes_k \mathrm{Ext}^i_{\mathcal{C}_k}(V, W) \to \mathrm{Ext}^i_{\mathcal{C}_l}(l \otimes_k V, l \otimes_k W)$$

is a natural isomorphism.

Proof. The case $i = 0$ is true since V is finitely generated over a Noetherian quotient of $k[KZ]$. By dimension shifting, it then suffices to prove that $l \otimes_k I$ is injective if I is injective in $\mathcal{C}_k$.

The usual argument for proving Baer's criterion shows that a smooth KZ-representation I over k, with fixed central character ζ, is injective if and only $\mathrm{Ext}^1_{\mathcal{C}_k}(V, I) = 0$ for each finite-dimensional object V of $\mathcal{C}_k$. Since some open subgroup H of K acts trivially on V, we deduce that I is injective if and only if its submodule of invariants I^H is injective as a k-representation of KZ/H, with fixed central character, for each open subgroup H of K.

Assume that this holds. Then $(l \otimes_k I)^H = l \otimes_k I^H$, by the case $i = 0$, and Lemma 2.5.10 shows that $l \otimes_k I^H$ is injective as an l-representation of KZ/H, with central character ζ. We conclude that $l \otimes_k I$ is an injective object of $\mathcal{C}_l$, as claimed. $\square$

Corollary 2.5.12. *If $\mathcal{O} \subset \mathcal{O}'$ is an unramified extension, and if V is a finite dimensional object of $\mathcal{C}_{\mathbf{F}}$, then for any object W of $\mathcal{C}_{\mathbf{F}}$, the base-change map*

$$\mathbf{F}' \otimes_{\mathbf{F}} \mathrm{Ext}^i_{\mathcal{C}_{\mathcal{O}}}(V, W) \to \mathrm{Ext}^i_{\mathcal{C}_{\mathcal{O}'}}(\mathbf{F}' \otimes_{\mathbf{F}} V, \mathbf{F}' \otimes_{\mathbf{F}} W)$$

is a natural isomorphism.

Proof. For any i and j, there is a base-change isomorphism

$$\begin{aligned}\mathbf{F}' \otimes_{\mathbf{F}} \mathrm{Ext}^i_{\mathcal{C}_{\mathbf{F}}}\big(V, \mathrm{Ext}^j_{\mathcal{O}}(\mathbf{F}, W)\big) &\xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{C}_{\mathbf{F}'}}\big(\mathbf{F}' \otimes_{\mathbf{F}} V, \mathbf{F}' \otimes_{\mathcal{O}} \mathrm{Ext}^j(\mathbf{F}, W)\big)\\ &= \mathrm{Ext}^i_{\mathcal{C}_{\mathbf{F}'}}\big(\mathbf{F}' \otimes_{\mathbf{F}} V, \mathrm{Ext}^j_{\mathcal{O}'}(\mathbf{F}', \mathbf{F}' \otimes_{\mathbf{F}} W)\big),\end{aligned}$$

the first isomorphism following from Corollary 2.5.11, and the equality holding because $\mathcal{O}'$ is unramified over $\mathcal{O}$, so that the derived functors $\mathrm{RHom}_{\mathcal{O}'}(\mathbf{F}', \mathcal{O}' \otimes_{\mathcal{O}} -)$ and $\mathcal{O}' \otimes_{\mathcal{O}} \mathrm{RHom}_{\mathcal{O}}(\mathbf{F}, -)$ coincide on the category of $\mathcal{O}$-modules; both are computed by the complex

$$\mathcal{O}' \otimes_{\mathcal{O}} (-) \xrightarrow{\varpi \cdot} \mathcal{O}' \otimes_{\mathcal{O}} (-)$$

(where ϖ is a uniformizer of $\mathcal{O}$).

If we now consider the spectral sequence (2.5.5) for each of $\mathcal{O}$ and $\mathcal{O}'$, we find that these base-change isomorphisms abut to the base-change morphism in the statement of the present corollary, showing that this morphism is also an isomorphism, as claimed. $\square$

Proposition 2.5.13. *If l is an extension of k, and if π and π' are objects of $\mathcal{A}_k$, with π being finitely generated, then there is a natural isomorphism*

$$l \otimes_k \mathrm{Ext}^i_{\mathcal{A}_k}(\pi, \pi') \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{A}_l}(l \otimes_k \pi, l \otimes_k \pi'),$$

for each $i \geq 0$.

Similarly, if π and π' are objects of $\mathcal{A}_{\mathbf{F}}$ with π finitely generated, and $\mathcal{O} \subset \mathcal{O}'$ is an unramified extension, then there is a natural isomorphism

$$\mathbf{F}' \otimes_{\mathbf{F}} \operatorname{Ext}^i_{\mathcal{A}_{\mathcal{O}}}(\pi, \pi') \xrightarrow{\sim} \operatorname{Ext}^i_{\mathcal{A}_{\mathcal{O}'}}(\mathbf{F}' \otimes_{\mathbf{F}} \pi, \mathbf{F}' \otimes_{\mathbf{F}} \pi')$$

Remark 2.5.14. If we were working in the category of all $k[G]$-modules (with fixed central character ζ), then this result would be straightforward, since we would be able to compute the Ext's using a resolution of π by finite rank free $k[G]/I_\zeta$-modules. It is the fact that we are working in the category $\mathcal{A}_k$, i.e. that we have imposed smoothness, which makes the result less obvious.

We also remark that in the case when π is assumed to be of finite length, the result is proved by Paškūnas [Paš13, Proposition 5.33]

Proof of Prop. 2.5.13. Using Theorem 2.4.1 and dimension shifting, one easily reduces to the case when $\pi = c\text{-}\operatorname{Ind}_{KZ}^G V$ for some finitely generated smooth KZ-representation V. Since compact induction is compatible with extension of scalars, the proposition follows from Corollary 2.5.11, Corollary 2.5.12 and (2.5.1). $\square$

Remark 2.5.15. Our proof of Proposition 2.5.13 uses as input the fact that finitely generated objects of $\mathcal{A}_k$ and $\mathcal{A}_{\mathcal{O}}$ are Noetherian (i.e. Theorem 2.4.1). Bearing in mind Remark 2.4.2, we see that the argument, and perhaps also the result, won't extend to more general p-adic Lie groups as stated. It seems plausible that it should at least hold in general with "finitely generated" replaced by "finitely presented", and it's likely that Shotton's results [Sho20] about smooth finitely presented $\mathrm{GL}_2(F)$-representations can be used to show this for $\mathrm{GL}_2(F)$, for arbitrary p-adic fields F. (Note that for $\mathrm{GL}_2(\mathbf{Q}_p)$ all finitely generated representations are finitely presented by Theorem 2.4.1.)

2.6. **Bernstein centres of blocks.** The results in this section might be of independent interest, but they are quite technical in nature and will only be applied in Section 3.8. Let x denote a block of absolutely irreducible representations of $\mathcal{A}^{\mathrm{l.adm}}$, and write $\mathcal{A}_x$ for the category $\mathcal{A}_Y$ with $Y = \{x\}$. The paper [Paš13] describes an equivalence of $\mathcal{A}_x$ with the category of modules over a ring $\widetilde{E}_x$ defined as follows. Let $\pi_x = \bigoplus_{\pi \in x} \pi$ and choose an injective envelope $\pi_x \to J_x$, so that $P_x = J_x^\vee$ is a projective envelope of $\pi_x^\vee$. Let $\widetilde{E}_x$ be the endomorphism ring of J_x, which is naturally a topological ring. Then the functor $\operatorname{Hom}_G(-, J_x)$ is an equivalence of $\mathcal{A}_x$ with the category of compact left $\widetilde{E}_x$-modules, and so it defines an isomorphism between the Bernstein centre of $\mathcal{A}_x$ and the centre Z_x of $\widetilde{E}_x$.

Remark 2.6.1. If $\tau \in \mathcal{A}_x$ and J is an injective object in $\mathcal{A}_x$, and $E = \operatorname{End}_G(J)$, then E is a compact ring and $\operatorname{Hom}_G(\tau, J)$ is a compact E-module: this follows from [Gab62, Section IV.4]. Hence its Pontrjagin dual $\operatorname{Hom}_G(\tau, J)^\vee$ is a discrete E-module.

On the other hand, $\operatorname{Hom}_G(\tau, J)$ still makes sense for general $\tau \in \mathcal{A}$. If $\tau = c\text{-}\operatorname{Ind}_{KZ}^G \lambda$ for a finite length KZ-representation λ then the action of E on $\operatorname{Hom}_G(\tau, J)$ is continuous for the discrete topology on this module. This follows from [Paš15, Lemma 2.1]. Hence $\operatorname{Hom}_G(\tau, J)^\vee$ is a compact E-module.

If $\pi \in x$ is an irreducible object we will write $\pi \to J_\pi$ for an injective envelope of π in $\mathcal{A}_x$. It is often (but not always) the case that Z_x is isomorphic to $\operatorname{End}(J_\pi)$

for an irreducible object $\pi \in x$. More precisely, we have the following result of Paškūnas.

Theorem 2.6.2. *Let x be a block of absolutely irreducible objects of $\mathcal{A}$, and choose $\pi \in x$. Then the natural map $Z_x \to \mathrm{End}_G(J_\pi)$ is an isomorphism if x is a block of type (1), (2) or (4).*

Proof. This is [Paš13, Proposition 6.3] for type (1), [Paš13, Corollary 8.7] for type (2), and [Paš13, Corollary 10.78, Theorem 10.87] for type (4). $\square$

We will often apply this result together with the following proposition.

Proposition 2.6.3. *Let π be an absolutely irreducible object of $\mathcal{A}$. Let x be the block of $\mathcal{A}$ containing π, and choose an injective envelope $\pi \to J_\pi$ in $\mathcal{A}_x$. If σ is a Serre weight, then*

$$\mathrm{Hom}_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma, J_\pi)^\vee$$

is a cyclic module over the endomorphism ring $E_\pi = \mathrm{End}_{\mathcal{A}_x}(J_\pi)$.

Proof. This is proved in most cases in [HT15, Proposition 2.9]. We give a different, uniform proof as follows. Let $\mathfrak{m}_\pi$ be the Jacobson radical of E_π. By Nakayama's lemma for compact modules, it suffices to prove that $\mathrm{Hom}_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma, J_\pi)^\vee/\mathfrak{m}_\pi$ is at most one-dimensional. We know that

$$\mathrm{Hom}_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma, J_\pi)^\vee/\mathfrak{m}_\pi \cong \mathrm{Hom}_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma, J_\pi[\mathfrak{m}_\pi])^\vee$$

and the representation $J_\pi[\mathfrak{m}_\pi]$ contains π as a subquotient with multiplicity one. Then the proposition follows from the fact that $c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma$ has at most one locally admissible quotient X with socle π appearing with multiplicity one in X.

To see this, observe that X is necessarily a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma/f(T)$ for some Hecke operator $f(T)$, and since the socle of X is absolutely irreducible we have $f(T) = (T-\lambda)^n$ for some n. Since X contains π as a subquotient with multiplicity one, we have $n = 1$. Hence X is irreducible except possibly in the case that σ is a twist of Sym^0 or Sym^{p-1} and $\lambda = \pm 1$. However, in this case there are only two possibilities for X, and they are distinguished by their G-socle. $\square$

Now let σ be a Serre weight and choose $\lambda \in \mathbf{F}$. In Section 3.8 we will be working with the inverse system $\{\tau_n\}$ of finite length G-representations defined by

$$\tau_n = c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma/(T-\lambda)^n.$$

These are all contained in the same block, which we will denote by x. There is an action of the Hecke algebra $\mathcal{H}(\sigma)$ on τ_n for all n, and the transition maps are equivariant for this action. In fact, the action extends to the completion of $\mathcal{H}(\sigma)$ at the maximal ideal generated by $(T-\lambda)$, which is isomorphic to a power series ring in one variable. We will show that in most cases this action is induced by the Bernstein centre of the block $\mathcal{A}_x$.

Proposition 2.6.4. *In the notation of the previous paragraph, if x has type (1), (2) or (4) then there exists an element of the Bernstein centre Z_x of $\mathcal{A}_x$ inducing the Hecke operator T on τ_n for all n.*

Proof. Assume first that $\pi = \tau_1$ is irreducible, so that every composition factor of τ_i is isomorphic to π. If π' is another irreducible object of $\mathcal{A}_x$ and $\pi' \to J_{\pi'}$ is an injective envelope in $\mathcal{A}_x$ then the space $\mathrm{Hom}_{\mathcal{A}}(\tau_i, J_{\pi'})$ vanishes for all i, since $\mathrm{soc}_G(J_{\pi'}) \cong \tau_i$ is not a composition factor of τ_i. Hence the functor

$$M_\pi(-) = \mathrm{Hom}_{\mathcal{A}}(-, J_\pi)^\vee$$

is fully faithful on the full subcategory of $\mathcal{A}_x$ generated by the representations $\{\tau_n\}$. By Proposition 2.6.3 the E_π-module $M_\pi(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is cyclic. Hence there exists $T_\pi \in E_\pi$ such that $T = T_\pi$ as endomorphisms of $M_\pi(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$. By Theorem 2.6.2 the natural map $Z_x \to E_\pi$ is an isomorphism. Since $M_\pi(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ surjects onto $M_\pi(\tau_i)$ equivariantly for T and T_π, we deduce that there exists $T_\pi \in Z_x$ such that $T = T_\pi$ as endomorphisms of $M_\pi(\tau_i)$. Since the natural map

$$\mathrm{End}_{\mathcal{A}_x}(\tau_i) \to \mathrm{End}_{E_\pi}(M_\pi(\tau_i))$$

is an isomorphism, the proposition follows in the case that τ_1 is irreducible.

After twisting there remains to prove the proposition in the case that $\sigma = \mathrm{Sym}^0, \mathrm{Sym}^{p-1}$ and $\lambda = 1$. Hence τ_1 is a nonsplit extension of 1 by St and the block containing τ_1 is $x = \{1, \mathrm{St}, \pi\}$, where $\pi = \mathrm{Ind}_B^G(\omega \otimes \omega^{-1})$. We will follow [Paš13, Section 10.4] and work in a quotient category of $\mathcal{A}_x$. By [Paš13, Lemma 10.84] the kernel of the functor

$$\mathrm{Hom}_{\mathcal{A}_x}(-, J_\pi \oplus J_{\mathrm{St}})^\vee$$

is the category $\mathfrak{T}(k)$ of modules with trivial G-action. By [Gab62, IV.4, Théorème 4] this functor defines an equivalence of the quotient category $\mathfrak{Q}(k) = \mathcal{A}_x/\mathfrak{T}(k)$ with the category of left discrete $\mathrm{End}_G(J_\pi \oplus J_{\mathrm{St}})$-modules. (Compare the proof of [Paš13, Corollary 10.85], and note that our categories are opposite to those denoted in [Paš13] by the same symbols.) More precisely, the functor

$$\mathrm{Hom}_{\mathfrak{Q}(k)}(-, J_\pi \oplus J_{\mathrm{St}})$$

is an equivalence, but by the equivalence of (b) and (c) in [Gab62, III.2, Lemme 1] the natural map

$$\mathrm{Hom}_{\mathcal{A}_x}(X, J_\pi \oplus J_{\mathrm{St}}) \to \mathrm{Hom}_{\mathfrak{Q}(k)}(X, J_\pi \oplus J_{\mathrm{St}})$$

is an isomorphism for every object X of $\mathcal{A}_x$.

Since $\mathrm{Hom}_{\mathcal{A}}(\tau_i, J_\pi) = 0$ for all i, the functor $M_{\mathrm{St}}(-) = \mathrm{Hom}_{\mathcal{A}}(-, J_{\mathrm{St}})^\vee$ induces a fully faithful functor on the full subcategory of $\mathfrak{Q}(k)$ generated by the τ_i. By Proposition 2.6.3 we know that $M_{\mathrm{St}}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is cyclic over the endomorphism ring E_{St} of J_{St}. Arguing as before, it follows from Theorem 2.6.2 that there exists $T_x \in E_{\mathrm{St}} \cong Z_x$ such that $T = T_x$ as endomorphisms of $\mathrm{Hom}_{\mathcal{A}_x}(\tau_i, J_{\mathrm{St}})$. The faithfulness of $M_{\mathrm{St}}(-)$ implies that $T = T_x$ as elements of $\mathrm{End}_{\mathfrak{Q}(k)}(\tau_i)$. This implies that the image in $\mathcal{A}_x$ of

$$(T - T_x) : \tau_i \to \tau_i$$

has trivial G-action. This is true for all i, and the only submodule of τ_i with trivial G-action has length at most one: this is an immediate consequence of Lemma 2.1.12. Using this for τ_{i+1} together with the fact that $T - T_x$ commutes with the projection $\tau_{i+1} \to \tau_i$ it follows that $T - T_x = 0$ as elements of $\mathrm{End}_G(\pi_i)$. □

Blocks of type (3) are an exception to Theorem 2.6.2 and Proposition 2.6.4 because the rings $\mathrm{End}_G(J_\pi)$ are no longer commutative. The following weaker result will suffice for our purposes; by a more detailed analysis of the endomorphism rings E_x, taking into account their relationship with Galois pseudodeformation rings, it should actually be possible to compute explicit integral dependence equations for Hecke operators over the centre of the block.

Proposition 2.6.5. *Let σ be a twist of Sym^{p-2}, let $\lambda = \pm 1$, and let*

$$\tau_i = c\text{-}\mathrm{Ind}_{KZ}^G \sigma/(T-\lambda)^i.$$

Let x be the block of type (3) containing the irreducible representation $\pi = \tau_1$. Then there exist a positive integer n and central elements $z_0, z_1, \ldots, z_n \in Z_x$ contained in the unique maximal ideal $\mathfrak{m}_x \subset Z_x$ such that

$$(T-\lambda)^{n+1} + z_n(T-\lambda)^n + \cdots + z_1(T-\lambda) + z_0 = 0$$

holds in $\mathrm{End}_G(\tau_i)$ for all i.

Proof. Fix an injective envelope $\pi \to J_\pi$ in $\mathcal{A}_x$ and let $E_\pi = \mathrm{End}_G(J_\pi)$. The functor

$$M_\pi(-) = \mathrm{Hom}_G(-, J_\pi)^\vee$$

is defined on $\mathcal{A}$, and it restricts to an equivalence of $\mathcal{A}_x$ with the category of discrete left E_π-modules. Hence the natural map $Z_x \to E_\pi$ induces an isomorphism $Z_x \to Z(E_\pi)$. By [Paš13, Corollary 9.25], the ring E_π is a free module of rank 4 over its centre, which is a local ring. By Proposition 2.6.3, the module

$$M = M_\pi(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$$

is cyclic over E_π. It follows that M is finite over Z_x. (See the proof of [San16, Lemma 2.6] for more details.)

The Hecke algebra $\mathcal{H}(\sigma)$ acts on M by Z_x-linear endomorphisms. The endomorphism induced by $(T-\lambda)$ on $M^\vee$ is locally nilpotent, because every G-linear map $c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to J_\pi$ factors through a representation of finite length, in fact through one of the τ_i. The quotient $M/\mathfrak{m}_x M$ is a finite-dimensional vector space over the residue field of Z_x, and it is dual to $M^\vee[\mathfrak{m}_x]$, hence $(T-\lambda)$ induces a nilpotent endomorphism of $M/\mathfrak{m}_x M$. It follows that there exists $n > 0$ such that $(T-\lambda)^n(M) \subset \mathfrak{m}_x M$. This implies that there exist $z_0, z_1, \ldots, z_n \in \mathfrak{m}_x$ such that

$$(T-\lambda)^{n+1} + z_n(T-\lambda)^n + \cdots + z_1(T-\lambda) + z_0 = 0$$

as endomorphisms of M. Hence the same relation holds as endomorphisms of $M_\pi(\tau_i)$ for all i, since they are quotients of M. Since the functor $M_\pi(-)$ is fully faithful on $\mathcal{A}_x$, this implies the proposition. □

3. Localization of smooth $\mathrm{GL}_2(\mathbf{Q}_p)$-representations

3.1. **Localization.** In this subsection we begin to explain how we localize $\mathcal{A}$ over X. The fundamental input is that the closed points of X are in bijection with the blocks of $\mathcal{A}_{\mathcal{O}}^{\mathrm{l.adm}}$.

Definition 3.1.1. If Y is a closed subset of X, then we let $\mathcal{A}_Y$ denote the subcategory of $\mathcal{A}$ consisting of those representations all of whose irreducible subquotients lie in a block corresponding to a closed point of Y.

We refer to Appendix A for the basic definitions and properties of localizing categories.

Lemma 3.1.2. *The subcategory $\mathcal{A}_Y$ is localizing.*

Proof. To see that $\mathcal{A}_Y$ is a Serre subcategory, it suffices to check that given an exact sequence

$$0 \to \pi_1 \to \pi \to \pi_2 \to 0$$

in $\mathcal{A}$ the set $\mathrm{JH}(\pi)$ of irreducible subquotients of π coincides with $\mathrm{JH}(\pi_1) \cup \mathrm{JH}(\pi_2)$. Since it is immediate that $\mathrm{JH}(\pi_1) \cup \mathrm{JH}(\pi_2) \subseteq \pi$, let V be an irreducible subquotient of π. Then there are subspaces $W_1 \subset W_2 \subset \pi$ such that $V \cong W_2/W_1$, and V is a subquotient of π_2 unless $W_1 + \pi_1 = W_2 + \pi_1$. Similarly, V is a subquotient of π_1 unless $W_1 \cap \pi_1 = W_2 \cap \pi_1$. Since $W_1 \neq W_2$, these two equalities cannot hold at the same time.

