



# Generalizations of the General Lotto and Colonel Blotto games

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## Abstract

In this paper, we generalize the General Lotto game (budget constraints satisfied in expectation) and the Colonel Blotto game (budget constraints hold with probability one) to allow for battlefield valuations that are heterogeneous across battlefields and asymmetric across players and for the players to have asymmetric resource constraints. We completely characterize Nash equilibrium in the generalized version of the General Lotto game and find that there exist sets of nonpathological parameter configurations of positive Lebesgue measure with multiple payoff nonequivalent equilibria. Across equilibria each player achieves a higher payoff when he more aggressively attacks battlefields in which he has lower relative valuations. Hence, the best defense is a good offense. We, then, show how this characterization can be applied to identify equilibria in the Colonel Blotto version of the game.

**Keywords** Colonel Blotto game · General Lotto game · Multi-battle contest · Redistributive politics · All-pay auction

**JEL Classification** C72 · D72 · D74

## 1 Introduction

The Colonel Blotto game is a two-player resource allocation game in which each player is endowed with a level of a resource to allocate across a set of battlefields;

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within each battlefield the player that allocates the higher level of the resource wins the battlefield, and each player's payoff is the sum of the valuations of the battlefields won. This simple game, which originates with Borel (1921), illustrates some of the fundamental strategic considerations that arise in conflicts or competition involving multi-dimensional resource allocation such as political campaigns, research and development competition (where innovation may involve obtaining a collection of interrelated patents), and military and systems defense.

In this paper, we examine generalized formulations of the General Lotto (budget constraints hold in expectation) and Colonel Blotto (budget constraints hold with probability one) games in which battlefield (or component contest) valuations may be heterogeneous across battlefields and asymmetric across players, and the players may face asymmetric resource constraints.<sup>1</sup> We completely characterize Nash equilibrium in the General Lotto game. This generalizes the symmetric resource constraint versions of the General Lotto game examined in Bell and Cover (1980), Myerson (1993), Sahuguet and Persico (2006), and Hart (2008), where the valuation in the single battlefield<sup>2</sup> is symmetric across players, as well as Kovenock and Roberson (2008) and Washburn (2013), in which battlefield valuations are heterogeneous across battlefields but symmetric across players.<sup>3</sup> We then show how this characterization can be applied over a subset of the parameter space to identify equilibria in generalizations of the Colonel Blotto game (in which budget constraints hold with probability one) that have hitherto been unexplored.

In contrast to constant-sum formulations of General Lotto and Colonel Blotto games, we find that the Nash equilibrium set of univariate marginal distributions is not unique in the generalized (non-constant-sum) versions of the games examined here. We also show that in the generalized versions of both the General Lotto and Colonel Blotto games there exist sets of nonpathological parameter configurations of positive Lebesgue measure with multiple payoff nonequivalent equilibria.

To provide intuition for why multiple payoff nonequivalent equilibria arise, consider a single all-pay auction in which one player,  $A$ , has a low valuation  $v_A$  and one player  $B$  has a high valuation  $v_B$ , i.e.,  $v_B > v_A$ . It is well known that in the unique Nash equilibrium<sup>4</sup>  $B$ 's expected payoff is  $v_B - v_A$  and  $A$ 's expected payoff is 0. Now suppose  $v_B$  decreases and  $v_A$  increases maintaining the inequality  $v_B \geq v_A$ . In this case, the contest becomes more competitive, which results in higher equilibrium

<sup>1</sup> Other notable formulations of Blotto-type games include Friedman (1958), which introduces a version of the game with the lottery contest success function [see also Robson (2005)], and Hart (2008) which introduces a version of the game in which resource allocations are restricted to be nonnegative integers [see also Hortala-Vallve and Llorente-Saguer (2012) and Dziubiński (2013)].

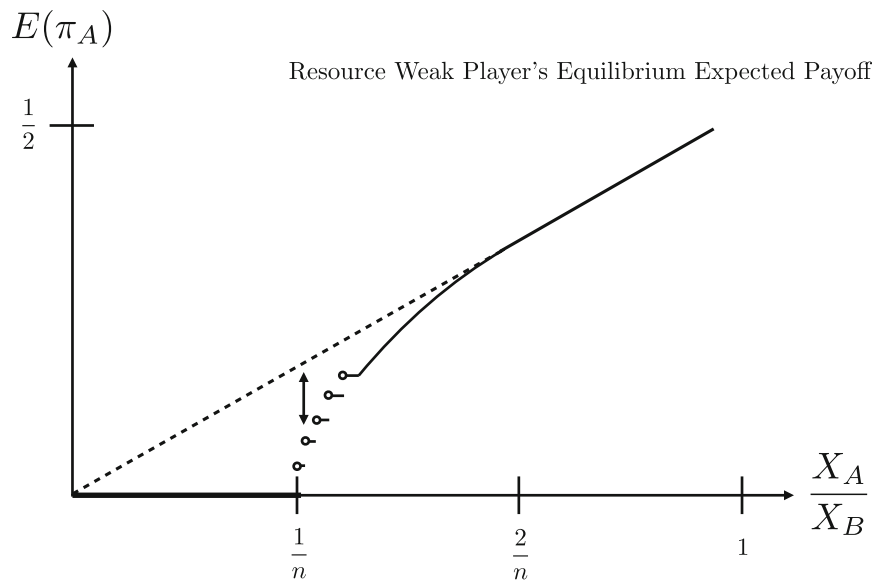
<sup>2</sup> The models in these papers may either be interpreted as having a single battlefield, where each player's allocation of the resource to this battlefield is drawn from his univariate distribution function that is budget balancing on average, or as a continuum of homogeneous battlefields, where each point in the support of a player's univariate distribution function represents an allocation of the resource to a battlefield, and the budget constraint is on the average resource allocation. In this paper, we focus on the first interpretation.

<sup>3</sup> Following Myerson (1993), the General Lotto game has become a benchmark model of redistributive politics. Related political economy applications include Lizzeri and Persico (2001, 2005), Sahuguet and Persico (2006), Roberson (2008), and Crutzen and Sahuguet (2009). See also Laslier and Picard (2002) for a similar application of the Colonel Blotto game.

<sup>4</sup> See Baye et al. (1996) for further details.

expenditures for both players and a lower equilibrium expected payoff for the high valuation player ( $B$ ). In the General Lotto game, the two players  $A$  and  $B$  each have a (normalized) value for each battlefield  $j$ ,  $v_{i,j} > 0$ , where  $\sum_{j=1}^n v_{i,j} = 1$  for each player  $i = A, B$ . The players are also resource constrained, where the expectation of player  $i$ 's total expenditure across battlefields must be less than or equal to  $X_i$ , for  $i = A, B$ , and player  $B$  is assumed to have a resource advantage,  $X_B \geq X_A > 0$ . Given an equilibrium mixed strategy of the rival player, the budget constrained optimization problem of player  $i$  yields a Lagrange multiplier  $\lambda_i$ —the shadow value of an increment to  $i$ 's budget—that serves as a unit cost to an incremental allocation to each battlefield  $j$ . Player  $i$  maximizes his payoff by acting as if he is playing in an all-pay auction in each battlefield with constant unit cost equal to the multiplier  $\lambda_i$ . Because of invariance of behavior with respect to affine transformations of utility, this implies that, in an equilibrium generating the pair of player multipliers  $(\lambda_A, \lambda_B)$ , the two players  $A$  and  $B$  behave in battlefield  $j$  as if they are engaged in an all-pay auction with constant unit cost equal to one and valuations  $\frac{v_{A,j}}{\lambda_A}$  and  $\frac{v_{B,j}}{\lambda_B}$ , respectively. In this all-pay auction, player  $A$  has a higher valuation than player  $B$  if  $\frac{v_{A,j}}{v_{B,j}} > \frac{\lambda_A}{\lambda_B} \equiv \gamma$  and  $B$  has a higher value than  $A$  if the inequality is reversed. This implies that in each equilibrium of the General Lotto game there exists a cut point,  $\gamma$ , equal to the ratio of the two multipliers  $\lambda_A$  and  $\lambda_B$  induced by the equilibrium, such that for each battlefield  $j$  with  $\gamma > \frac{v_{A,j}}{v_{B,j}}$  player  $B$  utilizes a strategy similar to that of the high valuation player in the all-pay auction and player  $A$  utilizes a strategy similar to that of the low valuation player in the all-pay auction. Similarly, for each battlefield  $j$  with  $\frac{v_{A,j}}{v_{B,j}} > \gamma$ , the roles are reversed with player  $A$  being the high valuation player and  $B$  being the low valuation player. Multiple equilibria arise in this setting because there exist multiple pairs of shadow values  $(\lambda_A, \lambda_B)$ , or alternatively cut points  $\gamma$ , that generate budget-balancing strategies that are mutual best responses and therefore form an equilibrium. In moving across equilibria, as  $\gamma$  increases the set of battlefields for which  $\frac{v_{A,j}}{v_{B,j}} > \gamma$  (weakly) shrinks. Furthermore, in the remaining set of battlefields for which  $\frac{v_{A,j}}{v_{B,j}} > \gamma$  each battlefield becomes more competitive, thereby increasing both players' expected expenditures in those battlefields and lowering player  $A$ 's expected payoff for those battlefields. Conversely, each battlefield  $j$  with  $\gamma > \frac{v_{A,j}}{v_{B,j}}$  becomes less competitive as  $\gamma$  increases, thereby decreasing both players' expected expenditures in those battlefields and increasing player  $B$ 's expected payoff across such battlefields. Therefore, in moving from an equilibrium in which  $\frac{\lambda_A}{\lambda_B} = \gamma$  to an equilibrium in which  $\frac{\lambda_A}{\lambda_B} = \gamma' > \gamma$ , the increased allocation to battlefields  $j$  with higher values of  $\frac{v_{A,j}}{v_{B,j}}$  exactly offsets the reduced allocation to battlefields with lower values of  $\frac{v_{A,j}}{v_{B,j}}$ , so that budget balance holds.

When battlefield valuations are homogeneous across battlefields and symmetric across players, it is known that for sufficiently asymmetric resource endowments the relationship between the equilibria in the General Lotto and Colonel Blotto games



**Fig. 1** The weak player A's equilibrium expected payoff in the General Lotto (dashed line) and Colonel Blotto (solid line) games as a function of the ratio of the weak player's resource endowment ( $X_A$ ) to the strong player's resource endowment ( $X_B$ )

breaks down.<sup>5</sup> In Fig. 1, the dashed line illustrates the (resource endowment) weak player A's maximal expected proportion of battlefields won in the General Lotto game with  $n$  battlefields, and the solid curve is the weak player's expected proportion of battlefields won in the corresponding Colonel Blotto game, both as a function of the ratio of the weak player's and the strong player's budgets ( $\frac{X_A}{X_B}$ ). Note that the point of departure between the weak player's expected payoffs in the General Lotto game and those arising in the Colonel Blotto game occurs at a ratio  $\frac{X_A}{X_B} = \frac{2}{n}$ . As the ratio of the weak player's budget to the strong player's budget ( $\frac{X_A}{X_B}$ ) decreases, the weak player focuses his resources in smaller and smaller random subsets of battlefields. When the ratio  $\frac{X_A}{X_B}$  is less than or equal to  $\frac{2}{n}$ , the fact that budgets bind with certainty in the Colonel Blotto game results in a situation in which the weak player has exhausted his ability to shrink the size of the subset of battlefields in which he focuses his resources. This binding constraint is what causes the weak player's expected payoffs to decrease below the dashed line. Furthermore, once  $\frac{X_A}{X_B}$  is less than  $\frac{1}{n}$ , the strong player has the ability to outbid the weak player on each and every battlefield. This issue arises because, with sufficiently asymmetric resource endowments, the Colonel Blotto game's binding budget constraint makes it infeasible for the weak player to play a multi-dimensional mixed strategy that is budget-balancing with certainty and that provides the same set of univariate marginals as in the corresponding General Lotto game in which the budget only holds in expectation.

<sup>5</sup> For asymmetric resource endowments, the characterization of the equilibrium payoffs in the General Lotto game is due to Sahuguet and Persico (2006) and for the Colonel Blotto game is due to Roberson (2006).

For the generalized version of the Colonel Blotto game examined in this paper, we provide a sufficient condition for the existence of an equilibrium in which each player's univariate marginal distributions coincide with those of an equilibrium in the corresponding General Lotto version of the game and a sufficient condition for the equilibrium sets of univariate marginal distributions in the two versions of the game to differ across all equilibria. Because the battlefield valuations may be heterogeneous across battlefields and asymmetric across players, this condition involves the players' resource endowments, as well as their  $n$ -tuples of battlefield valuations. Furthermore, equilibrium univariate marginal distributions in the Colonel Blotto and General Lotto games may differ in the case of symmetric resource endowments, if the players have sufficiently different battlefield valuation  $n$ -tuples. That is, asymmetries in players' valuations alone may be sufficient for the equilibria in the Colonel Blotto and General Lotto games to differ.

Table 1 summarizes the branch of the Colonel Blotto literature that assumes an auction contest success function, a finite number of battlefields, resource endowments that are continuously divisible and are use-it-or-lose-it in the sense that unused resources have no value (i.e., the per unit cost of allocating the resource is 0 up to the budget constraint),<sup>6</sup> and that each player's payoff is the sum of the battlefield valuations in the battlefields won.<sup>7</sup> The type of player objective varies across rows and the cost structure varies across columns. In describing the type of player objective, linear pure count refers to games in which each player's payoff is the sum of the battlefield valuations in the battlefields won, where battlefield valuations are homogeneous across battlefields and symmetric across players, so that each player's payoff is linear in the number, or pure count, of battlefields won. Linear heterogeneous symmetric (asymmetric) is similar, except that battlefield valuations are now heterogeneous across battlefields but symmetric (asymmetric) across players, and each player's payoff is equal to the sum of the player's valuations in the battlefields won.

As shown in Table 1, this paper provides a partial result for each of the three checked cells: linear heterogeneous symmetric objective with asymmetric budget constraints, and linear heterogeneous asymmetric objective with both symmetric and asymmetric budget constraints. Section 4 provides a detailed discussion of the literature in Table 1, including a discussion of the relationship between the equilibrium joint distributions that have been constructed for those formulations and the distributions that we utilize in our analysis of the Generalized Colonel Blotto game. As a point of reference, Table 1 also includes the corresponding results for the discrete version of the Colonel Blotto game (in which the feasible sets of bids of the players are discrete). Lastly, the case of  $n = 2$  places a severe restriction on the set of available joint distributions, which leads to a distinct set of strategic considerations. For both symmetric and asymmetric budgets and both homogeneous and heterogeneous (symmetric) battlefield valuations, the case of  $n = 2$  was first examined in Gross and Wagner (1950). Macdonell and Mastronardi (2015) complete the characterization for the case of  $n = 2$  and pro-

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<sup>6</sup> For alternative cost functions see Kvasov (2007) and Roberson and Kvasov (2012).

<sup>7</sup> For alternative definitions of success see Szentes and Rosenthal (2003a, b), Golman and Page (2009), Tang et al. (2010), Rinott et al. (2012), Barelli et al. (2014), and Kovenock and Roberson (2018).

