Companion forms over totally real fields, II

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Abstract

We prove a companion forms theorem for mod l Hilbert modular forms. This work generalises results of Gross and Coleman–Voloch for modular forms over $\mathbf{Q}$, and gives a new proof of their results in many cases.

1 Introduction

If $f \in S_k(\Gamma_1(N); \overline{\mathbf{F}}_p)(\epsilon)$ is a mod l cuspidal eigenform, where $l \nmid N$, there is a continuous, odd, semisimple Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_l)$$

attached to f. A famous conjecture of Serre predicts that all continuous odd irreducible mod l representations should arise in this fashion. Furthermore, the "strong Serre conjecture" predicts a minimal weight k_ρ and level N_ρ, in the sense that $\rho \cong \rho_g$ for some eigenform g of weight k_ρ and level N_ρ (prime to l), and if $\rho \cong \rho_f$ for some eigenform f of weight k and level N prime to l then $N_\rho | N$ and $k \geq k_\rho$. The question as to whether all continuous odd irreducible mod l Galois representations are modular in this sense is still open, but the implication "weak Serre $\Rightarrow$ strong Serre" is essentially known (aside from a few cases where $l = 2$).

In solving the problem of weight optimisation it becomes necessary to consider the companion forms problem; that is, the question of when it can occur that we have $f = \sum a_n q^n$ of weight $2 \leq k \leq l$ with $a_l \neq 0$, and an eigenform $g = \sum b_n q^n$ of weight $k' = l + 1 - k$ such that $n a_n = n^k b_n$ for all n. Serre conjectured that this can occur if and only if the representation ρ_f is tamely ramified above l. This conjecture has been settled in most cases in the papers of Gross ([Gro90]) and Coleman-Voloch ([CV92]).

Our earlier paper [Gee04] generalised these results to the case of parallel weight Hilbert modular forms over totally real fields F in which l splits completely, by generalising the methods of [CV92]. In this paper we take a completely different and rather more conceptual approach; we construct our companion form by using a method of Ramakrishna to find an appropriate characteristic zero Galois representation, and then use recent work of Kisin ([Kis04]) to prove that the representation is modular. Note that our companion form is not necessarily of minimal prime-to-l level, but that this is irrelevant for applications to Artin's conjecture, and that in many cases a form of minimal level may be obtained from ours by the methods of [Jar99], [SW01], [Raj01] and [Fuj99]. In the case of weight l forms, we avoid potential difficulties with weight 1 forms by constructing a companion form in weight l.

2 Statement of the main results

Let $l > 2$ be a prime, and let F be a totally real field. We assume that if $l > 3$, $[F(\zeta_l) : F] > 3$ (note that this is automatic if l is unramified in F). Let ϵ denote both the l-adic and mod l cyclotomic characters; this should cause no confusion. Let $\rho : G_K \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation, where is K a finite extension of $\mathbf{Q}_l$, and $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbf{Q}_l$. We say that ρ is *ordinary* if it is Barsotti-Tate, coming from an l-divisible group which is an extension of an étale group by a multiplicative group, each of rank one as $\mathcal{O}$-modules. We say that it is *potentially ordinary* if it becomes ordinary upon restriction to an open subgroup of G_K. We say that a Hilbert modular form of parallel weight 2 is *(potentially) ordinary* at a place $v|l$ if its associated Galois representation is (potentially) ordinary at v. These definitions agree with those in [Kis04]; they are slightly non-standard, but note that if the level is prime to l then this is equivalent to the U_v-eigenvalue being an l-adic unit. We say that a Hilbert modular form of parallel weight k, $3 \leq k \leq l$ is *ordinary* at a place $v|l$ if its U_v-eigenvalue is an l-adic unit. Finally, we say that a modular form is *(potentially) ordinary* if it is (potentially) ordinary at all places $v|l$.

