SERRE WEIGHTS FOR RANK TWO UNITARY GROUPS.

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ABSTRACT. We study the weight part of (a generalisation of) Serre's conjecture for mod l Galois representations associated to automorphic representations on rank two unitary groups for odd primes l. We propose a conjectural set of Serre weights, agreeing with all conjectures in the literature, and under a mild assumption on the image of the mod l Galois representation we are able to show that any modular representation is modular of each conjectured weight. We make no assumptions on the ramification or inertial degrees of l. Our main innovations are the use of outer forms of GL_2 to avoid the parity difficulties with inner forms, and the use of the lifting techniques and "change of weight" results introduced in [BLGG11], [BLGG12] and [BLGGT10].

CONTENTS

1. INTRODUCTION.

1.1. Serre's conjecture ([Ser87]), now a theorem of Khare–Wintenberger and Kisin, predicts that certain mod l Galois representations are modular, in the sense that they arise from modular forms. There is a folklore generalisation of this conjecture to Hilbert modular forms (see e.g. the introduction to [BDJ10]), which appears to be beyond reach at present. However, in recent years there has been considerable progress on proving generalisations of the weight part of Serre's conjecture for mod l representations corresponding to automorphic representations of GL_2; that is, of determining the possible weights of automorphic forms giving rise to a Galois representation which is assumed to be modular.

Such a generalisation was initially formulated in [BDJ10], for Hilbert modular forms over a totally real field F^+ in which l is unramified, and was largely proved in [Gee11b]. A generalisation of the conjecture of [BDJ10] for tamely ramified Galois representations was proposed in [Sch08], and in the case that l is totally ramified

in F^+ this conjecture was mostly proved in [GS]. In his University of Arizona PhD thesis, Ryan Smith used essentially the same argument to prove some cases when the inertial and ramification indexes are both two.

While these results represent a considerable advance on our understanding of 2-dimensional mod l Galois representations, they are limited in several respects. Firstly, it seems to be hopeless to expect to be able to push the methods of proof to work over a general totally real field. This is not merely aesthetically unsatisfactory; it also limits the applicability of the results, for example limiting the options of combining them with base change techniques, or of applying them to generalisations of the arguments of Khare and Wintenberger which proved Serre's conjecture over $\mathbb{Q}$. Secondly, the techniques of [Gee11b] do not allow one to prove results for all weights, but only for weights which are sufficiently regular; in applications, for example to modularity lifting theorems and the Breuil-Mezard conjecture (cf. [Kis10]), one often needs a result for all weights. Finally, the methods employed in these earlier papers entail some exceedingly unpleasant combinatorial and p-adic Hodge theoretic calculations.

In the present paper we resolve most of these difficulties, proving a very general theorem about the weight part of Serre's conjecture for rank two unitary groups. These groups are outer forms of GL_2 over totally real fields, as opposed to the inner forms studied in the papers discussed above. We choose to use these groups for two reasons. Firstly, we have developed a considerable body of material on automorphy lifting theorems for these groups in our recent work ([BLGG11], [BLGG12], [BLGGT10]). Secondly, the relationship between the weights of mod l Galois representations and l-adic Galois representations is simpler than for the inner forms, because there is no obstruction coming from the units in the totally real field (this can already be seen for GL_1: one has considerably more flexibility to choose the weights of an algebraic character over an imaginary CM field than over a totally real field).

Our main theorem is as follows (see Theorem 5.1.3). Given a modular representation $\bar{r}$, we define a set of Serre weights $W^{\mathrm{explicit}}(\bar{r})$, which is the set of predicted weights for $\bar{r}$ from the papers [BDJ10], [Sch08] and [EGHS13].

Theorem A. *Let F be an imaginary CM field with maximal totally real subfield F^+. Assume that F does not contain a primitive l-th root of unity ζ_l, that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Assume that $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let a be a Serre weight. Assume that $a \in W^{\mathrm{explicit}}(\bar{r})$. Then $\bar{r}$ is modular of weight a.

(See Sections 2 and 4 for any unfamiliar terminology.) Note in particular that if $l \geq 7$, the hypothesis that $\bar{r}(G_{F(\zeta_l)})$ is adequate may be replaced by the usual Taylor-Wiles assumption that $\bar{r}|_{G_{F(\zeta_l)}}$ is irreducible.

Our approach is related to that of [Gee11b], in that we prove that a mod l Galois representation is modular of a given weight by producing l-adic lifts with certain properties. In [Gee11b] we were forced to work with potentially Barsotti-Tate lifts, due to our dependence on the modularity lifting theorems proved in [Kis09] and [Gee06]. This led to much of the combinatorial difficulties mentioned above, which in turn limited us to working over a totally real field in which l is unramified.

Thanks to the techniques developed in our previous papers, and in particular the lifting theorems proved in [BLGGT10], in the present paper we are able to produce lifts of arbitrary weight. This, and our use of outer forms of GL_2 rather than inner forms, completely removes the combinatorial difficulties, as we now explain.

Let F be an imaginary CM field with maximal totally real subfield F^+. Assume that F/F^+ is unramified at all finite places and split at all places lying over l, and that $[F^+ : \mathbb{Q}]$ is even. In section 2 below we define a certain rank two unitary group G over F^+, which is compact at all infinite places and quasisplit at all finite places, and split over F. It is thus split at all places dividing l, so there is a natural notion of a Serre weight a, which is an irreducible representation of the product of the $GL_2(k_v)$, where v runs over the places of F dividing l. We have a notion of an irreducible mod l Galois representation $\bar{r} : G_F \to GL_2(\overline{\mathbb{F}}_l)$ being modular of some Serre weight, in terms of algebraic modular forms on G. An elementary, but extremely useful, fact is that any Serre weight a can be lifted to a characteristic 0 weight λ (that is, to an irreducible algebraic representation of $GL_2(\mathcal{O}_{F^+,l})$). Since G is compact, it is easy to check that $\bar{r}$ being modular of weight a is equivalent to $\bar{r}$ having a lift which corresponds to an automorphic representation of weight λ and level prime to l, and by the theory of base change this is equivalent to $\bar{r}$ having a lift which corresponds to a conjugate-self dual automorphic representation of $GL_2(\mathbb{A}_F)$ of weight λ and level prime to l. It is key here that we are working with outer forms of GL_2; if we were using Hilbert modular forms over F^+, the usual parity issues would intervene. We remark that the results of this paper have subsequently been used, in combination with the results of [GLS12a], to prove the Buzzard–Diamond–Jarvis conjecture for inner forms of GL_2 ([GK12],[New]).

The weight part of Serre's conjecture thus reduces to a question about the existence of automorphic lifts of $\bar{r}$ with specific local properties; the condition that the corresponding automorphic representation has weight λ and level prime to l translates to the condition that the Galois representation be crystalline with Hodge–Tate weights determined by λ. This gives an obvious necessary condition for $\bar{r}$ to be modular of weight a: for each place $v|l$ of F, $\bar{r}|_{G_{F_v}}$ must have a crystalline lift of the appropriate Hodge–Tate weights. Following [Gee11a], we conjecture that this condition is also sufficient.

Our main result in this direction, which we prove via the "change of weight" techniques of [BLGGT10], is that subject to mild hypotheses on the image $\bar{r}(G_F)$, if $\bar{r}$ is assumed to be modular and if for each place $v|l$ of F, $\bar{r}|_{G_{F_v}}$ has a potentially diagonalizable crystalline lift of the appropriate Hodge–Tate weights, then $\bar{r}$ is modular of weight a. We refer the reader to section 3 for the definition of the term "potentially diagonalizable", which was introduced in [BLGGT10]. This result is a straightforward consequence of the above discussion and the results of [BLGGT10], together with the results of [Kis09] and [Gee06] (which show that $\bar{r}$ necessarily has some automorphic lift which is potentially diagonalizable).

Since we do not know if every crystalline representation is potentially diagonalizable, it is not immediately clear how useful the above result is. Accordingly, we examine the explicit conjectures made in [BDJ10], [Sch08] and [EGHS13], and note that in (almost) every case, whenever the conjectures made in those papers suggest that $\bar{r}$ should be modular of weight a, we can find potentially diagonalizable crystalline lifts of the correct Hodge–Tate weights. Indeed, we can find potentially

diagonalizable lifts of a particularly simple kind: they are either an extension of two characters, or are induced from a character.

Accordingly, we have reduced the weight part of Serre's conjecture in this setting to a purely local question, of determining whether if a mod l Galois representation has a crystalline lift with specified Hodge–Tate weights (constrained to lie in a particular range), it has one which is furthermore potentially diagonalizable. We strongly suspect that this question has an affirmative answer. In the 2-dimensional cases at hand, this is presumably accessible via a brute force calculation in integral p-adic Hodge theory. We do, however, completely determine the list of weights when the absolute ramification index of each prime v of F dividing l is at least l, and for each such v the representation $\bar{r}|_{G_{F_v}}$ is semisimple. Note that one can always reduce to this case by base change, which may make this result particularly valuable in applications.

We remark that some of the above discussion carries over to rank n unitary groups for arbitrary n. However, there are several difficulties with obtaining results as strong as those obtained here. Firstly, the correspondence between weights in characteristic 0 and characteristic l is less simple: there are irreducible $\overline{\mathbb{F}}_l$-representations of $\mathrm{GL}_n(\mathbb{F}_l)$ which do not lift to irreducible $\overline{\mathbb{Q}}_l$-representations. Secondly, we do not know that every modular $\bar{r}$ has an automorphic lift which is potentially diagonalizable. Nonetheless, our methods give non-trivial results for general n, which we will explain in a subsequent paper.

We now explain the structure of this paper. In section 2, we define the unitary groups that we use, and recall some basic facts about the automorphic representations and Galois representations that we use. In section 3 we deduce the main lifting theorem that we need from the results of [BLGGT10]. In section 4 we explain the explicit Serre weight conjectures in the literature, and write down various explicit potentially diagonalizable representations. In Section 5 we deduce our main explicit theorems. Finally, in Appendix A we discuss the adequate subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$ for $l = 3$ and $l = 5$, and we improve on a result of [BLGGT10]; this allows us to treat the cases $l = 3, 5$ in this paper, whereas a direct appeal to the results of [BLGGT10] would force us to assume that $l \geq 7$. The results proved in the appendix are used in the proof of Theorem 3.1.2, which in turn is an input to Theorem 4.1.9, which in turn is applied to prove Theorem A.

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1.2. **Notation and conventions.** If M is a field, we let G_M denote its absolute Galois group. We write all matrix transposes on the left; so ^tA is the transpose of A. Let ϵ_l denote the l-adic cyclotomic character, and $\bar{\epsilon}_l$ or ω_l the mod l cyclotomic character. If M is a finite extension of $\mathbb{Q}_p$ for some prime p, we write I_M for the inertia subgroup of G_M. If M and K are algebraic extensions of $\mathbb{Q}_p$, then all homomorphisms $M \to K$ are assumed to be continuous for the p-adic topology. If R is a local ring we write $\mathfrak{m}_R$ for the maximal ideal of R.

If K is a finite extension of $\mathbb{Q}_p$, we will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible admissible complex representation of $\mathrm{GL}_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Weil-Deligne representation of the Weil group W_K. We will write rec for rec_K when the choice of K is clear. We write $\mathrm{Art}_K : K^\times \to W_K$

for the isomorphism of local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements. If (V, r, N) is a Weil-Deligne representation of W_K over some algebraically closed field of characteristic zero, then we define its Frobenius semisimplification $(V, r, N)^{F-ss}$ (resp. its semisimplification $(V, r, N)^{ss}$) as in section 1 of [TY07].

Let W be a continuous finite-dimensional representation of G_K over $\overline{\mathbb{Q}}_l$ for some prime l. If $p = l$, assume that W is de Rham. Then we denote by $\mathrm{WD}(W)$ the Weil-Deligne representation associated to W. Assume now that $p = l$. If $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ is a continuous embedding, then by definition the multiset $\mathrm{HT}_\tau(W)$ of Hodge–Tate weights of W with respect to τ contains i with multiplicity $\dim_{\overline{\mathbb{Q}}_l}(W \otimes_{\tau, K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example $\mathrm{HT}_\tau(\epsilon_l) = \{-1\}$.

We will let ζ_l denote a primitive l-th root of unity. If F is an imaginary CM field with maximal totally real subfield F^+, we will write c for the nontrivial element of $\mathrm{Gal}\,(F/F^+)$, which is given by complex conjugation.

2. Definitions

2.1.   Let $l > 2$ be a prime, and let F be an imaginary CM field with maximal totally real field subfield F^+. We assume throughout this paper that:

- F/F^+ is unramified at all finite places.
- Every place $v|l$ of F^+ splits in F.
- $[F^+ : \mathbb{Q}]$ is even.

Under these hypotheses, there is a reductive algebraic group G/F^+ with the following properties:

- G is an outer form of GL_2, with $G_{/F} \cong \mathrm{GL}_{2/F}$.
- If v is a finite place of F^+, G is quasi-split at v.
- If v is an infinite place of F^+, then $G(F_v^+) \cong U_2(\mathbb{R})$.

To see that such a group exists, one may argue as follows. Let B denote the matrix algebra $M_2(F)$. An involution $\ddagger$ of the second kind on B gives a reductive group $G_\ddagger$ over F^+ by setting

$$G_\ddagger(R) = \{g \in B \otimes_{F^+} R : g^\ddagger g = 1\}$$

for any F^+-algebra R. Any such $G_\ddagger$ is an outer form of GL_2, with $G_{\ddagger/F} \cong \mathrm{GL}_{2/F}$. One can choose $\ddagger$ such that

- If v is a finite place of F^+, $G_\ddagger$ is quasi-split at v.
- If v is an infinite place of F^+, then $G_\ddagger(F_v^+) \cong U_2(\mathbb{R})$.

To see this, one uses the argument of Lemma I.7.1 of [HT01]; it is here that we require the hypotheses that F/F^+ is unramified at all finite places, and that $[F^+ : \mathbb{Q}]$ is even. We then fix some choice of $\ddagger$ as above, and take $G = G_\ddagger$.

As in section 3.3 of [CHT08] we define a model for G over $\mathcal{O}_{F^+}$ in the following way. We choose an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^\ddagger = \mathcal{O}_B$, and $\mathcal{O}_{B,w}$ is a maximal order in B_w for all places w of F which are split over F^+ (see section 3.3 of [CHT08] for a proof that such an order exists). Then we can define G over $\mathcal{O}_{F^+}$ by setting

$$G(R) = \{g \in \mathcal{O}_B \otimes_{\mathcal{O}_{F^+}} R : g^\ddagger g = 1\}$$

for any $\mathcal{O}_{F^+}$-algebra R.

If v is a place of F^+ which splits as ww^c over F, then we choose an isomorphism

$$\iota_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_2(\mathcal{O}_{F,v}) = M_2(\mathcal{O}_{F_w}) \oplus M_2(\mathcal{O}_{F_{w^c}})$$

such that $\iota_v(x^\ddagger) = {}^t\iota_v(x)^c$. This gives rise to an isomorphism

$$\iota_w : G(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_2(\mathcal{O}_{F_w})$$

sending $\iota_v^{-1}(x, {}^t x^{-c})$ to x.

Let K be an algebraic extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ which contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$, let $\mathcal{O}$ denote the ring of integers of K, and let k denote the residue field of K. Let S_l denote the set of places of F^+ lying over l, and for each $v \in S_l$ fix a place $\widetilde{v}$ of F lying over v. Let $\tilde{S}_l$ denote the set of places $\widetilde{v}$ for $v \in S_l$.

Let W be an $\mathcal{O}$-module with an action of $G(\mathcal{O}_{F^+,l})$, and let $U \subset G(\mathbb{A}_{F^+}^\infty)$ be a compact open subgroup with the property that for each $u \in U$, if u_l denotes the projection of u to $G(F_l^+)$, then $u_l \in G(\mathcal{O}_{F_l^+})$. Let $S(U, W)$ denote the space of algebraic modular forms on G of level U and weight W, i.e. the space of functions

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty) \to W$$

with $f(gu) = u_l^{-1} f(g)$ for all $u \in U$.

Let $\widetilde{I}_l$ denote the set of embeddings $F \hookrightarrow K$ giving rise to a place in $\widetilde{S}_l$. For any $\widetilde{v} \in \tilde{S}_l$, let $\widetilde{I}_{\widetilde{v}}$ denote the set of elements of $\widetilde{I}_l$ lying over $\widetilde{v}$. We can naturally identify $\widetilde{I}_{\widetilde{v}}$ with $\mathrm{Hom}\,(F_{\widetilde{v}}, \overline{\mathbb{Q}}_l)$. Let $\mathbb{Z}_+^2$ denote the set of pairs (λ_1, λ_2) of integers with $\lambda_1 \geq \lambda_2$. If Ω is an algebraically closed field of characteristic 0 we write $(\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F,\Omega)}$ for the subset of elements $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(F,\Omega)}$ such that

$$\lambda_{\tau,1} + \lambda_{\tau \circ c, 2} = 0$$

for all τ. Note that we can identify $(\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ with $(\mathbb{Z}_+^2)^{\widetilde{I}_l}$ in a natural fashion. If λ is an element of $(\mathbb{Z}_+^2)^{\widetilde{I}_l}$ (resp. $(\mathbb{Z}_+^2)^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$) and $w \in \widetilde{S}_l$ (resp. $w|l$) is a place of F, we define $\lambda_w \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(F_w, K)}$ to be $(\lambda_\sigma)_\sigma$ with σ running over all embeddings $F \hookrightarrow K$ inducing w.

If $w|l$ is a place of F and $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(F_w, \overline{\mathbb{Q}}_l)}$, let W_λ be the free $\mathcal{O}$-module with an action of $\mathrm{GL}_2(\mathcal{O}_{F_w})$ given by

$$W_\lambda := \otimes_{\tau \in \mathrm{Hom}\,(F_w, \overline{\mathbb{Q}}_l)} \det{}^{\lambda_{\tau,2}} \otimes \mathrm{Sym}^{\lambda_{\tau,1} - \lambda_{\tau,2}} \mathcal{O}_{F_w}^2 \otimes_{\mathcal{O}_{F_w}, \tau} \mathcal{O}.$$

If $v = w|_{F^+}$, we give this an action of $G(\mathcal{O}_{F^+, v})$ via ι_w. If $\lambda \in (\mathbb{Z}_+^2)^{\widetilde{I}_l}$, we let W_λ be the free $\mathcal{O}$-module with an action of $G(\mathcal{O}_{F^+, l})$ given by

$$W_\lambda := \otimes_{\widetilde{v} \in \tilde{S}_l} W_{\lambda_{\widetilde{v}}}.$$

If A is an $\mathcal{O}$-module we let

$$S_\lambda(U, A) := S(U, W_\lambda \otimes_{\mathcal{O}} A).$$

For any compact open subgroup U as above of $G(\mathbb{A}_{F^+}^\infty)$ we may write $G(\mathbb{A}_{F^+}^\infty) = \coprod_i G(F^+) t_i U$ for some finite set $\{t_i\}_{i \in I}$. Then there is an isomorphism

$$S(U, W) \to \oplus_{i \in I} W^{U \cap t_i^{-1} G(F^+) t_i}$$

given by $f \mapsto (f(t_i))_{i \in I}$. We say that U is *sufficiently small* if for some finite place v of F^+ the projection of U to $G(F_v^+)$ contains no element of finite order other than the identity. Suppose that U is sufficiently small. Then for each i as above we have $U \cap t_i^{-1} G(F^+) t_i = \{1\}$, so taking $W = W_\lambda \otimes_{\mathcal{O}} A$ we see that for any $\mathcal{O}$-algebra A, we have

$$S_\lambda(U, A) \cong S_\lambda(U, \mathcal{O}) \otimes_{\mathcal{O}} A.$$

We note when U is not sufficiently small, we still have $S_\lambda(U, A) \cong S_\lambda(U, \mathcal{O}) \otimes_\mathcal{O} A$ whenever A is $\mathcal{O}$-flat.

We now recall the relationship between our spaces of algebraic modular forms and the space of automorphic forms on G. Write $S_\lambda(\overline{\mathbb{Q}}_l)$ for the direct limit of the spaces $S_\lambda(U, \overline{\mathbb{Q}}_l)$ over compact open subgroups U as above (with the transition maps being the obvious inclusions $S_\lambda(U, \overline{\mathbb{Q}}_l) \subset S_\lambda(V, \overline{\mathbb{Q}}_l)$ whenever $V \subset U$). Concretely, $S_\lambda(\overline{\mathbb{Q}}_l)$ is the set of functions

$$f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \to W_\lambda \otimes_\mathcal{O} \overline{\mathbb{Q}}_l$$

such that there is a compact open subgroup U of $G(\mathbb{A}_{F^+}^{\infty,l}) \times G(\mathcal{O}_{F^+,l})$ with

$$f(gu) = u_l^{-1} f(g)$$

for all $u \in U$, $g \in G(\mathbb{A}_{F^+}^\infty)$. This space has a natural left action of $G(\mathbb{A}_{F^+}^\infty)$ via

$$(g \cdot f)(h) := g_l f(hg).$$

Fix an isomorphism $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$, there is a unique embedding $\tilde{\tau} : F \hookrightarrow \mathbb{C}$ extending τ such that $\imath^{-1}\tilde{\tau} \in \tilde{I}_l$. Let $\sigma_{\imath\lambda}$ denote the representation of $G(F_\infty^+)$ given by $W_\lambda \otimes_\mathcal{O} \overline{\mathbb{Q}}_l \otimes_{\overline{\mathbb{Q}}_l,\imath} \mathbb{C}$, with an element $g \in G(F_\infty^+)$ acting via $\otimes_\tau \tilde{\tau}(\iota_{\tilde{\tau}}(g))$. Let $\mathcal{A}$ denote the space of automorphic forms on $G(F^+)\backslash G(\mathbb{A}_{F^+})$. From the proof of Proposition 3.3.2 of [CHT08], one easily obtains the following.