To see that $\mathcal{A}_Y$ is localizing it thus suffices to check that it is closed under direct sums. Assume that $\pi_i \in \mathcal{A}_Y$ and V is an irreducible submodule of a quotient of $\bigoplus_{i \in I} \pi_i$. Since V is cyclic over $\mathcal{O}[G]$, there exists a finite subset $J \subset I$ such that V is a submodule of a quotient of $\bigoplus_{i \in J} \pi_i$. By the previous discussion, $V \in \cup_{i \in J} \mathrm{JH}(\pi_i) \subset \mathcal{A}_Y$, implying the claim. □

Lemma 3.1.3. *If $Y = \{y_1, \ldots, y_n\}$ is a finite closed subset of X, then $\mathcal{A}_Y$ is the product of the finitely many blocks corresponding to the points y_i. In particular, $\mathcal{A}_Y$ is contained in $\mathcal{A}^{\mathrm{l.adm}}$, and is an Artinian category.*

Proof. If we write $Y = \{y_1, \ldots, y_n\}$, then $\mathcal{A}_Y$ consists of objects whose subquotients lie in the blocks corresponding to one of the points y_i. If π is such an object, let σ be a Serre weight, and let $c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to \pi$ be a morphism; then since $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ has subquotients lying in infinitely many different blocks, this morphism must factor through a non-trivial, and hence finite length, quotient of $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$. Thus any finitely generated subrepresentation of π is of finite length, and hence locally admissible, so that $\mathcal{A}_Y \subseteq \mathcal{A}^{\mathrm{l.adm}}$. The lemma then follows from the theory of blocks for $\mathcal{A}^{\mathrm{l.adm}}$. □

Definition 3.1.4. If U is an open subset of X, then we write $\mathcal{A}_U := \mathcal{A}/\mathcal{A}_Y$, where $Y := X \setminus U$.

Since $\mathcal{A}_Y$ is localizing there is a fully faithful right adjoint $(j_U)_* : \mathcal{A}_U \to \mathcal{A}$ to the canonical quotient functor $(j_U)^* : \mathcal{A} \to \mathcal{A}_U$. Where U is understood, we write j_* for $(j_U)_*$, and j^* for $(j_U)^*$.

Lemma 3.1.5.

(1) *The right adjoint j_* commutes with filtered colimits.*
(2) *The category $\mathcal{A}_U$ is locally Noetherian.*
(3) *If we let $\mathcal{A}_U^{\mathrm{fg}}$ denote the essential image of $\mathcal{A}^{\mathrm{fg}}$ under the localization functor, then $\mathcal{A}_U^{\mathrm{fg}}$ is precisely the subcategory of Noetherian objects in $\mathcal{A}_U$.*

Proof. This follows from Lemma A.2.8 and the fact that $\mathcal{A}$ and $\mathcal{A}_Y$ are both generated by finitely generated objects, which are Noetherian objects in $\mathcal{A}$ by Corollary 2.4.4. □

Lemma 3.1.6. *Let σ be a Serre weight, let Y be a closed subset of X, and choose $g \in \mathcal{H}(\sigma)$ such that $f_\sigma^{-1}(Y) = V(g)$ (a closed subset of $\operatorname{Spec} \mathcal{H}(\sigma)$). Then for any object π of $\mathcal{A}_Y$, we have $\mathrm{Ext}^i_{\mathcal{A}}\big(\pi, (c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]\big) = 0$ for all i.*

Proof. Applying Lemma A.2.7, we see that it suffices to prove the claimed vanishing in the case when π is finitely generated, which we assume from now on. By dévissage we can furthermore assume that the maximal ideal of $\mathcal{O}$ acts trivially on π. Using the spectral sequence (2.5.5) and the fact that $\mathcal{O}$ is a DVR, it suffices to prove that $\mathrm{Ext}^i_{\mathcal{A}_{\mathbf{F}}}\big(\pi, (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/g]\big) = 0$ for all i. The formation of this Ext^i-module is compatible with extension of scalars, by Proposition 2.5.13, and so we may and do replace $\mathbf{F}$ by some uncountable extension. We thus assume that $\mathbf{F}$ is uncountable for the remainder of the argument. Also, if $g = 0$ then $(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/g] = 0$, and the claimed vanishing follows immediately. Thus we may and do assume for the remainder of the proof that $g \neq 0$.

We first suppose that Y is a finite union of closed points. In this case, Lemma 3.1.3 shows that any object π of $\mathcal{A}_Y$ decomposes as a direct sum of objects lying in the various blocks corresponding to the points of Y. It will be useful in this case to prove a slightly stronger statement, namely that $\mathrm{Ext}^i_{\mathcal{A}}\big(\pi, (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/h]\big) = 0$ for any non-zero multiple h of g. To this end, we note that if $f \in \mathcal{H}(\sigma)$ is coprime to h (and so in particular coprime to g), then $(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)/f(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)$ is of finite length, and has no Jordan–Hölder factor lying in $\mathcal{A}_Y$ (this last claim following from the relationship between the morphisms f_σ and the blocks of $\mathcal{A}^{\mathrm{l.adm}}$). Thus by Lemma 2.5.8 we have $\mathrm{Ext}^i_{\mathcal{A}}\big(\pi, (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)/f(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)\big) = 0$ for all i, and a consideration of the long exact Ext sequence arising from the short exact sequence

$$0 \to (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/h] \xrightarrow{f\cdot} (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/h] \to (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)/f(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) \to 0$$

shows that f acts invertibly on $\mathrm{Ext}^i_{\mathcal{A}}\big(\pi, (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/h]\big)$. Certainly h also acts invertibly, and so this Ext module is a vector space over the fraction field of $\mathcal{H}(\sigma)$, which is isomorphic to $\mathbf{F}(T)$, a field of uncountable dimension over $\mathbf{F}$. On the other hand, Lemma 2.5.7 shows that this Ext module is of countable dimension over $\mathbf{F}$. Thus it must vanish, as claimed.

We now consider the general case. Since we are assuming that π is finitely generated, it is a quotient of a representation of the form $c\text{-}\mathrm{Ind}_{KZ}^G\,V$, for some finite dimensional KZ-representation V. A dévissage using the long exact Ext sequence reduces to the case when π is a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G\,\tau$, for some Serre weight τ. We consider two cases: (i) π is equal to $c\text{-}\mathrm{Ind}_{KZ}^G\,\tau$, or (ii) π is a proper quotient of $c\text{-}\mathrm{Ind}_{KZ}^G\,\tau$. In this latter case, π is of finite length, and so lies in $\mathcal{A}_{Y_0}$ for some finite closed subset Y_0 of Y. The claimed vanishing then follows from the special case already proved, and so we now put ourselves in case (i).

The assumption that $c\text{-}\mathrm{Ind}_{KZ}^G\,\tau$ is an object of $\mathcal{A}_Y$ implies that for each non-zero element $q \in \mathcal{H}(\tau)$, the (finite length) quotient $(c\text{-}\mathrm{Ind}_{KZ}^G\,\tau)/q(c\text{-}\mathrm{Ind}_{KZ}^G\,\tau)$ lies in $\mathcal{A}_{Y_0}$ for some finite closed subset Y_0 of Y. The result we already proved then shows that $\mathrm{Ext}^i_{\mathcal{A}}\big((c\text{-}\mathrm{Ind}_{KZ}^G\,\tau)/q(c\text{-}\mathrm{Ind}_{KZ}^G\,\tau), (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/g]\big) = 0$ for each i, and a consideration of the long exact Ext sequence arising from the short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G\,\tau \xrightarrow{q\cdot} c\text{-}\mathrm{Ind}_{KZ}^G\,\tau \to (c\text{-}\mathrm{Ind}_{KZ}^G\,\tau)/q(c\text{-}\mathrm{Ind}_{KZ}^G) \to 0$$

shows that q acts invertibly on $\mathrm{Ext}^i_{\mathcal{A}}\big(c\text{-}\mathrm{Ind}_{KZ}^G\,\tau, (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/g]\big)$. This Ext module is thus a vector space over the fraction field of $\mathcal{H}(\tau)$, while also being a countable dimensional $\mathbf{F}$-vector space, by Lemma 2.5.7. Since the fraction field of $\mathcal{H}(\tau)$ is isomorphic to $\mathbf{F}(T)$, which is of uncountable $\mathbf{F}$-dimension when $\mathbf{F}$ itself is uncountable, we find that this Ext module vanishes, as claimed. □

Proposition 3.1.7. *Let Y be a closed subset of X, and write $U := X \setminus Y$.*

(1) *If σ is a Serre weight, and if $f_\sigma^{-1}(Y) = V(g)$ (a closed subset of* $\operatorname{Spec} \mathcal{H}(\sigma)$*) for some $g \in \mathcal{H}(\sigma)$, then the natural map*

$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to (c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]$$

can be identified with the unit morphism

$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to j_{U*} j_U^* c\text{-}\mathrm{Ind}_{KZ}^G \sigma.$$

(2) *The functor j_{U*} is exact.*

Proof. We first put ourselves in the situation of (1). If $g = 0$ then $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$ is an object of $\mathcal{A}_Y$, and both $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]$ and $j_* j^*(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ vanish, so that (1) follows immediately. If $g \neq 0$, then the embedding

$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma \hookrightarrow (c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g] \tag{3.1.8}$$

is the colimit of the embeddings

$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to \frac{1}{g^n}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma).$$

The cokernel of this latter morphism is isomorphic to $\big(c\text{-}\mathrm{Ind}_{KZ}^G \sigma\big)/g^n\big(c\text{-}\mathrm{Ind}_{KZ}^G \sigma\big)$, which is of finite length, and lies in $\mathcal{A}_Y$. Thus each of these inclusions induces an isomorphism after applying $j_* j^*(-)$, and hence the colimiting inclusion (3.1.8) also induces an isomorphism after applying this functor. Thus it suffices to show that $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]$ is in the image of j_*, or equivalently, by Lemma A.2.2, that $\mathrm{Ext}^i_{\mathcal{A}}\big(\pi, (c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]\big) = 0$ for $i = 0, 1$, whenever π is an object of $\mathcal{A}_Y$. This follows from Lemma 3.1.6.

To prove (2), it suffices, by Corollary A.2.6, along with Lemmas 2.5.6 and 3.1.5, to show that $\mathrm{Ext}^2_{\mathcal{A}}(\pi, j_* j^* \pi') = 0$, as π runs over the objects of $\mathcal{A}_Y$ and π' runs over a collection of generators of $\mathcal{A}$. We let π' range over the representations $c\text{-}\mathrm{Ind}_{KZ}^G V$, where V is a finite length smooth $\mathcal{O}[KZ]$-representation. A dévissage then reduces us to checking the required vanishing when $V = \sigma$ is a Serre weight. In this case, the required vanishing again follows from Lemma 3.1.6. ∎

Corollary 3.1.9. *Let Y be a closed subset of X, and write $U := X \setminus Y$. If π is an object of $\mathcal{A}_Y$, and π' is any object of $\mathcal{A}$, then $\mathrm{Ext}^i(\pi, j_{U*} j_U^* \pi') = 0$ for all i.*

Proof. This follows directly from Proposition 3.1.7 (2), together with Corollary A.2.6. In fact, it was essentially proved directly in the course of proving Proposition 3.1.7 (2). ∎

Lemma 3.1.10. *If σ is a Serre weight, and if $c\text{-}\mathrm{Ind}_{KZ}^G \sigma \hookrightarrow \pi$ is an essential embedding whose cokernel is of finite length, then there exists a non-zero $g \in \mathcal{H}(\sigma)$ such that π (thought of as an overmodule of $c\text{-}\mathrm{Ind}_{KZ} \sigma$) is contained in $\frac{1}{g} \cdot c\text{-}\mathrm{Ind}_{KZ}^G \sigma$.*

Proof. Consider the short exact sequence $0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to \pi \to \pi' \to 0$. Since π' is of finite length, it is supported on some finite closed subset Y of X. Write $U := X \setminus Y$. Then $j_{U*} j_U^* \pi = 0$, and so $j_{U*} c\text{-}\mathrm{Ind}_{KZ}^G \sigma \xrightarrow{\sim} j_{U*} \pi$. Proposition 3.1.7 (1) shows that $j_{U*} c\text{-}\mathrm{Ind}_{KZ}^G \sigma = (c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]$ for some non-zero element g of $\mathcal{H}(\sigma)$.

Consider the commutative square

$$\begin{tikzcd} c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma \arrow[r] \arrow[d] & \pi \arrow[d] \\ (c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)[1/g] \arrow[r, equal] & j_{U*}j_U^*\pi \end{tikzcd}$$

Since the left-hand vertical arrow is injective, and since the upper horizontal arrow is an essential embedding, we find that the right-hand vertical arrow is injective. Thus we find that π embeds as a submodule of $(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)[1/g]$ containing $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$. Since π is finitely generated (being an extension of a finite length module by a finitely generated module), we see that if we replace g by a sufficiently large power, then in fact $\pi \subseteq \frac{1}{g}\cdot c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$, as claimed. $\square$

Remark 3.1.11. An immediate consequence of Lemma 3.1.10 is that if $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma \to \pi$ is an essential embedding with cokernel of finite length then the maximal ideal of $\mathcal{O}$ acts trivially on π. This can also be seen directly: if $\pi_E \in \mathcal{O}$ is a uniformizer, the image of $\pi_E : \pi \to \pi$ has finite length, and so intersects $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$ trivially. Since the embedding $c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma \to \pi$ is essential, this implies that $\pi_E : \pi \to \pi$ is the zero map.

Remark 3.1.12. Note that if σ is not isomorphic to a twist of Sym^0 or Sym^{p-1}, then $(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)/h(c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$ is irreducible, for each irreducible $h \in \mathcal{H}(\sigma)$. In this case, by choosing g in Lemma 3.1.10 appropriately, we may even assume that $\pi = \frac{1}{g}\cdot c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$. In the exceptional cases, we get a counterexample from the exact sequence (3.2.2) below.

Remark 3.1.13. Another way to prove Lemma 3.1.10 is to use the isomorphism

$$\mathrm{Hom}(\pi', (c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)[1/g]/c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma) \xrightarrow{\sim} \mathrm{Ext}^1(\pi', c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma)$$

of Section 3.6 below.

3.2. **Čech resolutions.** If $\{U_0, \ldots, U_n\}$ is any finite open cover of X, then for any object π of $\mathcal{A}$, we obtain a functorial Čech resolution

$$0 \to \pi \to \prod_i (j_i)_*(j_i)^*\pi \to \cdots \to (j_{0,\ldots,n})_*(j_{0,\ldots,n})^*\pi \to 0 \tag{3.2.1}$$

where as usual we write $U_{i\ldots k}$ for $U_i \cap \cdots \cap U_k$, we have written $j_{i\ldots k}$ for $j_{U_{i\ldots k}}$, and the differentials are given by the usual formulas (see e.g. [Sta, Tag 01FG]). (In the terminology of the Stacks Project, we are working with the *ordered Čech complex*.) We are going to prove that the complex (3.2.1) is always exact. To do so we will use another resolution of smooth $\mathrm{GL}_2(\mathbf{Q}_p)$-representations, arising from the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbf{Q}_p)$. Let N be the normalizer of the Iwahori subgroup Iw. Write $\delta : N \to \mathcal{O}^\times$ for the orientation character, which is trivial on $\mathrm{Iw}Z$ and takes value -1 at $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. We identify the representation space of δ with $\mathcal{O}$. Then we have an exact sequence of $\mathcal{O}[G]$-representations

$$0 \to c\text{-}\mathrm{Ind}_{N}^{G}(\delta) \xrightarrow{\partial} c\text{-}\mathrm{Ind}_{KZ}^{G}(\mathrm{triv}) \xrightarrow{\mathrm{sum}} \mathrm{triv} \to 0, \tag{3.2.2}$$

see for example [Dot, Section 2.4.14].

Proposition 3.2.3. *For any object π of $\mathcal{A}$, and any finite open cover $\{U_i\}$ of X, the resolution* (3.2.1) *is acyclic.*

Proof. By Frobenius reciprocity there is a short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_N^G \,\delta\pi \to c\text{-}\mathrm{Ind}_{KZ}^G \,\pi \to \pi \to 0$$

where δ is the nontrivial quadratic character of $N/\mathrm{Iw}Z$.

Since $p > 2$, $c\text{-}\mathrm{Ind}_N^G \,\delta\pi$ is a direct summand of $c\text{-}\mathrm{Ind}_{\mathrm{Iw}Z}^G \,\pi = c\text{-}\mathrm{Ind}_{KZ}^G(c\text{-}\mathrm{Ind}_{\mathrm{Iw}Z}^{KZ} \,\pi)$. Since the formation of each of the terms in (3.2.1) is exact, we reduce to verifying the claim of the proposition in the case when π is of the form $c\text{-}\mathrm{Ind}_{KZ}^G \,V$ for some smooth $\mathcal{O}[KZ]$-representation V. Since both compact induction, and the formation of the terms in (3.2.1), are compatible with passing to filtered colimits (use Lemma A.2.8 (1)), we then reduce to the case when V is finitely generated. Finally, since compact induction is exact, we reduce to the case when π is of the form $c\text{-}\mathrm{Ind}_{KZ}^G \,\sigma$ for some Serre weight σ.

The open cover $\{U_i\}$ pulls back via f_σ to an open cover $D(g_i)$ of $\mathrm{Spec}\,\mathcal{H}(\sigma)$. Proposition 3.1.7 then implies that (3.2.1) may be identified with the tensor product

$$(c\text{-}\mathrm{Ind}_{KZ}^G \,\sigma) \otimes_{\mathcal{H}(\sigma)} K^\bullet(g_0, \ldots, g_n), \tag{3.2.4}$$

where $K^\bullet(g_0, \ldots, g_n)$ denotes the usual Čech complex for a finite open cover of Spec of a ring by distinguished opens associated to the sequence $\{g_0, \ldots, g_n\}$. Since $\{D(g_i)\}$ is an open cover of the affine scheme $\mathrm{Spec}\,\mathcal{H}(\sigma)$, the complex $K^\bullet(g_0, \ldots, g_n)$ is acyclic, and remains acyclic after tensoring with any $\mathcal{H}(\sigma)$-module. We consequently find that (3.2.4) is acyclic, and thus that (3.2.1) is acyclic, as claimed. □

Corollary 3.2.5. *If $Y \subset U$ is an inclusion of a closed subset of X in an open subset of X, and if π is an object of $\mathcal{A}_Y$, then the natural morphism $\pi \to (j_U)_*(j_U)^*\pi$ is an isomorphism; in other words, π lies in the essential image of $(j_U)_*$.*

Proof. This follows from the acyclicity of the Čech resolution of π with respect to the open cover $\{U, V := X \setminus Y\}$ of X, together with the fact that, by definition, $(j_V)^*\pi = (j_{U\cap V})^*\pi = 0$. □

Corollary 3.2.6. *Let Y and W be disjoint closed subset of X. If π is an object of $\mathcal{A}_Y$, and π' is an object of $\mathcal{A}_W$, then $\mathrm{Ext}^i(\pi, \pi') = 0$ for all i.*

Proof. This follows immediately from Corollaries 3.1.9 and 3.2.5 (the latter being applied to the inclusion $W \subseteq X \setminus Y$). □

Lemma 3.2.7. *Let Y be a closed subset of X, let σ be a Serre weight, and write $f_\sigma^{-1}(Y) = V(g)$ (a closed subset of $\mathrm{Spec}\,\mathcal{H}(\sigma)$) for some $g \in \mathcal{H}(\sigma)$. Then for any object π of $\mathcal{A}_Y$, each element of $\mathrm{Ext}^i_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G \,\sigma, \pi)$ is annihilated by some power of g.*

Proof. By Lemma 2.5.6 we can assume that π is finitely generated, and by dévissage we can assume that the maximal ideal of $\mathcal{O}$ acts trivially on π. Using (2.5.5) it suffices to prove the claim for $\mathrm{Ext}^i_{\mathcal{A}_{\mathbf{F}}}$ instead of $\mathrm{Ext}^i_{\mathcal{A}}$. By Proposition 2.5.13, we may verify this after extending scalars to an uncountable extension of $\mathbf{F}$. Thus we may and do assume that $\mathbf{F}$ is uncountable.

If $h \in \mathcal{H}(\sigma)$ is coprime to g, then $(c\text{-}\mathrm{Ind}_{KZ}^G \,\sigma)/h(c\text{-}\mathrm{Ind}_{KZ}^G \,\sigma)$ is of finite length, and lies in $\mathcal{A}_W$, for some finite closed subset W of $X \setminus Y$. If we set $U := X \setminus W$, then $\pi = (j_U)_*(j_U)^*\pi$, by Corollary 3.2.5 (note that $Y \subseteq U$). Thus

$\mathrm{Ext}^i_{\mathcal{A}}\big((c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)/h(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma),\pi\big)$ vanishes for every i, by Corollary 3.1.9, and so a consideration of the long exact sequence of Ext's arising from the short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \xrightarrow{h\cdot} c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \to (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)/h(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) \to 0$$

shows that h acts invertibly on $\mathrm{Ext}^i_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma,\pi)$; thus the $\mathcal{H}(\sigma)$-module structure on $\mathrm{Ext}^i_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma,\pi)$ extends to a $\mathcal{H}(\sigma)_S$-module structure, where S denotes the multiplicative subset of elements coprime to g. Since $\mathbf{F}$ is uncountable, the ring $\mathcal{H}(\sigma)_S$ has uncountable $\mathbf{F}$-dimension, while Lemma 2.5.7 shows that $\mathrm{Ext}^i_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma,\pi)$ has countable $\mathbf{F}$-dimension. Thus $\mathrm{Ext}^i_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma,\pi)$ cannot contain any $\mathcal{H}(\sigma)_S$ torsion-free submodule. The lemma follows. □

Remark 3.2.8. We explain an alternate proof of Lemma 3.2.7 in the case when Y is finite. In this case Lemma 3.1.3 shows that $\mathcal{A}_Y$ is a product of blocks in $\mathcal{A}^{\mathrm{l.adm}}$, and it follows from [Paš13, Cor. 5.18] that any injective resolution of π in $\mathcal{A}_Y$ also provides an injective resolution in $\mathcal{A}$. But if $I^\bullet$ is such a resolution in $\mathcal{A}_Y$, then any element of $\mathrm{Hom}_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma, I^\bullet)$ factors through $(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)/g^n(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)$, for some n.