Our main theorem is the following:

Theorem 2.1. *Let g be an ordinary Hilbert modular eigenform of parallel weight k, $2 \leq k \leq l$, and level coprime to l. Let its associated Galois representation be $\rho_g : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_l)$, so that (by Theorem 2 of [Wil88]) we have, for all places $v|l$,*

$$\rho_g|_{G_v} \simeq \begin{pmatrix} \epsilon^{k-1}\psi_{v,1} & * \\ 0 & \psi_{v,2} \end{pmatrix}$$

for unramified characters $\psi_{v,1}$, $\psi_{v,2}$. Suppose that the residual representation $\overline{\rho}_g : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_l)$ is absolutely irreducible. Assume further that for all $v|l$ we have that $\epsilon^{k-1}\overline{\psi}_{v,1} \neq \overline{\psi}_{v,2}$, and that the representation $\overline{\rho}_g|_{G_v}$ is tamely ramified, so that

$$\overline{\rho}_g|_{G_v} \simeq \begin{pmatrix} \epsilon^{k-1}\overline{\psi}_{v,1} & 0 \\ 0 & \overline{\psi}_{v,2} \end{pmatrix}.$$

Assume in addition that if $\epsilon^{k-2}\overline{\psi}_{v,1} = \overline{\psi}_{v,2}$, then the absolute ramification index of F_v is less than $l-1$. If $k = l$ then let $k' = l$, and otherwise let $k' = l+1-k$. Then there is a Hilbert modular form g' of parallel weight k' and level coprime to l satisfying

$$\overline{\rho}_{g'} \simeq \overline{\rho}_g \otimes \epsilon^{k'-1}$$

and the U_v-eigenvalue of g' is a lift of $\overline{\psi}_{v,1}(\mathrm{Frob}_v)$.

In fact, we work throughout with forms of parallel weight 2, and we use Hida theory to treat forms of more general (parallel) weight. In the case where $\overline{\rho}_g(G_F)$ is soluble the Langlands-Tunnell theorem makes the proof straightforward, so we concentrate on the insoluble case, where we prove:

Theorem 2.2. *Let $\overline{\rho}_f : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_l)$ be an absolutely irreducible modular representation, coming from a Hilbert eigenform f of parallel weight 2, with associated Galois representation $\rho_f : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_l)$. Suppose that $\overline{\rho}_f(G_F)$ is* *insoluble. Suppose also that for every place v of F dividing l $\rho_f|_{G_v}$ is potentially ordinary, and we have*

$$\overline{\rho}_f|_{G_v} \simeq \begin{pmatrix} \epsilon^{k-1}\overline{\psi}_{v,1} & 0 \\ 0 & \overline{\psi}_{v,2} \end{pmatrix}$$

where $\overline{\psi}_{v,1}$, $\overline{\psi}_{v,2}$ are unramified characters, with $\epsilon^{k-1}\overline{\psi}_{v,1} \neq \overline{\psi}_{v,2}$. Assume in addition that if $\epsilon^{k-2}\overline{\psi}_{v,1} = \overline{\psi}_{v,2}$, then the absolute ramification index of F_v is less than $l-1$.

If $k = l$ then let $k' = l$, and otherwise let $k' = l + 1 - k$. Then there is an eigenform f' of parallel weight 2 which is potentially ordinary at all places $v|l$ such that the mod l Galois representation $\overline{\rho}_{f'}$ associated to f' satisfies

$$\overline{\rho}_{f'} \simeq \overline{\rho}_f \otimes \epsilon^{k'-1},$$

and such that at all places $v|l$ we have

$$\rho_{f'}|_{G_v} \simeq \begin{pmatrix} \epsilon\omega^{k'-2}\psi_{v,2} & * \\ 0 & \psi_{v,1} \end{pmatrix}$$

with $\psi_{v,i}$ an unramified lift of $\overline{\psi}_{v,i}$ for i=1, 2, and ω the Teichmuller lift of ϵ.

3 Lifting theorems

Firstly, we prove a straightforward generalisation of the results of [Ram02] and [Tay03] to totally real fields. We begin by analysing the local representation theory at primes not dividing l. The next lemma is essentially contained in [Dia97]:

Lemma 3.1. *Let $p \neq l$ be a prime, and let K be a finite extension of $\mathbf{Q}_p$. Let I_K denote the inertia subgroup of G_K. Let $\sigma : G_K \to \mathrm{GL}_2(k)$ be a continuous representation, with k a finite field of characteristic l, and assume that $l|\#\sigma(I_K)$.*

Then either $p = 2$, $l = 3$, and $\operatorname{proj}\sigma(G_K) \simeq A_4$ or S_4, or

$$\sigma \simeq \begin{pmatrix} \epsilon\overline{\chi} & * \\ 0 & \overline{\chi} \end{pmatrix}$$

with respect to some basis for some character $\overline{\chi}$.