**Table 1** Blotto game variations with linear count objectives and use-it-or-lose-it resources

Objective	Costs	
	Symmetric budget Use-it-or-lose-it resources	Asymmetric budget Use-it-or-lose-it resources
Linear pure-count	Continuous	Continuous
	Borel and Ville (1938) [ $n = 3$ ]	Gross and Wagner (1950) [ $n = 2$ ]
	Gross and Wagner (1950) [ $n \geq 2$ ]	Roberson (2006) [ $n \geq 3$ ]
	Weinstein (2012) [ $n \geq 3$ ]	Macdonell and Mastronardi (2015) [ $n = 2$ ]
	Discrete	Discrete
	Hart (2008)	Hart (2008) [partial result]
	Continuous	Continuous
	Gross (1950)	Gross and Wagner (1950) [ $n = 2$ ]
	Laslier (2002)	✓
	Thomas (2018)	[partial result]
Linear heterogeneous symmetric	Discrete	Schwartz et al. (2014) [ $n \geq 3$ ]
	Hortala-Vallve and Llorente-Saguer (2012)	[partial result]
	[partial result]	Macdonell and Mastronardi (2015) [ $n = 2$ ]
	Continuous	Continuous
	✓	✓
	[partial result]	[partial result]
	Discrete	
	Hortala-Vallve and Llorente-Saguer (2012)	
	[partial result]	
Linear heterogeneous asymmetric		

vide a characterization of equilibrium in a version of the heterogeneous (symmetric) battlefield valuation game with nonlinear asymmetric budgets.

The rest of the paper is organized as follows. In Sect. 2, we provide a description of the General Lotto model. Section 3 provides the results for the General Lotto game and an example of multiple payoff nonequivalent equilibria. Section 4 examines the Generalized Colonel Blotto game and the relationship between our results and the existing literature. Section 5 concludes.

## 2 The model

Two risk-neutral players,  $A$  and  $B$ , simultaneously allocate a resource across a positive integer number  $n \geq 1$  of independent battlefields. Battlefield  $j$  has a (normalized) value of  $v_{i,j} > 0$ , where  $\sum_{j=1}^n v_{i,j} = 1$ , for player  $i = A, B$ . Each player has a fixed level of the available resource (or budget),  $X_i$  for  $i = A, B$ . Let  $X_B \geq X_A > 0$ , and let  $\mathbf{x}_i$  denote player  $i$ 's allocation of the resource  $(x_{i,1}, \dots, x_{i,j}, \dots, x_{i,n})$  across the  $n$ -battlefields. In each battlefield, the player with the higher resource expenditure wins, and in the event of a tie<sup>8</sup> each player wins the battlefield with probability  $\frac{1}{2}$ .

In each battlefield  $j$ , the payoff to player  $i$  for a resource expenditure of  $x_{i,j}$  is given by

$$\pi_{i,j}(x_{i,j}, x_{-i,j}) = \begin{cases} v_{i,j} & \text{if } x_{i,j} > x_{-i,j} \\ \frac{v_{i,j}}{2} & \text{if } x_{i,j} = x_{-i,j} \\ 0 & \text{if } x_{i,j} < x_{-i,j} \end{cases}$$

Each player's payoff across all  $n$  battlefields is the sum of the payoffs across the individual battlefields.

We now define the Generalized Colonel Blotto and General Lotto games.

### 2.1 The Generalized Colonel Blotto game

The level of the resource allocated to each battlefield must be nonnegative. For player  $i$ , the set of feasible resource allocations across the  $n$  battlefields is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}_+^n \mid \sum_{j=1}^n x_{i,j} \leq X_i \right\}.$$

A mixed strategy, which we term a *distribution of resources*, for player  $i$  is an  $n$ -variate distribution function  $P_i : \mathbb{R}_+^n \rightarrow [0, 1]$  with support (denoted  $\text{Supp}(P_i)$ ) contained in

<sup>8</sup> The choice of tie-breaking rule is not critical for any of our results. This is generally true in the General Lotto game and is true for the corresponding parameter ranges covered in our treatment of the Colonel Blotto game. More generally, in the Colonel Blotto game the choice of a tie-breaking rule is important for the parameter range in which the correspondence between General Lotto and Colonel Blotto breaks down. In this range, the tie-breaking rule in the Colonel Blotto game must be chosen judiciously in order to avoid the need for  $\epsilon$ -equilibrium arguments. See Roberson (2006).



player  $i$ 's set of feasible bids  $\mathfrak{B}_i$  and with one-dimensional marginal distribution functions  $\{F_{i,j}\}_{j=1}^n$ , one univariate marginal distribution function for each battlefield  $j$ . Player  $i$ 's allocation of the resource across the  $n$  battlefields is a random  $n$ -tuple drawn from the  $n$ -variate distribution function  $P_i$ .

The *Generalized Colonel Blotto game*, which we abbreviate as the GCB game and formally label as

$$\text{GCB}\left\{X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n\right\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of the resource subject to their budget constraints; each battlefield is won by the player that allocates the higher level of the resource to that battlefield (where in the case of a tie the tie-breaking rule described above applies), and each player's payoff is the sum of the values of the individual battlefields that he wins.

## 2.2 The Generalized General Lotto game

In the Generalized General Lotto game, a mixed strategy for player  $i$  is still an  $n$ -variate distribution function  $P_i : \mathbb{R}_+^n \rightarrow [0, 1]$  with one-dimensional marginal distribution functions  $\{F_{i,j}\}_{j=1}^n$ , one univariate marginal distribution function for each battlefield  $j$ . And, the level of the resource allocated to each battlefield must be nonnegative,  $F_{i,j}(x) = 0$  for all  $x < 0$ . The General Lotto game differs from the Colonel Blotto game in that each player  $i$ 's budget must hold in expectation,  $\sum_{j=1}^n E_{F_{i,j}}(x) \leq X_i$ .

The *Generalized General Lotto game*, which we abbreviate as the GGL game and formally label as

$$\text{GGL}\left\{X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n\right\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of the resource subject to their budget constraints, each battlefield is won by the player that allocates the higher level of the resource to that battlefield (where in the case of a tie the tie-breaking rule described above applies), and each player's payoff is the sum of the values of the individual battlefields that he wins.

## 3 Generalized General Lotto results

In order to provide intuition for our main results, we begin this section with a few informal insights regarding the necessary conditions for equilibrium in the GGL game. First, note that any joint distribution may be broken into a set of univariate marginal distribution functions and an  $n$ -copula, the function that maps the univariate marginal distribution functions into a joint distribution function.<sup>9</sup> Given that player  $-i$ 's strategy is given by the  $n$ -variate distribution function  $P_{-i}$  with the set of univariate marginal

<sup>9</sup> See Nelsen (1999) or Schweizer and Sklar (1983) for an introduction to copulas.



distribution functions  $\{F_{-i,j}\}_{j=1}^n$ , note that player  $i$ 's expected payoff<sup>10</sup> for any feasible  $n$ -variate distribution function  $P_i$  with the set of univariate marginal distribution functions  $\{F_{i,j}\}_{j=1}^n$  is

$$\pi_i\left(\{F_{i,j}, F_{-i,j}\}_{j=1}^n\right) = \sum_{j=1}^n \left[ \int_0^\infty v_{i,j} F_{-i,j}(x_{i,j}) dF_{i,j} \right]. \quad (1)$$

Recalling that the budget constraint holds in expectation, player  $i$ 's constrained optimization problem may be written as

$$\max_{\{F_{i,j}\}_{j=1}^n} \sum_{j=1}^n \left[ \int_0^\infty [v_{i,j} F_{-i,j}(x_{i,j}) - \lambda_i x_{i,j}] dF_{i,j} \right] + \lambda_i X_i, \quad (2)$$

where  $\lambda_i$  is the multiplier on player  $i$ 's expected resource expenditure constraint. Note that for the GGL game both the expected payoff in (1) and the budget constraint depend on only the sets of univariate marginal distribution functions and not the joint distribution function. That is, in the GGL game, any  $n$ -copula may be used to map a set of equilibrium univariate marginal distribution functions into an equilibrium joint distribution function. However, because the budget constraint in the GCB game holds with probability one, the choice of a set of univariate marginal distribution functions in that game is constrained in the sense that there must exist an  $n$ -copula for which the resulting joint distribution is budget-balancing with probability one. We will return to this issue in Sect. 4.

For each  $j = 1, \dots, n$  the corresponding first-order condition provides a necessary condition for equilibrium and is given by

$$\frac{d}{dx_{i,j}} [v_{i,j} F_{-i,j}(x_{i,j}) - \lambda_i x_{i,j}] = 0. \quad (3)$$

Dividing both sides of (3) by  $\lambda_i > 0$ , we see that (3) is equivalent to the necessary condition for a single all-pay auction, without a budget constraint, and in which player  $i$ 's value for the prize is  $\frac{v_{i,j}}{\lambda_i}$ . In such an all-pay auction, the unique equilibrium<sup>11</sup> is described as follows. If  $\frac{v_{i,j}}{\lambda_i} \geq \frac{v_{-i,j}}{\lambda_{-i}}$ , then

$$\begin{aligned} F_{-i,j}(x) &= \left( \frac{\frac{v_{i,j}}{\lambda_i} - \frac{v_{-i,j}}{\lambda_{-i}}}{\frac{v_{i,j}}{\lambda_i}} \right) + \frac{x}{\frac{v_{i,j}}{\lambda_i}} \quad x \in \left[ 0, \frac{v_{-i,j}}{\lambda_{-i}} \right] \\ F_{i,j}(x) &= \frac{x}{\frac{v_{-i,j}}{\lambda_{-i}}} \quad x \in \left[ 0, \frac{v_{-i,j}}{\lambda_{-i}} \right]. \end{aligned} \quad (4)$$

Next, to solve for the multipliers  $(\lambda_A, \lambda_B)$ , let  $\gamma \equiv \frac{\lambda_A}{\lambda_B}$  and let  $\Omega_A(\gamma)$  denote the set of battlefields in which  $\frac{v_{A,j}}{v_{B,j}} > \gamma$ , or equivalently  $\frac{v_{A,j}}{\lambda_A} > \frac{v_{B,j}}{\lambda_B}$ . The combination of (4)

<sup>10</sup> This expression is for the case in which none of player  $-i$ 's univariate marginal distributions contains a mass point.

<sup>11</sup> For more details, see Baye et al. (1996).

and budget-balance implies the following system of equations, which we refer to as (★):

$$\sum_{j \in \Omega_A(\gamma)} \frac{v_{B,j}}{2\lambda_B} + \sum_{j \notin \Omega_A(\gamma)} \frac{\left(\frac{v_{A,j}}{\lambda_A}\right)^2}{2\left(\frac{v_{B,j}}{\lambda_B}\right)} = X_A \quad (5)$$

$$\sum_{j \in \Omega_A(\gamma)} \frac{\left(\frac{v_{B,j}}{\lambda_B}\right)^2}{2\left(\frac{v_{A,j}}{\lambda_A}\right)} + \sum_{j \notin \Omega_A(\gamma)} \frac{v_{A,j}}{2\lambda_A} = X_B. \quad (6)$$

$\lambda_A^*$  and  $\lambda_B^*$  are implicitly defined by Eqs. (5) and (6), henceforth referred to as a solution to system (★).

Our first result is that, for every feasible configuration of battlefield values  $\{v_{A,j}, v_{B,j}\}_{j=1}^n$  and resource endowments  $\{X_A, X_B\}$ , there exists at least one solution to system (★).

**Proposition 1** *For any feasible configuration of battlefield values  $\{v_{A,j}, v_{B,j}\}_{j=1}^n$  and resource endowments  $\{X_A, X_B\}$  there exists a solution to system (★). If  $v_{A,j} = v_{B,j}$  for all  $j$ , then there exists a unique solution to system (★).*

**Proof** We begin with the proof that there exists a solution to system (★) and then examine the issue of uniqueness in constant-sum versions of the game. Recall that  $\gamma = \frac{\lambda_A}{\lambda_B}$  and that  $\Omega_A(\gamma)$  denotes the set of battlefields in which  $\frac{v_{A,j}}{v_{B,j}} > \gamma$ . Let  $\partial\Omega_A(\gamma)$  denote the (possibly empty) set of battlefields for which  $\frac{v_{A,j}}{v_{B,j}} = \gamma$  and let  $\hat{\Gamma} \subset \mathbb{R}_+$  denote the set of  $\gamma$  such that  $\partial\Omega_A(\gamma) \neq \emptyset$ —that is the set of  $\gamma$  that satisfy  $\gamma = \frac{v_{A,j}}{v_{B,j}}$  for some  $j \in \{1, \dots, n\}$ .  $\hat{\Gamma}$  corresponds to the set of  $\gamma$  at which  $\Omega_A(\gamma)$  ‘changes.’

Let

$$\underline{\gamma} \equiv \min \left\{ \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}, \frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1} \right\} > 0$$

and let

$$\bar{\gamma} \equiv \max \left\{ \frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}, \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} \right\} < \infty.$$

We first bound the set of potential solutions by showing that if there exists a solution to system (★), then  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ .

Suppose that  $\gamma \geq \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ . Because  $\Omega_A(\gamma)$  is the set of  $j$  for which  $\frac{v_{A,j}}{v_{B,j}} > \gamma$ , it follows that if  $\gamma \geq \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$  then there exists no  $j$  such that  $\frac{v_{A,j}}{v_{B,j}} > \gamma$  and  $\Omega_A(\gamma) = \emptyset$ . Then, because  $\Omega_A(\gamma) = \emptyset$  Eqs. (5) and (6) may be written as

$$\frac{\lambda_B}{2\lambda_A^2} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} = X_A \quad \text{and} \quad \frac{1}{2\lambda_A} = X_B. \quad (7)$$

The unique solution to the system in (7) is  $\lambda_A^* = \frac{1}{2X_B}$ ,  $\lambda_B^* = \frac{X_A}{2X_B^2} \left( \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} \right)^{-1}$ , and  $\gamma^* = \frac{\lambda_A^*}{\lambda_B^*} = \frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}$ . If  $\frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} \geq \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , then there exists a unique solution to system (★) for  $\gamma \geq \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ ,  $\gamma^* = \frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}$ . If  $\frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} < \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , then there exists no solution to system (★) with  $\gamma \geq \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , and so, in any solution to system (★)  $\gamma < \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ .

Now, suppose that  $\gamma < \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ . Because  $\Omega_A(\gamma)$  is the set of  $j$  for which  $\frac{v_{A,j}}{v_{B,j}} > \gamma$ , it follows that if  $\gamma < \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$  then  $\frac{v_{A,j}}{v_{B,j}} > \gamma$  for all  $j$  and  $\Omega_A(\gamma) = \{1, 2, \dots, n\}$ . Then, because  $\Omega_A(\gamma) = \{1, 2, \dots, n\}$  Eqs. (5) and (6) may be written as

$$\frac{1}{2\lambda_B} = X_A \quad \text{and} \quad \frac{\lambda_A}{2\lambda_B^2} \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} = X_B. \quad (8)$$

The unique solution to the system in (8) is  $\lambda_A^* = \frac{X_B}{2X_A^2} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1}$ ,  $\lambda_B^* = \frac{1}{2X_A}$ , and  $\gamma^* = \frac{\lambda_A^*}{\lambda_B^*} = \frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1}$ . If  $\frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1} < \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , then there exists a unique solution to system (★) for  $\gamma < \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ ,  $\gamma^* = \frac{\lambda_A^*}{\lambda_B^*} = \frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1}$ . If  $\frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1} \geq \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , then there exists no solution to system (★) with  $\gamma < \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , and so, in any solution to system (★)  $\gamma \geq \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ . This completes the proof that if there exists a solution to system (★), then  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ .

We now show that for any feasible configuration of battlefield values  $\{v_{A,j}, v_{B,j}\}_{j=1}^n$  and resource endowments  $\{X_A, X_B\}$ , there exists a solution to system (★) with  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ . Because  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ , it follows directly that  $\lambda_A, \lambda_B \in (0, \infty)$ . Multiplying both sides of (5) by  $\lambda_B$  and both sides of (6) by  $\lambda_A$  yields

$$\lambda_B X_A = \frac{1}{2} \sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{2\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{(v_{A,j})^2}{v_{B,j}} \quad (9)$$

$$\lambda_A X_B = \frac{\gamma^2}{2} \sum_{j \in \Omega_A(\gamma)} \frac{(v_{B,j})^2}{v_{A,j}} + \frac{1}{2} \sum_{j \notin \Omega_A(\gamma)} v_{A,j}. \quad (10)$$

Then dividing (10) by (9), we have:

$$\frac{X_B \gamma}{X_A} = \frac{\gamma^2 \sum_{j \in \Omega_A(\gamma)} \frac{(v_{B,j})^2}{v_{A,j}} + \sum_{j \notin \Omega_A(\gamma)} v_{A,j}}{\sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{(v_{A,j})^2}{v_{B,j}}}. \quad (11)$$

The right-hand side of (11) is continuous with respect to  $\gamma$ . In particular, for each  $\hat{\gamma} \in \hat{\Gamma}$ ,  $\hat{\gamma} v_{B,k} = v_{A,k}$  for each  $k \in \partial \Omega_A(\hat{\gamma})$ . Thus, for each  $\hat{\gamma} \in \hat{\Gamma}$

$$\lim_{\gamma \rightarrow \hat{\gamma}^+} \frac{\gamma^2 \sum_{j \in \Omega_A(\gamma)} \frac{(v_{B,j})^2}{v_{A,j}} + \sum_{j \notin \Omega_A(\gamma)} v_{A,j}}{\sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{(v_{A,j})^2}{v_{B,j}}} = \lim_{\gamma \rightarrow \hat{\gamma}^-} \frac{\gamma^2 \sum_{j \in \Omega_A(\gamma)} \frac{(v_{B,j})^2}{v_{A,j}} + \sum_{j \notin \Omega_A(\gamma)} v_{A,j}}{\sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{(v_{A,j})^2}{v_{B,j}}}.$$

Next, note that if  $\frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} \geq \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$  then  $\gamma^* = \frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}$  is a solution to (11) in which  $\Omega_A(\gamma^*) = \emptyset$ , and the result follows directly. Similarly, if  $\frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1} < \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$  then  $\gamma^* = \frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1}$  is a solution to (11) in which  $\Omega_A(\gamma^*) = \{1, \dots, n\}$  and the result follows directly.

We now examine the case in which  $\frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} < \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$  and  $\frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1} \geq \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ . Note first that if  $\min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} = \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , then  $\frac{v_{A,j}}{v_{B,j}} = 1$  for all  $j$  and the first inequality becomes  $\frac{X_B}{X_A} < 1$ , which is violated by the assumption that  $\frac{X_B}{X_A} \geq 1$ . Hence, in this case  $\min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} < \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ . To verify that a solution in  $\gamma$  to (11) exists multiply both sides of (11) by  $\frac{X_A}{X_B}$ . The left-hand side of (11) then equals  $\gamma$  and the right-hand side equals the following continuous and increasing function:

$$f(\gamma) = \left( \frac{X_A}{X_B} \right) \left( \frac{\gamma^2 \sum_{j \in \Omega_A(\gamma)} \frac{(v_{B,j})^2}{v_{A,j}} + \sum_{j \notin \Omega_A(\gamma)} v_{A,j}}{\sum_{j \in \Omega_A(\gamma)} v_{B,j} + \frac{1}{\gamma^2} \sum_{j \notin \Omega_A(\gamma)} \frac{(v_{A,j})^2}{v_{B,j}}} \right).$$

Because, by assumption,  $\frac{X_B}{X_A} \sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}} < \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , it follows that

$$f\left(\max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}\right) = \left( \frac{X_A}{X_B} \right) \left( \frac{\left( \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} \right)^2}{\sum_{j=1}^n \frac{(v_{A,j})^2}{v_{B,j}}} \right) > \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} \quad (12)$$

and, as  $\frac{X_B}{X_A} \left( \sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}} \right)^{-1} \geq \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}$ , it follows that

$$f\left(\min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}\right) = \left(\frac{X_A}{X_B}\right) \left(\min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\}\right)^2 \left(\sum_{j=1}^n \frac{(v_{B,j})^2}{v_{A,j}}\right) \leq \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} \quad (13)$$

Combining (12), (13), with the continuity of  $f(\gamma)$ , it follows that there exists at least one point  $\gamma^* \in [\underline{\gamma}, \bar{\gamma}]$  such that  $f(\gamma^*) = \gamma^*$ . This completes the proof of the existence of a  $\gamma^*$  that solves (11), and then given a solution  $\gamma^*$ , (9) and (10) can be used to solve for  $\lambda_B$  and  $\lambda_A$  (a solution to system  $(\star)$ ), respectively.

For uniqueness in the constant-sum game, note that when  $v_{A,j} = v_{B,j}$  for all  $j$  then  $\max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} = \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} = 1$  for all  $j$  and  $\underline{\gamma} = 1 \leq \bar{\gamma} = \frac{X_B}{X_A}$ . Consequently,  $\Omega_A(\gamma^*) = \emptyset$  and (11) becomes  $\gamma^* = \frac{X_B}{X_A}$ .  $\square$

Although there exists a unique solution to system  $(\star)$  when the game is constant-sum, there may exist multiple solutions to system  $(\star)$  in non-constant-sum versions of the game, and these multiple solutions give rise to multiple equilibria. Following the statement and proof of Theorem 1, we provide an example in which there are multiple payoff nonequivalent equilibria.

We now examine equilibrium in the general case of the linear heterogeneous asymmetric objective and, then, discuss the special case of the linear heterogeneous symmetric objective.

**Theorem 1** *For each solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ , each player in the GGL game has a unique set of Nash equilibrium univariate marginals. If  $v_{i,j}/\lambda_i^* \geq v_{-i,j}/\lambda_{-i}^*$ , then*

$$F_{-i,j}(x) = \left( \frac{\frac{v_{i,j}}{\lambda_i^*} - \frac{v_{-i,j}}{\lambda_{-i}^*}}{\frac{v_{i,j}}{\lambda_i^*}} \right) + \frac{x}{\frac{v_{i,j}}{\lambda_i^*}} \quad x \in \left[ 0, \frac{v_{-i,j}}{\lambda_{-i}^*} \right]$$

$$F_{i,j}(x) = \frac{x}{\frac{v_{-i,j}}{\lambda_{-i}^*}} \quad x \in \left[ 0, \frac{v_{-i,j}}{\lambda_{-i}^*} \right].$$

*Conversely, for each equilibrium of the GGL game, there exists a corresponding solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ . For each solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ , the expected payoff for player A is  $\sum_{j \in \Omega_A(\gamma^*)} \left( v_{A,j} - \frac{\gamma^* v_{B,j}}{2} \right) + \sum_{j \notin \Omega_A(\gamma^*)} \left( \frac{v_{A,j}^2}{2\gamma^* v_{B,j}} \right)$  and the expected payoff for player B is  $\sum_{j \notin \Omega_A(\gamma^*)} \left( v_{B,j} - \frac{v_{A,j}}{2\gamma^*} \right) + \sum_{j \in \Omega_A(\gamma^*)} \left( \frac{\gamma^* v_{B,j}^2}{2v_{A,j}} \right)$ .*

**Proof** We now show that for each solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$  any pair of joint distributions  $(P_A, P_B)$  with the sets of univariate marginals specified in Theorem 1 is a Nash equilibrium of the GGL game. In Appendix 1, we show that: (i) for each equilibrium of the GGL game, there exists a corresponding solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$  and (ii) for each solution  $(\lambda_A^*, \lambda_B^*)$  each player in the GGL game has a unique set of Nash equilibrium univariate marginals.

For the proof that for each solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$  any pair of joint distributions  $(P_A, P_B)$  with the sets of univariate marginals specified in Theorem 1 is a Nash equilibrium of the GGL game, we focus on player  $A$ , and note that the argument for player  $B$  is symmetric. First, observe that because  $(\lambda_A, \lambda_B)$  is a solution to  $(\star)$ , this is a feasible strategy for player  $A$ :

$$\sum_{j=1}^n \int_0^\infty x dF_{A,j} = \sum_{j \in \Omega_A(\gamma^*)} \frac{v_{B,j}}{2\lambda_B^*} + \sum_{j \notin \Omega_A(\gamma^*)} \frac{\left(\frac{v_{A,j}}{\lambda_A^*}\right)^2}{2\left(\frac{v_{B,j}}{\lambda_B^*}\right)} = X_A.$$

Then, given that player  $B$  is following the equilibrium strategy, player  $A$ 's payoff from an arbitrary strategy with the set of univariate marginals  $\{\bar{F}_{A,j}\}_{j=1}^n$  is:

$$\pi_A(\{\bar{F}_{A,j}, F_{B,j}\}_{j=1}^n) = \sum_{j=1}^n \int_0^\infty v_{A,j} F_{B,j}(x) d\bar{F}_{A,j}(x).$$

Because it is never a best response for player  $A$  to place strictly positive mass at zero in any battlefield  $j \in \Omega_A(\gamma^*)$  nor to provide offers outside of the support of any of player  $B$ 's univariate marginal distributions, we have:

$$\begin{aligned} \pi_A(\{\bar{F}_{A,j}, F_{B,j}\}_{j=1}^n) &= \sum_{j \in \Omega_A(\gamma^*)} \left[ \left( v_{A,j} - \frac{v_{B,j} \lambda_A^*}{\lambda_B^*} \right) + \int_0^{\frac{v_{B,j}}{\lambda_B^*}} x \lambda_A^* d\bar{F}_{A,j}(x) \right] \\ &\quad + \sum_{j \notin \Omega_A(\gamma^*)} \int_0^{\frac{v_{A,j}}{\lambda_A^*}} x \lambda_A^* d\bar{F}_{A,j}(x). \end{aligned}$$

But from the budget constraint, it follows that

$$\pi_A(\{\bar{F}_{A,j}, F_{B,j}\}_{j=1}^n) \leq \sum_{j \in \Omega_A(\gamma^*)} \left( v_{A,j} - \frac{\lambda_A^* v_{B,j}}{\lambda_B^*} \right) + \lambda_A^* X_A$$

which, together with (5), yields

$$\pi_A(\{\bar{F}_{A,j}, F_{B,j}\}_{j=1}^n) \leq \sum_{j \in \Omega_A(\gamma^*)} \left( v_{A,j} - \frac{\gamma^* v_{B,j}}{2} \right) + \sum_{j \notin \Omega_A(\gamma^*)} \left( \frac{v_{A,j}^2}{2\gamma^* v_{B,j}} \right),$$

which holds with equality if  $\{\bar{F}_{A,j}\}_{j=1}^n$  is the equilibrium strategy. This completes the proof that there are no payoff increasing deviations for player  $A$ . A symmetric argument applies to player  $B$ , and thus, any pair of joint distributions  $(P_A, P_B)$  providing the sets of univariate marginal distributions  $(\{F_{A,j}, F_{B,j}\}_{j=1}^n)$  is an equilibrium.  $\square$

Proposition 1 guarantees at least one solution to system  $(\star)$  and Theorem 1 demonstrates that corresponding to every such solution there is a unique set of Nash equilibrium univariate marginal distributions in the General Lotto game. When there

are multiple solutions to system ( $\star$ ), Theorem 1 establishes that the payoff to player A decreases (and the payoff to player B increases) across equilibria as the corresponding  $\gamma^*$  increases.

If the game is constant-sum (i.e., the players' battlefield valuations are symmetric for all battlefields), the unique solution to system ( $\star$ )<sup>12</sup> is  $\lambda_A^* = \frac{1}{2X_B}$  and  $\lambda_B^* = \frac{X_A}{2X_B^2}$ . We then have the following corollary, which appears in a closely related form in Bell and Cover (1980), Sahuguet and Persico (2006), and Washburn (2013).

**Corollary 1** *If  $v_{A,j} = v_{B,j} \equiv v_j$  for all  $j$ , then the unique set of Nash equilibrium univariate marginal distributions of the GGL game are, for all  $j \in \{1, \dots, n\}$ :*

$$\begin{aligned} F_{A,j}(x) &= \left(1 - \frac{X_A}{X_B}\right) + \frac{x}{2v_j X_B} \left(\frac{X_A}{X_B}\right) & x \in [0, 2v_j X_B] \\ F_{B,j}(x) &= \frac{x}{2v_j X_B} & x \in [0, 2v_j X_B] \end{aligned}$$

*The expected payoff for player A is  $\frac{X_A}{2X_B}$  and the expected payoff for player B is  $1 - \frac{X_A}{2X_B}$ .*

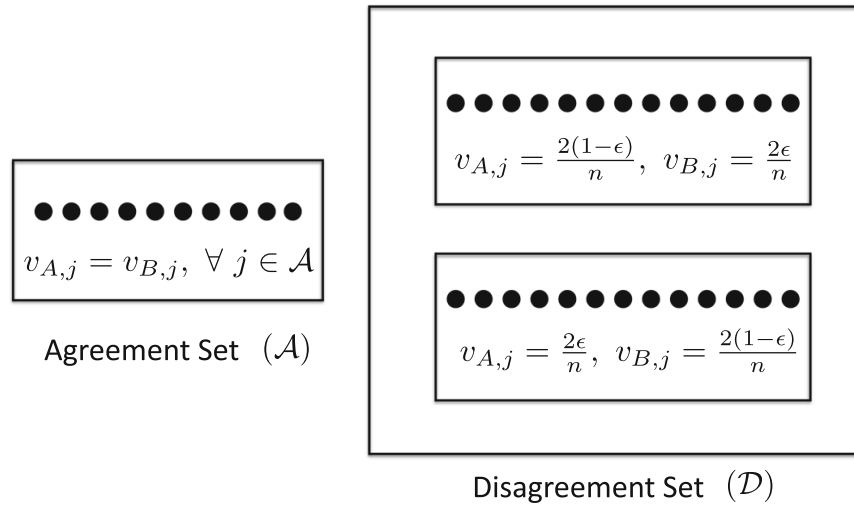
We now examine a special class of GGL games, labeled Quasi-Symmetric General Lotto games, in which the players have symmetric resource endowments and the players' sets of battlefield valuations feature a form of symmetry with respect to the ratio  $\frac{v_{A,j}}{v_{B,j}}$ . Even in this simple case, we find that there arise multiple equilibria with distinct payoffs and sets of Nash equilibrium univariate marginal distributions. The parametric form used for battlefield valuations is useful in that it makes the calculation of the set  $\Omega_A(\gamma)$  easier, thereby simplifying the problem of solving system ( $\star$ ). In moving from this example to an arbitrary configuration of battlefield valuations, the calculation of the set  $\Omega_A(\gamma)$  becomes more involved.