3.3. **A stack of abelian categories.** For each open subset U of X we have the localized category $\mathcal{A}_U$, and for each open subset $V \subset U$ we have the natural localization functor $j^*_{UV} : \mathcal{A}_U \to \mathcal{A}_V$.

Theorem 3.3.1. *The collection $\{\mathcal{A}_U\}$ together with the localization functors $j^*_{UV} : \mathcal{A}_U \to \mathcal{A}_V$ for $V \subset U$ forms a stack (of abelian categories) over the Zariski site of X.*

Remark 3.3.2. By a stack in abelian categories over X we simply mean a stack over the Zariski site of X whose fiber categories are abelian. There are further properties one could ask of such an object, such as exactness of all the pullback functors, and indeed the stack determined by the $\{\mathcal{A}_U\}$ has a lot more structure, such as pushforward functors and acyclic Čech resolutions. This is very similar to the formalism underlying cohomological descent [Sta, Tag 0D8D], as might be expected taking into account the connection between the categories $\mathcal{A}_U$ and sheaves over a stack of (φ,Γ)-modules explained in the introduction.

Proof of Theorem 3.3.1. Since the localization functors $j^*_{UV} : \mathcal{A}_U \to \mathcal{A}_V$ are the identity on objects, it is straightforward to check that $U \mapsto \mathcal{A}_U$ defines a pseudofunctor in the sense of [Vis05, Definition 3.10]. More precisely, the universal property of localization specifies a natural isomorphism $j^*_{UW} \cong j^*_{VW} j^*_{UV}$ associated to any composite inclusion $W \subset V \subset U$ of subsets of V, such that j^*_{UU} is the identity and Properties (a) and (b) in [Vis05, Definition 3.10] are satisfied. By the procedure explained in [Vis05, Section 3.1.3] we obtain a fibered category $\mathcal{A}_\bullet \to X$.

Following [Vis05, Definition 4.6] we need to prove that for any covering $\{U_i \to U\}_{i\in I}$ the functor from $\mathcal{A}_U$ to the category $\mathcal{A}_\bullet(\{U_i \to U\})$ of objects with descent data is an equivalence. Since X is quasicompact, we can assume without loss of generality that the indexing set is equal to $\{0, 1\ldots, n\}$. We will deduce the result from Proposition 3.2.3. In what follows we will write $U_{ij} = U_i \cap U_j$ and abbreviate the pullback and pushforward associated to the inclusion $U_{ij} \subset U_i$ by $j^*_{i,ij}$ and $j_{i,ij*}$, and those associated to the inclusion $U_{ij} \subset U$ by j^*_{ij} and j_{ij*}.

Choose objects $\pi_i \in \mathcal{A}_{U_i}$ and isomorphisms

$$j^*_{i,ij}\pi_i \xrightarrow{\sim} j^*_{j,ij}\pi_j \tag{3.3.3}$$

in $\mathcal{A}_{U_{ij}}$ for all i, j, satisfying the cocycle condition after pullback to U_{ijk}. Fix indices $i < j$ and define $\pi_{ij} = j^*_{i,ij}\pi_i$. The adjunction between pullback and push-forward defines a map $u_{ij} : \pi_i \to j_{i,ij*}\pi_{ij}$, and pushing forward to U we find a map

$$u_{ij} : j_{i*}\pi_i \to j_{ij*}\pi_{ij}.$$

Similarly, the isomorphism (3.3.3) yields a map

$$u^+_{ij} : j_{j*}\pi_j \to j_{ij*}\pi_{ij}.$$

Putting these together we obtain

$$u^+_{ij} - u_{ij} : \prod_i j_{i*}\pi_i \to \prod_{i<j} j_{ij*}\pi_{ij}. \tag{3.3.4}$$

If we are in the special case that $U_{i_0} = U$ for some index i_0, the cocycle condition implies that the maps $j^*_{i_0,i_0i}\pi_{i_0} \to \pi_i = j^*_{i,i_0i}\pi_i$ define an isomorphism in $\mathcal{A}_\bullet(\{U_i \to U\})$. Since the formation of (3.3.4) defines a functor on $\mathcal{A}_\bullet(\{U_i \to U\})$, it follows that the complex (3.3.4) is isomorphic to the first truncation of the Čech resolution of π_{i_0}. Furthermore, the formation of (3.3.4) is compatible with pullback for any open inclusion $V \subset U$. It follows from this discussion together with Proposition 3.2.3 and the exactness of pullback functors that if we define π by the exact sequence

$$0 \to \pi \to \prod_i j_{i*}\pi_i \xrightarrow{(3.3.4)} \prod_{i<j} j_{ij*}\pi_{ij}$$

then $j^*_i(\pi) \cong \pi_i$, proving essential surjectivity of $\mathcal{A}_U \to \mathcal{A}_\bullet(\{U_i \to U\})$.

Now let $\alpha : \pi_1 \to \pi_2$ be a morphism in $\mathcal{A}_U$. It follows immediately from Proposition 3.2.3 and functoriality of the Čech resolutions that if $j^*_i(\alpha) = 0$ for all i then $\alpha = 0$, which proves that $\mathcal{A}_U \to \mathcal{A}_\bullet(\{U_i \to U\})$ is faithful. To prove that it is full we begin with representations $\pi, \tau \in \mathcal{A}_U$ together with morphisms of descent data $\alpha_i : j^*_i\pi \to j^*_i\tau$. The argument for essential surjectivity implies that the α_i induce a morphism on the first-truncated Čech resolutions of π and τ, and so a morphism $\alpha : \pi \to \tau$. Since the formation of the Čech resolution commutes with j^*_i we deduce that that $j^*(\alpha) = \alpha_i$, which concludes the proof. □

3.4. **Further results about j_*.** As we see from part (1) of Proposition 3.1.7, it is not typically the case that $j_*j^*\pi$ is finitely generated, even if π is. Indeed we have the following result.

Lemma 3.4.1. *If U is an open subset of X, and if $j_{U*}j^*_U\pi$ is finitely generated for some object π of $\mathcal{A}$, then the natural morphism $\pi \to j_{U*}j^*_U\pi$ is surjective.*

Proof. We may write π as the filtered colimit of its finitely generated submodules π_i, and then Lemma 3.1.5 shows that $j_*j^*\pi$ is the filtered colimit of the $j_*j^*(\pi_i)$. Since $j_*j^*\pi$ is finitely generated, and so Noetherian, by assumption, we find that $j_*j^*(\pi_i) = j_*j^*\pi$ for some value of i. Replacing π by π_i, we may thus assume that π is finitely generated.

Choose a surjection $c\text{-Ind}^G_{KZ} V \to \pi$, for some finite length $\mathcal{O}[KZ]$-representation V. We proceed by induction on the length of V (the case $V = 0$ being trivial). Let σ be an irreducible subrepresentation of V, and let π' denote the image of $c\text{-Ind}^G_{KZ} \sigma$

in π. If $\pi' = 0$ then we replace V by V/π', and we are done by induction. Otherwise, since j_* is exact, we see that $j_*j^*(\pi/\pi')$ is a quotient of $j_*j^*\pi$, and so is finitely generated. Thus our inductive hypothesis shows that $(\pi/\pi') \to j_*j^*(\pi/\pi')$ is surjective. Also, since $j_*j^*\pi'$ is a subobject of $j_*j^*\pi$, and since finitely generated G-representations are Noetherian, we see that $j_*j^*\pi'$ is finitely generated.

Now either $c\text{-Ind}_{KZ}^G\,\sigma \xrightarrow{\sim} \pi'$, or else π' is a proper quotient of $c\text{-Ind}_{KZ}^G\,\sigma$, in which case it is of finite length. In the latter case, it is automatic that $\pi' \to j_*j^*\pi'$ is surjective. In the former case, a consideration of the formula of Proposition 3.1.7 (1) shows that since $j_*j^*\pi' = j_*j^*(c\text{-Ind}_{KZ}^G\,\sigma)$ is finitely generated, the natural morphism $\pi' \to j_*j^*\pi'$ is an isomorphism, and so in particular is surjective. Again using the fact that j_* is exact, we see that we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi' \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d] & (\pi/\pi') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j_*j^*\pi' \arrow[r] & j_*j^*\pi \arrow[r] & j_*j^*(\pi/\pi') \arrow[r] & 0
\end{tikzcd}$$

in which the outer two vertical arrows are surjections. The middle vertical arrow is thus a surjection as well. □

Corollary 3.4.2. *If π is finitely generated and lies in the essential image of j_*, then the same is true for any subquotient of π.*

Proof. The assumption on π is equivalent to asking that $\pi \xrightarrow{\sim} j_*j^*\pi$ and that $j_*j^*\pi$ be finitely generated (see the proof of Lemma A.2.2). Bearing this in mind, since j_* is exact, it suffices to prove the statement of the lemma for subobjects of π, since it then follows for quotients, and so also for subquotients. If π' is any subobject of π, we see that $j_*j^*\pi'$ is a subobject of $j_*j^*\pi$, and so finitely generated; thus Lemma 3.4.1 implies that the natural morphism

$$\pi' \to j_*j^*\pi' \tag{3.4.3}$$

is surjective. The left-exactness of j_*, and the fact that the corresponding map for π is an isomorphism, implies that (3.4.3) is also injective. Thus in fact (3.4.3) is an isomorphism, and so π' also lies in the essential image of j_*. □

Remark 3.4.4. The preceding corollary is not true in general without the assumption of finite generation. Indeed, if π is finitely generated while $j_*j^*\pi$ is not finitely generated (Proposition 3.1.7 provides plenty of examples of such π), then if we let π' denote the image of π in $j_*j^*\pi$, we have that $\pi' \subsetneq j_*j^*\pi$, while $j_*j^*\pi' = j_*j^*\pi$. Thus $j_*j^*\pi$ is an object in the essential image of j_*, while its subobject π' is *not* in this essential image.

Corollary 3.4.5. *Let Y be a closed subset of X, and write $U := X \setminus Y$. If π is a finitely generated object lying in the essential image of j_{U*}, and if π' is an object of $\mathcal{A}_Y$, then $\mathrm{Ext}^i_{\mathcal{A}}(\pi, \pi') = 0$ for all i.*

Proof. Arguing by dévissage, and taking into account Corollary 3.4.2, we may assume that π is a quotient of $c\text{-Ind}_{KZ}^G\,\sigma$, for some Serre weight σ. If it is a proper quotient, then it is of finite length, and so lies in $\mathcal{A}_W$, for some finite closed subset of W of U, while $\pi' = (j_V)_*(j_V)^*\pi$, if $V = X \setminus W$ (by Corollary 3.2.5; note that $Y \subseteq V$). In this case the lemma follows from Corollary 3.1.9. Otherwise, we may assume that $\pi = c\text{-}\mathrm{Ind}_{KZ}^{G}\,\sigma$, in which case the claimed vanishing follows from Lemma 3.2.7. ☐

We may use the preceding result to strengthen Lemma 3.4.1.

Corollary 3.4.6. *Let Y be a closed subset of X, and write $U := X \setminus Y$. If $j_{U*}j_U^*\pi$ is finitely generated, then the natural morphism $\pi \to j_{U*}j_U^*\pi$ is a split surjection.*

Proof. Lemma 3.4.1 ensures that the morphism $\pi \to j_*j^*\pi$ is surjective. If we denote its kernel by π', then π' lies in $\mathcal{A}_Y$. The short exact sequence

$$0 \to \pi' \to \pi \to j_*j^*\pi \to 0$$

represents an element of $\mathrm{Ext}^1_{\mathcal{A}}(j_*j^*\pi, \pi')$, and Corollary 3.4.5 shows that this Ext module vanishes. Thus the short exact sequence splits, as claimed. ☐

3.5. **Completion.** If Y is a closed subset of X, then $\mathcal{A}_Y$ is an abelian subcategory (indeed a localizing subcategory) of $\mathcal{A}$. Thus the discussion of Section A.3.17 applies, and shows that the inclusion $\mathrm{Pro}(\mathcal{A}_Y) \hookrightarrow \mathrm{Pro}(\mathcal{A})$ admits a left adjoint, which we denote by $\pi \mapsto \widehat{\pi}_Y$. Being a left adjoint, this functor preserves colimits when they exist. In particular, it is right exact. We sometimes write simply $\widehat{\pi}$ if Y is understood from the context.

The following lemma describes $\widehat{\pi}_Y$ explicitly when $\pi \in \mathcal{A}$.

Lemma 3.5.1. *If π is an object of $\mathcal{A}$, there is a natural isomorphism*

$$\widehat{\pi}_Y \xrightarrow{\sim} \text{“}\varprojlim\text{”}\,\pi',$$

where π' runs over the cofiltered directed set of quotients of π lying in $\mathcal{A}_Y$.

Proof. This is a particular instance of Lemma A.3.18. ☐

Recall that $\mathcal{A}^{\mathrm{fg}}$ denotes the full subcategory of $\mathcal{A}$ consisting of finitely generated objects. Similarly, for any closed subset Y of X, we write $\mathcal{A}_Y^{\mathrm{fg}}$ to denote the full subcategory of $\mathcal{A}_Y$ consisting of finitely generated objects. If Y is a finite closed subset of X, then Lemma 3.1.3 shows that $\mathcal{A}_Y^{\mathrm{fg}}$ may equally well be described as the subcategory of $\mathcal{A}_Y$ consisting of finite length objects.

Corollary 3.5.2. *If π is an object of $\mathcal{A}^{\mathrm{fg}}$, then $\widehat{\pi}_Y$ is an object of $\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$.*

Proof. This follows immediately from Lemma 3.5.1. ☐

Proposition 3.5.3. *Let Y be a closed subset of X, and suppose that*

$$0 \to \pi' \to \pi \to \pi'' \to 0$$

is a short exact sequence in $\mathcal{A}^{\mathrm{fg}}$, with π' being an object of $\mathcal{A}_Y^{\mathrm{fg}}$. Then we may find a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi' \arrow[r] \arrow[d, equal] & \pi \arrow[r] \arrow[d] & \pi'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi' \arrow[r] & \widetilde{\pi} \arrow[r] & \widetilde{\pi}'' \arrow[r] & 0 \end{tikzcd}$$

in which the right-hand (and hence also the middle) vertical arrow is surjective, and in which $\widetilde{\pi}''$ (and hence also $\widetilde{\pi}$) are objects of $\mathcal{A}_Y^{\mathrm{fg}}$.

Proof. By assumption we may find a surjection $c\text{-}\mathrm{Ind}_{KZ}^G \tau \to \pi''$, for some finite length representation τ of $\mathcal{O}[KZ]$, and we argue by induction on the KZ-length of τ. Thus, to begin with, we assume that $\tau = \sigma$ is a Serre weight. If the surjection $c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to \pi''$ is not an isomorphism, then π'' is of finite length, and so it is an object of $\mathcal{A}_W$, for some finite closed subset W of X. Write $W' = W \cap Y$ and $W'' = W \setminus Y$, so that W is the disjoint union of its closed subsets W' and W''. Correspondingly, we may write $\pi'' = \pi_1'' \oplus \pi_2''$, where π_1'' is an object of $\mathcal{A}_{W'} \subseteq \mathcal{A}_Y$, and π_2'' is an object of $\mathcal{A}_{W''}$. Since Y and W'' are disjoint, it follows from Corollary 3.2.6 that $\mathrm{Ext}^1_{\mathcal{A}}(\pi_2'', \pi') = 0$. Thus the natural map

$$\mathrm{Ext}^1_{\mathcal{A}}(\pi_1'', \pi') \to \mathrm{Ext}^1_{\mathcal{A}}(\pi'', \pi')$$

given by pullback along the surjection $\pi'' \to \pi_1''$, is an isomorphism, and so π is pulled back from an extension π_1 of π_1'' by π'.

Suppose, then, that $c\text{-}\mathrm{Ind}_{KZ}^G \sigma \xrightarrow{\sim} \pi''$. Write $f_\sigma^{-1}(Y) = V(g)$, for some $g \in \mathcal{H}(\sigma)$. If $g = 0$, then π'' itself is an object of $\mathcal{A}_Y^{\mathrm{fg}}$, and there is nothing to prove. Otherwise, g is non-zero. The extension π represents a class of $\mathrm{Ext}^1_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma, \pi')$, and Lemma 3.2.7 shows that some power g^n of g annihilates this class. A consideration of the long exact Ext sequence associated to the short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma \xrightarrow{g^n \cdot} c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to (c\text{-}\mathrm{Ind}_{KZ}^G \sigma)/g^n(c\text{-}\mathrm{Ind}_{KZ}^G) \to 0$$

then shows that the class of π arises by pullback along the surjection $c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to (c\text{-}\mathrm{Ind}_{KZ}^G \sigma)/g^n(c\text{-}\mathrm{Ind}_{KZ}^G)$, whose target is an object of $\mathcal{A}_Y$.

We now turn to the general case, when τ is not necessarily irreducible. Our inductive hypothesis allows us to choose a subrepresentation π_0'' of π'' such that the statement of the proposition holds for each of π_0'' and π''/π_0'' in place of π''. Pulling back the extension given by π along the inclusion $\pi_0'' \subseteq \pi''$, we obtain a short exact sequence

$$0 \to \pi' \to \pi_0 \to \pi_0'' \to 0,$$

and by assumption we may find a subrepresentation $\pi_2 \subseteq \pi_0$ such that $\pi' \cap \pi_2 = 0$, and such that π_0/π_2 is an object of $\mathcal{A}_Y^{\mathrm{fg}}$. Next consider the short exact sequence

$$0 \to \pi_0/\pi_2 \to \pi/\pi_2 \to \pi/\pi_0 (\xrightarrow{\sim} \pi''/\pi_0'') \to 0.$$

Again by assumption, we may find a subrepresentation π_1 of π containing π_2 such that $\pi_0 \cap \pi_1 = \pi_2$, and such that π/π_1 $(= (\pi/\pi_2)/(\pi_1/\pi_2))$ is an object of $\mathcal{A}_Y^{\mathrm{fg}}$. Then $\pi' \cap \pi_1 = \pi' \cap \pi_0 \cap \pi_1 = \pi' \cap \pi_2 = 0$, and so setting $\widetilde{\pi} = \pi/\pi_1$ does the job. $\square$

Corollary 3.5.4. *The restriction to $\mathcal{A}^{\mathrm{fg}}$ of the functor $\pi \mapsto \widehat{\pi}_Y$ is exact.*

Proof. If $\pi' \subseteq \pi$ is an inclusion of finitely generated objects of $\mathcal{A}$, we must prove that the induced morphism $\widehat{\pi'}_Y \to \widehat{\pi}_Y$ is a monomorphism. By Lemma A.3.9 and the definition of the functor $\widehat{(-)}_Y$ as an adjoint, it suffices to show any morphism $\pi' \to \overline{\pi}'$ whose target lies in $\mathcal{A}_Y$ can be placed in a commutative diagram

$$\begin{tikzcd} \pi' \arrow[r] \arrow[d] & \pi \arrow[d] \\ \overline{\pi}' \arrow[r] & \overline{\pi}, \end{tikzcd}$$

where $\overline{\pi}$ also lies in $\mathcal{A}_Y$ and in which the bottom arrow is a monomorphism. In fact, it suffices to do this with $\overline{\pi}'$ replaced by the image $\widetilde{\pi}'$ of π' in $\overline{\pi}'$. (Indeed, if we then find a requisite monomorphism embedding $\widetilde{\pi}' \hookrightarrow \widetilde{\pi}$, we can take $\overline{\pi}$ to be the coproduct of $\widetilde{\pi}$ and $\overline{\pi}'$ over $\widetilde{\pi}'$.)

If we pushout the short exact sequence

$$0 \to \pi' \to \pi \to \pi/\pi' \to 0$$

along the surjection $\pi' \to \widetilde{\pi}'$, and then apply Proposition 3.5.3, we obtain the required quotient $\widetilde{\pi}$ of π. □

If π is an object of $\mathcal{A}$, and π' is an object of $\mathcal{A}_Y$, and if we write $\widehat{\pi}_Y = $ "$\varprojlim_I$ "π_i, then the adjunction property of completion can be expressed as

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}_Y)}(\widehat{\pi}_Y, \pi') \xrightarrow{\sim} \varinjlim_I \mathrm{Hom}_{\mathcal{A}_Y}(\pi_i, \pi') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}_Y}(\pi, \pi').$$

(The first isomorphism is an instance of (A.3.1), and the second isomorphism is induced by the maps $\pi \to \pi_i$ induced by the unit of adjunction $\pi \to \widehat{\pi}_Y$.) The following lemma extends this result to higher Ext.

Lemma 3.5.5. *Let π be an object of $\mathcal{A}^{\mathrm{fg}}$, and write $\widehat{\pi}_Y =$ "$\varprojlim_I$ "π_i, where (by Corollary 3.5.2) the π_i are objects of $\mathcal{A}_Y^{\mathrm{fg}}$. If π' is an object of $\mathcal{A}_Y$, then, for any value of n, the natural morphism*

$$\mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{A}_Y)}(\widehat{\pi}_Y, \pi') \xrightarrow{\sim} \varinjlim_I \mathrm{Ext}^n_{\mathcal{A}_Y}(\pi_i, \pi') \to \mathrm{Ext}^n_{\mathcal{A}}(\pi, \pi')$$

is an isomorphism.

Proof. We note that the first isomorphism in the displayed chain of isomorphisms is an instance of Lemma A.3.13.

We may write π' as the filtered colimit of its finitely generated subobjects. Lemma 2.5.6, along with its analogue for $\mathcal{A}_Y$, shows that each side of the natural morphism under consideration commutes with the passage to filtered colimits in π', and thus we may and do assume for the remainder of the proof that π' is finitely generated.