Proof. Note that $l|\#\sigma(I_K)$ if and only if $l|\#\operatorname{proj}\sigma(I_K)$. We must have $\sigma|_{I_K}$ indecomposable. If σ is reducible, then σ is a twist of a representation $\left(\begin{smallmatrix} \psi & u \\ 0 & 1 \end{smallmatrix}\right)$ for some character ψ, with u a cocycle representing a class in $\mathrm{H}^1(G_K, k(\psi))$ whose image in $\mathrm{H}^1(I_K, k(\psi))^{G_K}$ is non-zero; but the latter group is zero unless $\psi = \epsilon$.

If instead σ is irreducible but $\sigma|_{I_K}$ is reducible, then $\sigma|_{I_K}$, being indecomposable, must fix precisely one element of $\mathbf{P}^1(k)$. But then σ would also have to fix this element, a contradiction.

Assume now that $\sigma|_{I_K}$ is irreducible, and that $\sigma|_{P_K}$ is reducible, where P_K is the wild inertia subgroup of I_K. Then P_K must fix precisely two elements of $\mathbf{P}^1(k)$ (as $\sigma|_{I_K}$ is irreducible), so σ is induced from a character on a ramified quadratic extension of K, and thus $\sigma(I_K)$ has order $2p^r$ for some $r \geq 1$, a contradiction.

Finally, if $\sigma|_{P_K}$ is irreducible we must have $p = 2$. That $\operatorname{proj}\sigma(G_K) \simeq A_4$ or S_4 follows from the same argument as in the proof of Proposition 2.4 of [Dia97]. That $l = 3$ follows from $l | \#\sigma(I_K)$. $\square$

Let $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ be continuous, odd, and absolutely irreducible, with k a finite field of characteristic l. Let S denote a finite set of finite places of F which contains all places dividing l and all places where $\overline{\rho}$ is ramified, and let G_S denote the Galois group of the maximal extension of F unramified outside S. A *deformation* of $\overline{\rho}$ is a complete noetherian local ring $(R, \mathfrak{m})$ with residue field k and a continuous representation $\rho : G_S \to \mathrm{GL}_2(R)$ such that $(\rho \bmod \mathfrak{m}) = \overline{\rho}$ and $\epsilon^{-1} \det \rho$ has finite order prime to l. We define deformations of $\overline{\rho}|_{G_v}$ in a similar fashion.

Suppose that for each $v \in S$ we have a pair $(\mathcal{C}_v, L_v)$ satisfying the properties P1-P7 listed in section 1 of [Tay03]. Define $\mathrm{H}^1_{\{L_v\}}(G_S, \operatorname{ad}^0 \overline{\rho})$ and $\mathrm{H}^1_{\{L_v^\perp\}}(G_S, \operatorname{ad}^0 \overline{\rho})$ in the usual way.

Lemma 3.2. *If* $\mathrm{H}^1_{\{L_v^\perp\}}(G_S, \operatorname{ad}^0 \overline{\rho}) = (0)$ *then there is an S-deformation* $(W(k), \rho)$ *of $\overline{\rho}$ such that for all $v \in S$ we have* $(W(k), \overline{\rho}|_{G_v}) \in \mathcal{C}_v$.

Proof. Identical to the proof of Lemma 1.1 of [Tay03]. $\square$

Lemma 3.3. *Suppose that* $\sum_{v \in S} \dim L_v \geq \sum_{v \in S \cup \{\infty\}} \dim \mathrm{H}^0(G_v, \operatorname{ad}^0 \overline{\rho})$. *Then we can find a finite set of places $T \supset S$ and data $(\mathcal{C}_v, L_v)$ for $v \in T - S$ satisfying conditions P1-P7 and such that* $\mathrm{H}^1_{\{L_v^\perp\}}(G_T, \operatorname{ad}^0 \overline{\rho}) = (0)$.

