LOCAL GEOMETRY OF MODULI STACKS OF TWO-DIMENSIONAL GALOIS REPRESENTATIONS

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ABSTRACT. We construct moduli stacks of two-dimensional mod p representations of the absolute Galois group of a p-adic local field, as well as their resolutions by moduli stacks of two-dimensional Breuil–Kisin modules with tame descent data. We study the local geometry of these moduli stacks by comparing them with local models of Shimura varieties at hyperspecial and Iwahori level.

CONTENTS

1. INTRODUCTION

1.1. **Moduli of Galois representations.** Let $K/\mathbf{Q}_p$ be a finite extension, let $\overline{K}$ be an algebraic closure of K, and let $\bar{r} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ be a continuous representation. The theory of deformations of $\bar{r}$ — that is, liftings of $\bar{r}$ to continuous representations $r : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_d(A)$, where A is a complete local ring with residue field $\overline{\mathbf{F}}_p$ — is extremely important in the Langlands program, and in particular is crucial for proving automorphy lifting theorems via the Taylor–Wiles method. Proving such theorems often comes down to studying the moduli spaces of those deformations which satisfy various p-adic Hodge-theoretic conditions.

From the point of view of algebraic geometry, it seems unnatural to study only formal deformations of this kind, and Kisin observed about fifteen years ago that results on the reduction modulo p of two-dimensional crystalline representations suggested that there should be moduli spaces of p-adic representations in which

the residual representations $\overline{r}$ should be allowed to vary. In particular, the special
fibres of these moduli spaces would be moduli spaces of mod p representations of
$\mathrm{Gal}(\bar{K}/K)$.

In this paper and its sequels [CEGS20c, CEGS20a] we construct such a space
(or rather stack) $\mathcal{Z}$ of mod p representations in the case $d = 2$, and describe its
geometry. In particular, over the course of these three papers we show that their
irreducible components are naturally labelled by Serre weights, and that our spaces
give a geometrisation of the weight part of Serre's conjecture.

In the present paper we prove in particular the following theorem (see Theorem 5.1.2 and Proposition 5.2.20, as well as Definition A.3 for the notion of a très
ramifiée representation).

Theorem 1.1.1. *There is an algebraic stack $\mathcal{Z}$ of finite type over $\overline{\mathbf{F}}_p$, which is
equidimensional of dimension $[K : \mathbf{Q}_p]$, and whose $\overline{\mathbf{F}}_p$-points are in natural bijection
with the continuous representations $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which are not très ramifiée.*

1.2. **The construction.** The reason that we restrict to the case of two-dimensional
representations is that in this case one knows that most mod p representations are
"tamely potentially finite flat"; that is, after restriction to a finite tamely ramified
extension, they come from the generic fibres of finite flat group schemes. Indeed, the
only representations not of this form are the so-called très ramifiée representations,
which are twists of certain extensions of the trivial character by the mod p cyclotomic
character, and can be described explicitly in terms of Kummer theory. (This is a
local Galois-theoretic analogue of the well-known fact that, up to twist, modular
forms of any weight and level $\Gamma_1(N)$, with N prime to p, are congruent modulo p to
modular forms of weight two and level $\Gamma_1(Np)$; the corresponding modular curves
acquire semistable reduction over a tamely ramified extension of $\mathbf{Q}_p$.)

These Galois representations, and the corresponding finite flat group schemes,
can be described in terms of semilinear algebra data. Such descriptions also exist
for more general p-adic Hodge theoretic conditions (such as being crystalline of
prescribed Hodge–Tate weights), although they are more complicated, and can be
used to construct analogues for higher dimensional representations of the moduli
stacks we construct here; this construction is the subject of [EG22].

The semilinear algebra data that we use in the present paper are Breuil–Kisin
modules and étale φ-modules. A Breuil–Kisin module is a module with Frobenius
over a power series ring, satisfying a condition on the cokernel of the Frobenius
which depends on a fixed integer, called the *height* of the Breuil–Kisin module.
Inverting the formal variable in the power series ring gives a functor from the
category of Breuil–Kisin modules to the category of étale φ-modules. By Fontaine's
theory [Fon90], these étale φ-modules correspond to representations of $\mathrm{Gal}(\overline{K}/K_\infty)$,
where K_∞ is an infinite non-Galois extension of K obtained by extracting p-power
roots of a uniformiser. By work of Breuil and Kisin (in particular [Kis09]), for
étale φ-modules that arise from a Breuil–Kisin module of height at most 1 the
corresponding representations admit a natural extension to $\mathrm{Gal}(\overline{K}/K)$, and in this
way one obtains precisely the finite flat representations. This is the case that we will
consider throughout this paper, extended slightly to incorporate descent data from
a finite tamely ramified extension K'/K and thereby allowing us to study tamely
potentially finite flat representations.

Following Pappas and Rapoport [PR09], we then consider the moduli stack $\mathcal{C}^{\mathrm{dd}}$
of rank two projective Breuil–Kisin modules with descent data, and the moduli

stack $\mathcal{R}^{\mathrm{dd}}$ of étale φ-modules with descent data, together with the natural map $\mathcal{C}^{\mathrm{dd}} \to \mathcal{R}^{\mathrm{dd}}$. We deduce from the results of [PR09] that the stack $\mathcal{C}^{\mathrm{dd}}$ is algebraic (that is, it is an Artin stack); however $\mathcal{R}$ is not algebraic, and indeed is infinite-dimensional. (In fact, we consider versions of these stacks with p-adic coefficients, in which case $\mathcal{C}^{\mathrm{dd}}$ is a p-adic formal algebraic stack, but we suppress this for the purpose of this introduction.) The analogous construction without tame descent data was considered in [EG21], where it was shown that one can define a notion of the "scheme-theoretic image" of the morphism $\mathcal{C}^{\mathrm{dd}} \to \mathcal{R}^{\mathrm{dd}}$, and that the scheme-theoretic image is algebraic. Using similar arguments, we define our moduli stack $\mathcal{Z}^{\mathrm{dd}}$ of two-dimensional Galois representations to be the scheme-theoretic image, under the morphism $\mathcal{C}^{\mathrm{dd}} \to \mathcal{R}^{\mathrm{dd}}$, of a substack of $\mathcal{C}^{\mathrm{dd}}$ cut out by a condition modeling that of being tamely potentially Barsotti–Tate.

By construction, we know that the finite type points of $\mathcal{Z}^{\mathrm{dd}}$ are in bijection with the (non-très ramifiée) representations $\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, and by using standard results on the corresponding formal deformation problems, we deduce that $\mathcal{Z}^{\mathrm{dd}}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$. The finite type points of $\mathcal{C}^{\mathrm{dd}}$ correspond to potentially finite flat models of these Galois representations, and we are able to deduce that $\mathcal{C}^{\mathrm{dd}}$ is also equidimensional of dimension $[K : \mathbf{Q}_p]$ (at least morally, this is by Tate's theorem on the uniqueness of prolongations of p-divisible groups).

An important tool in our proofs is that $\mathcal{C}^{\mathrm{dd}}$ has rather mild singularities, and in particular is Cohen–Macaulay and reduced. We show this by relating the singularities of (certain substacks of) $\mathcal{C}^{\mathrm{dd}}$ to the local models at Iwahori level of Shimura varieties of GL_2-type; such a relationship was first found in [Kis09] (in the context of formal deformation spaces, with no descent data) and [PR09] (in the context of stacks of Breuil–Kisin modules, although again without descent data) and developed further by AC and Levin in [CL18].

1.4. **Notation and conventions.**

Topological groups. If M is an abelian topological group with a linear topology, then as in [Sta13, Tag 07E7] we say that M is *complete* if the natural morphism $M \to \varprojlim_i M/U_i$ is an isomorphism, where $\{U_i\}_{i \in I}$ is some (equivalently any) fundamental system of neighbourhoods of 0 consisting of subgroups. Note that in some other references this would be referred to as being *complete and separated*. In particular, any p-adically complete ring A is by definition p-adically separated.

Galois theory and local class field theory. If M is a field, we let G_M denote its absolute Galois group. If M is a global field and v is a place of M, let M_v denote the completion of M at v. If M is a local field, we write I_M for the inertia subgroup of G_M.

Let p be a prime number. Fix a finite extension $K/\mathbf{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. Let e and f be the ramification and inertial degrees of K, respectively, and write $\#k = p^f$ for the cardinality of k. Let K'/K be a finite tamely ramified Galois extension. Let k' be the residue field of K', and let e', f' be the ramification and inertial degrees of K' respectively.

Our representations of G_K will often have coefficients in $\overline{\mathbf{Q}}_p$, a fixed algebraic closure of $\mathbf{Q}_p$ whose residue field we denote by $\overline{\mathbf{F}}_p$. Let E be a finite extension of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$ and containing the image of every embedding of K' into $\overline{\mathbf{Q}}_p$. Let $\mathcal{O}$ be the ring of integers in E, with uniformiser ϖ and residue field $\mathbf{F} \subset \overline{\mathbf{F}}_p$.

Fix an embedding $\sigma_0 : k' \hookrightarrow \mathbf{F}$, and recursively define $\sigma_i : k' \hookrightarrow \mathbf{F}$ for all $i \in \mathbf{Z}$ so that $\sigma_{i+1}^p = \sigma_i$; of course, we have $\sigma_{i+f'} = \sigma_i$ for all i. We let $e_i \in k' \otimes_{\mathbf{F}_p} \mathbf{F}$ denote the idempotent satisfying $(x \otimes 1)e_i = (1 \otimes \sigma_i(x))e_i$ for all $x \in k'$; note that $\varphi(e_i) = e_{i+1}$. We also denote by e_i the natural lift of e_i to an idempotent in $W(k') \otimes_{\mathbf{Z}_p} \mathcal{O}$. If M is an $W(k') \otimes_{\mathbf{Z}_p} \mathcal{O}$-module, then we write M_i for $e_i M$.

p-adic Hodge theory. We normalise Hodge–Tate weights so that all Hodge–Tate weights of the cyclotomic character are equal to -1. We say that a potentially crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has *Hodge type* 0, or is *potentially Barsotti–Tate*, if for each $\varsigma : K \hookrightarrow \overline{\mathbf{Q}}_p$, the Hodge–Tate weights of ρ with respect to ς are 0 and 1. (Note that this is a more restrictive definition of potentially Barsotti–Tate than is sometimes used; however, we will have no reason to deal with representations with non-regular Hodge-Tate weights, and so we exclude them from consideration. Note also that it is more usual in the literature to say that ρ is potentially Barsotti–Tate if it is potentially crystalline, and $\rho^\vee$ has Hodge type 0.)

We say that a potentially crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has *inertial type* τ if the traces of elements of I_K acting on τ and on

$$\mathrm{D}_{\mathrm{pcris}}(\rho) = \varinjlim_{K'/K} (\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V_\rho)^{G_{K'}}$$

are equal (here V_ρ is the underlying vector space of V_ρ). A representation $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ *has a potentially Barsotti–Tate lift of type* τ if and only if $\bar{r}$ admits a lift to a representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$ of Hodge type 0 and inertial type τ.

Stacks. We will use the terminology and results theory of algebraic stacks as developed in [Sta13], and that of formal algebraic stacks developed in [Eme]. For an overview of this material we refer the reader to [EG22, App. A]. For a commutative ring A, an *fppf stack over A* (or *fppf A-stack*) is a stack fibred in groupoids over the big *fppf* site of $\mathrm{Spec}\, A$.

Scheme-theoretic images. We briefly remind the reader of some definitions from [EG21, §3.2]. Let $\mathcal{X} \to \mathcal{F}$ be a proper morphism of stacks over a locally Noetherian base-scheme S, where $\mathcal{X}$ is an algebraic stack which is locally of finite presentation over S, and the diagonal of $\mathcal{F}$ is representable by algebraic spaces and locally of finite presentation.

We refer to [EG21, Defn. 3.2.8] for the definition of the *scheme-theoretic image $\mathcal{Z}$* of the proper morphism $\mathcal{X} \to \mathcal{F}$. By definition, it is a full subcategory in groupoids

of $\mathcal{F}$, and in fact by [EG21, Lem. 3.2.9] it is a Zariski substack of $\mathcal{F}$. By [EG21, Lem. 3.2.14], the finite type points of $\mathcal{Z}$ are precisely the finite type points of $\mathcal{F}$ for which the corresponding fibre of $\mathcal{X}$ is nonzero.

The results of [EG21, §3.2] give criteria for $\mathcal{Z}$ to be an algebraic stack, and prove a number of associated results (such as universal properties of the morphism $\mathcal{Z} \to \mathcal{F}$, and a description of versal deformation rings for $\mathcal{Z}$); rather than recalling these results in detail here, we will refer to them as needed in the body of the paper.

2. Integral p-adic Hodge theory with tame descent data

In this section we introduce various objects in semilinear algebra which arise in the study of potentially Barsotti–Tate Galois representations with tame descent data. Much of this material is standard, and none of it will surprise an expert, but we do not know of a treatment in the literature in the level of generality that we require; in particular, we are not aware of a treatment of the theory of tame descent data for Breuil–Kisin modules with coefficients. However, the arguments are almost identical to those for strongly divisible modules and Breuil modules, so we will be brief.

The various equivalences of categories between the objects we consider and finite flat group schemes or p-divisible groups will not be relevant to our main arguments, except at a motivational level, so we largely ignore them.

2.1. Breuil–Kisin modules and φ-modules with descent data.

Recall that we have a finite tamely ramified Galois extension K'/K. Suppose further that there exists a uniformiser π' of $\mathcal{O}_{K'}$ such that $\pi := (\pi')^{e(K'/K)}$ is an element of K, where $e(K'/K)$ is the ramification index of K'/K. Recall that k' is the residue field of K', while e', f' are the ramification and inertial degrees of K' respectively. Let $E(u)$ be the minimal polynomial of π' over $W(k')[1/p]$.

Let φ denote the arithmetic Frobenius automorphism of k', which lifts uniquely to an automorphism of $W(k')$ that we also denote by φ. Define $\mathfrak{S} := W(k')[[u]]$, and extend φ to $\mathfrak{S}$ by

$$\varphi\left(\sum a_i u^i\right) = \sum \varphi(a_i) u^{pi}.$$

By our assumptions that $(\pi')^{e(K'/K)} \in K$ and that K'/K is Galois, for each $g \in \mathrm{Gal}(K'/K)$ we can write $g(\pi')/\pi' = h(g)$ with $h(g) \in \mu_{e(K'/K)}(K') \subset W(k')$, and we let $\mathrm{Gal}(K'/K)$ act on $\mathfrak{S}$ via

$$g\left(\sum a_i u^i\right) = \sum g(a_i) h(g)^i u^i.$$

Let A be a p-adically complete $\mathbf{Z}_p$-algebra, set $\mathfrak{S}_A := (W(k') \otimes_{\mathbf{Z}_p} A)[[u]]$, and extend the actions of φ and $\mathrm{Gal}(K'/K)$ on $\mathfrak{S}$ to actions on $\mathfrak{S}_A$ in the obvious (A-linear) fashion.

The actions of φ and $\mathrm{Gal}(K'/K)$ on $\mathfrak{S}_A$ extend to actions on $\mathfrak{S}_A[1/u] = (W(k') \otimes_{\mathbf{Z}_p} A)((u))$ in the obvious way. It will sometimes be necessary to consider the subring $\mathfrak{S}_A^0 := (W(k) \otimes_{\mathbf{Z}_p} A)[[v]]$ of $\mathfrak{S}_A$ consisting of power series in $v := u^{e(K'/K)}$, on which $\mathrm{Gal}(K'/K)$ acts trivially.

Definition 2.1.1. Fix a p-adically complete $\mathbf{Z}_p$-algebra A. A *weak Breuil–Kisin module with A-coefficients and descent data from K' to K* is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \{\hat{g}\}_{g \in \mathrm{Gal}(K'/K)})$ consisting of a $\mathfrak{S}_A$-module $\mathfrak{M}$ and a φ-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that:

    — the $\mathfrak{S}_A$-module $\mathfrak{M}$ is finitely generated and u-torsion free, and

— the induced map $\Phi_{\mathfrak{M}} = 1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is an isomorphism after inverting $E(u)$ (here as usual we write $\varphi^*\mathfrak{M} := \mathfrak{S}_A \otimes_{\varphi, \mathfrak{S}_A} \mathfrak{M}$),

together with additive bijections $\hat{g} : \mathfrak{M} \to \mathfrak{M}$, satisfying the further properties that the maps $\hat{g}$ commute with $\varphi_{\mathfrak{M}}$, satisfy $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$, and have $\hat{g}(sm) = g(s)\hat{g}(m)$ for all $s \in \mathfrak{S}_A$, $m \in \mathfrak{M}$. We say that $\mathfrak{M}$ is has *height at most* h if the cokernel of $\Phi_{\mathfrak{M}}$ is killed by $E(u)^h$.

If $\mathfrak{M}$ as above is projective as an $\mathfrak{S}_A$-module (equivalently, if the condition that the $\mathfrak{M}$ is u-torsion free is replaced with the condition that $\mathfrak{M}$ is projective) then we say that $\mathfrak{M}$ is a *Breuil–Kisin module with A-coefficients and descent data from* K' *to* K, or even simply that $\mathfrak{M}$ is a *Breuil–Kisin module*.

The Breuil–Kisin module $\mathfrak{M}$ is said to be of rank d if the underlying finitely generated projective $\mathfrak{S}_A$-module has constant rank d. It is said to be free if the underlying $\mathfrak{S}_A$-module is free.

A morphism of (weak) Breuil–Kisin modules with descent data is a morphism of $\mathfrak{S}_A$-modules that commutes with φ and with the $\hat{g}$. In the case that $K' = K$ the data of the $\hat{g}$ is trivial, so it can be forgotten, giving the category of (*weak*) *Breuil–Kisin modules with A-coefficients.* In this case it will sometimes be convenient to elide the difference between a Breuil–Kisin module with trivial descent data, and a Breuil–Kisin module without descent data, in order to avoid making separate definitions in the case of Breuil–Kisin modules without descent data; the same convention will apply to the étale φ-modules considered below.

Lemma 2.1.2. *Suppose either that A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 1$, or that A is p-adically complete and $\mathfrak{M}$ is projective. Then in Definition 2.1.1 the condition that $\Phi_{\mathfrak{M}}$ is an isomorphism after inverting $E(u)$ may equivalently be replaced with the condition that $\Phi_{\mathfrak{M}}$ is injective and its cokernel is killed by a power of $E(u)$.*

Proof. If A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 0$, then $E(u)^h$ divides $u^{e(a+h-1)}$ in $\mathfrak{S}_A$ (see [EG21, Lem. 5.2.6] and its proof), so that $\mathfrak{M}[1/u]$ is étale in the sense that the induced map

$$\Phi_{\mathfrak{M}}[1/u] : \varphi^*\mathfrak{M}[1/u] \to \mathfrak{M}[1/u]$$

is an isomorphism. The injectivity of $\Phi_{\mathfrak{M}}$ now follows because $\mathfrak{M}$, and therefore $\varphi^*\mathfrak{M}$, is u-torsion free.

If instead A is p-adically complete, then no Eisenstein polynomial over $W(k')$ is a zero divisor in $\mathfrak{S}_A$: this is plainly true if p is nilpotent in A, from which one deduces the same for p-adically complete A. Assuming that $\mathfrak{M}$ is projective, it follows that the maps $\mathfrak{M} \to \mathfrak{M}[1/E(u)]$ and $\varphi^*\mathfrak{M} \to (\varphi^*\mathfrak{M})[1/E(u)]$ are injective, and we are done. $\square$

Definition 2.1.3. If Q is any (not necessarily finitely generated) A-module, and $\mathfrak{M}$ is an $A[[u]]$-module, then we let $\mathfrak{M} \widehat{\otimes}_A Q$ denote the u-adic completion of $\mathfrak{M} \otimes_A Q$.

Lemma 2.1.4. *If $\mathfrak{M}$ is a Breuil–Kisin module and B is an A-algebra, then the base change $\mathfrak{M} \widehat{\otimes}_A B$ is a Breuil–Kisin module.*

Proof. We claim that $\mathfrak{M} \widehat{\otimes}_A B \cong \mathfrak{M} \otimes_{A[[u]]} B[[u]]$ for any finitely generated projective $A[[u]]$-module; the lemma then follows immediately from Definition 2.1.1.

To check the claim, we must see that the finitely generated $B[[u]]$-module $\mathfrak{M} \otimes_{A[[u]]} B[[u]]$ is u-adically complete. But $\mathfrak{M}$ is a direct summand of a free $A[[u]]$-module of

finite rank, in which case $\mathfrak{M} \otimes_{A[[u]]} B[[u]]$ is a direct summand of a free $B[[u]]$-module of finite rank and hence is u-adically complete. $\qquad\square$

Remark 2.1.5. If $I \subset A$ is a finitely generated ideal then $A[[u]] \otimes_A A/I \cong (A/I)[[u]]$, and $\mathfrak{M} \otimes_A A/I \cong \mathfrak{M} \otimes_{A[[u]]} (A/I)[[u]] \cong \mathfrak{M} \widehat{\otimes}_A A/I$; so in this case $\mathfrak{M} \otimes_A A/I$ itself is a Breuil–Kisin module.

Note that the base change (in the sense of Definition 2.1.3) of a weak Breuil–Kisin module may not be a weak Breuil–Kisin module, because the property of being u-torsion free is not always preserved by base change.

Definition 2.1.6. Let A be a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 1$. A *weak étale φ-module with A-coefficients and descent data from K' to K* is a triple $(M, \varphi_M, \{\hat{g}\})$ consisting of:

— a finitely generated $\mathfrak{S}_A[1/u]$-module M;

— a φ-semilinear map $\varphi_M : M \to M$ with the property that the induced map

$$\Phi_M = 1 \otimes \varphi_M : \varphi^*M := \mathfrak{S}_A[1/u] \otimes_{\varphi, \mathfrak{S}_A[1/u]} M \to M$$

is an isomorphism,

together with additive bijections $\hat{g} : M \to M$ for $g \in \mathrm{Gal}(K'/K)$, satisfying the further properties that the maps $\hat{g}$ commute with φ_M, satisfy $\hat{g_1} \circ \hat{g_2} = \widehat{g_1 \circ g_2}$, and have $\hat{g}(sm) = g(s)\hat{g}(m)$ for all $s \in \mathfrak{S}_A[1/u]$, $m \in M$.

If M as above is projective as an $\mathfrak{S}_A[1/u]$-module then we say simply that M is an étale φ-module. The étale φ-module M is said to be of rank d if the underlying finitely generated projective $\mathfrak{S}_A[1/u]$-module has constant rank d.

Remark 2.1.7. We could also consider étale φ-modules for general p-adically complete $\mathbf{Z}_p$-algebras A, but we would need to replace $\mathfrak{S}_A[1/u]$ by its p-adic completion. As we will not need to consider these modules in this paper, we do not do so here, but we refer the interested reader to [EG22].

A morphism of weak étale φ-modules with A-coefficients and descent data from K' to K is a morphism of $\mathfrak{S}_A[1/u]$-modules that commutes with φ and with the $\hat{g}$. Again, in the case $K' = K$ the descent data is trivial, and we obtain the usual category of étale φ-modules with A-coefficients.

Note that if A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra, and $\mathfrak{M}$ is a Breuil–Kisin module (resp., weak Breuil–Kisin module) with descent data, then $\mathfrak{M}[1/u]$ naturally has the structure of an étale φ-module (resp., weak étale φ-module) with descent data.

Suppose that A is an $\mathcal{O}$-algebra (where $\mathcal{O}$ is as in Section 1.4). In making calculations, it is often convenient to use the idempotents e_i (again as in Section 1.4). In particular if $\mathfrak{M}$ is a Breuil–Kisin module, then writing as usual $\mathfrak{M}_i := e_i\mathfrak{M}$, we write $\Phi_{\mathfrak{M},i} : \varphi^*(\mathfrak{M}_{i-1}) \to \mathfrak{M}_i$ for the morphism induced by $\Phi_{\mathfrak{M}}$. Similarly if M is an étale φ-module then we write $M_i := e_iM$, and we write $\Phi_{M,i} : \varphi^*(M_{i-1}) \to M_i$ for the morphism induced by Φ_M.

2.2. **Dieudonné modules.** Let A be a $\mathbf{Z}_p$-algebra. We define a *Dieudonné module of rank d with A-coefficients and descent data from K' to K* to be a finitely generated projective $W(k') \otimes_{\mathbf{Z}_p} A$-module D of constant rank d on $\mathrm{Spec}\, A$, together with:

— A-linear endomorphisms F, V satisfying $FV = VF = p$ such that F is φ-semilinear and V is φ^{-1}-semilinear for the action of $W(k')$, and

— a $W(k') \otimes_{\mathbf{Z}_p} A$-semilinear action of $\mathrm{Gal}(K'/K)$ which commutes with F and V.

Definition 2.2.1. If $\mathfrak{M}$ is a Breuil–Kisin module of height at most 1 and rank d with descent data, then there is a corresponding Dieudonné module $D = D(\mathfrak{M})$ of rank d defined as follows. We set $D := \mathfrak{M}/u\mathfrak{M}$ with the induced action of $\mathrm{Gal}(K'/K)$, and F given by the induced action of φ. The endomorphism V is determined as follows. Write $E(0) = cp$, so that we have $p \equiv c^{-1}E(u) \pmod{u}$. The condition that the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)$ allows us to factor the multiplication-by-$E(u)$ map on $\mathfrak{M}$ uniquely as $\mathfrak{V} \circ \varphi$, and V is defined to be $c^{-1}\mathfrak{V}$ modulo u.

2.3. **Galois representations.**

The theory of fields of norms [FW79] was used in [Fon90] to relate étale φ-modules with descent data to representations of a certain absolute Galois group; not the group G_K, but rather the group G_{K_∞}, where K_∞ is a certain infinite extension of K (whose definition is recalled below). Breuil–Kisin modules of height $h \leq 1$ are closely related to finite flat group schemes (defined over $\mathcal{O}_{K'}$, but with descent data to K on their generic fibre). Passage from a Breuil–Kisin module to its associated étale φ-module can morally be interpreted as the passage from a finite flat group scheme (with descent data) to its corresponding Galois representation (restricted to G_{K_∞}). Since the generic fibre of a finite flat group scheme over $\mathcal{O}_{K'}$, when equipped with descent data to K, in fact gives rise to a representation of G_K, in the case $h = 1$ we may relate Breuil–Kisin modules with descent data (or, more precisely, their associated étale φ-modules), not only to representations of G_{K_∞}, but to representations of G_K.

In this subsection, we recall some results coming from this connection, and draw some conclusions for Galois deformation rings.

2.3.1. *From étale φ-modules to G_{K_∞}-representations.*

We begin by recalling from [Kis09] some constructions arising in p-adic Hodge theory and the theory of fields of norms, which go back to [Fon90]. Following Fontaine, we write $R := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\bar{K}}/p$. Fix a compatible system $(\sqrt[p^n]{\pi})_{n \geq 0}$ of p^nth roots of π in $\bar{K}$ (compatible in the obvious sense that $(\sqrt[p^{n+1}]{\pi})^p = \sqrt[p^n]{\pi}$), and let $K_\infty := \cup_n K(\sqrt[p^n]{\pi})$, and also $K'_\infty := \cup_n K'(\sqrt[p^n]{\pi})$. Since $(e(K'/K), p) = 1$, the compatible system $(\sqrt[p^n]{\pi})_{n \geq 0}$ determines a unique compatible system $(\sqrt[p^n]{\pi'})_{n \geq 0}$ of p^nth roots of π' such that $(\sqrt[p^n]{\pi'})^{e(K'/K)} = \sqrt[p^n]{\pi}$. Write $\underline{\pi}' = (\sqrt[p^n]{\pi'})_{n \geq 0} \in R$, and $[\underline{\pi}'] \in W(R)$ for its image under the natural multiplicative map $R \to W(R)$. We have a Frobenius-equivariant inclusion $\mathfrak{S} \hookrightarrow W(R)$ by sending $u \mapsto [\underline{\pi}']$. We can naturally identify $\mathrm{Gal}(K'_\infty/K_\infty)$ with $\mathrm{Gal}(K'/K)$, and doing this we see that the action of $g \in G_{K_\infty}$ on u is via $g(u) = h(g)u$.

We let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}[1/u]$, and let $\mathcal{E}$ be the field of fractions of $\mathcal{O}_{\mathcal{E}}$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathcal{E} \hookrightarrow W(\mathrm{Frac}(R))[1/p]$. Let $\mathcal{E}^{\mathrm{nr}}$ be the maximal unramified extension of $\mathcal{E}$ in $W(\mathrm{Frac}(R))[1/p]$, and let $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}} \subset W(\mathrm{Frac}(R))$ denote its ring of integers. Let $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ be the p-adic completion of $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}$. Note that $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ is stable under the action of G_{K_∞}.

Definition 2.3.2. Suppose that A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 1$. If M is a weak étale φ-module with A-coefficients and descent data, set $T_A(M) := \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}[1/u]} M\right)^{\varphi=1}$, an A-module with a G_{K_∞}-action (via the diagonal action on

$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ and M, the latter given by the $\hat{g}$.) If $\mathfrak{M}$ is a weak Breuil–Kisin module with A-coefficients and descent data, set $T_A(\mathfrak{M}) := T_A(\mathfrak{M}[1/u])$.

Lemma 2.3.3. *Suppose that A is a local $\mathbf{Z}_p$-algebra and that $|A| < \infty$. Then T_A induces an equivalence of categories from the category of weak étale φ-modules with A-coefficients and descent data to the category of continuous representations of G_{K_∞} on finite A-modules. If $A \to A'$ is finite, then there is a natural isomorphism $T_A(M) \otimes_A A' \xrightarrow{\sim} T_{A'}(M \otimes_A A')$. A weak étale φ-module with A-coefficients and descent data M is free of rank d if and only if $T_A(M)$ is a free A-module of rank d.*

Proof. This is due to Fontaine [Fon90], and can be proved in exactly the same way as [Kis09, Lem. 1.2.7]. $\qquad\square$

We will frequently simply write T for T_A. Note that if we let M' be the étale φ-module obtained from M by forgetting the descent data, then by definition we have $T(M') = T(M)|_{G_{K'_\infty}}$.

2.3.4. *Relationships between G_K-representations and G_{K_∞}-representations.* We will later need to study deformation rings for representations of G_K in terms of the deformation rings for the restrictions of these representations to G_{K_∞}. Note that the representations of G_{K_∞} coming from Breuil–Kisin modules of height at most 1 admit canonical extensions to G_K by [Kis09, Prop. 1.1.13].