Lemma 2.5.9 shows that any object of $\mathrm{Ext}^n_{\mathcal{A}}(\pi, \pi')$ may be represented as a length n Yoneda extension involving finitely generated objects of $\mathcal{A}$. Taking completions along Y, and taking into account Corollary 3.5.4, we obtain a length n Yoneda Ext in $\mathrm{Pro}(\mathcal{A}_Y)$ between $\widehat{\pi}_Y$ and π'. Thus we find that the morphism

$$\mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{A}_Y)}(\widehat{\pi}_Y, \pi') \to \mathrm{Ext}^n_{\mathcal{A}}(\pi, \pi')$$

is surjective, for every n. It now follows by a formal argument that this map is in fact an isomorphism for every n.

In more detail, since the formation of filtered colimits is exact, we obtain a morphism

$$\mathrm{Ext}^{\bullet}_{\mathrm{Pro}(\mathcal{A}_Y)}(\widehat{\pi}_Y, –) \to \mathrm{Ext}^{\bullet}_{\mathcal{A}}(\pi, –)$$

of δ-functors on $\mathcal{A}_Y$. This morphism is an isomorphism in degree zero, is surjective in every degree, and its source is an effaceable δ-functor. (If π' is an object of $\mathcal{A}_Y$, we can embed it into an injective object of $\mathcal{A}_Y$, which by Lemma A.3.11 remains injective in $\mathrm{Pro}(\mathcal{A}_Y)$ and so effaces the higher $\mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{A}_Y)}(\widehat{\pi}_Y, –)$.) An easy dimension-shifting argument shows that this morphism is then an isomorphism, as claimed. □

We have the following corollary, which in the case of finite Y also follows directly from Lemma 3.1.3 together with [Paš13, Cor. 5.18] (as was already noted in Remark 3.2.8).

Corollary 3.5.6. *The inclusion $\mathcal{A}_Y \hookrightarrow \mathcal{A}$ preserves injectives. Consequently, if π and π' are objects of $\mathcal{A}_Y$, then the natural morphism*

$$\mathrm{Ext}^n_{\mathcal{A}_Y}(\pi, \pi') \to \mathrm{Ext}^n_{\mathcal{A}}(\pi, \pi')$$

is an isomorphism, for every value of n.

Proof. To show that an object π' of $\mathcal{A}$ is injective, it suffices to show that $\mathrm{Ext}^i_{\mathcal{A}}(\pi, \pi') = 0$ for all $i > 0$, for finitely generated objects π. If π' is an injective object of $\mathcal{A}_Y$, and if π is finitely generated, it follows from Lemma 3.5.5 that indeed these higher Ext do vanish. Thus π' is an injective object of $\mathcal{A}$. The claimed isomorphism of Ext groups follows immediately. ☐

3.5.7. *Ind-completion.* Corollary 3.5.4 shows that $\widehat{(-)}_Y$ is particularly well-behaved on $\mathcal{A}^{\mathrm{fg}}$, and this suggests the following modification of its definition. Namely, we recall (see e.g. Section A.1.2) the equivalence $\mathrm{Ind}(\mathcal{A}^{\mathrm{fg}}) \xrightarrow{\sim} \mathcal{A}$, and then extend the exact (by Corollary 3.5.4) functor $\widehat{(-)}_Y : \mathcal{A}^{\mathrm{fg}} \to \mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)$ formally to an exact colimit-preserving functor $\mathrm{Ind}(\mathcal{A}^{\mathrm{fg}}) \to \mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)$. We continue to denote this functor by $\widehat{(-)}_Y$, relying on context to distinguish it from the functor defined on $\mathcal{A}$ earlier.

There is a canonical functor $\mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y) \to \mathrm{Pro}\,\mathrm{Ind}(\mathcal{A}^{\mathrm{fg}}_Y)$, while the equivalence $\mathrm{Ind}(\mathcal{A}^{\mathrm{fg}}_Y) \to \mathcal{A}_Y$ induces an equivalence $\mathrm{Pro}\,\mathrm{Ind}(\mathcal{A}^{\mathrm{fg}}_Y) \xrightarrow{\sim} \mathrm{Pro}(\mathcal{A}_Y)$; composing these gives a functor

$$\mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y) \to \mathrm{Pro}(\mathcal{A}_Y). \tag{3.5.8}$$

We then have the following commutative diagram comparing our "redefined" $\widehat{(-)}_Y$ to our original definition:

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{A}^{\mathrm{fg}}) \arrow[r, "\widehat{(-)}_Y"] \arrow[d, "\sim"] & \mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y) \arrow[d, "(3.5.8)"] \\
\mathcal{A} \arrow[r, "\widehat{(-)}_Y"] & \mathrm{Pro}(\mathcal{A}_Y)
\end{tikzcd}$$

3.5.9. *Further results for finite Y.* We present some additional results that hold in the case when Y is finite.

Lemma 3.5.10. *Assume that Y is finite. If π is an object of $\mathcal{A}^{\mathrm{fg}}_Y$, and if "$\varprojlim_m$"$\pi_m$ is a countably indexed object of $\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)$, then the natural map*

$$\mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)}(\pi, \text{“}\varprojlim_m\text{”}\pi_m) \to \varprojlim_m \mathrm{Ext}^n_{\mathcal{C}}(\pi, \pi_m)$$

is an isomorphism for every n.

Proof. This follows from Lemma A.3.14 and the fact that all Hom modules in $\mathcal{A}^{\mathrm{fg}}_Y$ are of finite length. ☐

Our next result has some formal similarity to Lemma 3.5.5, but seems to lie deeper: its proof relies on Propositions 2.6.5 and 2.6.4 above, and these results require Paškūnas' work [Paš13] on the p-adic local Langlands correspondence for their proof.

Proposition 3.5.11. *Let Y be a finite subset of X. If π is an object of $\mathcal{A}_Y^{\mathrm{fg}}$ and π' is an object of $\mathcal{A}^{\mathrm{fg}}$, then passing to the completion along Y induces isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{A}^{\mathrm{fg}}}(\pi, \pi') \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})}(\pi, \widehat{\pi}'_Y).$$

Proof. Although the category $\mathcal{A}^{\mathrm{fg}}$ does not have enough injectives or projectives, the map in the statement of the proposition is well-defined by Corollary 3.5.4, viewing the Ext^i-groups as morphisms in the derived category, or equivalently as groups of Yoneda extensions.

Since Y is finite, $\mathcal{A}_Y^{\mathrm{fg}}$ is a finite length category, and so it decomposes as a direct product of blocks. Writing π as a direct sum according to this decomposition we see that it suffices to prove the proposition when $Y = \{x\}$ consists of a single closed points of X, possibly not defined over $\mathbf{F}$.

On the other hand, every object of $\mathcal{A}^{\mathrm{fg}}$ has a finite filtration whose graded pieces are either finite length representations or isomorphic to $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$ for a Serre weight σ. Since $\pi' \mapsto \widehat{\pi}'_Y$ is exact, it follows by dévissage that it suffices to prove the claim when π is irreducible, and either $\pi' = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$ or π' has finite length. When π' has finite length the result is immediate from Lemma A.3.12.

Assume now that $\pi' = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$. By the discussion in Section 2.1, there exists a polynomial $f_x(T) \in \mathcal{H}_G(\sigma) \cong \mathbf{F}[T]$ such that the inverse system

$$\left\{\pi'_n = \left(c\text{-}\mathrm{Ind}_{KZ}^G\, \sigma\right) / f_x^n\right\}$$

is cofinal in the diagram defining $\widehat{\pi}'_Y$. By construction, π'_1 is the largest multiplicity-free quotient of $c\text{-}\mathrm{Ind}_{KZ}^G\, \sigma$ contained in $\mathcal{A}_Y^{\mathrm{fg}}$.

It follows from the above and Lemma 3.5.10 that it suffices to prove that the map

$$\mathrm{Ext}^i(\pi, c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)) \to \varprojlim_n \mathrm{Ext}^i(\pi, \pi'_n)$$

induced by the projections $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma) \to \pi'_n$ is an isomorphism. Notice furthermore that f_x is regular on the compact induction, and so we have short exact sequences

$$0 \to \pi'_m \to \pi'_{n+m} \to \pi'_n \to 0$$

for all positive integers m, n.

As explained at the end of Section 3.6, it follows from Corollary 3.1.9 that we have an isomorphism

$$\varinjlim_n \mathrm{Ext}^i_{\mathcal{A}}(\pi, \pi'_n) \xrightarrow{\sim} \mathrm{Ext}^{i+1}_{\mathcal{A}}(\pi, c\text{-}\mathrm{Ind}_{KZ}^G(\sigma))$$

induced by connecting homomorphism of the exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G(\sigma) \to c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)[1/f_x] \to \varinjlim \pi'_n \to 0 \tag{3.5.12}$$

and the fact (proved in Lemma 2.5.6) that Ext^i commutes with filtered colimits.

Pulling back (3.5.12) along $\pi'_m \to \varinjlim \pi'_n$, and then pushing forward along $c\text{-Ind}_{KZ}^G(\sigma) \to \pi'_n$, we rephrase our goal as proving that the map

$$\varinjlim \mathrm{Ext}^i_{\mathcal{A}}(\pi, \pi'_m) \to \varprojlim \mathrm{Ext}^{i+1}_{\mathcal{A}}(\pi, \pi'_m)$$

induced by the connecting homomorphisms of the exact sequences

$$0 \to \pi'_m \to \pi'_{n+m} \to \pi'_n \to 0$$

is an isomorphism.

Making a finite unramified extension $\mathcal{O}'/\mathcal{O}$ and applying Proposition 2.5.13, and noting that all terms of the inverse limit have finite $\mathcal{O}$-length, we can furthermore assume that $Y = \{x\}$ is a block of absolutely irreducible representations. Hence the polynomial f_x can be taken to be linear in almost all cases: the only exception is for blocks of type (2) consisting of quotients of $c\text{-Ind}_{KZ}^G(\sigma)$ when σ is a twist of Sym^{p-2}. Indeed, when $\lambda \neq \pm 1$ the representation $c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda)(T-\lambda^{-1})$ is multiplicity-free and contained in a single block, and so we need to take $f_x = (T-\lambda)(T-\lambda^{-1})$, which is quadratic in T. However, the representation π'_m is a direct sum

$$\pi'_m = c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda)^m \oplus c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda^{-1})^m,$$

and the exact sequence $0 \to \pi'_m \to \pi'_{n+m} \to \pi'_n \to 0$ is also a direct sum of exact sequences. Hence the proposition follows from the next result. $\square$

Proposition 3.5.13. *Let σ be a Serre weight and $\lambda \in \mathbf{F}$. Define*

$$\tau_n = c\text{-Ind}_{KZ}^G(\sigma)/(T-\lambda)^n$$

and let π be an irreducible object of the block x containing τ_n. Then the connecting homomorphisms of the exact sequence

$$0 \to \tau_m \xrightarrow{(T-\lambda)^n} \tau_{n+m} \to \tau_n \to 0$$

induces an isomorphism

$$\varinjlim \mathrm{Ext}^i_{\mathcal{A}}(\pi, \tau_m) \to \varprojlim \mathrm{Ext}^{i+1}_{\mathcal{A}}(\pi, \tau_n)$$

for all $i \geq 0$.

Proof. Let $J_x = \bigoplus_{\pi \in x} J_\pi$ be the direct sum of injective envelopes of all the irreducible objects of $\mathcal{A}_x$. Then the functor

$$M_x(-) = \mathrm{Hom}_{\mathcal{A}}(-, J_x)^\vee$$

is an equivalence of $\mathcal{A}_x$ with the category of discrete left modules over $E_x = \mathrm{End}_{\mathcal{A}}(J_x)$. Hence it suffices to prove the proposition after applying $M_x(-)$.

Define $M = \varprojlim_i M_x(\tau_i)$, which is a module over E_x (in fact, since the $M_x(\tau_i)$ have finite length, this is actually a compact E_x-module). The Hecke operator $(T-\lambda)$ induces compatible endomorphisms of all the $M(\tau_i)$, hence also of M. There is an exact sequence

$$0 \to M \to M[1/(T-\lambda)] \to \varinjlim_i M_x(\tau_i) \to 0.$$

Since the groups

$$\mathrm{Ext}^i_{E_x}(M_x(\pi), M_x(\tau_j)) \cong \mathrm{Ext}^i_{\mathcal{A}_x}(\pi, \tau_j)$$

have finite dimension over k, the same argument as in Lemma A.3.14 implies that the natural map

$$\operatorname{Ext}^{i+1}_{E_x}(M_x(\pi), M) \xrightarrow{\sim} \varprojlim_j \operatorname{Ext}^{i+1}_{E_x}(M_x(\pi), M_x(\tau_j))$$

is an isomorphism. So we obtain a connecting homomorphism

$$\varinjlim_k \operatorname{Ext}^{i}_{E_x}(M_x(\pi), M_x(\tau_k)) \to \varprojlim_j \operatorname{Ext}^{i+1}_{E_x}(M_x(\pi), M_x(\tau_j))$$

which can be checked to coincide with the map induced by the connecting homomorphisms of the short exact sequences

$$0 \to M_x(\tau_m) \to M_x(\tau_{n+m}) \to M_x(\tau_n) \to 0$$

So it suffices to prove that

$$E^i = \operatorname{Ext}^i_{E_x}(M_x(\pi), M[1/(T-\lambda)]) = 0 \tag{3.5.14}$$

for all i. We are going to deduce this from Propositions 2.6.4 and 2.6.5, using the fact that this Ext^i-group is a module over the centre Z_x of E_x in a unique way under the actions on either factor.

Assume first that x is a block of type (1), (2) or (4). Then it follows from Proposition 2.6.4 that there exists an element $T_x - \lambda \in Z_x$ inducing the operator $T - \lambda$ on $M_x(\tau_i)$ for all i, and so on M too. Since π is an absolutely irreducible object of $\mathcal{A}_x$ and $T_x - \lambda = 0$ on τ_1, we also have $T_x - \lambda = 0$ on π. Hence the vanishing of (3.5.14) follows from the fact that $T_x - \lambda$ is simultaneously invertible and zero on E^i.

Assume now that x has type (3). Then E^i has an invertible endomorphism $T - \lambda$ arising from the second factor. Furthermore, by Proposition 2.6.5 there are central elements $z_0, \dots, z_n \in Z_x$ such that

$$(T-\lambda)^{n+1} + z_n(T-\lambda)^n + \cdots + z_1(T-\lambda) + z_0 = 0$$

as endomorphisms of $M_x(\tau_i)$, hence as endomorphisms of M and E^i. In addition, the z_i are contained in the maximal ideal $\mathfrak{m}_x$, and so they are zero on $M_x(\pi)$, which is the only irreducible object of the block. So they are zero on E^i. It follows that $(T-\lambda)^{n+1}$ is simultaneously invertible and zero on E^i, and so $E^i = 0$. $\square$

3.5.15. *From formal to literal completions.* When Y is finite, since the objects of $\mathcal{A}^{\mathrm{fg}}_Y$ are Artinian when regarded as objects of $\mathcal{A}$, it follows that the functor

$$\varprojlim : \operatorname{Pro}(\mathcal{A}^{\mathrm{fg}}_Y) \to \mathcal{O}[G]\text{-Mod},$$

given by mapping a pro-object "$\varprojlim_I$"π_i to its literal projective limit $\varprojlim_i \pi_i$ in the category of $\mathcal{O}[G]$-modules, is faithful and exact [Jen72, Cor. 7.2].

If we compose this with the exact (by Corollary 3.5.4) completion functor $\mathcal{A}^{\mathrm{fg}} \to \operatorname{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)$, we obtain the exact "literal completion along Y" functor

$$\varprojlim \widehat{(-)}_Y : \mathcal{A}^{\mathrm{fg}} \to \mathcal{O}[G]\text{-Mod},$$

which we denote by $\pi \mapsto \varprojlim \widehat{\pi}_Y$,

Lemma 3.5.16. *For any object π of $\mathcal{A}^{\mathrm{fg}}$, there is a natural morphism of $\mathcal{O}[G]$-modules $\pi \to \varprojlim \widehat{\pi}_Y$.*

Proof. As discussed in A.3.17, the unit of adjunction induces a morphism $\pi \to \widehat{\pi}_Y$ in $\mathrm{Pro}(\mathcal{A})$. Regarding this as a morphism in the larger category $\mathrm{Pro}(\mathcal{O}[G])$-Mod, and them passing to literal projective limits in $\mathcal{O}[G]$-Mod (and recalling that π is constant as a pro-object), we obtain the desired morphism. $\square$

The discussion at end of A.3.17 shows that, if we apply Lemma 3.5.1 to write $\widehat{\pi}_Y \xrightarrow{\sim}$ "$\varprojlim$"π', where π' runs over the cofiltered directed set of quotients of π', so that $\varprojlim \widehat{\pi}_Y = \varprojlim \pi'$, then the morphism of the preceding lemma is the one induced by the collection of quotient morphisms $\pi \to \pi'$.

If we pass to Ind-categories, then "literal projective limit" formally extends to an exact and faithful functor $\varprojlim : \mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}}) \to \mathrm{Ind}\,\mathcal{O}[G]$-Mod. Thus if π is an object of $\mathcal{A} \xrightarrow{\sim} \mathrm{Ind}\,\mathcal{A}^{\mathrm{fg}}$, we may form the "literal completion" $\varprojlim \widehat{\pi}$, which is an object of $\mathrm{Ind}\,\mathcal{O}[G]$-Mod. There is again a natural morphism $\pi \to \varprojlim \widehat{\pi}$.

3.5.17. *Some analogues for $\mathcal{A}_U$.* Suppose that $Y \subseteq U$ for some open subset U of X. Write $Z := X \setminus U$. If π and π' are objects of $\mathcal{A}_Z$ and $\mathcal{A}_Y$ respectively, then

$$\mathrm{Hom}_{\mathcal{A}_Y}(\widehat{\pi}_Y, \pi') = \mathrm{Hom}_{\mathcal{A}}(\pi, \pi') = 0$$

by Corollary 3.2.6. Thus $\widehat{(-)}_Y$ vanishes on $\mathcal{A}_Z$, and so the functor $\widehat{\pi}_Y$ factors through $\mathcal{A}_U$, and then extends canonically to a functor $\mathrm{Pro}(\mathcal{A}_U) \to \mathrm{Pro}(\mathcal{A}_Y)$ preserving cofiltered limits. We use the same notation $\widehat{(-)}_Y$ for this induced functor, so that by definition we have

$$\widehat{j_U^*\pi}_Y = \widehat{\pi}_Y, \tag{3.5.18}$$

for objects π of $\mathrm{Pro}(\mathcal{A})$.

Lemma 3.5.19. *If π_U is an object of $\mathrm{Pro}(\mathcal{A}_U)$, then there is a natural isomorphism $\widehat{j_*(\pi_U)}_Y \xrightarrow{\sim} \widehat{(\pi_U)}_Y$.*

Proof. Since the functors $\widehat{(-)}_Y$ and j_* are extended from $\mathcal{A}_U$ to $\mathrm{Pro}\,\mathcal{A}_U$ via the universal property of the latter, and similarly $\widehat{(-)}_Y$ is extended from $\mathcal{A}$ to $\mathrm{Pro}(\mathcal{A})$ via the universal property of the latter, it suffices to construct the natural isomorphism for objects π_U of $\mathcal{A}_U$.

Now the counit of adjunction gives an isomorphism $j_U^* j_{U*}\pi_U \xrightarrow{\sim} \pi_U$, and the required isomorphism follows from (3.5.18). $\square$

Recall from Corollary 3.2.5 that the unit of adjunction $\mathrm{id} \to j_{U*}j_U^*$ restricts to an isomorphism on $\mathcal{A}_Y$, so that j_U^* embeds $\mathcal{A}_Y$ fully faithfully into $\mathcal{A}_U$. Thus, if π_U is an object of $\mathrm{Pro}(\mathcal{A}_U)$ and π' is an object of $\mathrm{Pro}(\mathcal{A}_Y)$, we find that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}_Y)}(\widehat{(\pi_U)}_Y, \pi') &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}_Y)}(\widehat{(j_*\pi_U)}_Y, \pi') \\ &= \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(j_*\pi_U, \pi') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(j_*\pi_U, j_*j^*\pi') \\ &= \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}_U)}(\pi_U, j^*\pi').\end{aligned}$$

(The first isomorphism is given by Lemma 3.5.19, while, as already noted, the second isomorphism follows from (3.5.18). The equalities are instances of the various adjunctions that we've established.) Thus $\widehat{(-)}_Y$ on $\mathrm{Pro}(\mathcal{A}_U)$ satisfies an analogous adjunction to the one satisfied by $\widehat{(-)}_Y$ on $\mathrm{Pro}(\mathcal{A})$.

We now show that $\widehat{(-)}_Y$ on $\mathcal{A}_U$ satisfies analogues of the results we proved for $\widehat{(-)}_Y$ on $\mathcal{A}$.