Lemma 2.1.1. *There is an isomorphism of $G(\mathbb{A}_{F^+}^\infty)$-modules*

$$\imath S_\lambda(\overline{\mathbb{Q}}_l) \xrightarrow{\sim} \mathrm{Hom}_{G(F_\infty^+)}(\sigma_{\imath\lambda}^\vee, \mathcal{A}).$$

In particular, we note that $S_\lambda(\overline{\mathbb{Q}}_l)$ is a semisimple admissible $G(\mathbb{A}_{F^+}^\infty)$-module.

We now recall from [CHT08] the notion of a RACSDC automorphic representation. We say that an automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ is

- *regular algebraic* if π_∞ has the same infinitesimal character as some irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$;
- *conjugate self dual* if $\pi^c \cong \pi^\vee$.

If π satisfies both of these properties and is also cuspidal, we well say that π is *RACSDC* (regular, algebraic, conjugate self dual and cuspidal). We say that π *has level prime to l* if π_v is unramified for all $v|l$.

If $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(F,\mathbb{C})}$ we write Σ_λ for the irreducible algebraic representation of $\mathrm{GL}_2^{\mathrm{Hom}\,(F,\mathbb{C})} \cong \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2 \times_\mathbb{Q} \mathbb{C}$ given by the tensor product over τ of the irreducible representations with highest weights λ_τ; i.e. of the representations

$$\det{}^{\lambda_{\tau,2}} \otimes \mathrm{Sym}^{\lambda_{\tau,1}-\lambda_{\tau,2}} \mathbb{C}^2.$$

We say that a RACSDC automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ *has weight* $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(F,\mathbb{C})}$ if π_∞ has the same infinitesimal character as $\Sigma_\lambda^\vee$. If this is the case then necessarily $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$.

Theorem 2.1.2. *If π is a RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight λ, then there is a continuous irreducible representation*

$$r_{l,\imath}(\pi) : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$$

such that

(1) $r_{l,\imath}(\pi)^c \cong r_{l,\imath}(\pi)^\vee \otimes \epsilon_l^{-1}$.

(2) *The representation $r_{l,\iota}(\pi)$ is de Rham, and is crystalline if π has level prime to l. If $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ then*

$$\mathrm{HT}_\tau(r_{l,\iota}(\pi)) = \{\lambda_{\iota\tau,1} + 1, \lambda_{\iota\tau,2}\}.$$

(3) *If $v \nmid l$ then*

$$\iota\mathrm{WD}(r_{l,\iota}(\pi)|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}(\pi_v \otimes |\det|^{-1/2}).$$

(4) *If $v | l$ then*

$$\iota\mathrm{WD}(r_{l,\iota}(\pi)|_{G_{F_v}})^{ss} \cong \mathrm{rec}(\pi_v \otimes |\det|^{-1/2})^{ss}.$$

Proof. This follows immediately (as the special case $n = 2$ of the general results proved for RACSDC representations of GL_n) from the main results of [Shi11], [CH09], [Car12] and [BLGGT12]. □

After conjugating, we may assume that $r_{l,\iota}(\pi)$ takes values in $\mathrm{GL}_2(\mathcal{O}_{\overline{\mathbb{Q}}_l})$. Composing with the map $\mathrm{GL}_2(\mathcal{O}_{\overline{\mathbb{Q}}_l}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ and semisimplifying, we obtain a representation $\overline{r}_{l,\iota}(\pi) : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ which is independent of any choices made.

We say that a continuous irreducible representation $r : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ (respectively $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$) is *automorphic* if $r \cong r_{l,\iota}(\pi)$ (respectively $\overline{r} \cong \overline{r}_{l,\iota}(\pi)$) for some RACSDC representation π of $\mathrm{GL}_2(\mathbb{A}_F)$. We say that a continuous irreducible representation $r : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ is *automorphic of weight* $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ if $r \cong r_{l,\iota}(\pi)$ for some RACSDC representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\iota\lambda$. By Théorème 3.13 of [Clo90], these notions do not depend on the choice of ι.

The theory of base change gives a close relationship between automorphic representations of $G(\mathbb{A}_{F^+})$ and automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. For example, one has the following consequences of Corollaire 5.3 and Théorème 5.4 of [Lab09].

Theorem 2.1.3. *Suppose that π is a RACSDC representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$. Then there is an automorphic representation Π of $G(\mathbb{A}_{F^+})$ such that*

(1) *For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and each $\hat{\tau} \hookrightarrow \mathbb{C}$ extending τ, we have $\Pi_\tau \cong \Sigma_{\lambda_{\hat{\tau}}}^\vee \circ \iota_{\hat{\tau}}$.*

(2) *If v is a finite place of F^+ which splits as ww^c in F, then $\Pi_v \cong \pi_w \circ \iota_w$.*

(3) *If v is a finite place of F^+ which is inert in F, and π_v is unramified, then Π_v has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^+)$.*

Theorem 2.1.4. *Suppose that Π is an automorphic representation of $G(\mathbb{A}_{F^+})$. Then there is a regular algebraic, conjugate self dual automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of some weight $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$ such that either*

(a) *π is cuspidal, or*

(b) *$\pi = \chi_1 \boxplus \chi_2$ is the isobaric direct sum of characters $\chi_1, \chi_2 : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ (cf. Definition 1.2 of [Clo90] for this notion),*

and in either case we have:

(1) *For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and each $\hat{\tau} \hookrightarrow \mathbb{C}$ extending τ, we have $\Pi_\tau \cong \Sigma_{\lambda_{\hat{\tau}}}^\vee \circ \iota_{\hat{\tau}}$.*

(2) *If v is a finite place of F^+ which splits as ww^c in F, then $\Pi_v \cong \pi_w \circ \iota_w$.*

(3) If v is a finite place of F^+ which is inert in F, and Π_v has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^+)$, then π_v is unramified.

We now wish to define what it means for an irreducible representation $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ to be modular of some weight. In order to do so, we return to the spaces of algebraic modular forms considered before. For each place $w|l$ of F, let k_w denote the residue field of F_w. If w lies over a place v of F^+, write $v = ww^c$. Let $(\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ denote the subset of $(\mathbb{Z}_+^2)^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ consisting of elements a such that for each $w|l$, if $\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)$ then

$$a_{\sigma,1} + a_{\sigma c,2} = 0.$$

If $a \in (\mathbb{Z}_+^2)^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ and $w|l$ is a place of F, then we denote by a_w the element $(a_\sigma)_{\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)} \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$.

If $\mathbb{F}$ is a finite extension of $\mathbb{F}_l$, we say that an element $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)}$ is a *Serre weight* if for each $\sigma \in \mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)$ we have

$$l - 1 \geq a_{\sigma,1} - a_{\sigma,2}.$$

If $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)}$ is a Serre weight then we define an irreducible $\overline{\mathbb{F}}_l$-representation F_a of $\mathrm{GL}_2(\mathbb{F})$ by

$$F_a := \otimes_{\tau \in \mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)} \det{}^{a_{\tau,2}} \otimes \mathrm{Sym}^{a_{\tau,1} - a_{\tau,2}} \mathbb{F}^2 \otimes_{\mathbb{F}, \tau} \overline{\mathbb{F}}_l.$$

We say that two Serre weights a and b are *equivalent* if and only if $F_a \cong F_b$ as representations of $\mathrm{GL}_2(\mathbb{F})$. This is equivalent to demanding that for each $\sigma \in \mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)$, we have

$$a_{\sigma,1} - a_{\sigma,2} = b_{\sigma,1} - b_{\sigma,2},$$

and the character $\mathbb{F}^\times \to \overline{\mathbb{F}}_l^\times$ given by

$$x \mapsto \prod_{\sigma \in \mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)} \sigma(x)^{a_{\sigma,2} - b_{\sigma,2}}$$

is trivial. If L is a finite extension of $\mathbb{Q}_l$ with residue field $\mathbb{F}$, we say that an element $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(L, \overline{\mathbb{Q}}_l)}$ is a *lift* of an element $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)}$ if for each $\sigma \in \mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)$ there is an element $\tau \in \mathrm{Hom}\,(L, \overline{\mathbb{Q}}_l)$ lifting σ such that $\lambda_\tau = a_\sigma$, and for all other $\tau' \in \mathrm{Hom}\,(L, \overline{\mathbb{Q}}_l)$ lifting σ we have $\lambda_{\tau'} = 0$.

We say that an element $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ is a *Serre weight* if a_w is a Serre weight for each $w|l$. If $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ is a Serre weight, we define an irreducible $\overline{\mathbb{F}}_l$-representation F_a of $G(\mathcal{O}_{F^+,l})$ as follows: we define

$$F_a = \otimes_{\widetilde{v} \in \widetilde{S}_l} F_{a_{\widetilde{v}}},$$

an irreducible representation of $\prod_{\widetilde{v} \in \widetilde{S}_l} \mathrm{GL}_2(k_{\widetilde{v}})$, and we let $G(\mathcal{O}_{F^+,l})$ act on $F_{a_{\widetilde{v}}}$ by the composition of $\iota_{\widetilde{v}}$ and the map $\mathrm{GL}_2(\mathcal{O}_{F_{\widetilde{v}}}) \to \mathrm{GL}_2(k_{\widetilde{v}})$. Again, we say that two Serre weights a and b are equivalent if and only if $F_a \cong F_b$ as representations of $G(\mathcal{O}_{F^+,l})$. This is equivalent to demanding that for each place $w|l$ and each $\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)$ we have

$$a_{\sigma,1} - a_{\sigma,2} = b_{\sigma,1} - b_{\sigma,2},$$

and the character $k_w^\times \to \overline{\mathbb{F}}_l^\times$ given by

$$x \mapsto \prod_{\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)} \sigma(x)^{a_{\sigma_2} - b_{\sigma_2}}$$

is trivial. We say that a weight $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)}$ is a *lift* of a Serre weight $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ if for each $w|l$, λ_w is a lift of a_w.

For the rest of this section, fix $K = \overline{\mathbb{Q}}_l$.

Definition 2.1.5. We say that a compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$ is *good* if $U = \prod_v U_v$ with U_v a compact open subgroup of $G(F_v^+)$ such that:

- $U_v \subset G(\mathcal{O}_{F_v^+})$ for all v which split in F;
- $U_v = G(\mathcal{O}_{F_v^+})$ if $v|l$;
- U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$ if v is inert in F.

Let U be a good compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$. Let T be a finite set of finite places of F^+ which split in F, containing S_l and all the places v which split in F for which $U_v \neq G(\mathcal{O}_{F_v^+})$. We let $\mathbb{T}^{T,univ}$ be the commutative $\mathcal{O}$-polynomial algebra generated by formal variables $T_w^{(j)}$ for all $1 \leq j \leq 2$, w a place of F lying over a place v of F^+ which splits in F and is not contained in T. For any $\lambda \in (\mathbb{Z}_+^n)^{\tilde{I}_l}$ (resp. any Serre weight $a \in (\mathbb{Z}_+^2)_0^{\coprod_{v|l} \mathrm{Hom}\,(k_v, \overline{\mathbb{F}}_l)}$), the algebra $\mathbb{T}^{T,univ}$ acts on $S_\lambda(U, \mathcal{O})$ (resp. $S(U, F_a)$) via the Hecke operators

$$T_w^{(j)} := \iota_w^{-1} \left[GL_2(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{2-j} \end{pmatrix} GL_2(\mathcal{O}_{F_w}) \right]$$

for $w \notin T$ and ϖ_w a uniformiser in $\mathcal{O}_{F_w}$.

Suppose that $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}^{T,univ}$ with residue field $\overline{\mathbb{F}}_l$ such that $S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} \neq 0$. Then (cf. Proposition 3.4.2 of [CHT08]) by Lemma 2.1.1, Theorem 2.1.4, and Theorem 2.1.2, there is a continuous semisimple representation

$$\bar{r}_\mathfrak{m} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$$

associated to $\mathfrak{m}$, which is uniquely determined by the properties that:

- $\bar{r}_\mathfrak{m}^c \cong \bar{r}_\mathfrak{m}^\vee \bar{\epsilon}_l^{-1}$,
- for all finite places w of F not lying over T, $\bar{r}_\mathfrak{m}|_{G_{F_w}}$ is unramified, and
- if w is a finite place of F which doesn't lie over T and which splits over F^+, then the characteristic polynomial of $\bar{r}_\mathfrak{m}(\mathrm{Frob}_w)$ is

$$X^2 - T_w^{(1)} X + (\mathbf{N}w) T_w^{(2)}.$$

Lemma 2.1.6. *Suppose that U is sufficiently small, and let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_\lambda^{T,univ}$ with residue field $\overline{\mathbb{F}}_l$. Suppose that $a \in (\mathbb{Z}_+^2)_0^{\coprod_{v|l} \mathrm{Hom}\,(k_v, \overline{\mathbb{F}}_l)}$ is a Serre weight, and that $\lambda \in (\mathbb{Z}_+^2)^{\tilde{I}_l}$ is a lift of a. Then*

$$S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} \neq 0$$

if and only if

$$S(U, F_a)_\mathfrak{m} \neq 0.$$

Proof. Since $\overline{\mathbb{Q}}_l$ is l-torsion free, we have $S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} = S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_\mathfrak{m} \otimes \overline{\mathbb{Q}}_l$, so $S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} \neq 0$ if and only if $S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_\mathfrak{m} \neq 0$. Since U is sufficiently small, $S_\lambda(U, \overline{\mathbb{F}}_l)_\mathfrak{m} \neq 0$ if and only if $S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_\mathfrak{m} \neq 0$, so that $S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} \neq 0$ if and only if $S_\lambda(U, \overline{\mathbb{F}}_l)_\mathfrak{m} \neq 0$.

It then suffices to note that there is a natural isomorphism of $G(\mathcal{O}_{F^+, l})$-representations $W_\lambda \otimes_{\mathcal{O}_{\overline{\mathbb{Q}}_l}} \overline{\mathbb{F}}_l \xrightarrow{\sim} F_a$, so that we obtain a $\mathbb{T}^{T, univ}$-equivariant isomorphism $S_\lambda(U, \overline{\mathbb{F}}_l) \xrightarrow{\sim} S(U, F_a)$. $\square$

We have the following definitions.

Definition 2.1.7. If R is a commutative ring and $r : G_F \to \mathrm{GL}_2(R)$ is a representation, we say that r has *split ramification* if $r|_{G_{F_w}}$ is unramified for any finite place $w \in F$ which does not split over F^+.

Definition 2.1.8. If π is a RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, we say that π has *split ramification* if π_w is unramified for any finite place $w \in F$ which does not split over F^+.

Definition 2.1.9. Suppose that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation. Then we say that $\bar{r}$ *is modular of weight* $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ if a is a Serre weight and there is a sufficiently small, good level U, a set of places T as above and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T, univ}$ with residue field $\overline{\mathbb{F}}_l$ such that

- $S(U, F_a)_\mathfrak{m} \neq 0$, and
- $\bar{r} \cong \bar{r}_\mathfrak{m}$.

(Note that $\bar{r}_\mathfrak{m}$ exists by Lemma 2.1.6.) We say that $\bar{r}$ is *modular* if it is modular of some weight.

Remark 2.1.10. Note that if $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is modular then $\bar{r}$ must have split ramification and $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{-1}$. Note also that this definition is independent of the choice of $\tilde{S}_l$ (because $F_{a_{\tilde{v}}} \circ \iota_{\tilde{v}} \cong F_{a_{c\tilde{v}}} \circ \iota_{c\tilde{v}}$, we see that F_a itself is independent of the choice of $\tilde{S}_l$).

Lemma 2.1.11. *Suppose that* $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ *is a continuous irreducible representation with split ramification. Let* $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ *be a Serre weight, and let* $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)}$ *be a lift of* a. *Then* $\bar{r}$ *is modular of weight* $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ *if and only if there is a RACSDC automorphic representation* π *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *of weight* $\iota\lambda$ *and level prime to* l *which has split ramification, and which satisfies* $\bar{r}_{l,\iota}(\pi) \cong \bar{r}$.

Proof. Suppose firstly that $\bar{r}$ is modular of weight a. Then by definition there is a good U and a T as above with U sufficiently small, and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T, univ}$ with residue field $\overline{\mathbb{F}}_l$ such that

- $S(U, F_a)_\mathfrak{m} \neq 0$, and
- $\bar{r} \cong \bar{r}_\mathfrak{m}$.

By Lemma 2.1.6, the first condition is equivalent to $S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} \neq 0$. Define a compact open subgroup $U' = \prod_w U'_w$ of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ by

- $U'_w = \mathrm{GL}_2(\mathcal{O}_{F_w})$ if w is not split over F^+.
- $U'_w = \iota_w(U_{w|_{F^+}})$ if w splits over F^+.

By Lemma 2.1.1, Theorem 2.1.4, and Theorem 2.1.2, there is a RACSDC automorphic representation π of weight $\imath\lambda$ which satisfies $\bar{r}_{l,\iota}(\pi) \cong \bar{r}$, and $\pi_w^{U'_w} \neq 0$ for all finite places w of F. Since U is good, we see that π has level prime to l, and it has split ramification, as required.

Conversely, suppose that there is a RACSDC automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\imath\lambda$ which has split ramification and level prime to l with $\bar{r}_{l,\iota}(\pi) \cong \bar{r}$. Let $U = \prod_v U_v$ be a compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$ such that:

- $U_v = G(\mathcal{O}_{F_v^+})$ if v is inert in F;
- if v splits as $v = ww^c$ in F, then $\pi_w^{\iota_w(U_v)} \neq (0)$;
- there is a finite place v' of F which splits as $w'w'^c$ in F and is such that
  - v' lies above a rational prime p with $[F(\zeta_p) : F] > 2$, and
  - $\iota_{w'}(U_{v'}) = \ker(\mathrm{GL}_2(\mathcal{O}_{w'}) \to \mathrm{GL}_2(k_{w'}))$.

The third bullet point implies that U is sufficiently small. Then by Lemma 2.1.1 and Theorem 2.1.3 we have $S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} \neq 0$. The result follows from Lemma 2.1.6. □

3. A LIFTING THEOREM

3.1. We recall some terminology from [BLGGT10], specialized to the crystalline (as opposed to potentially crystalline) case. Fix a prime l. Let K be a finite extension of $\mathbb{Q}_l$, and let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$, with residue field k. Assume that for each continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_l$, the image is contained in the field of fractions of $\mathcal{O}$.

Let $\overline{\rho} : G_K \to \mathrm{GL}_n(k)$ be a continuous representation, and let $R_{\mathcal{O},\overline{\rho}}^\square$ be the universal $\mathcal{O}$-lifting ring. Let $\{H_\tau\}$ be a collection of n-element multisets of integers parametrized by $\tau \in \mathrm{Hom}(K, \overline{\mathbb{Q}}_l)$. Then $R_{\mathcal{O},\overline{\rho}}^\square$ has a unique quotient $R_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}}^\square$ which is reduced and without l-torsion and such that a $\overline{\mathbb{Q}}_l$-point of $R_{\mathcal{O},\overline{\rho}}^\square$ factors through $R_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}}^\square$ if and only if it corresponds to a representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ which is crystalline and has $\mathrm{HT}_\tau(\rho) = H_\tau$ for all $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$. We will write $R_{\overline{\rho},\{H_\tau\},\mathrm{cris}}^\square \otimes \overline{\mathbb{Q}}_l$ for $R_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}}^\square \otimes_\mathcal{O} \overline{\mathbb{Q}}_l$. This definition is independent of the choice of $\mathcal{O}$. The scheme $\mathrm{Spec}\,(R_{\overline{\rho},\{H_\tau\},\mathrm{cris}}^\square \otimes \overline{\mathbb{Q}}_l)$ is formally smooth over $\overline{\mathbb{Q}}_l$. (See [Kis08].)

If $\rho_i : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ are continuous representations for $i = 1, 2$, we say that ρ_1 *connects to* ρ_2, which we denote $\rho_1 \sim \rho_2$, if and only if

- the reduction $\overline{\rho}_1 := \rho_1 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$ is equivalent to the reduction $\overline{\rho}_2 := \rho_2 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$;
- ρ_1 and ρ_2 are both crystalline;
- for each $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\rho_1) = \mathrm{HT}_\tau(\rho_2)$;
- and ρ_1 and ρ_2 define points on the same irreducible component of the scheme $\mathrm{Spec}\,(R_{\overline{\rho}_1,\{\mathrm{HT}_\tau(\rho_1)\},\mathrm{cris}}^\square \otimes \overline{\mathbb{Q}}_l)$.