Let $\bar{r} : G_K \to \mathrm{GL}_2(\mathbf{F})$ be a continuous representation, let $R_{\bar{r}}$ denote the universal framed deformation $\mathcal{O}$-algebra for $\bar{r}$, and let $R_{\bar{r}}^{[0,1]}$ be the quotient with the property that if A is an Artinian local $\mathcal{O}$-algebra with residue field $\mathbf{F}$, then a local $\mathcal{O}$-morphism $R_{\bar{r}} \to A$ factors through $R_{\bar{r}}^{[0,1]}$ if and only if the corresponding G_K-module (ignoring the A-action) admits a G_K-equivariant surjection from a potentially crystalline $\mathcal{O}$-representation all of whose Hodge–Tate weights are equal to 0 or 1, and whose restriction to $G_{K'}$ is crystalline. (The existence of this quotient follows as in [Kim11, §2.1].)

Let $R_{\bar{r}|_{G_{K_\infty}}}$ be the universal framed deformation $\mathcal{O}$-algebra for $\bar{r}|_{G_{K_\infty}}$, and let $R_{\bar{r}|_{G_{K_\infty}}}^{\leq 1}$ denote the quotient with the property that if A is an Artinian local $\mathcal{O}$-algebra with residue field $\mathbf{F}$, then a morphism $R_{\bar{r}|_{G_{K_\infty}}} \to A$ factors through $R_{\bar{r}|_{G_{K_\infty}}}^{\leq 1}$ if and only if the corresponding G_{K_∞}-module is isomorphic to $T(\mathfrak{M})$ for some weak Breuil–Kisin module $\mathfrak{M}$ of height at most one with A-coefficients and descent data from K' to K. (The existence of this quotient follows exactly as for [Kim11, Thm. 1.3].)

Proposition 2.3.5. *The natural map induced by restriction from G_K to G_{K_∞} induces an isomorphism $\mathrm{Spec}\, R_{\bar{r}}^{[0,1]} \to \mathrm{Spec}\, R_{\bar{r}|_{G_{K_\infty}}}^{\leq 1}$.*

Proof. This can be proved in exactly the same way as [Kim11, Cor. 2.2.1] (which is the case that $E = \mathbf{Q}_p$ and $K' = K$). $\qquad\square$

3. Moduli stacks of Breuil–Kisin modules with descent data

In this section we introduce certain moduli stacks of Breuil–Kisin modules and étale φ-modules with tame descent data, following [PR09, EG21] (which consider the case without descent data).

The definitions and basic properties of our moduli stacks of Breuil–Kisin modules with tame descent data, and of étale φ-modules with tame descent data, are contained

in Section 3.1. In Section 3.3 we explain how to decompose the Breuil–Kisin moduli stacks as a disjoint union of substacks according to the *type* of the descent data, making use of results in Section 3.2 on the stacks classifying representations of tame groups.

3.1. **Moduli stacks of Breuil–Kisin modules and φ-modules with descent data.** We begin by defining the moduli stacks of Breuil–Kisin modules, with and without descent data.

Definition 3.1.1. For each integer $a \geq 1$, we let $\mathcal{C}^{\mathrm{dd},a}_{d,h,K'}$ be the *fppf* stack over $\mathcal{O}/\varpi^a$ which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{C}^{\mathrm{dd},a}_{d,h,K'}(A)$ of rank d Breuil–Kisin modules of height at most h with A-coefficients and descent data from K' to K.

By [Sta13, Tag 04WV], we may also regard each of the stacks $\mathcal{C}^{\mathrm{dd},a}_{d,h,K'}$ as an *fppf* stack over $\mathcal{O}$, and we then write $\mathcal{C}^{\mathrm{dd}}_{d,h,K'} := \varinjlim_a \mathcal{C}^{\mathrm{dd},a}_{d,h,K'}$; this is again an *fppf* stack over $\mathcal{O}$.

We will frequently omit any (or all) of the subscripts d, h, K' from this notation when doing so will not cause confusion. We write $\mathcal{C}^a$ for $\mathcal{C}^{\mathrm{dd},a}$ and $\mathcal{C}$ for $\mathcal{C}^{\mathrm{dd}}$ for the stacks with descent data for the trivial extension K'/K'.

The natural morphism $\mathcal{C}^{\mathrm{dd}} \to \operatorname{Spec}\mathcal{O}$ factors through $\operatorname{Spf}\mathcal{O}$, and by construction, there is an isomorphism $\mathcal{C}^{\mathrm{dd},a} \xrightarrow{\sim} \mathcal{C}^{\mathrm{dd}} \times_{\operatorname{Spf}\mathcal{O}} \operatorname{Spec}\mathcal{O}/\varpi^a$, for each $a \geq 1$; in particular, each of the morphisms $\mathcal{C}^{\mathrm{dd},a} \to \mathcal{C}^{\mathrm{dd},a+1}$ is a thickening (in the sense that its pullback under any test morphism $\operatorname{Spec}A \to \mathcal{C}^{\mathrm{dd},a+1}$ becomes a thickening of schemes, as defined in [Sta13, Tag 04EX] [1]). In Corollary 3.1.8 below we show that for each integer $a \geq 1$, $\mathcal{C}^{\mathrm{dd},a}$ is in fact an algebraic stack of finite type over $\operatorname{Spec}\mathcal{O}/\varpi^a$. The stack $\mathcal{C}^{\mathrm{dd}}$ is then *a priori* an Ind-algebraic stack, endowed with a morphism to $\operatorname{Spf}\mathcal{O}$ which is representable by algebraic stacks; we show that it is in fact a ϖ-*adic formal algebraic stack*, in the sense of the following definition whose basic theory has been developed by Emerton [Eme].

Definition 3.1.2. An *fppf* stack in groupoids $\mathcal{X}$ over a scheme S is called a *formal algebraic stack* if there is a morphism $U \to \mathcal{X}$, whose domain U is a formal algebraic space over S (in the sense of [Sta13, Tag 0AIL]), and which is representable by algebraic spaces, smooth, and surjective.

A formal algebraic stack $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}$ is called ϖ-*adic* if the canonical map $\mathcal{X} \to \operatorname{Spec}\mathcal{O}$ factors through $\operatorname{Spf}\mathcal{O}$, and if the induced map $\mathcal{X} \to \operatorname{Spf}\mathcal{O}$ is algebraic, i.e. representable by algebraic stacks (in the sense of [Sta13, Tag 06CF] and [Eme, Def. 3.1]).

Our approach will be to deduce the statements in the case with descent data from the corresponding statements in the case with no descent data, which follow from the methods of Pappas and Rapoport [PR09]. More precisely, in that reference it is proved that each $\mathcal{C}^a$ is an algebraic stack over $\mathcal{O}/\varpi^a$ [PR09, Thm. 0.1 (i)], and thus that $\mathcal{C} := \varinjlim_a \mathcal{C}^a$ is a ϖ-adic Ind-algebraic stack (in the sense that it is an Ind-algebraic stack with a morphism to $\operatorname{Spf}\mathcal{O}$ that is representable by algebraic stacks). (In [PR09] the stack $\mathcal{C}$ is described as being a p-adic formal algebraic stack.

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1. Note that for morphisms of algebraic stacks — and we will see below that $\mathcal{C}^{\mathrm{dd},a}$ and $\mathcal{C}^{\mathrm{dd},a+1}$ *are* algebraic stacks — this notion of thickening coincides with the notion defined in [Sta13, Tag 0BPP], by [Sta13, Tag 0CJ7].

However, in that reference, this term is used synonomously with our notion of a p-adic Ind-algebraic stack; the question of the existence of a smooth cover of $\mathcal{C}$ by a p-adic formal algebraic space is not discussed. As we will see, though, the existence of such a cover is easily deduced from the set-up of [PR09].)

We thank Brandon Levin for pointing out the following result to us. The proof is essentially (but somewhat implicitly) contained in the proof of [CL18, Thm. 3.5], but we take the opportunity to make it explicit. Note that it could also be directly deduced from the results of [PR09] using [EG22, Prop. A.10], but the proof that we give has the advantage of giving an explicit cover by a formal algebraic space.

Proposition 3.1.3. *For any choice of d, h, $\mathcal{C}$ is a ϖ-adic formal algebraic stack of finite type over $\mathrm{Spf}\,\mathcal{O}$ with affine diagonal.*

Proof. We begin by recalling some results from [PR09, §3.b] (which is where the proof that each $\mathcal{C}^a$ is an algebraic stack of finite type over $\mathcal{O}/\varpi^a$ is given). If A is an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, then we set

$$L^+G(A) := GL_d(\mathfrak{S}_A),$$

$$LG^{h,K'}(A) := \{X \in M_d(\mathfrak{S}_A) \mid X^{-1} \in E(u)^{-h} M_d(\mathfrak{S}_A)\},$$

and let $g \in L^+G(A)$ act on the right on $LG^{h,K'}(A)$ by φ-conjugation as $g^{-1} \cdot X \cdot \varphi(g)$. Then we may write

$$\mathcal{C} = [LG^{h,K'}/_\phi\, L^+G].$$

For each $n \geq 1$ we have the principal congruence subgroup U_n of L^+G given by $U_n(A) = I + u^n \cdot M_d(\mathfrak{S}_A)$. As in [PR09, §3.b.2], for any integer $n(a) > eah/(p-1)$ we have a natural identification

$$(3.1.4) \qquad [LG^{h,K'}/_\phi\, U_{n(a)}]_{\mathcal{O}/\varpi^a} \cong [LG^{h,K'}/U_{n(a)}]_{\mathcal{O}/\varpi^a}$$

where the $U_{n(a)}$-action on the right hand side is by left translation by the inverse; moreover this quotient stack is represented by a finite type scheme $(X_{n(a)}^{h,K'})_{\mathcal{O}/\varpi^a}$, and we find that

$$\mathcal{C}^a \cong [(X_{n(a)}^{h,K'})_{\mathcal{O}/\varpi^a}/_\phi\, (\mathcal{G}_{n(a)})_{\mathcal{O}/\varpi^a}],$$

where $(\mathcal{G}_{n(a)})_{\mathcal{O}/\varpi^a} = (L^+G/U_{n(a)})_{\mathcal{O}/\varpi^a}$ is a smooth finite type group scheme over $\mathcal{O}/\varpi^a$.

Now define $Y_a := [(X_{n(a)}^{h,K'})_{\mathcal{O}/\varpi^a}/_\phi\, (U_{n(1)})_{\mathcal{O}/\varpi^a}]$. If $a \geq b$, then there is a natural isomorphism $(Y_a)_{\mathcal{O}/\varpi^b} \cong Y_b$. Thus we may form the ϖ-adic Ind-algebraic stack $Y := \varinjlim_a Y_a$. Since $Y_1 := (X_{n(1)}^{h,K'})_{\mathbf{F}}$ is a scheme, each Y_a is in fact a scheme [Sta13, Tag 0BPW], and thus Y is a ϖ-adic formal scheme. (In fact, it is easy to check directly that $U_{n(1)}$ acts freely on $X_{n(a)}^{h,K'}$, and thus to see that Y_a is an algebraic space.) The natural morphism $Y \to \mathcal{C}$ is then representable by algebraic spaces; indeed, any morphism from an affine scheme to $\mathcal{C}$ factors through some $\mathcal{C}^a$, and representability by algebraic spaces then follows from the representability by algebraic spaces of $Y_a \to \mathcal{C}^a$, and the Cartesianness of the diagram

$$\begin{array}{ccc} Y_a & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{C}^a & \longrightarrow & \mathcal{C} \end{array}$$

Similarly, the morphism $Y \to \mathcal{C}$ is smooth and surjective, and so witnesses the claim that $\mathcal{C}$ is a ϖ-adic formal algebraic stack.

To check that $\mathcal{C}$ has affine diagonal, it suffices to check that each $\mathcal{C}^a$ has affine diagonal, which follows from the fact that $(\mathcal{G}_{n(a)})_{\mathcal{O}/\varpi^a}$ is in fact an affine group scheme over $\mathcal{O}/\varpi^a$ (indeed, as in [PR09, §2.b.1], it is a Weil restriction of GL_d). $\square$

We next introduce the moduli stack of étale φ-modules, again both with and without descent data.

Definition 3.1.5. For each integer $a \geq 1$, we let $\mathcal{R}^{\mathrm{dd},a}_{d,K'}$ be the *fppf* $\mathcal{O}/\varpi^a$-stack which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{R}^{\mathrm{dd},a}_{d,K'}(A)$ of rank d étale φ-modules with A-coefficients and descent data from K' to K.

By [Sta13, Tag 04WV], we may also regard each of the stacks $\mathcal{R}^{\mathrm{dd},a}_{d,K'}$ as an *fppf* $\mathcal{O}$-stack, and we then write $\mathcal{R}^{\mathrm{dd}} := \varinjlim_a \mathcal{R}^{\mathrm{dd},a}$, which is again an *fppf* $\mathcal{O}$-stack.

We will omit d, K' from the notation wherever doing so will not cause confusion. We write $\mathcal{R}$ and $\mathcal{R}^a$ for $\mathcal{R}^{\mathrm{dd}}$ and $\mathcal{R}^{\mathrm{dd},a}$ when the descent data is for the trivial extension K'/K'.

Just as in the case of $\mathcal{C}^{\mathrm{dd}}$, the morphism $\mathcal{R}^{\mathrm{dd}} \to \operatorname{Spec} \mathcal{O}$ factors through $\operatorname{Spf} \mathcal{O}$, and for each $a \geq 1$, there is a natural isomorphism $\mathcal{R}^{\mathrm{dd},a} \xrightarrow{\sim} \mathcal{R}^{\mathrm{dd}} \times_{\operatorname{Spf} \mathcal{O}} \operatorname{Spec} \mathcal{O}/\varpi^a$. Thus each of the morphisms $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a+1}$ is a thickening.

There is a natural morphism $\mathcal{C}^{\mathrm{dd}}_{d,h,K'} \to \mathcal{R}^{\mathrm{dd}}_d$, defined via

$$(\mathfrak{M}, \varphi, \{\hat{g}\}_{g \in \operatorname{Gal}(K'/K)}) \mapsto (\mathfrak{M}[1/u], \varphi, \{\hat{g}\}_{g \in \operatorname{Gal}(K'/K)}),$$

and natural morphisms $\mathcal{C}^{\mathrm{dd}} \to \mathcal{C}$ and $\mathcal{R}^{\mathrm{dd}} \to \mathcal{R}$ given by forgetting the descent data. In the optic of Section 2.3, the stack $\mathcal{R}^{\mathrm{dd}}_d$ may morally be thought of as a moduli of G_{K_∞}-representations, and the morphisms $\mathcal{C}^{\mathrm{dd}}_{d,h,K'} \to \mathcal{R}^{\mathrm{dd}}_d$ correspond to passage from a Breuil–Kisin module to its underlying Galois representation.

Proposition 3.1.6. *For each $a \geq 1$, the natural morphism $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^a$ is representable by algebraic spaces, affine, and of finite presentation.*

Proof. To see this, consider the pullback along some morphism $\operatorname{Spec} A \to \mathcal{R}^a$ (where A is a $\mathcal{O}/\varpi^a$-algebra); we must show that given an étale φ-module M of rank d without descent data, the data of giving additive bijections $\hat{g} : M \to M$, satisfying the further property that:

   — the maps $\hat{g}$ commute with φ, satisfy $\hat{g_1} \circ \hat{g_2} = \widehat{g_1 \circ g_2}$, and we have $\hat{g}(sm) =$
    $g(s)\hat{g}(m)$ for all $s \in \mathfrak{S}_A[1/u]$, $m \in M$

is represented by an affine algebraic space (i.e. an affine scheme!) of finite presentation over A.

To see this, note first that such maps $\hat{g}$ are by definition $\mathfrak{S}^0_A[1/v]$-linear. The data of giving an $\mathfrak{S}^0_A[1/v]$-linear automorphism of M which commutes with φ is representable by an affine scheme of finite presentation over A by [EG21, Prop. 5.4.8] and so the data of a finite collection of automorphisms is also representable by a finitely presented affine scheme over A. The further commutation and composition conditions on the $\hat{g}$ cut out a closed subscheme, as does the condition of $\mathfrak{S}_A[1/u]$-semi-linearity, so the result follows. $\square$

Corollary 3.1.7. *The diagonal of $\mathcal{R}^{\mathrm{dd}}$ is representable by algebraic spaces, affine, and of finite presentation.*

Proof. Since $\mathcal{R}^{\mathrm{dd}} = \varinjlim_a \mathcal{R}^{\mathrm{dd},a} \xrightarrow{\sim} \varinjlim_a \mathcal{R}^{\mathrm{dd}} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}/\varpi^a$, and since the transition morphisms are closed immersions (and hence monomorphisms), we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{R}^{\mathrm{dd},a} & \longrightarrow & \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{O}/\varpi^a} \mathcal{R}^{\mathrm{dd},a} \\ \downarrow & & \downarrow \\ \mathcal{R}^{\mathrm{dd}} & \longrightarrow & \mathcal{R}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{R}^{\mathrm{dd}} \end{array}$$

for each $a \geq 1$, and the diagonal morphism of $\mathcal{R}^{\mathrm{dd}}$ is the inductive limit of the diagonal morphisms of the various $\mathcal{R}^{\mathrm{dd},a}$. Any morphism from an affine scheme T to $\mathcal{R}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{R}^{\mathrm{dd}}$ thus factors through one of the $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{O}/\varpi^a} \mathcal{R}^{\mathrm{dd},a}$, and the fibre product $\mathcal{R}^{\mathrm{dd}} \times_{\mathcal{R}^{\mathrm{dd}} \times \mathcal{R}^{\mathrm{dd}}} T$ may be identified with $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^{\mathrm{dd},a} \times \mathcal{R}^{\mathrm{dd},a}} T$. It is thus equivalent to prove that each of the diagonal morphisms $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{O}/\varpi^a} \mathcal{R}^{\mathrm{dd},a}$ is representable by algebraic spaces, affine, and of finite presentation.

The diagonal of $\mathcal{R}^{\mathrm{dd},a}$ may be obtained by composing the pullback over $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{O}}$ $\mathcal{R}^{\mathrm{dd},a}$ of the diagonal $\mathcal{R}^a \to \mathcal{R}^a \times_{\mathcal{O}} \mathcal{R}^a$ with the relative diagonal of the morphism $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^a$. The first of these morphisms is representable by algebraic spaces, affine, and of finite presentation, by [EG21, Thm. 5.4.11 (2)], and the second is also representable by algebraic spaces, affine, and of finite presentation, since it is the relative diagonal of a morphism which has these properties, by Proposition 3.1.6. $\quad\square$

Corollary 3.1.8.   *(1) For each $a \geq 1$, $\mathcal{C}^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$, with affine diagonal.*

    (2) The Ind-algebraic stack $\mathcal{C}^{\mathrm{dd}} := \varinjlim_a \mathcal{C}^{\mathrm{dd},a}$ is furthermore a ϖ-adic formal algebraic stack.

    (3) The morphism $\mathcal{C}_h^{\mathrm{dd}} \to \mathcal{R}^{\mathrm{dd}}$ is representable by algebraic spaces and proper.

Proof. By Proposition 3.1.3, $\mathcal{C}^a$ is an algebraic stack of finite type over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$ with affine diagonal. In particular it has quasi-compact diagonal, and so is quasi-separated. Since $\mathcal{O}/\varpi^a$ is Noetherian, it follows from [Sta13, Tag 0DQJ] that $\mathcal{C}^a$ is in fact of finite presentation over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$.

By Proposition 3.1.6, the morphism $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a \to \mathcal{C}^a$ is representable by algebraic spaces and of finite presentation, so it follows from [Sta13, Tag 05UM] that $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a$ is an algebraic stack of finite presentation over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$. In order to show that $\mathcal{C}^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$, it therefore suffices to show that the natural monomorphism

$$(3.1.9) \qquad\qquad\qquad \mathcal{C}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a$$

is representable by algebraic spaces and of finite presentation. We will in fact show that it is a closed immersion (in the sense that its pull-back under any morphism from a scheme to its target becomes a closed immersion of schemes); since the target is locally Noetherian, and closed immersions are automatically of finite type and quasi-separated, it follows from [Sta13, Tag 0DQJ] that this closed immersion is of finite presentation, as required.

By [Sta13, Tag 0420], the property of being a closed immersion can be checked after pulling back to an affine scheme, and then working *fpqc*-locally. The claim then follows easily from the proof of [EG21, Prop. 5.4.8], as *fpqc*-locally the condition that a lattice in an étale φ-module of rank d with descent data is preserved by the

action of the $\hat{g}$ is determined by the vanishing of the coefficients of negative powers of u in a matrix.

To complete the proof of (1), it suffices to show that the diagonal of $\mathcal{C}^{\mathrm{dd},a}$ is affine. Since (as we have shown) the morphism (3.1.9) is a closed immersion, and thus a monomorphism, it is equivalent to show that the diagonal of $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a$ is affine. To ease notation, we denote this fibre product by $\mathcal{Y}$. We may then factor the diagonal of $\mathcal{Y}$ as the composite of the pull-back over $\mathcal{Y} \times_{\mathcal{O}/\varpi^a} \mathcal{Y}$ of the diagonal morphism $\mathcal{C}^a \to \mathcal{C}^a \times_{\mathcal{O}/\varpi^a} \mathcal{C}^a$ and the relative diagonal $\mathcal{Y} \to \mathcal{Y} \times_{\mathcal{C}^a} \mathcal{Y}$. The former morphism is affine, by [EG21, Thm. 5.4.9 (1)], and the latter morphism is also affine, since it is the pullback via $\mathcal{C}^a \to \mathcal{R}^a$ of the relative diagonal morphism $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{R}^{\mathrm{dd},a}$, which is affine (as already observed in the proof of Corollary 3.1.7).

To prove (2), consider the morphism $\mathcal{C}^{\mathrm{dd}} \to \mathcal{C}$. This is a morphism of ϖ-adic Ind-algebraic stacks, and by what we have already proved, it is representable by algebraic spaces. Since the target is a ϖ-adic formal algebraic stack, it follows from [Eme, Lem. 7.9] that the source is also a ϖ-adic formal algebraic stack, as required.

To prove (3), since each of $\mathcal{C}^{\mathrm{dd},a}$ and $\mathcal{R}^{\mathrm{dd},a}$ is obtained from $\mathcal{C}^{\mathrm{dd}}$ and $\mathcal{R}^{\mathrm{dd}}$ via pull-back over $\mathcal{O}/\varpi^a$, it suffices to prove that each of the morphisms $\mathcal{C}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ is representable by algebraic spaces and proper. Each of these morphisms factors as

$$\mathcal{C}^{\mathrm{dd},a} \overset{(3.1.9)}{\longrightarrow} \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a \overset{\mathrm{proj.}}{\longrightarrow} \mathcal{R}^{\mathrm{dd},a}.$$

We have already shown that the first of these morphisms is a closed immersion, and hence representable by algebraic spaces and proper. The second morphism is also representable by algebraic spaces and proper, since it is a base-change of the morphism $\mathcal{C}^a \to \mathcal{R}^a$, which has these properties by [EG21, Thm. 5.4.11 (1)]. $\square$

The next lemma gives a concrete interpretation of the points of $\mathcal{C}^{\mathrm{dd}}$ over ϖ-adically complete $\mathcal{O}$-algebras, extending the tautological interpretation of the points of each $\mathcal{C}^{\mathrm{dd},a}$ prescribed by Definition 3.1.1.

Lemma 3.1.10. *If A is a ϖ-adically complete $\mathcal{O}$-algebra then the $\mathrm{Spf}(A)$-points of $\mathcal{C}^{\mathrm{dd}}$ are the Breuil–Kisin modules of rank d and height h with A-coefficients and descent data.*

Proof. Let $\mathfrak{M}$ be a Breuil–Kisin module of rank d and height h with A-coefficients and descent data. Then the sequence $\{\mathfrak{M}/\varpi^a\mathfrak{M}\}_{a \geq 1}$ defines a $\mathrm{Spf}(A)$-point of $\mathcal{C}^{\mathrm{dd}}$ (*cf.* Remark 2.1.5), and since $\mathfrak{M}$ is ϖ-adically complete it is recoverable from the sequence $\{\mathfrak{M}/\varpi^a\mathfrak{M}\}_{a \geq 1}$.

In the other direction, suppose that $\{\mathfrak{M}_a\}$ is a $\mathrm{Spf}(A)$-point of $\mathcal{C}^{\mathrm{dd}}$, so that $\mathfrak{M}_a \in \mathcal{C}^{\mathrm{dd},a}(A/\varpi^a)$. Define $\mathfrak{M} = \varprojlim_a \mathfrak{M}_a$, and similarly define $\varphi_{\mathfrak{M}}$ and $\{\hat{g}\}$ as inverse limits. Observe that $\varphi^*\mathfrak{M} = \varprojlim_a \varphi^*\mathfrak{M}_a$ (since $\varphi : \mathfrak{S}_A \to \mathfrak{S}_A$ makes $\mathfrak{S}_A$ into a free $\mathfrak{S}_A$-module). Since each $\Phi_{\mathfrak{M}_a}$ is injective with cokernel killed by $E(u)^h$ the same holds for $\Phi_{\mathfrak{M}}$.

Since the required properties of the descent data are immediate, to complete the proof it remains to check that $\mathfrak{M}$ is a projective $\mathfrak{S}_A$-module (necessarily of rank d, since its rank will equal that of $\mathfrak{M}_1$), which is a consequence of [GD71, Prop. 0.7.2.10(ii)]. $\square$

We now temporarily reintroduce h to the notation.

Definition 3.1.11. For each $h \geq 0$, write $\mathcal{R}_h^a$ for the scheme-theoretic image of $\mathcal{C}_h^a \to \mathcal{R}^a$ in the sense of [EG21, Defn. 3.2.8]; then by [EG21, Thms. 5.4.19, 5.4.20], $\mathcal{R}_h^a$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, the morphism $\mathcal{C}_h^a \to \mathcal{R}^a$ factors through $\mathcal{R}_h^a$, and we may write $\mathcal{R}^a \cong \varinjlim_h \mathcal{R}_h^a$ as an inductive limit of closed substacks, the natural transition morphisms being closed immersions.

We similarly write $\mathcal{R}_h^{\mathrm{dd},a}$ for the scheme-theoretic image of the morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ in the sense of [EG21, Defn. 3.2.8].

Theorem 3.1.12. *For each $a \geq 1$, $\mathcal{R}^{\mathrm{dd},a}$ is an Ind-algebraic stack. Indeed, we can write $\mathcal{R}^{\mathrm{dd},a} = \varinjlim_h \mathcal{X}_h$ as an inductive limit of algebraic stacks of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, the transition morphisms being closed immersions.*

Proof. As we have just recalled, by [EG21, Thm. 5.4.20] we can write $\mathcal{R}^a = \varinjlim_h \mathcal{R}_h^a$, so that if we set $\mathcal{X}_h^{\mathrm{dd},a} := \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{R}_h^a$, then $\mathcal{R}^{\mathrm{dd},a} = \varinjlim_h \mathcal{X}_h^{\mathrm{dd},a}$, and the transition morphisms are closed immersions. Since $\mathcal{R}_h^a$ is of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, and a composite of morphisms of finite presentation is of finite presentation, it follows from Proposition 3.1.6 and [Sta13, Tag 05UM] that $\mathcal{X}_h^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, as required. $\square$

Theorem 3.1.13. *$\mathcal{R}_h^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$. It is a closed substack of $\mathcal{R}^{\mathrm{dd},a}$, and the morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through a morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}_h^{\mathrm{dd},a}$ which is representable by algebraic spaces, proper, and scheme-theoretically dominant in the sense of [EG21, Def. 3.1.3].*

Proof. As in the proof of Theorem 3.1.12, if we set $\mathcal{X}_h^{\mathrm{dd},a} := \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{R}_h^a$, then $\mathcal{X}_h^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, and the natural morphism $\mathcal{X}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ is a closed immersion. The morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through $\mathcal{X}_h^{\mathrm{dd},a}$ (because the morphism $\mathcal{C}_h^a \to \mathcal{R}^a$ factors through its scheme-theoretic image $\mathcal{R}_h^a$), so by [EG21, Prop. 3.2.31], $\mathcal{R}_h^{\mathrm{dd},a}$ is the scheme-theoretic image of the morphism of algebraic stacks $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{X}_h^{\mathrm{dd},a}$. The required properties now follow from [EG21, Lem. 3.2.29] (using representability by algebraic spaces and properness of the morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$, as proved in Corollary 3.1.8 (3), to see that the induced morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}_h^{\mathrm{dd},a}$ is representable by algebraic spaces and proper, along with [Sta13, Tag 0DQJ], and the fact that $\mathcal{X}_h^{\mathrm{dd},a}$ is of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, to see that $\mathcal{R}_h^{\mathrm{dd},a}$ is of finite presentation). $\square$

3.2. **Representations of tame groups.** Let G be a finite group.

Definition 3.2.1. We let $\operatorname{Rep}_d(G)$ denote the algebraic stack classifying d-dimensional representations of G over $\mathcal{O}$: if X is any $\mathcal{O}$-scheme, then $\operatorname{Rep}_d(G)(X)$ is the groupoid consisting of locally free sheaves of rank d over X endowed with an $\mathcal{O}_X$-linear action of G (rank d locally free G-sheaves, for short); morphisms are G-equivariant isomorphisms of vector bundles.

We now suppose that G is tame, i.e. that it has prime-to-p order. In this case (taking into account the fact that $\mathbf{F}$ has characteristic p, and that $\mathcal{O}$ is Henselian), the isomorphism classes of d-dimensional G-representations of G over E and over $\mathbf{F}$ are in natural bijection. Indeed, any finite-dimensional representation τ of G over E contains a G-invariant $\mathcal{O}$-lattice τ°, and the associated representation of G over $\mathbf{F}$ is given by forming $\overline{\tau} := \mathbf{F} \otimes_{\mathcal{O}} \tau^\circ$.

Lemma 3.2.2. *Suppose that G is tame, and that E is chosen large enough so that each irreducible representation of G over E is absolutely irreducible (or, equivalently, so that each irreducible representation of G over $\mathbf{F}$ is absolutely irreducible), and so that each irreducible representation of G over $\overline{\mathbf{Q}}_p$ is defined over E (equivalently, so that each irreducible representation of G over $\overline{\mathbf{F}}_p$ is defined over $\mathbf{F}$).*

(1) $\mathrm{Rep}_d(G)$ is the disjoint union of irreducible components $\mathrm{Rep}_d(G)_\tau$, where τ ranges over the finite set of isomorphism classes of d-dimensional representations of G over E.