Proof. The proof of this lemma is almost identical to that of Lemma 1.2 of [Tay03]. We sketch a few of the less obvious details. In the case $l = 5$, $\operatorname{ad}^0 \overline{\rho}(G_F) \simeq A_5$, we choose $w \notin S$ such that $\mathbf{N}w \equiv 1 \bmod 5$ and $\operatorname{ad}^0 \overline{\rho}(\mathrm{Frob}_w)$ has order 5 (such a w exists by Cebotarev's theorem). Adding w to S with the pair $(\mathcal{C}_w, L_w)$ of type E3 (see below), we may assume $\mathrm{H}^1_{\{L_v^\perp\}}(G_S, \operatorname{ad}^0 \overline{\rho}) \cap \mathrm{H}^1(\operatorname{ad}^0 \overline{\rho}(G_F), \operatorname{ad}^0 \overline{\rho}) = (0)$.

From here on, almost exactly the same argument as in [Tay03] applies, the only difference being that one must replace every occurence of "$\mathbf{Q}$" with "F". Let $K = F(\operatorname{ad}^0 \overline{\rho}, \mu_l)$. The argument is essentially formal once one knows that there is an element $\sigma \in \mathrm{Gal}(K/F)$ such that $\operatorname{ad}^0 \overline{\rho}(\sigma)$ has an eigenvalue $\epsilon(\sigma) \not\equiv 1 \bmod l$, that $\operatorname{ad}^0 \overline{\rho}$ is absolutely irreducible, and that $\operatorname{ad}^0 \overline{\rho}$ is not isomorphic to $(\operatorname{ad}^0 \overline{\rho})(1)$. All of these assertions follow from our assumption that $[F(\zeta_l) : F] > 3$ if $l > 3$, with the proofs being similar to those in [Ram99] (note that one may replace the assumption that $\overline{\rho}(G_{\mathbf{Q}}) \supseteq \mathrm{SL}_2(k)$ in [Ram99] with the assumption that $\operatorname{proj}\overline{\rho}(G_{\mathbf{Q}}) \supseteq \mathrm{PSL}_2(k)$ without affecting the proofs). For example, to check that $\operatorname{ad}^0 \overline{\rho}$ is not isomorphic to $(\operatorname{ad}^0 \overline{\rho})(1)$ it is enough to prove that there is an element $\sigma' \in \mathrm{Gal}(K/F)$ such that all of the eigenvalues of $\operatorname{ad}^0 \overline{\rho}$ are 1, and $\epsilon(\sigma') \neq 1$. The existence of σ and σ' follows exactly as in the proof of Theorem 2 of [Ram99]. $\square$

We now give examples of pairs $(\mathcal{C}_v, L_v)$. Again, our pairs are very similar to those in section 1 of [Tay03], and the verification of the required properties is almost identical. We use the notation of [Tay03] for ease of comparison with that paper.

- E1. Suppose that $v \nmid l$ and that $l \nmid \#\overline{\rho}(I_v)$. Take $\mathcal{C}_v$ to be the class of lifts of $\overline{\rho}|_{G_v}$ which factor through $G_v/(I_v \cap \ker \overline{\rho})$ and let L_v be $\mathrm{H}^1(G_v/I_v, (\mathrm{ad}^0\,\overline{\rho})^{I_v})$. Then it is straightforward to see that properties P1-P7 are satisfied, and that

  - $\mathrm{H}^2(G_v/(I_v \cap \ker \overline{\rho}), \mathrm{ad}^0\,\overline{\rho}) \simeq \mathrm{H}^2(G_v/I_v, (\mathrm{ad}^0\,\overline{\rho})^{I_v}) = (0)$, (as $G_v/I_v \simeq \hat{\mathbf{Z}}$ has cohomological dimension 1),
  - $\mathrm{H}^1(G_v/(I_v \cap \ker \overline{\rho}), \mathrm{ad}^0\,\overline{\rho}) = L_v \subset \mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho})$,
  - $\dim L_v = \dim \mathrm{H}^0(G_v, \mathrm{ad}^0\,\overline{\rho})$ (by the local Euler characteristic formula).