**Quasi-Symmetric General Lotto and Colonel Blotto games** *Consider a GGL [GCB] game in which  $X_A = X_B = 1$ , and the  $n(> 0)$  battlefields may be partitioned into an agreement set, denoted  $\mathcal{A}$ , in which  $v_{A,j} = v_{B,j} \equiv \frac{1}{n}$  for each  $j \in \mathcal{A}$  and  $\sum_{j \in \mathcal{A}} v_{A,j} = \frac{n_{\mathcal{A}}}{n}$ , where  $n_{\mathcal{A}} \geq 0$  is the number of battlefields in the agreement set, and a disagreement set, denoted  $\mathcal{D}$ , with an even number  $n_{\mathcal{D}} \geq 0$  of battlefields, where for the first  $\frac{n_{\mathcal{D}}}{2}$  battlefields  $v_{A,j} = \frac{2(1-\epsilon)}{n}$  and  $v_{B,j} = \frac{2\epsilon}{n}$  and for the last  $\frac{n_{\mathcal{D}}}{2}$  battlefields  $v_{A,j} = \frac{2\epsilon}{n}$  and  $v_{B,j} = \frac{2(1-\epsilon)}{n}$ , with  $\epsilon \in (0, \frac{1}{2})$ . A GGL [GCB] game with such a configuration of battlefield values, for some  $(\epsilon, n_{\mathcal{A}}, n_{\mathcal{D}})$ , and resource endowments  $X_A = X_B = 1$  is labeled a Quasi-Symmetric General Lotto game [Quasi-Symmetric Colonel Blotto game], which we abbreviate as a Quasi-Symmetric GL game [Quasi-Symmetric CB game].*

Figure 2 illustrates a Quasi-Symmetric GL game configuration of battlefield values. Given any Quasi-Symmetric GL game endowments and configuration of battlefield valuations, it follows directly from system ( $\star$ ) that for all  $\epsilon \in (0, 0.5)$ ,  $n_{\mathcal{D}} \geq 0$ , and  $n_{\mathcal{A}} \geq 0$  Eq. (11) has a solution at  $\gamma = 1$ . Furthermore, it follows from Theorem 1 that the  $\gamma = \frac{\lambda_A}{\lambda_B} = 1$  equilibrium respects the quasi-symmetry, in the sense that for each  $j$  in the first  $\frac{n_{\mathcal{D}}}{2}$  battlefields in the disagreement set and corresponding mirror battle  $j'$  in the last  $\frac{n_{\mathcal{D}}}{2}$  battlefields in the disagreement set, player A's equilibrium univariate

<sup>12</sup> As  $\frac{X_A}{X_B} \leq 1$  it must be the case that  $\lambda_B \leq \lambda_A$ .





**Fig. 2** Quasi-Symmetric GL game battlefield configuration [ $\epsilon \in (0, 0.5)$ ]

marginal distribution in battlefield  $j$ ,  $F_{A,j}(\cdot)$ , is the same as player  $B$ 's equilibrium univariate marginal distribution in the corresponding mirror battlefield  $j'$ ,  $F_{B,j'}(\cdot)$ , and vice versa.

Depending on the values of  $\epsilon$ ,  $n_D$ , and  $n_A$  there may exist multiple solutions and thus multiple equilibria. To solve for all possible solutions to system  $(\star)$ , note that (11) may be written as

$$\gamma^3 \sum_{j \in \Omega_A(\gamma)} \frac{(v_{B,j})^2}{v_{A,j}} - \frac{X_B \gamma^2}{X_A} \sum_{j \in \Omega_A(\gamma)} v_{B,j} + \gamma \sum_{j \notin \Omega_A(\gamma)} v_{A,j} - \frac{X_B}{X_A} \sum_{j \notin \Omega_A(\gamma)} \frac{(v_{A,j})^2}{v_{B,j}} = 0. \quad (14)$$

Next, note that with symmetric resource constraints it must be the case that either  $\frac{1-\epsilon}{\epsilon} > \gamma \geq 1$  or  $1 > \gamma \geq \frac{\epsilon}{1-\epsilon}$ .<sup>13</sup> If  $1 > \gamma \geq \frac{\epsilon}{1-\epsilon}$ , then  $\Omega_A(\gamma)$  includes  $\mathcal{A}$  and the portion of  $\mathcal{D}$  in which  $v_{A,j} = \frac{2(1-\epsilon)}{n}$  and  $v_{B,j} = \frac{2\epsilon}{n}$ , and (14) may be written as

$$\gamma^3 \left( \frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_D}{n} + \frac{n_A}{n} \right) - \gamma^2 \left( \epsilon \cdot \frac{n_D}{n} + \frac{n_A}{n} \right) + \gamma \left( \epsilon \cdot \frac{n_D}{n} \right) - \left( \frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_D}{n} \right) = 0. \quad (15)$$

Similarly, if  $\frac{1-\epsilon}{\epsilon} > \gamma \geq 1$ , then  $\Omega_A(\gamma)$  includes only the portion of  $\mathcal{D}$  in which  $v_{A,j} = \frac{2(1-\epsilon)}{n}$  and  $v_{B,j} = \frac{2\epsilon}{n}$  and (14) may be written as

$$\gamma^3 \left( \frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_D}{n} \right) - \gamma^2 \left( \epsilon \cdot \frac{n_D}{n} \right) + \gamma \left( \epsilon \cdot \frac{n_D}{n} + \frac{n_A}{n} \right) - \left( \frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_D}{n} + \frac{n_A}{n} \right) = 0. \quad (16)$$

<sup>13</sup> If  $\gamma < \min_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} = \frac{\epsilon}{1-\epsilon}$  then  $\Omega_A(\gamma) = \{1, \dots, n\}$ , and if  $\gamma \geq \max_j \left\{ \frac{v_{A,j}}{v_{B,j}} \right\} = \frac{1-\epsilon}{\epsilon}$  then  $\Omega_A(\gamma) = \emptyset$ . In either case, one player has a weakly higher expected expenditure of the resource in every battlefield and a strictly higher expenditure in a nonempty subset of battlefields. With symmetric budget constraints, it is clear that this is not possible.

**Table 2** Multiple equilibria in example ( $\epsilon = 0.10$ ,  $(n_A/n) = 0.1$ , and  $(n_D/n) = 0.9$ )

$\gamma^*$	$\lambda_A^*$	$\lambda_B^*$	$\pi_A^*$	$\pi_B^*$
0.1604	0.0464	0.2893	0.9259	0.5383
0.5669	0.0627	0.1106	0.8650	0.7618
1.00	0.10	0.10	0.82	0.82
1.7640	0.1106	0.0627	0.7618	0.8650
6.2362	0.2893	0.0464	0.5383	0.9259

We now consider a Quasi-Symmetric GL game example that demonstrates the ‘quasi-symmetry’ of the  $\gamma = 1$  equilibrium and, with five solutions to system  $(\star)$ , demonstrates the multiplicity of equilibria issue.

**Quasi-Symmetric Example** If  $\epsilon = 0.10$ ,  $(n_A/n) = 0.1$ , and  $(n_D/n) = 0.9$ , then there are five solutions to system  $(\star)$ —Eq. (15) has two real roots for  $1 > \gamma \geq \frac{\epsilon}{1-\epsilon} = \frac{1}{9}$  and Eq. (16) has three real roots for  $9 = \frac{1-\epsilon}{\epsilon} > \gamma \geq 1$ —and Theorem 1 provides the corresponding equilibrium expected payoffs and sets of univariate marginal distributions. These five equilibria are summarized in Table 2.

For the two solutions with  $1 > \gamma \geq \frac{1}{9}$  equilibrium is described as follows: for all battlefields  $j \in \mathcal{A}$  let  $v_j \equiv v_{A,j} = v_{B,j}$

$$F_{B,j}(x) = \left(1 - \frac{\lambda_A}{\lambda_B}\right) + \frac{\lambda_A x}{v_j} \quad x \in \left[0, \frac{v_j}{\lambda_B}\right]$$

$$F_{A,j}(x) = \frac{\lambda_B x}{v_j} \quad x \in \left[0, \frac{v_j}{\lambda_B}\right],$$

for  $j \in \mathcal{D}$  such that  $v_{A,j} = \frac{9}{5n}$  and  $v_{B,j} = \frac{1}{5n}$

$$F_{B,j}(x) = \left(1 - \frac{\lambda_A}{9\lambda_B}\right) + \frac{\lambda_A x}{(9/5n)} \quad x \in \left[0, \frac{1}{5n\lambda_B}\right]$$

$$F_{A,j}(x) = \frac{\lambda_B x}{(1/5n)} \quad x \in \left[0, \frac{1}{5n\lambda_B}\right],$$

and for  $j \in \mathcal{D}$  such that  $v_{A,j} = \frac{1}{5n}$  and  $v_{B,j} = \frac{9}{5n}$

$$F_{A,j}(x) = \left(1 - \frac{\lambda_B}{9\lambda_A}\right) + \frac{\lambda_B x}{(9/5n)} \quad x \in \left[0, \frac{1}{5n\lambda_A}\right]$$

$$F_{B,j}(x) = \frac{\lambda_A x}{(1/5n)} \quad x \in \left[0, \frac{1}{5n\lambda_A}\right].$$

The expected payoff for player A is  $\frac{91}{100} - \frac{19\gamma}{200} + \frac{1}{200\gamma}$  and the expected payoff for player B is  $\frac{81}{100} - \frac{9}{200\gamma} + \frac{11\gamma}{200}$ . Similarly, for the three solutions with  $9 > \gamma \geq 1$  equilibrium is described as follows: for all battlefields  $j \in \mathcal{A}$

$$F_{A,j}(x) = \left(1 - \frac{\lambda_B}{\lambda_A}\right) + \frac{\lambda_B x}{v_j} \quad x \in \left[0, \frac{v_j}{\lambda_A}\right]$$

$$F_{B,j}(x) = \frac{\lambda_A x}{v_j} \quad x \in \left[0, \frac{v_j}{\lambda_A}\right],$$

for  $j \in \mathcal{D}$  such that  $v_{A,j} = \frac{9}{5n}$  and  $v_{B,j} = \frac{1}{5n}$

$$F_{B,j}(x) = \left(1 - \frac{\lambda_A}{9\lambda_B}\right) + \frac{\lambda_A x}{(9/5n)} \quad x \in \left[0, \frac{1}{5n\lambda_B}\right]$$

$$F_{A,j}(x) = \frac{\lambda_B x}{(1/5n)} \quad x \in \left[0, \frac{1}{5n\lambda_B}\right],$$

and for  $j \in \mathcal{D}$  such that  $v_{A,j} = \frac{1}{5n}$  and  $v_{B,j} = \frac{9}{5n}$

$$F_{A,j}(x) = \left(1 - \frac{\lambda_B}{9\lambda_A}\right) + \frac{\lambda_B x}{(9/5n)} \quad x \in \left[0, \frac{1}{5n\lambda_A}\right]$$

$$F_{B,j}(x) = \frac{\lambda_A x}{(1/5n)} \quad x \in \left[0, \frac{1}{5n\lambda_A}\right].$$

The expected payoff for player A is  $\frac{81}{100} - \frac{9\gamma}{200} + \frac{11}{200\gamma}$  and the expected payoff for player B is  $\frac{91}{100} - \frac{19}{200\gamma} + \frac{\gamma}{200}$ . Note also that in the equilibrium with  $\lambda_A = \lambda_B = 0.10$ , for each  $j$  in the first  $\frac{n_D}{2}$  battlefields in the disagreement set and corresponding mirror battle  $j'$  in the last  $\frac{n_D}{2}$  battlefields in the disagreement set, player A's equilibrium univariate marginal distribution in battlefield  $j$ ,  $F_{A,j}(\cdot)$ , is the same as player B's equilibrium univariate marginal distribution in the corresponding mirror battlefield  $j'$ ,  $F_{B,j'}(\cdot)$ , and vice versa.

Although this example is a simple one in which only three values of the ratio  $\frac{v_{A,j}}{v_{B,j}}$  arise, there continues to exist a multiplicity of payoff nonequivalent equilibria even when all of the parameters of the example are slightly perturbed, so that the ratio  $\frac{v_{A,j}}{v_{B,j}}$  may be distinct for every battlefield  $j$ . In fact, fixing  $n$  and taking the relevant space of parameters to be  $(X_A, X_B, \{v_{A,j}\}_{j=1}^n, \{v_{B,j}\}_{j=1}^n) \in \mathbb{R}_{++}^2 \times [\text{Int}(S^{n-1})]^2$ , where  $\text{Int}(S^{n-1})$  is the interior of the  $n - 1$  dimensional unit simplex containing the values  $\{v_{i,j}\}_{j=1}^n, i = A, B$ , there is a set of positive Lebesgue measure in  $\mathbb{R}_{++}^2 \times [\text{Int}(S^{n-1})]^2$  which contains the parameters in our example and in which such a multiplicity exists.<sup>14</sup>

<sup>14</sup> For a fixed number of battlefields  $n$ , the equilibrium values  $\gamma^*$  are the solutions in  $\gamma$  to Eq. (14). If the set of indices  $\Omega_A(\gamma)$  is invariant over an interval of  $\gamma$ 's, the left-hand side of (14) is a cubic in  $\gamma$  over that interval. In our specific numerical example with  $\epsilon = 0.1$ ,  $\frac{n_A}{n} = 0.1$ , and  $\frac{n_D}{n} = 0.9$ , the sets of indices  $\Omega_A(\gamma)$  are invariant in each of two adjacent domains of  $\gamma$ ,  $1 > \gamma \geq \frac{1}{9}$  and  $9 > \gamma \geq 1$ , but differ across the domains (represented, respectively, by Eqs. (15) and (16)). More generally, because the set of indices  $\Omega_A(\gamma)$  changes only at values of  $\gamma$  for which  $\gamma = \frac{v_{A,j}}{v_{B,j}}$  for some  $j$ , the coefficients of  $\gamma$  in the cubic are fixed over distinct intervals between adjacent values of  $\frac{v_{A,j}}{v_{B,j}}$  and the left-hand side of (14) is, in fact, continuous in  $\gamma$  over  $[\underline{\gamma}, \overline{\gamma}]$ , including at values of  $\gamma$  at which the set of indices  $\Omega_A(\gamma)$  changes. Moreover, the left-hand side of (14) is also continuous in the  $2n + 2$ -tuple of parameters  $(X_A, X_B, \{v_{A,j}\}_{j=1}^n, \{v_{B,j}\}_{j=1}^n)$  over the relevant domain. In the numerical example, two of the five solutions  $\gamma^*$  to (14) identified in Table 2 are interior to  $[\frac{1}{9}, 1)$  and two are interior to  $[1, 9)$ . (The remaining solution  $\gamma^* = 1$  is on the boundary of the two sets.) It is easily verified that none of the four solutions to (14) that are interior to  $[\frac{1}{9}, 1)$  or  $[1, 9)$  are multiple roots of the polynomial in  $\gamma$  (for the fixed set of indices  $\Omega_A(\gamma)$  applicable over the interval). Therefore, they cannot represent tangencies to the  $\gamma$ -axis of the applicable polynomial, but rather represent values of  $\gamma$  where the left-hand side of (14) cuts the origin. As a consequence, for sufficiently small perturbations of the  $2n + 2$ -tuple of parameters chosen in the example, for each of these four values of  $\gamma^*$  there exists a neighborhood about  $\gamma^*$  such that the set of indices contained in  $\Omega_A(\gamma^*)$  coincides with the set of indices in the example and for that fixed set  $\Omega_A(\gamma^*)$ , the polynomial in  $\gamma$  given by the left-hand side of (14) has a root within the neighborhood. That is, there is an open set of parameters  $(X_A, X_B, \{v_{A,j}\}_{j=1}^n, \{v_{B,j}\}_{j=1}^n)$

That is, a multiplicity of payoff nonequivalent equilibria should not be viewed as an anomaly.

## 4 Generalized Colonel Blotto results

As pointed out in Hart (2008, 2016), the General Lotto game can be used as an intermediate step in solving the Colonel Blotto game. In moving from the GGL game to the GCB game, we face the added requirement that for each player a joint distribution exists that satisfies the player's budget constraint with probability one and not just in expectation. From this, it is clear that if a pair of joint distributions is found that yields for each player the set of univariate marginal distributions corresponding to an equilibrium in the GGL game and each joint distribution satisfies the constraint that the budget holds with probability one, then this pair will also be an equilibrium of the Colonel Blotto version of the game.

In Sect. 4.1, we begin by identifying a sufficient condition for an equilibrium set of univariate marginal distributions in a GGL game to be able to map, via a copula, into an equilibrium joint distribution in the corresponding GCB game and then examine the converse issue of a sufficient condition for an equilibrium set of univariate marginal distributions in the GGL game to not be able to map into an equilibrium joint distribution in the corresponding GCB game. These two results only cover a subset of the parameter space, and we conclude Sect. 4.1 by examining this gap in the case of the Quasi-Symmetric GL game. In Sect. 4.2, we examine how the results in Sect. 4.1 relate to the existing literature on the Colonel Blotto game.