(2) A morphism $X \to \mathrm{Rep}_d(G)$ factors through $\mathrm{Rep}_d(G)_\tau$ if and only if the associated locally free G-sheaf on X is Zariski locally isomorphic to $\tau^\circ \otimes_\mathcal{O} \mathcal{O}_X$.

(3) If we write $G_\tau := \mathrm{Aut}_{\mathcal{O}[G]}(\tau^\circ)$, then G_τ is a smooth (indeed reductive) group scheme over $\mathcal{O}$, and $\mathrm{Rep}_d(G)_\tau$ is isomorphic to the classifying space $[\mathrm{Spec}\,\mathcal{O}/G_\tau]$.

Proof. Since G has order prime to p, the representation $P := \oplus_\sigma \sigma^\circ$ is a projective generator of the category of $\mathcal{O}[G]$-modules, where σ runs over a set of representatives for the isomorphism classes of irreducible E-representations of G. (Indeed, each σ° is projective, because the fact that G has order prime to p means that all of the Ext^1s against σ° vanish. To see that $\oplus_\sigma \sigma^\circ$ is a generator, we need to show that every $\mathcal{O}[G]$-module admits a non-zero map from some σ°. We can reduce to the case of a finitely generated module M, and it is then enough (by projectivity) to prove that $M \otimes_\mathcal{O} \mathbf{F}$ admits such a map, which is clear.) Our assumption that each σ is absolutely irreducible furthermore shows that $\mathrm{End}_G(\sigma^\circ) = \mathcal{O}$ for each σ, so that $\mathrm{End}_G(P) = \prod_\sigma \mathcal{O}$.

Standard Morita theory then shows that the functor $M \mapsto \mathrm{Hom}_G(P, M)$ induces an equivalence between the category of $\mathcal{O}[G]$-modules and the category of $\prod_\sigma \mathcal{O}$-modules. Of course, a $\prod_\sigma \mathcal{O}$-module is just given by a tuple $(N_\sigma)_\sigma$ of $\mathcal{O}$-modules, and in this optic, the functor $\mathrm{Hom}_G(P, -)$ can be written as $M \mapsto \left(\mathrm{Hom}_G(\sigma^\circ, M)\right)_\sigma$, with a quasi-inverse functor being given by $(N_\sigma) \mapsto \bigoplus_\sigma \sigma^\circ \otimes_\mathcal{O} N_\sigma$. It is easily seen (just using the fact that $\mathrm{Hom}_G(P, -)$ induces an equivalence of categories) that M is a finitely generated projective A-module, for some $\mathcal{O}$-algebra A, if and only if each $\mathrm{Hom}_G(\sigma^\circ, M)$ is a finitely generated projective A-module.

The preceding discussion shows that giving a rank d representation of G over an $\mathcal{O}$-algebra A amounts to giving a tuple $(N_\sigma)_\sigma$ of projective A-modules, of ranks n_σ, such that $\sum_\sigma n_\sigma \dim \sigma = d$. For each such tuple of ranks (n_σ), we obtain a corresponding moduli stack $\mathrm{Rep}_{(n_\sigma)}(G)$ classifying rank d representations of G which decompose in this manner, and $\mathrm{Rep}_d(G)$ is isomorphic to the disjoint union of the various stacks $\mathrm{Rep}_{(n_\sigma)}(G)$.

If we write $\tau = \oplus_\sigma \sigma^{n_\sigma}$, then we may relabel $\mathrm{Rep}_{(n_\sigma)}(G)$ as $\mathrm{Rep}_\tau(G)$; statements (1) and (2) are then proved. By construction, there is an isomorphism

$$\mathrm{Rep}_\tau(G) = \mathrm{Rep}_{(n_\sigma)}(G) \xrightarrow{\sim} \prod_\sigma [\mathrm{Spec}\,\mathcal{O}/\mathrm{GL}_{n_\sigma}].$$

Noting that $G_\tau := \mathrm{Aut}(\tau) = \prod_\sigma \mathrm{GL}_{n_\sigma/\mathcal{O}}$, we find that statement (3) follows as well. $\square$

For each τ, it follows from the identification of $\mathrm{Rep}_d(G)_\tau$ with $[\mathrm{Spec}\,\mathcal{O}/G_\tau]$ that there is a natural map $\mathrm{Rep}_d(G)_\tau \to \mathrm{Spec}\,\mathcal{O}$. We let $\pi_0(\mathrm{Rep}_d(G))$ denote the disjoint union of copies of $\mathrm{Spec}\,\mathcal{O}$, one for each isomorphism class τ; then there

is a natural map $\mathrm{Rep}_d(G) \to \pi_0(\mathrm{Rep}_d(G))$. While we do not want to develop a general theory of the étale π_0 groups of algebraic stacks, we note that it is natural to regard $\pi_0(\mathrm{Rep}_d(G))$ as the étale π_0 of $\mathrm{Rep}_d(G)$.

3.3. **Tame inertial types.** Write $I(K'/K)$ for the inertia subgroup of $\mathrm{Gal}(K'/K)$. Since we are assuming that E is large enough that it contains the image of every embedding $K' \hookrightarrow \overline{\mathbf{Q}}_p$, it follows in particular that every $\overline{\mathbf{Q}}_p$-character of $I(K'/K)$ is defined over E.

Recall from Subsection 1.4 that if A is an $\mathcal{O}$-algebra, and $\mathfrak{M}$ is a Breuil–Kisin module with A-coefficients, then we write $\mathfrak{M}_i$ for $e_i\mathfrak{M}$. Since $I(K'/K)$ acts trivially on $W(k')$, the $\hat{g}$ for $g \in I(K'/K)$ stabilise each $\mathfrak{M}_i$, inducing an action of $I(K'/K)$ on $\mathfrak{M}_i/u\mathfrak{M}_i$.

Write

$$\mathrm{Rep}_{d,I(K'/K)} := \prod_{i=0}^{f'-1} \mathrm{Rep}_d\big(I(K'/K)\big),$$

the fibre product being taken over $\mathcal{O}$. If $\{\tau_i\}$ is an f'-tuple of isomorphism classes of d-dimensional representations of $I(K'/K)$, we write

$$\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}} := \prod_{i=0}^{f'-1} \mathrm{Rep}_d\big(I(K'/K)\big)_{\tau_i}.$$

Lemma 3.2.2 shows that we may write

$$\mathrm{Rep}_{d,I(K'/K)} = \coprod_{\{\tau_i\}} \mathrm{Rep}_{d,I(K'/K),\tau_i}.$$

Note that since K'/K is tamely ramified, $I(K'/K)$ is abelian of prime-to-p order, and each τ_i is just a sum of characters. If all of the τ_i are equal to some fixed τ, then we write $\mathrm{Rep}_{d,I(K'/K),\tau}$ for $\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}}$. We have corresponding stacks $\pi_0(\mathrm{Rep}_{d,I(K'/K)})$, $\pi_0(\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}})$ and $\pi_0(\mathrm{Rep}_{d,I(K'/K),\tau})$, defined in the obvious way.

If $\mathfrak{M}$ is a Breuil–Kisin module of rank d with descent data and A-coefficients, then $\mathfrak{M}_i/u\mathfrak{M}_i$ is projective A-module of rank d, endowed with an A-linear action of $I(K'/K)$, and so is an A-valued point of $\mathrm{Rep}_d\big(I(K'/K)\big)$. Thus we obtain a morphism

$$(3.3.1) \qquad\qquad \mathcal{C}_d^{\mathrm{dd}} \to \mathrm{Rep}_{d,I(K'/K)},$$

defined via $\mathfrak{M} \mapsto (\mathfrak{M}_0/u\mathfrak{M}_0, \ldots, \mathfrak{M}_{f'-1}/u\mathfrak{M}_{f'-1})$.

Definition 3.3.2. Let A be an $\mathcal{O}$-algebra, and let $\mathfrak{M}$ be a Breuil–Kisin module of rank d with A-coefficients. We say that $\mathfrak{M}$ has *mixed type* $(\tau_i)_i$ if the composite $\mathrm{Spec}\, A \to \mathcal{C}_d^{\mathrm{dd}} \to \mathrm{Rep}_{d,I(K'/K)}$ (the first arrow being the morphism that classifies $\mathfrak{M}$, and the second arrow being (3.3.1)) factors through $\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}}$. Concretely, this is equivalent to requiring that, Zariski locally on $\mathrm{Spec}\, A$, there is an $I(K'/K)$-equivariant isomorphism $\mathfrak{M}_i/u\mathfrak{M}_i \cong A \otimes_{\mathcal{O}} \tau_i^\circ$ for each i.

If each $\tau_i = \tau$ for some fixed τ, then we say that the type of $\mathfrak{M}$ is unmixed, or simply that $\mathfrak{M}$ has *type* τ.

Remark 3.3.3. If $A = \mathcal{O}$ then a Breuil–Kisin module necessarily has some (unmixed) type τ, since after inverting $E(u)$ and reducing modulo u the map $\Phi_{\mathfrak{M},i}$ gives an $I(K'/K)$-equivariant E-vector space isomorphism $\varphi^*(\mathfrak{M}_{i-1}/u\mathfrak{M}_{i-1})[\frac{1}{p}] \xrightarrow{\sim}$

$(\mathfrak{M}_i/u\mathfrak{M}_i)[\frac{1}{p}]$. However if $A = \mathbf{F}$ there are Breuil–Kisin modules which have a genuinely mixed type; indeed, it is easy to write down examples of free Breuil–Kisin modules of rank one of any mixed type (see also [CL18, Rem. 3.7]), which necessarily cannot lift to characteristic zero. This shows that $\mathcal{C}_d^{\mathrm{dd}}$ is not flat over $\mathcal{O}$. In the following sections, when $d = 2$ and $h = 1$ we define a closed substack $\mathcal{C}_d^{\mathrm{dd,BT}}$ of $\mathcal{C}^{\mathrm{dd}}$ which *is* flat over $\mathcal{O}$, and can be thought of as taking the Zariski closure of $\overline{\mathbf{Q}}_p$-valued Galois representations that become Barsotti–Tate over K' and such that all pairs of labeled Hodge–Tate weights are $\{0, 1\}$ (see Remark 4.2.8 below).

Definition 3.3.4. Let $\mathcal{C}_d^{(\tau_i)}$ be the étale substack of $\mathcal{C}_d^{\mathrm{dd}}$ which associates to each $\mathcal{O}$-algebra A the subgroupoid $\mathcal{C}_d^{(\tau_i)}(A)$ of $\mathcal{C}_d^{\mathrm{dd}}(A)$ consisting of those Breuil–Kisin modules which are of mixed type (τ_i). If each $\tau_i = \tau$ for some fixed τ, we write $\mathcal{C}_d^{\tau}$ for $\mathcal{C}_d^{(\tau_i)}$.

Proposition 3.3.5. *Each $\mathcal{C}_d^{(\tau_i)}$ is an open and closed substack of $\mathcal{C}_d^{\mathrm{dd}}$, and $\mathcal{C}_d^{\mathrm{dd}}$ is the disjoint union of its substacks $\mathcal{C}_d^{(\tau_i)}$.*

Proof. By Lemma 3.2.2, $\mathrm{Rep}_{d,I(K'/K)}$ is the disjoint union of its open and closed substacks $\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}}$. By definition $\mathcal{C}_d^{(\tau_i)}$ is the preimage of $\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}}$ under the morphism (3.3.1); the lemma follows. $\square$

4. Local geometry of Breuil–Kisin moduli stacks

Our aim in this section is to use local models of Shimura varieties at hyperspecial and Iwahori level to study the geometry of the moduli stacks of Breuil–Kisin modules described in Section 3, in the case where the height h is equal to 1.

We begin in Section 4.1 by explaining how to construct a smooth morphism from the stack $\mathcal{C}_{1,K'}^{\mathrm{dd}}$ (and thus its closed substacks $\mathcal{C}^{(\tau_i)}$) to a suitable local model stack. When $d = 2$, the main case of interest in this paper, we want to consider stacks of Breuil–Kisin modules of whose $\mathrm{Spf}(\mathcal{O})$-points correspond to crystalline representations that are not just of height 1, but are actually potentially Barsotti–Tate (i.e., whose Hodge–Tate weights are regular). This is done by imposing Kottwitz-type determinant conditions, which we discuss in Section 4.2. The resulting stacks will be denoted $\mathcal{C}^{\tau,\mathrm{BT}}$.

In Section 4.4 we explain how to relate our local models to the naive local models of [PRS13] at hyperspecial and Iwahori level, and finally in Section 4.5 we deduce our main results on the local geometry of the stacks $\mathcal{C}^{\tau,\mathrm{BT}}$ by comparison with known results on the geometry of these local models. Section 4.3 contains a brief interlude on the behaviour of our constructions under change of base, which will allow us to unify the discussion of principal series and cuspidal types in Section 5.

Finally, suppose that $d = 2$ and that the tame type $\tau = \eta \oplus \eta'$ has $\eta \neq \eta'$. In Section 4.6 we construct a morphism from $\mathcal{C}^{\tau,\mathrm{BT}}$ to an auxiliary *Dieudonné stack* $\mathcal{D}_\eta$. This construction is not needed elsewhere in the present paper, but will be used in [CEGS20a, CEGS20c].

4.1. **Local models: generalities.** Throughout this section we allow d to be arbitrary; in Section 4.2 we specialise to the case $d = 2$, where we relate the local models considered in this section to the local models considered in the theory of Shimura varieties. We will usually omit d from our notation, writing for example $\mathcal{C}$

for $\mathcal{C}_d$, without any further comment. We begin with the following lemma, for which we allow h to be arbitrary.

Lemma 4.1.1. *Let $\mathfrak{M}$ be a rank d Breuil–Kisin module of height h with descent data over an $\mathcal{O}$-algebra A. Assume further either that A is p^n-torsion for some n, or that A is Noetherian and p-adically complete. Then $\operatorname{im}\Phi_{\mathfrak{M}}/E(u)^h\mathfrak{M}$ is a finite projective A-module, and is a direct summand of $\mathfrak{M}/E(u)^h\mathfrak{M}$ as an A-module.*

Proof. We follow the proof of [Kis09, Lem. 1.2.2]. We have a short exact sequence

$$0 \to \operatorname{im}\Phi_{\mathfrak{M}}/E(u)^h\mathfrak{M} \to \mathfrak{M}/E(u)^h\mathfrak{M} \to \mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}} \to 0$$

in which the second term is a finite projective A-module (since it is a finite projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module), so it is enough to show that the third term is a projective A-module. It is therefore enough to show that the finitely generated A-module $\mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}}$ is finitely presented and flat.

To see that it is finitely presented, note that we have the equality

$$\mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}} = (\mathfrak{M}/E(u)^h)/(\operatorname{im}\Phi_{\mathfrak{M}}/E(u)^h),$$

and the right hand side admits a presentation by finitely generated projective A-modules

$$\varphi^*(\mathfrak{M}/E(u)^h) \to \mathfrak{M}/E(u)^h \to (\mathfrak{M}/E(u)^h)/(\operatorname{im}\Phi_{\mathfrak{M}}/E(u)^h) \to 0.$$

To see that it is flat, it is enough to show that for every finitely generated ideal I of A, the map

$$I \otimes_A \mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}} \to \mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}}$$

is injective. It follows easily (for example, from the snake lemma) that it is enough to check that the complex

$$0 \to \varphi^*\mathfrak{M} \to \mathfrak{M} \to \mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}} \to 0,$$

which is exact by Lemma 2.1.2, remains exact after tensoring with A/I. Since I is finitely generated we have $\mathfrak{M} \otimes_A A/I \cong \mathfrak{M}\widehat{\otimes}_A A/I$ by Remark 2.1.5, and the desired exactness amounts to the injectivity of $\Phi_{\mathfrak{M}\widehat{\otimes}_A A/I}$ for the Breuil–Kisin module $\mathfrak{M}\widehat{\otimes}_A A/I$. This follows immediately from Lemma 2.1.2 in the case that A is killed by p^n, and otherwise follows from the same lemma once we check that A/I is p-adically complete, which follows from the Artin–Rees lemma (as A is assumed Noetherian and p-adically complete). $\square$

We assume from now on that $h = 1$, but we continue to allow arbitrary d. We allow K'/K to be any Galois extension such that $[K' : K]$ is prime to p (so in particular K'/K is tame).

Definition 4.1.2. We let $\mathcal{M}_{\mathrm{loc}}^{K'/K,a}$ be the algebraic stack of finite presentation over $\operatorname{Spec}\mathcal{O}/\varpi^a$ defined as follows: if A is an $\mathcal{O}/\varpi^a$-algebra, then $\mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ is the groupoid of tuples $(\mathfrak{L}, \mathfrak{L}^+)$, where:

— $\mathfrak{L}$ is a rank d projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module, with a $\operatorname{Gal}(K'/K)$-semilinear, A-linear action of $\operatorname{Gal}(K'/K)$;

— $\mathfrak{L}^+$ is an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-submodule of $\mathfrak{L}$, which is locally on $\operatorname{Spec} A$ a direct summand of $\mathfrak{L}$ as an A-module (or equivalently, for which $\mathfrak{L}/\mathfrak{L}^+$ is projective as an A-module), and is preserved by $\operatorname{Gal}(K'/K)$.

We set $\mathcal{M}_{\mathrm{loc}}^{K'/K} := \varinjlim_a \mathcal{M}_{\mathrm{loc}}^{K'/K,a}$, so that $\mathcal{M}_{\mathrm{loc}}^{K'/K}$ is a ϖ-adic formal algebraic stack, of finite presentation over $\mathrm{Spf}\,\mathcal{O}$ (indeed, it is easily seen to be the ϖ-adic completion of an algebraic stack of finite presentation over $\mathrm{Spec}\,\mathcal{O}$).

In Section 4.4 we will explain how to relate these local model stacks (or more precisely, their refinements satisfying a certain determinant condition that will be developed in Section 4.2) to local models of Shimura varieties. Lemma 4.1.1 shows that they are also connected to our stacks of Breuil–Kisin modules, as follows.

Definition 4.1.3. By Lemma 4.1.1, we have a natural morphism $\Psi : \mathcal{C}_{1,K'}^{\mathrm{dd}} \to \mathcal{M}_{\mathrm{loc}}^{K'/K}$, which takes a Breuil–Kisin module with descent data $\mathfrak{M}$ of height 1 to the pair

$$(\mathfrak{M}/E(u)\mathfrak{M}, \mathrm{im}\,\Phi_{\mathfrak{M}}/E(u)\mathfrak{M}).$$

Remark 4.1.4. The definition of the stack $\mathcal{M}_{\mathrm{loc}}^{K'/K,a}$ does not include any condition that mirrors the commutativity between the Frobenius and the descent data on a Breuil–Kisin module, and so in general the morphism $\Psi_A : \mathcal{C}_{1,K'}^{\mathrm{dd}}(A) \to \mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ cannot be essentially surjective.

It will be convenient to consider the twisted group rings $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$ and $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A)[\mathrm{Gal}(K'/K)]$, in which the elements $g \in \mathrm{Gal}(K'/K)$ obey the following commutation relation with elements $s \in \mathfrak{S}_A$ (resp. $s \in \mathcal{O}_{K'}$):

$$g \cdot s = g(s) \cdot g.$$

(In the literature these twisted group rings are more often written as $\mathfrak{S}_A * \mathrm{Gal}(K'/K)$, $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A) * \mathrm{Gal}(K'/K)$, in order to distinguish them from the usual (untwisted) group rings, but as we will only use the twisted versions in this paper, we prefer to use this notation for them.)

By definition, endowing a finitely generated $\mathfrak{S}_A$-module P with a semilinear $\mathrm{Gal}(K'/K)$-action is equivalent to giving it the structure of a left $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module. If P is projective as an $\mathfrak{S}_A$-module, then it is also projective as an $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module. Indeed, $\mathfrak{S}_A$ is a direct summand of $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$ as a $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module, given by the central idempotent $\frac{1}{|\mathrm{Gal}(K'/K)|} \sum_{g \in \mathrm{Gal}(K'/K)} g$, so P is a direct summand of the projective module $\mathfrak{S}_A[\mathrm{Gal}(K'/K)] \otimes_{\mathfrak{S}_A} P$. Similar remarks apply to the case of $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A)[\mathrm{Gal}(K'/K)]$-modules.

Theorem 4.1.5. *The morphism* $\Psi : \mathcal{C}_{1,K'}^{\mathrm{dd}} \to \mathcal{M}_{\mathrm{loc}}^{K'/K}$ *is representable by algebraic spaces and smooth.*

Proof. We first show that the morphism Ψ is formally smooth, in the sense that it satisfies the infinitesimal lifting criterion for nilpotent thickenings of affine test objects [EG21, Defn. 2.4.2]. For this, we follow the proof of [Kis09, Prop. 2.2.11] (see also the proof of [CL18, Thm. 4.9]). Let A be an $\mathcal{O}/\varpi^a$-algebra and $I \subset A$ be a nilpotent ideal. Suppose that we are given $\mathfrak{M}_{A/I} \in \mathcal{C}_{1,K'}^{\mathrm{dd}}(A/I)$ and $(\mathfrak{L}_A, \mathfrak{L}_A^+) \in \mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ together with an isomorphism

$$\Psi(\mathfrak{M}_{A/I}) \xrightarrow{\sim} (\mathfrak{L}_A, \mathfrak{L}_A^+) \otimes_A A/I =: (\mathfrak{L}_{A/I}, \mathfrak{L}_{A/I}^+).$$

We must show that there exists $\mathfrak{M}_A \in \mathcal{C}_{1,K'}^{\mathrm{dd}}(A)$ together with an isomorphism $\Psi(\mathfrak{M}_A) \xrightarrow{\sim} (\mathfrak{L}_A, \mathfrak{L}_A^+)$ lifting the given isomorphism.

As explained above, we can and do think of $\mathfrak{M}_{A/I}$ as a finite projective $\mathfrak{S}_{A/I}[\mathrm{Gal}(K'/K)]$-module, and $\mathfrak{L}_A$ as a finite projective $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A)[\mathrm{Gal}(K'/K)]$-module. Since the

closed 2-sided ideal of $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$ generated by I consists of nilpotent elements, we may lift $\mathfrak{M}_{A/I}$ to a finite projective $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module $\mathfrak{M}_A$. This is presumably standard, but for lack of a reference we indicate a proof. In fact the proof of [Sta13, Tag 0D47] goes over unchanged to our setting. Writing $\mathfrak{M}_{A/I}$ as a direct summand of a finite free $\mathfrak{S}_{A/I}[\mathrm{Gal}(K'/K)]$-module F, it is enough to lift the corresponding idempotent in $\mathrm{End}_{\mathfrak{S}_{A/I}[\mathrm{Gal}(K'/K)]}(F)$, which is possible by [Sta13, Tag 05BU]. (See also [Lam91, Thm. 21.28] for another proof of the existence of lifts of idempotents in this generality.) Note that since $\mathfrak{M}_{A/I}$ is of rank d as a projective $\mathfrak{S}_{A/I}$-module, $\mathfrak{M}_A$ is of rank d as a projective $\mathfrak{S}_A$-module.

Since $\mathfrak{M}_A/E(u)\mathfrak{M}_A$ is a projective $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A)[\mathrm{Gal}(K'/K)]$-module, we may lift the composite

$$\mathfrak{M}_A/E(u)\mathfrak{M}_A \twoheadrightarrow \mathfrak{M}_{A/I}/E(u)\mathfrak{M}_{A/I} \xrightarrow{\sim} \mathfrak{L}_{A/I}$$

to a morphism $\theta : \mathfrak{M}_A/E(u)\mathfrak{M}_A \to \mathfrak{L}_A$. Since the composite of θ with $\mathfrak{L}_A \twoheadrightarrow \mathfrak{L}_{A/I}$ is surjective, it follows by Nakayama's lemma that θ is surjective. But a surjective map of projective modules of the same rank is an isomorphism, and so θ is an isomorphism lifting the given isomorphism $\mathfrak{M}_{A/I}/E(u)\mathfrak{M}_{A/I} \xrightarrow{\sim} \mathfrak{L}_{A/I}$.

We let $\mathfrak{M}_A^+$ denote the preimage in $\mathfrak{M}_A$ of $\theta^{-1}(\mathfrak{L}_A^+)$. The image of the induced map $f : \mathfrak{M}_A^+ \subset \mathfrak{M}_A \twoheadrightarrow \mathfrak{M}_A \widehat{\otimes}_A A/I \cong \mathfrak{M}_{A/I}$ is precisely $\mathrm{im}\,\Phi_{\mathfrak{M}_{A/I}}$, since the same is true modulo $E(u)$ and because $\mathfrak{M}_A^+$, $\mathrm{im}\,\Phi_{\mathfrak{M}_{A/I}}$ contain $E(u)\mathfrak{M}_A$, $E(u)\mathfrak{M}_{A/I}$ respectively. Observing that

$$\mathfrak{M}_A/\mathfrak{M}_A^+ \cong (\mathfrak{M}_A/E(u)\mathfrak{M}_A)/(\mathfrak{M}_A^+/E(u)\mathfrak{M}_A) \cong \mathfrak{L}_A/\mathfrak{L}_A^+$$

we deduce that $\mathfrak{M}_A/\mathfrak{M}_A^+$ is projective as an A-module, whence $\mathfrak{M}_A^+$ is an A-module direct summand of $\mathfrak{M}_A$. By the same argument $\mathrm{im}\,\Phi_{\mathfrak{M}_{A/I}}$ is an A-module direct summand of $\mathfrak{M}_{A/I}$, and we conclude that the map $\mathfrak{M}_A^+ \widehat{\otimes}_A A/I \to \mathrm{im}\,\Phi_{\mathfrak{M}_{A/I}}$ induced by f is an isomorphism.

Finally, we have the diagram

$$
\begin{array}{ccc}
\varphi^*\mathfrak{M}_A & & \mathfrak{M}_A^+ \\
\downarrow & & \downarrow \\
\varphi^*\mathfrak{M}_{A/I} & \longrightarrow & \mathrm{im}\,\Phi_{\mathfrak{M}_{A/I}}
\end{array}
$$

where the horizontal arrow is given by $\Phi_{\mathfrak{M}_{A/I}}$, and the right hand vertical arrow is f. Since $\varphi^*\mathfrak{M}_A$ is a projective $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module, we may find a morphism of $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-modules $\varphi^*\mathfrak{M}_A \to \mathfrak{M}_A^+$ which fills in the commutative square. Since the composite $\varphi^*\mathfrak{M}_A \to \mathfrak{M}_A^+ \to \mathrm{im}\,\Phi_{\mathfrak{M}_{A/I}} \cong \mathfrak{M}_A^+ \widehat{\otimes}_A A/I$ is surjective, it follows by Nakayama's lemma that $\varphi^*\mathfrak{M}_A \to \mathfrak{M}_A^+$ is also surjective, and the composite $\varphi^*\mathfrak{M}_A \to \mathfrak{M}_A^+ \subset \mathfrak{M}_A$ gives a map $\Phi_{\mathfrak{M}_A}$. Since $\Phi_{\mathfrak{M}_A}[1/E(u)]$ is a surjective map of projective modules of the same rank, it is an isomorphism, and we see that $\mathfrak{M}_A$ together with $\Phi_{\mathfrak{M}_A}$ is our required lifting to a Breuil–Kisin module of rank d with descent data.

Since the source and target of Ψ are of finite presentation over $\mathrm{Spf}\,\mathcal{O}$, and ϖ-adic, we see that Ψ is representable by algebraic spaces (by [Eme, Lem. 7.10]) and locally of finite presentation (by [EG21, Cor. 2.1.8] and [Sta13, Tag 06CX]). Thus Ψ is in fact smooth (being formally smooth and locally of finite presentation). $\square$

We now show that the inertial type of a Breuil–Kisin module is visible on the local model.

Lemma 4.1.6. *There is a natural morphism* $\mathcal{M}_{\mathrm{loc}}^{K'/K} \to \pi_0(\mathrm{Rep}_{I(K'/K)})$.

Proof. The morphism $\mathcal{M}_{\mathrm{loc}}^{K'/K} \to \pi_0(\mathrm{Rep}_{I(K'/K)})$ is defined by sending $(\mathfrak{L}, \mathfrak{L}^+) \mapsto$ $\mathfrak{L}/\pi'$. More precisely, $\mathfrak{L}/\pi'$ is a rank d projective $k' \otimes_{\mathbf{Z}_p} A$-module with a linear action of $I(K'/K)$, so determines an A/p-point of $\pi_0(\mathrm{Rep}_{I(K'/K)}) = \prod_{i=0}^{f'-1} \pi_0(\mathrm{Rep}_d(I(K'/K)))$. Since the target is a disjoint union of copies of $\mathrm{Spec}\,\mathcal{O}$, the morphism $\mathrm{Spec}\,A/p \to \pi_0(\mathrm{Rep}_{I(K'/K)})$ lifts uniquely to a morphism $\mathrm{Spec}\,A \to \pi_0(\mathrm{Rep}_{I(K'/K)})$, as required. $\square$

Definition 4.1.7. We let
$$\mathcal{M}_{\mathrm{loc}}^{(\tau_i)} := \mathcal{M}_{\mathrm{loc}}^{K'/K} \otimes_{\pi_0(\mathrm{Rep}_{I(K'/K)})} \pi_0(\mathrm{Rep}_{I(K'/K),\{\tau_i\}}).$$

If each $\tau_i = \tau$ for some fixed τ, we write $\mathcal{M}_{\mathrm{loc}}^{\tau}$ for $\mathcal{M}_{\mathrm{loc}}^{(\tau_i)}$. By Lemma 3.2.2, $\mathcal{M}_{\mathrm{loc}}^{K'/K}$ is the disjoint union of the open and closed substacks $\mathcal{M}_{\mathrm{loc}}^{(\tau_i)}$.

Lemma 4.1.8. *We have* $\mathcal{C}^{(\tau_i)} = \mathcal{C}^{\mathrm{dd}} \times_{\mathcal{M}_{\mathrm{loc}}^{K'/K}} \mathcal{M}_{\mathrm{loc}}^{(\tau_i)}$.

Proof. This is immediate from the definitions. $\square$

In particular, $\mathcal{C}^{(\tau_i)}$ is a closed substack of $\mathcal{C}^{\mathrm{dd}}$.