- E2. (Note that our definitions here differ slightly from those in [Tay03]; we thank Richard Taylor for explaining this modification to us.) Suppose that $l = 3$, that $v|2$, and that $(\mathrm{ad}^0(\overline{\rho})(G_v) \xrightarrow{\sim} S_4$. Take $\mathcal{C}_v$ to be the class of lifts of $\overline{\rho}|_{G_v}$ which factor through $G_v/(I_v \cap \ker \overline{\rho})$ and let L_v be $\mathrm{H}^1(G_v/I_v, (\mathrm{ad}^0\,\overline{\rho})^{I_v})$. The verification of properties P1-P7 is then as in [Tay03], except that to check that $\mathrm{H}^i(\overline{\rho}(I_v), \mathrm{ad}^0\,\overline{\rho}) = (0)$ for all $i \geq 0$ one uses the Hochschild-Serre spectral sequence and the fact that $\mathrm{H}^i(C_2 \times C_2, \mathrm{ad}^0\,\overline{\rho}) = (0)$ for all $i \geq 0$.

- E3. Suppose that $v \neq l$, that either $\mathbf{N}v \not\equiv 1 \pmod{l}$ or $l|\#\overline{\rho}(G_v)$, and that with respect to some basis e_1, e_2 of k^2 the restriction $\overline{\rho}|_{G_v}$ has the form
$$\begin{pmatrix} \epsilon\overline{\chi} & * \\ 0 & \overline{\chi} \end{pmatrix}.$$
Take $\mathcal{C}_v$ to be the class of deformations of the form (with respect to some basis)
$$\begin{pmatrix} \epsilon\chi & * \\ 0 & \chi \end{pmatrix}$$
with χ lifting $\overline{\chi}$, and take L_v to be the image of
$$\mathrm{H}^1(G_v, \mathrm{Hom}(ke_2, ke_1)) \to \mathrm{H}^1(G_v, (\mathrm{ad}^0\,\overline{\rho})).$$
That the pair $(\mathcal{C}_v, L_v)$ satisfies the properties P1-P7 follows from an identical argument to that in [Tay03]. An identical calculation to that in [Tay03] shows that $\dim L_v = \dim \mathrm{H}^0(G_v, \mathrm{ad}^0\,\overline{\rho})$.

- E4. Suppose that $v|l$ and that with respect to some basis e_1, e_2 of k^2 $\overline{\rho}|_{G_v}$ has the form
$$\begin{pmatrix} \epsilon\overline{\chi_1} & 0 \\ 0 & \overline{\chi_2} \end{pmatrix}.$$
Suppose also that $\overline{\chi}_1 \neq \overline{\chi}_2$ and that $\epsilon\overline{\chi}_1 \neq \overline{\chi}_2$. Take $\mathcal{C}_v$ to consist of all deformations of the form
$$\begin{pmatrix} \epsilon\chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$
where χ_1, χ_2 are tamely ramified lifts of $\overline{\chi}_1$, $\overline{\chi}_2$ respectively. Let $U^0 = \mathrm{Hom}(ke_2, ke_1)$, and let L_v be the kernel of the map $\mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}) \to \mathrm{H}^1(I_v, \mathrm{ad}^0\,\overline{\rho}/U^0)^{G_v/I_v}$. The verification of properties P1-P7 follows as

in [Tay03], and we may compute $\dim L_v$ via a similar computation to that in the proof of Lemma 5 of [Ram02].