(In this last bullet point, we mean that ρ_1 and $A\rho_2 A^{-1}$ define points on the same irreducible component of $\mathrm{Spec}\,(R_{\overline{\rho}_1,\{\mathrm{HT}_\tau(\rho_1)\},\mathrm{cris}}^\square \otimes \overline{\mathbb{Q}}_l)$ where $A \in \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ is such that $A\overline{\rho}_2 A^{-1} = \overline{\rho}_1$. This condition is independent of the choice of A by Lemma 1.2.1 of [BLGGT10].) As in section 1.4 of [BLGGT10], we have the following:

(1) The relation $\rho_1 \sim \rho_2$ does not depend on the $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugacy class of ρ_1 or ρ_2.
(2) $\sim$ is symmetric and transitive.

(3) If K'/K is a finite extension and $\rho_1 \sim \rho_2$ then $\rho_1|_{G_{K'}} \sim \rho_2|_{G_{K'}}$.

(4) If $\rho_1 \sim \rho_2$ and $\rho_1' \sim \rho_2'$ then $\rho_1 \oplus \rho_1' \sim \rho_2 \oplus \rho_2'$ and $\rho_1 \otimes \rho_1' \sim \rho_2 \otimes \rho_2'$ and $\rho_1^\vee \sim \rho_2^\vee$.

(5) If $\mu : G_K \to \overline{\mathbb{Q}}_l^\times$ is a continuous unramified character with $\overline{\mu} = 1$ and ρ_1 is crystalline then $\rho_1 \sim \rho_1 \otimes \mu$.

(6) Suppose that ρ_1 crystalline and that $\overline{\rho}_1$ is semisimple. Let Fil^i be an invariant filtration on ρ_1 by $\mathcal{O}_{\overline{\mathbb{Q}}_l}$ direct summands. Then $\rho_1 \sim \bigoplus_i \mathrm{gr}^i \rho_1$.

We will call a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *diagonal* if it is of the form $\chi_1 \oplus \cdots \oplus \chi_n$ with $\chi_i : G_K \to \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$. We will call a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *diagonalizable* if it connects to some diagonal representation. We will call a representation $\rho : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *potentially diagonalizable* if there is a finite extension K'/K such that $\rho|_{G_{K'}}$ is diagonalizable. Note that if K''/K is a finite extension and ρ is diagonalizable (resp. potentially diagonalizable) then $\rho|_{G_{K''}}$ is diagonalizable (resp. potentially diagonalizable).

As in [BLGGT10], we make the following convention: Suppose that F is a global field and that $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ is a continuous representation with irreducible reduction $\overline{r}$. In this case there is model $r^\circ : G_F \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of r, which is unique up to $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugation. If $v|l$ is a place of F we write $r|_{G_{F_v}} \sim \rho_2$ to mean $r^\circ|_{G_{F_v}} \sim \rho_2$. We will also say that $r|_{G_{F_v}}$ is (potentially) diagonalizable to mean that $r^\circ|_{G_{F_v}}$ is.

Let F be an imaginary CM field with maximal totally real subfield F^+. We now demonstrate that any irreducible modular representation $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is, after a solvable base change, congruent to an automorphic Galois representation which is diagonalizable at all places dividing l. The argument is similar to that of the proof of Lemma 6.1.1 of [BLGG12], which proves an analogous result over totally real fields. By the results of [Kis09] and [Gee06], it suffices to produce a congruence (after a solvable base change) to an automorphic representation of weight 0 and level prime to l, which is easily achieved by standard arguments.

Lemma 3.1.1. *Suppose that π is a RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ and that $\overline{r}_{l,\imath}(\pi)$ is irreducible. Let $F^{(\mathrm{avoid})}/F$ be a finite extension. Then there is a finite solvable extension of CM imaginary fields F'/F and a RACSDC automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_{F'})$ such that*

- *F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F.*
- *π' has weight 0.*
- *$\overline{r}_{l,\imath}(\pi') \cong \overline{r}_{l,\imath}(\pi)|_{G_{F'}}$.*
- *For each place $w|l$ of F', $r_{l,\imath}(\pi')|_{G_{F'_w}}$ is diagonalizable.*

Proof. We first show that after a solvable base change, $\overline{r}_{l,\imath}(\pi)$ has a lift which is automorphic of weight 0. (This is presumably true without making a base change but the weaker statement will suffice for our purposes and allows us to transfer to a definite unitary group.) Choose a finite solvable extension of CM fields F_1/F such that

- *F_1 is linearly disjoint from $F^{(\mathrm{avoid})}\overline{F}^{\mathrm{ker}\, \overline{r}_{l,\imath}(\pi)}$ over F.*
- *F_1/F_1^+ is unramified at all finite places.*
- *$[F_1^+ : \mathbb{Q}]$ is even.*
- *Every place $v|l$ of F_1^+ splits in F_1.*

- If π_1 denotes the base change of π to F_1, then π_1 is unramified at all finite places of F_1 lying over an inert place of F_1^+.
- If $v|l$ is a place of F_1 such that π_v is ramified, then π_v is an unramified twist of the Steinberg representation.

As in section 2, we can choose a rank two unitary group G/F_1^+ which is quasisplit at all finite places, compact at all infinite places, and is split over F_1. Fix a model for G over $\mathcal{O}_{F_1^+}$ as in section 2. We will freely use the notation introduced in section 2 to describe spaces of algebraic modular forms on G.

Suppose that π_1 has weight $a \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F_1,\mathbb{C})}$. By Theorem 2.1.3 there is an automorphic representation Π of G such that

- If v is a finite place of F_1^+ which is inert in F_1, then Π_v has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_{1,v}^+)$.
- If v is a finite place of F_1^+ which splits as ww^c in F_1, then $\Pi_v \cong \pi_{1,w} \circ \iota_w$.
- For each embedding $\tau : F_1^+ \hookrightarrow \mathbb{R}$ and each $\tilde\tau$ extending τ, we have $\Pi_\tau \cong \Sigma_{a_\tau}^\vee \circ \iota_{\tilde\tau}$.

Define a representation W of $G(\mathcal{O}_{F_1^+,l})$ on a finite-dimensional $\overline{\mathbb{Q}}_l$-vector space as follows. Let S_l denote the set of places of F_1^+ lying over l, and let $\widetilde{S}_l$ denote a subset of the places of F_1 lying over l consisting of exactly one place $\widetilde{v}$ lying over each place $v \in S_l$. Let $\widetilde{I}_l$ denote the set of embeddings $F_1 \hookrightarrow \overline{\mathbb{Q}}_l$ giving rise to a place in $\widetilde{S}_l$, and for each $\widetilde{v} \in \widetilde{S}_l$ let $\widetilde{I}_{\widetilde{v}}$ denote the subset of $\widetilde{I}_l$ of elements lying over $\widetilde{v}$. Let V_a be the $\overline{\mathbb{Q}}_l$-vector space with an action of $G(\mathcal{O}_{F_{1,l}^+})$ given by $W_a \otimes_{\mathcal{O}_{\overline{\mathbb{Q}}_l}} \overline{\mathbb{Q}}_l$, where W_a is defined as in section 2. Let V_l be the $\overline{\mathbb{Q}}_l$-vector space with an action of $G(\mathcal{O}_{F_{1,l}^+})$ given by

$$V_l := \otimes_{v \in S_l} V_v$$

where V_v is the (self-dual) representation $\mathrm{Ind}_{I(\widetilde{v})}^{\mathrm{GL}_2(\mathcal{O}_{F_{1,\widetilde{v}}})} 1_{\overline{\mathbb{Q}}_l}$, where $1_{\overline{\mathbb{Q}}_l}$ is the trivial $\overline{\mathbb{Q}}_l$-representation of the standard Iwahori subgroup $I(\widetilde{v})$ of $\mathrm{GL}_2(\mathcal{O}_{F_{1,\widetilde{v}}})$, and $G(\mathcal{O}_{F_{1,l}^+})$ acts on V_v via $\iota_{\widetilde{v}}$. Finally, let $W := V_a \otimes V_l$, and let W° be a $G(\mathcal{O}_{F_{1,l}^+})$-stable $\mathcal{O}_{\overline{\mathbb{Q}}_l}$-lattice in W.

Lemma 2.1.1 and the existence of Π imply that there is a compact open subgroup $U \in G(\mathbb{A}_{F_1^+}^\infty)$ which is good in the sense of Definition 2.1.5 and is sufficiently small, together with a finite set of places T of F_1^+ as in section 2, such that there is a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T,univ}$ with:

- $S(U,W)_\mathfrak{m} \neq 0$.
- $\bar{r}_\mathfrak{m} \cong \bar{r}_{l,\iota}(\pi_1)$.

Since U is sufficiently small, we see (as in the proof of Lemma 2.1.6) that $S(U,W^\circ \otimes_{\mathcal{O}_{\overline{\mathbb{Q}}_l}} \overline{\mathbb{F}}_l)_\mathfrak{m} \neq 0$. Thus there is a Jordan-Hölder factor F of the $G(\mathcal{O}_{F_{1,l}^+})$-representation $W^\circ \otimes_{\mathcal{O}_{\overline{\mathbb{Q}}_l}} \overline{\mathbb{F}}_l$ such that $S(U,F)_\mathfrak{m} \neq 0$. There is a smooth irreducible $\overline{\mathbb{Q}}_l$-representation W_{sm} of $G(\mathcal{O}_{F_{1,l}^+})$ containing a stable $\mathcal{O}_{\overline{\mathbb{Q}}_l}$-lattice W_{sm}° such that F is a Jordan-Hölder factor of $W_{sm}^\circ \otimes_{\mathcal{O}_{\overline{\mathbb{Q}}_l}} \overline{\mathbb{F}}_l$ (this follows from the fact that F is a subquotient of $\otimes_{\widetilde{v} \in \widetilde{S}_l} \mathrm{Ind}_1^{\mathrm{GL}_2(k_{\widetilde{v}})} 1_{\overline{\mathbb{F}}_l}$, so we may take W_{sm} to be a subquotient of the representation $\otimes_{\widetilde{v} \in \widetilde{S}_l} \mathrm{Ind}_{K_{\widetilde{v},1}}^{\mathrm{GL}_2(\mathcal{O}_{F_{1,\widetilde{v}}})} 1_{\overline{\mathbb{Q}}_l}$ where $K_{\widetilde{v},1} = \ker(\mathrm{GL}_2(\mathcal{O}_{F_{1,\widetilde{v}}}) \to \mathrm{GL}_2(k_{\widetilde{v}}))$). Since U is sufficiently small, we see that $S(U,W_{sm}^\circ)_\mathfrak{m} \neq 0$, so $S(U,W_{sm})_\mathfrak{m} \neq 0$. Again,

by Lemma 2.1.1 and Theorem 2.1.4 we see that there is a RACSDC automorphic representation π_1' of $\mathrm{GL}_2(\mathbb{A}_{F_1})$ of weight 0 such that $\bar{r}_{l,\imath}(\pi_1') \cong \bar{r}_{\mathfrak{m}}$ (the fact that $\bar{r}_{\mathfrak{m}}$ is irreducible allows us to deduce that π_1' is cuspidal). After possibly making a further solvable base change, we can assume that in addition to the properties of F_1 listed above,

- if $v|l$ is a place of F_1 such that $\pi_{1,v}'$ is ramified, then $\pi_{1,v}'$ is an unramified twist of the Steinberg representation.

We now repeat the argument above with π_1 replaced by π_1' and hence with a replaced by 0. By Lemma 3.1.5 of [Kis09], we can choose W_{sm}° to be of the form $W_{sm} = \otimes_{v \in S_l} W_{sm,v}^\circ \circ \imath_{\widetilde{v}}$ where each $W_{sm,v}^\circ$ is a cuspidal $F_{\widetilde{v}}$-type (in the sense of *loc. cit.*). We see that there is a RACSDC automorphic representation π_1'' of $\mathrm{GL}_2(\mathbb{A}_{F_1})$ of weight 0 such that $\bar{r}_{l,\imath}(\pi_1'') \cong \bar{r}_{\mathfrak{m}}$, and for each place $v|l$, $\pi_{1,v}''$ is supercuspidal; so, after making another solvable base change, we may assume that

- $\pi_{1,v}''$ is unramified for all $v|l$.

Summarising, we have obtained a solvable extension F_1/F of CM fields, and a RACSDC automorphic representation π_1'' of $\mathrm{GL}_2(\mathbb{A}_{F_1})$ such that

- F_1 is linearly disjoint from $F^{(\mathrm{avoid})}$ over F.
- π_1'' has weight 0.
- $\bar{r}_{l,\imath}(\pi_1'') \cong \bar{r}_{l,\imath}(\pi)|_{G_{F_1}}$.

By Theorem 2.1.2, we see that for each place $v|l$ we have

- $r_{l,\imath}(\pi_1'')|_{G_{F_{1,v}}}$ is crystalline, and for each embedding $\tau : F_{1,v} \hookrightarrow \overline{\mathbb{Q}}_l$, we have $\mathrm{HT}_\tau(r_{l,\imath}(\pi_1'')|_{G_{F_{1,v}}}) = \{0,1\}$.

Making a further base change, we may assume in addition that

- For each place $v|l$ of F_1, $\bar{r}_{l,\imath}(\pi_1'')|_{G_{F_{1,v}}}$ is trivial, and there are crystalline representations ρ_1, $\rho_2 : G_{F_{1,v}} \to \mathrm{GL}_2(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ such that
  - $\overline{\rho}_1 = \overline{\rho}_2$ is the trivial representation.
  - ρ_1 and ρ_2 are both diagonal.
  - ρ_1 is ordinary, and ρ_2 is non-ordinary.
  - For each $\tau : F_{1,v} \hookrightarrow \overline{\mathbb{Q}}_l$, $\mathrm{HT}_\tau(\rho_1) = \mathrm{HT}_\tau(\rho_2) = \{0,1\}$.

From the existence of ρ_1 and ρ_2, Proposition 2.3 of [Gee06], and Corollary 2.5.16 of [Kis09], it follows that

- For each place $v|l$, $r_{l,\imath}(\pi_1'')|_{G_{F_{1,v}}}$ is diagonalizable.

The result follows, taking $F' = F_1$ and $\pi' = \pi_1''$. $\qquad\square$

The following Theorem will allow us to "change the weight" of a modular Galois representation. For the notion of an *adequate* subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$, which was originally defined in [Tho12], we refer the reader to Appendix A, where a detailed discussion of this condition is given. In particular, we remind the reader that if $l \geq 7$, any irreducible subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is adequate.

Theorem 3.1.2. *Let $l > 2$ be prime and let F be a CM field with maximal totally real subfield F^+. Assume that $\zeta_l \notin F$ and that the extension F/F^+ is split at all places dividing l. Let S be a finite set of finite places of F^+ which split in F and assume that S contains all the places dividing l. For each $v \in S$ choose a place $\widetilde{v}$ of F above v.*

Suppose that

$$\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$$

is an irreducible representation which is unramified at all places not lying over S and which satisfies the following properties.

    (1) $\bar{r}$ is automorphic.
    (2) $\bar{r}(G_{F(\zeta_l)})$ is adequate.

For each $v \in S$, let $\rho_{\widetilde{v}} : G_{F_{\widetilde{v}}} \to \mathrm{GL}_2(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ be a lift of $\bar{r}|_{G_{F_{\widetilde{v}}}}$. Assume that

- *if $v|l$, then $\rho_{\widetilde{v}}$ is crystalline and potentially diagonalizable, and if $\tau : F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$ is any embedding, then $\mathrm{HT}_\tau(\rho_{\widetilde{v}})$ consists of two distinct integers.*

    Then there is a RACSDC automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of level prime to l such that

- *$\bar{r} \cong \bar{r}_{l,\iota}(\pi)$.*
- *π_v is unramified for all v not lying over a place of S, so that $r_{l,\iota}(\pi_v)$ is unramified at all such v.*
- *$r_{l,\iota}(\pi)|_{G_{F_{\widetilde{v}}}} \sim \rho_{\widetilde{v}}$ for all $v \in S$. In particular, for each place $v|l$, $r_{l,\iota}(\pi)|_{G_{F_{\widetilde{v}}}}$ is crystalline and for each embedding $\tau : F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$, $\mathrm{HT}_\tau(r_{l,\iota}(\pi)|_{G_{F_{\widetilde{v}}}}) = \mathrm{HT}_\tau(\rho_{\widetilde{v}})$.*

Proof. Let $\mathcal{G}_2$ be the group scheme over $\mathbb{Z}$ defined in section 2.1 of [CHT08]. Then by the main result of [BC11], $\bar{r}$ extends to a representation $\bar{\rho} : G_{F^+} \to \mathcal{G}_2(\overline{\mathbb{F}}_l)$ with multiplier $\bar{\epsilon}_l^{-1}$.

By Lemma 3.1.1, we may find a finite solvable extension F'/F of CM fields and a RACSDC automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_{F'})$ such that

- $\bar{r}_{l,\iota}(\pi') \cong \bar{r}|_{G_{F'}}$.
- F' is linearly disjoint from $\bar{F}^{\ker \bar{r}}(\zeta_l)$ over F.
- π' is unramified at all finite places.
- For each place $w|l$ of F', $r_{l,\iota}(\pi')|_{G_{F_w}}$ is crystalline and potentially diagonalizable.

We now apply Theorem A.4.1 below, with

- F, F', S and l as in the present setting,
- $n = 2$,
- $\bar{r}$ our present $\bar{\rho}$,
- ρ_v our $\rho_{\widetilde{v}}$,
- $\mu = \epsilon_l^{-1}$.

We conclude that there is a lift $r : G_F \to \mathrm{GL}_2(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ (the restriction to G_F of the representation r produced by Theorem A.4.1) of $\bar{r}$ such that

- $r^c \cong r^\vee \epsilon_l^{-1}$,
- if $v \in S$ then $r|_{G_{F_{\widetilde{v}}}} \sim \rho_{\widetilde{v}}$,
- r is unramified outside S.
- $r|_{G_{F'}}$ is automorphic of level potentially prime to l.

Since the extension F'/F is solvable, we deduce that r is automorphic. Let π be the RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with $r_{l,\iota}(\pi) \cong r$. By Theorem 2.1.2, we see that (since $r|_{G_{F_w}}$ is crystalline for all $w|l$, and unramified at all places w not lying over a place in S) π_w is unramified for all $w|l$ and all w not lying over a place in S, as required. $\square$

4. SERRE WEIGHT CONJECTURES

4.1. We now recall various formulations of Serre weight conjectures for GL_2, following [BDJ10], [Sch08], [Gee11a], and [EGHS13]. These conjectures were formulated for various inner forms of GL_2 (indefinite and definite quaternion algebras), but it is widely believed that they should also apply to outer forms of GL_2, such as the groups considered in the present paper. These conjectures all consist of purely local descriptions of sets of weights, in a sense which we will now explain (as in the rest of the paper, we work with unitary groups, but the local formulations are the same as for inner forms of GL_2 which are split at all places lying over l).

Let K be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field k. Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be a continuous representation. Then it is a folklore conjecture that there is a set $W(\overline{\rho})$ of Serre weights of $\mathrm{GL}_2(k)$ with the property that if F is a CM field and $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation (so in particular it is conjugate self-dual), and $v|l$ is a place of F, then $\overline{r}$ is modular of some Serre weight $\sigma_w \otimes_{\overline{\mathbb{F}}_l} \sigma^w$ (where σ_w is a representation of $\mathrm{GL}_2(k_w)$) for some σ^w if and only if $\sigma_w \in W(\overline{r}|_{G_{F_w}})$.

It is natural to believe that there is a description of $W(\overline{\rho})$ in terms of the existence of crystalline lifts with particular Hodge–Tate weights, as we now explain. This is one of the motivations for the general Serre weight conjectures explained in [EGHS13].

Definition 4.1.1. Let $K/\mathbb{Q}_l$ be a finite extension, let $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$, and let $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ be a de Rham representation. Then we say that ρ has *Hodge type* λ if for each $\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$, we have $\mathrm{HT}_\tau(\rho) = \{\lambda_{\tau,1} + 1, \lambda_{\tau,2}\}$.

Remark 4.1.2. As an immediate consequence of the definition and of Theorem 2.1.2, we see that if π is a RACSDC automorphic representation of weight $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$, then for each place $w|l$, $r_{l,\imath}(\pi)|_{G_{F_w}}$ has Hodge type $(\imath^{-1}\lambda)_w$.

Lemma 4.1.3. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that $l > 2$, and that $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ be a Serre weight, and let $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ be a lift of a. If $\overline{r}$ is modular of weight a, then for each place $w|l$ there is a continuous lift $r_w : G_{F_w} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ of $\overline{r}|_{G_{F_w}}$ such that*

- *r_w is crystalline.*
- *r_w has Hodge type λ_w.*

Proof. By Lemma 2.1.11 there is a RACSDC automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, which has level prime to l and weight $\imath\lambda$, such that $\overline{r}_{l,\imath}(\pi) \cong \overline{r}$. Then we may take $r_w := r_{l,\imath}(\pi)|_{G_{F_w}}$, which satisfies the above conditions by Remark 4.1.2. $\square$

This suggests the following definition, first made in [Gee11a].