4.2. **Local models: determinant conditions.** Write $N = K \cdot W(k')[1/p]$, so that K'/N is totally ramified. Since $I(K'/K)$ is cyclic of order prime to p and acts trivially on $\mathcal{O}_N$, we may write

$$(4.2.1) \qquad\qquad (\mathfrak{L}, \mathfrak{L}^+) = \oplus_\xi (\mathfrak{L}_\xi, \mathfrak{L}_\xi^+)$$

where the sum is over all characters $\xi : I(K'/K) \to \mathcal{O}^\times$, and $\mathfrak{L}_\xi$ (resp. $\mathfrak{L}_\xi^+$) is the $\mathcal{O}_N \otimes A$-submodule of $\mathfrak{L}$ (resp. of $\mathfrak{L}^+$) on which $I(K'/K)$ acts through ξ.

Definition 4.2.2. We say that an object $(\mathfrak{L}, \mathfrak{L}^+)$ of $\mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ *satisfies the strong determinant condition* if Zariski locally on $\mathrm{Spec}\,A$ the following condition holds: for all $a \in \mathcal{O}_N$ and all ξ, we have

$$(4.2.3) \qquad\qquad \det{}_A(a|\mathfrak{L}_\xi^+) = \prod_{\psi:N\hookrightarrow E} \psi(a)$$

as polynomial functions on $\mathcal{O}_N$ in the sense of [Kot92, §5].

Remark 4.2.4. An explicit version of this determinant condition is stated, in this generality, in [Kis09, §2.2], specifically in the proof of [Kis09, Prop. 2.2.5]. We recall this here, with our notation. We have a direct sum decomposition

$$\mathcal{O}_N \otimes_{\mathbf{Z}_p} A \xrightarrow{\sim} \bigoplus_{\sigma_i:k'\hookrightarrow \mathbf{F}} \mathcal{O}_N \otimes_{W(k'),\sigma_i} A$$

Recall that $e_i \in \mathcal{O}_N \otimes_{\mathbf{Z}_p} \mathcal{O}$ denotes the idempotent that identifies $e_i \cdot \mathcal{O}_N \otimes_{\mathbf{Z}_p} A$ with the summand $\mathcal{O}_N \otimes_{W(k'),\sigma_i} A$. For $j = 0, 1, \ldots, e-1$, let X_{j,σ_i} be an indeterminate. Then the strong determinant condition on $(\mathfrak{L}, \mathfrak{L}^+)$ is that for all ξ, we have

$$(4.2.5) \qquad \det_A \left(\sum_{j,\sigma_i} e_i \pi^j X_{j,\sigma_i} \mid \mathfrak{L}_\xi^+ \right) = \prod_\psi \sum_{j,\sigma_i} \left(\psi(e_i \pi^j) X_{j,\sigma_i} \right),$$

where j runs over $0, 1, \ldots, e-1$, σ_i over embeddings $k' \hookrightarrow \mathbf{F}$, and ψ over embeddings $\mathcal{O}_N \hookrightarrow \mathcal{O}$. Note that $\psi(e_i) = 1$ if $\psi|_{W(k')}$ lifts σ_i and is equal to 0 otherwise.

Definition 4.2.6. We write $\mathcal{M}_{\mathrm{loc}}^{K'/K,\mathrm{BT}}$ for the substack of $\mathcal{M}_{\mathrm{loc}}^{K'/K}$ given by those $(\mathfrak{L}, \mathfrak{L}^+)$ which satisfy the strong determinant condition. For each (possibly mixed) type (τ_i), we write $\mathcal{M}_{\mathrm{loc}}^{(\tau_i),\mathrm{BT}} := \mathcal{M}_{\mathrm{loc}}^{(\tau_i)} \times_{\mathcal{M}_{\mathrm{loc}}^{K'/K}} \mathcal{M}_{\mathrm{loc}}^{K'/K,\mathrm{BT}}$.

Suppose for the remainder of this section that $d = 2$ and $h = 1$, so that $\mathcal{C}^{\mathrm{dd}}$ consists of Breuil–Kisin modules of rank two and height at most 1. We then set $\mathcal{C}^{\mathrm{dd},\mathrm{BT}} := \mathcal{C}^{\mathrm{dd}} \times_{\mathcal{M}_{\mathrm{loc}}^{K'/K}} \mathcal{M}_{\mathrm{loc}}^{K'/K,\mathrm{BT}}$, and $\mathcal{C}^{(\tau_i),\mathrm{BT}} := \mathcal{C}^{(\tau_i)} \times_{\mathcal{M}_{\mathrm{loc}}^{(\tau_i)}} \mathcal{M}_{\mathrm{loc}}^{(\tau_i),\mathrm{BT}}$; we also write $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} := \mathcal{C}^{\mathrm{dd},\mathrm{BT}} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ and $\mathcal{C}^{(\tau_i),\mathrm{BT},a} := \mathcal{C}^{(\tau_i),\mathrm{BT}} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$.

A Breuil–Kisin module $\mathfrak{M} \in \mathcal{C}^{\mathrm{dd}}(A)$ is said to satisfy the strong determinant condition if and only if its image $\Psi(\mathfrak{M}) \in \mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ does, i.e. if and only if it lies in $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$.

Proposition 4.2.7. $\mathcal{C}^{(\tau_i),\mathrm{BT}}$ (*resp.* $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$) *is a closed substack of* $\mathcal{C}^{(\tau_i)}$ (*resp.* $\mathcal{C}^{\mathrm{dd}}$); *in particular, it is a* ϖ-*adic formal algebraic stack of finite presentation over* $\mathcal{O}$.

Proof. This is immediate from Corollary 3.1.8 and the definition of the strong determinant condition as an equality of polynomial functions. $\qquad \square$

Remark 4.2.8. The motivation for imposing the strong determinant condition is as follows. One can take the flat part (in the sense of [Eme, Ex. 9.11]) of the ϖ-adic formal stack $\mathcal{C}^{\mathrm{dd}}$, and on this flat part, one can impose the condition that the corresponding Galois representations have all pairs of labelled Hodge–Tate weights equal to $\{0,1\}$; that is, we can consider the substack of $\mathcal{C}^{\mathrm{dd}}$ corresponding to the Zariski closure of the these Galois representations.

We will soon see that $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ is flat (Corollary 4.5.3). By Lemma 4.2.16 below, it follows that the substack of the previous paragraph is equal to $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$; so we may think of the strong determinant condition as being precisely the condition which imposes this condition on the labelled Hodge–Tate weights, and results in a formal stack which is flat over $\mathrm{Spf}\,\mathcal{O}$. Since the inertial types of p-adic Galois representations are unmixed, it is natural from this perspective to expect that $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ should be the disjoint union of the stacks $\mathcal{C}^{\tau,\mathrm{BT}}$ for *unmixed* types, and indeed this will be proved shortly at Corollary 4.2.13.

To compare the strong determinant condition to the condition that the type of a Breuil–Kisin module is unmixed, we make some observations about these conditions in the case of finite field coefficients.

Lemma 4.2.9. *Let* $\mathbf{F}'/\mathbf{F}$ *be a finite extension, and let* $(\mathfrak{L}, \mathfrak{L}^+)$ *be an object of* $\mathcal{M}_{\mathrm{loc}}^{K'/K}(\mathbf{F}')$. *Then* $(\mathfrak{L}, \mathfrak{L}^+)$ *satisfies the strong determinant condition if and only if the following property holds: for each* i *and for each* $\xi : I(K'/K) \to \mathcal{O}^\times$ *we have* $\dim_{\mathbf{F}'}(\mathfrak{L}_i^+)_\xi = e$.

Proof. This is proved in a similar way to [Kis09, Lemma 2.5.1], using the explicit formulation of the strong determinant condition from Remark 4.2.4. In the notation

of that remark, we see that the strong determinant condition holds at ξ if and only if for each embedding $\sigma_i : k' \hookrightarrow \mathbf{F}$ we have

$$(4.2.10) \qquad \det_A \left(\sum_j \pi^j X_{j,\sigma_i} \mid (\mathfrak{L}_i^+)_\xi \right) = \prod_\psi \sum_j \left(\psi(\pi)^j X_{j,\sigma_i} \right),$$

where the product runs over the embeddings $\psi : \mathcal{O}_N \hookrightarrow \mathcal{O}$ with the property that $\psi|_{W(k')}$ lifts σ_i. Since π induces a nilpotent endomorphism of $(\mathfrak{L}_i^+)_\xi$ the left-hand side of (4.2.10) evaluates to $X_{0,\sigma_i}^{\dim_{\mathbf{F}'}(\mathfrak{L}_i^+)_\xi}$ while the right-hand side, which can be viewed as a norm from $\mathcal{O}_N \otimes_{\mathbf{Z}_p} \mathbf{F}'$ down to $W(k') \otimes_{\mathbf{Z}_p} \mathbf{F}'$, is equal to X_{0,σ_i}^e. $\square$

Lemma 4.2.11. *Let $\mathbf{F}'/\mathbf{F}$ be a finite extension, and let $\mathfrak{M}$ be a Breuil–Kisin module of rank 2 and height at most one with $\mathbf{F}'$-coefficients and descent data.*

(1) *$\mathfrak{M}$ satisfies the strong determinant condition if and only if the following property holds: for each i and for each $\xi : I(K'/K) \to \mathcal{O}^\times$ we have $\dim_{\mathbf{F}'}(\operatorname{im} \Phi_{\mathfrak{M},i}/E(u)\mathfrak{M}_i)_\xi = e$.*

(2) *If $\mathfrak{M}$ satisfies the strong determinant condition, then the determinant of $\Phi_{\mathfrak{M},i}$ with respect to some choice of basis has u-adic valuation e'.*

Proof. The first part is immediate from Lemma 4.2.9. For the second part, let $\Phi_{\mathfrak{M},i,\xi}$ be the restriction of $\Phi_{\mathfrak{M},i}$ to $\varphi^*(\mathfrak{M}_{i-1})_\xi$. We think of $\mathfrak{M}_i$ and $\varphi^*(\mathfrak{M}_{i-1})$ as free $\mathbf{F}'[\![v]\!]$-modules of rank $2e(K'/K)$, where $v = u^{e(K'/K)}$. We have

$$\det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i}) = \left(\det_{\mathbf{F}'[\![u]\!]}(\Phi_{\mathfrak{M},i}) \right)^{e(K'/K)}.$$

Since $\Phi_{\mathfrak{M},i}$ commutes with the descent datum, we also have

$$\det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i}) = \prod_\xi \det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i,\xi}),$$

where ξ runs over the $e(K'/K)$ characters $I(K'/K) \to \mathcal{O}^\times$.

The proof of the second part of [Kis09, Lemma 2.5.1] implies that, for each ξ, $\det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i,\xi})$ is $v^e = u^{e'}$ times a unit. Indeed, each $\mathfrak{M}_{i,\xi}$ is a free $\mathbf{F}'[\![v]\!]$-module of rank 2. It admits a basis $\{e_{1,\xi}, e_{2,\xi}\}$ such that $\operatorname{im} \Phi_{\mathfrak{M},i,\xi} = \langle v^i e_{1,\xi}, v^j e_{2,\xi} \rangle$ for some non-negative integers i, j. The strong determinant condition on $(\operatorname{im} \Phi_{\mathfrak{M},1,\xi}/v^e \mathfrak{M}_{i,\xi})$ implies that $i + j = 2e - e = e$, and this is precisely the v-adic valuation of $\det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i,\xi})$. We deduce that the u-adic valuation of $\left(\det_{\mathbf{F}'[\![u]\!]}(\Phi_{\mathfrak{M},i}) \right)^{e(K'/K)}$ is $e(K'/K) \cdot e'$, which implies the second part of the lemma. $\square$

By contrast, we have the following criterion for the type of a Breuil–Kisin module to be unmixed.

Proposition 4.2.12. *Let $\mathbf{F}'/\mathbf{F}$ be a finite extension, and let $\mathfrak{M}$ be a Breuil–Kisin module of rank 2 and height at most one with $\mathbf{F}'$-coefficients and descent data. Then the type of $\mathfrak{M}$ is unmixed if and only if $\dim_{\mathbf{F}'}(\operatorname{im} \Phi_{\mathfrak{M},i}/E(u)\mathfrak{M}_i)_\xi$ is independent of ξ for each fixed i. In particular, if $\mathfrak{M}$ satisfies the strong determinant condition, then the type of $\mathfrak{M}$ is unmixed.*

Proof. We begin the proof of the first part with the following observation. Let Λ be a rank two free $\mathbf{F}'[[u]]$-module with an action of $I(K'/K)$ that is $\mathbf{F}'$-linear and u-semilinear with respect to a character χ (i.e., such that $g(u\lambda) = \chi(g)ug(\lambda)$ for $\lambda \in \Lambda$). In particular $I(K'/K)$ acts on $\Lambda/u\Lambda$ through a pair of characters which we

call η and η'. Let $\Lambda' \subset \Lambda$ be a rank two $I(K'/K)$-sublattice. We claim that there are integers $m, m' \geq 0$ such that the multiset of characters of $I(K'/K)$ occurring in Λ/Λ' has the shape

$$\{\eta\chi^i : 0 \leq i < m\} \cup \{\eta'\chi^i : 0 \leq j < m'\}$$

and the multiset of characters occurring in $\Lambda'/u\Lambda'$ is $\{\eta\chi^m, \eta'\chi^{m'}\}$.

To check the claim we proceed by induction on $\dim_{\mathbf{F}'} \Lambda/\Lambda'$, the case $\Lambda = \Lambda'$ being trivial. Suppose $\dim_{\mathbf{F}'} \Lambda/\Lambda' = 1$, so that Λ' lies between Λ and $u\Lambda$. Consider the chain of containments $\Lambda \supset \Lambda' \supset u\Lambda \supset u\Lambda'$. If without loss of generality $I(K'/K)$ acts on Λ/Λ' via η, then it acts on $\Lambda'/u\Lambda$ by η' and $u\Lambda/u\Lambda'$ by $\chi\eta$, proving the claim with $m = 1$ and $m' = 0$. The general case follows by iterated application of the case $\dim_k \Lambda/\Lambda' = 1$, noting that since $I(K'/K)$ is abelian the quotient Λ/Λ' has a filtration by $I(K/K')$-submodules whose graded pieces have dimension 1.

Now return to the statement of the proposition. Let (τ_i) be the mixed type of $\mathfrak{M}$ and write $\tau_{i-1} = \eta \oplus \eta'$. We apply the preceding observation with $\Lambda = \operatorname{im} \Phi_{\mathfrak{M},i}$ and $\Lambda' = E(u)\mathfrak{M}_i = u^{e'}\mathfrak{M}_i$. Note that χ is a generator of the cyclic group $I(K'/K)$ of order e'/e. Since $\Phi_{\mathfrak{M}}$ commutes with descent data, the group $I(K'/K)$ acts on $\Lambda/u\Lambda$ via η and η'. Then the the multiset

$$\{\eta\chi^i : 0 \leq i < m\} \cup \{\eta'\chi^i : 0 \leq j < m'\}$$

contains each character of $I(K'/K)$ with equal multiplicity if and only if one of η, η' is the successor to $\eta\chi^{m-1}$ in the list $\eta, \eta\chi, \eta\chi^2, \ldots$, and the other is the successor to $\eta'\chi^{m'-1}$ in the list $\eta', \eta'\chi, \eta'\chi^2, \ldots$, i.e., if and only if $\{\eta\chi^m, \eta'\chi^{m'}\} = \{\eta, \eta'\}$. Since $\mathfrak{M}_i/u\mathfrak{M}_i \cong u^{e'}\mathfrak{M}_i/u^{e'+1}\mathfrak{M}_i = \Lambda'/u\Lambda'$, this occurs if and only if that $\tau_i = \tau_{i-1}$.

Finally, the last part of the proposition follows immediately from the first part and Lemma 4.2.11. $\qquad\square$

Corollary 4.2.13. *$\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ is the disjoint union of its closed substacks $\mathcal{C}^{\tau,\mathrm{BT}}$.*

Proof. This follows from Propositions 3.3.5 and 4.2.12. Indeed, from Proposition 3.3.5, it suffices to show that if (τ_i) is a mixed type, and $\mathcal{C}^{(\tau_i),\mathrm{BT}}$ is nonzero, then (τ_i) is in fact an unmixed type. Indeed, note that if $\mathcal{C}^{(\tau_i),\mathrm{BT}}$ is nonzero, then it contains a dense set of finite type points, so in particular contains an $\mathbf{F}'$-point for some finite extension $\mathbf{F}'/\mathbf{F}$. It follows from Proposition 4.2.12 that the type is unmixed, as required. $\qquad\square$

Remark 4.2.14. Since our primary interest is in Breuil–Kisin modules, we will have no further need to consider the stacks $\mathcal{M}_{\mathrm{loc}}^{(\tau_i),\mathrm{BT}}$ or $\mathcal{C}^{(\tau_i),\mathrm{BT}}$ for types that are not unmixed.

Let τ be a tame type; since $I(K'/K)$ is cyclic, we can write $\tau = \eta \oplus \eta'$ for (possibly equal) characters $\eta, \eta' : I(K'/K) \to \mathcal{O}^\times$. Let $(\mathfrak{L}, \mathfrak{L}^+)$ be an object of $\mathcal{M}_{\mathrm{loc}}^\tau(A)$. Suppose that $\xi \neq \eta, \eta'$. Then elements of $\mathfrak{L}_\xi$ are divisible by π' in $\mathfrak{L}$, and so multiplication by π' induces an isomorphism of projective $e_i(\mathcal{O}_N \otimes A)$-modules of equal rank

$$p_{i,\xi} : e_i\mathfrak{L}_{\xi\chi_i^{-1}} \xrightarrow{\sim} e_i\mathfrak{L}_\xi$$

where $\chi_i : I(K'/K) \to \mathcal{O}^\times$ is the character sending $g \mapsto \sigma_i(h(g))$. The induced map

$$p_{i,\xi}^+ : e_i\mathfrak{L}_{\xi\chi_i^{-1}}^+ \longrightarrow e_i\mathfrak{L}_\xi^+$$

is in particular an injection. The following lemma will be useful for checking the strong determinant condition when comparing various different stacks of local model stacks.

Lemma 4.2.15. *Let $(\mathfrak{L}, \mathfrak{L}^+)$ be an object of $\mathcal{M}_{\mathrm{loc}}^\tau(A)$. Then $(\mathfrak{L}, \mathfrak{L}^+)$ is an object of $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}(A)$ if and only if both*

> *(1) the condition (4.2.3) holds for $\xi = \eta, \eta'$, and*
>
> *(2) the injections $p_{i,\xi}^+ : e_i \mathfrak{L}_{\xi\chi_i^{-1}}^+ \xrightarrow{\pi'} e_i \mathfrak{L}_\xi^+$ are isomorphisms for all $\xi \neq \eta, \eta'$ and for all i.*

The second condition is equivalent to

> *(2′) we have $(\mathfrak{L}^+/\pi'\mathfrak{L}^+)_\xi = 0$ for all $\xi \neq \eta, \eta'$.*

Proof. The equivalence between (2) and (2′) is straightforward. Suppose now that $\xi \neq \eta, \eta'$. Locally on $\operatorname{Spec} A$ the module $e_i \mathfrak{L}_{\xi\chi_i^{-1}}^+$ is by definition a direct summand of $e_i \mathfrak{L}_{\xi\chi_i^{-1}}$. Since $p_{i,\xi}$ is an isomorphism, the image of $p_{i,\xi}^+$ is locally on $\operatorname{Spec} A$ a direct summand of $e_i \mathfrak{L}_\xi^+$. Under the assumption that (4.2.10) holds for i and $\xi\chi_i^{-1}$, the condition (4.2.10) for i and ξ is therefore equivalent to the surjectivity of $p_{i,\xi}^+$. The lemma follows upon noting that χ_i is a generator of the group of characters $I(K'/K) \to \mathcal{O}^\times$. $\square$

To conclude this section we describe the $\mathcal{O}_{E'}$-points of $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$, for E'/E a finite extension; recall that our convention is that a two-dimensional Galois representation is Barsotti–Tate if all its labelled pairs of Hodge–Tate weights are equal to $\{0, 1\}$ (and not just that all of the labelled Hodge–Tate weights are equal to 0 or 1).

Lemma 4.2.16. *Let E'/E be a finite extension. Then the $\operatorname{Spf}(\mathcal{O}_{E'})$-points of $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ correspond precisely to the potentially Barsotti–Tate Galois representations $G_K \to \mathrm{GL}_2(\mathcal{O}_{E'})$ which become Barsotti–Tate over K'; and the $\operatorname{Spf}(\mathcal{O}_{E'})$-points of $\mathcal{C}^{\tau,\mathrm{BT}}$ correspond to those representations which are potentially Barsotti–Tate of type τ.*

Proof. In light of Lemma 3.1.10 and the first sentence of Remark 3.3.3, we are reduced to checking that a Breuil–Kisin module of rank 2 and height 1 with $\mathcal{O}_{E'}$-coefficients and descent data corresponds to a potentially Barsotti–Tate representation if and only if it satisfies the strong determinant condition, as well as checking that the descent data on the Breuil–Kisin module matches the type of the corresponding Galois representation.

Let $\mathfrak{M}_{\mathcal{O}_{E'}} \in \mathcal{C}^{\mathrm{dd},\mathrm{BT}}(\operatorname{Spf}(\mathcal{O}_{E'}))$. Plainly $\mathfrak{M}_{\mathcal{O}_{E'}}$ satisfies the strong determinant condition if and only if $\mathfrak{M} := \mathfrak{M}_{\mathcal{O}_{E'}}[1/p]$ satisfies the strong determinant condition (with the latter having the obvious meaning). Consider the filtration

$$\mathrm{Fil}^1(\varphi^*(\mathfrak{M}_i)) := \{m \in \varphi^*(\mathfrak{M}_i) \mid \Phi_{\mathfrak{M},i+1}(m) \in E(u)\mathfrak{M}_{i+1}\} \subset \varphi^*\mathfrak{M}_i$$

inducing

$$\mathrm{Fil}_i^1 \subset \varphi^*(\mathfrak{M}_i)/E(u)\varphi^*(\mathfrak{M}_i).$$

Note that $\varphi^*(\mathfrak{M}_i)/E(u)\varphi^*(\mathfrak{M}_i)$ is isomorphic to a free $K' \otimes_{W(k'),\sigma_i} E'$-module of rank 2. Then $\mathfrak{M}$ corresponds to a Barsotti–Tate Galois representation

$$G_{K'} \to \mathrm{GL}_2(E')$$

if and only, if for every i, Fil_i^1 is isomorphic to $K' \otimes_{W(k'),\sigma_i} E'$ as a $K' \otimes_{W(k'),\sigma_i} E'$-submodule of $\varphi^* \mathfrak{M}_i / E(u) \varphi^* \mathfrak{M}_i$. This follows, for example, by specialising the proof of [Kis08, Cor. 2.6.2] to the Barsotti–Tate case (taking care to note that the conventions of *loc. cit.* for Hodge–Tate weights and for Galois representations associated to Breuil–Kisin modules are both dual to ours).

Let $\xi : I(K'/K) \to \mathcal{O}^\times$ be a character. Consider the filtration

$$\mathrm{Fil}_{i,\xi}^1 \subset \varphi^*(\mathfrak{M}_i)_\xi / E(u)\varphi^*(\mathfrak{M}_i)_\xi \simeq N^2 \otimes_{W(k'),\sigma_i} E'$$

induced by Fil_i^1. The strong determinant condition on $(\mathrm{im}\,\Phi_{\mathfrak{M},i+1}/E(u)\mathfrak{M}_{i+1})_\xi$ holds if and only if $\mathrm{Fil}_{i,\xi}^1$ is isomorphic to $N \otimes_{W(k'),\sigma_i} E'$. By [CL18, Lemma 5.10], we have an isomorphism of $K' \otimes_{W(k'),\sigma_i} E'$-modules

$$\mathrm{Fil}_i^1 \simeq K' \otimes_N \mathrm{Fil}_{i,\xi}^1.$$

This, together with the previous paragraph, allows us to conclude. Note that, since u acts invertibly when working with E'-coefficients and after quotienting by $E(u)$, the argument is independent of the choice of character ξ.

For the statement about types, let $S_{K_0'}$ be Breuil's period ring (see e.g. [Bre00, §5.1]) endowed with the evident action of $\mathrm{Gal}(K'/K)$ compatible with the embedding $\mathfrak{S} \hookrightarrow S_{K_0'}$. Here K_0' is the maximal unramified extension in K'. Recall that by [Liu08, Cor. 3.2.3] there is a canonical (φ, N)-module isomorphism

$$(4.2.17) \qquad S_{K_0'} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \cong S_{K_0'} \otimes_{K_0'} D_{\mathrm{pcris}}(T(\mathfrak{M})).$$

One sees from its construction that the isomorphism (4.2.17) is in fact equivariant for the action of $I(K'/K)$, and the claim follows by reducing modulo u and all its divided powers. $\qquad\square$

4.3. **Change of extensions.** We now discuss the behaviour of various of our constructions under change of K'. Let L'/K' be an extension of degree prime to p such that L'/K is Galois. We suppose that we have fixed a uniformiser π'' of L' such that $(\pi'')^{e(L'/K')} = \pi'$. Let $\mathfrak{S}_A' := (W(l') \otimes_{\mathbf{Z}_p} A)[[u]]$, where l' is the residue field of L', and let $\mathrm{Gal}(L'/K)$ and φ act on $\mathfrak{S}_A'$ via the prescription of Section 2.1 (with π'' in place of π').

There is a natural injective ring homomorphism $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A \to \mathcal{O}_{L'} \otimes_{\mathbf{Z}_p} A$, which is equivariant for the action of $\mathrm{Gal}(L'/K)$ (acting on the source via the natural surjection $\mathrm{Gal}(L'/K) \to \mathrm{Gal}(K'/K)$). There is also an obvious injective ring homomorphism $\mathfrak{S}_A \to \mathfrak{S}_A'$ sending $u \mapsto u^{e(L'/K')}$, which is equivariant for the actions of φ and $\mathrm{Gal}(L'/K)$; we have $(\mathfrak{S}_A')^{\mathrm{Gal}(L'/K')} = \mathfrak{S}_A$. If τ is an inertial type for $I(K'/K)$, we write τ' for the corresponding type for $I(L'/K)$, obtained by inflation.

For any $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^{K'/K}$, we define $(\mathfrak{L}', (\mathfrak{L}')^+) \in \mathcal{M}_{\mathrm{loc}}^{L'/K}$ by

$$(\mathfrak{L}', (\mathfrak{L}')^+) := \mathcal{O}_{L'} \otimes_{\mathcal{O}_{K'}} (\mathfrak{L}, \mathfrak{L}^+),$$

with the diagonal action of $\mathrm{Gal}(L'/K)$. Similarly, for any $\mathfrak{M} \in \mathcal{C}^{\mathrm{dd}}(A)$, we let $\mathfrak{M}' := \mathfrak{S}_A' \otimes_{\mathfrak{S}_A} \mathfrak{M}$, with φ and $\mathrm{Gal}(L'/K)$ again acting diagonally.

Proposition 4.3.1.

  (1) *The assignments $(\mathfrak{L}, \mathfrak{L}^+) \to (\mathfrak{L}', (\mathfrak{L}')^+)$ and $\mathfrak{M} \mapsto \mathfrak{M}'$ induce compatible monomorphisms $\mathcal{M}_{\mathrm{loc}}^{K'/K} \to \mathcal{M}_{\mathrm{loc}}^{L'/K}$ and $\mathcal{C}_{K'}^{\mathrm{dd}} \to \mathcal{C}_{L'}^{\mathrm{dd}}$, i.e., as functors they are fully faithful.*

(2) *The monomorphism $\mathcal{C}_{K'}^{\mathrm{dd}} \to \mathcal{C}_{L'}^{\mathrm{dd}}$ induces an isomorphism $\mathcal{C}^\tau \to \mathcal{C}^{\tau'}$, as well as a monomorphism $\mathcal{C}_{K'}^{\mathrm{dd,BT}} \to \mathcal{C}_{L'}^{\mathrm{dd,BT}}$ and an isomorphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{C}^{\tau',\mathrm{BT}}$.*

Proof. (1) One checks easily that the assignments $(\mathfrak{L}, \mathfrak{L}^+) \to (\mathfrak{L}', (\mathfrak{L}')^+)$ and $\mathfrak{M} \mapsto \mathfrak{M}'$ are compatible. For the rest of the claim, we consider the case of the functor $\mathfrak{M} \mapsto \mathfrak{M}'$; the arguments for the local models case are similar but slightly easier, and we leave them to the reader. Let A be a ϖ-adically complete $\mathcal{O}$-algebra. If $\mathfrak{N}$ is a rank d Breuil–Kisin module with descent data from L' to K, consider the Galois invariants $\mathfrak{N}^{\mathrm{Gal}(L'/K')}$. Since $(\mathfrak{S}'_A)^{\mathrm{Gal}(L'/K')} = \mathfrak{S}_A$, these invariants are naturally an $\mathfrak{S}_A$-module, and moreover they naturally carry a Frobenius and descent data satisfying the conditions required of a Breuil–Kisin module of height at most h. In general the invariants need not be projective of rank d, and so need not be a rank d Breuil–Kisin module with descent data from K' to K. However, in the case $\mathfrak{N} = \mathfrak{M}'$ we have

$$(\mathfrak{M}')^{\mathrm{Gal}(L'/K')} = (\mathfrak{S}'_A \otimes_{\mathfrak{S}_A} \mathfrak{M})^{\mathrm{Gal}(L'/K')} = (\mathfrak{S}'_A)^{\mathrm{Gal}(L'/K')} \otimes_{\mathfrak{S}_A} \mathfrak{M} = \mathfrak{M}.$$

Here the second equality holds e.g. because $\mathrm{Gal}(L'/K')$ has order prime to p, so that taking $\mathrm{Gal}(L'/K')$-invariants is exact, and indeed is given by multiplication by an idempotent $i \in \mathfrak{S}'_A$ (use the decomposition $\mathfrak{S}'_A = i\mathfrak{S}'_A \oplus (1-i)\mathfrak{S}'_A$ and note that i kills the latter summand). It follows immediately that the functor $\mathfrak{M} \to \mathfrak{M}'$ is fully faithful, so $\mathcal{C}_{K'}^{\mathrm{dd},a} \to \mathcal{C}_{L'}^{\mathrm{dd},a}$ is a monomorphism.