### 4.1 Sufficient conditions for arbitrary parameter configurations

The following proposition provides a sufficient condition for the set of univariate marginal distributions corresponding to an equilibrium in the GGL game given in Theorem 1 to be generated by a pair of joint distributions that balance the players' respective budgets with probability one. That is, it provides a sufficient condition for an equilibrium set of univariate marginal distributions in the GGL game to be attainable in equilibrium in the GCB game. In the analysis that follows, consider a partition of the battlefields into subsets based on distinct pairs of valuations  $v_{A,j}$  and  $v_{B,j}$  so that two battlefields  $h$  and  $m$ , where  $h, m \in \{1, \dots, n\}$ , are in the same set in the partition if and only if  $v_{A,h} = v_{A,m}$  and  $v_{B,h} = v_{B,m}$ . Let  $k \leq n$  denote the number of subsets in this partition,  $j \in \{1, \dots, k\}$  index the distinct pairs of battlefield valuations  $(v_{A,j}, v_{B,j})$ ,  $\Psi_j$  denote the set of battlefields with battlefield valuations  $(v_{A,j}, v_{B,j})$ , and  $n_j \geq 1$  denote the number of battlefields in  $\Psi_j$ .

**Proposition 2** *Given a solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ , if for each distinct pair of battlefield valuations  $(v_{A,j}, v_{B,j})$  with  $\frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*} \leq 1$ , for some  $i \in \{A, B\}$ , it is the case that  $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$ , then there exists a Nash equilibrium of the GCB game with the same*

Footnote 14 continued

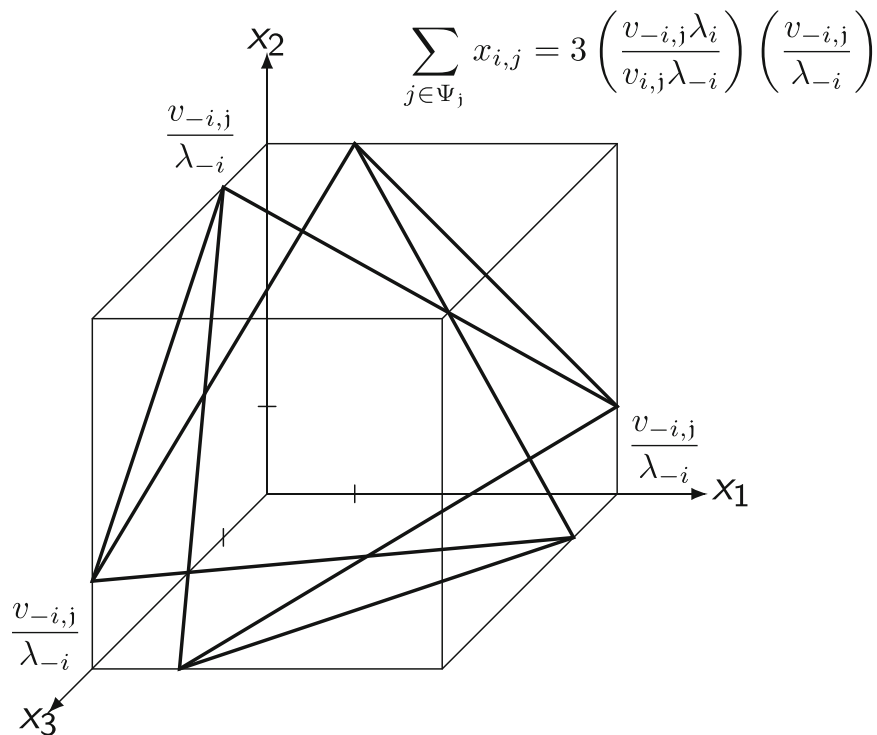
containing those in the example for which there are solutions to (14) 'close' to the four values of  $\gamma^*$  identified in the interior of  $[\frac{1}{9}, 1)$  and  $[1, 9)$ .

set of univariate marginal distributions and expected payoffs as in the corresponding equilibrium in Theorem 1.

Given the  $k \leq n$  distinct pairs of battlefield valuations, we can form independent  $n_j$ -variate marginal distributions on each of the  $j = 1, \dots, k$  subsets of battlefields with a distinct pair of valuations, where the budget constraint for each subset  $j$  is equal to the expected expenditure from the Theorem 1 set of univariate marginal distribution functions on that subset of battlefields. For example, if  $\frac{v_{i,j}}{\lambda_i^*} \geq \frac{v_{-i,j}}{\lambda_{-i}^*}$ , then from Theorem 1 it follows that player  $-i$ 's expected expenditure on the  $j$ th set of battlefields is  $\frac{n_j}{2} \left( \frac{v_{-i,j}}{\lambda_{-i}^*} \right)^2 / \left( \frac{v_{i,j}}{\lambda_i^*} \right)$  and  $i$ 's expected expenditure on the  $j$ th set of battlefields is  $n_j \left( \frac{v_{-i,j}}{2\lambda_{-i}^*} \right)$ . Then, the problem of constructing an equilibrium  $n$ -variate joint distribution  $P_i$ , for each player  $i$ , that is budget balancing with probability one and that provides the univariate marginals given in Theorem 1 is replaced by the problem of constructing  $k$   $n_j$ -variate marginal distributions, denoted  $P_{i,j}$ , one for each of the  $k$  sets of battlefields with distinct valuations, and then letting each player  $i$ 's joint distribution be defined as  $P_i(x) = \prod_{j=1}^k P_{i,j}(x_j)$  where  $x_j$  is the restriction of  $x$  to the battlefields in set  $j$ . For each of the sets of battlefields  $j \in \{1, \dots, k\}$  if  $\frac{v_{i,j}}{\lambda_i^*} \geq \frac{v_{-i,j}}{\lambda_{-i}^*}$  and it is the case that  $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$ , then player  $i$ 's  $n_j$ -variate marginal distribution  $P_{i,j}$  may be formed by deterministically allocating  $n_j \left( \frac{v_{-i,j}}{2\lambda_{-i}^*} \right)$  of  $i$ 's budget to the subset  $j$  of battlefields and then constructing  $P_{i,j}$  using the existing construction methods in Gross and Wagner (1950), Roberson (2006), or Weinstein (2012). Similarly, player  $-i$ 's  $n_j$ -variate marginal distribution  $P_{-i,j}$  may be formed by deterministically allocating  $\frac{n_j}{2} \left( \frac{v_{-i,j}}{\lambda_{-i}^*} \right)^2 / \left( \frac{v_{i,j}}{\lambda_i^*} \right)$  of  $-i$ 's budget to the subset  $j$  of battlefields and then constructing  $P_{-i,j}$  using the distribution from Roberson (2006, Theorem 4 p.9).<sup>15</sup> Any such construction provides the necessary univariate marginals characterized in Theorem 1 and the resulting joint distributions  $P_i$  and  $P_{-i}$  are budget-balancing with probability one.

To fix ideas regarding the multi-variate marginal distributions that are utilized in the construction method summarized above, it is instructive to briefly examine an example of such a multi-variate marginal. For the case of  $n_j = 3$  and  $\frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*} > \frac{2}{n_j}$ , the support of  $P_{-i,j}$  is given in Fig. 3. Note that the support of  $P_{-i,j}$  lies on the budget hyperplane  $\sum_{i=1}^3 x_i = 3 \left( \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*} \right) \left( \frac{v_{-i,j}}{2\lambda_{-i}^*} \right)$  and, as is shown in Roberson (2006), there exists a distribution of mass across this support that provides the set of univariate marginals specified by Theorem 1. The role of the condition that  $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$  for each  $j \in \{1, \dots, k\}$  can be seen in Fig. 3, where the condition implies that that budget hyperplane cuts through the  $n_j$ -box, or hypercube, above its intercepts and thus the upper bound

<sup>15</sup> In Roberson (2006), the construction is carried out with respect to the players' aggregate resource endowments  $X_A \leq X_B$ . Note that in this paper's subset  $j$  of battlefields player  $-i$ 's budget is  $X_{-i,j} \equiv \frac{n_j}{2} \left( \frac{v_{-i,j}}{\lambda_{-i}^*} \right)^2 / \left( \frac{v_{i,j}}{\lambda_i^*} \right)$  and player  $i$ 's budget is  $X_{i,j} \equiv n_j \left( \frac{v_{-i,j}}{2\lambda_{-i}^*} \right)$ , where  $X_{-i,j} \leq X_{i,j}$ . To apply the construction in Roberson (2006) to an  $n_j$ -variate marginal distribution in this paper, substitute player  $-i$  and  $X_{-i,j}$  for player  $A$  and  $X_A$ , respectively, and player  $i$  and  $X_{i,j}$  for player  $B$  and  $X_B$ , respectively.



**Fig. 3** Support of  $P_{-i,j}$  for the case of  $\frac{v_{i,j}}{\lambda_i^*} \geq \frac{v_{-i,j}}{\lambda_{-i}^*}$

of the support of each of the univariate marginals is feasible. Furthermore, Roberson (2006) shows that, as long as the hyperplane cuts through the hypercube above its intercepts, the constraint that the support of the joint distribution satisfies the budget constraint with probability one is satisfied and the results in Theorem 1 extend directly to the Colonel Blotto version of the game. Thus, the conditions of Proposition 2 are sufficient for the existence of budget-balancing joint distributions, one for each player, that provide the sets of equilibrium univariate marginal distributions in Theorem 1. It follows directly that a necessary condition for the existence of such a joint distribution is  $X_i \geq \max_j \left\{ \min \left\{ \frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*} \right\} \right\}$  for each player  $i$ . This necessary condition states that each player's budget-balancing hyperplane cuts through the  $n$ -box formed by the supports of each of the  $n$  univariate marginals specified by Theorem 1 above its intercepts.

Proposition 2 provides a sufficient condition for any particular equilibrium set of univariate marginal distributions in a GGL game to be able to map, via a copula, into an equilibrium joint distribution in the corresponding GCB game. Note that if the set of battlefields is replicated  $\alpha \in \mathbb{Z}_+$  times, then: (i) each  $n_j$  increases to  $\alpha n_j$  but (ii) the proportion of battlefields of type  $j$  remains constant. Thus, the set of solutions  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$  is invariant to any  $\alpha$ -replication of the set of battlefields. It follows that if the Proposition 2 condition of  $\frac{2}{n_j} \leq \frac{v_{-i,j} \lambda_i^*}{v_{i,j} \lambda_{-i}^*}$  fails to hold for some  $j \in \{1, \dots, k\}$ , then there exists an  $\alpha \in \mathbb{Z}_+$  such that  $\frac{2}{\alpha n_j} \leq \frac{v_{-i,j} \lambda_i^*}{v_{i,j} \lambda_{-i}^*}$ , and Proposition 2 applies to the  $\alpha$ -replication of the set of battlefields.

We now examine the issue of a sufficient condition for any particular equilibrium set of univariate marginal distributions in the GGL game to not be able to map into



an equilibrium joint distribution in the corresponding GCB game. If  $X_B > nX_A$  then, clearly, in any equilibrium of the GCB game player  $B$  wins every battlefield with certainty. But, it turns out that there exists a stronger condition that can be invoked. Recall that given a solution to system  $(\star)$  a necessary condition for the existence of a budget-balancing joint distribution that provides the Theorem 1 sets of univariate marginals is  $X_i \geq \max_j \left\{ \min \left\{ \frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*} \right\} \right\}$  for each player  $i$ . Thus, if in a GCB game  $\text{GCB} \left\{ X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n \right\}$ , it is the case that for each solution to system  $(\star)$  there exists a player  $i$  such that  $X_i < \max_j \left\{ \min \left\{ \frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*} \right\} \right\}$ , then there exists no equilibrium in which both players utilize joint distributions providing the Theorem 1 sets of univariate marginals. That is, the constraint on the joint distribution function is binding. This result is summarized in the following corollary.

**Corollary 2** *Given a solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ , if there exists a player  $i \in \{A, B\}$  such that  $X_i < \max_j \left\{ \min \left\{ \frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*} \right\} \right\}$ , then there does not exist a Nash equilibrium of the GCB game with the same set of univariate marginal distributions and expected payoffs as in the corresponding equilibrium in Theorem 1.*

To illustrate that the Proposition 2 and Corollary 2 conditions only partially answer the question of whether or not there exists an equilibrium set of univariate marginal distributions in a GGL game that can be mapped into an equilibrium joint distribution of the corresponding GCB game, consider the Quasi-Symmetric CB game from Sect. 3 for which we have the following result.

**Proposition 3** *In a Quasi-Symmetric CB game with  $(\epsilon, n_A, n_D)$ :*

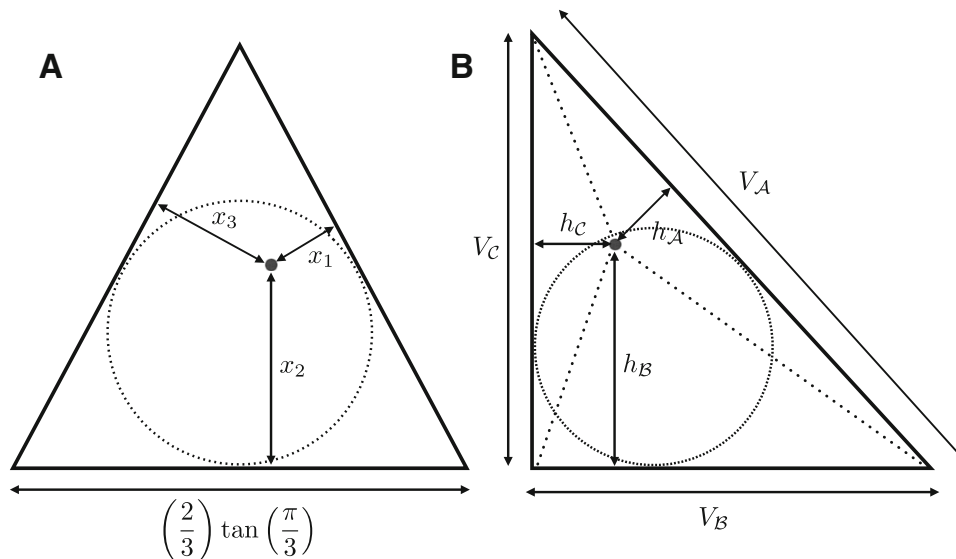
- (1) *There exists a solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$  that satisfies the condition in Proposition 2 if and only if one of the following three conditions holds:*
  - (i)  $n_A = 0$  and  $n_D \geq \frac{4}{\epsilon} - 4$
  - (ii)  $n_A \geq 2$  and  $n_D = 0$
  - (iii)  $n_A \geq 2$  and  $n_D \geq \frac{4}{\epsilon} - 4$ .
- (2) *If  $n_A + \frac{\epsilon n_D}{(1-\epsilon)} \geq 2$ , then there exists no solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$  that satisfies the condition in Corollary 2.*

The proof of Proposition 3 is provided in Appendix 2. From Proposition 3, it follows that in the Quasi-Symmetric CB game there is a gap between the Proposition 2 and Corollary 2 conditions. For example, the parameter configuration  $n_A = 1, n_D = 2$ , and  $\epsilon \in \left[ \frac{1}{3}, \frac{1}{2} \right)$  does not satisfy the conditions for either Proposition 2 or Corollary 2. That is, the Proposition 2 and Corollary 2 conditions do not completely answer the question of whether or not there exists an equilibrium set of univariate marginal distributions in the Quasi-Symmetric GL game that can be mapped into an equilibrium joint distribution of the corresponding Quasi-Symmetric CB game.