Corollary 3.5.20. *If π is an object of $\mathcal{A}_U^{\mathrm{fg}}$, then $\widehat{\pi}_Y$ is an object of $\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$.*

Proof. This follows directly from Lemma 3.1.5(3) and Corollary 3.5.2. $\square$

Lemma 3.5.21. *The restriction of $\widehat{(-)}_Y$ to $\mathcal{A}_U^{\mathrm{fg}}$ is exact.*

Proof. Any exact sequence in $\mathcal{A}_U^{\mathrm{fg}}$ may be written as the image under j_U^* of an exact sequence in $\mathcal{A}^{\mathrm{fg}}$. The claim then follows from Corollary 3.5.4 and (3.5.18). $\square$

Lemma 3.5.22. *Let π_U be an object of $\mathcal{A}_U^{\mathrm{fg}}$, and (via Corollary (3.5.20)) write $\widehat{(\pi_U)}_Y =$ "$\varprojlim_I$"π_i, where the π_i are objects of $\mathcal{A}_Y^{\mathrm{fg}}$. If π' is an object of $\mathcal{A}_Y$, then, for any value of n, there are natural isomorphisms*

$$\mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{A}_Y)}(\widehat{(\pi_U)}_Y, \pi') \xrightarrow{\sim} \varinjlim_I \mathrm{Ext}^n_{\mathcal{A}_Y}(\pi_i, \pi') \to \mathrm{Ext}^n_{\mathcal{A}_U}(\pi_U, j^*\pi').$$

Proof. If we write $\pi_U = j^*\pi$ for some object π of $\mathcal{A}$, take into account (3.5.18) and apply Lemma 3.5.5, then the lemma follows from the chain of isomorphisms and adjunctions

$$\mathrm{Ext}^n_{\mathcal{A}}(\pi, \pi') \xrightarrow{\sim} \mathrm{Ext}^n_{\mathcal{A}}(\pi, j_*j^*\pi') = \mathrm{Ext}^n_{\mathcal{A}_U}(j^*\pi, j^*\pi') = \mathrm{Ext}^n_{\mathcal{A}_U}(\pi_U, j^*\pi')$$

(the isomorphism being the consequence of Corollary 3.2.5 that we've already noted). $\square$

3.6. **The functors Γ_Y and $R^1\Gamma_Y$.** If Y is a closed subset of X, then we define the functor $\Gamma_Y : \mathcal{A} \to \mathcal{A}_Y$ as follows: $\Gamma_Y(\pi)$ is the maximal subobject of π lying in $\mathcal{A}_Y$. Equivalently, $\Gamma_Y(\pi)$ is the kernel of the natural morphism $\pi \to j_*j^*\pi$ (where, as usual, we write $U := X \setminus Y$). The functor Γ_Y is right adjoint to the inclusion of $\mathcal{A}_Y$ in $\mathcal{A}$; since this latter functor is exact, we see that Γ_Y takes injectives in $\mathcal{A}$ to injectives in $\mathcal{A}_Y$. Corollary 3.5.6 then shows that Γ_Y in fact takes injectives in $\mathcal{A}$ to injectives in $\mathcal{A}$.

We may consider the derived functors $R^i\Gamma_Y$ of Γ_Y. Their computation is facilitated by the following result.

Lemma 3.6.1. *If I is an injective object of $\mathcal{A}$, then the natural map $I \to j_*j^*I$ is surjective.*

Proof. As already noted, it follows from Corollary 3.5.6 that $\Gamma_Y(I)$ is again an injective object of $\mathcal{A}$. Thus the inclusion $\Gamma_Y(I) \hookrightarrow I$ is split, and so the image J of I in j_*j^*I is yet again injective. The inclusion $J \hookrightarrow j_*j^*I$ is thus also split. The characterization given by Lemma A.2.2 then shows that J lies in the essential image of j_*, and thus that $J = j_*j^*J$. Thus the surjection $I \to J$ induces an identification $j_*j^*I \to j_*j^*J = J$, and the lemma is proved. $\square$

Corollary 3.6.2. *For any object π of $\mathcal{A}$, the derived functors of Γ_Y are computed by the complex*

$$\pi \to j_*j^*\pi.$$

In particular, the only non-trivial higher derived functor is $R^1\Gamma_Y$.

Proof. If $I^\bullet$ is an injective resolution of π, then the various $R^\bullet\Gamma_Y(\pi)$ are computed as the cohomology of the kernel of the morphism of complexes $I^\bullet \to j_*j^*I^\bullet$. Lemma 3.6.1 shows that this kernel coincides with the cone (up to a shift). Now since j_* is exact, the complex $j_*j^*I^\bullet$ resolves $j_*j^*\pi$, and so this shifted cone is quasi-isomorphic to the complex $\pi \to j_*j^*\pi$, proving the lemma. □

Since Γ_Y preserves injectives, we obtain, for any objects π of $\mathcal{A}_Y$ and π' of $\mathcal{A}$, a spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_{\mathcal{A}_Y}\big(\pi, R^q\Gamma_Y(\pi')\big) \implies \mathrm{Ext}^{p+q}_{\mathcal{A}}(\pi,\pi'). \tag{3.6.3}$$

In the case when $\pi' \to j_*j^*\pi'$ is injective, i.e. when $\Gamma_Y(\pi') = 0$, this simplifies to the formula

$$\mathrm{Ext}^p_{\mathcal{A}_Y}\big(\pi, R^1\Gamma_Y(\pi')\big) = \mathrm{Ext}^{p+1}_{\mathcal{A}}(\pi,\pi'),$$

which can be obtained directly by computing the long exact sequence of Ext's arising from the short exact sequence

$$0 \to \pi' \to j_*j^*\pi' \to R^1\Gamma_Y(\pi') \to 0,$$

and taking into account Corollary 3.1.9.

Lemma 3.4.1 shows that if π and $R^1\Gamma_Y(\pi)$ are both finitely generated, then in fact $R^1\Gamma_Y(\pi) = 0$. Thus $R^1\Gamma_Y(\pi)$ is typically *not* finitely generated. Nevertheless, we can gain some control over it, using the following results. (By contrast, if π is finitely generated then so is $\Gamma_Y(\pi)$, by Theorem 2.4.1, since it is a submodule of π.)

Lemma 3.6.4. *Suppose that Y is a closed subset of X, and that σ is a Serre weight. As in the statement of Proposition* 3.1.7, *write $f_\sigma^{-1}(Y) = V(g)$ for some $g \in \mathcal{H}(\sigma)$. If $g = 0$, then $\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) = c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma$ and $R^1\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) = 0$, while if $g \neq 0$, then $\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) = 0$, while $R^1\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) = (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/g]/(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)$.*

Proof. Proposition 3.1.7 shows that the exact sequence

$$0 \to \Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) \to c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \to j_*j^*c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \to R^1\Gamma_Y(\sigma) \to 0$$

reduces to either

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \to c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \to 0 \to 0 \to 0$$

or

$$0 \to 0 \to c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma \to (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/g] \to (c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma)[1/g]/(c\text{-}\mathrm{Ind}_{KZ}^G\,\sigma) \to 0,$$

depending on whether $g = 0$ or not. □

Lemma 3.6.5. *If Y is a closed subset of X and π is of finite length, then $R\Gamma_Y^1(\pi) = 0$.*

Proof. Since π is of finite length, it is an object of $\mathcal{A}_Z$ for some finite closed set Z of X, which we write in the form $Z := Z_1 \coprod Z_2$, where $Z_1 = Y \cap Z$ and $Z_2 = Z \setminus Z_1$. Correspondingly we may write $\pi = \pi_1 \oplus \pi_2$ with π_i an object of $\mathcal{A}^{\mathrm{fl}}_{Y_i}$. The complexes

$$0 \to \Gamma_Y(\pi_i) \to \pi_i \to j_*j^*\pi_i \to R^1\Gamma_Y(\pi_i) \to 0$$

then reduce to

$$0 \to \pi_1 \to \pi_1 \to 0 \to 0 \to 0,$$

respectively

$$0 \to 0 \to \pi_2 \to \pi_2 \to 0 \to 0,$$

in the case $i = 1$, respectively 2. In particular, in either case we have $R^1\Gamma_Y(\pi_i) = 0$, and so $R^1\Gamma_Y(\pi) = 0$. $\square$

We may now prove the following result.

Lemma 3.6.6. *If Y is a closed subset of X and π is finitely generated, then $R\Gamma^1_Y(\pi)$ is an Artinian object of $\mathcal{A}_Y$.*

Proof. By the usual dévissage, we reduce to the case when either π is of finite length, or π is of the form $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$. In the finite length case, Lemma 3.6.5 shows that $R^1\Gamma_Y$ even vanishes. In the case of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$, Lemma 3.6.4 reduces us to showing that if g is a non-zero element of $\mathcal{H}(\sigma)$, then $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)[1/g]/(c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is Artinian, which is Lemma 2.1.10. $\square$

Definition 3.6.7. If π is any object of $\mathcal{A}$, then we let $\pi_{\mathrm{f.l.}}$ denote the maximal subobject of π which is locally of finite length; equivalently, this is the maximal subobject of π which is locally admissible.

We may write $\pi_{\mathrm{f.l.}} = \varinjlim_Y \Gamma_Y(\pi)$, where Y runs over all finite closed subsets of X. We then see that this functor has a single non-vanishing higher derived functor, namely $\varinjlim_Y R^1\Gamma_Y$.

3.7. **More on** $\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$**.** We now extend some of the preceding constructions to the context of $\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$, since we will need them for the Beauville–Laszlo-type results of the following section.

We may form the quotient category $\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})_U := \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})/\mathcal{A}_Y^{\mathrm{fg}}$, and just as in the case of $\mathcal{A}$ considered above, we let $j^* : \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}}) \to \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})_U$ denote the canonical functor. Since this functor is exact, its Ind-extension admits a right adjoint (applying the Ind-analogue of the discussion of A.3.7), which we denote by $j_* : \mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})_U \to \mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$. We will only consider the restriction of this functor to $\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})_U$. (But note that the target of this restriction is still $\mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$.)

The completion functor $\widehat{(-)} : \mathcal{A}^{\mathrm{fg}} \to \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$ induces a corresponding functor

$$\mathcal{A}_U^{\mathrm{fg}} := \mathcal{A}^{\mathrm{fg}}/\mathcal{A}_Y^{\mathrm{fg}} \to \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})/\mathcal{A}_Y^{\mathrm{fg}} =: \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})_U,$$

which we again denote by $\widehat{(-)}$. By construction, then, the diagram

(3.7.1)
$$\begin{tikzcd} \mathcal{A}^{\mathrm{fg}} \arrow[r, "\widehat{(-)}"] \arrow[d, "j^*"] & \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}}) \arrow[d, "j^*"] \\ \mathcal{A}_U^{\mathrm{fg}} \arrow[r, "\widehat{(-)}"] & \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})_U \end{tikzcd}$$

commutes up to natural transformation.

We also have a diagram

(3.7.2)
$$\begin{tikzcd} \mathcal{A}_U^{\mathrm{fg}} \arrow[rr, "\widehat{(-)}"] \arrow[d, "j_*"] & & \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})_U \arrow[d, "j_*"] \\ \mathcal{A} \arrow[r, "\sim"] & \mathrm{Ind}\,\mathcal{A}^{\mathrm{fg}} \arrow[r, "\widehat{(-)}"] & \mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}}) \end{tikzcd}$$

(where we have extended $\widehat{(-)} : \mathcal{A}^{\mathrm{fg}} \to \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$ to the corresponding Ind-categories, as in 3.5.7 above), but the commutativity of this diagram is less obvious, and in fact we only establish it under the additional hypothesis that Y is finite. In that case, it is more-or-less the content of the following proposition.

Proposition 3.7.3. *Assume that Y is finite. If π is an object of $\mathcal{A}^{\mathrm{fg}}$, then there is a natural isomorphism $\widehat{j_*j^*\pi} \xrightarrow{\sim} j_*j^*\widehat{\pi}$.*

Proof. From (3.7.1) (and its formal extension to Ind-objects), we obtain the first and third of the following sequence of isomorphisms

$$j^*\widehat{j_*j^*\pi} \xrightarrow{\sim} \widehat{j^*j_*j^*\pi} \xrightarrow{\sim} \widehat{j^*\pi} \xrightarrow{\sim} j^*\widehat{\pi},$$

the second being clear. Thus, by adjunction, we obtain a morphism

$$\widehat{j_*j^*\pi} \to j_*j^*\widehat{\pi}, \tag{3.7.4}$$

which we must show is an isomorphism.

Completing the natural (unit to the adjunction) morphism $\pi \to j_*j^*\pi$ induces a morphism $\widehat{\pi} \to \widehat{j_*j^*\pi}$. There is also the natural (unit to the adjunction) morphism $\widehat{\pi} \to j_*j^*\widehat{\pi}$. The resulting diagram

$$\begin{tikzcd} \widehat{\pi} \arrow[r] \arrow[d, equal] & \widehat{j_*j^*\pi} \arrow[d, "(3.7.4)"] \\ \widehat{\pi} \arrow[r] & j_*j^*\widehat{\pi} \end{tikzcd} \tag{3.7.5}$$

is easily checked to commute. (By its construction, it suffices to check commutativity after applying j^*, in which case all the vertices become $\widehat{j^*\pi}$ and all the arrows become the identity morphism of $\widehat{j^*\pi}$ to itself.)

Our next step is to embed (3.7.5) in a larger commutative diagram. To begin with, we consider the tautological exact sequence

$$0 \to \Gamma_Y(\pi) \to \pi \to j_*j^*\pi \to R^1\Gamma_Y(\pi) \to 0,$$

with $\Gamma_Y(\pi)$ an object of $\mathcal{A}_Y^{\mathrm{fg}}$, π an object of $\mathcal{A}^{\mathrm{fg}}$, $j_*j^*\pi$ an object of $\mathcal{A} \xrightarrow{\sim} \mathrm{Ind}\,\mathcal{A}^{\mathrm{fg}}$, and $R^1\Gamma_Y(\pi)$ an object of $\mathcal{A}_Y \xrightarrow{\sim} \mathrm{Ind}\,\mathcal{A}_Y^{\mathrm{fg}}$. Applying the completion functor (in its Ind-category incarnation following 3.5.7 above), which is exact by Corollary 3.5.4, we obtain the exact sequence

$$0 \to \Gamma_Y(\pi) \to \widehat{\pi} \to \widehat{j_*j^*\pi} \to R^1\Gamma_Y(\pi) \to 0,$$

where we still have $\Gamma_Y(\pi)$ being an object of $\mathcal{A}_Y^{\mathrm{fg}}$, we have $\widehat{\pi}$ being an object of $\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$, we have $\widehat{j_*j^*\pi}$ being an object of $\mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})$, and we have $R^1\Gamma_Y(\pi)$ being an object of $\mathrm{Ind}\,\mathcal{A}_Y^{\mathrm{fg}}$.

We next define V and W to be the kernel and cokernel respectively of the natural morphism $\widehat{\pi} \to j_*j^*\widehat{\pi}$. The diagram (3.7.5) can then be placed in the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_Y(\pi) \arrow[r] \arrow[d] & \widehat{\pi} \arrow[r] \arrow[d, equal] & \widehat{j_*j^*\pi} \arrow[r] \arrow[d, "(3.7.4)"] & R^1\Gamma_Y(\pi) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V \arrow[r] & \widehat{\pi} \arrow[r] & j_*j^*\widehat{\pi} \arrow[r] & W \arrow[r] & 0 \end{tikzcd} \tag{3.7.6}$$

Now, by its construction, $\Gamma_Y(\pi)$ can be characterized as the object of $\mathcal{A}_Y^{\mathrm{fg}}$ that represents the functor $\mathrm{Hom}_{\mathcal{A}^{\mathrm{fg}}}(-,\pi)$ on $\mathcal{A}_Y^{\mathrm{fg}}$. Similarly, V can be characterized as the object of $\mathcal{A}_Y^{\mathrm{fg}}$ that represents the functor $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})}(-,\widehat{\pi})$ on $\mathcal{A}_Y^{\mathrm{fg}}$. The Ext^0 case of Proposition 3.5.11 then implies that the morphism $\Gamma_Y(\pi)\to V$ is an isomorphism.

A similar argument shows that the morphism $R^1\Gamma_Y(\pi)\to W$ is an isomorphism. Indeed, the long exact Ext sequences attached to the short exact sequences

$$0\to\pi/\Gamma_Y(\pi)\to j_*j^*\pi\to R^1\Gamma_Y(\pi)\to 0$$

and

$$0\to\widehat{\pi}/V\to j_*j^*\widehat{\pi}\to W\to 0$$

give rise to the horizontal arrows of the commutative square of functors (on $\mathcal{A}_Y^{\mathrm{fg}}$)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}\big(-,R^1\Gamma_Y(\pi)\big) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{A}}\big(-,\pi/\Gamma_Y(\pi)\big) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})}(-,W) \arrow[r] & \mathrm{Ext}^1_{\mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})}(-,\widehat{\pi}/V)
\end{tikzcd}$$

in which the horizontal arrows are monomorphisms. In fact, the discussion following Corollary 3.6.2 shows that the upper horizontal arrow is an isomorphism, and the Ext^1 case of Proposition 3.5.11 shows that the right-hand vertical arrow is also an isomorphism. It follows that the other two arrows — in particular, the left-hand vertical arrow — are isomorphisms. This implies that $R^1\Gamma_Y(\pi)\to W$ is an isomorphism.

We have now seen that the outer vertical arrows in (3.7.6) are isomorphisms. The five lemma then implies that (3.7.4) is an isomorphism, as required. $\square$

We finish this section by stating a result related to the proof of the preceding proposition. In order to make the statement, we first recall (continuing to assume that Y is finite) from 3.5.15 that we have the "literal projective limit" functor $\varprojlim:\mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})\to\mathrm{Ind}\,\mathcal{O}[G]\text{-Mod}$, and there is a natural morphism $\pi\to\varprojlim\widehat{\pi}$.

Lemma 3.7.7. *Assume Y is finite. If π is an object of $\mathcal{A}^{\mathrm{fg}}$, then there is a natural short exact sequence*

$$0\to\pi\to\widehat{\pi}\times j_*j^*\pi\to\widehat{j_*j^*\pi}\to 0$$

Proof. As in the proof of Proposition 3.7.3, we may complete the tautological exact sequence

$$0\to\Gamma_Y(\pi)\to\pi\to j_*j^*\pi\to R^1\Gamma_Y(\pi)\to 0$$

to obtain an exact sequence

$$0\to\Gamma_Y(\pi)\to\widehat{\pi}\to\widehat{j_*j^*\pi}\to R^1\Gamma_Y(\pi)\to 0.$$

Now formation of literal projective limits is exact; also both $\Gamma_Y(\pi)$ and $R^1\Gamma_Y(\pi)$ are in $\mathrm{Ind}\,\mathcal{A}_Y^{\mathrm{fg}}$ (i.e. are trivial in the Pro-aspect), and so $\varprojlim$ acts identically on each of them. Thus we obtain an exact sequence in $\mathrm{Ind}\,\mathcal{O}[G]$-Mod

$$0\to\Gamma_Y(\pi)\to\varprojlim\widehat{\pi}\to\varprojlim\widehat{j_*j^*\pi}\to R^1\Gamma_Y(\pi)\to 0.$$

Now consider the square

$$\begin{tikzcd} \pi \arrow[r] \arrow[d] & j_*j^*\pi \arrow[d] \\ \widehat{\pi} \arrow[r] & \widehat{j_*j^*\pi} \end{tikzcd}$$

If we regard this as a morphism (in the vertical direction) of complexes (in the horizontal direction), then the discussion of the preceding paragraph shows that the vertical morphisms induce a quasi-isomorphism of complexes. Passing to the associated total complex yields the required exact sequence. □

3.8. **Beauville–Laszlo-type gluing.** The results of this section are not used elsewhere in this paper, but they are crucially applied in [DEG]. The categorical p-adic Langlands correspondence developed in [DEG] relates smooth representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ to sheaves on stacks, and the arguments made in [DEG] rely on being able to glue representations in the same way that we can glue sheaves. In fact, in order to make use of the results of [Paš13], we need an analogue of Beauville–Laszlo gluing of sheaves (i.e. of gluing sheaves over formal completions at closed loci to sheaves on the complementary open substack). Theorem 3.8.1 provides such a result.

Theorem 3.8.1. *Suppose that Y is a finite closed subset of X, with open complement U. The canonical functor*

$$\mathcal{A}^{\mathrm{fg}} \longrightarrow \mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}}) \times_{\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}})_U} \mathcal{A}_U^{\mathrm{fg}},$$

induced by completion along Y and by localization over U, is an equivalence of categories.

Proof. As usual, it suffices to show that the functor is fully faithful and essentially surjective. We begin by showing faithfulness. To this end, suppose that $f, g : \pi_1 \rightrightarrows \pi_2$ are morphisms in $\mathcal{A}^{\mathrm{fg}}$, for which the induced morphisms $\widehat{\pi_1} \to \widehat{\pi_2}$ and $j^*\pi_1 \to j^*\pi_2$ coincide. We then find that the morphisms

$$\varprojlim \widehat{\pi_1} \times j_*j^*\pi_1 \to \varprojlim \widehat{\pi_2} \times j_*j^*\pi_2$$

coincide. The injectivity aspect of Lemma 3.7.7 shows that f and g in fact coincide.

We turn to proving fullness. Thus we suppose given a pair π_1, π_2 of objects of $\mathcal{A}^{\mathrm{fg}}$, together with morphisms $g : \widehat{\pi}_1 \to \widehat{\pi}_2$ and $h : j^*\pi_1 \to j^*\pi_2$, such that $j^*g = \widehat{h}$ as morphisms

$$\widehat{j^*\pi_1} = j^*\widehat{\pi_1} \to j^*\widehat{\pi}_2 = \widehat{j^*\pi_2}.$$

Applying j_*, we then find that j_*j^*g and $j_*\widehat{h}$ coincide as morphisms

$$\widehat{j_*j^*\pi_1} = j_*j^*\widehat{\pi_1} \to j_*j^*\widehat{\pi}_2 = \widehat{j_*j^*\pi_2}$$

(where we are using the identifications given by Proposition 3.7.3).

"Literal projective limit" formally extends to an exact and faithful functor $\varprojlim : \mathrm{Ind}\,\mathrm{Pro}(\mathcal{A}_Y^{\mathrm{fg}}) \to \mathrm{Ind}\,\mathcal{O}[G]\text{-Mod}$. Thus if π is an object of $\mathcal{A} \xrightarrow{\sim} \mathrm{Ind}\,\mathcal{A}^{\mathrm{fg}}$, we may form the "literal completion" $\varprojlim \widehat{\pi}$, which is an object of $\mathrm{Ind}\,\mathcal{O}[G]\text{-Mod}$. There is a natural morphism $\pi \to \varprojlim \widehat{\pi}$.