(2) Suppose now that $\mathfrak{N}$ has type τ'. In view of what we have proven so far, in order to prove that $\mathcal{C}^\tau \to \mathcal{C}^{\tau'}$ is an isomorphism, it is enough to show that $\mathfrak{N}^{\mathrm{Gal}(L'/K')}$ is a rank d Breuil–Kisin module of type τ, and that the natural map of $\mathfrak{S}'_A$-modules

$$(4.3.2) \qquad \mathfrak{S}'_A \otimes_{\mathfrak{S}_A} \mathfrak{N}^{\mathrm{Gal}(L'/K')} \to \mathfrak{N}$$

is an isomorphism. For the remainder of this proof, for clarity we write $u_{K'}$, $u_{L'}$ instead of u for the variables of $\mathfrak{S}_A$ and $\mathfrak{S}'_A$ respectively. Since the type τ' of $\mathfrak{N}$ is inflated from τ, the action of $\mathrm{Gal}(L'/K')$ on $\mathfrak{N}/u_{L'}\mathfrak{N}$ factors through $\mathrm{Gal}(l'/k')$; noting that $W(l')$ has a normal basis for $\mathrm{Gal}(l'/k')$ over $W(k')$, we obtain an isomorphism

$$(4.3.3) \qquad W(l') \otimes_{W(k')} (\mathfrak{N}/u_{L'}\mathfrak{N})^{\mathrm{Gal}(L'/K')} \xrightarrow{\sim} \mathfrak{N}/u_{L'}\mathfrak{N}.$$

In particular the $W(k') \otimes_{\mathbf{Z}_p} A$-module $(\mathfrak{N}/u_{L'}\mathfrak{N})^{\mathrm{Gal}(L'/K')}$ is projective of rank d.

Observe however that $(\mathfrak{N}/u_{K'}\mathfrak{N})^{\mathrm{Gal}(L'/K')} = (\mathfrak{N}/u_{L'}\mathfrak{N})^{\mathrm{Gal}(L'/K')}$. To see this, by the exactness of taking $\mathrm{Gal}(L'/K')$ invariants it suffices to check that $u_{L'}^i \mathfrak{N}/u_{L'}^{i+1}$ has trivial $\mathrm{Gal}(L'/K')$-invariants for $0 < i < e(L'/K')$. Multiplication by $u_{L'}^i$ gives an isomorphism $\mathfrak{N}/u_{L'}\mathfrak{N} \cong u_{L'}^i\mathfrak{N}/u_{L'}^{i+1}$, so that for i in the above range, the inertia group $I(L'/K')$ acts linearly on $u_{L'}^i\mathfrak{N}/u_{L'}^{i+1}$ through a twist of τ' by a nontrivial character; so there are no $I(L'/K')$-invariants, and thus no $\mathrm{Gal}(L'/K')$-invariants either.

It follows that the isomorphism (4.3.3) is the map (4.3.2) modulo $u_{L'}$. By Nakayama's lemma it follows that (4.3.2) is surjective. Since $\mathfrak{N}$ is projective, the surjection (4.3.2) is split, and is therefore an isomorphism, since it is an isomorphism modulo $u_{L'}$. This isomorphism exhibits $\mathfrak{N}^{\mathrm{Gal}(L'/K')}$ as a direct summand (as an $\mathfrak{S}_A$-module) of the projective module $\mathfrak{N}$, so it is also projective; and it is projective of rank d, since this holds modulo $u_{K'}$.

Finally, we need to check the compatibility of these maps with the strong determinant condition. By Corollary 4.2.13, it is enough to prove this for the case of the morphism $\mathcal{C}^\tau \to \mathcal{C}^{\tau'}$ for some τ; by the compatibility in part (1), this comes down to the same for the corresponding map of local model stacks $\mathcal{M}_{\mathrm{loc}}^\tau \to \mathcal{M}_{\mathrm{loc}}^{\tau'}$. If $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^\tau$, it therefore suffices to show to show that conditions (1) and (2′) of Lemma 4.2.15 for $(\mathfrak{L}, \mathfrak{L}^+)$ and $(\mathfrak{L}', (\mathfrak{L}')^+) := \mathcal{O}_{L'} \otimes_{\mathcal{O}_{K'}} (\mathfrak{L}, \mathfrak{L}^+)$ are equivalent. This is immediate for condition (2′), since we have $(\mathfrak{L}')^+/\pi''(\mathfrak{L}')^+ \cong l' \otimes_{k'} (\mathfrak{L}^+/\pi'\mathfrak{L}^+)$ as $I(L'/K)$-representations.

Writing $\tau = \eta \oplus \eta'$, it remains to relate the strong determinant conditions on the η, η'-parts over both K' and L'. Unwinding the definitions using Remark 4.2.4, one finds that the condition over L' is a product of $[l' : k']$ copies (with different sets of variables) of the condition over K'. Thus the strong determinant condition over K' implies the condition over L', while the condition over L' implies the condition over K' up to an $[l' : k']$th root of unity. Comparing the terms involving only copies of X_{0,σ_i}'s shows that this root of unity must be 1. $\qquad\square$

Remark 4.3.4. The morphism of local model stacks $\mathcal{M}_{\mathrm{loc}}^\tau \to \mathcal{M}_{\mathrm{loc}}^{\tau'}$ is not an isomorphism (provided that the extension L'/K' is nontrivial). The issue is that, as we observed in the preceding proof, local models $(\mathfrak{L}', (\mathfrak{L}')^+)$ in the image of the morphism $\mathcal{M}_{\mathrm{loc}}^\tau \to \mathcal{M}_{\mathrm{loc}}^{\tau'}$ can have $((\mathfrak{L}')^+/\pi''(\mathfrak{L}')^+)_\xi \neq 0$ only for characters $\xi : I(L'/K) \to \mathcal{O}^\times$ that are inflated from $I(K'/K)$. However, one does obtain an isomorphism from the substack of $\mathcal{M}_{\mathrm{loc}}^\tau$ of pairs $(\mathfrak{L}, \mathfrak{L}^+)$ satisfying condition (2) of Lemma 4.2.15 to the analogous substack of $\mathcal{M}_{\mathrm{loc}}^{\tau'}$; therefore the induced map $\mathcal{M}^{\tau,\mathrm{BT}} \to \mathcal{M}^{\tau',\mathrm{BT}}$ *will* also be an isomorphism. Analogous remarks will apply to the maps of local model stacks in §4.4.

4.4. **Explicit local models.** We now explain the connection between the moduli stacks $\mathcal{C}^\tau$ and local models for Shimura varieties at hyperspecial and Iwahori level. This idea has been developed in [CL18] for Breuil–Kisin modules of arbitrary rank with tame principal series descent data, inspired by [Kis09], which analyses the case without descent data.

The results of [CL18] relate the moduli stacks $\mathcal{C}^\tau$ (in the case that τ is a principal series type) via a local model diagram to a mixed-characteristic deformation of the affine flag variety, introduced in this generality by Pappas and Zhu [PZ13]. The local models in [PZ13, §6] are defined in terms of Schubert cells in the generic fibre of this mixed-characteristic deformation, by taking the Zariski closure of these Schubert cells. The disadvantage of this approach is that it does not give a direct moduli-theoretic interpretation of the special fibre of the local model. Therefore, it is hard to check directly that our stack $\mathcal{C}^{\tau,\mathrm{BT}}$, which has a moduli-theoretic definition, corresponds to the desired local model under the diagram introduced in [CL18, Cor. 4.11] [2].

In our rank 2 setting, the local models admit a much more explicit condition, using lattice chains and Kottwitz's determinant condition, and in the cases of nonscalar types, we will relate our local models to the naive local model at Iwahori level for the Weil restriction of GL_2, in the sense of [PRS13, §2.4].

We begin with the simpler case of scalar inertial types. Suppose that K'/K is totally ramified, and that τ is a scalar inertial type, say $\tau = \eta \oplus \eta$. In this

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2. One should be able to check this by adapting the ideas in [HN02, §2.1] and [PZ13, Prop. 6.2] to $\mathrm{Res}_{K/\mathbf{Q}_p} \mathrm{GL}_n$ where $K/\mathbf{Q}_p$ can be ramified.

case we define the local model stack $\mathcal{M}_{\mathrm{loc,hyp}}$ ("hyp" for "hyperspecial") to be the *fppf* stack over $\mathrm{Spf}\,\mathcal{O}$ (in fact, a p-adic formal algebraic stack), which to each p-adically complete $\mathcal{O}$-algebra A associates the groupoid $\mathcal{M}_{\mathrm{loc,hyp}}(A)$ consisting of pairs $(\mathfrak{L}, \mathfrak{L}^+)$, where

— $\mathfrak{L}$ is a rank 2 projective $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-module, and

— $\mathfrak{L}^+$ is an $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-submodule of $\mathfrak{L}$, which is locally on $\mathrm{Spec}\,A$ a direct summand of $\mathfrak{L}$ as an A-module (or equivalently, for which the quotient $\mathfrak{L}/\mathfrak{L}^+$ is projective as an A-module).

We let $\mathcal{M}_{\mathrm{loc,hyp}}^{\mathrm{BT}}$ be the substack of pairs $(\mathfrak{L}, \mathfrak{L}^+)$ with the property that for all $a \in \mathcal{O}_K$, we have

$$(4.4.1) \qquad \det_A(a|\mathfrak{L}^+) = \prod_{\psi:K\hookrightarrow E} \psi(a)$$

as polynomial functions on $\mathcal{O}_K$.

Lemma 4.4.2. *The functor $(\mathfrak{L}', (\mathfrak{L}')^+) \mapsto ((\mathfrak{L}')_\eta, (\mathfrak{L}')_\eta^+)$ defines a morphism $\mathcal{M}_{\mathrm{loc}}^\tau \to \mathcal{M}_{\mathrm{loc,hyp}}$ which induces an isomorphism $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}} \to \mathcal{M}_{\mathrm{loc,hyp}}^{\mathrm{BT}}$. (We remind the reader that K'/K is assumed totally ramified, and that τ is assumed to be a scalar inertial type associated to the character η.)*

Proof. If $(\mathfrak{L}', (\mathfrak{L}')^+)$ is an object of $\mathcal{M}_{\mathrm{loc}}^\tau$, the proof that $((\mathfrak{L}')_\eta, (\mathfrak{L}')_\eta^+)$ is indeed an object of $\mathcal{M}_{\mathrm{loc,hyp}}(A)$ is very similar to the proof of Proposition 4.3.1, and is left to the reader. Similarly, the reader may verify that the functor

$$(\mathfrak{L}, \mathfrak{L}^+) \mapsto (\mathfrak{L}', (\mathfrak{L}')^+) := \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} (\mathfrak{L}, \mathfrak{L}^+),$$

where the action of $\mathrm{Gal}(K'/K)$ is given by the tensor product of the natural action on $\mathcal{O}_{K'}$ with the action on $(\mathfrak{L}, \mathfrak{L}^+)$ given by the character η, defines a morphism $\mathcal{M}_{\mathrm{loc,hyp}} \to \mathcal{M}_{\mathrm{loc}}^\tau$.

The composition $\mathcal{M}_{\mathrm{loc,hyp}} \to \mathcal{M}_{\mathrm{loc}}^\tau \to \mathcal{M}_{\mathrm{loc,hyp}}$ is evidently the identity. The composition in the other order is not, in general, naturally equivalent to the identity morphism, because for $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^\tau(A)$ one cannot necessarily recover $\mathfrak{L}^+$ from the projection to its η-isotypic part. However, this *will* hold if $\mathfrak{L}^+$ satisfies condition (2) of Lemma 4.2.15 (and so in particular will hold after imposing the strong determinant condition).

Indeed, suppose $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^\tau(A)$. Then there is a natural $\mathrm{Gal}(K'/K)$-equivariant map of projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules

$$(4.4.3) \qquad \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathfrak{L}_\eta \to \mathfrak{L}$$

of the same rank (in which $\mathrm{Gal}(K'/K)$ acts by η on $\mathfrak{L}_\eta$). This map is surjective because it is surjective on η-parts and the maps $p_{i,\xi}$ are surjective for all $\xi \neq \eta$; therefore it is an isomorphism. One further has a a natural $\mathrm{Gal}(K'/K)$-equivariant map of $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules

$$(4.4.4) \qquad \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathfrak{L}_\eta^+ \to \mathfrak{L}^+$$

that is injective because locally on $\mathrm{Spec}(A)$ it is a direct summand of the isomorphism (4.4.3). If one further assumes that $\mathfrak{L}^+$ satisfies condition (2) of Lemma 4.2.15 then (4.4.4) is an isomorphism, as claimed.

It remains to check the compatibility of these maps with the strong determinant condition. If $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc,hyp}}$ then certainly condition (2) of Lemma 4.2.15

holds for $(\mathfrak{L}', (\mathfrak{L}')^+) := \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} (\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^\tau$. By Lemma 4.2.15 the strong determinant condition holds for $(\mathfrak{L}', (\mathfrak{L}')^+)$ if and only if (4.2.3) holds for $\mathfrak{L}'$ with $\xi = \eta$; but this is exactly the condition (4.4.1) for $\mathfrak{L}$, as required. $\qquad\square$

Next, we consider the case of principal series types. We suppose that K'/K is totally ramified. We begin by defining a p-adic formal algebraic stack $\mathcal{M}_{\mathrm{loc,Iw}}$ over $\mathrm{Spf}\,\mathcal{O}$ ("Iw" for Iwahori). For each complete $\mathcal{O}$-algebra A, we let $\mathcal{M}_{\mathrm{loc,Iw}}(A)$ be the groupoid of tuples $(\mathfrak{L}_1, \mathfrak{L}_1^+, \mathfrak{L}_2, \mathfrak{L}_2^+, f_1, f_2)$, where

— $\mathfrak{L}_1$, $\mathfrak{L}_2$ are rank 2 projective $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-modules,

— $f_1 : \mathfrak{L}_1 \to \mathfrak{L}_2$, $f_2 : \mathfrak{L}_2 \to \mathfrak{L}_1$ are morphisms of $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-modules, satisfying $f_1 \circ f_2 = f_2 \circ f_1 = \pi$,

— both $\mathrm{coker}\, f_1$ and $\mathrm{coker}\, f_2$ are rank one projective $k \otimes_{\mathbf{Z}_p} A$-modules,

— $\mathfrak{L}_1^+$, $\mathfrak{L}_2^+$ are $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-submodules of $\mathfrak{L}_1$, $\mathfrak{L}_2$, which are locally on $\mathrm{Spec}\,A$ direct summands as A-modules (or equivalently, for which the quotients $\mathfrak{L}_i/\mathfrak{L}_i^+$ ($i = 1, 2$) are projective A-modules), and moreover for which the morphisms f_1, f_2 restrict to morphisms $f_1 : \mathfrak{L}_1^+ \to \mathfrak{L}_2^+$, $f_2 : \mathfrak{L}_2^+ \to \mathfrak{L}_1^+$.

We let $\mathcal{M}_{\mathrm{loc,Iw}}^{\mathrm{BT}}$ be the substack of tuples with the property that for all $a \in \mathcal{O}_K$ and $i = 1, 2$, we have

$$(4.4.5) \qquad \det_A(a|\mathfrak{L}_i^+) = \prod_{\psi: K \hookrightarrow E} \psi(a)$$

as polynomial functions on $\mathcal{O}_K$.

Write $\tau = \eta \oplus \eta'$ with $\eta \neq \eta'$. Recall that the character $h : \mathrm{Gal}(K'/K) = I(K'/K) \to W(k)^\times$ is given by $h(g) = g(\pi')/\pi'$. Since we are assuming that $\eta \neq \eta'$, for each embedding $\sigma : k \hookrightarrow \mathbf{F}$ (which we also think of as $\sigma : W(k) \hookrightarrow \mathcal{O}$) there are integers $0 < a_\sigma, b_\sigma < e(K'/K)$ with the properties that $\eta'/\eta = \sigma \circ h^{a_\sigma}$, $\eta/\eta' = \sigma \circ h^{b_\sigma}$; in particular, $a_\sigma + b_\sigma = e(K'/K)$. Recalling that $e_\sigma \in W(k) \otimes_{\mathbf{Z}_p} \mathcal{O} \subset \mathcal{O}_K$ is the idempotent corresponding to σ, we set $\pi_1 = \sum_\sigma (\pi')^{a_\sigma} e_\sigma$, $\pi_2 = \sum_\sigma (\pi')^{b_\sigma} e_\sigma$; so we have $\pi_1 \pi_2 = \pi$, and $\pi_1 \in (\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} \mathcal{O})^{I(K'/K) = \eta'/\eta}$, $\pi_2 \in (\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} \mathcal{O})^{I(K'/K) = \eta/\eta'}$.

We define a morphism $\mathcal{M}_{\mathrm{loc}}^\tau \to \mathcal{M}_{\mathrm{loc,Iw}}$ as follows. Given a pair $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^\tau(A)$, we set $(\mathfrak{L}_1, \mathfrak{L}_1^+) = (\mathfrak{L}_\eta, \mathfrak{L}_\eta^+)$, $(\mathfrak{L}_2, \mathfrak{L}_2^+) = (\mathfrak{L}_{\eta'}, \mathfrak{L}_{\eta'}^+)$, and we let f_1, f_2 be given by multiplication by π_1, π_2 respectively. The only point that is perhaps not immediately obvious is to check that $\mathrm{coker}\, f_1$ and $\mathrm{coker}\, f_2$ are rank one projective $k \otimes_{\mathbf{Z}_p} A$-modules. To see this, note that by [Sta13, Tag 05BU], we can lift $(\mathfrak{L}/\pi'\mathfrak{L})_{\eta'}$ to a rank one summand U of the projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module $\mathfrak{L}$. Let U_η be the direct summand of $\mathfrak{L}_\eta$ obtained by projection of U to the η-eigenspace $\mathfrak{L}_\eta$; then the projective $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-module U_η has rank one, as can be checked modulo π. Similarly we may lift $(\mathfrak{L}/\pi'\mathfrak{L})_{\eta'}$ to a rank one summand V of $\mathfrak{L}$, and we let $V_{\eta'}$ be the projection to the η'-part.

The natural map

$$(4.4.6) \qquad \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} (U_\eta \oplus V_{\eta'}) \to \mathfrak{L}$$

is an isomorphism, since both sides are projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules of rank two, and the given map is an isomorphism modulo π'. It follows immediately from (4.4.6) that $\mathfrak{L}_\eta = U_\eta \oplus \pi_2 V_{\eta'}$ and $\mathfrak{L}_{\eta'} = \pi_1 U_\eta \oplus V_{\eta'}$, so that $\mathrm{coker}\, f_1$ and $\mathrm{coker}\, f_2$ are projective of rank one, as claimed.

Proposition 4.4.7. *The morphism* $\mathcal{M}_{\mathrm{loc}}^{\tau} \to \mathcal{M}_{\mathrm{loc,Iw}}$ *induces an isomorphism* $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}} \to \mathcal{M}_{\mathrm{loc,Iw}}^{\mathrm{BT}}$. (*We remind the reader that* K'/K *is assumed totally ramified, and that* τ *is assumed to be a principal series inertial type.*)

Proof. We begin by constructing a morphism $\mathcal{M}_{\mathrm{loc,Iw}} \to \mathcal{M}_{\mathrm{loc}}^{\tau}$, inspired by the arguments of [RZ96, App. A]. We define an $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-module $\mathfrak{L}$ by

$$\mathfrak{L} = \oplus_\sigma \left(\oplus_{i=0}^{a_\sigma - 1} e_\sigma \mathfrak{L}_1^{(i)} \oplus_{j=0}^{b_\sigma - 1} e_\sigma \mathfrak{L}_2^{(j)} \right),$$

where the $\mathfrak{L}_1^{(i)}$'s and $\mathfrak{L}_2^{(j)}$'s are copies of $\mathfrak{L}_1$, $\mathfrak{L}_2$ respectively.

We can upgrade the $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-module structure on $\mathfrak{L}$ to that of an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module by specifying how π' acts. If $i < a_\sigma - 1$, then we let $\pi' : e_\sigma \mathfrak{L}_1^{(i)} \to e_\sigma \mathfrak{L}_1^{(i+1)}$ be the map induced by the identity on $\mathfrak{L}_1$, and if $j < b_\sigma - 1$, then we let $\pi' : e_\sigma \mathfrak{L}_2^{(j)} \to e_\sigma \mathfrak{L}_2^{(j+1)}$ be the map induced by the identity on $\mathfrak{L}_2$. We let $\pi' : e_\sigma \mathfrak{L}_1^{(a_\sigma - 1)} \to e_\sigma \mathfrak{L}_2^{(0)}$ be the map induced by $f_1 : \mathfrak{L}_1 \to \mathfrak{L}_2$, and we let $\pi' : e_\sigma \mathfrak{L}_2^{(b_\sigma - 1)} \to e_\sigma \mathfrak{L}_1^{(0)}$ be the map induced by $f_2 : \mathfrak{L}_2 \to \mathfrak{L}_1$. That this indeed gives $\mathfrak{L}$ the structure of an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module follows from our assumption that $f_1 \circ f_2 = f_2 \circ f_1 = \pi$. We give $\mathfrak{L}$ a semilinear action of $\mathrm{Gal}(K'/K) = I(K'/K)$ by letting it act via $(\sigma \circ h)^i \cdot \eta$ on each $e_\sigma \mathfrak{L}_1^{(i)}$ and via $(\sigma \circ h)^j \cdot \eta'$ on each $e_\sigma \mathfrak{L}_2^{(j)}$.

We claim that $\mathfrak{L}$ is a rank 2 projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module. Since $\mathrm{coker}\, f_2$ is projective by assumption, we can choose a section to the $k \otimes_{\mathbf{Z}_p} A$-linear morphism $\mathfrak{L}_1/\pi \to \mathrm{coker}\, f_2$, with image $\overline{U}_\eta$, say. Similarly we choose a section to $\mathfrak{L}_2/\pi \to \mathrm{coker}\, f_1$ with image $\overline{V}_{\eta'}$. We choose lifts U_η, $V_{\eta'}$ of $\overline{U}_\eta$, $\overline{V}_{\eta'}$ to direct summands of the $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-modules $\mathfrak{L}_1$, $\mathfrak{L}_2$ respectively. There is a map of $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules

$$(4.4.8) \qquad\qquad \lambda : \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} (U_\eta \oplus V_{\eta'}) \to \mathfrak{L}$$

induced by the map identifying $U_\eta, V_{\eta'}$ with their copies in $\mathfrak{L}_1^{(0)}$ and $\mathfrak{L}_2^{(0)}$ respectively. The map λ is surjective modulo π' by construction, hence surjective by Nakayama's lemma. Regarding λ as a map of projective $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-modules of equal rank $2e(K'/K)$, we deduce that λ is an isomorphism. Since the source of λ is a projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module, the claim follows.

We now set

$$\mathfrak{L}^+ = \oplus_\sigma \left(\oplus_{i=0}^{a_\sigma - 1} e_\sigma (\mathfrak{L}_1^{(i)})^+ \oplus_{j=0}^{b_\sigma - 1} e_\sigma (\mathfrak{L}_2^{(j)})^+ \right) \subset \mathfrak{L}$$

It is immediate from the construction that $\mathfrak{L}^+$ is preserved by $\mathrm{Gal}(K'/K)$. The hypothesis that f_1, f_2 and preserve $\mathfrak{L}_1^+, \mathfrak{L}_2^+$ implies that $\mathfrak{L}^+$ is an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-submodule of $\mathfrak{L}$, while the hypothesis that each $\mathfrak{L}_i/\mathfrak{L}_i^+$ is a projective A-module implies the same for $\mathfrak{L}/\mathfrak{L}^+$. This completes the construction of our morphism $\mathcal{M}_{\mathrm{loc,Iw}} \to \mathcal{M}_{\mathrm{loc}}^{\tau}$.

Just as in the proof of Proposition 4.4.2, the morphism $\mathcal{M}_{\mathrm{loc,Iw}} \to \mathcal{M}_{\mathrm{loc}}^{\tau}$ followed by our morphism $\mathcal{M}_{\mathrm{loc}}^{\tau} \to \mathcal{M}_{\mathrm{loc,Iw}}$ is the identity, while the composition in the other order is not, in general, naturally equivalent to the identity morphism. However, it follows immediately from Lemma 4.2.15 and the construction of $\mathfrak{L}^+$ that our morphisms $\mathcal{M}_{\mathrm{loc,Iw}} \to \mathcal{M}_{\mathrm{loc}}^{\tau}$ and $\mathcal{M}_{\mathrm{loc}}^{\tau} \to \mathcal{M}_{\mathrm{loc,Iw}}$ respect the strong determinant condition, and so induce maps $\mathcal{M}_{\mathrm{loc,Iw}}^{\mathrm{BT}} \to \mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}$ and $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}} \to \mathcal{M}_{\mathrm{loc,Iw}}^{\mathrm{BT}}$. To see that the composite $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}} \to \mathcal{M}_{\mathrm{loc,Iw}}^{\mathrm{BT}} \to \mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}$ is naturally equivalent to the identity, suppose that $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^{\tau}(A)$ and observe that there is a natural

$\mathrm{Gal}(K'/K)$-equivariant isomorphism of $\mathcal{O}_K \otimes_A \mathbf{Z}_p$-modules

$$(4.4.9) \qquad \oplus_\sigma \left(\oplus_{i=0}^{a_\sigma - 1} e_\sigma \mathfrak{L}_\eta^{(i)} \oplus_{j=0}^{b_\sigma - 1} e_\sigma \mathfrak{L}_{\eta'}^{(j)} \right) \xrightarrow{\sim} \mathfrak{L}$$

induced by the maps $\mathfrak{L}_\eta^{(i)} \xrightarrow{(\pi')^i} (\pi')^i \mathfrak{L}_\eta$ and $\mathfrak{L}_{\eta'}^{(j)} \xrightarrow{(\pi')^j} (\pi')^j \mathfrak{L}_{\eta'}$. The commutativity of the diagram

$$
\begin{array}{ccc}
e_\sigma \mathfrak{L}_\eta^{(a_\sigma - 1)} & \xrightarrow{(\pi')^{a_\sigma - 1}} & e_\sigma (\pi')^{a_\sigma - 1} \mathfrak{L}_\eta \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi'} \\
e_\sigma \mathfrak{L}_{\eta'}^{(0)} & \xrightarrow[\mathrm{id}]{} & e_\sigma \mathfrak{L}_{\eta'}
\end{array}
$$

implies that the map in (4.4.9) is in fact an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module isomorphism. The map (4.4.9) induces an inclusion

$$\oplus_\sigma \left(\oplus_{i=0}^{a_\sigma - 1} e_\sigma (\mathfrak{L}_\eta^{(i)})^+ \oplus_{j=0}^{b_\sigma - 1} e_\sigma (\mathfrak{L}_{\eta'}^{(j)})^+ \right) \to \mathfrak{L}^+.$$

If furthermore $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^{\tau, \mathrm{BT}}$ then this is an isomorphism because $\mathfrak{L}^+$ satisfies condition (2) of Lemma 4.2.15. $\qquad \square$

Finally, we turn to the case of a cuspidal type. Let L as usual be a quadratic unramified extension of K, and set $K' = L(\pi^{1/(p^{2f}-1)})$. The field N continues to denote the maximal unramified extension of K in K', so that $N = L$. Let τ be a cuspidal type, so that $\tau = \eta \oplus \eta'$, where $\eta \neq \eta'$ but $\eta' = \eta^{p^f}$.

Proposition 4.4.10. *There is a morphism $\mathcal{M}_{\mathrm{loc}}^{\tau, \mathrm{BT}} \to \mathcal{M}_{\mathrm{loc}, \mathrm{Iw}}^{\mathrm{BT}}$ which is representable by algebraic spaces and smooth. (We remind the reader that τ is now assumed to be a cuspidal inertial type.)*

Proof. Let τ' be the type τ, considered as a (principal series) type for the totally ramified extension K'/N. Let $c \in \mathrm{Gal}(K'/K)$ be the unique element which fixes $\pi^{1/(p^{2f}-1)}$ but acts nontrivially on N. For any map $\alpha : X \to Y$ of $\mathcal{O}_N$-modules we write α^c for the twist $1 \otimes \alpha : \mathcal{O}_N \otimes_{\mathcal{O}_N, c} X \to \mathcal{O}_N \otimes_{\mathcal{O}_N, c} Y$.

We may think of an object $(\mathfrak{L}, \mathfrak{L}^+)$ of $\mathcal{M}_{\mathrm{loc}}^{\tau, \mathrm{BT}}$ as an object $(\mathfrak{L}', (\mathfrak{L}')^+)$ of $\mathcal{M}_{\mathrm{loc}}^{\tau', \mathrm{BT}}$ equipped with the additional data of an isomorphism of $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules $\theta : \mathcal{O}_{K'} \otimes_{\mathcal{O}_{K'}, c} \mathfrak{L}' \to \mathfrak{L}'$ which is compatible with $(\mathfrak{L}')^+$, which satisfies $\theta \circ \theta^c = \mathrm{id}$, and which is compatible with the action of $\mathrm{Gal}(K'/N) = I(K'/N)$ in the sense that $\theta \circ (1 \otimes g) = g^{p^f} \circ \theta$.

Employing the isomorphism of Proposition 4.4.7, we think of $(\mathfrak{L}', (\mathfrak{L}')^+)$ as a tuple $(\mathfrak{L}'_1, (\mathfrak{L}'_1)^+, \mathfrak{L}'_2, (\mathfrak{L}'_2)^+, f_1, f_2)$, where the $\mathfrak{L}'_i, (\mathfrak{L}'_i)^+$ are $\mathcal{O}_N \otimes_{\mathbf{Z}_p} A$-modules; by construction, the map θ induces isomorphisms $\theta_1 : \mathcal{O}_N \otimes_{\mathcal{O}_N, c} \mathfrak{L}'_1 \xrightarrow{\sim} \mathfrak{L}'_2$, $\theta_2 : \mathcal{O}_N \otimes_{\mathcal{O}_N, c} \mathfrak{L}'_2 \xrightarrow{\sim} \mathfrak{L}'_1$, which are compatible with $(\mathfrak{L}'_1)^+, (\mathfrak{L}'_2)^+$ and f_1, f_2, and satisfy $\theta_1 \circ \theta_2^c = \mathrm{id}$.

Choose for each embedding $\sigma : k \hookrightarrow \mathbf{F}$ an extension to an embedding $\sigma^{(1)} : k' \hookrightarrow \mathbf{F}$, set $e_1 = \sum_\sigma e_{\sigma^{(1)}}$, and write $e_2 = 1 - e_1$. Then the map θ_1 induces isomorphisms $\theta_{11} : e_1 \mathfrak{L}'_1 \xrightarrow{\sim} e_2 \mathfrak{L}'_2$ and $\theta_{12} : e_2 \mathfrak{L}'_1 \xrightarrow{\sim} e_1 \mathfrak{L}'_2$, while θ_2 induces isomorphisms $\theta_{21} : e_1 \mathfrak{L}'_2 \xrightarrow{\sim} e_2 \mathfrak{L}'_1$ and $\theta_{22} : e_2 \mathfrak{L}'_2 \xrightarrow{\sim} e_1 \mathfrak{L}'_1$. The condition that $\theta_1 \circ \theta_2^c = \mathrm{id}$ translates to $\theta_{22} = \theta_{11}^{-1}$ and $\theta_{21} = \theta_{12}^{-1}$, and compatibility with $(\mathfrak{L}'_1)^+, (\mathfrak{L}'_2)^+$ implies that θ_{11}, θ_{21} induce isomorphisms $e_1 (\mathfrak{L}'_1)^+ \xrightarrow{\sim} e_2 (\mathfrak{L}'_2)^+$ and $e_1 (\mathfrak{L}'_2)^+ \xrightarrow{\sim} e_2 (\mathfrak{L}'_1)^+$ respectively.