Definition 4.1.4. Let K be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field k. Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be a continuous representation. Then we let $W^{\mathrm{cris}}(\overline{\rho})$ be the set of Serre weights $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ with the property that there is a crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ lifting $\overline{\rho}$, such that

- ρ has Hodge type λ for some lift $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)}$ of a.

The results of section 3 inspire the following definition.

Definition 4.1.5. Let K be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field k. Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be a continuous representation. Then we let $W^{\mathrm{diag}}(\overline{\rho})$ be the set of Serre weights $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)}$ with the property that there is a continuous potentially diagonalizable crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ lifting $\overline{\rho}$, such that

- ρ has Hodge type λ for some lift $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)}$ of a.

Remark 4.1.6. If a and b are equivalent Serre weights, then $a \in W^{\mathrm{cris}}(\overline{\rho})$ (respectively $W^{\mathrm{diag}}(\overline{\rho})$) if and only if $b \in W^{\mathrm{cris}}(\overline{\rho})$ (respectively $W^{\mathrm{diag}}(\overline{\rho})$). This is an easy consequence of Lemma 4.1.15 below, which provides a crystalline character with trivial reduction by which one can twist the crystalline Galois representations of Hodge type some lift of a to obtain crystalline representations of Hodge type some lift of b. The same remarks apply to the set $W^{\mathrm{explicit}}(\overline{\rho})$ defined below.

By definition we have $W^{\mathrm{diag}}(\overline{\rho}) \subset W^{\mathrm{cris}}(\overline{\rho})$. We "globalise" these definitions in the obvious way:

Definition 4.1.7. Let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be a continuous representation with $\overline{r}^c \cong \overline{r}^\vee \overline{\epsilon}_l^{-1}$. Then we let $W^{\mathrm{cris}}(\overline{r})$ (respectively $W^{\mathrm{diag}}(\overline{r})$) be the set of Serre weights $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ such that for each place $w|l$, the corresponding Serre weight $a_w \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ is an element of $W^{\mathrm{cris}}(\overline{r}|_{G_{F_w}})$ (respectively $W^{\mathrm{diag}}(\overline{r}|_{G_{F_w}})$).

The point of these definitions is the following corollary and theorem.

Corollary 4.1.8. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that $l > 2$, and that $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_v, \overline{\mathbb{F}}_l)}$ be a Serre weight. If $\overline{r}$ is modular of weight a, then $a \in W^{\mathrm{cris}}(\overline{r})$.*

Proof. This is an immediate consequence of Lemma 4.1.3 and Definition 4.1.7. $\square$

Theorem 4.1.9. *Let F be an imaginary CM field with maximal totally real subfield F^+. Assume that $\zeta_l \notin F$, that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that $l > 2$, and that $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Assume that $\overline{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight. Assume that $a \in W^{\mathrm{diag}}(\overline{r})$. Then $\overline{r}$ is modular of weight a.

Proof. By the assumption that $a \in W^{\mathrm{diag}}(\overline{r})$, there is a lift λ of a such that for each $w|l$ there is a potentially diagonalizable crystalline lift $\rho_w : G_{F_w} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ of $\overline{r}|_{G_{F_w}}$ of Hodge type λ_w.

By Theorem 3.1.2 (applied with the set S of that theorem being the set of places dividing l together with the places at which $\overline{r}$ is ramified, and taking the lifts $\rho_{\widetilde{v}}$ to be those defined in the previous paragraph for $v|l$ and arbitrary for v not dividing l, noting that the fact that $\overline{r}$ is modular guarantees the existence of lifts), there is a

RACSDC automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\imath\lambda$, of level prime to l and with split ramification, such that $\bar{r}_{l,\imath}(\pi) \cong \bar{r}$. The result follows from Lemma 2.1.11. $\qquad\square$

The majority of the rest of this paper will be devoted to making this theorem more explicit. We believe that in fact $W^{\mathrm{diag}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ in all cases, and we are able to show strong results in this direction. In addition, we exhibit many explicit weights in $W^{\mathrm{diag}}(\bar{r})$ (and again, conjecturally all such weights). In view of Corollary 4.1.8 and Theorem 4.1.9 (and the trivial inclusion $W^{\mathrm{diag}}(\bar{r}) \subset W^{\mathrm{cris}}(\bar{r})$), we are reduced to purely local questions, so we return to the setting of a finite extension $K/\mathbb{Q}_l$ with residue field k and absolute ramification index e, and we fix a continuous representation $\bar{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$. We then consider the following two questions:

- What is a good lower bound for the set $W^{\mathrm{diag}}(\bar{\rho})$?
- What is a good upper bound for the set $W^{\mathrm{cris}}(\bar{\rho})$?

If these two questions have the same answer, then the above work gives a complete determination of the Serre weights of 2-dimensional mod l Galois representations. In particular, we conjecture (following [EGHS13]) that the lower bound we provide for $W^{\mathrm{diag}}(\bar{\rho})$ is also an upper bound for $W^{\mathrm{cris}}(\bar{\rho})$.

The papers [BDJ10], [Sch08] and [EGHS13] all give explicit conjectural descriptions of $W^{\mathrm{cris}}(\bar{\rho})$ in increasing orders of generality. [Strictly speaking, [BDJ10] and [Sch08] do not phrase their conjectures in the language of crystalline lifts, but the results above make it reasonable to discuss their descriptions in this optic; that is, we would like to see whether their lists of weights can be proved to be lower bounds for $W^{\mathrm{diag}}(\bar{\rho})$ or upper bounds for $W^{\mathrm{cris}}(\bar{\rho})$. We will see that the lower bound we provide for $W^{\mathrm{diag}}(\bar{\rho})$ agrees with the sets of weights predicted in [BDJ10] and [Sch08] in most cases, and conjecturally in all cases.] We now recall these conjectures.

We begin by defining the fundamental characters of the inertia group of a finite extension K of $\mathbb{Q}_l$. For each $\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)$ we define the fundamental character ω_σ corresponding to σ to be the composite

$$I_{K^{\mathrm{ab}}/K} \xrightarrow{\mathrm{Art}_K^{-1}} \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\sigma^{-1}} \overline{\mathbb{F}}_l{}^\times.$$

Let K' denote the quadratic unramified extension of K inside $\overline{\mathbb{Q}}_l$, with residue field $k' \subset \overline{\mathbb{F}}_l$.

We now recall a slight variant of the conjectures of [BDJ10], who associate a set of weights to any continuous representation $\bar{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ in the case that $K/\mathbb{Q}_l$ is unramified. We define a set of weights $W^{\mathrm{BDJ}}(\bar{\rho})$ as follows:

Definition 4.1.10. Assume that $K/\mathbb{Q}_l$ is unramified. If $\bar{\rho}$ is irreducible, then a Serre weight $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)}$ is in $W^{\mathrm{BDJ}}(\bar{\rho})$ if and only if there is a subset $J \subset \mathrm{Hom}\,(k', \overline{\mathbb{F}}_l)$ consisting of exactly one embedding extending each element of $\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)$, such that if we write $\mathrm{Hom}\,(k', \overline{\mathbb{F}}_l) = J \coprod J^c$, then (where here and below, if $\sigma \in \mathrm{Hom}\,(k', \overline{\mathbb{F}}_l)$ we write $a_{\sigma,i}$ for $a_{\sigma|_k,i}$)

$$\bar{\rho}|_{I_K} \cong \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}} & 0 \\ 0 & \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,1}+1} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,2}} \end{pmatrix}.$$

If $\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$, we let $\bar{\tau}$ be the induced element of $\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)$.

Definition 4.1.11. Assume that $K/\mathbb{Q}_l$ is unramified. If $\overline{\rho}$ is reducible, then a Serre weight $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ is in $W^{\mathrm{BDJ}}(\overline{\rho})$ if and only if there is a subset $J \subset \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$ such that $\overline{\rho}$ has a crystalline lift of the form

$$\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$$

where $\mathrm{HT}_\tau(\psi_1) = a_{\overline{\tau},1} + 1$ if $\tau \in J$ and $a_{\overline{\tau},2}$ otherwise, and $\mathrm{HT}_\tau(\psi_2) = a_{\overline{\tau},2}$ if $\tau \in J$, and $a_{\overline{\tau},1} + 1$ otherwise. In particular, if we write $\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l) = J \coprod J^c$ and $a \in W^{\mathrm{BDJ}}(\overline{\rho})$ then we necessarily have

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}} & * \\ 0 & \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,1}+1} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,2}} \end{pmatrix}.$$

[The description of $\overline{\rho}|_{I_K}$ in the reducible case is immediate from Lemma 4.1.15 below (see also Lemma 4.1.16). To see the relationship of these definitions to those of [BDJ10] is straightforward. In the irreducible case, it follows at once from equation 3.1(1) of [BDJ10] that our description agrees with that of [BDJ10] (where the set that we denote $W^{\mathrm{BDJ}}(\overline{\rho})$ is called $W_{\mathfrak{p}}(\rho)$).

In the reducible case, it is possible that our set $W^{\mathrm{BDJ}}(\overline{\rho})$ differs from the set proposed in [BDJ10], although it is conjectured in [BDJ10] that this is not the case, and in any case we shall see below that $W^{\mathrm{BDJ}}(\overline{\rho}) \subset W^{\mathrm{diag}}(\overline{\rho})$. Suppose firstly that $\overline{\rho}$ is not a twist of an extension of the trivial character by either the trivial character or the cyclotomic character. Then the definition of $W_{\mathfrak{p}}(\rho)$ in [BDJ10] agrees with our $W^{\mathrm{BDJ}}(\overline{\rho})$, except that [BDJ10] make an additional prescription on the character $\psi_1\psi_2^{-1}$ (they demand that it takes a certain value on a fixed Frobenius element). However, Remark 3.10 of [BDJ10] explains that in most cases these two formulations are equivalent, and conjectures that they are always equivalent.

In the remaining cases, it is not immediately clear that our definitions agree, although the authors of [BDJ10] have indicated to us that they conjecture that they agree, and that their definition is intended as a refinement of the definition given here. The definition given in [BDJ10] is better suited to comparisons of the sets $W^{\mathrm{BDJ}}(\overline{\rho})$ as $\overline{\rho}$ varies over representations with the same semisimplification.]

We now turn to the formulation given in [Sch08]. We drop the assumption that $K/\mathbb{Q}_l$ is unramified, but assume instead that $\overline{\rho}|_{I_K}$ is semisimple. In this case a set $W^{\mathrm{Sch}}(\overline{\rho})$ of Serre weights is proposed in [Sch08] as follows.

Definition 4.1.12. If $\overline{\rho}$ is irreducible, then a Serre weight $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ is in $W^{\mathrm{Sch}}(\overline{\rho})$ if and only if there is a subset $J \in \mathrm{Hom}\,(k',\overline{\mathbb{F}}_l)$ consisting of exactly one embedding extending each element of $\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$, and for each $\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$ an integer $0 \le \delta_\sigma \le e - 1$ such that if we write $\mathrm{Hom}\,(k',\overline{\mathbb{F}}_l) = J \coprod J^c$, then (where here and below we write δ_σ for $\delta_{\sigma|k}$)

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma} & 0 \\ 0 & \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma} \end{pmatrix}.$$

Definition 4.1.13. If $\overline{\rho}$ is reducible and $\overline{\rho}|_{I_K}$ is semisimple, then a Serre weight $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ is in $W^{\mathrm{Sch}}(\overline{\rho})$ if and only if there is a subset $J \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$, and for each $\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$ an integer $0 \le \delta_\sigma \le e - 1$ such that if we write

$\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l) = J \coprod J^c$, then

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}+\delta_\sigma} & 0 \\ 0 & \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,1}+e-\delta_\sigma} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma} \end{pmatrix}.$$

[That these agree with the definitions of [Sch08] is immediate from the statements of Theorems 2.4 and 2.5 of [Sch08] (after replacing δ_σ by $e - 1 - \delta_\sigma$ in the case $\sigma \in J^c$).] Finally, following [EGHS13], we define an explicit set of weights $W^{\mathrm{GHS}}(\overline{\rho})$ in the case that $\overline{\rho}$ is reducible but not necessarily decomposable when restricted to I_K (without assuming that $K/\mathbb{Q}_l$ is unramified).

Definition 4.1.14. If $\overline{\rho}$ is reducible, then a Serre weight $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)}$ is in $W^{\mathrm{GHS}}(\overline{\rho})$ if and only if $\overline{\rho}$ has a crystalline lift of the form

$$\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$$

which has Hodge type λ for some lift $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)}$ of a. In particular, if $a \in W^{\mathrm{GHS}}(\overline{\rho})$ then it is necessarily the case that there is a decomposition $\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l) = J \coprod J^c$ and for each $\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)$ there is an integer $0 \le \delta_\sigma \le e - 1$ such that

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}+\delta_\sigma} & * \\ 0 & \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,1}+e-\delta_\sigma} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma} \end{pmatrix}.$$

[Again, the form of $\overline{\rho}|_{I_K}$ is immediate from Lemma 4.1.15 below.] In order to see the relationship between these definitions, we now study the question of when it is "obvious" that one can write down a crystalline lift with specified Hodge–Tate weights of a given $\overline{\rho}$. If χ is a character of G_K or I_K valued in $\mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$, we denote its reduction mod l by $\overline{\chi}$.

Lemma 4.1.15. *Let* $A = \{a_\tau\}_{\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)}$ *be a set of integers. Then there is a crystalline character* ϵ_A *of* G_K *such that* $\mathrm{HT}_\tau(\epsilon_A) = a_\tau$ *for all* $\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$, *and* ϵ_A *is unique up to unramified twist. Furthermore,* $\overline{\epsilon}_A|_{I_K} = \prod_{\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)} \omega_\sigma^{b_\sigma}$, *where*

$$b_\sigma = \sum_{\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l): \overline{\tau} = \sigma} a_\tau.$$

Proof. This is Lemma 6.2 of [GS]. [Note that the definitions of fundamental characters in this paper are the inverse of those defined in section 5 of [GS]; this is because our conventions for Hodge–Tate weights are the opposite of those of [GS].] $\square$

Lemma 4.1.16. *Suppose that* $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)}$ *is a Serre weight, and that* $\overline{\rho}: G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ *is a continuous representation which is a direct sum of two characters. Suppose that there is a decomposition* $\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l) = J \coprod J^c$ *and for each* $\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)$ *there is an integer* $0 \le \delta_\sigma \le e - 1$ *with*

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}+\delta_\sigma} & 0 \\ 0 & \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,1}+e-\delta_\sigma} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma} \end{pmatrix}.$$

Then for any $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)}$ *lifting* a, $\overline{\rho}$ *has a diagonal crystalline lift of Hodge type* λ.

Proof. We define sets $B = \{b_\tau\}_{\tau \in \mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$ and $C = \{c_\tau\}_{\tau \in \mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$ of integers as follows. For each $\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$, let S_σ be the subset of $\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)$ consisting of those τ with $\overline{\tau} = \sigma$. By definition, for each σ there is a distinguished element $\tilde{\sigma}$ of S_σ with $\lambda_{\tilde{\sigma},i} = a_{\sigma,i}$, and for each element $\tau \neq \tilde{\sigma}$ of S_σ we have $\lambda_{\tau,i} = 0$. Choose a subset K_σ of $S_\sigma \backslash \{\tilde{\sigma}\}$ of size δ_σ.

Suppose $\sigma \in J$. We let $b_{\tilde{\sigma}} = a_{\sigma,1} + 1$, we let $b_\tau = 1$ if $\tau \in K_\sigma$, and $b_\tau = 0$ for all other $\tau \in S_\sigma$. Similarly, we let $c_{\tilde{\sigma}} = a_{\sigma,2}$, we let $c_\tau = 1$ if $\tau \in S_\sigma \backslash (K_\sigma \cup \{\tilde{\sigma}\})$ and $c_\tau = 0$ for $\tau \in K_\sigma$.

Suppose $\sigma \notin J$. We let $c_{\tilde{\sigma}} = a_{\sigma,1} + 1$, we let $c_\tau = 1$ if $\tau \in K_\sigma$, and $c_\tau = 0$ for all other elements of S_σ. Similarly, we let $b_{\tilde{\sigma}} = a_{\sigma,2}$, we let $b_\tau = 1$ if $\tau \in S_\sigma \backslash (K_\sigma \cup \{\tilde{\sigma}\})$ and $b_\tau = 0$ for $\tau \in K_\sigma$.

Then by Lemma 4.1.15, $\overline{\rho}$ has a lift given by the direct sum of an unramified twist of ϵ_B and an unramified twist of ϵ_C. By definition, this is a diagonal crystalline lift of Hodge type λ. $\qquad\square$

Corollary 4.1.17. *Suppose that $e \geq l$, and $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is a continuous representation which is a direct sum of two characters. Suppose that $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ is a Serre weight such that*

$$\det \overline{\rho}|_{I_K} = \prod_{\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1} + a_{\sigma,2} + e}.$$

Then for any $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$ lifting a, $\overline{\rho}$ has a diagonal crystalline lift of Hodge type λ.

Proof. Suppose that $\overline{\rho} \cong \psi_1 \oplus \psi_2$. Since any representation as in the statement of Lemma 4.1.16 has $\det \overline{\rho}|_{I_K} = \prod_{\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1} + a_{\sigma,2} + e}$, it suffices to show that we can choose J and δ_σ as in the statement of Lemma 4.1.16 such that

$$\psi_1|_{I_K} = \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1} + 1 + \delta_\sigma} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2} + \delta_\sigma}.$$

Take $J = \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$, and write $\psi_1|_{I_K} \prod_{\sigma \in J} \omega_\sigma^{-(a_{\sigma,1}+1)}$ in the form $\prod_{\sigma \in J} \omega_\sigma^{c_\sigma}$ with $0 \leq c_\sigma \leq l - 1$. Then we may take $\delta_\sigma = c_\sigma$. $\qquad\square$

Remark 4.1.18. Contrary to the claim made in the introduction to [Sch08], once $e = l-1$, it is no longer the case that for every $\overline{\rho}$ with determinant $\prod_{\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1} + a_{\sigma,2} + e}$ can we can apply Lemma 4.1.16 to find a crystalline diagonal lift of Hodge type lifting a. For a counterexample, take $l = 7$, $[k : \mathbb{F}_l] = 2$, and label the two embeddings $k \hookrightarrow \overline{\mathbb{F}}_l$ as σ_1 and σ_2. Then take

$$
\begin{aligned}
a_{\sigma_1,1} &= l - 1, & a_{\sigma_2,1} &= 1, \\
a_{\sigma_1,2} &= a_{\sigma_2,2} = 0, & \overline{\rho} &= \psi_1 \oplus \psi_2, \\
\text{where} \qquad \psi_1 &= \omega_{\sigma_1}^{l-1} \omega_{\sigma_2}^4 & \psi_2 &= \omega_{\sigma_1}^{l-1} \omega_{\sigma_2}^{l-4}.
\end{aligned}
$$

Then it is easy to see (by considering all 4 possible sets J) that we can never choose δ_σ to make $\overline{\rho}$ equivalent to the representation in Lemma 4.1.16.

We now consider the case of irreducible representations $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$. Recall that K' denotes the unique unramified quadratic extension of K and k' denotes its residue field. Then $\overline{\rho}$ is induced from a character of $G_{K'}$, and $\overline{\rho}|_{I_K}$ decomposes as a sum of characters.

Lemma 4.1.19. *Suppose that $a \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ is a Serre weight, and that $\overline{\rho}$: $G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation. Suppose that there is a decomposition $\mathrm{Hom}\,(k',\overline{\mathbb{F}}_l) = J \coprod J^c$ such that J contains exactly one embedding extending each element of $\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$, and for each $\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$ there is an integer $0 \le \delta_\sigma \le e-1$ with*

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma} & 0 \\ 0 & \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma} \end{pmatrix}.$$

Then for any $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$ lifting a, $\overline{\rho}$ has a potentially diagonalizable crystalline lift of Hodge type λ which becomes diagonal when restricted to $G_{K'}$.

Proof. We may write

$$\overline{\rho} \cong \mathrm{Ind}_{G_{K'}}^{G_K} \psi$$

for some character $\psi : G_{K'} \to \overline{\mathbb{F}}_l^\times$ which satisfies

$$\psi|_{I_{K'}} = \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma}.$$

We define a set $B = \{b_\tau\}_{\tau \in \mathrm{Hom}\,(K',\overline{\mathbb{Q}}_l)}$ as follows.

For each $\tau \in \mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)$, we denote the two extensions of τ to elements of $\mathrm{Hom}\,(K',\overline{\mathbb{Q}}_l)$ by τ_1 and τ_2, where $\overline{\tau}_1 \in J$ and $\overline{\tau}_2 \in J^c$.