We now make use of the "literal projective limit" of 3.5.15, and the natural morphism $\pi \to \varprojlim \widehat{\pi}$, where π is an object of $\mathcal{A}$. We will apply this with $\pi = j_*j^*\pi_i$ $(i = 1, 2)$, and so obtain a commutative square

$$\begin{tikzcd} j_*j^*\pi_1 \arrow[r] \arrow[d, "j_*h"] & \varprojlim \widehat{j_*j^*\pi_1} \arrow[d, "\varprojlim j_*j^*g = \varprojlim \widehat{j_*h}"] \\ j_*j^*\pi_2 \arrow[r] & \varprojlim \widehat{j_*j^*\pi_2} \end{tikzcd} \tag{3.8.2}$$

We may also apply $\varprojlim$ to the natural morphisms $\widehat{\pi}_i \to \widehat{j_*j^*\pi}_i = j_*j^*\widehat{\pi}_i$, to obtain a commutative square

$$\begin{tikzcd} \varprojlim \widehat{\pi}_1 \arrow[r] \arrow[d, "\varprojlim g"] & \varprojlim \widehat{j_*j^*\pi_1} \arrow[d, "\varprojlim j_*j^*g = \varprojlim \widehat{j_*h}"] \\ \varprojlim \widehat{\pi}_2 \arrow[r] & \varprojlim \widehat{j_*j^*\pi_2} \end{tikzcd} \tag{3.8.3}$$

We may then "glue" the two squares (3.8.2) and (3.8.3) into the following commutative square, in which the horizontal arrows are given by taking the difference of the horizontal arrows in those preceding two squares:

$$\begin{tikzcd} \varprojlim \widehat{\pi}_1 \times j_*j^*\pi_1 \arrow[r] \arrow[d, "\varprojlim g \times j_*h"] & \varprojlim \widehat{j_*j^*\pi_1} \arrow[d, "\varprojlim j_*j^*g = \varprojlim \widehat{j_*h}"] \\ \varprojlim \widehat{\pi}_2 \times j_*j^*\pi_2 \arrow[r] & \varprojlim \widehat{j_*j^*\pi}_2 \end{tikzcd}$$

Now Lemma 3.7.7 shows that the vertical morphisms in this square induce a morphism $f : \pi_1 \to \pi_2$, which in turn gives rise to the morphisms $\varprojlim g$ and j_*h after literally completing, respectively applying j_*j^*. Since the formation of literal projective limits is faithful, we see that f gives rise to g after completion. We also see that

$$j^*f = j^*j_*j^*f = j^*j_*h = h.$$

Thus $f : \pi_1 \to \pi_2$ induces $(g, h) : (\widehat{\pi}_1, j_*\pi_1) \to (\widehat{\pi}_2, j_*\pi_2)$, and fullness is proved.

It remains to prove essential surjectivity. To this end, suppose that we are given a pair $\big(\widehat{\pi}_1, (\pi_2)_U\big)$, where $\widehat{\pi}_1$ is an object of $\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)$ and π_2 is an object of $\mathcal{A}^{\mathrm{fg}}$, with image $(\pi_2)_U := j^*\pi_2$ in the quotient category $\mathcal{A}^{\mathrm{fg}}_U$. We are furthermore given an isomorphism of these objects in the quotient category $\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)_U$, which we can represent by a "roof", i.e. a pair of morphisms $\widehat{\pi}_1, \widehat{\pi}_2 \rightrightarrows \widehat{\pi}_3$, where $\widehat{\pi}_2$ denotes the completion of π_2 along Y, and $\widehat{\pi}_3$ is another object of $\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)$. This pair of morphisms has the property that their kernels and cokernels lie in $\mathcal{A}^{\mathrm{fg}}_Y$. (This is why they give rise to an isomorphism in $\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)_U$.)

In order to show that the pair $\big(\widehat{\pi}_1, (\pi_2)_U\big)$ is in the image of the functor, we need in particular to find some object π_1 of $\mathcal{A}$ whose completion along Y is $\widehat{\pi}_1$. We will do this by repeatedly applying Proposition 3.5.11, and it will turn out that our construction also guarantees that $(\pi_1)_U = (\pi_2)_U$.

The kernel π' of $\widehat{\pi}_2 \to \widehat{\pi}_3$ is supported on Y by assumption, and so lies in $\Gamma_Y(\widehat{\pi}_2)$, which by Proposition 3.5.11 is equal to $\Gamma_Y(\pi_2)$. In other words, π' is actually a subobject of π_2, and the exactness of completion ensures that $(\pi_2/\pi')\widehat{} = \widehat{\pi}_2/\pi'$.

Again by assumption, the cokernel π'' of the morphism $\widehat{\pi}_2 \to \widehat{\pi}_3$ is supported on Y. Thus $\widehat{\pi}_3$ represents an element of $\mathrm{Ext}^1_{\mathrm{Pro}(\mathcal{A}^{\mathrm{fg}}_Y)}(\pi'', \widehat{\pi}_2/\pi')$. By Proposition 3.5.11, this corresponds to an element of $\mathrm{Ext}^1_{\mathcal{A}^{\mathrm{fg}}}(\pi'', \pi_2/\pi')$, i.e. we can find an object π_3 of $\mathcal{A}^{\mathrm{fg}}$ that contains π_2/π', with $(\pi_3/(\pi_2/\pi') = \pi''$, and whose completion along Y coincides with $\widehat{\pi}_3$.

Now let π''' denote the cokernel of the morphism $\widehat{\pi}_1 \to \widehat{\pi}_3$, which is yet again supported on Y. By Lemma 3.5.5, the surjection $\widehat{\pi}_3 \to \pi'''$ then induces and is induced by a surjection $\pi_3 \to \pi'''$. We denote the kernel by π_4. If we let $\pi^{(4)}$ denote the kernel of $\widehat{\pi}_1 \to \widehat{\pi}_3$, then we get a short exact sequence

$$0 \to \pi^{(4)} \to \widehat{\pi}_1 \to \widehat{\pi}_4 \to 0,$$

which (by an application of Proposition 3.5.11 as in the preceding paragraph) arises by completing a short exact sequence

$$0 \to \pi^{(4)} \to \pi_1 \to \pi_4 \to 0$$

of objects in $\mathcal{A}^{\mathrm{fg}}$. In other words, $\widehat{\pi}_1$ is the completion of an object π_1, admitting a morphism to π_3, which induces the given morphism $\widehat{\pi}_1 \to \widehat{\pi}_3$ after completion.

We claim that the functor of the proposition takes π_1 to the pair $\big(\widehat{\pi}_1, (\pi_2)_U\big)$. Indeed the completion of π_1 is $\widehat{\pi}_1$ by construction, while chasing through the construction above we find in turn that

$$(\pi_1)_U = (\pi_4)_U = (\pi_3)_U = (\pi_2)_U,$$

as required. □

Remark 3.8.4. It is likely that Theorem 3.8.1 holds for arbitrary closed subsets Y of X, and indeed a proof in this case should reduce to the finite case via an appropriate application of the gluing result for the (finite!) closed subset $\overline{U} \cap Y$. Since the case of finite Y suffices for our envisaged applications, we haven't tried to write up the details of this reduction.

4. Applications and examples

4.1. **The structure of smooth representations.** In many of our arguments we have used dévissage to reduce to the cases of irreducible representations, and representations of the form $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$. We can use our results to make more precise the way in which these representations interact. We begin with the following structural results.

Proposition 4.1.1. *Every object of $\mathcal{A}/\mathcal{A}^{\mathrm{l.adm}}$ is locally of finite length.*

Proof. Since $\mathcal{A}^{\mathrm{l.adm}}$ is localizing, the quotient functor $\mathcal{A} \to \mathcal{A}^{\mathrm{l.adm}}$ preserves colimits. It follows that it suffices to prove that $\pi = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$ is irreducible in the quotient category for all Serre weights σ. In order to do this, it suffices to prove that every nonzero arrow $X \to \pi$ in the quotient category is surjective. Writing X as a colimit of finitely generated representations, it suffices to prove this when X is finitely generated.

We can represent the arrow by a "roof diagram" in $\mathcal{A}$, i.e. a pair of arrows $X \to Z, \pi \to Z$ for some object Z, such that $\pi \to Z$ has locally admissible kernel and cokernel. Replacing Z with the subrepresentation generated by the images of X and π, we can assume that Z is finitely generated. Hence $\pi \to Z$ has kernel and cokernel of finite length.

Since π has no subobjects of finite length, it follows that $\pi \to Z$ is injective. Quotienting out by $Z_{\mathrm{f.l.}}$, we can assume that $\pi \to Z$ is furthermore an essential embedding. It follows from Lemma 3.1.10 that Z is isomorphic to a submodule of π. Since every nonzero submodule of π has cofinite length in π, it follows that every nonzero submodule of Z has cofinite length in Z. Hence the map $X \to Z$ has cokernel of finite length, and so it is surjective in the quotient category $\mathcal{A}/\mathcal{A}^{\mathrm{l.adm}}$, which was to be proved. $\square$

Remark 4.1.2. By definition (see e.g. [Kan21, §2.4]), Proposition 4.1.1 says that the Krull–Gabriel dimension of $\mathcal{A}$ is 1.

We have the following more concrete variant of Proposition 4.1.1. Recall from Definition 3.6.7 that we write $\pi_{\mathrm{f.l.}}$ for the maximal subobject of π which is locally of finite length (equivalently, locally admissible).

Proposition 4.1.3. *If $\pi \in \mathcal{A}$ is finitely generated, then $\pi/\pi_{\mathrm{f.l.}}$ is a successive extension of representations isomorphic to submodules of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$, for various Serre weights σ.*

Proof. We suppose that $\pi_{\mathrm{f.l.}} = 0$, and show that π is a successive extension of submodules of representations of the form $c\text{-}\mathrm{Ind}_{KZ}^G \sigma$. We write π as the quotient of $c\text{-}\mathrm{Ind}_{KZ}^G V$ for some smooth $\mathcal{O}[KZ]$-representation V of finite $\mathcal{O}[KZ]$-length, and argue by induction on the length of V. Let σ be an irreducible subrepresentation of V, and consider the induced morphism $c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to \pi$. If this morphism is zero, we may replace V by V/σ, and thus we may assume it is non-zero. Since $\pi_{\mathrm{f.l.}} = 0$ by assumption, it is then furthermore injective. Now consider the short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to \pi \to \pi/c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to 0$$

Pulling back along the finite length part of the target, we obtain a short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma \to \pi' \to (\pi/c\text{-}\mathrm{Ind}_{KZ}^G \sigma)_{\mathrm{f.l.}} \to 0$$

which is furthermore essential, since $\pi'_{\mathrm{f.l.}} \subseteq \pi_{\mathrm{f.l.}} = 0$. Lemma 3.1.10 and Remark 3.1.12 then show that π' is isomorphic to a submodule of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma'$ for some σ'. By construction, $(\pi/\pi')_{\mathrm{f.l.}} = 0$, and π/π' is a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G(V/\sigma)$. The result follows by induction. $\square$

Remark 4.1.4. If σ is not isomorphic to a twist of Sym^0 or Sym^{p-1}, it follows from Lemma 2.1.3 that every submodule of $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$ is isomorphic to $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$, and in fact coincides with the image of a Hecke operator in $\mathcal{H}(\sigma)$.

4.2. **Extensions between irreducible representations and compact inductions.** It is possible to use our results to compute Ext groups between irreducible representations and compact inductions in complete generality. We begin with the following qualitative statement, whose proof depends upon the main results of [Paš13].

Lemma 4.2.1. *If π is of finite length, and π' is finitely generated, then the $\mathcal{O}$-module $\mathrm{Ext}^i_{\mathcal{A}}(\pi, \pi')$ is of finite length for all $i \geq 0$.*

Proof. By dévissage it suffices to prove the claim when π' has finite length or π' is the compact induction of a weight. The first case is a consequence of [Paš13], so we assume that $\pi' = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$. The representation π is an object of $\mathcal{A}_Y$ for some finite Y, and the spectral sequence (3.6.3) implies that

$$\mathrm{Ext}^{i+1}_{\mathcal{A}}(\pi, \pi') \cong \mathrm{Ext}^i_{\mathcal{A}_Y}(\pi, R^1\Gamma_Y(\pi')).$$

By Lemma 3.6.4 and Lemma 2.1.10 we know that $R^1\Gamma_Y(\pi')$ is Artinian, and so it suffices to prove that $\mathrm{Ext}^i_{\mathcal{A}_Y}(\pi, \tau)$ has finite $\mathcal{O}$-length whenever τ is an Artinian object of $\mathcal{A}_Y$. However, the socle of τ has finite length, hence by Proposition 2.1.11 there exists an exact sequence

$$0 \to \tau \to J \to \tau' \to 0$$

where J is an injective Artinian object of $\mathcal{A}_Y$. It follows that τ' is also Artinian, and so by dimension shifting and induction it suffices to prove that $\mathrm{Hom}_{\mathcal{A}_Y}(\pi, \tau')$ has finite $\mathcal{O}$-length whenever τ' is Artinian. This follows again from Proposition 2.1.11. □

For extensions of absolutely irreducible representations and compact inductions of Serre weights we can be a lot more precise, as in the following proposition.

Proposition 4.2.2. *Let σ be a Serre weight and π an absolutely irreducible object of $\mathcal{A}$. Then $\dim \mathrm{Ext}^1_{\mathcal{A}}(\pi, c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)) \leq 1$, and if it is not zero then one of the following is true:*

(1) π is a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$ by some $T-\lambda$, and the extension is isomorphic to

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G(\sigma) \xrightarrow{T-\lambda} c\text{-}\mathrm{Ind}_{KZ}^G(\sigma) \to \pi \to 0.$$

(2) $\pi = \chi \circ \det$, $\sigma \cong \mathrm{Sym}^{p-1} \otimes (\chi \circ \det)|_{KZ}$, and the extension is isomorphic to

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G(\sigma) \to c\text{-}\mathrm{Ind}_{N}^G(\chi \circ \det) \to \pi \to 0.$$

(3) $\pi = (\chi \circ \det) \otimes \mathrm{St}_G$, $\sigma \cong (\chi \circ \det)|_{KZ}$, and the extension is isomorphic to the preimage of π under

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G(\sigma) \to c\text{-}\mathrm{Ind}_{KZ}^G(\mathrm{Sym}^{p-1} \otimes (\chi \circ \det)|_{KZ}) \to \pi \oplus (\pi \otimes (\mathrm{nr}_{-1} \circ \det)) \to 0.$$

Proof. Let x be the block of π and write $Y = \{x\}$. Since $\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)) = 0$ the spectral sequence (3.6.3) yields an isomorphism

$$\mathrm{Hom}(\pi, R^1\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma))) \xrightarrow{\sim} \mathrm{Ext}^1(\pi, c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)).$$

By Lemma 2.1.10 we know that the socle of $R^1\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma))$ has multiplicity one, which implies the assertion on dimensions.

Assume that π is not a character or a Steinberg twist. Then Lemma 2.1.10 implies that the socle of $R^1\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma))$ is isomorphic to a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$, and so if $\mathrm{Ext}^1(\pi, c\text{-}\mathrm{Ind}_{KZ}^G(\sigma))$ does not vanish then π is a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$. Since we have already proved that there is only one isomorphism class of nonsplit extensions, it follows that it must be the one given in the proposition.

Assume now that π is a character; the case of Steinberg twists is treated similarly. Twisting by $\chi^{-1} \circ \det$ we can assume that $\pi = 1$ is trivial and then we deduce from Lemma 2.1.10 that

$$\mathrm{soc}(R^1\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G(\mathrm{Sym}^{p-1}))) = \mathrm{soc}(c\text{-}\mathrm{Ind}_{KZ}^G(\mathrm{Sym}^{p-1})/(T-1)) \cong 1$$
$$\mathrm{soc}(R^1\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G(1))) = \mathrm{soc}(c\text{-}\mathrm{Ind}_{KZ}^G(1)/(T-1)) \cong \mathrm{St}_G$$

hence if $\mathrm{Ext}^1(\pi, c\text{-}\mathrm{Ind}_{KZ}^G(\sigma))$ does not vanish then $\sigma = \mathrm{Sym}^{p-1}$. To find a representative for the extension class it suffices to start from the exact sequences

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \xrightarrow{\beta} c\text{-}\mathrm{Ind}_{KZ}^G 1 \to 1 \oplus \mathrm{nr}_{-1} \to 0$$
$$0 \to c\text{-}\mathrm{Ind}_N^G(\delta) \to c\text{-}\mathrm{Ind}_{KZ}^G 1 \to 1 \to 0$$

defined in Lemma 2.1.3 and (3.2.2), and to twist the second one by nr_{-1}. □

Remark 4.2.3. Let Y be the block of the trivial representation of G. Using Lemma 2.1.12 we can refine the conclusion of Proposition 4.2.2: if τ is an extension of 1 by π which is not a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G(1)$ then

$$\mathrm{Hom}_G(\tau, R^1\Gamma_Y(c\text{-}\mathrm{Ind}_{KZ}^G(1))) = 0$$

and so there are no extensions of τ by $c\text{-}\mathrm{Ind}_{KZ}^G(1)$.

Using our results in Section 2.6 it is also possible to compute the Ext^1-groups in the other direction. For simplicity, we will only deal with compact inductions of generic Serre weights.

Proposition 4.2.4. *Let σ be a Serre weight, let π be an absolutely irreducible object of $\mathcal{A}$, and assume that σ is not a twist of $\mathrm{Sym}^0, \mathrm{Sym}^{p-2}$ or Sym^{p-1}. Then $\mathrm{Ext}^1_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma, \pi)$ is not zero if and only if the completion of $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$ at the block of π is not zero. If this is the case then*

$$\begin{aligned} \dim \mathrm{Ext}^1_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma, \pi) &= 2 \text{ if } \pi \text{ is supersingular} \\ &= 1 \text{ otherwise.} \end{aligned}$$

Proof. If the completion is zero then

$$\mathrm{Ext}^i_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma), \pi) = 0$$

for all i by Lemma 3.5.5. Otherwise, let $\tau = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)/(T-\lambda)$ be the only irreducible quotient of $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)$ in the same block of π, and let $\tau_i = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma)/(T-\lambda)^i$. By Lemma 3.5.5 there is an isomorphism

$$\varinjlim_i \mathrm{Ext}^1_{\mathcal{A}_Y}(\tau_i, \pi) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma), \pi)$$

which is equivariant for the action of $(T-\lambda)$ on the first factor. However, by Proposition 2.6.4 the Hecke operator $(T-\lambda)$ is induced by an element of the Bernstein centre of $\mathcal{A}_Y$, which must act by zero on π since it does so on τ. It follows that $(T-\lambda)$ is zero on $\mathrm{Ext}^1_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma), \pi)$, and so we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{A}}(\tau, \pi) \to \mathrm{Hom}(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma), \pi) \to \mathrm{Hom}(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma), \pi) \to \\ \to \mathrm{Ext}^1_{\mathcal{A}}(\tau, \pi) \to \mathrm{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G(\sigma), \pi) \to 0. \end{aligned}$$

The proposition follows from this together with the computation of Ext^1-groups between absolutely irreducible objects of $\mathcal{A}$. More specifically, we know that $\mathrm{Ext}^1_{\mathcal{A}}(\pi, \pi)$ has dimension 3 if π is supersingular [Paš10, Theorem 10.13] and dimension 2 otherwise [Paš13, Section 8]. On the other hand, if τ is in the same block of π but is not isomorphic to π then π is not supersingular, and $\dim \mathrm{Ext}^1_{\mathcal{A}}(\tau, \pi) = 1$, see [Paš13, Section 8]. □

4.3. **Extensions between compact inductions.** We can also consider the Ext groups between full compact inductions. We have the following general result.

Lemma 4.3.1. *Let σ_0 and σ_1 be Serre weights, and assume they are not twists of Sym^{p-1}. If $\mathrm{Ext}^i(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ is non-zero for some i, then either σ_0 and σ_1 are isomorphic, or they are the Jordan–Hölder factors of the reduction of a tame type.*

Proof. For $i = 0, 1$ let τ_i be the unique cuspidal type containing σ_i in its semisimplified mod p reduction. If $\tau_0 = \tau_1$ then $\sigma_0 = \sigma_1$ or, by definition, there exists a tame type whose reduction has Jordan–Hölder factors equal to σ_0 and σ_1.

Assume that $\tau_0 \neq \tau_1$. If τ_0 and τ_1 are not adjacent then the intersection $X(\tau_0) \cap X(\tau_1)$ is empty, by Proposition 2.2.7. Let j_i be the inclusion of the complement of $X(\tau_i)$ in X. By Proposition 3.1.7 (1), the natural maps

$$c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_0) \to j_{1*}j_1^* c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_0) \tag{4.3.1}$$

$$c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_1) \to j_{0*}j_0^* c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_1) \tag{4.3.1}$$

are isomorphisms. On the other hand, $c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_i)$ is an object of $\mathcal{A}_{X(\tau_i)}$ by Definition 3.1.1. Hence

$$\mathrm{Ext}^i(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0) = 0$$

by Corollary 3.1.9 and Corollary 3.4.5.

Finally, if τ_0 and τ_1 are adjacent but σ_0 and σ_1 are not constituents of a principal series type then the same argument as above shows that $\mathrm{Ext}^i(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ vanishes. Indeed, the image of f_{σ_0} does not intersect $X(\tau_1)$, and similarly the image of f_{σ_1} does not intersect $X(\tau_0)$. Hence the localization morphisms (4.3.1) are still isomorphisms. ☐

Remark 4.3.2. The same argument proves that if $\mathrm{Ext}^i(c\text{-}\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1}, c\text{-}\mathrm{Ind}_{KZ}^G \sigma)$ is non-zero for some i then $\sigma \in \{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}, \mathrm{Sym}^{p-3} \otimes \det\}$.

The computation of Ext^0 (i.e. of Hom) is due to Barthel and Livné [BL94] and is treated in Lemma 2.1.3. In the remainder of this section we explicitly compute some instances of the $\mathcal{H}(\sigma_0) \otimes_k \mathcal{H}(\sigma_1)$-module $\mathrm{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$. If we write $\mathcal{H}(\sigma_0) = \mathbf{F}[S]$ and $\mathcal{H}(\sigma_1) = \mathbf{F}[T]$, then $\mathcal{H}(\sigma_0) \otimes_{\mathbf{F}} \mathcal{H}(\sigma_1) = \mathbf{F}[S, T]$. (Note that even if σ_0 and σ_1 are isomorphic, the Hecke operators S and T give distinct actions on Ext^i, at least *a priori*.)