Furthermore f_1, f_2 induce maps $e_1 f_1 : e_1 \mathfrak{L}'_1 \to e_1 \mathfrak{L}'_2$, $e_1 g : e_1 \mathfrak{L}'_2 \to e_1 \mathfrak{L}'_1$. It follows that there is a map $\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}} \to \mathcal{M}_{\mathrm{loc,Iw}}$, sending $(\mathfrak{L}, \mathfrak{L}^+)$ to the tuple $(e_1 \mathfrak{L}'_1, e_1(\mathfrak{L}'_1)^+, e_1 \mathfrak{L}'_2, e_1(\mathfrak{L}'_2)^+, e_1 f_1, e_1 f_2)$. To see that it respects the strong determinant condition, one has to check that the conditions on $\mathfrak{L}^+_1$, $\mathfrak{L}^+_2$ imply those for $e_1(\mathfrak{L}'_1)^+$, $e_2(\mathfrak{L}'_2)^+$ coincide, and this follows from the definitions (via Remark 4.2.4). We therefore obtain a map $\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}} \to \mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,Iw}}$,

Since this morphism is given by forgetting the data of $e_2 \mathfrak{L}'_1, e_2 \mathfrak{L}'_2$ and the pair of isomorphisms θ_{11}, θ_{21}, it is evidently formally smooth. It is also a morphism between ϖ-adic formal algebraic stacks that are locally of finite presentation, and so is representable by algebraic spaces (by [Eme, Lem. 7.10]) and locally of finite presentation (by [EG21, Cor. 2.1.8] and [Sta13, Tag 06CX]). Thus this morphism is in fact smooth. $\qquad\square$

4.5. **Local models: local geometry.** We now deduce our main results on the local structure of our moduli stacks from results in the literature on local models for Shimura varieties.

Proposition 4.5.1. *We can identify $\mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,Iw}}$ with the quotient of (the p-adic formal completion of) the naive local model for $\mathrm{Res}_{K/\mathbf{Q}_p} \mathrm{GL}_2$ (as defined in [PRS13, §2.4]) by a smooth group scheme over $\mathcal{O}$.*

Proof. Let $\widetilde{\mathcal{M}}^{\mathrm{BT}}_{\mathrm{loc,Iw}}$ be the p-adic formal completion of the naive local model for $\mathrm{Res}_{K/\mathbf{Q}_p} \mathrm{GL}_2$ corresponding to a standard lattice chain $\mathcal{L}$, as defined in [PRS13, §2.4]. By [RZ96, Prop. A.4], the automorphisms of the standard lattice chain $\mathcal{L}$ are represented by a smooth group scheme $\mathcal{P}_{\mathcal{L}}$ over $\mathcal{O}$. (This is in fact a parahoric subgroup scheme of $\mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}_p} \mathrm{GL}_2$, and in particular it is affine.) Also by *loc. cit.*, every lattice chain of type $(\mathcal{L})$ is Zariski locally isomorphic to $\mathcal{L}$. By comparing the two moduli problems, we see that $\widetilde{\mathcal{M}}^{\mathrm{BT}}_{\mathrm{loc,Iw}}$ is a $\mathcal{P}_{\mathcal{L}}$-torsor over $\mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,Iw}}$ for the Zariski topology and the proposition follows. $\qquad\square$

The following theorem describes the various regularity properties of local models. Since we are working in the context of formal algebraic stacks, we use the terminology developed in [Eme, §8] (see in particular [Eme, Rem. 8.21] and [Eme, Def. 8.35]).

Theorem 4.5.2. *Suppose that $d = 2$ and that τ is a tame inertial type. Then*

(1) *$\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}}$ is residually Jacobson and analytically normal, and Cohen–Macaulay.*

(2) *The special fibre $\mathcal{M}^{\tau,\mathrm{BT},1}_{\mathrm{loc}}$ is reduced.*

(3) *$\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}}$ is flat over $\mathcal{O}$.*

Proof. For scalar types, this follows from [Kis09, Prop. 2.2.2] by Lemma 4.4.2, and so we turn to studying the case of a non-scalar type. The properties in question can be checked smooth locally (see [Eme, §8] for (1), and [Sta13, Tag 04YH] for (2); for (3), note that morphisms that are representable by algebraic spaces and smooth are also flat, and take into account the final statement of [Eme, Lem. 8.34]), and so by Propositions 4.4.7 and 4.4.10 we reduce to checking the assertions of the theorem for $\mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,Iw}}$. Proposition 4.5.1 then reduces us to checking these assertions for the ϖ-adic completion of the naive local model at Iwahori level for $\mathrm{Res}_{K/\mathbf{Q}_p} \mathrm{GL}_2$.

Since this naive local model is a scheme of finite presentation over $\mathcal{O}$, its special fibre (i.e. its base-change to $\mathbf{F}$) is Jacobson, and it is excellent; thus its ϖ-adic completion satisfies the properties of (1) if and only if the naive local model itself is

normal and Cohen–Macaulay. The special fibre of its ϖ-adic completion is of course just equal to its own special fibre, and so verifying (2) for the first of these special fibres is equivalent to verifying it for the second. Finally, the ϖ-adic completion of an $\mathcal{O}$-flat Noetherian ring is again $\mathcal{O}$-flat, and so the ϖ-adic completion of the naive local model will be $\mathcal{O}$-flat if the naive local model itself is.

All the properties of the naive local model that are described in the preceding paragraph, other than the Cohen–Macaulay property, are contained in [PRS13, Thm. 2.4.3], if we can identify the naive local models at Iwahori level with the vertical local models at Iwahori level, in the sense of [PR03, §8] (see also the discussion above *loc. cit.* in [PRS13]).

The vertical local models are obtained by intersecting the preimages at Iwahori level of the flat local models at hyperspecial level. Although naive local models are not flat in general, for the Weil restriction of GL_2 at hyperspecial level they are, by a special case of [PR03, Cor. 4.3]. Thus the naive local models at Iwahori level are identified with the vertical ones and [PRS13, Thm. 2.4.3] applies to them directly. (To be precise, the results of [PR03] apply to restrictions of scalars $\mathrm{Res}_{F/F_0} \mathrm{GL}_2$ with F/F_0 totally ramified. However, thanks to the decomposition $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A \cong \oplus_{\sigma: W(k) \hookrightarrow \mathcal{O}} \mathcal{O}_K \otimes_{W(k), \sigma} A$ the local model for $\mathrm{Res}_{K/\mathbf{Q}_p} \mathrm{GL}_2$ decomposes as a product of local models for totally ramified extensions.)

Finally, Cohen–Macaulayness can be proved as in [Gör01, Prop. 4.24] and the discussion immediately following. We thank U. Görtz for explaining this argument to us. As in the previous paragraph we reduce to the case of local models at Iwahori level for $\mathrm{Res}_{F/F_0} \mathrm{GL}_2$ with F/F_0 totally ramified of degree e. In this setting the admissible set M for the coweight $\mu = (e, 0)$ has precisely one element of length 0 and two elements of each length between 1 and e. Moreover, for elements $x, y \in M$ we have $x < y$ in the Bruhat order if and only if $\ell(x) < \ell(y)$. One checks easily that M is e-Cohen–Macaulay in the sense of [Gör01, Def. 4.23], and we conclude by [Gör01, Prop. 4.24]. Alternatively, this also follows from the much more general recent results of Haines–Richarz [HR19]. □

Corollary 4.5.3. *Suppose that $d = 2$ and that τ is a tame inertial type. Then*

(1) $\mathcal{C}^{\tau, \mathrm{BT}}$ *is analytically normal, and Cohen–Macaulay.*

(2) *The special fibre $\mathcal{C}^{\tau, \mathrm{BT}, 1}$ is reduced.*

(3) $\mathcal{C}^{\tau, \mathrm{BT}}$ *is flat over $\mathcal{O}$.*

Proof. This follows from Theorems 4.1.5 and 4.5.2, since all of these properties can be verified smooth locally (as was already noted in the proof of the second of these theorems). □

Remark 4.5.4. There is another structural result about vertical local models that is proved in [PR03, §8] but which we haven't incorporated into our preceding results, namely, that the irreducible components of the special fibre are normal. Since the notion of irreducible component is not étale local (and so in particular not smooth local), this statement does not imply the corresponding statement for the special fibres of $\mathcal{M}_{\mathrm{loc}}^{\tau, \mathrm{BT}}$ or $\mathcal{C}^{\tau, \mathrm{BT}}$. Rather, it implies the weaker, and somewhat more technical, statement that each of the analytic branches passing through each closed point of the special fibre of these ϖ-adic formal algebraic stacks is normal. We won't discuss this further here, since we don't need this result.

4.6. **The Dieudonné stack.** In this subsection we assume that $\eta \neq \eta'$. As previously mentioned, the construction described in this section will not be needed elsewhere in the present paper; however, it is convenient to include it here because of its reliance on the discussion of determinant conditions in Section 4.2.

Let $\mathfrak{M}$ be a Breuil–Kisin module with A-coefficients and descent data of type τ and height at most 1, and let $D := \mathfrak{M}/u\mathfrak{M}$ be its corresponding Dieudonné module as in Definition 2.2.1. If we write $D_i := e_i D$, then this Dieudonné module is given by rank two projective modules D_j over A ($j = 0, \ldots, f' - 1$) with linear maps $F : D_j \to D_{j+1}$ and $V : D_j \to D_{j-1}$ (subscripts understood modulo f') such that $FV = VF = p$.

Now, $I(K'/K)$ is abelian of order prime to p, so we can write $D = D_\eta \oplus D_{\eta'}$, where D_η is the submodule on which $I(K'/K)$ acts via η. Since $\mathfrak{M}_\eta$ is obtained from the projective $\mathfrak{S}_A$-module $\mathfrak{M}$ by applying a projector, each $D_{\eta,j}$ is an invertible A-module, and F, V induce linear maps $F : D_{\eta,j} \to D_{\eta,j+1}$ and $V : D_{\eta,j+1} \to D_{\eta,j}$ such that $FV = VF = p$.

We can of course apply the same construction with η' in the place of η, obtaining a Dieudonné module $D_{\eta'}$. We now prove some lemmas relating these various Dieudonné modules. We will need to make use of a variant of the strong determinant condition, so we begin by discussing this and its relationship to the strong determinant condition of Subsection 4.2.

Definition 4.6.1. Let $(\mathfrak{L}, \mathfrak{L}^+)$ be a pair consisting of a rank two projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module $\mathfrak{L}$, and an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-submodule $\mathfrak{L}^+ \subset \mathfrak{L}$, such that Zariski locally on $\operatorname{Spec} A$, $\mathfrak{L}^+$ is a direct summand of $\mathfrak{L}$ as an A-module.

Then we say that the pair $(\mathfrak{L}, \mathfrak{L}^+)$ *satisfies the Kottwitz determinant condition over K'* if for all $a \in \mathcal{O}_{K'}$, we have

$$\det\nolimits_A(a|\mathfrak{L}^+) = \prod_{\psi : K' \hookrightarrow E} \psi(a)$$

as polynomial functions on $\mathcal{O}_{K'}$ in the sense of [Kot92, §5].

There is a finite type stack $\mathcal{M}_{K',\det}$ over $\operatorname{Spec} \mathcal{O}$, with $\mathcal{M}_{K',\det}(\operatorname{Spec} A)$ being the groupoid of pairs $(\mathfrak{L}, \mathfrak{L}^+)$ as above which satisfy the Kottwitz determinant condition over K'. As we have seen above, by a result of Pappas–Rapoport, this stack is flat over $\operatorname{Spec} \mathcal{O}$ (see [Kis09, Prop. 2.2.2]).

Lemma 4.6.2. *If A is an E-algebra, then a pair $(\mathfrak{L}, \mathfrak{L}^+)$ as in Definition 4.6.1 satisfies the Kottwitz determinant condition over K' if and only if $\mathfrak{L}^+$ is a rank one projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module.*

Proof. We may write $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A = K' \otimes_{\mathbf{Q}_p} A \cong \prod_{\psi : K' \hookrightarrow E} A$, where the embedding $\psi : K' \hookrightarrow E$ corresponds to an idempotent $e_\psi \in K' \otimes_{\mathbf{Q}_p} A$. Decomposing $\mathfrak{L}^+$ as $\oplus_\psi e_\psi \mathfrak{L}^+$, the left-hand side of the Kottwitz determinant condition becomes $\prod_\psi \det_A(a|e_\psi \mathfrak{L}^+) = \prod_\psi \psi(a)^{\operatorname{rk}_A e_\psi \mathfrak{L}^+}$. It follows that the Kottwitz determinant condition is satisfied if and only if the projective A-module $e_\psi \mathfrak{L}^+$ has rank one for all ψ, which is equivalent to $\mathfrak{L}^+$ being a rank one projective $K' \otimes_{\mathbf{Q}_p} A$-module, as required. $\square$

Proposition 4.6.3. *If $\mathfrak{M}$ is an object of $\mathcal{C}^{\tau,\mathrm{BT}}(A)$, then the pair*

$$(\mathfrak{M}/E(u)\mathfrak{M}, \operatorname{im} \Phi_{\mathfrak{M}}/E(u)\mathfrak{M})$$

satisfies the Kottwitz determinant condition for K'.

Proof. Let $\mathcal{C}^{\tau,\mathrm{BT}'}$ be the closed substack of $\mathcal{C}^\tau$ consisting of those $\mathfrak{M}$ for which the pair $(\mathfrak{M}/E(u)\mathfrak{M}, \operatorname{im} \Phi_\mathfrak{M}/E(u)\mathfrak{M})$ satisfies the Kottwitz determinant condition for K'. We need to show that $\mathcal{C}^{\tau,\mathrm{BT}}$ is a closed substack of $\mathcal{C}^{\tau,\mathrm{BT}'}$. Since $\mathcal{C}^{\tau,\mathrm{BT}}$ is flat over $\operatorname{Spf}\mathcal{O}$ by Corollary 4.5.3, it is enough to show that if A° is a flat $\mathcal{O}$-algebra, then $\mathcal{C}^{\tau,\mathrm{BT}}(A^\circ) = \mathcal{C}^{\tau,\mathrm{BT}'}(A^\circ)$.

To see this, let $\mathfrak{M}^\circ$ be an object of $\mathcal{C}^\tau(A^\circ)$. Since A° is flat over $\mathcal{O}$, the Kottwitz determinant conditions for $\mathfrak{M}$ to be an object of $\mathcal{C}^{\tau,\mathrm{BT}'}(A^\circ)$ or of $\mathcal{C}^{\tau,\mathrm{BT}}(A^\circ)$ can be checked after inverting p. Accordingly, we write $A = A^\circ[1/p]$, and write $\mathfrak{M}$ for the base change of $\mathfrak{M}^\circ$ to A. By Lemma 4.6.2, $\mathfrak{M}^\circ$ is an object of $\mathcal{C}^{\tau,\mathrm{BT}'}(A^\circ)$ if and only if $\operatorname{im} \Phi_\mathfrak{M}/E(u)\mathfrak{M}$ is a rank one projective $K' \otimes_{\mathbf{Q}_p} A$-module. Similarly, $\mathfrak{M}^\circ$ is an object of $\mathcal{C}^{\tau,\mathrm{BT}}(A^\circ)$ if and only if for each ξ, $(\operatorname{im} \Phi_\mathfrak{M})_\xi/E(u)\mathfrak{M}_\xi$ is a rank one projective $N \otimes_{\mathbf{Q}_p} A$-module. Since

$$\operatorname{im} \Phi_\mathfrak{M}/E(u)\mathfrak{M} = \oplus_\xi (\operatorname{im} \Phi_\mathfrak{M})_\xi/E(u)\mathfrak{M}_\xi,$$

the equivalence of these two conditions is clear. $\qquad\square$

Lemma 4.6.4. *If $(\mathfrak{L}, \mathfrak{L}^+)$ is an object of $\mathcal{M}_{K',\det}(A)$ (i.e. satisfies the Kottwitz determinant condition over K'), then the morphism $\bigwedge^2_{\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A} \mathfrak{L}^+ \to \bigwedge^2_{\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A} \mathfrak{L}$ induced by the inclusion $\mathfrak{L}^+ \subset \mathfrak{L}$ is identically zero.*

Remark 4.6.5. Note that, although $\mathfrak{L}^+$ need not be locally free over $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$, its exterior square is nevertheless defined, so that the statement of the lemma makes sense.

Proof of Lemma 4.6.4. Since $\mathcal{M}_{K',\det}$ is $\mathcal{O}$-flat, it is enough to treat the case that A is $\mathcal{O}$-flat. In this case $\mathfrak{L}$, and thus also $\bigwedge^2 \mathfrak{L}$, are $\mathcal{O}$-flat. Given this additional assumption, it suffices to prove that the morphism of the lemma becomes zero after tensoring with $\mathbf{Q}_p$ over $\mathbf{Z}_p$. This morphism may naturally be identified with the morphism $\bigwedge^2_{K' \otimes_{\mathbf{Z}_p} A} \mathfrak{L}^+ \to \bigwedge^2_{K' \otimes_{\mathbf{Z}_p} A} \mathfrak{L}$ induced by the injection $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathfrak{L}^+ \hookrightarrow \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathfrak{L}$. Locally on $\operatorname{Spec} A$, this is the embedding of a free $K' \otimes_{\mathbf{Z}_p} A$-module of rank one as a direct summand of a free $K' \otimes_{\mathbf{Z}_p} A$-module of rank two. Thus $\bigwedge^2$ of the source in fact vanishes, and hence so does $\bigwedge^2$ of the embedding. $\qquad\square$

Lemma 4.6.6. *If $\mathfrak{M}$ is an object of $\mathcal{C}^{\tau,\mathrm{BT}}(A)$, then $\bigwedge^2 \Phi_\mathfrak{M} : \bigwedge^2 \varphi^*\mathfrak{M} \hookrightarrow \bigwedge^2 \mathfrak{M}$ is exactly divisible by $E(u)$, i.e. can be written as $E(u)$ times an isomorphism of $\mathfrak{S}_A$-modules.*

Proof. It follows from Proposition 4.6.3 and Lemma 4.6.4 that the reduction of $\bigwedge^2 \Phi_\mathfrak{M}$ modulo $E(u)$ vanishes, so we can think of $\bigwedge^2 \Phi_\mathfrak{M}$ as a morphism $\bigwedge^2 \varphi^*\mathfrak{M} \to E(u) \bigwedge^2 \mathfrak{M}$. We need to show that the cokernel X of this morphism vanishes. Since $\operatorname{im} \Phi_\mathfrak{M} \supseteq E(u)\mathfrak{M}$, X is a finitely generated A-module, so that in order to prove that it vanishes, it is enough to prove that $X/pX = 0$.

Since the formation of cokernels is compatible with base change, this means that we can (and do) assume that A is an $\mathbf{F}$-algebra. Since the special fibre $\mathcal{C}^{\tau,\mathrm{BT}}$ is of finite type over $\mathbf{F}$, we can and do assume that A is furthermore of finite type over $\mathbf{F}$. The special fibre of $\mathcal{C}^{\tau,\mathrm{BT}}$ is reduced by Corollary 4.5.3, so we may assume that A is reduced, and it is therefore enough to prove that X vanishes modulo each maximal ideal of A. Since the residue fields at such maximal ideals are finite, we are reduced to the case that A is a finite field, when the result follows from [Kis09, Lem. 2.5.1]. $\qquad\square$

Lemma 4.6.7. *There is a canonical isomorphism*

$$\text{``}(F \otimes F)/p\text{''}: D_{\eta,j} \otimes_A D_{\eta',j} \xrightarrow{\sim} D_{\eta,j+1} \otimes_A D_{\eta',j+1},$$

characterised by the fact that it is compatible with change of scalars, and that

$$p \cdot \text{``}(F \otimes F)/p\text{''} = F \otimes F.$$

Proof. Since $\mathcal{C}^{\mathrm{BT}}$ is flat over $\mathcal{O}$, we see that in the universal case, the formula

$$p \cdot \text{``}(F \otimes F)/p\text{''} = F \otimes F$$

uniquely determines the isomorphism "$(F \otimes F)/p$" (if it exists). Since any Breuil–Kisin module with descent data is obtained from the universal case by change of scalars, we see that the isomorphism "$(F \otimes F)/p$" is indeed characterised by the properties stated in the lemma, provided that it exists.

To check that the isomorphism exists, we can again consider the universal case, and hence assume that A is a flat $\mathcal{O}$-algebra. In this case, it suffices to check that the morphism $F \otimes F : D_{\eta,j} \otimes_A D_{\eta',j} \to D_{\eta,j+1} \otimes_A D_{\eta,j+1}$ is divisible by p, and that the formula $(F \otimes F)/p$ is indeed an isomorphism. Noting that the direct sum over $j = 0, \ldots, f'-1$ of these morphisms may be identified with the reduction modulo u of the morphism $\bigwedge^2 \Phi_{\mathfrak{M}} : \bigwedge^2 \varphi^* \mathfrak{M} \to \bigwedge^2 \mathfrak{M}$, this follows from Lemma 4.6.6. $\qquad\square$

The isomorphism "$(F \otimes F)/p$" of the preceding lemma may be rewritten as an isomorphism of invertible A-modules

$$(4.6.8) \qquad\qquad \mathrm{Hom}_A(D_{\eta,j}, D_{\eta,j+1}) \xrightarrow{\sim} \mathrm{Hom}_A(D_{\eta',j+1}, D_{\eta',j}).$$

Lemma 4.6.9. *The isomorphism* (4.6.8) *takes F to V.*

Proof. The claim of the lemma is equivalent to showing that the composite

$$D_{\eta,j} \otimes_A D_{\eta',j+1} \xrightarrow{\mathrm{id} \otimes V} D_{\eta,j} \otimes_A D_{\eta',j} \xrightarrow{\text{``}(F \otimes F)/p\text{''}} D_{\eta,j+1} \otimes_A D_{\eta',j+1}$$

coincides with the morphism $F \otimes \mathrm{id}$. It suffices to check this in the universal case, and thus we may assume that p is a non-zero divisor in A, and hence verify the required identity of morphisms after multiplying each of them by p. The identity to be verified then becomes

$$(F \otimes F) \circ (\mathrm{id} \otimes V) \overset{?}{=} p(F \otimes \mathrm{id}),$$

which follows immediately from the formula $FV = p$. $\qquad\square$

We now consider the moduli stacks classifying the Dieudonné modules with the properties we have just established, and the maps from the moduli stacks of Breuil–Kisin modules to these stacks. In this discussion we specialise the choice of K' in the following way. Choose a tame inertial type $\tau = \eta \oplus \eta'$ with $\eta \neq \eta'$. Fix a uniformiser π of K. If τ is a tame principal series type, we take $K' = K(\pi^{1/(p^f-1)})$, while if τ is a tame cuspidal type, we let L be an unramified quadratic extension of K, and set $K' = L(\pi^{1/(p^{2f}-1)})$. Let N be the maximal unramified extension of K in K'. In either case K'/K is a Galois extension; in the principal series case, we have $e' = (p^f - 1)e$, $f' = f$, and in the cuspidal case we have $e' = (p^{2f} - 1)e$, $f' = 2f$.

Suppose first that we are in the principal series case. Then there is a moduli stack classifying the data of the $D_{\eta,j}$ together with the F and V, namely the stack

$$\mathcal{D}_\eta := \left[\left(\mathrm{Spec}\, W(k)[X_0, Y_0, \ldots, X_{f-1}, Y_{f-1}]/(X_j Y_j - p)_{j=0,\ldots,f-1} \right) / \mathbf{G}_m^f \right],$$

where the f copies of $\mathbf{G}_m$ act as follows:

$$(u_0, \ldots, u_{f-1}) \cdot (X_j, Y_j) \mapsto (u_j u_{j+1}^{-1} X_j, u_{j+1} u_j^{-1} Y_j).$$

To see this, recall that the stack

$$[\text{point}/\mathbf{G}_m]$$

classifies line bundles, so the f copies of $\mathbf{G}_m$ in $\mathcal{D}_\eta$ correspond to f line bundles, which are the line bundles $D_{\eta,j}$ $(j = 0, \ldots, f-1)$. If we locally trivialise these line bundles, then the maps $F : D_{\eta,j} \to D_{\eta,j+1}$ and $V : D_{\eta,j+1} \to D_{\eta,j}$ act by scalars, which we denote by X_j and Y_j respectively. The f copies of $\mathbf{G}_m$ are then encoding possible changes of trivialisation, by units u_j, which induce the indicated changes on the X_j's and Y_j's.

There is then a natural map

$$\mathcal{C}^{\tau,\text{BT}} \to \mathcal{D}_\eta,$$

classifying the Dieudonné modules underlying the Breuil–Kisin modules with descent data.

There is a more geometric way to think about what $\mathcal{D}_\eta$ classifies. To begin with, we just rephrase what we've already indicated: it represents the functor which associates to a $W(k)$-scheme the groupoid whose objects are f-tuples of line bundles $(D_{\eta,j})_{j=0,\ldots,f-1}$ equipped with morphisms $X_j : D_{\eta,j} \to D_{\eta,j+1}$ and $Y_j : D_{\eta,j+1} \to D_{\eta,j}$ such that $Y_j X_j = p$. (Morphisms in the groupoid are just isomorphisms between collections of such data.) Equivalently, we can think of this as giving the line bundle $D_{\eta,0}$, and then the f line bundles $\mathcal{D}_j := D_{\eta,j+1} \otimes D_{\eta,j}^{-1}$, equipped with sections $X_j \in \mathcal{D}_j$ and $Y_j \in \mathcal{D}_j^{-1}$ whose product in $\mathcal{D}_j \otimes \mathcal{D}_j^{-1} = \mathcal{O}$ (the trivial line bundle) is equal to the element p. Note that it superficially looks like we are remembering $f + 1$ line bundles, rather than f, but this is illusory, since in fact $\mathcal{D}_0 \otimes \cdots \otimes \mathcal{D}_{f-1}$ is trivial; indeed, the isomorphism $\mathcal{D}_0 \otimes \cdots \otimes \mathcal{D}_{f-1} \xrightarrow{\sim} \mathcal{O}$ is part of the data we should remember.

We now turn to the case that τ is a cuspidal type. In this case our Dieudonné modules have unramified as well as inertial descent data; accordingly, we let φ^f denote the element of $\text{Gal}(K'/K)$ which acts trivially on $\pi^{1/(p^{2f}-1)}$ and non-trivially on L. Then the descent data of φ^f induces isomorphisms $D_j \xrightarrow{\sim} D_{j+f}$, which are compatible with the F, V, and which identify $D_{\eta,j}$ with $D_{\eta',f+j}$.

If we choose local trivialisations of the line bundles $D_{\eta,0}, \ldots, D_{\eta,f}$, then the maps $F : D_{\eta,j} \to D_{\eta,j+1}$ and $V : D_{\eta,j+1} \to D_{\eta,j}$ for $0 \leq j \leq f-1$ are given by scalars X_j and Y_j respectively. The identification of $D_{\eta,j}$ and $D_{\eta',f+j}$ given by φ^f identifies $D_{\eta,j} \otimes D_{\eta,j+1}^{-1}$ with $D_{\eta',f+j} \otimes D_{\eta',f+j+1}^{-1}$, which via the isomorphsim (4.6.8) is identified with $D_{\eta,f+j+1} \otimes D_{\eta,f+j}^{-1}$. It follows that for $0 \leq j \leq f-2$ the data of $D_{\eta,j}$, $D_{\eta,j+1}$ and $D_{\eta,f+j}$ recursively determines $D_{\eta,f+j+1}$. From Lemma 4.6.9 we see, again recursively for $0 \leq j \leq f-2$, that there are unique trivialisations of $D_{\eta,f+1}, \ldots, D_{\eta,2f-1}$ such that $F : D_{\eta,f+j} \to D_{\eta,f+j+1}$ is given by Y_j, and $V : D_{\eta,f+j+1} \to D_{\eta,f+j}$ is given by X_j. Furthermore, there is some unit α such that $F : D_{\eta,2f-1} \to D_{\eta,0}$ is given by αY_{f-1}, and $V : D_{\eta,0} \to D_{\eta,2f-1}$ is given by $\alpha^{-1} X_{f-1}$. Note that the map $F^{2f} : D_{\eta,0} \to D_{\eta,0}$ is precisely $p^f \alpha$.

Consequently, we see that the data of the $D_{\eta,j}$ (together with the F, V) is classified by the stack

$$\mathcal{D}_\eta := \left[\left(\operatorname{Spec} W(k)[X_0, Y_0, \ldots, X_{f-1}, Y_{f-1}]/(X_j Y_j - p)_{j=0,\ldots,f-1} \right) \times \mathbf{G}_m \right) / \mathbf{G}_m^{f+1} \right],$$

where the $f + 1$ copies of $\mathbf{G}_m$ act as follows:

$$(u_0, \ldots, u_{f-1}, u_f) \cdot ((X_j, Y_j), \alpha) \mapsto ((u_j u_{j+1}^{-1} X_j, u_{j+1} u_j^{-1} Y_j), \alpha).$$

One again there is a natural map

$$\mathcal{C}^{\tau, \mathrm{BT}} \to \mathcal{D}_\eta,$$

classifying the Dieudonné modules underlying the Breuil–Kisin modules with descent data.

5. Galois moduli stacks

We are finally ready to discuss the algebraic stacks $\mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau,a}$, which are defined in Section 5.1 as the scheme-theoretic images of the morphisms $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ respectively. We informally think of these stacks as being moduli stacks of two-dimensional representations of G_K.

In Section 5.2 we relate the versal rings of the stacks $\mathcal{Z}^\tau$ to potentially crystalline deformation rings, and use this to show that the stacks $\mathcal{Z}^{\mathrm{dd},a}$, $\mathcal{Z}^{\tau,a}$, and $\mathcal{C}^{\tau,\mathrm{BT},a}$ are equidimensional of dimension $[K : \mathbf{Q}_p]$ (see Propositions 5.2.20 and 5.2.21).