## 4.2 Relationship to the Colonel Blotto literature

As indicated in Table 1 of Sect. 1, Proposition 2 provides a partial characterization of two variations of the (continuous) Colonel Blotto game that have not previously been





**Fig. 4** Arbitrary points in a regular and an irregular 3-gon

examined. We now briefly summarize how our results relate to the literature on the four variations that have been previously examined (linear-pure count objective with symmetric and asymmetric budgets, and the linear heterogeneous symmetric objective with symmetric and asymmetric budgets) and focus, in particular, on the equilibrium joint distribution functions previously identified in the literature. For a more complete summary of the related literature, see the survey Kovenock and Roberson (2012).

Because all of the constructions of the equilibrium joint distributions in the first two rows of the symmetric budget column of Table 1—with the exception of Weinstein (2012), which we will return to below—involve randomizing on the surface of an  $n$ -gon, the two following properties of regular  $n$ -gons are worth noting: (1) the sum of the perpendiculars from any point in a regular  $n$ -gon to the sides of the regular  $n$ -gon is equal to  $n$  times the inradius, i.e., the radius of the incircle (the largest circle that can be inscribed in the  $n$ -gon) and (2) if each side of the regular  $n$ -gon has length  $(2/n) \tan(\pi/n)$ , then the inradius is equal to  $(1/n)$ . Normalizing the symmetric budget to one unit of a (use-it-or-lose-it) resource, these two properties of regular  $n$ -gons imply that any arbitrary point in a regular  $n$ -gon with side length of  $(2/n) \tan(\pi/n)$  is budget balancing in that the perpendiculars sum to one, and, for the case of  $n = 3$ , this is illustrated in panel A of Fig. 4 where  $x_1 + x_2 + x_3 = 1$ . For  $n = 3$  any distribution on the surface of a regular 3-gon with side lengths  $(2/3) \tan(\pi/3)$  that generates uniform marginal distributions on  $[0, 2/3]$  for each of the three battlefields is an equilibrium joint distribution, and Borel and Ville (1938) provide two such equilibrium joint distributions.<sup>16</sup> Gross and Wagner (1950), making use of the two properties of regular  $n$ -gons listed above, show that both types of equilibria in Borel and Ville (1938) for the linear pure-count objective game with symmetric budgets and  $n = 3$  can be directly extended to  $n > 3$ . They also provide a new fractal equilibrium.

<sup>16</sup> Borel (1921), a paper on mixed strategies in zero-sum games, introduces the Colonel Blotto game as an example, but does not provide a solution.

For the case of symmetric budgets, the regular  $n$ -gon approach can be modified to allow for battlefield valuations to be symmetric across players, but heterogeneous across battlefields, i.e., the linear heterogeneous symmetric objective game with  $n \geq 3$  as in the second row of the first column of Table 1. Thomas (2018) provides an interesting new construction method for the linear heterogeneous symmetric objective game with symmetric budgets and  $n \geq 3$ , which utilizes an irregular  $n$ -gon in which the number of sides equals the number battlefields. Here we focus on Gross (1950) and Laslier (2002), where the modification involves partitioning the  $n$  battlefields into three sets, denoted  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ , and then randomizing on the surface of the irregular triangle with the three side lengths equal to the total valuations of each of the three sets of battlefields, henceforth denoted  $V_{\mathcal{A}}$ ,  $V_{\mathcal{B}}$ , and  $V_{\mathcal{C}}$ , respectively.<sup>17</sup> Then, as illustrated in panel B of Fig. 4, for each point on the surface of this irregular triangle the sum across the three sides of the product of each perpendicular and the length of its corresponding side is equal to a constant. That is,  $h_{\mathcal{A}}V_{\mathcal{A}} + h_{\mathcal{B}}V_{\mathcal{B}} + h_{\mathcal{C}}V_{\mathcal{C}}$  is equal to twice the surface area of the triangle which, with  $V_{\mathcal{A}} + V_{\mathcal{B}} + V_{\mathcal{C}} = 1$ , is equal to the inradius. Furthermore, note that  $h_i \leq 2r$  for all  $i$ , where  $r$  denotes the inradius. Thus, for any tri-variate distribution on the incircle the random variable  $\tilde{h}_i$  is contained in the interval  $[0, 2r]$  for each  $i = \mathcal{A}, \mathcal{B}, \mathcal{C}$ . Thus, we can construct an  $n$ -variate distribution function where the random variable  $\tilde{h}_{\mathcal{A}}$  is transformed into  $\tilde{x}_j \equiv \frac{\tilde{h}_{\mathcal{A}}v_j}{r}$  for each  $j \in \mathcal{A}$ , and a similar transformation is carried out for each  $j \in \mathcal{B}$  and  $j \in \mathcal{C}$ . The resulting  $n$ -variate distribution function is budget-balancing with probability one ( $\sum_{j=1}^n x_j = 1$ ) and for each  $j = 1, \dots, n$ , the random variable  $\tilde{x}_j$  is contained in  $[0, 2v_j]$ . Lastly, as shown in Gross (1950) and Laslier (2002), one of the Borel and Ville (1938) solutions can be used for the tri-variate distribution of the  $\tilde{h}_i$  variables. In this case, each  $\tilde{h}_i$  is uniformly distributed on the interval  $[0, 2r]$ —so that each  $\tilde{x}_j$  is uniformly distributed on the interval  $[0, 2v_j]$  for each battlefield  $j$  where  $v_j$  is the value of battlefield  $j$ —and with symmetric budgets, equilibrium in the linear heterogeneous symmetric objective game requires, utilizing a similar argument as the linear-pure count game, that the univariate marginal distribution functions are uniform on  $[0, 2v_j]$  for each battlefield  $j$ .

A drawback of using  $n$ -gons to construct budget-balancing joint distribution functions is that this method reduces the dimensionality of the set of points that can be used to form the support of the joint distribution function. With symmetric budgets and symmetric battlefield valuations, this reduction in dimensionality does not preclude the construction of equilibrium joint distribution functions. However, with asymmetric budgets and/or asymmetric battlefield valuations it is easier, if not necessary, to work directly with the budget hyperplane in  $\mathbb{R}^n$ , as in Roberson (2006) and Weinstein (2012). In this paper, we utilized this full dimensionality approach to examine a subset of possible parameter configurations for each of the three checked cells in Table 1. Closely related is Schwartz et al. (2014)<sup>18</sup> who, for the case of the linear heterogeneous

<sup>17</sup> This construction, and the following discussion, is for the case in which no battlefield has a value that is over half of the total value of all battlefields and for which it is not possible to combine battlefields into four groups with equal sums of valuations. For more details on the remaining two special cases, see Laslier (2002).

<sup>18</sup> An earlier version of our paper circulated under the title ‘Generalizations on the Colonel Blotto Game’ and was first presented at the 13th SAET Conference at MINES ParisTech in July of 2013. Following the first circulated version of our paper, Schwartz et al. (2014) independently derived the special case of our

symmetric objective with  $n \geq 3$ , show how in that constant-sum case where battlefield valuations are heterogeneous across battlefields but symmetric across players, the construction utilized in Roberson (2006) can be extended, along the lines described above, to construct a Nash equilibrium of the GCB game with the same set of equilibrium univariate marginal distributions and expected payoffs as in the corresponding equilibrium in Corollary 1.

For the first two rows of the asymmetric budget column of Table 1, the case of  $n = 2$ —where the Blotto game's binding budget constraint implies that an increase in the allocation of the resource in one battlefield necessarily implies a corresponding decrease in the allocation of the resource to the remaining battlefield—leads to a substantively different set of strategic considerations than those arising in the case of  $n \geq 3$ . For  $n = 2$ , Gross and Wagner (1950) provide an equilibrium for all feasible parameter configurations in the first two rows of the asymmetric budget column of Table 1. Macdonell and Mastronardi (2015) complete the characterization of equilibrium and examine the case of nonlinear budget constraints.

Also relevant is the related work on discrete versions of the Colonel Blotto game. Table 1 lists two such papers, Hart (2008) and Hortala-Vallve and Llorente-Saguer (2012). Hart (2008) examines the relationship between the discrete versions of the General Lotto game and the Colonel Blotto game with symmetric and asymmetric budgets. The characterization of the asymmetric-budget, discrete General Lotto game is completed in Dziubiński (2013). In a related work, Hart (2016) examines the relationship between a discrete expenditure version of the all-pay auction and the discrete General Lotto game. Hortala-Vallve and Llorente-Saguer (2012) also examine the discrete Colonel Blotto game with asymmetric budgets but allow for heterogeneous and asymmetric battlefield valuations, with a focus on pure-strategy equilibria.

## 5 Conclusion

In this paper, we provide a complete characterization of the set of Nash equilibria in the Generalized General Lotto game in which battlefield valuations may be heterogeneous across battlefields and asymmetric across players, and in which players' budgets may be asymmetric. We demonstrate that there exist nonpathological parameter configurations for which multiple payoff nonequivalent equilibria exist.

We then show that this characterization may be applied to extend the existing analysis of equilibrium in the Colonel Blotto game to incorporate a range of parameter configurations with heterogeneous battlefield valuations and asymmetric valuations and budgets across players. For the Generalized Colonel Blotto game we provide sufficient conditions for the existence of an equilibrium pair of joint distributions with univariate marginal distributions that coincide with those of an equilibrium in the Generalized General Lotto game. Characterization of Colonel Blotto equilibria for the remaining subset of parameter configurations remains an open question but, for this region, we provide a sufficient condition for the sets of equilibrium univariate

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Footnote 18 continued

construction for the constant-sum game with the linear heterogeneous symmetric objective and asymmetric budgets.

marginal distributions to differ from those arising in any equilibrium of the General Lotto game.

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## Appendix 1

Appendix 1 contains the remaining two parts of the proof of Theorem 1: (i) for each equilibrium of the GGL game there exists a corresponding solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$  and (ii) for each solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$  each player in the GGL game has a unique set of Nash equilibrium univariate marginal distributions. We begin with the proof of part (i) and then conclude with the proof of part (ii).

The proof of the converse claim in Theorem 1, that for each equilibrium of the GGL game there exists a corresponding solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ , extends the arguments in Hart (2008) on the continuous General Lotto game and Hart (2016) on the relationship between the all-pay auction and the continuous General Lotto game. We begin by noting that the standard constant-sum continuous General Lotto game, denoted  $L\{X_A, X_B\}$ , is a special case of the GGL game in which  $n = 1$ ,  $v_A = v_B = 1$ , and a strategy is a univariate distribution function denoted  $F_i$ , for  $i = A, B$ , with  $E_{F_i}(x) \leq X_i$ . Let  $\tilde{x}_i$  denote the realization of a random variable distributed according to the distribution function  $F_i$ . Player A's expected payoff in the General Lotto game is given by

$$\pi_A(F_A, F_B) = \Pr(\tilde{x}_A > \tilde{x}_B) + \frac{1}{2}\Pr(\tilde{x}_A = \tilde{x}_B)$$

and player B's expected payoff is given by

$$\pi_B(F_B, F_A) = \Pr(\tilde{x}_B > \tilde{x}_A) + \frac{1}{2}\Pr(\tilde{x}_B = \tilde{x}_A) = 1 - \Pr(\tilde{x}_A > \tilde{x}_B) - \frac{1}{2}\Pr(\tilde{x}_A = \tilde{x}_B).$$

In this constant-sum game, player A chooses  $F_A$  to maximize  $\Pr(\tilde{x}_A > \tilde{x}_B) + \frac{1}{2}\Pr(\tilde{x}_A = \tilde{x}_B)$  and player B chooses  $F_B$  to minimize  $\Pr(\tilde{x}_A > \tilde{x}_B) + \frac{1}{2}\Pr(\tilde{x}_A = \tilde{x}_B)$ .

Equilibrium in the (continuous) General Lotto game with strictly positive budgets is characterized by Sahuguet and Persico (2006) and Hart (2008). The following theorem extends that characterization to allow for one or both of the players to have a budget of 0. Unlike the case of  $X_B \geq X_A > 0$ , if either  $X_A = 0$  and  $X_B > 0$  or  $X_A > 0$  and  $X_B = 0$ , then there are multiple equilibria.<sup>19</sup> However, because the game is constant sum, the equilibrium expected payoffs are unique for all possible resource endowments  $(X_A, X_B)$ .

<sup>19</sup> For the player  $i$  with  $X_i = 0$ , the unique equilibrium strategy is  $F_{i,j}^*(0) = 1$ , but for player  $-i$  with  $X_{-i} > 0$  any distribution function with  $F_{-i,j}^*(0) = 0$  and  $E_{F_{-i,j}}(x) \leq X_{-i}$  is an equilibrium strategy.

**Theorem 2** For the General Lotto game  $L\{X_A, X_B\}$  with  $X_B \geq X_A > 0$ , the unique equilibrium strategies are

$$F_A(x) = \left(1 - \frac{X_A}{X_B}\right) + \frac{x \cdot X_A}{2X_B^2} \quad \text{for } x \in [0, 2X_B]$$

$$F_B(x) = \frac{x}{2X_B} \quad \text{for } x \in [0, 2X_B]$$

and the equilibrium expected payoffs are  $\frac{X_A}{2X_B}$  for player A and  $1 - \frac{X_A}{2X_B}$  for player B. For the General Lotto game  $L\{X_A, X_B\}$  with  $X_B = 0$  and/or  $X_A = 0$ :

1. If  $X_A = 0$  and  $X_B > 0$ , then the unique equilibrium expected payoffs are 0 for player A and 1 for player B.
2. If  $X_A > 0$  and  $X_B = 0$ , then the unique equilibrium expected payoffs are 1 for player A and 0 for player B.
3. If  $X_A = 0$  and  $X_B = 0$ , then the unique equilibrium strategies are  $F_{A,j}^*(0) = F_{B,j}^*(0) = 1$  and the equilibrium expected payoffs are  $\frac{1}{2}$  for player A and  $\frac{1}{2}$  for player B.

In moving from the General Lotto game  $L\{X_A, X_B\}$  to the GGL game  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , recall that in the GGL game a strategy is an  $n$ -variate distribution function,  $P_i$  for  $i = A, B$ , that satisfies the constraint that  $\sum_{j=1}^n E_{F_{i,j}}(x) \leq X_i$ , where  $F_{i,j}$  is the univariate marginal distribution of  $P_i$  for battlefield  $j$ . Let  $\tilde{x}_{i,j}$  denote the realization of a random variable distributed according to the univariate marginal distribution  $F_{i,j}$ . Then, given the strategy profile  $(P_A, P_B)$ , player A's expected payoff is given by

$$\pi_A(P_A, P_B) = \sum_{j=1}^n v_{A,j} \left( \Pr(\tilde{x}_{A,j} > \tilde{x}_{B,j}) + \frac{1}{2} \Pr(\tilde{x}_{A,j} = \tilde{x}_{B,j}) \right)$$

and player B's expected payoff is given by

$$\pi_B(P_B, P_A) = \sum_{j=1}^n v_{B,j} \left( 1 - \Pr(\tilde{x}_{A,j} > \tilde{x}_{B,j}) - \frac{1}{2} \Pr(\tilde{x}_{A,j} = \tilde{x}_{B,j}) \right).$$

Given an equilibrium  $(P_A^*, P_B^*)$ , let  $X_{i,j}^* \equiv E_{F_{i,j}^*}(x)$  for  $i = A, B$  denote player  $i$ 's expected allocation of the resource to battlefield  $j$  under the strategy  $P_i^*$ .

**Lemma 1** If  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , then within each battlefield  $j$ ,  $(F_{A,j}^*, F_{B,j}^*)$  is an equilibrium of  $L(X_{A,j}^*, X_{B,j}^*)$ .