For each $\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$, let S_σ be the subset of $\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)$ consisting of those τ with $\overline{\tau} = \sigma$. By definition, for each $\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)$ there is a distinguished element $\tilde{\sigma}$ of S_σ with $\lambda_{\tilde{\sigma},i} = a_{\sigma,i}$, and for each element $\tau \ne \tilde{\sigma}$ of S_σ we have $\lambda_{\tau,i} = 0$. Choose a subset K_σ of $S_\sigma \backslash \{\tilde{\sigma}\}$ of size δ_σ.

Then we let $b_{\tilde{\sigma}_1} = a_{\sigma,1}+1$, and $b_{\tilde{\sigma}_2} = a_{\sigma,2}$. If $\tau \in K_\sigma$, we let $b_{\tau_1} = 1$ and $b_{\tau_2} = 0$. If $\tau \in S_\sigma \backslash (\{\tilde{\sigma}\} \cup K_\sigma)$, we let $b_{\tau_1} = 0$ and $b_{\tau_2} = 1$.

Then by Lemma 4.1.15 there is a crystalline character $\tilde{\psi}$ of $G_{K'}$ lifting ψ, which is an unramified twist of the character ϵ_B. The representation $\mathrm{Ind}_{G_{K'}}^{G_K} \tilde{\psi}$ gives the required lift. $\qquad\square$

Corollary 4.1.20. *Suppose that $e \ge l$, and $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation. Suppose that $a \in (\mathbb{Z}^2)_+^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ is a Serre weight such that*

$$\det \overline{\rho}|_{I_K} = \prod_{\sigma \in \mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1}+a_{\sigma,2}+e}.$$

Then for any weight $\lambda \in (\mathbb{Z}^2)_+^{\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$ lifting a, $\overline{\rho}$ has a potentially diagonalizable crystalline lift of Hodge type λ, which becomes diagonal upon restriction to $G_{K'}$.

Proof. We can write $\overline{\rho} \cong \mathrm{Ind}_{G_{K'}}^{G_K} \phi$ for some character $\phi : G_{K'} \to \overline{\mathbb{F}}_l^\times$. The condition on the determinant of $\overline{\rho}$ tells us that

$$(\phi\phi^c)|_{I_{K'}} = \prod_{\sigma \in \mathrm{Hom}\,(k',\overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1}+a_{\sigma,2}+e},$$

where c denotes the nontrivial element of $\mathrm{Gal}\,(K'/K)$ and ϕ^c denotes ϕ conjugated by c.

On the other hand, by Lemma 4.1.19 and the first line of its proof, we know that if we choose J and $(\delta_\sigma)_{\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)}$ as in the statement of that lemma and write

$$\psi = \prod_{\sigma \in J} \omega_\sigma^{a_{\sigma,1}+1+\delta_\sigma} \prod_{\sigma \in J^c} \omega_\sigma^{a_{\sigma,2}+e-1-\delta_\sigma},$$

then for any $\lambda \in (\mathbb{Z}_+^2)_0^{\mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)}$ lifting a and any representation $G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ agreeing with $\mathrm{Ind}_{I_{K'}}^{I_K} \psi$ on I_K, that representation has a potentially diagonalizable crystalline lift of Hodge type λ which becomes diagonal when restricted to $G_{K'}$.

Thus to prove the present corollary it suffices to show that, for an appropriate choice of J and $(\delta_\sigma)_{\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)}$, we can arrange for ψ to equal $\phi|_{I_{K'}}$.

Let $f = [k : \mathbb{F}_l]$, and let $\{\sigma_1, \ldots, \sigma_{2f}\}$ denote the embeddings $k' \hookrightarrow \overline{\mathbb{F}}_l$. We will take the labels mod $2f$, and we can and do choose the labelling such that

- $\omega_{\sigma_i} = \omega_{\sigma_{i+1}}^l$, and
- $\omega_{\sigma_{i+f}|k} = \omega_{\sigma_i|k}$. (In fact, this second point will follow from the first.)

We will write ω_i for ω_{σ_i} (thus the i here is taken mod $2f$); and we will write δ_i for $\delta_{\sigma_i|k}$ and $a_{i,j}$ for $a_{\sigma_i|k,j}$ (thus the i here is taken mod f). We will choose $J = \{\sigma_1, \ldots, \sigma_f\}$, and we see that this contains, as is required, exactly one embedding extending each element of $\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)$.

We let

$$\phi' := \phi|_{I_{K'}} \prod_{i=1}^f \omega_i^{-a_{i,1}-1} \prod_{i=f+1}^{2f} \omega_i^{-a_{i,2}-e+l},$$

and we write $\phi' = \prod_{\sigma \in \mathrm{Hom}\,(k', \overline{\mathbb{F}}_l)} \omega_\sigma^{\eta_\sigma}$, where $0 \le \eta_\sigma \le l-1$ for each η_σ. This expression is unique except that the special case where all the η are 0 is indistinguishable from the case when they are all $l-1$. Let us assume for the moment that we are not in this special case and thus the expression is genuinely unique.

We write η_i for η_{σ_i}. We then calculate that

$$\phi'(\phi')^c = (\phi\phi^c)|_{I_{K'}} \prod_{i=1}^{2f} \omega_i^{-a_{i,1}-1} \prod_{i=1}^{2f} \omega_i^{-a_{i,2}-e+l}$$

$$= \prod_{i=1}^{2f} \omega_i^{a_{i,1}+a_{i,2}+e} \prod_{i=1}^{2f} \omega_i^{-a_{i,1}-a_{i,2}-e+l-1}$$

$$= \prod_{i=1}^{2f} \omega_i^{l-1} = 1,$$

so that

$$\phi' = ((\phi')^c)^{-1} = \left(\prod_{i=1}^{2f} \omega_i^{\eta_{i+f}}\right)^{-1} = \prod_{i=1}^{2f} \omega_i^{-\eta_{i+f}} = \prod_{i=1}^{2f} \omega_i^{l-1-\eta_{i+f}}.$$

It follows from the uniqueness discussed above that $\eta_{i+f} = l - 1 - \eta_i$. For $i = 1, \ldots, f$, we let $\delta_i = \eta_i$. Then we see that with this choice of J and $(\delta_\sigma)_{\sigma \in \mathrm{Hom}(k, \overline{\mathbb{F}}_l)}$,

$$\psi = \prod_{i=1}^{f} \omega_i^{a_{i,1}+1+\delta_i} \prod_{i=f+1}^{2f} \omega_i^{a_{i,2}+e-1-\delta_i} = \prod_{i=1}^{f} \omega_i^{a_{i,1}+1+\eta_i} \prod_{i=f+1}^{2f} \omega_i^{a_{i,2}+e-1-(l-1-\eta_i)}$$

$$= \prod_{i=1}^{f} \omega_i^{a_{i,1}+1} \prod_{i=f+1}^{2f} \omega_i^{a_{i,2}+e-l} \prod_{i=1}^{2f} \omega_i^{\eta_i} = (\phi/\phi') \prod_{i=1}^{2f} \omega_i^{\eta_i}$$

So $\psi = (\phi|_{I_{K'}}/\phi')\phi' = \phi|_{I_{K'}}$, as we required.

Thus we are done apart from considering the special case we deferred earlier, where $\phi' = 1$. Assume we are in this case, and put

$$\phi'' := \phi' \omega_0^{-a_{0,1}-1+a_{0,2}} \omega_f^{a_{0,1}+1-a_{0,2}}.$$

We claim that ϕ'' does not equal 1. To see this, since $\phi' = 1$, we must show that $\phi''/\phi' \neq 1$. We recall that $1 \leq a_{0,1} + 1 - a_{0,2} \leq l$; since

$$\phi''/\phi' = \omega_0^{-a_{0,1}-1+a_{0,2}} \omega_f^{a_{0,1}+1-a_{0,2}} = \omega_0^{(l^f-1)(a_{0,1}+1-a_{0,2})}$$

and ω_0 has order $l^{2f} - 1$, the claim follows. Write $\phi'' = \prod_{\sigma \in \mathrm{Hom}(k', \overline{\mathbb{F}}_l)} \omega_\sigma^{\eta_\sigma''}$, where $0 \leq \eta_\sigma'' \leq l - 1$ for each η_σ''. This expression is unique, since $\phi'' \neq 1$. Now, we calculate that

$$(\phi'')(\phi'')^c = \phi'(\phi')^c \omega_f^{a_{f,1}+1-a_{f,2}} \omega_0^{-a_{f,1}-1+a_{f,2}} \omega_0^{a_{f,1}+1-a_{f,2}} \omega_f^{-a_{f,1}-1+a_{f,2}} = 1.$$

We conclude that $\eta_i'' = l - 1 - \eta_{i+f}''$ in the same way as we saw the corresponding fact for η above.

We now take $J = \{0, \ldots, f-1\}$, $\delta_i = \eta_i''$ for $i = 1, \ldots, f-1$ and $\delta_0 = e - 1 - \eta_f''$. Then

$$\psi = \prod_{i=0}^{f-1} \omega_i^{a_{i,1}+1+\delta_i} \prod_{i=f}^{2f-1} \omega_i^{a_{i,2}+e-1-\delta_i}$$

$$= \omega_0^{a_{0,1}+1+e-1-\eta_f''} \left(\prod_{i=1}^{f-1} \omega_i^{a_{i,1}+1+\eta_i''} \right) \omega_f^{a_{f,2}+e-1-(e-1-\eta_f'')} \left(\prod_{i=f+1}^{2f-1} \omega_i^{a_{i,2}+e-1-(l-1-\eta_i'')} \right)$$

$$= \omega_0^{a_{0,1}+1-a_{0,2}} \omega_{2f}^{a_{0,2}+e-1-(l-1-\eta_{2f}'')} \left(\prod_{i=1}^{f-1} \omega_i^{a_{i,1}+1+\eta_i''} \right)$$

$$\omega_f^{a_{0,2}-1-a_{0,1}} \omega_f^{a_{0,1}+1+\eta_f''} \left(\prod_{i=f+1}^{2f-1} \omega_i^{a_{i,2}+e-1-(l-1-\eta_i'')} \right)$$

$$= \omega_0^{a_{0,1}+1-a_{0,2}} \omega_f^{a_{0,2}-1-a_{0,1}} \prod_{i=1}^{f} \omega_i^{a_{i,1}+1+\eta_i''} \prod_{i=f+1}^{2f} \omega_i^{a_{i,2}+e-1-(l-1-\eta_i'')}$$

$$= \omega_0^{a_{0,1}+1-a_{0,2}} \omega_f^{a_{0,2}-1-a_{0,1}} \prod_{i=1}^{f} \omega_i^{a_{i,1}+1} \prod_{i=f+1}^{2f} \omega_i^{a_{i,2}+e-l} \prod_{i=1}^{2f} \omega_i^{\eta_i''}$$

$$= (\phi'/\phi'')(\phi|_{I_{K'}}/\phi') \prod_{i=1}^{2f} \omega_i^{\eta_i''}.$$

So $\psi = (\phi'/\phi'')(\phi|_{I_{K'}}/\phi')\phi'' = \phi|_{I_{K'}}$, as we required. $\qquad\square$

Remark 4.1.21. Again, if $e = l - 1$, it is no longer the case that for every $\overline{\rho}$ with determinant $\prod_{\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1} + a_{\sigma,2} + e}$ can we can apply Lemma 4.1.19 to find a crystalline diagonal lift of Hodge type lifting a. For a counterexample, take $l = 7$, $[k : \mathbb{F}_l] = 2$, and label the two embeddings $k \hookrightarrow \overline{\mathbb{F}}_l$ as σ_1 and σ_2. Then take

$$a_{\sigma_1,1} = l - 1, \qquad\qquad a_{\sigma_2,1} = 1,$$
$$a_{\sigma_1,2} = a_{\sigma_2,2} = 0, \qquad\qquad \overline{\rho} = \mathrm{Ind}\,_{G_K}^{G_{K'}} \psi$$

where $\psi : G_{K'} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ has

$$\psi|_{I_{K'}} = \omega_{\tilde{\sigma}_2}^{l^3(l-1) + l^2 4 + l(l-1) + (l-4)}$$

for $\tilde{\sigma}_2 : k' \to \overline{\mathbb{F}}_l$ an embedding extending σ_2.

Lemma 4.1.22. *If $K/\mathbb{Q}_l$ is unramified and $\overline{\rho}$ is semisimple, then $W^{\mathrm{BDJ}}(\overline{\rho}) = W^{\mathrm{Sch}}(\overline{\rho})$. Similarly, if $K/\mathbb{Q}_l$ is unramified and $\overline{\rho}$ is reducible, then $W^{\mathrm{BDJ}}(\overline{\rho}) = W^{\mathrm{GHS}}(\overline{\rho})$, and if K is arbitrary, $\overline{\rho}$ is reducible and $\overline{\rho}$ is semisimple, then $W^{\mathrm{Sch}}(\overline{\rho}) = W^{\mathrm{GHS}}(\overline{\rho})$.*

Proof. This follows immediately from Lemmas 4.1.16 and 4.1.19, together with the definitions of $W^{\mathrm{BDJ}}(\overline{\rho})$, $W^{\mathrm{Sch}}(\overline{\rho})$ and $W^{\mathrm{GHS}}(\overline{\rho})$. $\qquad\square$

This motivates the following definition.

Definition 4.1.23. Suppose that $K/\mathbb{Q}_l$ is a finite extension, and that $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is a continuous representation. Then we define a set $W^{\mathrm{explicit}}(\overline{\rho})$ of Serre weights as follows:

- If $\overline{\rho}$ is irreducible, we set $W^{\mathrm{explicit}}(\overline{\rho}) := W^{\mathrm{Sch}}(\overline{\rho})$.
- If $\overline{\rho}$ is reducible, we set $W^{\mathrm{explicit}}(\overline{\rho}) := W^{\mathrm{GHS}}(\overline{\rho})$.

Remark 4.1.24. It is an immediate consequence of Lemma 4.1.22 that if $\overline{\rho}$ is semisimple then $W^{\mathrm{explicit}}(\overline{\rho}) = W^{\mathrm{Sch}}(\overline{\rho})$.

Proposition 4.1.25. *We have $W^{\mathrm{explicit}}(\overline{\rho}) \subset W^{\mathrm{diag}}(\overline{\rho})$.*

Proof. If $\overline{\rho}$ is irreducible, this follows from Lemma 4.1.19. If $\overline{\rho}$ is reducible, then this follows from the definition of $W^{\mathrm{GHS}}(\overline{\rho})$, together with point (6) of the list of properties of $\sim$ in section 3. $\qquad\square$

Having obtained a lower bound on $W^{\mathrm{diag}}(\overline{\rho})$, we now consider whether there are any obvious upper bounds. Here our results are rather less complete. Firstly, we have the following conjecture.

Conjecture 4.1.26. *([EGHS13]) $W^{\mathrm{cris}}(\overline{\rho}) = W^{\mathrm{explicit}}(\overline{\rho})$.*

By Proposition 4.1.25 we have $W^{\mathrm{explicit}}(\overline{\rho}) \subset W^{\mathrm{cris}}(\overline{\rho})$, so to prove this conjecture it would be enough to show that $W^{\mathrm{cris}}(\overline{\rho}) \subset W^{\mathrm{explicit}}(\overline{\rho})$. This is presumably accessible to the techniques of integral l-adic Hodge theory, but in the absence of any further insight we suspect that an attempt to prove the result would result in extensive unpleasant computation. In lieu of such calculations, we recall what is known in the case that $K/\mathbb{Q}_l$ is unramified or highly ramified.

Lemma 4.1.27. *Suppose that $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation, and that $a \in W^{\mathrm{cris}}(\overline{\rho})$ is a Serre weight. Then*

$$\det \overline{\rho}|_{I_K} = \prod_{\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1} + a_{\sigma,2} + e}.$$

Proof. This follows immediately from the definition of $W^{\mathrm{cris}}(\overline{\rho})$ and Lemma 4.1.15. $\square$

Lemma 4.1.28. *Suppose that K has absolute ramification index $e \geq l$, and that $\overline{\rho}$ is semisimple. Then $W^{\mathrm{cris}}(\overline{\rho}) \subset W^{\mathrm{explicit}}(\overline{\rho})$, so that $W^{\mathrm{cris}}(\overline{\rho}) = W^{\mathrm{diag}}(\overline{\rho}) = W^{\mathrm{explicit}}(\overline{\rho})$, and all three sets consist of precisely the set of Serre weights a with*

$$\det \overline{\rho}|_{I_K} = \prod_{\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1} + a_{\sigma,2} + e}.$$

Proof. This is an immediate consequence of Lemma 4.1.27, and Corollaries 4.1.17 and 4.1.20. $\square$

Theorem 4.1.29. *If K is absolutely unramified then $W^{\mathrm{cris}}(\overline{\rho}) = W^{\mathrm{explicit}}(\overline{\rho})$.*

Proof. This is the main result of [GLS12a]. $\square$

Remark 4.1.30. It is also possible to argue globally to obtain bounds on the set of Serre weights by considering lifts of weight 0 and nontrivial type, as was done in [Gee11b] and [Sch08]. In [GLS12b] these methods are combined with the results of this paper to completely determine the set of Serre weights in the totally ramified case; see Theorem 5.1.6 below.

Definition 4.1.31. Let e be a positive integer. We say that a Serre weight $a \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)}$ is *e-regular* if $a_{\sigma,1} - a_{\sigma,2} \leq l - 1 - e$ for all $\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)$.

The arguments of [Sch08] can presumably be carried over to the present setting to prove the following analogue of Theorem 3.4 of [Sch08]. Suppose that $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible and modular of some Serre weight $a \in (\mathbb{Z}_+^2)_0^{\coprod_{v|l} \mathrm{Hom}\,(k_v, \overline{\mathbb{F}}_l)}$. Let $v|l$ be a place of F such that $\overline{r}|_{G_{F_v}}$ is irreducible and the corresponding weight $a_v \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k_v, \overline{\mathbb{F}}_l)}$ is e-regular. Then $a_v \in W^{\mathrm{explicit}}(\overline{r}|_{G_{F_v}})$.

5. The main theorems

5.1. We now combine the results of the previous sections to prove a variety of concrete theorems.

Fix an imaginary CM field F with maximal totally real subfield F^+, such that

- F/F^+ is unramified at all finite places.
- Every place $v|l$ of F^+ splits in F.
- $[F^+ : \mathbb{Q}]$ is even.

Let $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be a continuous irreducible representation which is modular in the sense of Definition 2.1.9. In particular, $\overline{r}$ has split ramification in the sense of Definition 2.1.7, and $\overline{r}^c \cong \overline{r}^\vee \overline{\epsilon}_l^{-1}$. We define sets $W^{\mathrm{BDJ}}(\overline{r})$ and $W^{\mathrm{explicit}}(\overline{r})$ of Serre weights as follows (cf. Definition 4.1.7). The set $W^{\mathrm{BDJ}}(\overline{r})$ is only defined if l is unramified in F.

Definition 5.1.1. $W^{\mathrm{explicit}}(\bar{r})$ (respectively $W^{\mathrm{BDJ}}(\bar{r})$) is the set of Serre weights $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ such that for each place $w|l$, the corresponding Serre weight $a_w \in (\mathbb{Z}_+^2)^{\mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ is an element of $W^{\mathrm{explicit}}(\bar{r}|_{G_{F_w}})$ (respectively $W^{\mathrm{BDJ}}(\bar{r}|_{G_{F_w}})$).

Remark 5.1.2. In fact $W^{\mathrm{BDJ}}(\bar{r}) = W^{\mathrm{explicit}}(\bar{r})$ when both are defined, but as the definition of $W^{\mathrm{BDJ}}(\bar{r})$ is perhaps more familiar to the reader, we prefer to separate them.

Theorem 5.1.3. *Let F be an imaginary CM field with maximal totally real subfield F^+. Assume that $\zeta_l \notin F$, that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Assume that $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight. Assume that $a \in W^{\mathrm{explicit}}(\bar{r})$. Then $\bar{r}$ is modular of weight a.

Proof. By Proposition 4.1.25, $a \in W^{\mathrm{diag}}(\bar{r})$, so the result follows from Theorem 4.1.9. $\square$

We can make this result particularly explicit in the cases where l is either unramified or highly ramified in F. The following theorem is the main global result of [GLS12a].

Theorem 5.1.4. *Let F be an imaginary CM field with maximal totally real subfield F^+. Assume that $\zeta_l \notin F$, that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that $[F^+ : \mathbb{Q}]$ is even. Assume that l is unramified in F. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Assume that $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight. Then $\bar{r}$ is modular of weight a if and only if $a \in W^{\mathrm{BDJ}}(\bar{r})$.