5.1. **Scheme-theoretic images.** We continue to fix $d = 2$, $h = 1$, and we set $K' = L(\pi^{1/p^{2f-1}})$, where L/K is the unramified quadratic extension, and π is a uniformiser of K. (This is the choice of K' that we made before in the cuspidal case, and contains the choice of K' that we made in the principal series case; since the category of Breuil–Kisin modules with descent data for the smaller extension is by Proposition 4.3.1 naturally a full subcategory of the category of Breuil–Kisin modules with descent data for the larger extension, we can consider both principal series and cuspidal types in this setting.)

Definition 5.1.1. For each $a \geq 1$ we write $\mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau,a}$ for the scheme-theoretic images (in the sense of [EG21, Defn. 3.2.8]) of the morphisms $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ respectively. We write $\overline{\mathcal{Z}}$, $\overline{\mathcal{Z}}^\tau$ for $\mathcal{Z}^1$, $\mathcal{Z}^{\tau,1}$ respectively.

The following theorem records some basic properties of these scheme-theoretic images. (See Definition A.3 for the notion of a très ramifiée representation.)

Theorem 5.1.2.

(1) *For each $a \geq 1$, $\mathcal{Z}^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\mathcal{O}/\varpi^a$, and is a closed substack of $\mathcal{R}^{\mathrm{dd},a}$. In turn, each $\mathcal{Z}^{\tau,a}$ is a closed substack of $\mathcal{Z}^{\mathrm{dd},a}$, and thus in particular is an algebraic stack of finite presentation over $\mathcal{O}/\varpi^a$; and $\mathcal{Z}^{\mathrm{dd},a}$ is the union of the $\mathcal{Z}^{\tau,a}$.*

(2) *The morphism $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through a morphism $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{Z}^{\mathrm{dd},a}$ which is representable by algebraic spaces, scheme-theoretically dominant, and proper. Similarly, the morphism $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through a morphism $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{Z}^{\tau,a}$ which is representable by algebraic spaces, scheme-theoretically dominant, and proper.*

(3) *The $\overline{\mathbf{F}}_p$-points of $\overline{\mathcal{Z}}$ are naturally in bijection with the continuous representations $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which are not a twist of a très ramifiée extension of the trivial character by the mod p cyclotomic character. Similarly, the $\overline{\mathbf{F}}_p$-points of $\overline{\mathcal{Z}}^\tau$ are naturally in bijection with the continuous representations $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which have a potentially Barsotti–Tate lift of type τ.*

Proof. Part (1) follows easily from Theorem 3.1.13. Indeed, by [EG21, Prop. 3.2.31] we may think of $\mathcal{Z}^{\mathrm{dd},a}$ as the scheme-theoretic image of the proper morphism of algebraic stacks $\mathcal{C}^{\mathrm{dd,BT},a} \to \mathcal{R}_1^{\mathrm{dd},a}$ and similarly for each $\mathcal{Z}^{\tau,a}$. The existence of the factorisations in (2) is then formal.

By [EG21, Lem. 3.2.14], for each finite extension $\mathbf{F}'/\mathbf{F}$, the $\mathbf{F}'$-points of $\overline{\mathcal{Z}}$ (respectively $\overline{\mathcal{Z}}^{\tau}$) correspond to the étale φ-modules with descent data of the form $\mathfrak{M}[1/u]$, where $\mathfrak{M}$ is a Breuil–Kisin module of rank 2 with descent data and $\mathbf{F}$-coefficients which satisfies the strong determinant condition (respectively, which satisfies the strong determinant condition and is of type τ). By Lemma 4.2.16 and Corollary 4.5.3, these precisely correspond to the Galois representations $\overline{r} : G_K \to \mathrm{GL}_2(\mathbf{F})$ which admit potentially Barsotti–Tate lifts of some tame type (respectively, of type τ). The result now follows from Lemma A.4. $\qquad\square$

The thickenings $\mathcal{C}^{\mathrm{dd,BT},a} \hookrightarrow \mathcal{C}^{\mathrm{dd,BT},a+1}$ and $\mathcal{R}^{\mathrm{dd},a} \hookrightarrow \mathcal{R}^{\mathrm{dd},a+1}$ induce closed immersions $\mathcal{Z}^{\mathrm{dd},a} \hookrightarrow \mathcal{Z}^{\mathrm{dd},a+1}$. Similarly, the thickenings $\mathcal{C}^{\tau,\mathrm{BT},a} \hookrightarrow \mathcal{C}^{\tau,\mathrm{BT},a+1}$ give rise to closed immersions $\mathcal{Z}^{\tau,a} \hookrightarrow \mathcal{Z}^{\tau,a+1}$.

Lemma 5.1.3. *Fix $a \geq 1$. Then the morphism $\mathcal{Z}^{\mathrm{dd},a} \hookrightarrow \mathcal{Z}^{\mathrm{dd},a+1}$ is a thickening, and for each tame type τ, the morphism $\mathcal{Z}^{\tau,a} \hookrightarrow \mathcal{Z}^{\tau,a+1}$ is a thickening.*

Proof. In each case, the claim of the lemma follows from the following more general statement: if

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}' \end{array}$$

is a diagram of morphisms of algebraic stacks in which the upper horizontal arrow is a thickening, the lower horizontal arrow is a closed immersion, and each of the vertical arrows is representable by algebraic spaces, quasi-compact, and scheme-theoretically dominant, then the lower horizontal arrow is also a thickening.

Since the property of being a thickening may be checked smooth locally, and since scheme-theoretic dominance of quasi-compact morphisms is preserved by flat base-change, we may show this after pulling the entire diagram back over a smooth surjective morphism $V' \to \mathcal{Y}$ whose source is a scheme, and thus reduce to the case in which the lower arrow is a morphism of schemes, and the upper arrow is a morphism of algebraic spaces. A surjecive étale morphism is also scheme-theoretically dominant, and so pulling back the top arrow over a surjective étale morphism $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ whose source is a scheme, we finally reduce to considering a diagram of morphisms of schemes

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array}$$

in which the top arrow is a thickening, the vertical arrows are quasi-compact and scheme-theoretically dominant, and the bottom arrow is a closed immersion.

Pulling back over an affine open subscheme of V', and then pulling back the top arrow over the disjoint union of the members of a finite affine open cover of the preimage of this affine open in U' (note that this preimage is quasi-compact), we

further reduce to the case when all the schemes involved are affine. That is, we have a diagram of ring morphisms

$$
\begin{array}{ccc}
A' & \longrightarrow & A \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B
\end{array}
$$

in which the vertical arrows are injective, the horizontal arrows are surjective, and the bottom arrow has nilpotent kernel. One immediately verifies that the top arrow has nilpotent kernel as well. $\qquad\square$

We write $\mathcal{C}^{\mathrm{dd,BT}} := \varinjlim_a \mathcal{C}^{\mathrm{dd,BT},a}$ and $\mathcal{Z}^{\mathrm{dd}} := \varinjlim_a \mathcal{Z}^{\mathrm{dd},a}$; we then have evident morphisms of Ind-algebraic stacks

$$\mathcal{C}^{\mathrm{dd,BT}} \to \mathcal{Z}^{\mathrm{dd}} \to \mathcal{R}^{\mathrm{dd}}$$

lying over $\mathrm{Spf}\,\mathcal{O}$, both representable by algebraic spaces, with the first being furthermore proper and scheme-theoretically dominant in the sense of [Eme, Def. 6.13], and the second being a closed immersion.

Similarly, for each choice of tame type τ, we set $\mathcal{C}^{\tau,\mathrm{BT}} = \varinjlim_a \mathcal{C}^{\tau,a}$, and $\mathcal{Z}^{\tau} := \varinjlim_a \mathcal{Z}^{\tau,a}$. We again have morphisms

$$\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{Z}^{\tau} \to \mathcal{R}^{\mathrm{dd}}$$

of Ind-algebraic stacks over $\mathrm{Spf}\,\mathcal{O}$, both being representable by algebraic spaces, the first being proper and scheme-theoretically dominant, and the second being a closed immersion. Note that by Corollary 4.2.13, $\mathcal{C}^{\mathrm{dd,BT}}$ is the disjoint union of the $\mathcal{C}^{\tau,\mathrm{BT}}$, so it follows that $\mathcal{Z}^{\mathrm{dd,BT}}$ is the union (but *not* the disjoint union) of the $\mathcal{Z}^{\tau}$.

Proposition 4.2.7 shows that $\mathcal{C}^{\tau,\mathrm{BT}}$ is a ϖ-adic formal algebraic stack of finite presentation over $\mathrm{Spf}\,\mathcal{O}$. Each $\mathcal{Z}^{\tau,a}$ is an algebraic stack of finite presentation over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$ by Theorem 5.1.2. Analogous remarks apply in the case of $\mathcal{C}^{\mathrm{dd,BT}}$ and $\mathcal{Z}^{\mathrm{dd}}$.

Proposition 5.1.4. $\mathcal{Z}^{\mathrm{dd}}$, *and each* $\mathcal{Z}^{\tau}$, *are* ϖ-*adic formal algebraic stacks, of finite presentation over* $\mathrm{Spf}\,\mathcal{O}$.

Proof. We give the argument for $\mathcal{Z}^{\mathrm{dd}}$, the argument for $\mathcal{Z}^{\tau}$ being identical. (Alternatively, this latter case follows from the former and the fact that the canonical morphism $\mathcal{Z}^{\tau} \hookrightarrow \mathcal{Z}^{\mathrm{dd}}$ is a closed immersion.) It follows from [EG22, Prop. A.10] that $\mathcal{Z}^{\mathrm{dd}}$ is a formal algebraic stack, and by construction it is locally Ind-finite type over $\mathrm{Spec}\,\mathcal{O}$. Since each $\mathcal{Z}^{\mathrm{dd},a}$ is of finite presentation over $\mathcal{O}/\varpi^a$, it follows from [EG22, Prop. A.21] that $\mathcal{Z}^{\mathrm{dd}}$ is a ϖ-adic formal algebraic stack.

The isomorphism $\mathcal{Z}^{\mathrm{dd}} \xrightarrow{\sim} \varinjlim_a \mathcal{Z}^{\mathrm{dd},a}$ induces an isomorphism

$$\mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^b \xrightarrow{\sim} \varinjlim_a \mathcal{Z}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{O}/\varpi^b,$$

for any fixed $b \geq 1$. Since $\mathcal{Z}^{\mathrm{dd}}$ is quasi-compact and quasi-separated, so is $\mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$, and thus this isomorphism factors through $\mathcal{Z}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$ for some a. Thus the directed system $\mathcal{Z}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$ in a eventually stabilises, and so we see that

$$\mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^b \xrightarrow{\sim} \mathcal{Z}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$$

for sufficiently large values of a. Since $\mathcal{Z}^{\mathrm{dd},a}$ is of finite presentation over $\mathcal{O}/\varpi^a$, we find that $\mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$ is of finite presentation over $\mathcal{O}/\varpi^b$. Consequently, we conclude that $\mathcal{Z}^{\mathrm{dd}}$ is of finite presentation over $\mathrm{Spf}\,\mathcal{O}$, as claimed. $\qquad\square$

Remark 5.1.5. The thickening $\mathcal{Z}^{\mathrm{dd},a} \hookrightarrow \mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ need not be an isomorphism *a priori*, and we have no reason to expect that it is. Nevertheless, in [CEGS20a] we prove that this thickening is *generically* an isomorphism for *every* value of $a \geq 1$, and we will furthermore show that each $(\mathcal{Z}^{\mathrm{dd},a})_{/\mathbf{F}}$ is generically reduced. The proof of this result depends on the detailed analysis of the irreducible components of the algebraic stacks $\mathcal{C}^{\tau,a}$ and $\mathcal{Z}^{\tau,a}$ that we make in [CEGS20c].

We conclude this subsection by establishing some basic lemmas about the reduced substacks underlying each of $\mathcal{C}^{\tau,\mathrm{BT}}$ and $\mathcal{Z}^\tau$.

Lemma 5.1.6. *Let $\mathcal{X}$ be an algebraic stack over $\mathcal{O}/\varpi^a$, and let $\mathcal{X}_{\mathrm{red}}$ be the underlying reduced substack of $\mathcal{X}$. Then $\mathcal{X}_{\mathrm{red}}$ is a closed substack of $\mathcal{X}_{/\mathbf{F}} := \mathcal{X} \times_{\mathcal{O}/\varpi^a} \mathbf{F}$.*

Proof. The structural morphism $\mathcal{X} \to \operatorname{Spec} \mathcal{O}/\varpi^a$ induces a natural morphism $\mathcal{X}_{\mathrm{red}} \to (\operatorname{Spec} \mathcal{O}/\varpi^a)_{\mathrm{red}} = \operatorname{Spec} \mathbf{F}$, so the natural morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ factors through $\mathcal{X}_{/\mathbf{F}}$. Since the morphisms $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ and $\mathcal{X}_{/\mathbf{F}} \to \mathcal{X}$ are both closed immersions, so is the morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}_{/\mathbf{F}}$. $\square$

Lemma 5.1.7. *If $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-compact morphism of algebraic stacks, and $\mathcal{W}$ is the scheme-theoretic image of f, then the scheme-theoretic image of the induced morphism of underlying reduced substacks $f_{\mathrm{red}} : \mathcal{X}_{\mathrm{red}} \to \mathcal{Y}_{\mathrm{red}}$ is the underlying reduced substack $\mathcal{W}_{\mathrm{red}}$.*

Proof. Since the definitions of the scheme-theoretic image and of the underlying reduced substack are both smooth local (in the former case see [EG21, Rem. 3.1.5(3)], and in the latter case it follows immediately from the construction in [Sta13, Tag 0509]), we immediately reduce to the case of schemes, which follows from [Sta13, Tag 056B]. $\square$

Lemma 5.1.8. *For each $a \geq 1$, $\mathcal{C}^{\mathrm{dd},\mathrm{BT},1}$ is the underlying reduced substack of $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a}$, and $\mathcal{Z}^{\mathrm{dd},1}$ is the underlying reduced substack of $\mathcal{Z}^{\mathrm{dd},a}$; consequently, $\mathcal{C}^{\mathrm{dd},\mathrm{BT},1}$ is the underlying reduced substack of $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$, and $\mathcal{Z}^{\mathrm{dd},1}$ is the underlying reduced substack of $\mathcal{Z}^{\mathrm{dd}}$. Similarly, for each tame type τ, $\mathcal{C}^{\tau,\mathrm{BT},1}$ is the underlying reduced substack of each $\mathcal{C}^{\tau,\mathrm{BT},a}$, and of $\mathcal{C}^{\tau,\mathrm{BT}}$, while $\mathcal{Z}^{\tau,1}$ is the underlying reduced substack of each $\mathcal{Z}^{\tau,a}$, and of $\mathcal{Z}^\tau$.*

Proof. The statements for the ϖ-adic formal algebraic stacks follow directly from the corresponding statements for the various algebraic stacks modulo ϖ^a, and so we focus on proving these latter statements, beginning with the case of $\mathcal{C}^{\tau,\mathrm{BT},a}$. Note that $\mathcal{C}^{\tau,\mathrm{BT},1} = \mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{O}/\varpi^a} \mathbf{F}$ is reduced by Corollary 4.5.3, so $\mathcal{C}^{\mathrm{dd},\mathrm{BT},1} = \mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \times_{\mathcal{O}/\varpi^a} \mathbf{F}$ is also reduced by Corollary 4.2.13. The claim follows for $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a}$ from Lemma 5.1.6.

The claims for $\mathcal{Z}^{\tau,a}$ and $\mathcal{Z}^{\mathrm{dd},a}$ are then immediate from Lemma 5.1.7, applied to the morphisms $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{Z}^{\tau,a}$ and $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{Z}^{\mathrm{dd},a}$. $\square$

5.2. **Versal rings and equidimensionality.** We now show that $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ and $\mathcal{Z}^{\mathrm{dd},\mathrm{BT}}$ (and their substacks $\mathcal{C}^{\tau,\mathrm{BT}}$, $\mathcal{Z}^\tau$) are equidimensional, and compute their dimensions, by making use of their versal rings. In [EG21, §5] these versal rings were constructed in a more general setting in terms of liftings of étale φ-modules; in our particular setting, we will find it convenient to interpret them as Galois deformation rings.

We remark that this equidimensionality is not immediate from the local model results of Section 4.5, because we have not shown that the fibres of the local model

map have constant dimensions; establishing this would be closely related to the arguments with versal rings that follow.

Fix a finite type point $x : \operatorname{Spec} \mathbf{F}' \to \mathcal{Z}^{\tau,a}$, where $\mathbf{F}'/\mathbf{F}$ is a finite extension; we also denote the induced finite type point of $\mathcal{R}^{\mathrm{dd},a}$ by x. Let $\overline{r} : G_K \to \operatorname{GL}_2(\mathbf{F}')$ be the Galois representation corresponding to x by Theorem 5.1.2 (3). Let E' be the compositum of E and $W(\mathbf{F}')[1/p]$, with ring of integers $\mathcal{O}_{E'}$ and residue field $\mathbf{F}'$.

We let $R_{\overline{r}}^{\square}$ denote the universal framed deformation $\mathcal{O}_{E'}$-algebra of $\overline{r}$, and we let $R_{\overline{r},0,\tau}^{\square}$ be the reduced and p-torsion free quotient of $R_{\overline{r}}^{\square}$ whose $\overline{\mathbf{Q}}_p$-points correspond to the potentially Barsotti–Tate lifts of $\overline{r}$ of type τ. In this section we will denote $R_{\overline{r},0,\tau}^{\square}$ by the more suggestive name $R_{\overline{r}}^{\tau,\mathrm{BT}}$. We recall, for instance from [BG19, Thm. 3.3.8], that the ring $R_{\overline{r}}^{\tau,\mathrm{BT}}[1/p]$ is regular.

As in Section 2.3, we write $R_{\overline{r}|_{G_{K_\infty}}}$ for the universal framed deformation $\mathcal{O}_{E'}$-algebra for $\overline{r}|_{G_{K_\infty}}$. By Lemma 2.3.3, we have a natural morphism

$$(5.2.1) \qquad \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \to \mathcal{R}^{\mathrm{dd}}.$$

Lemma 5.2.2. *The morphism* (5.2.1) *is versal* (*at* x).

Proof. By definition, it suffices to show that if $\rho : G_{K_\infty} \to \operatorname{GL}_d(A)$ is a representation with A a finite Artinian $\mathcal{O}_{E'}$-algebra, and if $\rho_B : G_{K_\infty} \to \operatorname{GL}_d(B)$ is a second representation, with B a finite Artinian $\mathcal{O}_{E'}$-algebra admitting a surjection onto A, such that the base change ρ_A of ρ_B to A is isomorphic to ρ (more concretely, so that there exists $M \in \operatorname{GL}_d(A)$ with $\rho = M\rho_A M^{-1}$), then we may find $\rho' : G_{K_\infty} \to \operatorname{GL}_d(B)$ which lifts ρ, and is isomorphic to ρ_B. This is straightforward: the natural morphism $\operatorname{GL}_d(B) \to \operatorname{GL}_d(A)$ is surjective, and so if M' is any lift of M to an element of $\operatorname{GL}_d(B)$, then we may set $\rho' = M'\rho_B(M')^{-1}$. $\qquad\square$

Definition 5.2.3. For any pro-Artinian $\mathcal{O}_{E'}$-algebra R with residue field $\mathbf{F}'$ we let $\widehat{\operatorname{GL}}_{2/R}$ denote the completion of $(\operatorname{GL}_2)_{/R}$ along the closed subgroup of its special fibre given by the centraliser of $\overline{r}|_{G_{K_\infty}}$.

Remark 5.2.4. For R as above we have $\widehat{\operatorname{GL}}_{2/R} = \operatorname{Spf} R \times_{\mathcal{O}_{E'}} \widehat{\operatorname{GL}}_{2/\mathcal{O}_{E'}}$. Indeed, if $R \cong \varprojlim_i A_i$, then $\operatorname{Spf} R \times_{\mathcal{O}_{E'}} \widehat{\operatorname{GL}}_{2/\mathcal{O}_{E'}} \cong \varinjlim_i \operatorname{Spec} A_i \times_{\mathcal{O}_{E'}} \widehat{\operatorname{GL}}_{2/\mathcal{O}_{E'}}$, and $\operatorname{Spec} A_i \times_{\mathcal{O}_{E'}} \widehat{\operatorname{GL}}_{2/\mathcal{O}_{E'}}$ agrees with the completion of $(\operatorname{GL}_2)_{/A_i}$ because A_i is a finite $\mathcal{O}_{E'}$-module.

It follows from this that $\widehat{\operatorname{GL}}_{2/R}$ has nice base-change properties more generally: if $R \to S$ is a morphism of pro-Artinian $\mathcal{O}_{E'}$-algebras each with residue field $\mathbf{F}'$, then there is an isomorphism $\widehat{\operatorname{GL}}_{2/S} \cong \operatorname{Spf} S \times_{\operatorname{Spf} R} \widehat{\operatorname{GL}}_{2/R}$. We apply this fact without further comment in various arguments below.

There is a pair of morphisms $\widehat{\operatorname{GL}}_{2/R_{\overline{r}|_{G_{K_\infty}}}} \rightrightarrows \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$, the first being simply the projection to $\operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$, and the second being given by "change of framing". Composing such changes of framing endows $\widehat{\operatorname{GL}}_{2/R_{\overline{r}|_{G_{K_\infty}}}}$ with the structure of a groupoid over $\operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$. Note that the two morphisms

$$\widehat{\operatorname{GL}}_{2/R_{\overline{r}|_{G_{K_\infty}}}} \rightrightarrows \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \xrightarrow{(5.2.1)} \mathcal{R}^{\mathrm{dd}}$$

coincide, since changing the framing does not change the isomorphism class (as a Galois representation) of a deformation of $\overline{r}|_{G_{K_\infty}}$. Thus there is an induced

morphism of groupoids over $\operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$

$$(5.2.5) \qquad \widehat{\operatorname{GL}_2}_{/R_{\overline{r}|_{G_{K_\infty}}}} \to \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}.$$

Lemma 5.2.6. *The morphism* (5.2.5) *is an isomorphism.*

Proof. If A is an Artinian $\mathcal{O}_{E'}$-algebra, with residue field $\mathbf{F}'$, then a pair of A-valued points of $\operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$ map to the same point of $\mathcal{R}^{\mathrm{dd}}$ if and only if they give rise to isomorphic deformations of $\overline{\rho}$, once we forget the framings. But this precisely means that the second of them is obtained from the first by changing the framing via an A-valued point of $\widehat{\operatorname{GL}_2}$. $\square$

It follows from Lemma 5.2.2 that, for each $a \geq 1$, the quotient $R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$ is a (non-Noetherian) versal ring for $\mathcal{R}^{\mathrm{dd},a}$ at x. By [EG21, Lem. 3.2.16], for each $a \geq 1$ a versal ring $R^{\tau,a}$ for $\mathcal{Z}^{\tau,a}$ at x is given by the scheme-theoretic image of the morphism

$$(5.2.7) \qquad \mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a \to \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a,$$

in the sense that we now explain.

In general, the notion of scheme-theoretic image for morphisms of formal algebraic stacks can be problematic; at the very least it should be handled with care. But in this particular context, a definition is given in [EG21, Def. 3.2.15]: we write $R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$ as an inverse limit of Artinian local rings A, form the corresponding scheme-theoretic images of the induced morphisms $\mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spec} A \to \operatorname{Spec} A$, and then take the inductive limit of these scheme-theoretic images; this is a formal scheme, which is in fact of the form $\operatorname{Spf} R^{\tau,a}$ for some quotient $R^{\tau,a}$ of $R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$ (where *quotient* should be understood in the sense of topological rings), and is by definition the scheme-theoretic image in question.

The closed immersions $\mathcal{C}^{\tau,\mathrm{BT},a} \hookrightarrow \mathcal{C}^{\tau,\mathrm{BT},a+1}$ induce corresponding closed immersions

$$\mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a \to \mathcal{C}^{\tau,\mathrm{BT},a+1} \times_{\mathcal{R}^{\mathrm{dd},a+1}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}/\varpi^{a+1},$$

and hence closed immersions of scheme-theoretic images $\operatorname{Spf} R^{\tau,a} \to \operatorname{Spf} R^{\tau,a+1}$, corresponding to surjections $R^{\tau,a+1} \to R^{\tau,a}$. (Here we are using the fact that an projective limit of surjections of finite Artin rings is surjective.) Thus we may form the pro-Artinian ring $\varprojlim_a R^{\tau,a}$. This projective limit is a quotient (again in the sense of topological rings) of $R_{\overline{r}|_{G_{K_\infty}}}$, and the closed formal subscheme $\operatorname{Spf}(\varprojlim R^{\tau,a})$ of $\operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$ is the scheme-theoretic image (computed in the sense described above) of the projection

$$(5.2.8) \qquad \mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \to \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$$

(This is a formal consequence of the construction of the $\operatorname{Spf} R^{\tau,a}$ as scheme-theoretic images, since any discrete Artinian quotient of $R_{\overline{r}|_{G_{K_\infty}}}$ is a discrete Artinian quotient of $R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$, for some $a \geq 1$.) It also follows formally (for example, by the same argument as in the proof of [EG21, Lem. 4.2.14]) that $\varprojlim R^{\tau,a}$ is a versal ring to $\mathcal{Z}^\tau$ at x.

Remark 5.2.9. The same method constructs a versal ring $R^{\mathrm{dd},a}$ for $\mathcal{Z}^{\mathrm{dd},a}$ at x, and each $R^{\tau,a}$ is a quotient of $R^{\mathrm{dd},a}$.

Our next aim is to identify $\varprojlim R^{\tau,a}$ with $R_{\overline{r}}^{\tau,\mathrm{BT}}$. Before we do this, we have to establish some preliminary facts related to the various objects and morphisms we have just introduced.

Lemma 5.2.10.

   (1) *Each of the rings $R^{\tau,a}$ is a complete local Noetherian ring, endowed with its $\mathfrak{m}$-adic topology, and the same is true of the inverse limit $\varprojlim_a R^{\tau,a}$.*

   (2) *For each $a \geq 1$, the morphism $\mathrm{Spf}\, R^{\tau,a} \to \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}$ induces an isomorphism*
   $$\mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \mathrm{Spf}\, R^{\tau,a} \xrightarrow{\sim} \mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a.$$

   (3) *For each $a \geq 1$, the morphism $\mathrm{Spf}\, R^{\tau,a} \to \mathcal{R}^{\mathrm{dd},a}$ is effective, i.e. may be promoted (in a unique manner) to a morphism $\mathrm{Spec}\, R^{\tau,a} \to \mathcal{R}^{\mathrm{dd},a}$, and the induced morphism*
   $$\mathcal{C}^{\tau,a} \times_{\mathcal{R}^{\mathrm{dd},a}} \mathrm{Spec}\, R^{\tau,a} \to \mathrm{Spec}\, R^{\tau,a}$$
   is proper and scheme-theoretically dominant.

   (4) *Each transition morphism $\mathrm{Spec}\, R^{\tau,a} \hookrightarrow \mathrm{Spec}\, R^{\tau,a+1}$ is a thickening.*

Proof. Recall that in Section 2.3.4 we defined a Noetherian quotient $R_{\overline{r}|_{G_{K_\infty}}}^{\leq 1}$ of $R_{\overline{r}|_{G_{K_\infty}}}$, which is naturally identified with the framed deformation ring $R_{\overline{r}}^{[0,1]}$ by Proposition 2.3.5. It follows from [EG21, Lem. 5.4.15] (via an argument almost identical to the one in the proof of [EG21, Prop. 5.4.17]) that the morphism $\mathrm{Spf}\, \varprojlim R^{\tau,a} \hookrightarrow \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}$ factors through $\mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}^{\leq 1} = \mathrm{Spf}\, R_{\overline{r}}^{[0,1]}$, and indeed that $\varprojlim R^{\tau,a}$ is a quotient of $\mathrm{Spf}\, R_{\overline{r}}^{[0,1]}$; this proves (1).

It follows by the very construction of the $R^{\tau,a}$ that the morphism (5.2.7) factors through the closed subscheme $\mathrm{Spf}\, R^{\tau,a}$ of $\mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$. The claim of (2) is a formal consequence of this.

We have already observed that the morphism $\mathrm{Spf}\, R^{\tau,a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through $\mathcal{Z}^{\tau,a}$. This latter stack is algebraic, and of finite type over $\mathcal{O}/\varpi^a$. It follows from [Sta13, Tag 07X8] that the morphism $\mathrm{Spf}\, R^{\tau,a} \to \mathcal{Z}^{\tau,a}$ is effective. Taking into account part (1) of the present lemma, we deduce from the theorem on formal functions that the formal completion of the scheme-theoretic image of the projection
$$\mathcal{C}^{\tau,a} \times_{\mathcal{R}^{\mathrm{dd},a}} \mathrm{Spec}\, R^{\tau,a} \to \mathrm{Spec}\, R^{\tau,a}$$
at the closed point of $\mathrm{Spec}\, R^{\tau,a}$ coincides with the scheme-theoretic image of the morphism
$$\mathcal{C}^{\tau,a} \times_{\mathcal{R}^{\mathrm{dd},a}} \mathrm{Spf}\, R^{\tau,a} \to \mathrm{Spf}\, R^{\tau,a}.$$
Taking into account (2), we see that this latter scheme-theoretic image coincides with $\mathrm{Spf}\, R^{\tau,a}$ itself. This completes the proof of (3).

The claim of (4) follows from a consideration of the diagram

$$
\begin{array}{ccc}
\mathcal{C}^{\tau,a} \times_{\mathcal{R}^{\mathrm{dd},a}} \mathrm{Spec}\, R^{\tau,a} & \longrightarrow & \mathcal{C}^{\tau,a+1} \times_{\mathcal{R}^{\mathrm{dd},a+1}} \mathrm{Spec}\, R^{\tau,a+1} \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, R^{\tau,a} & \longrightarrow & \mathrm{Spec}\, R^{\tau,a+1}
\end{array}
$$

just as in the proof of Lemma 5.1.3. $\qquad\square$

Lemma 5.2.11.