Note firstly that by local duality and the assumption that $\overline{\chi}_1 \neq \overline{\chi}_2$ we have $\mathrm{H}^2(G_v, U^0) = 0$. Thus the short exact sequence

$$0 \to U^0 \to \mathrm{ad}^0\,\overline{\rho} \to \mathrm{ad}^0\,\overline{\rho}/U^0 \to 0$$

yields an exact sequence

$$\mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}) \to \mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}/U^0) \to 0.$$

Inflation-restriction gives us an exact sequence

$$0 \to \mathrm{H}^1(G_v/I_v, (\mathrm{ad}^0\,\overline{\rho}/U^0)^{I_v}) \to \mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}/U^0) \to \mathrm{H}^1(I_v, \mathrm{ad}^0\,\overline{\rho}/U^0)^{G_v/I_v} \to 0,$$

and combining these two sequences shows that the map $\mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}) \to \mathrm{H}^1(I_v, \mathrm{ad}^0\,\overline{\rho}/U^0)^{G_v/I_v}$ is surjective. Thus

$$\begin{aligned}
\dim L_v &= \dim \mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}) - \dim \mathrm{H}^1(I_v, \mathrm{ad}^0\,\overline{\rho}/U^0)^{G_v/I_v} \\
&= \dim \mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}) - \dim \mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}/U^0) + \dim \mathrm{H}^1(G_v/I_v, (\mathrm{ad}^0\,\overline{\rho}/U^0)^{I_v}) \\
&= \dim \mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}) - \dim \mathrm{H}^1(G_v, \mathrm{ad}^0\,\overline{\rho}/U^0) \\
&\quad + \dim \mathrm{H}^0(G_v, \mathrm{ad}^0\,\overline{\rho}/U^0) \text{ (by Lemma 3 of [Ram02])} \\
&= \dim \mathrm{H}^0(G_v, \mathrm{ad}^0\,\overline{\rho}) + \dim \mathrm{H}^2(G_v, \mathrm{ad}^0\,\overline{\rho}) - \dim \mathrm{H}^2(G_v, \mathrm{ad}^0\,\overline{\rho}/U^0) \\
&\quad + [F_v : \mathbf{Q}_l] \text{ (local Euler characteristic)} \\
&= [F_v : \mathbf{Q}_l] + \dim \mathrm{H}^0(G_v, \mathrm{ad}^0\,\overline{\rho}).
\end{aligned}$$

- BT. Suppose that $v|l$ and that with respect to some basis e_1, e_2 of k^2 $\overline{\rho}|_{G_v}$ has the form
$$\begin{pmatrix} \epsilon\overline{\chi} & 0 \\ 0 & \overline{\chi} \end{pmatrix}$$
for some unramified character $\overline{\chi}$. Assume also that ϵ is not trivial (that is, that F_v does not contain $\mathbf{Q}_l(\zeta_l)$). Take $\mathcal{C}_v$ to consist of all flat deformations of the form
$$\begin{pmatrix} \epsilon\chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$
where χ_1, χ_2 are unramified lifts of $\overline{\chi}$, Then it follows from Corollary 2.5.16 of [Kis04] that there is an L_v of dimension $[F_v : \mathbf{Q}_l] + \dim \mathrm{H}^0(G_v, \mathrm{ad}^0\,\overline{\rho})$ so that properties P1-P7 are all satisfied.

Set $\overline{\rho} = \overline{\rho}_f \otimes \epsilon^{k'-1}$. We are now in a position to prove:

Theorem 3.4. *There is a deformation ρ of $\overline{\rho}$ to $W(k)$ such that at all places $v|l$ we have $\rho|_{G_v}$ potentially ordinary, and*

$$\rho|_{G_v} \simeq \begin{pmatrix} \epsilon\omega^{k'-2}\psi_{v,2} & * \\ 0 & \psi_{v,1} \end{pmatrix}$$

with $\psi_{v,i}$ an unramified lift of $\overline{\psi}_{v,i}$ for i=1, 2, and ω the Teichmuller lift of ϵ.

Proof. This follows almost at once from Lemma 3.3. By Lemma 3.1 we can choose $(\mathcal{C}_v, L_v)$ for all $v \nmid l$, with $\dim L_v = \dim \mathrm{H}^0(G_v, \mathrm{ad}^0 \overline{\rho})$ (simply choose as in examples E1 or E3). At places $v|l$, we choose $(\mathcal{C}_v, L_v)$ as in examples E4 or BT, so that $\dim L_v = [F_v : \mathbf{Q}_l] + \dim \mathrm{H}^0(G_v, \mathrm{ad}^0 \overline{\rho})$. Then as $\sum_{v|l}[F_v : \mathbf{Q}_l] = [F : \mathbf{Q}]$, we have $\sum_{v \in S} \dim L_v = \sum_{v \in S \cup \{\infty\}} \dim \mathrm{H}^0(G_v, \mathrm{ad}^0 \overline{\rho})$, so a deformation as in Lemma 3.3 exists. That the $\psi_{v,i}$ are unramified follows from the fact that they are tamely ramified lifts of unramified characters.