**Proof** If  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , then there are no payoff-increasing deviations for either player. But one feasible type of deviation for player  $i$  is to hold constant  $X_{i,j}^*$  on each battlefield  $j$  and choose a feasible deviation  $\hat{P}_i$  with the set of univariate marginals  $\{\hat{F}_{i,j}\}_{j=1}^n$  with  $E_{\hat{F}_{i,j}}(x) = X_{i,j}^*$  for all  $j$ . Let  $\hat{x}_{i,j}$  denote the realization of a random variable distributed according to the univariate

marginal distribution function  $\widehat{F}_{i,j}$ . Because in battlefield  $j$  each player  $i$  does not have a payoff increasing deviation  $\widehat{F}_{i,j}$  with  $E_{\widehat{F}_{i,j}}(x) = X_{i,j}^*$ , it follows that

$$v_{i,j} \left( \Pr(\widetilde{x}_{i,j} > \widetilde{x}_{-i,j}) + \frac{1}{2} \Pr(\widetilde{x}_{i,j} = \widetilde{x}_{-i,j}) \right) \geq v_{i,j} \left( \Pr(\widehat{x}_{i,j} > \widetilde{x}_{-i,j}) + \frac{1}{2} \Pr(\widehat{x}_{i,j} = \widetilde{x}_{-i,j}) \right) \quad (17)$$

for all possible univariate marginal distributions  $\widehat{F}_{i,j}$  with  $E_{\widehat{F}_{i,j}}(x) = X_{i,j}^*$ . But it follows directly from (17) that

$$\left( \Pr(\widetilde{x}_{i,j} > \widetilde{x}_{-i,j}) + \frac{1}{2} \Pr(\widetilde{x}_{i,j} = \widetilde{x}_{-i,j}) \right) \geq \left( \Pr(\widehat{x}_{i,j} > \widetilde{x}_{-i,j}) + \frac{1}{2} \Pr(\widehat{x}_{i,j} = \widetilde{x}_{-i,j}) \right) \quad (18)$$

for all possible deviations  $\widehat{F}_{i,j}$  with  $E_{\widehat{F}_{i,j}}(x) = X_{i,j}^*$ , and, thus,  $(F_{A,j}^*, F_{B,j}^*)$  is an equilibrium of  $L(X_{A,j}^*, X_{B,j}^*)$ .  $\square$

To complete the proof of the claim that if  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , then there exists a corresponding solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ , Lemmas 2-4 collectively establish that in any equilibrium  $(P_A^*, P_B^*)$  of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$  it must be the case that  $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ . Because  $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ , it follows from Lemma 1 and Theorem 2 that the equilibrium univariate marginal distributions are uniquely determined. Using the unique equilibrium univariate marginal distributions, Lemma 5 completes the proof that there exists a corresponding solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ .

**Lemma 2** *If  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , then*

$$\max\{X_{A,j}^*, X_{B,j}^*\} > 0 \quad \text{for all } j.$$

**Proof** By way of contradiction, suppose that there exists an equilibrium  $(P_A^*, P_B^*)$  in which for some battlefield  $k$   $\max\{X_{A,k}^*, X_{B,k}^*\} = 0$ , which implies that  $F_{A,k}^*(0) = F_{B,k}^*(0) = 1$ . We begin with the case in which  $\sum_{j=1}^n E_{F_{A,j}^*}(x) < X_A$  and then examine the case in which  $\sum_{j=1}^n E_{F_{A,j}^*}(x) = X_A$ . If  $\sum_{j=1}^n E_{F_{A,j}^*}(x) < X_A$ , then player A can increase his payoff by  $\frac{v_{A,k}}{2}$  by allocating a strictly positive level of the resource  $X_{A,k} \leq X_A - \sum_{j=1}^n E_{F_{A,j}^*}(x)$  to battlefield  $k$  and setting  $F_{A,k}(0) = 0$ , a contradiction.

For  $\sum_{j=1}^n E_{F_{A,j}^*}(x) = X_A > 0$ , there exists at least one battlefield  $j'$  in which  $X_{A,j'}^* > 0$  and there are two cases to consider: (i)  $X_{B,j'}^* = 0$  and (ii)  $X_{B,j'}^* > 0$ . In case (i), because  $\sum_{j=1}^n E_{F_{A,j}^*}(x) = X_A > 0$  and in battlefield  $j'$   $X_{A,j'}^* > 0$  and  $X_{B,j'}^* = 0$ , player A can increase his payoff by  $\frac{v_{A,k}}{2}$  by shifting  $X_{A,k} < X_{A,j'}^*$  of the resource from battlefield  $j'$  to battlefield  $k$  and setting  $F_{A,k}(0) = F_{A,j'}(0) = 0$ , a contradiction.

In case (ii),  $X_{A,j'}^* > 0$  and  $X_{B,j'}^* > 0$ , and it follows from Lemma 1 and Theorem 2 that  $F_{B,j'}^*(x)$  is the unique equilibrium strategy in the General Lotto game  $L(X_{A,j'}^*, X_{B,j'}^*)$ , where the support of  $F_{B,j'}^*(x)$ , denoted  $\text{supp}(F_{B,j'}^*(x))$ , is



$[0, 2 \max\{X_{A,j'}^*, X_{B,j'}^*\}]$ . Thus, player  $A$  can increase his total expected payoff by an amount arbitrarily close to  $\frac{v_{A,k}}{2}$  by shifting, for a sufficiently small  $\epsilon > 0$ ,  $\epsilon$  of the resource from battlefield  $j'$  to battlefield  $k$ , in battlefield  $k$  choosing a distribution function  $F_{A,k}(x)$  with  $F_{A,k}(0) = 0$  and  $E_{F_{A,k}}(x) = \epsilon$ , and in battlefield  $j'$  choosing a distribution function  $F_{A,j'}(x)$  with  $F_{A,j'}(0) = 0$ ,  $E_{F_{A,j'}}(x) = X_{A,j'}^* - \epsilon$ , and  $\text{supp}(F_{A,j'}) \subseteq \text{supp}(F_{B,j'}^*(x))$ . In battlefield  $j'$  player  $A$ 's expected payoff from the distribution function  $F_{A,j'}(x)$  when player  $B$ 's distribution function is  $F_{B,j'}^*(x)$  is given by

$$v_{A,j'} \int_0^\infty F_{B,j'}^*(x) dF_{A,j'}(x) = \begin{cases} v_{A,j'} \left( \left(1 - \frac{X_{B,j'}^*}{X_{A,j'}^*}\right) + \frac{(X_{A,j'}^* - \epsilon) X_{B,j'}^*}{2(X_{A,j'}^*)^2} \right) & \text{if } X_{A,j'}^* > X_{B,j'}^* \\ v_{A,j'} \left( \frac{(X_{A,j'}^* - \epsilon)}{2X_{B,j'}^*} \right) & \text{if } X_{A,j'}^* \leq X_{B,j'}^* \end{cases}$$

Thus, the loss in player  $A$ 's payoff in battlefield  $j'$  approaches 0 as  $\epsilon$  approaches 0, but the gain on battlefield  $k$  is  $\frac{v_{A,k}}{2}$  for all  $\epsilon > 0$ . This is a contradiction to the assumption that  $(P_A^*, P_B^*)$  is an equilibrium and completes the proof that if  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$  then  $\max\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ .  $\square$

**Lemma 3** *If  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , then  $\sum_{j=1}^n E_{F_{i,j}^*}(x) > 0$  for each player  $i = A, B$ .*

**Proof** By way of contradiction, suppose that there exists an equilibrium  $(P_A^*, P_B^*)$  in which  $\sum_{j=1}^n E_{F_{i,j}^*}(x) = 0$  for some player  $i$ . From Lemma 2, it follows that for player  $-i$ ,  $X_{-i,j}^* > 0$  for all  $j$ , which from Lemma 1 and Theorem 2 implies that player  $i$  earns an equilibrium expected payoff of 0. If  $i = B$ , then because  $X_B \geq X_A > 0$ , it is clear that player  $B$  has a payoff increasing deviation that involves mimicking player  $A$ 's strategy, which yields  $B$  an expected payoff of  $\frac{1}{2} \sum_{j=1}^n v_{B,j}$ . Hence a contradiction. If  $i = A$ , then player  $A$  can mimic player  $B$ 's strategy with probability  $\frac{X_A}{X_B}$  and bid 0 in every battlefield with probability  $\left(1 - \frac{X_A}{X_B}\right)$ , which similarly yields  $A$  an expected payoff of  $\frac{X_A}{2X_B} \sum_{j=1}^n v_{A,j}$ . This yields a contradiction and completes the proof.  $\square$

**Lemma 4** *If  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , then*

$$\min\{X_{A,j}^*, X_{B,j}^*\} > 0 \quad \text{for all } j.$$

**Proof** By way of contradiction, suppose that there exists an equilibrium  $(P_A^*, P_B^*)$  in which there is at least one battlefield  $k$  with  $\min\{X_{A,k}^*, X_{B,k}^*\} = 0$ . There are two cases to consider: (i)  $\min\{X_{A,j}^*, X_{B,j}^*\} = 0$  for all  $j$  or (ii)  $\min\{X_{A,j}^*, X_{B,j}^*\} = 0$  for at least one, but not all  $j$ . Beginning with case (i), because  $\min\{X_{A,j}^*, X_{B,j}^*\} = 0$  for all  $j$ , from Lemma 3  $\sum_{j=1}^n E_{F_{i,j}^*}(x) > 0$  for each player  $i$ , and from Lemma 2  $\max\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ , there exists at least one battlefield  $j'$  with  $X_{A,j'}^* > 0$  and  $X_{B,j'}^* = 0$  and at least one battlefield  $j''$  with  $X_{A,j''}^* = 0$  and  $X_{B,j''}^* > 0$ . But,



player  $B$  can strictly increase his total expected payoff by decreasing  $X_{B,j''}^*$  by an  $\epsilon \in (0, \min\{X_{B,j''}^*, X_{A,j'}^*\})$ , allocating  $\epsilon$  to battlefield  $j'$ , and utilizing a univariate marginal distribution on battlefield  $j'$  that places mass  $\left(1 - \frac{\epsilon}{X_{A,j'}^*}\right)$  on 0 and randomizes according to  $F_{A,j'}^*$  with probability  $\frac{\epsilon}{X_{A,j'}^*}$ . Such a deviation would increase player  $B$ 's expected payoff on battlefield  $j'$  by  $\frac{\epsilon v_{B,j'}}{2X_{A,j'}^*}$  with no decrease in the expected payoff on battlefield  $j''$ , a contradiction.

For case (ii), if  $\min\{X_{A,j}^*, X_{B,j}^*\} = 0$  for at least one, but not all  $j$ , then there exists at least one battlefield  $j'$  with  $\min\{X_{A,j'}^*, X_{B,j'}^*\} > 0$  and at least one battlefield  $k$  with  $\min\{X_{A,k}^*, X_{B,k}^*\} = 0$ . Because from Lemma 2  $\max\{X_{A,k}^*, X_{B,k}^*\} > 0$ , there exists a player  $i$  with  $X_{i,k} > 0$  and a player  $-i$  with  $X_{-i,k} = 0$ . Then, because from Lemma 1, each player's unique equilibrium univariate marginal distribution in battlefield  $j'$  is given by Theorem 2, player  $i$  has a payoff increasing deviation that involves shifting  $\epsilon \in (0, X_{i,k}^*)$  of the resource from battlefield  $k$  to battlefield  $j'$ , in battlefield  $k$  choosing a distribution function  $F_{i,k}(x)$  with  $F_{i,k}(0) = 0$  and  $E_{F_{i,k}}(x) = X_{i,k}^* - \epsilon$ , in battlefield  $j'$  choosing a distribution function  $F_{i,j'}(x)$  with  $F_{i,j'}(0) = 0$ ,  $E_{F_{i,j'}}(x) = X_{i,j'}^* + \epsilon$ , and  $\text{supp}(F_{i,j'}) \subseteq \text{supp}(F_{-i,j'}^*(x))$ . Such a deviation results in no loss to player  $i$ 's expected payoff in battlefield  $k$ . In battlefield  $j'$ , player  $i$ 's expected payoff from the distribution function  $F_{i,j'}(x)$  when player  $-i$ 's distribution function is  $F_{-i,j'}^*(x)$  is given by

$$v_{i,j'} \int_0^\infty F_{-i,j'}^*(x) dF_{i,j'}(x) = \begin{cases} v_{i,j'} \left( \left(1 - \frac{X_{-i,j'}^*}{X_{i,j'}^*}\right) + \frac{(X_{i,j'}^* + \epsilon)X_{-i,j'}^*}{2(X_{i,j'}^*)^2} \right) & \text{if } X_{i,j'}^* > X_{-i,j'}^* \\ v_{i,j'} \left( \frac{(X_{i,j'}^* + \epsilon)}{2X_{-i,j'}^*} \right) & \text{if } X_{i,j'}^* \leq X_{-i,j'}^* \end{cases}.$$

Thus, for all  $\epsilon \in (0, X_{i,k}^*)$  player  $i$ 's expected payoff in battlefield  $j'$  is strictly higher under the deviation, and there is no loss to player  $i$ 's expected payoff in battlefield  $k$ . This is a contradiction and completes the proof that if  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$  then  $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ .  $\square$

**Lemma 5** *If  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , then there exists a corresponding solution  $(\lambda_A^*, \lambda_B^*)$  to system  $(\star)$ .*

**Proof** From Lemma 4,  $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ . Then, because  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$  and  $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ , it follows from Lemma 1 that in each battlefield  $j$  the players' unique equilibrium univariate marginal distributions are given by Theorem 2. Because, the unique equilibrium univariate marginal distributions given by Theorem 2 are linear, it follows that for player  $A$  any deviation  $P_A$  that satisfies the following two conditions is payoff maximizing and feasible: (i) in each battlefield  $j$  the associated univariate marginal distribution function  $F_{A,j}(x)$  satisfies  $F_{A,j}(0) = 0$  if  $X_{A,j}^* > X_{B,j}^*$  and  $\text{supp}(F_{A,j}) \subseteq \text{supp}(F_{B,j}^*(x))$ , and (ii) across battlefields  $\sum_{j=1}^n E_{F_{A,j}}(x) = X_A$ . Letting  $X_{A,j} = E_{F_{A,j}}(x)$ , player  $A$ 's total expected payoff from such a joint distribution

function  $P_A$ , given that player  $B$  is using the joint distribution function  $P_B^*$ , is given by

$$\begin{aligned}\pi_A(P_A, P_B^*) &= \sum_{j=1}^n v_{A,j} \int_0^\infty F_{B,j}^*(x) dF_{A,j}(x) \\ &= \sum_{j|X_{A,j}^* > X_{B,j}^*} v_{A,j} \left[ \left( 1 - \frac{X_{B,j}^*}{X_{A,j}^*} \right) + \frac{X_{A,j} X_{B,j}^*}{2(X_{A,j}^*)^2} \right] + \sum_{j|X_{A,j}^* \leq X_{B,j}^*} v_{A,j} \left( \frac{X_{A,j}}{2X_{B,j}^*} \right).\end{aligned}\quad (19)$$

Similarly, for player  $B$  the maximum achievable total expected payoff from a feasible deviation  $P_B$  with  $\{X_{B,j}\}_{j=1}^n$  is given by

$$\begin{aligned}\pi_B(P_B, P_A^*) &= \sum_{j=1}^n v_{B,j} \int_0^\infty F_{A,j}^*(x) dF_{B,j}(x) \\ &= \sum_{j|X_{A,j}^* \geq X_{B,j}} v_{B,j} \left( \frac{X_{B,j}}{2X_{A,j}^*} \right) + \sum_{j|X_{A,j}^* < X_{B,j}} v_{B,j} \left[ \left( 1 - \frac{X_{A,j}^*}{X_{B,j}} \right) + \frac{X_{B,j} X_{A,j}^*}{2(X_{B,j})^2} \right]\end{aligned}\quad (20)$$

Because  $(P_A^*, P_B^*)$  is an equilibrium of  $\text{GGL}(X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n)$ , it must be the case that player  $A$  is maximizing Eq. (19) with respect to  $\{X_{A,j}\}_{j=1}^n$  and player  $B$  is maximizing Eq. (20) with respect to  $\{X_{B,j}\}_{j=1}^n$ . Then, because Eqs. (19) and (20) are concave and continuously differentiable with respect to  $\{X_{i,j}\}_{j=1}^n \in \mathbb{R}_+^n$ ,  $i = A, B$ , respectively, it follows that there exists a Lagrange multiplier  $\lambda_i^* \geq 0$  such that the Kuhn–Tucker first-order conditions hold:<sup>20</sup>

$$\begin{cases} v_{i,j} \left( \frac{1}{2X_{-i,j}^*} \right) - \lambda_i^* = 0 & \text{in each battlefield } j \text{ with } X_{i,j}^* \leq X_{-i,j}^* \\ v_{i,j} \left( \frac{X_{-i,j}^*}{2(X_{i,j}^*)^2} \right) - \lambda_i^* = 0 & \text{in each battlefield } j \text{ with } X_{i,j}^* > X_{-i,j}^* \end{cases}, \quad (21)$$

with complementary slackness condition  $\lambda_i^* \geq 0$ ,  $\sum_{j=1}^n X_{i,j}^* \leq X_i$ , and  $\lambda_i^* (\sum_{j=1}^n X_{i,j}^* - X_i) = 0$ . Complementary slackness is clearly satisfied because from (19) and (20) it is clearly suboptimal to set  $\sum_{j=1}^n X_{i,j} < X_i$ .