Lemma 4.3.3. *Suppose that σ_0 and σ_1 are the Jordan–Hölder factors of the reduction of an irreducible principal series type. Then $\mathrm{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ is one-dimensional, spanned by the class of $c\text{-}\mathrm{Ind}_{KZ}^G \tau$, where τ denotes the non-split extension of σ_1 by σ_0. As an $\mathcal{H}(\sigma_0) \otimes_k \mathcal{H}(\sigma_1)$-module it is isomorphic to $\mathbf{F}[S, T]/(S, T)$.*

Proof. Corollary 3.1.9 shows that $\mathrm{Ext}^i\big(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)[1/S]\big) = 0$ for all i, while Lemma 2.5.6 shows that

$$\mathrm{Ext}^i\big(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)[1/S]\big) = \mathrm{Ext}^i(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)[1/S];$$

thus we see that $\mathrm{Ext}^i(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ consists of S-torsion elements.

The long exact Ext sequence associated to the short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0 \xrightarrow{S^n} c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0 \to (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/S^n(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0) \to 0$$

yields an isomorphism

$$\operatorname{Hom}\bigl(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/S^n(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)\bigr) \xrightarrow{\sim} S^n\text{-torsion in } \operatorname{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0).$$

When $n = 1$, the quotients $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)/T(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)$ and $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/S(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ are isomorphic to the same supersingular irreducible representation π, but the thickenings

$$(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)/T^n(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)$$

and

$$(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/S^n(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$$

don't coincide to any higher order. More precisely, the elements of $\operatorname{Ext}^1(\pi, \pi)$ classifying these extensions for $n = 2$ are linearly independent: see [AB15, Theorem 1.2, Section 7.3] for a proof of this fact.

Hence

$$\operatorname{Hom}\bigl(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/S^n(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)\bigr)$$

is one-dimensional, no matter what the value of n is, because these morphisms have to factor through $c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1/T^n(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)$, and so through $c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1/T(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)$ by the discussion above. We find that $\operatorname{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ is one-dimensional, and annihilated by S.

If $g \in \mathcal{H}(\sigma_1)$ is coprime to T (i.e. does not vanish at $T = 0$), then Corollary 3.1.9 shows that $\operatorname{Ext}^i\bigl((c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)/g(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1), c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0\bigr) = 0$ for all i. A consideration of the long exact Ext sequence associated to the short exact sequence

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1 \xrightarrow{g\cdot} c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1 \to (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)/g(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1) \to 0$$

then shows that g acts invertibly on $\operatorname{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$. Thus this Ext module is annihilated by T as well, since it has dimension one over k.

It remains to show that $c\text{-}\mathrm{Ind}_{KZ}^G \tau$ is a non-split extension. Suppose otherwise. Then the surjection

$$c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0 \to (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/T(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$$

would extend to a surjection

$$c\text{-}\mathrm{Ind}_{KZ}^G \tau \to (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/T(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0),$$

inducing a KZ-equivariant embedding $\tau \hookrightarrow (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/T(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$. However, there is no such embedding, so $c\text{-}\mathrm{Ind}_{KZ}^G \tau$ is indeed non-split. $\square$

Remark 4.3.4. Here is another viewpoint on this computation. If we let τ' denote the non-split extension of σ_0 by σ_1, then there is an isomorphism $c\text{-}\mathrm{Ind}_{KZ}^G \tau \xrightarrow{\sim} c\text{-}\mathrm{Ind}_{KZ}^G \tau'$: indeed, the source is isomorphic to $c\text{-}\mathrm{Ind}_{IZ}^G \chi$ (for some character χ) and the target is isomorphic to $c\text{-}\mathrm{Ind}_{IZ}^G \chi^s$, and these are well-known to be isomorphic. Furthermore, the composite

$$0 \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0 \to c\text{-}\mathrm{Ind}_{KZ}^G \tau \xrightarrow{\sim} c\text{-}\mathrm{Ind}_{KZ}^G \tau' \to c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0 \to 0$$

is equal to the Hecke operator S. Thus we see that the element of the extension group $\operatorname{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ classified by $c\text{-}\mathrm{Ind}_{KZ}^G \tau$ is annihilated by S.

Remark 4.3.5. Let $\pi_i = c\text{-}\mathrm{Ind}_{KZ}^G(\sigma_i)$ where σ_0, σ_1 are as in Lemma 4.3.3. If we push out the extension $c\text{-}\mathrm{Ind}_{KZ}^G \tau$ along the inclusion

$$\pi_0 \hookrightarrow \pi_0[1/T],$$

then the resulting extension *does* split. Indeed, if ρ is the cuspidal type containing σ_1, then π_1 lies in $\mathcal{A}_{X(\rho)}$. On the other hand, Proposition 3.1.7 shows that $\pi_0[1/T] = j_{U*}j_U^*(\pi_0)$, where U denotes the complement in X of $X(\rho)$. So we see that $\mathrm{Ext}^1(\pi_1, \pi_0[1/T]) = 0$.

Concretely, this means that there is a KZ-equivariant embedding $\tau \hookrightarrow \pi_0[1/T]$ extending the canonical embedding $\sigma_0 \hookrightarrow \pi_0 \hookrightarrow \pi_0[1/T]$. This corresponds to the fact that $\pi_0[1/T]$ is a family of principal series representations of Serre weight σ_0.

Suppose now that σ_0, σ_1, are the constituents of a tame cuspidal type, and again write τ for the non-split extension of σ_1 by σ_0.

Lemma 4.3.6. *If σ_0, σ_1, are the constituents of a tame cuspidal type, then there are isomorphisms*

$$\begin{aligned}\mathrm{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0) & \\ &\xrightarrow{\sim} \mathrm{Ext}^1\big((c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)[1/T], c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0\big)\\ &\xrightarrow{\sim} \mathrm{Ext}^1\big(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)[1/S]\big)\\ &\qquad\xrightarrow{\sim} \mathrm{Ext}^1\big((c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)[1/T], (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)[1/S]\big).\end{aligned}$$

Proof. The usual argument shows that $\mathrm{Ext}^i_{\mathcal{A}}\big((c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)/T(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1), c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0\big) = 0$ for all i, so that multiplication by T induces an isomorphism from $\mathrm{Ext}^1_{\mathcal{A}}(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ to itself. Writing $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)[1/T]$ as the colimit of $c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1$ under multiplication by T, and using the spectral sequence

$$\begin{aligned}E_2^{p,q} := \varprojlim_{T\cdot}{}^{(p)} \mathrm{Ext}^q(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0) & \\ \implies \mathrm{Ext}^{p+q}\big((c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)[1/T], c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0\big)&\end{aligned}$$

(and recalling that for an inverse system of isomorphisms, the higher derived inverse limits all vanish), we obtain the first claimed isomorphism of the lemma. The remaining isomorphisms are proved similarly. $\square$

Notice that the localizations $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_i)[1/T]$ are algebraic families of principal series over the multiplicative group $\mathbf{G}_m$, a fact that goes back to [BL94]. It follows that the Ext groups in Lemma 4.3.6 can be computed using the techniques of [BP12, §6-7] to relate these Ext^1 of compact inductions to certain Ext^1 of Hecke modules. The result is the following analogue of Lemma 4.3.3, which would also follow by the method of [Eme10b, Section 4] if the δ-functor of derived ordinary parts had an extension to non-admissible representations. However, a more natural way to prove Lemma 4.3.7 (at least from the optic of this paper) is as a consequence of our expected results on Bernstein centres explained in Section 1.3, so we don't give a proof here.

Lemma 4.3.7. *If σ_0, σ_1, are the constituents of a tame cuspidal type, then as an $\mathbf{F}[S,T]$-module,* $\mathrm{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1, c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$ *is free of rank one over $\mathbf{F}[S,T]/(ST-1)$, generated by the class of* $c\text{-}\mathrm{Ind}_{KZ}^G \tau$.

Remark 4.3.8. In the context of Lemma 4.3.7 both compact inductions are objects of $\mathcal{A}_{X(\tau)}$ for the same cuspidal type τ. The compact induction $c\text{-}\mathrm{Ind}_{KZ}^G \tau$ is supported on the entirety of $X(\tau) \cong \mathbf{P}^1$, in the sense that it lies in $\mathcal{A}_{X(\tau)}$ and has irreducible quotients lying in every block parameterized by the closed points of $X(\tau)$. Indeed, its quotient $c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1$ has irreducible quotients lying over the whole of $f_{\sigma_1}(\mathbf{A}^1)$; the remaining point $X(\tau) \setminus f_{\sigma_1}(\mathbf{A}^1)$ corresponds to the supersingular quotient $(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)/T(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)$, which contains a copy of τ and is thus also a quotient of $c\text{-}\mathrm{Ind}_{KZ}^G \tau$.

If we let $U \cong \mathbf{G}_m$ denote the complement of the marked points in $X(\tau)$, then $j_{U*}j_U^*(c\text{-}\mathrm{Ind}_{KZ}^G \tau)$ "lies over" U in an intuitive sense, and gives a family of *atomes automorphes*. It is an extension of

$$j_{U*}j_U^*(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1) = (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1)[1/T]$$

by

$$j_{U*}j_U^*(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0) = (c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0)[1/S].$$

(Both the equalities follow from Proposition 3.1.7.)

To see the relationship between Lemma 4.3.7 and the Bernstein centre, notice that the equality $ST = 1$ in the endomorphism ring of

$$\mathrm{Ext}^1(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_1[1/T], c\text{-}\mathrm{Ind}_{KZ}^G \sigma_0[1/S])$$

implies that there exists an automorphism T^+ of $j_{U*}j_U^*(c\text{-}\mathrm{Ind}_{KZ}^G \tau)$ which induces S^{-1} on the subobject and T on the quotient. We expect the automorphism T^+ to be induced by the Bernstein centre of $\mathcal{A}_U$. Taking the cokernel of $T^+ - \lambda$ for $\lambda \in k^\times$ corresponds to taking the fibre at $\lambda \in \mathbf{G}_m$, and yields the corresponding *atome automorphe*.

Appendix A. Category-theoretic background

In this appendix we recall various results about abelian categories and their localizations. Much of this material goes back to Gabriel's thesis [Gab62], and most of the rest of it can be found in [KS06]. Many of the results that we need from the previous two references are collected in [Kan21, §2], which we frequently refer to for convenience.

In order to be able to use these references we need to fix a Grothendieck universe, which we do without further comment. Sets are *small* if they belong to this fixed universe, and all limits and colimits are assumed to be small, i.e. can be written as (co)limits over small indexing sets.

A.1. **Grothendieck and locally Noetherian categories.**

We recall that an abelian category $\mathcal{A}$ is a *Grothendieck category* if it satisfies (AB5) (which is to say that $\mathcal{A}$ is cocomplete and that the formation of filtered colimits in $\mathcal{A}$ is exact), and it furthermore admits a set of generators (i.e. a small set of objects G_i with the property that for any nonzero morphism $f : X \to Y$ in $\mathcal{A}$, there is a morphism $g : G_i \to X$ for some G_i such that $fg \neq 0$). Every object in a Grothendieck category admits an injective envelope [Gab62, §II.6 Thm. 2].

Recall that an object X of an abelian category is *Noetherian* if it satisfies the ascending chain condition on subobjects, and is *compact* if $\mathrm{Hom}(X, -)$ commutes with filtered colimits.

An abelian category is called *locally Noetherian* if it is Grothendieck, and furthermore admits a set of generators that are Noetherian.

Proposition A.1.1. *Suppose that $\mathcal{A}$ is a locally Noetherian category. Then*

1. *An object of $\mathcal{A}$ is compact if and only if it is Noetherian.*
2. *A filtered colimit of injective objects of $\mathcal{A}$ is injective.*
3. *If X is a Noetherian object of $\mathcal{A}$, then for each $n \geq 0$ the functor $\mathrm{Ext}^n_{\mathcal{A}}(X, -)$ commutes with filtered colimits.*

Proof. For (1), note that any Noetherian object of $\mathcal{A}$ is compact by [Gab62, §2.4 Cor. 1]. Conversely if X is compact then by writing $X = \varinjlim_i X_i$ as a filtered colimit of Noetherian objects, we see that we can factor the identity morphism 1_X through some X_i; so X is a retract of a Noetherian object and is thus itself Noetherian. Part (2) is part of [Gab62, §2.4 Cor. 1]. Part (3) is [Kan21, Prop. 2.7]. $\square$

A.1.2. *$\mathcal{A}$ vs. $\mathcal{A}^{\mathrm{fg}}$.* If $\mathcal{A}$ is a locally Noetherian abelian category, we write $\mathcal{A}^{\mathrm{fg}}$ for the full subcategory of compact (or equivalently, by the preceding result, Noetherian) objects. The canonical functor $\mathrm{Ind}(\mathcal{A}^{\mathrm{fg}}) \to \mathcal{A}$ given by evaluating inductive limits in $\mathcal{A}$ is an equivalence.

We can't expect the category $\mathcal{A}^{\mathrm{fg}}$ to have enough injectives, but can compute Exts in it via Yoneda Exts. We then have the following result.

Lemma A.1.3. *Suppose that $\mathcal{A}$ is locally Noetherian. If X and Y are objects of $\mathcal{A}^{\mathrm{fg}}$, then $\mathrm{Ext}^i_{\mathcal{A}^{\mathrm{fg}}}(X, Y) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ for all i.*

Proof. If $i = 1$, this reduces to the statement that $\mathcal{A}^{\mathrm{fg}}$ is closed under extensions in $\mathcal{A}$. If $i > 1$, then we may write a class in $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ as a Yoneda product $c \cup d$ for $c \in \mathrm{Ext}^{i-1}_{\mathcal{A}}(X, X')$ and $d \in \mathrm{Ext}^1_{\mathcal{A}}(X', Y)$ for some object X' of $\mathcal{A}$.

Writing X' as the filtered colimit of its compact subobjects, and taking into account Proposition A.1.1 3, we find an inclusion $i : X'' \hookrightarrow X$ of a compact object X'' and a class $c' \in \mathrm{Ext}^{i-1}_{\mathcal{A}}(X, X'')$ such that $c = i_* c'$. Then $c \cup d = i_* c' \cup d = c' \cup i^* d$, where now $i^* d \in \mathrm{Ext}^1_{\mathcal{A}}(X'', Y)$. The surjectivity of the morphism in the lemma follows by induction on i. The fact that it is an isomorphism follows by an easy δ-functor argument. $\square$

A.2. **Localizing categories.** Suppose that $\mathcal{A}$ is a Grothendieck category. If $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ (i.e. a non-empty full subcategory closed under the formation of subquotients and extensions in $\mathcal{A}$), then we may form the quotient category $\mathcal{A}/\mathcal{B}$. We are interested in the question of when the natural functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ has a right adjoint; if such a right adjoint exists, we say that $\mathcal{B}$ is a *localizing subcategory.* This right adjoint, if it exists, is necessarily left-exact: indeed, being a right adjoint, it preserves all limits. Furthermore, the counit of the adjunction is necessarily an isomorphism [Gab62, §III.2 Proposition 3], hence the right adjoint is fully faithful if it exists.

Since left adjoints necessarily preserve colimits, we see that for such a right adjoint to exist, the Serre subcategory $\mathcal{B}$ must be closed under the formation of (small) colimits in $\mathcal{A}$. (Since $\mathcal{B}$ is closed under forming quotients in $\mathcal{A}$, this is equivalent to asking that $\mathcal{B}$ be closed under the formation of arbitrary direct sums in $\mathcal{A}$.) It turns out that this condition is also sufficient for $\mathcal{B}$ to be localizing. The quotient category $\mathcal{A}/\mathcal{B}$ is then also a Grothendieck category. (See e.g. [Kan21, Rem. 2.21] for these facts.)

We note that the construction of the adjoint is not too difficult, modulo set-theoretic issues. Namely, if A is an object of $\mathcal{A}$, we consider the category of morphisms $A \to A'$ which project to an isomorphism in $\mathcal{A}/\mathcal{B}$ (equivalently, whose kernel and cokernel both lie in $\mathcal{B}$). Then taking the colimit of A' over this category gives the value of the adjoint on A.

We assume throughout the rest of this appendix that $\mathcal{B}$ is a localizing subcategory of $\mathcal{A}$, let $j^* : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ denote the natural projection, and let $j_* : \mathcal{A}/\mathcal{B} \to \mathcal{A}$ denote the right adjoint to j^*. As already noted, since j_* is a right adjoint, it is automatically left-exact. However, it needn't be exact in general. Another way to think of this is that the essential image of j_* is a full subcategory of $\mathcal{A}$ which is intrinsically an abelian category, but is not necessarily an abelian subcategory of $\mathcal{A}$. Indeed, this essential image will be an abelian subcategory of $\mathcal{A}$ if and only if j_* is actually exact.

Remark A.2.1. Our notation is motivated by the following example. If $j : U \subseteq X$ is the inclusion of a retrocompact open subset in a scheme X, then we have the adjoint pair $(j^*, j_*) : \mathrm{QCoh}(U) \to \mathrm{QCoh}(X)$, which realizes $\mathrm{QCoh}(U)$ as a Serre quotient category of $\mathrm{QCoh}(X)$: it is the quotient of $\mathrm{QCoh}(X)$ by the Serre subcategory $\mathrm{QCoh}_Z(X)$ consisting of quasicoherent sheaves, all of whose sections are supported on $Z := X \setminus U$. (Cf. [Sta, Tag 01PD].)

The essential image of j_* admits the following characterization:

Lemma A.2.2. *An object A of $\mathcal{A}$ lies in the essential image of j_* if and only if $\mathrm{Hom}(B, A) = \mathrm{Ext}^1(B, A) = 0$ for any object B of $\mathcal{B}$.*

Proof. The identity $j^*A = j^*A$ induces a morphism

$$A \to j_*j^*A \tag{A.2.3}$$

We note two basic facts:

(i) applying j^* to (A.2.3) recovers the identity morphism $j^*A = j^*A$;
(ii) the object A is in the essential image of j_* if and only if (A.2.3) is an isomorphism.

From (i), we see that the kernel and cokernel of (A.2.3) lie in $\mathcal{B}$. Thus if A satisfies the two conditions appearing in the statement of the lemma, we see that this morphism is indeed an isomorphism, so that by (ii), the object A lies in the essential image of j_*. (More precisely: the vanishing of the Hom gives that there is no kernel, and then the vanishing of the Ext^1 gives that $j_*j^*A = A \oplus B$, with B an object of $\mathcal{B}$. But $\mathrm{Hom}(B, j_*j^*A) = \mathrm{Hom}(j^*B, j^*A) = 0$ (because $j^*B = 0$); thus the inclusion of B in j_*j^*A is the zero map, so $B = 0$.)

Conversely, suppose that A lies in the essential image of j_*. Consider any morphism $B \to A$ whose source is an object of $\mathcal{B}$. We have a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ j_*j^*B \arrow[r] & j_*j^*A \end{tikzcd}$$

By assumption, the right-hand vertical arrow is an isomorphism, while $j^*B = 0$, since B is an object of $\mathcal{B}$, and thus also $j_*j^*B = 0$. Consequently the morphism $B \to A$ is equal to zero.

Now consider a short exact sequence $0 \to A \to E \to B \to 0$, with B an object of $\mathcal{B}$. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & j_*j^*A \arrow[r] & j_*j^*E \arrow[r] & j_*j^*B \arrow[r] & 0 \end{tikzcd}$$

Again, the left-hand vertical arrow is an isomorphism, and thus the top extension is pulled back from the bottom one. But also $j_*j^*B = 0$, so the bottom extension is trivial. Hence so is the top extension. □

Remark A.2.4. It is not hard to prove this result in a more constructive fashion, using the construction of j_* given above.

We now have the following result, which relates higher Ext's to higher direct images of j_*.

Lemma A.2.5. *Let A' be an object of $\mathcal{A}/\mathcal{B}$, and fix some $n \geq 2$. Then the following are equivalent:*

(1) $\mathrm{Ext}^i(B, j_*A') = 0$ for all objects B of $\mathcal{B}$, and all $0 \leq i \leq n$.
(2) $R^i j_*A' = 0$ for all $1 \leq i \leq n-1$.

Proof. Let $A' \hookrightarrow I^\bullet$ be an injective resolution (in the category $\mathcal{A}/\mathcal{B}$). Then $j_*I^\bullet$ computes the various $R^i j_*A'$. Since $j^*j_*I^\bullet \xrightarrow{\sim} I^\bullet$, we find that $j^*R^ij_*A' = 0$ for all $i > 0$, and thus that R^ij_*A' is an object of $\mathcal{B}$ for $i > 0$. In particular, we see that $R^ij_*A' = 0$ if and only if $\mathrm{Hom}_{\mathcal{A}}(B, R^ij_*A') = 0$ for all objects B of $\mathcal{B}$.

Now since j_* is right adjoint to an exact functor, it preserves injectives, and thus $\mathrm{Hom}_{\mathcal{A}}(B, j_*I^\bullet)$ computes $\mathrm{RHom}_{\mathcal{A}}(B, j_*I^\bullet)$, for any object B of $\mathcal{B}$. But this Hom is identically zero, by Lemma A.2.2, and so $\mathrm{RHom}_{\mathcal{A}}(B, j_*I^\bullet) = 0$. On the other hand, we have the usual E_2 spectral sequence computing this RHom, which thus becomes

$$E_2^{p,q} := \mathrm{Ext}^p_{\mathcal{A}}(B, R^qj_*A') \implies 0.$$

It is now easy to prove the lemma by analyzing this spectral sequence. In more detail, to see that the first condition implies the second, note that by the conclusion of the preceding paragraph, it is enough to prove that $E_2^{0,q}$ vanishes for $1 \leq q \leq n-1$; and if this holds for some q, then actually $E_2^{p,q} = 0$ for all p. In fact, if $E_2^{0,q} = 0$ for some q and for all objects B of $\mathcal{B}$ then $R^qj_*A' = 0$, again by the conclusion of the preceding paragraph, and this implies that $E_2^{p,q} = 0$ for all p. □

As a corollary, we have the following result.