Proof. By Definition 4.1.23 and Lemma 4.1.22, $W^{\mathrm{BDJ}}(\bar{r}) = W^{\mathrm{explicit}}(\bar{r})$. The result now follows from Theorem 5.1.3, Corollary 4.1.8 and Theorem 4.1.29. $\square$

Theorem 5.1.5. *Let F be an imaginary CM field with maximal totally real subfield F^+. Assume that $\zeta_l \notin F$, that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that $[F^+ : \mathbb{Q}]$ is even. Assume that for each place $w|l$ of F the absolute ramification index of F_w is at least l, and that $\bar{r}|_{G_{F_w}}$ is semisimple. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Assume that $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight. Then $\bar{r}$ is modular of weight a if and only if for each $w|l$,

$$\det \bar{r}|_{I_{F_w}} = \prod_{\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)} \omega_\sigma^{a_{\sigma,1} + a_{\sigma,2} + e}.$$

Proof. The necessity of the given condition follows from Corollary 4.1.8 and Lemma 4.1.28, and the sufficiency from Theorem 5.1.3 and Lemma 4.1.28 again. $\square$

Finally, using the results of this paper together with potential automorphy techniques and calculations with Breuil modules, the following theorem is proved in [GLS12b].

Theorem 5.1.6. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, that $\zeta_l \notin F$, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification such that $\bar{r}(G_{F(\zeta_l)})$ is adequate. Assume that for each place $w|l$ of F, $F_w/\mathbb{Q}_l$ is totally ramified.*

Let $a \in (\mathbb{Z}_+^2)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight. Then $a \in W^{\mathrm{explicit}}(\bar{r})$ if and only if $\bar{r}$ is modular of weight a.

Appendix A. Adequacy

In this section we will prove some slight improvements to results from [BLGGT10], which we apply in the main part of the paper. In order to do this, we build on the results of [GHTT10] (which proves a general criterion for adequacy of subgroups of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$) to give a precise description of the adequate subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$.

A.1. **The definition.**

Definition A.1.1. We call a finite subgroup $H \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ *adequate* if the following conditions are satisfied.

(1) H has no non-trivial quotient of l-power order (i.e. $H^1(H, \overline{\mathbb{F}}_l) = (0)$).
(2) $l \nmid n$.
(3) The elements of H with order coprime to l span $M_{n \times n}(\overline{\mathbb{F}}_l)$ over $\overline{\mathbb{F}}_l$. (This implies that $\overline{\mathbb{F}}_l^n$ is an irreducible representation of H.)
(4) $H^1(H, \mathfrak{gl}_n(\overline{\mathbb{F}}_l)) = (0)$.

(The notion of adequacy was introduced in [Tho12]. The formulation above is as in [BLGGT10], and while it is not identical to that in [Tho12], it is equivalent to it by the discussion following the definition of adequacy in Section 2.1 of [BLGGT10].)

Lemma A.1.2. *If $l \nmid \#H$ and H acts irreducibly, then H is adequate.*

Proof. The first statement in the definition of adequacy is trivial. For the second, observe that because $l \nmid \#H$, the tautological representation $H \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ will lift to characteristic zero, and hence n is the dimension of an irreducible characteristic zero representation of H and so divides $\#H$. It follows $l \nmid n$. For the third, we see that elements of H with order coprime to l will just be all the elements of H and will span $M_{n \times n}(\overline{\mathbb{F}}_l)$ over $\overline{\mathbb{F}}_l$ since H acts irreducibly. For the fourth, we use Corollary 1 of section VIII.2 of [Ser79]. $\square$

A small point of notation: throughout this section, we will be considering subgroups of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ for some n, and we will often find it useful to write V for the vector space $\overline{\mathbb{F}}_l^n$, especially considered as a representation of some subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ which should be clear from context.

The following lemmas will be useful. They were proved in the related context of bigness by Snowden and Wiles (see Propositions 2.1 and 2.2 of [SW10]), and the proofs generalize very straightforwardly.

Lemma A.1.3. *Suppose $H \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is a finite subgroup, and $N \triangleleft H$ is a normal subgroup which is adequate and has $[H:N]$ prime to l. Then H is adequate.*

Proof. Points (1), (2) and (3) are trivial. There is an exact sequence

$$H^1(H/N, \mathfrak{gl}_n(\overline{\mathbb{F}}_l)^N) \longrightarrow H^1(H, \mathfrak{gl}_n(\overline{\mathbb{F}}_l)) \longrightarrow H^1(N, \mathfrak{gl}_n(\overline{\mathbb{F}}_l))^{G/H}$$

Since N is adequate, $H^1(N, \mathfrak{gl}_n(\overline{\mathbb{F}}_l))$ is trivial and so the right term vanishes.

Since N is adequate, the standard representation of N is irreducible (by condition (3)), and thus $\mathfrak{gl}_n(\overline{\mathbb{F}}_l)^N = \overline{\mathbb{F}}_l \mathbf{1}$ (this uses $l \nmid n$). Then the left term in the exact sequence is just $H^1(H/N, \overline{\mathbb{F}}_l)$ and vanishes since H/N has order prime to l and hence no l power quotients. Thus the middle term in the exact sequence vanishes, establishing (4). □

Lemma A.1.4. *Suppose $H \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is a finite subgroup, and k is a finite extension of $\mathbb{F}_l$. Then H is adequate if and only if $k^\times H$ is adequate.*

Proof. Since H is a normal subgroup of $k^\times H$ of prime-to-l index, the 'only if' part follows from the previous lemma. We now prove the other direction, assuming $k^\times H$ is adequate, and showing H is adequate. Point (2) is trivial. For point (1), let K be a l-power order quotient of H. Since $k^\times \cap H$ has order prime to l, it has trivial image in K. Thus K is a quotient of the group $H/(H \cap k^\times) = k^\times H/k^\times$. By assumption, $k^\times H$ has no nontrivial l-power quotient so K is trivial and we have point (1). For point (3) note that the elements of $k^\times H$ of prime-to-l order will have the same $\overline{\mathbb{F}}_l$ span in $M_{n \times n}(\overline{\mathbb{F}}_l)$ as those of H.

For point (4), note that it will be enough to establish $H^1(H, \mathfrak{sl}_n(\overline{\mathbb{F}}_l)) = (0)$ (see the discussion immediately after the definition), and we may similarly assume $H^1(k^\times H, \mathfrak{sl}_n(\overline{\mathbb{F}}_l)) = (0)$. We have an exact sequence

$$1 \longrightarrow H \longrightarrow k^\times H \longrightarrow G \longrightarrow 1$$

for some quotient G of $k^\times$. We therefore have an exact sequence

$$H^1(k^\times H, \mathfrak{sl}_n(\overline{\mathbb{F}}_l)) \longrightarrow H^1(H, \mathfrak{sl}_n(\overline{\mathbb{F}}_l))^G \longrightarrow H^2(G, (\mathfrak{sl}_n(\overline{\mathbb{F}}_l))^H).$$

The left-hand group vanishes by our assumption that $k^\times H$ is adequate. Since $k^\times H$ is adequate, it acts irreducibly (by condition 3), and so (since $l \nmid n$) we have that $(\mathfrak{sl}_n(\overline{\mathbb{F}}_l))^H$ is trivial, thus the right hand group in the exact sequence vanishes. It follows that the middle term vanishes (alternatively, it vanishes because G has order prime to p). One easily checks that G acts trivially on $H^1(H, \mathfrak{sl}_n(\overline{\mathbb{F}}_l))$, so we are done. □

A.2. **Adequacy for** GL_2**.** The aim of this subsection is to explicate the notion of adequacy for subgroups of GL_2. Theorem 9 of [GHTT10] already tells us that in characteristic greater than 5, 'adequate' simply means 'acts irreducibly', but we would like to have results for characteristics 3 and 5. We prove that subgroups acting irreducibly are adequate apart from some explicit exceptions. More precisely, we prove the following proposition:

Proposition A.2.1. *Suppose that $l > 2$ is a prime, and that $G \leq \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is a finite subgroup which acts irreducibly on $\overline{\mathbb{F}}_l^2$. Then precisely one of the following is true:*

- *We have $l = 3$, and the image of G in $\mathrm{PGL}_2(\overline{\mathbb{F}}_3)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$.*
- *We have $l = 5$, and the image of G in $\mathrm{PGL}_2(\overline{\mathbb{F}}_5)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$.*

- *G is adequate.*

Remark A.2.2. For any G as in the theorem, its image in $\mathrm{PGL}_2(\overline{\mathbb{F}}_l)$, which we will call $\bar{G}$, either must be isomorphic to one of A_5, S_4, A_4, or a dihedral group of order coprime to l, or must be conjugate to $\mathrm{PSL}_2(k)$ or $\mathrm{PGL}_2(k)$ for some finite extension k of $\mathbb{F}_l$ (see Theorem 2.47 (b) of [DDT95]). We show in the course of the proof that if $l = 3$ (resp. $l = 5$) and if $\bar{G}$ is isomorphic to A_4 (resp. A_5) then in fact, $\bar{G}$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$ (resp. $\mathrm{PSL}_2(\mathbb{F}_5)$).

Proof. The proof will be a very straightforward case analysis. On the one hand, we have the list of possibilities for $\bar{G}$ recalled in the previous remark. We divide into cases according to which of these is true, further subdividing the $\mathrm{PSL}_2(k)$ and $\mathrm{PGL}_2(k)$ cases into the subcase where $|k| = l$ and the subcase where $|k| > l$. On the other hand, we divide into cases according to the value of l, considering the cases $l = 3$, $l = 5$ and $l \geq 7$. The resulting 'two dimensional' collection of cases is depicted in Figure 1. We will often give arguments which treat several cases at once, and the reader may find it useful to refer to Figure 1 which summarizes which argument is used in which case. We will number the various points of the argument to make them easier to refer to.

But before we move into the detailed consideration of the cases, it will be useful to discuss in a little more detail the cases where $\bar{G}$ is isomorphic to A_4 and A_5. Let us write $2.A_4$ (resp $2.A_5$) for the binary tetrahedral group (resp binary icosahedral group). (Thus if we consider A_5 as the group of symmetries of an icosahedron, a subgroup of $\mathrm{SO}(3)$, then $2.A_5$ is the inverse image of A_5 under the natural 2-to-1 map $\mathrm{SU}(2) \to \mathrm{SO}(3)$; and similarly for A_4 and the group of symmetries of the tetrahedron.) Specifically, it will be important to us to establish

Sublemma. Suppose that $\bar{G}$ is isomorphic to A_k for $k \in \{4, 5\}$. Then we can find some representation $\widetilde{\phi} : 2.A_k \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ such that $\overline{\mathbb{F}}_l^\times \widetilde{\phi}(2.A_k) = \overline{\mathbb{F}}_l^\times G$.

Proof. Before we can begin the proof proper, we must recall some general facts from the theory of projective modular representations of finite groups. Given any finite group H and prime l, we call a group $\widetilde{H}$ an *l-representation group* of H, if *(a)* $\widetilde{H}$ has a central subgroup A contained in the commutator subgroup $\widetilde{H}'$ of $\widetilde{H}$, *(b)* $\widetilde{H}/A \cong H$ and *(c)* $A \cong H^2(H, \overline{\mathbb{F}}_l^\times)$. We have the following facts. *(1.)* There always exists such a group (not necessarily unique). *(2.)* Given any such group $\widetilde{H}$, and given any homomorphism $\phi : H \to \mathrm{PGL}_n(\overline{\mathbb{F}}_l)$, there is a homomorphism $\widetilde{\phi} : \widetilde{H} \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ such that the maps $\widetilde{H} \twoheadrightarrow H \to \mathrm{PGL}_n(\overline{\mathbb{F}}_l)$ and $\widetilde{H} \to \mathrm{GL}_n(\overline{\mathbb{F}}_l) \twoheadrightarrow \mathrm{PGL}_n(\overline{\mathbb{F}}_l)$ agree. *(3.)* Finally, the group $H^2(H, \overline{\mathbb{F}}_l^\times)$ is just the prime-to-l part of $H^2(H, \overline{\mathbb{Q}}^\times)$, the *Schur multiplier* of H. [The original reference for these three facts is [AOT37], although the first two have older proofs in characteristic 0 which essentially go over unchanged to characteristic l. The authors found a more accessible 'reference' for the first (resp second) of these facts was to read the proof of Theorem 1.2 (resp 1.3) of [HH92], which proves these results in characteristic 0, and observe that the proof goes through in characteristic l. The third fact is [AOT37, Satz 1].]

We wish to apply these facts in the case where H is isomorphic to A_n for $n \geq 4$. By the last sentence of chapter 2 of [HH92] (on p23, just after the unnumbered remark after Theorem 2.12) we see the construction of a group, called there $\widetilde{A}_n$, which is a 'representation group' for A_n. [This means—see the definition at the bottom of [HH92, p6]—a group satisfying the properties *(a–c)* of the previous

	$l = 3$	$l = 5$	$l \geq 7$
dihedral	✓1	✓1	✓0
S_4	− 7	✓1	✓0
A_4	− 7	✓1	✓0
A_5	✓4	− 7	✓0
$\mathrm{PSL}_2(k),\ |k| = l$	× 6	× 5	✓0
$\mathrm{PGL}_2(k),\ |k| = l$	✓3	✓5	✓0
$\mathrm{PSL}_2(k),\ |k| \geq l^2$	✓2	✓2	✓0
$\mathrm{PGL}_2(k),\ |k| \geq l^2$	✓2	✓2	✓0

✓0	Always adequate by appeal to Theorem 9 of [GHTT10] (see point 0)
✓n	Always adequate; see point n.
× n	Never adequate; see point n.
− 7	This case is already included in other cases, and hence needs not be considered in its own right. See point 7.

FIGURE 1. The various cases for the proof of Proposition A.2.1

paragraph, except with $H^2(H, \overline{\mathbb{Q}}^\times)$ replacing $H^2(H, \overline{\mathbb{F}}_l^\times)$.] Given the construction there[1], $\widetilde{A}_n$ is a double cover of A_n, and we conclude that $H^2(A_n, \overline{\mathbb{Q}}^\times) = \mathbb{Z}/2\mathbb{Z}$. But then $H^2(A_n, \overline{\mathbb{Q}}^\times) \cong H^2(A_n, \overline{\mathbb{F}}_l^\times)$ (because $H^2(A_n, \overline{\mathbb{Q}}^\times) = \mathbb{Z}/2\mathbb{Z}$, $l > 2$ and using the fact (3) above) so $\widetilde{A}_n$ satisfies properties *(a–c)* of the previous paragraph. Thus $\widetilde{A}_n$ is in fact also an l-representation group of A_n, for $l > 2$.

Now we begin to the proof proper, and imagine that $\bar{G}$ is, as in the statement of the sublemma, isomorphic to A_k for $k \in \{4, 5\}$. By the discussion of the previous paragraph $\widetilde{A}_k$ is an l-representation group of $\bar{G}$, and so by fact (2) above applied with ϕ the natural inclusion $\bar{G} \hookrightarrow \mathrm{PGL}_2(\overline{\mathbb{F}}_l)$, there is a map $\widetilde{\phi} : \widetilde{A}_k \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ such that $\widetilde{A}_k \twoheadrightarrow A_k \xrightarrow{\sim} \bar{G} \hookrightarrow \mathrm{PGL}_2(\overline{\mathbb{F}}_l)$ and $\widetilde{A}_k \xrightarrow{\widetilde{\phi}} \mathrm{GL}_2(\overline{\mathbb{F}}_l) \twoheadrightarrow \mathrm{PGL}_2(\overline{\mathbb{F}}_l)$ agree, which means that $\overline{\mathbb{F}}_l^\times \widetilde{\phi}(\widetilde{A}_k) = \overline{\mathbb{F}}_l^\times G$.

This gives us everything we need, apart from checking this group $\widetilde{A}_k$ defined in [HH92], is isomorphic to the group $2.A_k$ as defined in the statement of the sublemma. To check this, observe $\widetilde{A}_n$ is defined in [HH92] as a certain subgroup of a certain group $\tilde{S}_n$, which is given a presentation just before Theorem 2.8 of *loc. cit.*, on p18. Comparing this presentation to the discussion in §2.7.2 of [Wil09], we see that $\widetilde{A}_n$ is the same group as the group called $2.A_n$ in [Wil09]. Examining the discussion in §5.6.8 and §5.6.2 of [Wil09], we see that the groups that book calls $2.A_5$ and $2.A_4$ are indeed respectively the binary icosahedral and tetrahedral groups. □

We are now ready to move on to the case analysis that is the proof proper.

Point 0. The majority of cases are handled by an appeal to Theorem 9 of [GHTT10]. In our present notation, this asserts *inter alia* that if we write G^0 for the subgroup of G generated by elements of l-power order and d for the maximal

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[1]Specifically, a group $\tilde{S}_n$ is constructed—see Theorem 2.8 of [HH92]— which is a double cover of S_n; $\widetilde{A}_n$ is defined as the inverse image of A_n under this map.

dimension of an irreducible G^0-submodule of $\overline{\mathbb{F}}_l^2$, then G is adequate so long as $l \geq 2(d+1)$. Since clearly $d \leq 2$, we immediately see that G is automatically adequate in any case with $l \geq 7$.

Point 1. Now we consider the case where either

- $l = 5$ and $\bar{G}$ is isomorphic to S_4 or A_4.
- $l = 3$ or $l = 5$ and $\bar{G}$ is a dihedral group of prime-to-l order

In either of these cases, the projective image of G has order coprime to l, whence G has order coprime to l, which is enough by Lemma A.1.2.

Point 2. Next we consider the case where $l = 3$ or 5 and the projective image of G is $\mathrm{PSL}_2(k)$ or $\mathrm{PGL}_2(k)$ for some k with $|k| \geq l^2$. We claim that G is adequate in this case.

If the projective image of G is $\mathrm{PSL}_2(k)$, then by applying Lemma A.1.4 we can replace G with $(k)^\times G = k^\times \mathrm{SL}_2(k)$, and by applying Lemma A.1.4 again we can replace G with $\mathrm{SL}_2(k)$. If the projective image of G is $\mathrm{PGL}_2(k)$, then by a similar argument we can replace G with $\mathrm{GL}_2(k)$ and then by applying Lemma A.1.3 we can again replace G with $\mathrm{SL}_2(k)$. Thus in either case we may assume that $G = \mathrm{SL}_2(k)$.

Let us verify the conditions for adequacy in turn:

- We see that G has no non-trivial quotient of l power order since the simplicity of $\mathrm{PSL}_2(\mathbb{F}_{3^n})$ and $\mathrm{PSL}_2(\mathbb{F}_{5^n})$ for $n \geq 2$ tells us G in fact has no Jordan Hölder constituent of l-power order.
- The fact that $l \nmid n = 2$ is trivial.
- Certainly the elements of $\mathrm{SL}_2(k)$ of order prime to l span $M_{2 \times 2}(\overline{\mathbb{F}}_l)$ as an $\overline{\mathbb{F}}_l$-vector space (one may use the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}$ for any $\alpha \in k^\times$, $\alpha \neq \pm 1$).
- To verify the fourth condition it will suffice to check $H^1(G, \mathfrak{sl}_n(\overline{\mathbb{F}}_l)) = (0)$. Since $G = \mathrm{SL}_2(k)$, this is just $H^1(\mathrm{SL}_2(k), \mathfrak{sl}_n(\overline{\mathbb{F}}_l)) = (0)$, which follows, under our present assumptions, from Lemma 2.48 of [DDT95].

Point 3. We now turn to the case where $l = 3$ and $\bar{G}$ is conjugate to $\mathrm{PGL}_2(\mathbb{F}_3)$. We claim that G is adequate in this case. Applying Lemma A.1.4 twice, we may assume that $G = \mathrm{GL}_2(\mathbb{F}_3)$. Since $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$, we see that G has no quotients of 3-power order. Indeed, S_4 has 3 subgroups of index 3 and they are all conjugate, being 2-Sylow subgroups. Thus the first condition for adequacy holds. The second condition holds trivially. For the third condition, we note that the elements $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{F}_3)$ are semi-simple and span $M_{2 \times 2}(\overline{\mathbb{F}}_3)$ as an $\overline{\mathbb{F}}_3$-vector space. To verify the fourth condition, we think of $\mathrm{SL}_2(\mathbb{F}_3)$ as a normal subgroup of $\mathrm{GL}_2(\mathbb{F}_3)$ with quotient Q of order 2, giving us an exact sequence

$$H^1(Q, \mathfrak{gl}_2(\overline{\mathbb{F}}_3)^{\mathrm{SL}_2(\mathbb{F}_3)}) \longrightarrow H^1(\mathrm{GL}_2(\mathbb{F}_3), \mathfrak{gl}_2(\overline{\mathbb{F}}_3)) \longrightarrow H^1(\mathrm{SL}_2(\mathbb{F}_3), \mathfrak{gl}_2(\overline{\mathbb{F}}_3))^Q.$$

The right term vanishes by appeal to Lemma 2.48 of [DDT95], which tells us that $H^1(\mathrm{SL}_2(\mathbb{F}_3), \mathfrak{sl}_2(\overline{\mathbb{F}}_3))$ is trivial. On the other hand $\mathfrak{gl}_2(\overline{\mathbb{F}}_3)^{\mathrm{SL}_2(\mathbb{F}_3)} = \mathfrak{sl}_2(\overline{\mathbb{F}}_3)^{\mathrm{SL}_2(\mathbb{F}_3)} \oplus (\mathbf{1}\overline{\mathbb{F}}_3)^{\mathrm{SL}_2(\mathbb{F}_3)} = \mathbf{1}\overline{\mathbb{F}}_3$ (since $\mathfrak{sl}_2(\overline{\mathbb{F}}_3)$ is irreducible and nontrivial under the action of

$\mathrm{SL}_2(\mathbb{F}_3)$); and $H^1(Q, \mathbf{1}\overline{\mathbb{F}}_3) = (0)$. So the left term vanishes too. Thus $H^1(\mathrm{GL}_2(\mathbb{F}_3), \mathfrak{gl}_2(\overline{\mathbb{F}}_3)) = (0)$; that is, $H^1(G, \mathfrak{gl}_2(\overline{\mathbb{F}}_3)) = (0)$, as required.