(1) *The projection* $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \mathrm{Spf}\, R_{\overline{\tau}|_{G_{K_\infty}}} \to \mathrm{Spf}\, R_{\overline{\tau}|_{G_{K_\infty}}}$ *factors through a morphism* $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \mathrm{Spf}\, R_{\overline{\tau}|_{G_{K_\infty}}} \to \mathrm{Spf}(\varprojlim R^{\tau,a})$, *which is scheme-theoretically dominant in the sense that its scheme-theoretic image* (*computed in the manner described above*) *is equal to its target.*

(2) *There is a projective morphism of schemes* $X_{\overline{\tau}} \to \mathrm{Spec}(\varprojlim R^{\tau,a})$, *which is uniquely determined, up to unique isomorphism, by the requirement that its* $\mathfrak{m}$-*adic completion* (*where* $\mathfrak{m}$ *denotes the maximal ideal of* $\varprojlim R^{\tau,a}$) *may be identified with the morphism* $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \mathrm{Spf}\, R_{\overline{\tau}|_{G_{K_\infty}}} \to \mathrm{Spf}(\varprojlim R^{\tau,a})$ *of (1).*

Proof. Part (1) follows formally from the various constructions and definitions of the objects involved (just like part (2) of Lemma 5.2.10).

We now consider the morphism

$$\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \mathrm{Spf}\, R_{\overline{\tau}|_{G_{K_\infty}}} \to \mathrm{Spf}(\varprojlim R^{\tau,a}).$$

Once we recall that $\varprojlim R^{\tau,a}$ is Noetherian, by Lemma 5.2.10 (1), it follows exactly as in the proof of [Kis09, Prop. 2.1.10] (which treats the case that τ is the trivial type), via an application of formal GAGA [Gro61, Thm. 5.4.5], that this morphism arises as the formal completion along the maximal ideal of $\varprojlim R^{\tau,a}$ of a projective morphism $X_{\overline{\tau}} \to \mathrm{Spec}(\varprojlim R^{\tau,a})$ (and $X_{\overline{\tau}}$ is unique up to unique isomorphism, by [Gro61, Thm. 5.4.1]). $\qquad\square$

We next establish various properties of the scheme $X_{\overline{\tau}}$ constructed in the previous lemma. To ease notation going forward, we write $\widehat{X}_{\overline{\tau}}$ to denote the fibre product $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \mathrm{Spf}\, R_{\overline{\tau}|_{G_{K_\infty}}}$ (which is reasonable, since this fibre product is isomorphic to the formal completion of $X_{\overline{\tau}}$).

Lemma 5.2.12. *The scheme* $X_{\overline{\tau}}$ *is Noetherian, normal, and flat over* $\mathcal{O}_{E'}$.

Proof. Since $X_{\overline{\tau}}$ is projective over the Noetherian ring $\varprojlim R^{\tau,a}$, it is Noetherian. The other claimed properties of $X_{\overline{\tau}}$ will be deduced from the corresponding properties of $\mathcal{C}^{\tau,\mathrm{BT}}$ that are proved in Corollary 4.5.3.

To this end, we first note that, since the morphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{R}^{\mathrm{dd}}$ factors through $\mathcal{Z}^\tau$, it follows (for example as in the proof of [EG21, Lem. 3.2.16]) that we have isomorphisms

$$\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^\tau} \mathrm{Spf}(\varprojlim R^{\tau,a}) \xrightarrow{\sim} \mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^\tau} \mathcal{Z}^\tau \times_{\mathcal{R}^{\mathrm{dd}}} \mathrm{Spf}\, R_{\overline{\tau}|_{G_{K_\infty}}}$$

$$\xrightarrow{\sim} \mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \mathrm{Spf}\, R_{\overline{\tau}|_{G_{K_\infty}}} =: \widehat{X}_{\overline{\tau}}.$$

In summary, we may identify $\widehat{X}_{\overline{\tau}}$ with the fibre product $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^\tau} \mathrm{Spf}(\varprojlim R^{\tau,a})$.

We now show that $\widehat{X}_{\overline{\tau}}$ is analytically normal. To see this, let $\mathrm{Spf}\, B \to \widehat{X}_{\overline{\tau}}$ be a morphism whose source is a Noetherian affine formal algebraic space, which is representable by algebraic spaces and smooth. We must show that the completion $\widehat{B}_{\mathfrak{n}}$ is normal, for each maximal ideal $\mathfrak{n}$ of B. In fact, it suffices to verify this for some collection of such $\mathrm{Spf}\, B$ which cover $\widehat{X}_{\overline{\tau}}$, and so without loss of generality we may choose our B as follows: first, choose a collection of morphisms $\mathrm{Spf}\, A \to \mathcal{C}^{\tau,\mathrm{BT}}$ whose sources are Noetherian affine formal algebraic spaces, and which are representable

by algebraic spaces and smooth, which, taken together, cover $\mathcal{C}^{\tau,\mathrm{BT}}$. Next, for each such A, choose a collection of morphisms

$$\mathrm{Spf}\, B \to \mathrm{Spf}_A \times_{\mathcal{C}^{\tau,\mathrm{BT}}} \widehat{X}_{\overline{r}}$$

whose sources are Noetherian affine formal algebraic spaces, and which are representable by algebraic spaces and smooth, which, taken together, cover the fibre product. Altogether (considering all such B associated to all such A), the composite morphisms

$$\mathrm{Spf}\, B \to \mathrm{Spf}_A \times_{\mathcal{C}^{\tau,\mathrm{BT}}} \widehat{X}_{\overline{r}} \to \widehat{X}_{\overline{r}}$$

are representable by algebraic spaces and smooth, and cover $\widehat{X}_{\overline{r}}$.

Now, let $\mathfrak{n}$ be a maximal ideal in one of these rings B, lying over a maximal ideal $\mathfrak{m}$ in the corresponding ring A. The extension of residue fields $A/\mathfrak{m} \to B/\mathfrak{n}$ is finite, and each of these fields is finite over $\mathbf{F}'$. Enlarging $\mathbf{F}'$ sufficiently, we may assume that in fact each of these residue fields coincides with $\mathbf{F}'$. (On the level of rings, this amounts to forming various tensor products of the form $- \otimes_{W(\mathbf{F}')} W(\mathbf{F}'')$, which doesn't affect the question of normality.) The morphism $\mathrm{Spf}\, B_\mathfrak{n} \to \mathrm{Spf}\, A_\mathfrak{n}$ is then seen to be smooth in the sense of [Sta13, Tag 06HG], i.e., it satisfies the infinitesimal lifting property for finite Artinian $\mathcal{O}'$-algebras with residue field $\mathbf{F}'$: this follows from the identification of $\widehat{X}_{\overline{r}}$ above as a fibre product, and the fact that $\mathrm{Spf}(\varprojlim R^{\tau,a}) \to \mathcal{Z}^\tau$ is versal at the closed point x. Thus $\mathrm{Spf}\, B_\mathfrak{n}$ is a formal power series ring over $\mathrm{Spf}\, A_\mathfrak{m}$, by [Sta13, Tag 06HL], and hence $\mathrm{Spf}\, B_\mathfrak{n}$ is indeed normal, since $\mathrm{Spf}\, A_\mathfrak{m}$ is so, by Corollary 4.5.3. By Lemma 5.2.15 below, this implies that the algebraization $X_{\overline{r}}$ of $\widehat{X}_{\overline{r}}$ is normal.

We next claim that the morphism

$$(5.2.13) \qquad\qquad \mathrm{Spf}(\varprojlim R^{\tau,a}) \to \mathcal{Z}^\tau$$

is a flat morphism of formal algebraic stacks, in the sense of [Eme, Def. 8.35]. Given this, we find that the base-changed morphism $\widehat{X}_{\overline{r}} \to \mathcal{C}^{\tau,\mathrm{BT}}$ is also flat. Since Corollary 4.5.3 shows that $\mathcal{C}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}_{E'}$, we conclude that the same is true of $\widehat{X}_{\overline{r}}$. Again, by Lemma 5.2.15, this implies that the algebraization $X_{\overline{r}}$ is also flat over $\mathcal{O}_{E'}$.

It remains to show the claimed flatness. To this end, we note first that for each $a \geq 1$, the morphism

$$(5.2.14) \qquad\qquad \mathrm{Spf}\, R^{\tau,a} \to \mathcal{Z}^{\tau,a}$$

is a versal morphism from a complete Noetherian local ring to an algebraic stack which is locally of finite type over $\mathcal{O}/\varpi^a$. We already observed in the proof of Lemma 5.2.10 (3) that (5.2.14) is effective, i.e. can be promoted to a morphism $\mathrm{Spec}\, R^{\tau,a} \to \mathcal{Z}^{\tau,a}$. It then follows from [Sta13, Tag 0DR2] that this latter morphism is flat, and thus that (5.2.14) is flat in the sense of [Eme, Def. 8.35]. It follows easily that the morphism (5.2.13) is also flat: use the fact that a morphism of ϖ-adically complete local Noetherian $\mathcal{O}$-algebras which becomes flat upon reduction modulo ϖ^a, for each $a \geq 1$, is itself flat, which follows from (for example) [Sta13, Tag 0523]. $\qquad\square$

The following lemma is standard, and is presumably well-known. We sketch the proof, since we don't know a reference.

Lemma 5.2.15. *If S is a complete Noetherian local $\mathcal{O}$-algebra and $Y \to \operatorname{Spec} S$ is a proper morphism of schemes, then Y is flat over $\operatorname{Spec} \mathcal{O}$ (resp. normal) if and only $\widehat{Y}$ (the $\mathfrak{m}_S$-adic completion of Y) is flat over $\operatorname{Spf} \mathcal{O}$ (resp. is analytically normal).*

Proof. The properties of Y that are in question can be tested by considering the various local rings $\mathcal{O}_{Y,y}$, as y runs over the points of Y; namely, we have to consider whether or not these rings are flat over $\mathcal{O}$, or normal. Since any point y specializes to a closed point y_0 of Y, so that $\mathcal{O}_{Y,y}$ is a localization of $\mathcal{O}_{Y,y_0}$, and thus $\mathcal{O}$-flat (resp. normal) if $\mathcal{O}_{Y,y_0}$ is, it suffices to consider the rings $\mathcal{O}_{Y,y_0}$ for closed points y_0 of Y. Note also that since Y is proper over $\operatorname{Spec} S$, any closed point of Y lies over the closed point of $\operatorname{Spec} S$.

Now let $\operatorname{Spec} A$ be an affine neighbourhood of a closed point y_0 of Y; let $\mathfrak{m}$ be the corresponding maximal ideal of A. As we noted, $\mathfrak{m}$ lies over $\mathfrak{m}_S$, and so gives rise to a maximal ideal $\widehat{\mathfrak{m}} := \mathfrak{m}\widehat{A}$ of $\widehat{A}$, the $\mathfrak{m}_S$-adic completion of A; and any maximal ideal of $\widehat{A}$ contains $\mathfrak{m}_S\widehat{A}$, and so arises from a maximal ideal of A in this manner (since $A/\mathfrak{m}_S \xrightarrow{\sim} \widehat{A}/\mathfrak{m}_S$). Write $\widehat{A}_{\mathfrak{m}}$ to denote the $\mathfrak{m}$-adic completion of A (which maps isomorphically to the $\widehat{\mathfrak{m}}$-adic completion of $\widehat{A}$). Then $\widehat{A}$ is faithfully flat over the localization $A_{\mathfrak{m}} = \mathcal{O}_{Y,y_0}$, and hence $A_{\mathfrak{m}}$ is flat over $\mathcal{O}$ if and only if $\widehat{A}_{\mathfrak{m}}$ is. Consequently we see that Y is flat over $\mathcal{O}$ if and only if, for each affine open subset $\operatorname{Spec} A$ of Y, the corresponding $\mathfrak{m}_S$-adic completion $\widehat{A}$ becomes flat over $\mathcal{O}$ after completing at each of its maximal ideals. Another application of faithful flatness of completions of Noetherian local rings shows that this holds if and only if each such $\widehat{A}$ is flat over $\mathcal{O}$ after localizing at each of its maximal ideals, which holds if and only each such $\widehat{A}$ is flat over $\mathcal{O}$. This is precisely what it means for $\widehat{Y}$ to be flat over $\mathcal{O}$.

The proof that analytic normality of $\widehat{Y}$ implies that Y is normal is similar. Indeed, analytic normality by definition means that the completion of $\widehat{A}$ at each of its maximal ideals is normal. This completion is faithfully flat over the localization of $\operatorname{Spec} A$ at its corresponding maximal ideal, and so [Sta13, Tag 033G] implies that this localization is also normal. The discussion of the first paragraph then implies that Y is normal. For the converse direction, we have to deduce normality of the completions $\widehat{A}_{\mathfrak{m}}$ from the normality of the corresponding localizations $A_{\mathfrak{m}}$. This follows from that fact that Y is an excellent scheme (being of finite type over the complete local ring S), so that each A is an excellent ring [Sta13, Tag 0C23]. $\square$

Proposition 5.2.16. *The projective morphism $X_{\overline{r}} \to \operatorname{Spec} R_{\overline{r}}^{[0,1]}$ factors through a projective and scheme-theoretically dominant morphism*

$$(5.2.17) \qquad\qquad X_{\overline{r}} \to \operatorname{Spec} R_{\overline{r}}^{\tau,\mathrm{BT}}$$

which becomes an isomorphism after inverting ϖ.

Proof. We begin by showing the existence of (5.2.17), and that it induces a bijection on closed points after inverting ϖ. Since $X_{\overline{r}}$ is $\mathcal{O}$-flat, by Lemma 5.2.12, it suffices to show that the induced morphism

$$\operatorname{Spec} E \times_{\mathcal{O}} X_{\overline{r}} \to \operatorname{Spec} R_{\overline{r}}^{[0,1]}[1/\varpi]$$

factors through a morphism

$$(5.2.18) \qquad\qquad \operatorname{Spec} E \times_{\mathcal{O}} X_{\overline{r}} \to \operatorname{Spec} R_{\overline{r}}^{\tau,\mathrm{BT}}[1/\varpi],$$

which induces a bijection on closed points.

This can be proved in exactly the same way as [Kis09, Prop. 2.4.8], which treats the case that τ is trivial. Indeed, the computation of the D_{cris} of a Galois representation in the proof of [Kis09, Prop. 2.4.8] goes over essentially unchanged to the case of a Galois representation coming from $\mathcal{C}^{\tau,\mathrm{BT}}$, and finite type points of $\operatorname{Spec} R_{\overline{r}}^{\tau,\mathrm{BT}}[1/\varpi]$ yield p-divisible groups and thus Breuil–Kisin modules exactly as in the proof of [Kis09, Prop. 2.4.8] (bearing in mind Lemma 4.2.16 above). The tame descent data comes along for the ride.

The morphism (5.2.18) is a projective morphism whose target is Jacobson, and which induces a bijection on closed points. It is thus proper and quasi-finite, and hence finite. Its source is reduced (being even normal, by Lemma 5.2.12), and its target is normal (as it is even regular, as we noted above). A finite morphism whose source is reduced, whose target is normal and Noetherian, and which induces a bijection on finite type points, is indeed an isomorphism. (The connected components of a normal scheme are integral, and so base-changing over the connected components of the target, we may assume that the target is integral. The source is a union of finitely many irreducible components, each of which has closed image in the target. Since the morphism is surjective on finite type points, it is surjective, and thus one of these closed images coincides with the target. The injectivity on finite type points then shows that the source is also irreducible, and thus integral, as it is reduced. It follows from [Sta13, Tag 0AB1] that the morphism is an isomorphism.) Thus (5.2.18) is an isomorphism. Finally, since $R_{\overline{r}}^{\tau,\mathrm{BT}}$ is also flat over $\mathcal{O}$ (by its definition), this implies that (5.2.17) is scheme-theoretically dominant. $\qquad\square$

Corollary 5.2.19. $\varprojlim R^{\tau,a} = R_{\overline{r}}^{\tau,\mathrm{BT}}$; thus $R_{\overline{r}}^{\tau,\mathrm{BT}}$ is a versal ring to $\mathcal{Z}^{\tau}$ at x.

Proof. The theorem on formal functions shows that if we write the scheme-theoretic image of (5.2.17) in the form $\operatorname{Spec} B$, for some quotient B of $R_{\overline{r}|G_{K_\infty}}$, then the scheme-theoretic image of the morphism (5.2.8) coincides with $\operatorname{Spf} B$. The corollary then follows from Proposition 5.2.16, which shows that (5.2.17) is scheme-theoretically dominant. $\qquad\square$

Proposition 5.2.20. *The algebraic stacks $\mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau,a}$ are equidimensional of dimension $[K:\mathbf{Q}_p]$.*

Proof. Let x be a finite type point of $\mathcal{Z}^{\tau,a}$, defined over some finite extension $\mathbf{F}'$ of $\mathbf{F}$, and corresponding to a Galois representation $\overline{r}$ with coefficients in $\mathbf{F}'$. By Corollary 5.2.19 the ring $R_{\overline{r}}^{\tau,\mathrm{BT}}$ coincides with the versal ring $\varprojlim_a R^{\tau,a}$ at x of the ϖ-adic formal algebraic stack $\mathcal{Z}^{\tau}$, and so $\operatorname{Spf} R^{\tau,a} \overset{\sim}{\longrightarrow} \operatorname{Spf} R_{\overline{r}}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^{\tau}} \mathcal{Z}^{\tau,a}$. Since $\mathcal{Z}^{\tau}$ is a ϖ-adic formal algebraic stack, the natural morphism $\mathcal{Z}^{\tau,1} \to \mathcal{Z}^{\tau} \times_{\operatorname{Spf}\mathcal{O}} \mathbf{F}$ is a thickening, and thus the same is true of the morphism $\operatorname{Spf} R^{\tau,1} \to \operatorname{Spf} R_{\overline{r}}^{\tau,\mathrm{BT}}/\varpi$ obtained by pulling the former morphism back over $\operatorname{Spf} R_{\overline{r}}^{\tau,\mathrm{BT}}/\varpi$.

Since $R_{\overline{r}}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}_{E'}$ and equidimensional of dimension $5+[K:\mathbf{Q}_p]$, it follows that $R^{\tau,1}$ is equidimensional of dimension $4+[K:\mathbf{Q}_p]$. The same is then true of each $R^{\tau,a}$, since these are thickenings of $R^{\tau,1}$, by Lemma 5.2.10 (4).

We have a versal morphism $\operatorname{Spf} R^{\tau,a} \to \mathcal{Z}^{\tau,a}$ at the finite type point x of $\mathcal{Z}^{\tau,a}$. It follows from Lemma 5.2.6 that

$$\widehat{\mathrm{GL}}_{2/\operatorname{Spf} R^{\tau,a}} \overset{\sim}{\longrightarrow} \operatorname{Spf} R^{\tau,a} \times_{\mathcal{Z}^{\tau,a}} \operatorname{Spf} R^{\tau,a}.$$

To find the dimension of $\mathcal{Z}^{\tau,a}$ it suffices to compute its dimension at finite type points (*cf.* [Sta13, Tag 0DRX], recalling the definition of the dimension of an

algebraic stack, [Sta13, Tag 0AFP]). It follows from [EG17, Lem. 2.40] applied to the presentation [Spf $R^{\tau,a}/\widehat{\mathrm{GL}}_{2/\mathrm{Spf}\,R^{\tau,a}}$] of $\widehat{\mathcal{Z}}_x^{\tau,a}$, together with Remark 5.2.4, that $\mathcal{Z}^{\tau,a}$ is equidimensional of dimension $[K:\mathbf{Q}_p]$. Since $\mathcal{Z}^{\mathrm{dd},a}$ is the union of the $\mathcal{Z}^{\tau,a}$ by Theorem 5.1.2, $\mathcal{Z}^{\mathrm{dd},a}$ is also equidimensional of dimension $[K:\mathbf{Q}_p]$ by [Sta13, Tag 0DRZ]. $\qquad\square$

Proposition 5.2.21. *The algebraic stacks $\mathcal{C}^{\tau,\mathrm{BT},a}$ are equidimensional of dimension $[K:\mathbf{Q}_p]$.*

Proof. Let x' be a finite type point of $\mathcal{C}^{\tau,\mathrm{BT},a}$, defined over some finite extension $\mathbf{F}'$ of $\mathbf{F}$, lying over the finite type point x of $\mathcal{Z}^{\tau,a}$. Let $\bar{r}$ be the Galois representation with coefficients in $\mathbf{F}'$ corresponding to x, and recall that $X_{\bar{r}}$ denotes a projective Spec $R_{\bar{r}}^{\tau,\mathrm{BT}}$-scheme whose pull-back $\widehat{X}_{\bar{r}}$ over Spf $R_{\bar{r}|_{G_{K_\infty}}}$ is isomorphic to $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}}$ Spf $R_{\bar{r}|_{G_{K_\infty}}}$. The point x' gives rise to a closed point $\tilde{x}$ of $X_{\bar{r}}$ (of which x' is the image under the morphism $X_{\bar{r}} \to \mathcal{C}^{\tau,\mathrm{BT}}$). Let $\widehat{\mathcal{O}}_{X_{\bar{r}},\tilde{x}}$ denote the complete local ring to $X_{\bar{r}}$ at the point $\tilde{x}$; then the natural morphism Spf $\widehat{\mathcal{O}}_{X_{\bar{r}},\tilde{x}} \to \mathcal{C}^{\tau,\mathrm{BT}}$ is versal at $\tilde{x}$, so that $\widehat{\mathcal{O}}_{X_{\bar{r}},\tilde{x}}/\varpi^a$ is a versal ring for the point x' of $\mathcal{C}^{\tau,\mathrm{BT},a}$.

The isomorphism (5.2.5) induces (after pulling back over $\mathcal{C}^{\tau,\mathrm{BT}}$) an isomorphism

$$\widehat{\mathrm{GL}}_{2/\widehat{X}_{\bar{r}}} \xrightarrow{\sim} \widehat{X}_{\bar{r}} \times_{\mathcal{C}^{\tau,\mathrm{BT}}} \widehat{X}_{\bar{r}},$$

and thence an isomorphism

$$\widehat{\mathrm{GL}}_{2/\widehat{\mathcal{O}}_{X_{\bar{r}},\tilde{x}}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X_{\bar{r}},\tilde{x}} \times_{\mathcal{C}^{\tau,\mathrm{BT}}} \widehat{\mathcal{O}}_{X_{\bar{r}},\tilde{x}}.$$

Since $R^{\tau,\mathrm{BT}}$ is equidimensional of dimension $5 + [K:\mathbf{Q}_p]$, it follows from Proposition 5.2.16 that $X_{\bar{r}}$ is equidimensional of dimension $5 + [K:\mathbf{Q}_p]$, and thus (taking into account the flatness statement of Lemma 5.2.12) that $\widehat{\mathcal{O}}_{X_{\bar{r}},\tilde{x}}/\varpi^a$ is equidimensional of dimension $4 + [K:\mathbf{Q}_p]$. As in the proof of Proposition 5.2.20, an application of [EG17, Lem. 2.40] shows that $\dim_{x'}\mathcal{C}^{\tau,\mathrm{BT},a}$ is equal to $[K:\mathbf{Q}_p]$. Since x' was an arbitrary finite type point, the result follows. $\qquad\square$

Appendix A. Très ramifiée representations and Serre weights

To complete our characterization in Theorem 5.1.2(3) of the $\mathbf{F}$-points of the stack $\mathcal{Z}^{\mathrm{dd},1}$, in this appendix we give a global argument that requires a brief detour into the theory of Serre weights.

By definition, a *Serre weight* is an irreducible $\mathbf{F}$-representation of $\mathrm{GL}_2(k)$, where we recall that k denotes the residue field of K. Concretely, such a representation is of the form

$$\overline{\sigma}_{\vec{t},\vec{s}} := \otimes_{j=0}^{f-1}(\det{}^{t_j}\mathrm{Sym}^{s_j}k^2) \otimes_{k,\sigma_j} \mathbf{F},$$

where $0 \le s_j, t_j \le p-1$ and not all t_j are equal to $p-1$. We say that a Serre weight is *Steinberg* if $s_j = p-1$ for all j, and *non-Steinberg* otherwise.

We now recall the definition of the set of Serre weights $W(\bar{r})$ associated to a representation $\bar{r}: G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$.

Definition A.1. *We say that a crystalline representation $r: G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has* type $\overline{\sigma}_{\vec{t},\vec{s}}$ *provided that for each embedding $\sigma_j: k \hookrightarrow \mathbf{F}$ there is an embedding $\widetilde{\sigma}_j: K \hookrightarrow \overline{\mathbf{Q}}_p$ lifting σ_j such that the $\widetilde{\sigma}_j$-labeled Hodge–Tate weights of r are $\{-s_j - t_j, 1 - t_j\}$, and the remaining $(e-1)f$ pairs of Hodge–Tate weights of r are*

all $\{0, 1\}$. (*In particular the representations of type $\overline{\sigma}_{\vec{0}, \vec{0}}$ (the trivial weight) are the same as those of Hodge type 0.*)

Definition A.2. *Given a representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ we define $W(\overline{r})$ to be the set of Serre weights $\overline{\sigma}$ such that $\overline{r}$ has a crystalline lift of type $\overline{\sigma}$.*

It follows easily from the formula $\overline{\sigma}_{\vec{t}, \vec{s}}^{\vee} = \overline{\sigma}_{-\vec{s}-\vec{t}, \vec{s}}$ that $\overline{\sigma} \in W(\overline{r})$ if and only if $\overline{\sigma}^{\vee}$ is in the set of Serre weights associated to $\overline{r}^{\vee}$ in [GLS15, Defn. 4.1.3].

Let $\sigma(\tau)$ denote $\mathrm{GL}_2(\mathcal{O}_K)$-representation associated to τ by Henniart's inertial local Langlands correspondence [BM02], and let $\overline{\sigma}(\tau)$ denote the special fibre of any lattice in $\sigma(\tau)$, well-defined up to semi-simplification.

Remark A.2.1. There are several definitions of the set $W(\overline{r})$ in the literature, which by the papers [BLGG13, GK14, GLS15] are known to be equivalent (up to normalisation). However the normalisations of Hodge–Tate weights and of inertial local Langlands used in [GK14, GLS15] are not all the same. Our conventions for Hodge–Tate weights and inertial types agree with those of [GK14], but our representation $\sigma(\tau)$ is the representation $\sigma(\tau^{\vee})$ of [GK14] (where $\tau^{\vee} = \eta^{-1} \oplus (\eta')^{-1}$); to see this, note the dual in the definition of $\sigma(\tau)$ in [GK14, Thm. 2.1.3].

On the other hand, our set of weights $W(\overline{r})$ is the set of duals of the weights $W(\overline{r})$ considered in [GK14]. In turn, the paper [GLS15] has the opposite convention for the signs of Hodge–Tate weights to our convention (and to the convention of [GK14]), so we find that our set of weights $W(\overline{r})$ is the set of duals of the weights $W(\overline{r}^{\vee})$ considered in [GLS15].

We have the following standard definition.

Definition A.3. *We say that $\overline{r}$ is* très ramifiée *if it is a twist of an extension of the trivial character by the mod p cyclotomic character, and if furthermore the splitting field of its projective image is* not *of the form $K(\alpha_1^{1/p}, \ldots, \alpha_s^{1/p})$ for some $\alpha_1, \ldots, \alpha_s \in \mathcal{O}_K^{\times}$.*

The following characterisation of très ramifiée representations presumably admits a purely local proof, but we find it convenient to make a global argument.

Lemma A.4.

    (1) If τ is a tame type, then $\overline{r}$ has a potentially Barsotti–Tate lift of type τ if and only if $W(\overline{r}) \cap \mathrm{JH}(\overline{\sigma}(\tau)) \neq 0$.

    (2) The following conditions are equivalent:

        (a) $\overline{r}$ admits a potentially Barsotti–Tate lift of some tame type.

        (b) $W(\overline{r})$ contains a non-Steinberg Serre weight.

        (c) $\overline{r}$ is not très ramifiée.

Proof. (1) By the main result of [GLS15], and bearing in mind the differences between our conventions and those of [GK14] as recalled in Remark A.2.1, we have $\overline{\sigma} \in W(\overline{r})$ if and only if $\overline{\sigma}^{\vee} \in W^{\mathrm{BT}}(\overline{r})$, where $W^{\mathrm{BT}}(\overline{r})$ is the set of weights defined in [GK14, §3]. By [GK14, Cor. 3.5.6] (bearing in mind once again the differences between our conventions and those of [GK14]), it follows that we have $W(\overline{r}) \cap \mathrm{JH}(\overline{\sigma}(\tau)) \neq 0$ if and only if $e(R^{\square, 0, \tau}/\pi) \neq 0$ in the notation of *loc. cit.*, and by definition $\overline{r}$ has a potentially Barsotti–Tate lift of type τ if and only if $R^{\square, 0, \tau} \neq 0$.

It follows from [EG14, Prop. 4.1.2] that $R^{\square,0,\tau} \neq 0$ if and only if $e(R^{\square,0,\tau}/\pi) \neq 0$, as required.

(2) By part (1), condition (a) is equivalent to $W(\bar{r})$ containing a Serre weight occurring as a Jordan–Hölder factor of $\bar{\sigma}(\tau)$ for some tame type τ. It is easy to see (from explicit tables of the Serre weights occurring as Jordan–Hölder factors of tame types, as computed e.g. in [Dia07]) that the Serre weights occurring as Jordan–Hölder factors of the $\bar{\sigma}(\tau)$ are precisely the non-Steinberg Serre weights, so (a) and (b) are equivalent.

Suppose that (a) holds; then $\bar{r}$ becomes finite flat over a tame extension. However the restriction to a tame extension of a très ramifiée representation is still très ramifiée, and therefore not finite flat, so (c) also holds. Conversely, suppose for the sake of contradiction that (c) holds, but that (b) does not hold, i.e. that $W(\bar{r})$ consists of a single Steinberg weight.

Twisting, we can without loss of generality assume that $W(\bar{r}) = \{\bar{\sigma}_{\vec{0},p\vec{-}1}\}$. By [GK14, Cor. A.5] we can globalise $\bar{r}$, and then the hypothesis that $W(\bar{r})$ contains $\bar{\sigma}_{\vec{0},p\vec{-}1}$ implies that it has a semistable lift of Hodge type 0. If this lift were in fact crystalline, then $W(\bar{r})$ would also contain the weight $\bar{\sigma}_{\vec{0},\vec{0}}$ by (1). So this lift is not crystalline, and in particular the monodromy operator N on the corresponding weakly admissible module is nonzero. But then $\ker(N)$ is a free filtered submodule of rank 1, and since the lift has Hodge type 0, $\ker(N)$ is in fact a weakly admissible submodule. It follows that the lift is an unramified twist of an extension of ε^{-1} by the trivial character, so that $\bar{r}$ is an unramified twist of an extension of $\bar{\varepsilon}^{-1}$ by the trivial character. But we are assuming that (c) holds, so $\bar{r}$ is finite flat, so that by (1), $W(\bar{r})$ contains the weight $\bar{\sigma}_{\vec{0},\vec{0}}$, a contradiction. $\qquad\square$

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