Corollary A.2.6. *The following are equivalent:*

(1) The functor j_* is exact.
(2) The derived functor R^1j_* is identically zero.
(3) For all objects A lying in the essential image of j_*, and for all objects B of $\mathcal{B}$, we have $\mathrm{Ext}^2(B, A) = 0$.

If these equivalent conditions hold, then in fact all the derived functors R^ij_* $(i \geq 1)$ vanish, and $\mathrm{Ext}^i(B, A) = 0$ for all $i \geq 0$ and objects A lying in the essential image of j_* and B lying in $\mathcal{B}$.

Proof. The equivalence of (1) and (2) is immediate. Lemma A.2.5 with $n = 2$ yields that (2) implies (3), and the converse implication follows from Lemma A.2.5 and Lemma A.2.2. □

We also note the following result, which in practice can simplify the checking of the various vanishing conditions introduced in the preceding results.

Lemma A.2.7. *If $\{B_j\}$ is a system of generators of $\mathcal{B}$, and A is an object of $\mathcal{A}$, then the following are equivalent:*

(1) $\mathrm{Ext}^i(B_j, A) = 0$ *for all B_j and all $i \leq n$.*
(2) $\mathrm{Ext}^i(B, A) = 0$ *for all objects B of $\mathcal{B}$, and all $i \leq n$.*

Proof. Clearly (2) implies (1), and so we focus on proving the converse. To say that $\{B_j\}$ is a system of generators of $\mathcal{B}$ is to say that for any object B of $\mathcal{B}$, we may find an epimorphism $\bigoplus_j B_j^{\oplus I_j} \to B$. Since $\mathcal{B}$ is a Serre subcategory which is closed under the formation of colimits in $\mathcal{A}$, the kernel of this epimorphism is again an object of $\mathcal{B}$. A straightforward dimension-shifting argument reduces us to checking the vanishing of (2) in the case when $B = \bigoplus_j B_j^{\oplus I_j}$. In this case, we immediately compute that $\mathrm{Ext}^i(B, A) = \prod_j \mathrm{Ext}^i(B_j, A)^{I_j}$. The vanishing claimed in (2) thus follows from the vanishing assumed in (1). □

Lemma A.2.8. *Suppose that $\mathcal{A}$ is locally Noetherian. Then:*

(1) j_* commutes with filtered colimits.
(2) Both $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are locally Noetherian.
(3) An object of $\mathcal{B}$ is Noetherian if and only if it is Noetherian as an object of $\mathcal{A}$, while an object of $\mathcal{A}/\mathcal{B}$ is Noetherian if and only if it is isomorphic to the image of a Noetherian object of $\mathcal{A}$.

Proof. The first two statements are [Gab62, §III.4 Cor. 1]. The third statement is immediate from [Kan21, Prop. 2.22]. □

A.3. **Pro-categories (and Ind-categories).** We now recall some standard properties of Pro- and Ind-categories. It suffices to develop the theory of Pro-categories, because for any category $\mathcal{C}$, there is an equivalence $\mathrm{Ind}(\mathcal{C}) \cong \mathrm{Pro}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$; in the body of the paper we make far more use of Pro-categories than Ind-categories, so we do not explicitly state results for Ind-categories in this appendix. (Note also that some of our references, in particular [KS06], take the opposite approach, so the corresponding statements for Ind-categories are often more readily available in the literature.)

If $\mathcal{C}$ is a category, one can define its associated pro-category $\mathrm{Pro}(\mathcal{C})$ to be the category whose objects are the diagrams $F : I \to \mathcal{C}$ indexed by a cofiltered small category I, with the morphisms between two diagrams $F : I \to \mathcal{C}, G : J \to \mathcal{C}$ being defined by the formula

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(F, G) = \varprojlim_J \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(F(i), G(j)).$$

We will sometimes denote the diagram $F : I \to \mathcal{C}$ via "$\varprojlim_I$"$F(i)$, and refer to it as a pro-object of $\mathcal{C}$. The point of this notation is to distinguish the pro-object from the limit $\varprojlim_I F(i)$ in $\mathcal{C}$ itself, if this limit happens to exist. Often, when employing this notation, we write X_i rather than $F(i)$, and so denote the object F of $\mathrm{Pro}(\mathcal{C})$ as "$\varprojlim_I$"X_i.

As a special case of the definition of morphisms, we note that if $X \in \mathcal{C}$ and $F : I \to \mathcal{C}$ is a cofiltered diagram, then the definitions imply that

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\text{“}\varprojlim_{I}\text{”} X_i, X) = \varinjlim_{I} \mathrm{Hom}_{\mathcal{C}}(X_i, X) \tag{A.3.1}$$

and

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, \text{“}\varprojlim_{I}\text{”} X_i) = \varprojlim_{I} \mathrm{Hom}_{\mathcal{C}}(X, X_i). \tag{A.3.2}$$

There is an equivalent definition of $\mathrm{Pro}(\mathcal{C})$, in which the object “$\varprojlim_I$”$F(i)$ is interpreted as the functor

$$\varinjlim_{i} \mathrm{Hom}_{\mathcal{C}}\big(F(i), –\big) \tag{A.3.3}$$

on $\mathcal{C}$. To give a little more detail: $\mathcal{C}$ admits its co-Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Sets})^{\mathrm{op}}$ via $X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, –)$. If $F : I \to \mathcal{C}$ is a diagram indexed by a cofiltered small category, then we can form the actual projective limit $\varprojlim_I F(i)$ in $\mathrm{Fun}(\mathcal{C}, \mathrm{Sets})^{\mathrm{op}}$; this yields precisely the functor (A.3.3). In this way we obtain a functor $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Sets})^{\mathrm{op}}$, which is fully faithful.

This functor $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Sets})^{\mathrm{op}}$ is an instance of a more general construction, coming from the universal mapping property that $\mathrm{Pro}(\mathcal{C})$ satisfies: namely, if $\mathcal{D}$ is a category that admits all cofiltered limits, then pullback of functors along the embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ induces an equivalence of categories of functors

$$\mathrm{Fun}'\big(\mathrm{Pro}(\mathcal{C}), \mathcal{D}\big) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D}), \tag{A.3.4}$$

where the domain denotes the full subcategory of $\mathrm{Fun}\big(\mathrm{Pro}(\mathcal{C}), \mathcal{D}\big)$ consisting of those functors that preserve cofiltered limits. (See e.g. [Lur09, Prop. 5.3.5.10].) The quasi-inverse generalizes the preceding construction: if $F : \mathcal{C} \to \mathcal{D}$ is a functor whose target admits cofiltered limits, we extend F to $\mathrm{Pro}(\mathcal{C})$ via

$$F(\text{“}\varprojlim_{I}\text{”} X_i) := \varprojlim_{I} F(X_i).$$

A.3.5. *Finite limits.* One useful fact [KS06, Cor. 6.1.14] is that any morphism in $\mathrm{Pro}(\mathcal{C})$ can be written as a cofiltered limit, over some small category I, of morphisms $X_i \to Y_i$ in $\mathcal{C}$. Similarly, any pair of morphisms having the same domain and codomain may be written as a cofiltered limit, over some small category I, of morphism pairs $X_i \rightrightarrows Y_i$ in $\mathcal{C}$ (see [KS06, Cor. 6.1.15]). Then, if $\mathcal{C}$ admits equalizers, the same is true of $\mathrm{Pro}(\mathcal{C})$, and furthermore we can compute the equalizer of “$\varprojlim_I$”$(X_i \rightrightarrows Y_i)$ as the limit in $\mathrm{Pro}(\mathcal{C})$ of the equalizers of the morphisms $X_i \rightrightarrows Y_i$ (see [AM69, Appendix, Prop. 4.1]). Likewise, if $\mathcal{C}$ admits finite products, then so does $\mathrm{Pro}(\mathcal{C})$, and these are computed “pointwise” in the same way as equalizers. It follows that if $\mathcal{C}$ admits finite limits then so does $\mathrm{Pro}(\mathcal{C})$, and then $\mathrm{Pro}(\mathcal{C})$ admits *all* small limits (since it admits finite limits and cofiltered limits). Compare [KS06, Proposition 6.1.18].

A.3.6. *Finite colimits.* If $\mathcal{C}$ admits finite colimits, the same is true of $\mathrm{Pro}(\mathcal{C})$, and finite coproduct and coequalizers can be computed “pointwise”, as for equalizers: this follows again from an application of [AM69, Appendix, Prop. 4.1]. More precisely, the coequalizer of “$\varprojlim_I$”$X_i \rightrightarrows Y_i$ is the limit in $\mathrm{Pro}(\mathcal{C})$ of the coequalizers of the various $X_i \rightrightarrows Y_i$, while

$$\text{“}\varprojlim_I\text{”} X_i \coprod \text{“}\varprojlim_J\text{”} Y_j = \text{“}\varprojlim_{I\times J}\text{”} X_i \coprod Y_j.$$

In particular, the inclusion $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ preserves finite colimits. Furthermore, finite colimits commute with cofiltered limits: compare [KS06, Proposition 6.1.19].

A.3.7. *Pro-adjoints.* Suppose that $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor preserving finite limits, between categories each of which admits finite limits. Then we obtain an induced functor $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}_1) \to \mathrm{Pro}(\mathcal{C}_2)$, which, by the discussions of Section A.3.5, also preserves equalizers and finite products, hence finite limits. The functor $\mathrm{Pro}(F)$ can also be regarded as corresponding, under the equivalence (A.3.4), to the composite $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathrm{Pro}(\mathcal{C}_2)$, and so is seen to preserve cofiltered limits. Thus $\mathrm{Pro}(F)$ preserves arbitrary limits, and hence admits a left adjoint $G : \mathrm{Pro}(\mathcal{C}_2) \to \mathrm{Pro}(\mathcal{C}_1)$. (This is a consequence of the special adjoint functor theorem, which applies because $\mathrm{Pro}(\mathcal{C}_i)$ is complete, by A.3.5, and has a cogenerator, see e.g. [Kan21, Thm. 2.39].)

A.3.8. *The abelian case.* We assume throughout this subsection that $\mathcal{C}$ is an abelian category (so that in particular all of the discussions above hold, in particular those of Section A.3.5). Then one can alternatively define $\mathrm{Pro}(\mathcal{C})$ as the opposite of the category of left-exact covariant functors $\mathcal{C} \to \mathrm{Ab}$, the point being that (A.3.3) is now abelian group-valued and left exact.

The category $\mathrm{Pro}(\mathcal{C})$ is again abelian and has exact cofiltered limits. (See e.g. [Kan21, Thm. 2.39].)

The following lemma provides a criterion for testing if a morphism in $\mathrm{Pro}(\mathcal{C})$ is a monomorphism.

Lemma A.3.9. *Suppose given a morphism $Y \to Z$ in $\mathrm{Pro}(\mathcal{C})$ with the property that, for any morphism $Y \to Y'$ with Y' an object of $\mathcal{C}$, we may find a commutative square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ Y' \arrow[r] & Z' \end{tikzcd} \tag{A.3.10}$$

in $\mathrm{Pro}(\mathcal{C})$ in which the bottom horizontal arrow is a monomorphism; then the given morphism $Y \to Z$ is a monomorphism.

Proof. If we let X denote the kernel of the morphism $Y \to Z$ in $\mathrm{Pro}(\mathcal{C})$, and write $Y = \text{“}\varprojlim_I\text{”} Y_i$ with the Y_i being objects of $\mathcal{C}$ — so that in fact $Y = \varprojlim_I Y_i$ in $\mathrm{Pro}(\mathcal{C})$ — then the exact sequence $0 \to X \to Y \to Z$ in $\mathrm{Pro}(\mathcal{C})$ induces exact sequences

$$0 \to X_i \to Y_i \to Z_i := Z \coprod_Y Y_i$$

in $\mathrm{Pro}(\mathcal{C})$, where X_i is defined to be the image (again, in $\mathrm{Pro}(\mathcal{C})$) of X in Y_i. We have $X = \varprojlim_I X_i$ and $Z = \varprojlim_i Z_i$, the limits being taken in $\mathrm{Pro}(\mathcal{C})$. (The second isomorphism is an instance of a finite colimit commuting with a cofiltered limit. The first isomorphism then follows from the second.) Thus we see that $X = 0$ if and only each $X_i = 0$.

Now form a diagram of the form (A.3.10) with $Y' = Y_i$. The morphism $Y_i = Y' \to Z'$ then factors through Z_i, and since the former morphism is a monomorphism by hypothesis, the morphism $Y_i \to Z_i$ must also be. Hence we find that $X_i = 0$, as required. $\square$

Lemma A.3.11. *If $X \in \mathcal{C}$ is an injective object then it remains injective in* $\mathrm{Pro}(\mathcal{C})$.

Proof. Let $\alpha : F \to G$ be a monomorphism in $\mathrm{Pro}(\mathcal{C})$. By [AM69, Appendix, Prop. 4.6] one can represent F and G by diagrams $F : I \to \mathcal{C}, G : I \to \mathcal{C}$ from the same index category I in such a way that α is represented by a natural transformation $\alpha : F \to G$ such that $\alpha_i : F_i \to G_i$ is a monomorphism for all $i \in I$.

Assume given a map $\lambda : F \to X$. By definition, it arises from a map $\lambda : F_i \to X$ for some i. Since α_i is a monomorphism and X is injective, we can extend λ to a map $G_i \to X$, which proves that X is injective in $\mathrm{Pro}(\mathcal{C})$. $\square$

Lemma A.3.12. *Suppose that $\mathcal{C}$ has enough injectives. If $X, Y \in \mathcal{C}$, then the natural map $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) \to \mathrm{Ext}^i_{\mathrm{Pro}(\mathcal{C})}(X, Y)$ is an isomorphism for all $i \geq 0$.*

Proof. This is a direct consequence of Lemma A.3.11 and the fact that $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ is fully faithful and exact. Indeed, Lemma A.3.11 and our assumption that $\mathcal{C}$ has enough injectives implies that Y admits an injective resolution $Y \to I^\bullet$ in $\mathcal{C}$ which is simultaneously an injective resolution in $\mathrm{Pro}(\mathcal{C})$, and then $\mathrm{Hom}_{\mathcal{C}}(X, I^\bullet)$ computes both $\mathrm{Ext}^i_{\mathcal{C}}(X, Y)$ and $\mathrm{Ext}^i_{\mathrm{Pro}(\mathcal{C})}(X, Y)$. $\square$

The preceding result admits the following evident extension.

Lemma A.3.13. *Assume that $\mathcal{C}$ has enough injectives. Let “$\varprojlim_I$ ”X_i be an object of* $\mathrm{Pro}(\mathcal{C})$*, and Y be an object of $\mathcal{C}$. Then the natural map*

$$\varinjlim_I \mathrm{Ext}^n_{\mathcal{C}}(X_i, Y) \to \mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{C})}(\text{“}\varprojlim_I\text{”} X_i, Y),$$

obtained by pullback and Lemma A.3.12, is an isomorphism for each $n \geq 0$.

Proof. Choose an injective resolution $Y \to J^\bullet$ in $\mathcal{C}$. By Lemma A.3.11 this is an injective resolution in $\mathrm{Pro}(\mathcal{C})$, and we have

$$\varinjlim_I \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X_i, J^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\text{“}\varprojlim_I\text{”} X_i, J^\bullet).$$

Since the passage to cohomology commutes with $\varinjlim$, the lemma follows. $\square$

In general, it's harder to say anything about Ext groups computed in the other direction, i.e. of the form $\mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{C})}(X, \text{“}\varprojlim_I\text{”} Y_i)$, but we have the following result which treats a special case.

Lemma A.3.14. *Assume that $\mathcal{C}$ is Artinian and $\mathcal{O}$-linear, for some commutative ring $\mathcal{O}$. If X is an object of $\mathcal{C}$ such that the $\mathcal{O}$-modules $\mathrm{Ext}^n(X, Y)$ are of finite length for all n and all objects Y of $\mathcal{C}$, and if “$\varprojlim_m$ ”Y_m is a countably indexed object of* $\mathrm{Pro}(\mathcal{C})$*, then the natural map*

$$\mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{C})}(X, \text{“}\varprojlim_m\text{”} Y_m) \to \varprojlim_m \mathrm{Ext}^n_{\mathcal{C}}(X, Y_m) \tag{A.3.15}$$

is an isomorphism for every n.

Proof. Replacing each Y_m by the "infimum" of the collection of images of transition morphisms $Y_{m'} \to Y_m$ for $m' \geq m$ (by Artinianness, this descending sequence of images stabilizes) — which replaces "$\varprojlim_n$"Y_m by an isomorphic pro-object — we may, and do, assume that the transition morphisms between the Y_m are epimorphisms. Since $\mathrm{Pro}(\mathcal{C})$ has enough projectives ([Oor64, p. 229]), we can compute $\mathrm{Ext}^n_{\mathrm{Pro}(\mathcal{C})}(X, \text{“}\varprojlim_m\text{”} Y_m)$ by taking a projective resolution $P_\bullet \to X$.

The projectivity of the terms of $P_\bullet$, together with the fact that the transition maps between the Y_m are epic, implies that the transition maps in the inverse system of complexes $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P_\bullet, Y_m)$ are surjective, and hence have vanishing $R^1 \varprojlim$. We thus find that

(A.3.16)
$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P_\bullet, \text{“}\varprojlim_m\text{”} Y_m) \xrightarrow{\sim} \varprojlim_m \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P_\bullet, Y_m) \xrightarrow{\sim} R\varprojlim_m \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P_\bullet, Y_m)$$

The composite isomorphism gives rise to a spectral sequence

$$\begin{aligned} E_2^{p,q} := R^p \varprojlim_m \mathrm{Ext}^q_{\mathcal{C}}(X, Y_m) \implies H^{p+q}\big(&\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P_\bullet, \text{“}\varprojlim_m\text{”} Y_m)\big) \\ &= \mathrm{Ext}^{p+q}_{\mathrm{Pro}(\mathcal{C})}(X, \text{“}\varprojlim_m\text{”} Y_m). \end{aligned}$$

Of course the $R^p \varprojlim_m$ vanish automatically for $p \geq 2$, but they also vanish for $p = 1$, since the inverse sytems $\mathrm{Ext}^q_{\mathcal{C}}(X, Y_m)$ are inverse systems of finite length $\mathcal{O}$-modules by assumption, and so satisfy the Mittag–Leffler condition. This proves the lemma. □

A.3.17. *Completion along an abelian subcategory.* Suppose that $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is an abelian subcategory (i.e. a full subcategory which is also abelian and for which the inclusion is exact). Then we may apply the discussion of Section A.3.7 to see that the induced inclusion $\mathrm{Pro}(\mathcal{C}_0) \hookrightarrow \mathrm{Pro}(\mathcal{C})$ (which is evidently fully faithful, and by the discussions of (A.3.5) and (A.3.6) is again exact) admits a left adjoint $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}_0)$, which we refer to as the functor of *completion* of $\mathcal{C}$ along $\mathcal{C}_0$, and which we denote by $X \mapsto \widehat{X}$.

The identity map from $\widehat{X}$ to itself induces, by the adjunction that defines $\widehat{X}$, a canonical morphism (the unit of the adjunction) $X \to \widehat{X}$ in $\mathrm{Pro}(\mathcal{C})$.

The following lemma describes $\widehat{X}$ explicitly for objects X of $\mathcal{C}$, in the case when $\mathcal{C}_0$ is furthermore closed under the formation of subobjects in $\mathcal{C}$ (e.g. a Serre subcategory of $\mathcal{C}$).

Lemma A.3.18. *If $\mathcal{C}_0$ is closed under the formation of subobjects in $\mathcal{C}$, then there is a natural isomorphism*

$$\widehat{X} \xrightarrow{\sim} \text{“}\varprojlim\text{”} X', \tag{A.3.19}$$

where X' runs over the cofiltered directed set of quotients of X lying in $\mathcal{C}_0$.

Proof. If $X \to X'$ is a surjection with X' lying in $\mathcal{C}_0$, then by adjunction there is an induced morphism $\widehat{X} \to X'$. These induced morphisms collectively give rise to the morphism (A.3.19), which we claim is an isomorphism. To see this, it suffices to show that the induced morphism of functors

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C}_0)}(\text{“}\varprojlim\text{”} X', -) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C}_0)}(\widehat{X}, -) \tag{A.3.20}$$

on $\mathcal{C}_0$ is an isomorphism, and this is what we will show.

This morphism is injective by construction. To see that it is surjective, write $\widehat{X} = \text{“}\varprojlim_I\text{”} X_i$, with X_i an object of $\mathcal{C}_0$, Each of the projection morphisms $\widehat{X} \to X_i$ (in $\mathrm{Pro}(\mathcal{C}_0)$) induces a morphism $X \to X_i$ in $\mathcal{C}$. If we let X_i' denote the image of X (which is again an object of $\mathcal{C}_0$, by assumption), then the factorization $X \to X_i' \to X_i$ induces a corresponding factorization $\widehat{X} \to X_i' \to X_i$, so that the projection $\widehat{X} \to X_i$ factors through the morphism (A.3.19). This shows that (A.3.20) is indeed surjective, and thus is an isomorphism. $\square$

In the context of the preceding lemma, the unit of the adjunction $X \to \widehat{X} = \text{“}\varprojlim\text{”} X'$ is the morphism induced by the various quotient morphisms $X \to X'$.

Concretely, if X is an object of $\mathcal{C}$ and if $\widehat{X}$ in $\mathrm{Pro}(\mathcal{C}_0)$ is its image under the adjoint, then there is a canonical morphism $X \to \widehat{X}$ (in $\mathrm{Pro}(\mathcal{C})$) through which any morphism $X \to X_0$, with X_0 an object of $\mathcal{C}_0$, uniquely factors.

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