Point 4. We now treat the case where $l = 3$ and $\bar{G} \cong A_5$. We claim G is adequate in this case. Applying the sublemma we can find some irreducible two-dimensional $\overline{\mathbb{F}}_l$-representation $\widetilde{\phi}$ of $2.A_5$ such that $\overline{\mathbb{F}}_l^{\times} \widetilde{\phi}(2.A_5) = \overline{\mathbb{F}}_l^{\times} G$. Having done this, by applying Lemma A.1.4 twice, we see that to show G adequate it suffices to show $\widetilde{\phi}(2.A_5)$ adequate. By consulting [JLPW95, p2], we see that $2.A_5$ has only two 2-dimensional irreducible mod 3 representations, corresponding to the Brauer characters ϕ_5 and ϕ_6 there. By comparing with [CCN+, p2] we see that these Brauer characters each come from characteristic 0 characters, viz the characters called χ_6 and χ_7 in [CCN+, p2], the first of which corresponds to $\rho_{\mathrm{nat},2.A_5}$, the natural representation we get by thinking of $2.A_5$ as the binary icosahedral group, and the second to $\rho_{\mathrm{nat},2.A_5}^{(12)}$. This means that $\widetilde{\phi}$ is either the reduction mod 3 of $\rho_{\mathrm{nat},2.A_5}$ or of $\rho_{\mathrm{nat},2.A_5}^{(12)}$. We will write $\bar{\rho}_{\mathrm{nat},2.A_5}$ and $\bar{\rho}_{\mathrm{nat},2.A_5}^{(12)}$ for these reductions.

We shall now verify that $\widetilde{\phi}(2.A_5)$ is adequate, verifying the conditions in turn.

- The first condition (no l-power order quotients) follows immediately from the simplicity of A_5, which shows $\widetilde{\phi}(2.A_5)$ can have no l-power order Jordan Hölder constituents.
- The second condition, $l \nmid n$, is trivial.
- Examining [JLPW95, p2], we see that the character ϕ_5 is real, so the dual representation of $\bar{\rho}_{\mathrm{nat},2.A_5}$ has the same character and $\mathrm{ad}^0 \bar{\rho}_{\mathrm{nat},2.A_5}$ has character $\phi_5^2 - 1$. We recognize this character as ϕ_2 from the table. Thus $\mathrm{ad}^0 \bar{\rho}_{\mathrm{nat}}$ is irreducible. Similarly $\mathrm{ad}^0 \bar{\rho}_{\mathrm{nat},2.A_5}^{(12)}$ has character ϕ_3, which is irreducible. It follows that $\mathrm{ad}^0 V = \mathfrak{sl}_2 \overline{\mathbb{F}}_l$ is irreducible. Choose $g \in \widetilde{\phi}(2.A_5)$ to be the image under $\widetilde{\phi}$ of some non-central element of $2.A_5$ of order prime to 3. Then g is not a scalar and acts semisimply, so is conjugate to $\mathrm{diag}(\alpha, \beta)$ where $\alpha \neq \beta$. Then it is easy to check that $\pi_{g,\alpha}\mathfrak{sl}_2(\overline{\mathbb{F}}_l)\iota_{h,\alpha} \neq (0)$. Thus we see that condition (C) of [GHTT10] holds, which is equivalent to the third condition for adequacy by Lemma 1 of [GHTT10].
- To verify the fourth condition it will suffice to check $H^1(\widetilde{\phi}(2.A_5), \mathfrak{sl}_2(\overline{\mathbb{F}}_l)) = (0)$. Recall that $\widetilde{\phi}$ is $\bar{\rho}_{\mathrm{nat},2.A_5}$ or $\bar{\rho}_{\mathrm{nat},2.A_5}^{(12)}$, both of which are easily seen to be injective. Thus we must show $H^1(2.A_5, \mathrm{ad}^0 \widetilde{\phi}) = (0)$ for $\widetilde{\phi} = \bar{\rho}_{\mathrm{nat},2.A_5}$ and $\widetilde{\phi} = \bar{\rho}_{\mathrm{nat},2.A_5}^{(12)}$. We give the argument for $\widetilde{\phi} = \bar{\rho}_{\mathrm{nat},2.A_5}$, the other case being entirely analogous. It is easy to see that $\mathrm{ad}^0 \bar{\rho}_{\mathrm{nat},2.A_5}$ is the natural three-dimensional representation $\bar{\rho}_{3,2.A_5}$ we get by mapping to A_5, realizing A_5 as the symmetries of a icosahedron, then reducing mod 3. By Proposition 46 in Section 16.4 of [Ser77], we see that $\mathrm{ad}^0 \bar{\rho}_{\mathrm{nat},2.A_5}$ is a projective $\overline{\mathbb{F}}_3[2.A_5]$-module, so it is the only simple module in its block, and in particular any extension of the trivial representation by $\mathrm{ad}^0 \bar{\rho}_{\mathrm{nat},2.A_5}$ splits, as required. (We thank Florian Herzig for supplying us with this argument.)

Point 5. Next we consider the case where $l = 5$ and $\bar{G}$ is $\mathrm{PSL}_2(\mathbb{F}_5)$ or $\mathrm{PGL}_2(\mathbb{F}_5)$. In the case where $\bar{G}$ is $\mathrm{PSL}_2(\mathbb{F}_5)$, Table 4.5 of [CPS75] tells us that $H^1(G, \mathfrak{gl}_2(\overline{\mathbb{F}}_l))$ is one dimensional, violating the fourth condition in the definition of adequacy. Thus in this case G will fail to be adequate. In the case where $\bar{G}$ is $\mathrm{PGL}_2(\mathbb{F}_5)$, we have $H^1(G, \mathfrak{gl}_n(\overline{\mathbb{F}}_l)) = (0)$ by [BP12, 5.6], and the first two conditions are trivially

verified. It remains to check the third condition, which is easily verified by hand, or by arguing as in the last paragraph of the proof of Theorem 9 of [GHTT10]. (We thank Robert Guralnick and Florian Herzig for pointing this out to us.)

Point 6. Next we consider the case where $l = 3$ and $\bar{G}$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$. We claim that G is not adequate in this case. Since $\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4$, it suffices to note that A_4 has a quotient of order 3, so that G must also have a quotient of order 3. This violates the first condition for adequacy.

Point 7. We now treat the remaining cases. We start with the case where where $l = 5$ and $\bar{G} \cong A_5$. It is obvious that this case *includes* the case already considered where we have (up to conjugation) an equality $\bar{G} = \mathrm{PSL}_2(\mathbb{F}_5)$ (rather than a mere isomorphism), since $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$. But we will show that in fact whenever $\bar{G} \cong A_5$ we must indeed have $\bar{G} = \mathrm{PSL}_2(\mathbb{F}_5)$ up to conjugation, thus reducing this case to a case we have already considered.

Applying the sublemma we can find some irreducible mod 5 representation $\widetilde{\phi}$ of $2.A_5$ such that $\overline{\mathbb{F}}_l^\times \widetilde{\phi}(2.A_5) = \overline{\mathbb{F}}_l^\times G$. Having done this, by applying Lemma A.1.4 twice, we see that to show G inadequate it suffices to show $\widetilde{\phi}(2.A_5)$ inadequate. By consulting [JLPW95, p2], we see that $2.A_5$ has only one mod 5 Brauer character of dimension 2. But $2.A_5 \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{F}_5) \hookrightarrow \mathrm{GL}_2(\mathbb{F}_5)$ (see [CCN⁺, p2]) is clearly an irreducible representation mod 5 of dimension 2, so we deduce that $\widetilde{\phi}$ must be exactly this map. This reduces us to the case $\bar{G} = \mathrm{PSL}_2(\mathbb{F}_5)$.

Similar arguments allow us to see that the (apparently more general) case where $l = 3$ and $\bar{G} \cong A_4$ is actually included in the case that $\bar{G}$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$.

Finally, again using similar arguments, we can reduce the case where $l = 3$ and $\bar{G} \cong S_4$ to the case where $\bar{G}$ is conjugate to $\mathrm{PGL}_2(\mathbb{F}_3)$.

$\square$

A.3. **Adequacy for tensor products.** We would like to thank Richard Taylor for allowing us to include the following lemma here; it was originally proved by him during the writing of [BLGGT10].

Lemma A.3.1. *Suppose that Γ is a group and that $r_i : \Gamma \to \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_l)$ is a representation of Γ for $i = 1, 2$. Suppose moreover that $r_1(\Gamma)$ is adequate, that $r_2|_{\ker r_1}$ is irreducible and that $r_2(\Gamma)$ has order prime to l. Then $(r_1 \otimes r_2)(\Gamma)$ is adequate.*

Proof. Write H_i for the image of r_i and H for the image of $r_1 \otimes r_2$. Write K_i for $r_i(\ker r_{3-i})$. Write Z for the set of $z \in \overline{\mathbb{F}}_l^\times$ for which there exists $\gamma \in \Gamma$ with $r_1(\gamma) = z$ and $r_2(\gamma) = z^{-1}$. Then there is a natural identification

$$H_1/K_1 = \Gamma/(\ker r_1).(\ker r_2) = H_2/K_2$$

and an exact sequence

$$\{1\} \longrightarrow Z \longrightarrow \{(h_1, h_2) \in H_1 \times H_2 : \ h_1 \bmod K_1 = h_2 \bmod K_2\} \longrightarrow H \longrightarrow \{1\}.$$

In particular there is an exact sequence

$$\{1\} \longrightarrow Z \longrightarrow H_1 \longrightarrow H/K_2 \longrightarrow \{1\}.$$

It is easy to check the first two conditions for H to be adequate. (Note that $\dim r_2 | \#H_2$, so that $l \nmid \dim r_2$, and that any l-power order quotient of H would yield an l-power order quotient of $H/K_2 \cong H_1/Z$ and thus of H_1, a contradiction.)

To check the third condition, suppose that $A_i \in M_{n_i \times n_i}(\overline{\mathbb{F}}_l)$. We can write

$$A_1 = \sum_i a_i r_1(\gamma_i)$$

for some $a_i \in \overline{\mathbb{F}}_l$ and $\gamma_i \in \Gamma$ with $r_1(\gamma_i)$ semi-simple. We can also write

$$r_2(\gamma_i^{-1})A_2 = \sum_j b_{ij} r_2(\delta_{ij})$$

for some $b_{ij} \in \overline{\mathbb{F}}_l$ and some $\delta_{ij} \in \ker r_1$. Then

$$
\begin{aligned}
& \sum_{i,j} a_i b_{ij} (r_1 \otimes r_2)(\gamma_i \delta_{ij}) \\
=\ & \sum_i a_i r_1(\gamma_i) \otimes (r_2(\gamma_i) \sum_j b_{ij} r_2(\delta_{ij})) \\
=\ & \sum_i a_i r_1(\gamma_i) \otimes A_2 \\
=\ & A_1 \otimes A_2.
\end{aligned}
$$

Moreover each $r_1(\gamma_i \delta_{ij}) = r_1(\gamma_i)$ is semi-simple by assumption and each $r_2(\gamma_i \delta_{ij})$ is semi-simple as H_2 has order prime to l. Thus H satisfies the third condition to be adequate.

To check the fourth condition it suffices by the Hochschild-Serre spectral sequence to check that $H^1(H/K_2, \operatorname{ad}(r_1 \otimes r_2)^{K_2}) = (0)$ and $H^1(K_2, \operatorname{ad}(r_1 \otimes r_2))^H = (0)$. However

$$H^1(H/K_2, \operatorname{ad}(r_1 \otimes r_2)^{K_2}) = H^1(H/K_2, \operatorname{ad} r_1) = H^1(H_1/Z, \operatorname{ad} r_1) = H^1(H_1, \operatorname{ad} r_1) = (0)$$

and

$$H^1(K_2, \operatorname{ad}(r_1 \otimes r_2))^H = ((\operatorname{ad} r_1) \otimes H^1(K_2, \operatorname{ad} r_2))^H = (0)$$

(since K_2 has order prime to l). The lemma follows. $\square$

A.4. **An improvement to a lifting result of** [BLGGT10]. We now prove a slight variant of Theorem 4.3.1 of [BLGGT10]. At the expense of assuming that the representation $\bar{r}$ admits a potentially automorphic lift, we are able to weaken the assumption on the prime l. We will follow the proof of Theorem 4.3.1 of [BLGGT10], and in particular we refer to [BLGGT10] for any notation not already defined in the present paper.

Theorem A.4.1. *Let n be a positive integer and l an odd prime. Suppose that F is a CM field not containing ζ_l and with maximal totally real subfield F^+. Let S be a finite set of finite places of F^+ which split in F and suppose that S includes all places above l. For each $v \in S$ choose a prime $\tilde{v}$ of F above v.*

Let $\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ be a continuous, totally odd, de Rham character unramified outside S. Also let

$$\bar{r} : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

be a continuous representation unramified outside S with $\nu \circ \bar{r} = \overline{\mu}$ and $\bar{r}^{-1} \mathcal{G}_n^0(\overline{\mathbb{F}}_l) = G_F$. Suppose that $\breve{\bar{r}}|_{G_{F(\zeta_l)}}$ is irreducible, and that $\breve{\bar{r}}(G_{F(\zeta_l)})$ is adequate.

For $v \in S$, let $\rho_v : G_{F_{\tilde{v}}} \to \operatorname{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ denote a lift of $\breve{\bar{r}}|_{G_{F_{\tilde{v}}}}$. If $v|l$ we assume that ρ_v is potentially diagonalizable and that, for all $\tau : F_{\tilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$, the multiset $\operatorname{HT}_\tau(\rho_v)$ consists of n distinct integers.

Assume further that there is a finite extension of CM fields F'/F and a RAECSDC automorphic representation (π', χ') of $\operatorname{GL}_n(\mathbb{A}_{F'})$ such that

- *F' does not contain ζ_l,*
- *(π', χ') is unramified outside the set of primes above S,*

- $(\bar{r}_{l,\imath}(\pi'), \bar{r}_{l,\imath}(\overline{\chi}')) \cong (\bar{r}|_{G_{F'}}, \overline{\mu}|_{G_{F'}})$,
- *for all places $w|l$ of F', $r_{l,\imath}(\pi')|_{G_{F'_w}}$ is potentially diagonalizable, and*
- $\breve{\bar{r}}(G_{F'(\zeta_l)})$ *is adequate.*

Then there is a lift

$$r : G_{F^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

of $\bar{r}$ such that

(1) $\nu \circ r = \mu$;
(2) *if $v \in S$ then $\breve{r}|_{G_{F_{\widetilde{v}}}} \sim \rho_v$;*
(3) *r is unramified outside S;*
(4) *$r|_{G_{F'}}$ is automorphic of level potentially prime to l.*

Proof. We begin the proof with some brief remarks that may help to orient the reader. In comparison to Theorem 4.3.1 of [BLGGT10], we have weakened the hypothesis that $l \geq 2(d+1)$, where d is the maximal dimension of an irreducible subrepresentation for the subgroup of $\bar{r}(G_{F(\zeta_l)})$ generated by elements of order l, to the hypothesis that $\bar{r}(G_{F(\zeta_l)})$ is adequate (this condition is implied by the assumption that $l \geq 2(d+1)$ by Theorem 9 of [GHTT10]). On the other hand, we have had to add the hypothesis that $\bar{r}|_{G_{F'}}$ is automorphic. In the proof of Theorem 4.3.1 of [BLGGT10], an appeal is made to Proposition 3.3.1 of *op. cit.*, which proves that $\bar{r}$ is potentially automorphic. We do not know whether Proposition 3.3.1 of [BLGGT10] can be proved using only the condition that $\bar{r}(G_{F(\zeta_l)})$ is adequate, rather than the condition that $l \geq 2(d+1)$; the difficulty lies in establishing when the induction of an adequate representation is adequate.

The proof below is essentially a combination of the proofs of Theorem 4.3.1 and Proposition 3.3.1 of [BLGGT10]. The reason that we need to incorporate details of the proof of Proposition 3.3.1 of [BLGGT10] is that in addition to proving the potential automorphy of $\bar{r}$, the Proposition also shows that $\bar{r}$ potentially admits an ordinary automorphic lift with prescribed behaviour at places not dividing l. In order to carry out the rest of the proof of Theorem 4.3.1 of [BLGGT10] in our setting, we need to produce such a lift of $\bar{r}|_{G_{F'}}$, possibly after making a further solvable base change. We can do this using the techniques of [BLGGT10].

In outline, we do the following: we choose a solvable CM extension F_1/F' with various helpful local properties. We then use the methods of [BLGGT10] to produce an ordinary automorphic lift r_1 of $\bar{r}|_{G_{F_1}}$. The arguments of [GG12], as refined in [Tho12] and [BLGGT10], allow us to replace this with an ordinary automorphic lift $r_{l,\imath}(\pi'_1)$ which has the behaviour prescribed for r at places not dividing l. The techniques of [BLGGT10] then allow us to produce the representation r, and the automorphicity of $r|_{G_{F'}}$ follows as a byproduct of the construction.

We now begin the proof proper. We may suppose that for $v \in S$ with $v \nmid l$ the representation ρ_v is robustly smooth (see Lemma 1.3.2 of [BLGGT10]) and hence lies on a unique component $\mathcal{C}_v$ of $R^{\square}_{\breve{\bar{r}}|_{G_{F_{\widetilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$. If $v|l$ is a place of F^+ then choose a finite extension $K_v/F_{\widetilde{v}}$ over which ρ_v becomes crystalline, and let $\mathcal{C}_v$ denote the unique component of $R^{\square}_{\breve{\bar{r}}|_{G_{F_{\widetilde{v}}}}, \{\mathrm{HT}_\tau(\rho_v)\}, K_v-\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$ on which ρ_v lies.

Let $\widetilde{\mu}$ denote the Teichmuller lift of $\overline{\mu}$. Choose a positive integer m which is greater than one plus the difference of every two Hodge–Tate numbers of ρ_v and of $r_{l,\imath}(\pi')|_{G_{F'_w}}$ for every place $v|l$ of F and every place $w|l$ of F'.

Choose a finite, soluble, Galois, CM extension F_1/F' which is linearly disjoint from $\overline{F}^{\ker \overline{r}|_{G_{F'}}}(\zeta_l)$ over F' such that

- for all u lying above S we have $\overline{r}(G_{F_{1,u}}) = \{1\}$;
- for all $u|l$ we have $\zeta_l \in F_{1,u}$;
- $\mu|_{G_{F_1^+}}$ is crystalline above l;
- $r_{l,\imath}(\chi')|_{G_{F_1^+}}$ is crystalline above l;
- if $u|\widetilde{v}|l$ with $v \in S$ then $\rho_v|_{G_{F_{1,u}}}$ is crystalline and $\rho_v|_{G_{F_{1,u}}} \sim \psi_1^{(u)} \oplus \cdots \oplus \psi_n^{(u)}$ with each $\psi_i^{(u)}$ a crystalline character;
- if $u|l$ then $r_{l,\imath}(\pi')|_{G_{F_{1,u}}}$ is crystalline and $r_{l,\imath}(\pi')|_{G_{F_{1,u}}} \sim \phi_1^{(u)} \oplus \cdots \oplus \phi_n^{(u)}$ with each $\phi_i^{(u)}$ a crystalline character.

We can and do assume that $(\phi_i^{(cu)})^c \phi_i^{(u)} = r_{l,\imath}(\chi')\epsilon^{1-n}|_{G_{F_{1,u}}}$. If $u|\widetilde{v}|l$ with $v \in S$, then for $i = 1, \ldots, n$, we define $\psi_i^{(cu)} : G_{F_{1,cu}} \to \overline{\mathbb{Q}}_l^\times$ by $(\psi_i^{(cu)})^c \psi_i^{(u)} = \mu|_{G_{F_{1,u}}}$.

Choose a CM extension M/F_1 such that

- M/F_1 is cyclic of degree n;
- M is linearly disjoint from $\overline{F}^{\ker \overline{r}|_{G_{F'}}}(\zeta_l)$ over F';
- and all primes of F_1 above l split completely in M.

Choose a prime u_q of F_1 above a rational prime q such that

- $q \neq l$ and q splits completely in M;
- $\overline{r}$ is unramified above q.

