AUTOMORPHY LIFTING FOR SMALL l - APPENDIX B TO "AUTOMORPHY OF $\mathrm{Symm}^5(\mathrm{GL}(2))$ AND BASE CHANGE"

LUIS DIEULEFAIT AND TOBY GEE

In this appendix we prove a slight generalization of Theorem 4.2.1 of [BLGGT10]. It strengthens *loc. cit.* in that it weakens the assumption that $l \geq 2(n+1)$ to an adequacy hypothesis (which is automatic if $l \geq 2(n+1)$ by the main result of [GHTT10]).

This theorem can be proved by a straightforward modification of the proof of Theorem 4.2.1 of [BLGGT10], using Lemma A.3.1 of [BLGG11] (which was proved by Richard Taylor during the writing of [BLGGT10]). In order to make the proof straightforward to read, rather than explaining how to modify the proof of Theorem 4.2.1 of [BLGGT10] using this Lemma, we combine Theorem A.4.1 of [BLGG11] (which is an improvement on Theorem 4.3.1 of [BLGGT10] in exactly the same way that Theorem 1 below is an improvement on Theorem 4.2.1 of [BLGGT10]) with Theorem 2.3.1 of [BLGGT10] (which is essentially Theorem 7.1 of [Tho12]).

We freely use the notation and terminology of [BLGGT10] without comment. We would like to thank Florian Herzig for his helpful comments on an earlier version of this appendix.

1. **Theorem.** *Let F be an imaginary CM field with maximal totally real subfield F^+ and let c denote the non-trivial element of $\mathrm{Gal}(F/F^+)$. Suppose that l is an odd prime, and that (r, μ) is a regular algebraic, irreducible, n-dimensional, polarized representation of G_F. Let $\overline{r}$ denote the semi-simplification of the reduction of r. Suppose that (r, μ) enjoys the following properties:*

(1) *$r|_{G_{F_v}}$ is potentially diagonalizable (and so in particular potentially crystalline) for all $v|l$.*
(2) *The restriction $\overline{r}(G_{F(\zeta_l)})$ is adequate, and $\zeta_l \notin F$.*
(3) *$(\overline{r}, \overline{\mu})$ is either ordinarily automorphic or potentially diagonalizably automorphic.*

Then (r, μ) is potentially diagonalizably automorphic (of level potentially prime to l).

Proof. By Lemma 2.2.2 of [BLGGT10] (base change) it is enough to prove the theorem after replacing F by a soluble CM extension which is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F. Thus we can and do suppose that all primes dividing l and all primes at which r ramifies are split over F^+.

Suppose firstly that we are in the case that $(\overline{r}, \overline{\mu})$ is ordinarily automorphic. Then by Hida theory we may reduce to the potentially diagonalizably automorphic case, because by (for example) Lemma 3.1.4 of [GG12] and Lemma 1.4.3(1) of [BLGGT10], every Hida family passes through points for which the associated l-adic Galois representation is potentially diagonalizable at all places dividing l.

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Suppose now that we are in the case that $(\overline{r}, \overline{\mu})$ is potentially diagonalizably automorphic. Applying Theorem A.4.1 of [BLGG11] (with $F = F'$, and the ρ_v being $r|_{G_{F_v}}$), we see that there is a regular algebraic, cuspidal, polarized automorphic representation (π, χ) of level potentially prime to l, such that

$$r_{l,\imath}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}$$

for each finite place v of F. The result then follows immediately from Theorem 2.3.1 of [BLGGT10]. $\square$

References

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E-mail address: `ldieulefait@ub.edu`

Facultat de Matemtiques, Universitat de Barcelona

E-mail address: `toby.gee@imperial.ac.uk`

Department of Mathematics, Imperial College London