It remains to check that $\rho|_{G_v}$ is potentially ordinary. By the remarks in section 2.4.15 of [Kis04] it suffices to check that it is potentially Barsotti-Tate. This is immediate if we are in the case BT, so suppose we are considering deformations as in E4. By the proposition in section 3.1 of [PR94], $\rho|_{G_v}$ is potentially semistable, and it clearly has Hodge-Tate weights in $\{0, 1\}$, so by Theorem 5.3.2 of [Bre00] it suffices to check that it is potentially crystalline. In order to check this, we consider the Weil-Deligne representation $WD(\rho|_{G_v})$ (see Appendix B of [CDT99] for the definition of $WD(\sigma)$ for any potentially semistable p-adic representation σ of G_v). We need to check that the associated nilpotent endomorphism N is zero. As is well-known, $N = 0$ unless $WD(\rho|_{G_v})$ is a twist of the Steinberg representation, which cannot happen because of our assumption that we are not in the BT case. □

Theorem 2.2 now follows immediately from:

Theorem 3.5. *The representation ρ is modular.*

Proof. This is an easy application of Theorem 3.5.5 of [Kis04]. We need to check that ρ is strongly residually modular. The representation $\rho_f \otimes \omega^{k'-1}$ (where ω is the Teichmuller lift of ϵ) is certainly modular, with residual representation $\overline{\rho}$; furthermore, it is automatically potentially ordinary at all places $v|l$ with $\epsilon^{k-2}\overline{\psi}_{v,1} \neq \overline{\psi}_{v,2}$. By Theorem 6.2 of [Jar04] and our assumption that if $\epsilon^{k-2}\overline{\psi}_{v,1} = \overline{\psi}_{v,2}$ the absolute ramification index of F_v is less than $l - 1$, we may replace $\rho_f \otimes \omega^{k'-1}$ with a modular lift of $\overline{\rho}$ which is potentially ordinary at all places $v|l$. By construction, ρ is potentially ordinary at all places $v|l$, so we are done. □

We now prove Theorem 2.1. Firstly, suppose that $\overline{\rho}_g(G_F)$ is insoluble. Then Hida theory (see [Wil88] or [Hid88]) provides us with a weight 2 form f which satisfies the hypotheses of Theorem 2.2, and which has $\overline{\rho}_f \simeq \overline{\rho}_g$ (that f is potentially ordinary follows as in the proof of Theorem 3.4). Then Theorem 2.2 provides us with a Hilbert modular form f' of parallel weight 2 with $\overline{\rho}_{f'} \simeq \overline{\rho}_f \otimes \epsilon^{k'-1}$ and

$$\rho_{f'}|_{G_v} \simeq \begin{pmatrix} \epsilon\omega^{k'-2}\psi_{v,2} & * \\ 0 & \psi_{v,1} \end{pmatrix}$$

for all places $v|l$, with $\psi_{v,1}$ an unramified lift of $\overline{\psi_{v,1}}$. Then Lemma 3.4.2 of [Kis04] shows that f' has U_v-eigenvalue $\psi_{v,1}(Frob_v)$, an l-adic unit. The existence of g' now follows from Hida theory.

Now suppose that $\overline{\rho}_f(G_F)$ is insoluble. Then there is a lift of $\overline{\rho}_f \otimes \epsilon^{k'-1}$ to a characteristic zero representation, which comes from a Hilbert modular form of parallel weight 1 by the Langlands-Tunnell theorem (see for example Lemma 5.2 of [Kha05]). Such a form is necessarily ordinary in the sense of Hida theory, and the theorem follows by Hida theory as in the insoluble case.

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