If $v|ql$ is a prime of F_1 we label the primes of M above v as $v_{M,1}, \ldots, v_{M,n}$ so that $(cv)_{M,i} = c(v_{M,i})$. Choose continuous characters

$$\theta, \theta', \theta'' : G_M \longrightarrow \overline{\mathbb{Q}}_l^\times$$

such that

- the reductions $\overline{\theta}$, $\overline{\theta}'$ and $\overline{\theta}''$ are equal;
- $\theta\theta^c = \widetilde{\mu}\omega_l^{(n-1)m}\epsilon_l^{(1-n)m}$, $\theta'(\theta')^c = \mu$, and $\theta''(\theta'')^c = r_{l,\imath}(\chi')\epsilon_l^{1-n}$;
- θ, θ' and θ'' are de Rham;
- if $\tau : M \hookrightarrow \overline{\mathbb{Q}}_l$ lies above a place $v_{M,i}|l$ of M then $\mathrm{HT}_\tau(\theta) = \{(i-1)m\}$, $\mathrm{HT}_\tau(\theta') = \mathrm{HT}_{\tau|_{F_1}}(\psi_i^{(v_{M,i}|F_1)})$ and $\mathrm{HT}_\tau(\theta'') = \mathrm{HT}_{\tau|_{F_1}}(\phi_i^{(v_{M,i}|F_1)})$;
- θ, θ' and θ'' are unramified at $u_{q,M,i}$ for $i > 1$, but q divides $\#\theta(I_{M_{u_{q,M,1}}})$, $\#\theta'(I_{M_{u_{q,M,1}}})$ and $\#\theta''(I_{M_{u_{q,M,1}}})$.

(Use Lemma 4.1.6 of [CHT08].)

Note the following:

- If $u|l$ is a place of F_1 and if $K/F_{1,u}$ is a finite extension over which θ, θ' and θ'' become crystalline and $\overline{\theta} = \overline{\theta}' = \overline{\theta}''$ become trivial, then

$$(\mathrm{Ind}_{G_M}^{G_{F_1}} \theta)|_{G_K} \sim 1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m},$$

$$(\mathrm{Ind}_{G_M}^{G_{F_1}} \theta')|_{G_K} \sim \psi_1^{(u|F_1)}|_{G_K} \oplus \cdots \oplus \psi_n^{(u|F_1)}|_{G_K},$$

and

$$(\mathrm{Ind}_{G_M}^{G_{F_1}} \theta'')|_{G_K} \sim \phi_1^{(u|F_1)}|_{G_K} \oplus \cdots \oplus \phi_n^{(u|F_1)}|_{G_K}.$$

- $(\mathrm{Ind}_{G_M}^{G_{F_1}} \theta)^c \cong (\mathrm{Ind}_{G_M}^{G_{F_1}} \theta)^\vee \otimes \widetilde{\mu}\omega_l^{(n-1)m}\epsilon_l^{(1-n)m}$, $(\mathrm{Ind}_{G_M}^{G_{F_1}} \theta')^c \cong (\mathrm{Ind}_{G_M}^{G_{F_1}} \theta')^\vee \otimes \mu$ and $(\mathrm{Ind}_{G_M}^{G_{F_1}} \theta'')^c \cong (\mathrm{Ind}_{G_M}^{G_{F_1}} \theta'')^\vee \otimes r_{l,\imath}(\chi')\epsilon_l^{1-n}$.

- The representation

$$(\mathrm{Ind}_{G_M}^{G_{F_1}}\overline{\theta})|_{\ker \breve{r}|_{G_{F_1(\zeta_l)}}}$$

is irreducible, and hence by Lemma A.3.1

$$(\breve{r}|_{G_{F_1}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\overline{\theta}))(G_{F_1(\zeta_l)})$$

is adequate.

[That $(\mathrm{Ind}_{G_M}^{G_{F_1}}\overline{\theta})|_{\ker \breve{r}|_{G_{F_1(\zeta_l)}}}$ is irreducible follows from looking at ramification above u_q, and noting that $\bar{r}$ is unramified at q, so that u_q is unramified in $\overline{F}^{\ker \breve{r}|_{G_{F_1(\zeta_l)}}}$.]

Let F_2/F_1 be a finite, soluble, Galois, CM extension linearly disjoint from $\overline{F_1}^{\ker \mathrm{Ind}_{G_M}^{G_{F_1}}\overline{\theta}}\overline{F}^{\ker \breve{r}|_{G_{F_1}}}(\zeta_l)$ over F_1 such that

- $\theta|_{G_{F_2 M}}$, $\theta'|_{G_{F_2 M}}$ and $\theta''|_{G_{F_2 M}}$ are crystalline above l and unramified away from l;
- MF_2/F_2 is unramified everywhere.

Then there is a RAECSDC automorphic representation (π_2, χ_2) of $GL_{n^2}(\mathbb{A}_{F_2})$ such that

- $r_{l,\imath}(\pi_2) \cong (r_{l,\imath}(\pi')|_{G_{F_1}} \otimes \mathrm{Ind}_{G_M}^{G_{F_1}}\theta)|_{G_{F_2}}$;
- $r_{l,\imath}(\chi_2) = \widetilde{\mu}\omega_l^{(n-1)m}\epsilon_l^{(n-1)(n-m)}r_{l,\imath}(\chi')\delta_{F_2/F_2^+}$;
- π_2 is unramified above l and outside S.

[The representation π_2 is the automorphic induction of $(\pi')_{MF_2}\otimes(\phi|\ |^{n(n-1)/2}\circ\det)$ to F_2, where $r_{l,\imath}(\phi) = \theta|_{G_{F_2 M}}$. The first two properties are clear. The third property follows by the choice of F_2 and local-global compatibility ([Car12], [BLGGT12]).]

Let $\widetilde{S}$ denote the set of $\widetilde{v}$ as v runs over S, let S_1 (resp. S_2) denote the primes of F_1^+ (resp. F_2^+) above S and $\widetilde{S}_1$ (resp. $\widetilde{S}_2$) the primes of F_1 (resp. F_2) above $\widetilde{S}$. If $v \in S_1$ (resp. S_2), let $\widetilde{v}$ denote the element of $\widetilde{S}_1$ (resp. $\widetilde{S}_2$) lying above it. For $v \in S_1$ with $v \nmid l$ (resp. $v|l$) let $\mathcal{C}_{1,v}$ denote the unique component of $R^{\square}_{\breve{r}|_{G_{F_1,\widetilde{v}}}} \otimes \overline{\mathbb{Q}}_l$ (resp. $R^{\square}_{\breve{r}|_{G_{F_1,\widetilde{v}}},\{0,m,2m,...,(n-1)m\},\mathrm{cris}}\otimes\overline{\mathbb{Q}}_l$) containing $r_{l,\imath}(\pi')|_{G_{F_1,\widetilde{v}}}$ (resp. $1\oplus\epsilon_l^{-m}\oplus\cdots\oplus\epsilon_l^{(1-n)m}$). For $v \in S_2$ with $v \nmid l$ (resp. $v|l$) let $\mathcal{C}_{2,v}$ denote the unique component of $R^{\square}_{\overline{r}_{l,\imath}(\pi_2)|_{G_{F_2,\widetilde{v}}}} \otimes \overline{\mathbb{Q}}_l$ (resp. $R^{\square}_{\overline{r}_{l,\imath}(\pi_2)|_{G_{F_2,\widetilde{v}}},\{\mathrm{HT}_\tau(r_{l,\imath}(\pi_2)|_{G_{F_2,\widetilde{v}}})\},\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$) containing $r_{l,\imath}(\pi_2)|_{G_{F_2,\widetilde{v}}}$. Choose a finite extension $L/\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ such that

- L contains the image of each embedding $F_2 \hookrightarrow \overline{\mathbb{Q}}_l$;
- L contains the image of θ;
- $r_{l,\imath}(\pi_2)$ is defined over L;
- each of the components $\mathcal{C}_{1,v}$ for $v \in S_1$ and $\mathcal{C}_{2,v}$ for $v \in S_2$ is defined over L.

Set

$$s = \mathrm{Ind}_{G_{M^+},G_M}^{G_{F_1^+},G_{F_1},r_{l,\imath}(\chi')\epsilon_l^{1-n}}(\theta'', r_{l,\imath}(\chi')\epsilon_l^{1-n}) : G_{F_1^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_L)$$

in the notation of section 1.1 of [BLGGT10] and section 2.1 of [CHT08]. Thus $\nu \circ s = r_{l,\imath}(\chi')\epsilon_l^{1-n}$. For $v \in S_1$ (resp. $v \in S_2$) let $\mathcal{D}_{1,v}$ (resp. $\mathcal{D}_{2,v}$) denote the

deformation problem for $\breve{r}|_{G_{F_{1,\widetilde{v}}}}$ (resp. $\overline{r}_{l,\imath}(\pi_2)|_{G_{F_{2,\widetilde{v}}}}$) over $\mathcal{O}_L$ corresponding to $\mathcal{C}_{1,v}$ (resp. $\mathcal{C}_{2,v}$). Also let

$$\mathcal{S}_1 = (F_1/F_1^+, S_1, \widetilde{S}_1, \mathcal{O}_L, \overline{r}|_{G_{F_1^+}}, \widetilde{\mu}|_{G_{F_2^+}}\omega_l^{(n-1)m}\epsilon_l^{(1-n)m}, \{\mathcal{D}_{1,v}\})$$

and

$$\mathcal{S}_2 = (F_2/F_2^+, S_2, \widetilde{S}_2, \mathcal{O}_L, \widetilde{\overline{r}}_{l,\imath}(\pi_2), \widetilde{\mu}|_{G_{F_2^+}}\omega_l^{(n-1)m}\epsilon_l^{(1-n)(m+1)}r_{l,\imath}(\chi')\delta_{F_2/F_2^+}, \{\mathcal{D}_{2,v}\}).$$

There is a natural map

$$R_{\mathcal{S}_2}^{\mathrm{univ}} \longrightarrow R_{\mathcal{S}_1}^{\mathrm{univ}}$$

induced by $r_{\mathcal{S}_1}^{\mathrm{univ}}|_{G_{F_2^+}} \otimes s|_{G_{F_2^+}}$. [We must check that if $u \in S_2$ then $\breve{r}_{\mathcal{S}}^{\mathrm{univ}}|_{G_{F_{2,\widetilde{u}}}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta'')|_{G_{F_{2,\widetilde{u}}}} \in \mathcal{D}_{2,u}$. Let $v = u|_{F^+}$ and let $\rho_{v,\mathcal{C}_{1,v}}^{\square}$ denote the universal lift of $\overline{r}|_{G_{F_{1,\widetilde{v}}}}$ to $R_{\mathcal{O}_L,\breve{r}|_{G_{F_{1,\widetilde{v}}}},\mathcal{C}_{1,v}}^{\square}$. It suffices to show that $\rho_{v,\mathcal{C}_{1,v}}^{\square}|_{G_{F_{2,\widetilde{u}}}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta'')|_{G_{F_{2,\widetilde{u}}}} \in \mathcal{D}_{2,u}$. For this, it suffices to show if $\rho : G_{F_{1,\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ is a lift of $\breve{r}|_{G_{F_{1,\widetilde{v}}}}$ lying on $\mathcal{C}_{1,v}$, then $\rho|_{G_{F_{2,\widetilde{u}}}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta'')|_{G_{F_{2,\widetilde{u}}}}$ lies on $\mathcal{C}_{2,u}$. If $u|l$, then $\rho|_{G_{F_{2,\widetilde{u}}}} \sim (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta)|_{G_{F_{2,\widetilde{u}}}}$ and $(\mathrm{Ind}_{G_M}^{G_{F_1}}\theta'')|_{G_{F_{2,\widetilde{u}}}} \sim r_{l,\imath}(\pi')|_{G_{F_{2,\widetilde{u}}}}$ and hence

$$\rho|_{G_{F_{2,\widetilde{u}}}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta'')|_{G_{F_{2,\widetilde{u}}}} \sim (r_{l,\imath}(\pi'_{F_1}) \otimes \mathrm{Ind}_{G_M}^{G_{F_1}}\theta)|_{G_{F_{2,\widetilde{u}}}} \cong r_{l,\imath}(\pi_2)|_{G_{F_{2,\widetilde{u}}}}.$$

If $u \nmid l$, then by definition $\rho|_{G_{F_{2,\widetilde{u}}}} \sim r_{l,\imath}(\pi')|_{G_{F_{2,\widetilde{u}}}}$. By the choice of F_2 we have $(\mathrm{Ind}_{G_M}^{G_{F_1}}\theta)|_{G_{F_{2,\widetilde{u}}}} \sim (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta'')|_{G_{F_{2,\widetilde{u}}}}$. Hence

$$\rho|_{G_{F_{2,\widetilde{u}}}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta'')|_{G_{F_{2,\widetilde{u}}}} \sim (r_{l,\imath}(\pi') \otimes \mathrm{Ind}_{G_M}^{G_{F_1}}\theta)|_{G_{F_{2,\widetilde{u}}}} \cong r_{l,\imath}(\pi_2)|_{G_{F_{2,\widetilde{u}}}}$$

and we are done.] It follows from Lemma 1.2.3 of [BLGGT10] that this map makes $R_{\mathcal{S}_1}^{\mathrm{univ}}$ a finite $R_{\mathcal{S}_2}^{\mathrm{univ}}$-module. By Theorem 2.3.2 of [BLGGT10], $R_{\mathcal{S}_2}^{\mathrm{univ}}$ is a finite $\mathcal{O}_L$-module, and hence $R_{\mathcal{S}_1}^{\mathrm{univ}}$ is a finite $\mathcal{O}_L$-module. On the other hand by Proposition 1.5.1 of [BLGGT10], $R_{\mathcal{S}_1}^{\mathrm{univ}}$ has Krull dimension at least 1. Hence $\mathrm{Spec}\, R_{\mathcal{S}_1}^{\mathrm{univ}}$ has a $\overline{\mathbb{Q}}_l$-point. This point gives rise to a lifting $r_1 : G_{F_1} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ of $\overline{r}|_{G_{F_1}}$ with the following properties:

- $\nu \circ r_1 = \widetilde{\mu}\omega_l^{(n-1)m}\epsilon_l^{(1-n)m}$,
- r_1 is unramified outside S,
- if $u|l$ then $r_1|_{G_{F_{1,u}}} \sim 1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m}$.

By Theorem 2.2.1 of [BLGGT10], Lemma 1.4 of [BLGHT11] and the construction of r_1, we also have that

- $r_1 \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta'')$ is automorphic of level prime to l.

It follows from Lemma 2.2.4 of [BLGGT10] that r_1 itself is automorphic of level prime to l, say $r_1 \cong r_{l,\imath}(\pi_1)$. By the main result of [Car12], π_1 is unramified outside of places lying over S, and by the main result of [BLGGT12] and Lemma 5.2.1 of [Ger09], we see that π_1 is $\imath$-ordinary of level prime to l. It then follows from Theorems 2.4.1 and 2.4.2 of [BLGGT10], which together strengthen Theorem 5.1.1 of [GG12], that we may find a RAECSDC automorphic representation (π'_1, χ'_1) of $\mathrm{GL}_n(\mathbb{A}_{F_1})$ such that

- $(\overline{r}_{l,\imath}(\pi'_1), r_{l,\imath}(\overline{\chi}'_1)) \cong (\overline{r}|_{G_{F_1}}, \overline{\mu}|_{G_{F_1}})$,
- π'_1 is $\imath$-ordinary, unramified at places dividing l, and unramified outside S,

- if $u|l$ then $r_{l,\imath}(\pi_1')|_{G_{F_{1,u}}} \sim 1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m}$,
- $r_{l,\imath}(\chi_1') = \widetilde{\mu}|_{G_{F_1^+}} \omega_l^{(n-1)m} \epsilon_l^{(1-n)(m-1)}$,
- if $\widetilde{u} \nmid l$ is a place in $\widetilde{S}_1$ lying over $v \in S$, then $r_{l,\imath}(\pi_1')|_{G_{F_{1,\widetilde{u}}}} \sim \rho_v|_{G_{F_{1,\widetilde{u}}}}$.

We now argue in a similar fashion to the above to construct the sought-after representation r.

There is a RAECSDC automorphic representation (π_2', χ_2') of $\mathrm{GL}_{n^2}(\mathbb{A}_{F_2})$ such that

- $r_{l,\imath}(\pi_2') \cong (r_{l,\imath}(\pi_1') \otimes \mathrm{Ind}_{G_M}^{G_{F_1}} \theta')|_{G_{F_2}}$;
- $r_{l,\imath}(\chi_2') = \mu\widetilde{\mu}\omega_l^{(n-1)m} \epsilon_l^{(1-n)(m+n+1)} \delta_{F_2/F_2^+}$;
- π_2' is unramified above l and outside S.

[The representation π_2' is the automorphic induction of $(\pi_1')_{MF_2} \otimes (\phi'| \ |^{n(n-1)/2} \circ \det)$ to F_2, where $r_{l,\imath}(\phi') = \theta'|_{G_{F_2M}}$. The first two properties are clear. The third property follows by the choice of F_2 and the fact that π_1' is unramified above l and outside S.]

For $v \in S_2$ with $v \nmid l$ (resp. $v|l$) let $\mathcal{C}_{2,v}'$ denote the unique component of $R_{\overline{r}_{l,\imath}(\pi_2')|_{G_{F_{2,\widetilde{v}}}}}^{\square} \otimes \overline{\mathbb{Q}}_l$ (resp. $R_{\overline{r}_{l,\imath}(\pi_2')|_{G_{F_{2,\widetilde{v}}}}, \{\mathrm{HT}_\tau(r_{l,\imath}(\pi_2')|_{G_{F_{2,\widetilde{v}}}})\}, \mathrm{cris}}^{\square} \otimes \overline{\mathbb{Q}}_l$) containing $r_{l,\imath}(\pi_2')|_{G_{F_{2,\widetilde{v}}}}$. Extending L if necessary we may further assume that

- L contains the image of μ;
- $r_{l,\imath}(\pi_2')$ is defined over L;
- each of the components $\mathcal{C}_v$ for $v \in S$ and $\mathcal{C}_{2,v}'$ for $v \in S_2$ is defined over L.

Set
$$s' = \mathrm{Ind}_{G_{M^+}, G_M}^{G_{F_1^+}, G_{F_1}, \widetilde{\mu}\omega_l^{(n-1)m}\epsilon_l^{(1-n)m}} (\theta, \widetilde{\mu}\omega_l^{(n-1)m}\epsilon_l^{(1-n)m}) : G_{F_1^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_L)$$

in the notation of section 1.1 of this paper and section 2.1 of [CHT08]. Thus $\nu \circ s' = \widetilde{\mu}\omega_l^{(n-1)m}\epsilon_l^{(1-n)m}$. For $v \in S$ (resp. $v \in S_2$) let $\mathcal{D}_v$ (resp. $\mathcal{D}_{2,v}'$) denote the deformation problem for $\breve{r}|_{G_{F_{\widetilde{v}}}}$ (resp. $\overline{r}_{l,\imath}(\pi_2)|_{G_{F_{2,\widetilde{v}}}}$) over $\mathcal{O}_L$ corresponding to $\mathcal{C}_v$ (resp. $\mathcal{C}_{2,v}'$). Also let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}_L, \overline{r}, \mu, \{\mathcal{D}_v\})$$

and

$$\mathcal{S}_2' = (F_2/F_2^+, S_2, \widetilde{S}_2, \mathcal{O}_L, \widetilde{\overline{r}}_{l,\imath}(\pi_2'), \mu\widetilde{\mu}|_{G_{F_1^+}}\omega_l^{(n-1)m}\epsilon_l^{(1-n)m}\delta_{F_2^+/F_2}, \{\mathcal{D}_{2,v}'\}).$$

As above, there is a natural map

$$R_{\mathcal{S}_2'}^{\mathrm{univ}} \longrightarrow R_{\mathcal{S}}^{\mathrm{univ}}$$

induced by $r_{\mathcal{S}}^{\mathrm{univ}}|_{G_{F_2^+}} \otimes s'|_{G_{F_2^+}}$. It follows from Lemma 1.2.3 of [BLGGT10] that this map makes $R_{\mathcal{S}}^{\mathrm{univ}}$ a finite $R_{\mathcal{S}_2'}^{\mathrm{univ}}$-module. By Theorem 2.3.2 of [BLGGT10], $R_{\mathcal{S}_2'}^{\mathrm{univ}}$ is a finite $\mathcal{O}_L$-module, and hence $R_{\mathcal{S}}^{\mathrm{univ}}$ is a finite $\mathcal{O}_L$-module. On the other hand by Proposition 1.5.1 of [BLGGT10], $R_{\mathcal{S}}^{\mathrm{univ}}$ has Krull dimension at least 1. Hence $\mathrm{Spec}\, R_{\mathcal{S}}^{\mathrm{univ}}$ has a $\overline{\mathbb{Q}}_l$-point. This point gives rise to the desired lifting r of $\overline{r}$. [To see that $r|_{G_{F'}}$ is automorphic, note that by Theorem 2.3.1 of [BLGGT10], $(r|_{G_{F_1}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta))|_{G_{F_2}}$ is automorphic, so by Lemma 1.4 of [BLGHT11] $r|_{G_{F_1}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_1}}\theta)$ is automorphic. It follows from Lemma 2.2.4 of [BLGGT10] that $r|_{G_{F_1}}$

is automorphic, and a further application of Lemma 1.4 of [BLGHT11] shows that $r|_{G_{F'}}$ is automorphic, as required.] $\qquad\square$

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