From the first-order conditions in (21) we see that in each battlefield  $j$  with  $X_{i,j}^* > X_{-i,j}^*$ : (i)  $X_{i,j}^* = \frac{v_{-i,j}}{2\lambda_{-i}^*}$  and (ii)  $X_{i,j}^* = \left( \frac{v_{i,j} X_{-i,j}^*}{2\lambda_i^*} \right)^{1/2}$  or equivalently  $X_{-i,j}^* = \frac{(v_{-i,j}/2\lambda_{-i}^*)^2}{(v_{i,j}/2\lambda_i^*)}$ . Combining (i) and (ii), it follows from budget balance that  $\lambda_A^*$  and  $\lambda_B^*$  solve

<sup>20</sup> For further details see p. 187 of Sundaram (1996).

$$\sum_{j|X_{A,j}^* > X_{B,j}^*} \frac{v_{B,j}}{2\lambda_B^*} + \sum_{j|X_{A,j}^* \leq X_{B,j}^*} \frac{\left(\frac{v_{A,j}}{\lambda_A^*}\right)^2}{2\left(\frac{v_{B,j}}{\lambda_B^*}\right)} = X_A \quad (22)$$

and

$$\sum_{j|X_{A,j}^* \geq X_{B,j}^*} \frac{\left(\frac{v_{B,j}}{\lambda_B^*}\right)^2}{2\left(\frac{v_{A,j}}{\lambda_A^*}\right)} + \sum_{j|X_{A,j}^* < X_{B,j}^*} \frac{v_{A,j}}{2\lambda_A^*} = X_B \quad (23)$$

Because  $X_{A,j}^* = \frac{v_{B,j}}{2\lambda_B^*}$  and  $X_{B,j}^* = \frac{(v_{B,j}/2\lambda_B^*)^2}{(v_{A,j}/2\lambda_A^*)}$  when  $X_{A,j}^* > X_{B,j}^*$ , and  $X_{A,j}^* = \frac{(v_{A,j}/2\lambda_A^*)^2}{(v_{B,j}/2\lambda_B^*)}$  and  $X_{B,j}^* = \frac{v_{A,j}}{2\lambda_A^*}$  when  $X_{A,j}^* \leq X_{B,j}^*$ , it follows that  $\frac{v_{A,j}}{\lambda_A^*} > \frac{v_{B,j}}{\lambda_B^*}$  if and only if  $X_{A,j}^* > X_{B,j}^*$ . Thus, the system (22) and (23) is equivalent to system (★).

This completes the proof of part (i), for each equilibrium of the GGL game, there exists a corresponding solution  $(\lambda_A^*, \lambda_B^*)$  to system (★).

We now conclude with the proof of part (ii), for each solution  $(\lambda_A^*, \lambda_B^*)$  to system (★) each player in the GGL game has a unique set of Nash equilibrium univariate marginal distributions. From the argument utilized in the proof of Lemma 5, it follows that for each solution  $(\lambda_A^*, \lambda_B^*)$  each player  $i$ 's  $n$ -tuple of the expected allocation of the resource to each of the  $n$  battlefields,  $\{X_{i,j}^*\}_{j=1}^n$ , is uniquely determined. Namely,  $X_{A,j}^* = \frac{v_{B,j}}{2\lambda_B^*}$  and  $X_{B,j}^* = \frac{(v_{B,j}/2\lambda_B^*)^2}{(v_{A,j}/2\lambda_A^*)}$  when  $X_{A,j}^* > X_{B,j}^*$ , and  $X_{A,j}^* = \frac{(v_{A,j}/2\lambda_A^*)^2}{(v_{B,j}/2\lambda_B^*)}$  and  $X_{B,j}^* = \frac{v_{A,j}}{2\lambda_A^*}$  when  $X_{A,j}^* \leq X_{B,j}^*$ . From Lemma 4, each player's expected allocation of the resource to each battlefield is strictly positive,  $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ . Then, because  $\{X_{A,j}^*, X_{B,j}^*\}_{j=1}^n$  is uniquely determined by  $(\lambda_A^*, \lambda_B^*)$  and  $\min\{X_{A,j}^*, X_{B,j}^*\} > 0$  for all  $j$ , it follows from Lemma 1 that in each battlefield  $j$  the players' unique equilibrium univariate marginal distributions are given by Theorem 2. That is, for each solution  $(\lambda_A^*, \lambda_B^*)$  each player in the GGL game has a unique set of Nash equilibrium univariate marginals, and this completes the proof of Theorem 1.

## Appendix 2

Appendix 2 contains the proof of Proposition 3. As demonstrated by the Quasi-Symmetric Example, there may exist multiple equilibria with distinct payoffs, but there always exists an equilibrium with  $\gamma^* = 1$ . Note that if  $\gamma^* = 1$ , then  $\lambda_A^* = \lambda_B^*$  which it will be convenient to denote by  $\lambda^1$ .

We begin with part (1) of Proposition 3, where we provide the proof for case (iii), i.e.,  $n_A \geq 2$  and  $n_D \geq \frac{4}{\epsilon} - 4$ . The proofs for the remaining two cases follow directly. Recall that Proposition 2 provides the following sufficient condition for an equilibrium set of univariate marginal distributions in a GGL game to be able to map, via a copula, into an equilibrium joint distribution in the corresponding GCB game:

if for each distinct pair of battlefield valuations  $(v_{A,j}, v_{B,j})$  with  $\frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*} \leq 1$ , for some  $i \in \{A, B\}$ , it is the case that  $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$ , then there exists a Nash equilibrium of the GCB game with the same set of univariate marginal distributions and expected payoffs as in the corresponding equilibrium in Theorem 1.

In a Quasi-Symmetric CB game, there are three distinct pairs of battlefield valuations to consider. We begin with the agreement set  $\mathcal{A}$  and then examine the two halves of the disagreement set  $\mathcal{D}$ . At each battlefield  $j \in \mathcal{A}$ ,  $(v_{A,j}, v_{B,j}) = (\frac{1}{n}, \frac{1}{n})$ . For  $\gamma^* = 1$ , the Proposition 2 condition for the agreement set  $\mathcal{A}$  may be stated as  $n_{\mathcal{A}} \geq 2$ . In the case that  $\gamma^* \neq 1$ , the Proposition 2 condition for the agreement set  $\mathcal{A}$  is  $n_{\mathcal{A}} \geq \frac{2}{\gamma^*}$  if  $\gamma^* \leq 1$  and  $n_{\mathcal{A}} \geq 2\gamma^*$  if  $\gamma^* \geq 1$ . Clearly the Proposition 2 condition for  $\mathcal{A}$  becomes more difficult to satisfy as  $\gamma^*$  moves away from the value of 1. That is, if the Proposition 2 condition for  $\mathcal{A}$  is not satisfied at  $\gamma^* = 1$ , then the Proposition 2 condition for  $\mathcal{A}$  will not be satisfied for any  $\gamma^* \neq 1$ .

Moving on to the disagreement set  $\mathcal{D}$  where either  $(v_{A,j}, v_{B,j}) = (\frac{2(1-\epsilon)}{n}, \frac{2\epsilon}{n})$  or  $(v_{A,j}, v_{B,j}) = (\frac{2\epsilon}{n}, \frac{2(1-\epsilon)}{n})$ , for  $\gamma^* = 1$  the Proposition 2 condition for both halves of the disagreement set  $\mathcal{D}$  may be stated as  $\frac{4}{n_{\mathcal{D}}} \leq \frac{\epsilon}{1-\epsilon}$ , or equivalently as  $n_{\mathcal{D}} \geq \frac{4}{\epsilon} - 4$ . In the case that  $\gamma^* \neq 1$ , there are two cases to consider, which we label  $(\diamond)$  for  $1 > \gamma^* \geq \frac{\epsilon}{1-\epsilon}$  and  $(\blacklozenge)$  for  $\frac{1-\epsilon}{\epsilon} > \gamma^* \geq 1$ . Note that in case  $(\diamond)$   $(\frac{\epsilon}{1-\epsilon})\gamma^* < 1$  and in case  $(\blacklozenge)$   $(\frac{\epsilon}{1-\epsilon})\frac{1}{\gamma^*} < 1$ . Thus for both cases  $(\diamond)$  and  $(\blacklozenge)$ , the Proposition 2 condition for half of the disagreement set  $\mathcal{D}$  is  $n_{\mathcal{D}} \geq (\frac{4}{\epsilon} - 4)\gamma^*$  and for the other half of the disagreement set  $\mathcal{D}$  is  $n_{\mathcal{D}} \geq (\frac{4}{\epsilon} - 4)\frac{1}{\gamma^*}$ . Because both of these Proposition 2 conditions for the disagreement set  $\mathcal{D}$  must be satisfied, it follows that regardless of the direction of the movement, one of these two conditions becomes more difficult to satisfy as  $\gamma^*$  moves away from the value of 1. This completes the proof of part (1) of Proposition 3.

For part (2) of Proposition 3, recall that Corollary 2 provides the following sufficient condition for an equilibrium set of univariate marginal distributions in the GGL game to not be able to map into an equilibrium joint distribution in the corresponding GCB game:  $X_i < \max_j \left\{ \min \left\{ \frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*} \right\} \right\}$ . For  $\gamma^* = 1$ , we can use Eq. (★) to solve for  $\lambda^1$  in the Quasi-Symmetric GL game, where

$$\frac{1}{\lambda^1} \left( \frac{n_{\mathcal{A}}}{2n} + \frac{n_{\mathcal{D}}}{2n} \left( \frac{\epsilon}{1-\epsilon} \right) \right) = 1. \quad (24)$$

Next, note that for the Quasi-Symmetric GL game with  $\gamma^* = 1$ , it follows that

$$X_i = 1 < \frac{1}{n\lambda^1} = \max_j \left\{ \min \left\{ \frac{v_{-i,j}}{\lambda^1}, \frac{v_{i,j}}{\lambda^1} \right\} \right\} \quad (25)$$

Combining  $\lambda^1$  from Eq. (24) with the Corollary 2 condition that  $1 < \frac{1}{n\lambda^1}$  from Eq. (25), it follows that if  $n_{\mathcal{A}} + \frac{\epsilon n_{\mathcal{D}}}{(1-\epsilon)} < 2$ , then for the  $\gamma^* = 1$  solution to system (★) the Quasi-Symmetric CB game with  $(\epsilon, n_{\mathcal{A}}, n_{\mathcal{D}})$  satisfies the condition in Corollary 2.

Conversely, if  $n_A + \frac{\epsilon n_D}{(1-\epsilon)} \geq 2$  then for the  $\gamma^* = 1$  solution to system  $(\star)$  the Corollary 2 condition is not satisfied. We now show that if  $n_A + \frac{\epsilon n_D}{(1-\epsilon)} \geq 2$ , then there exists no  $\gamma^* \neq 1$  solution to system  $(\star)$  that satisfies the Corollary 2 condition.

For  $\gamma^* \neq 1$  we have the two cases  $(\diamond)$  for  $1 > \gamma^* \geq \frac{\epsilon}{1-\epsilon}$  and  $(\blacklozenge)$  for  $\frac{1-\epsilon}{\epsilon} > \gamma^* \geq 1$ . Beginning with case  $(\diamond)$ , let  $\lambda_i^\diamond$  denote the value of the multiplier on player  $i$ 's expected resource expenditure constraint in a case  $(\diamond)$  equilibrium and let  $\gamma^\diamond = \frac{\lambda_A^\diamond}{\lambda_B^\diamond}$ . Because  $1 > \gamma^\diamond \geq \frac{\epsilon}{1-\epsilon}$  in case  $(\diamond)$ , we have the following

$$\begin{aligned} \min \left\{ \frac{1}{n\lambda_A^\diamond}, \frac{1}{n\lambda_B^\diamond} \right\} &= \frac{1}{n\lambda_B^\diamond}, \\ \min \left\{ \frac{2\epsilon}{n\lambda_A^\diamond}, \frac{2(1-\epsilon)}{n\lambda_B^\diamond} \right\} &= \frac{2\epsilon}{n\lambda_A^\diamond}, \\ \min \left\{ \frac{2(1-\epsilon)}{n\lambda_A^\diamond}, \frac{2\epsilon}{n\lambda_B^\diamond} \right\} &= \frac{2\epsilon}{n\lambda_B^\diamond}. \end{aligned} \quad (26)$$

It then follows from Eq. (26) that

$$\max_j \left\{ \min \left\{ \frac{v_{-i,j}}{\lambda_{-i}^\diamond}, \frac{v_{i,j}}{\lambda_i^\diamond} \right\} \right\} = \begin{cases} \frac{1}{n\lambda_B^\diamond} & \text{if } \gamma^\diamond \in (2\epsilon, 1] \\ \frac{2\epsilon}{n\lambda_A^\diamond} & \text{if } \gamma^\diamond \in \left[ \frac{\epsilon}{1-\epsilon}, 2\epsilon \right] \end{cases} \quad (27)$$

Recall that for the Quasi-Symmetric GL game Eq. (15) may be used to solve for all possible solutions to system  $(\star)$  with  $\gamma^\diamond \in \left[ \frac{\epsilon}{1-\epsilon}, 1 \right)$ . For each solution  $\gamma^\diamond$  to equation (15), we may use Eq.  $(\star)$  to solve for  $(\lambda_A^\diamond|_{\gamma^\diamond}, \lambda_B^\diamond|_{\gamma^\diamond})$ ,

$$\begin{aligned} \lambda_A^\diamond|_{\gamma^\diamond} &= (\gamma^\diamond)^2 \left( \frac{n_A}{2n} \right) + \frac{n_D}{2n} \left( \epsilon + (\gamma^\diamond)^2 \left( \frac{\epsilon^2}{(1-\epsilon)} \right) \right), \\ \lambda_B^\diamond|_{\gamma^\diamond} &= \frac{n_A}{2n} + \frac{n_D}{2n} \left( \epsilon + \left( \frac{1}{\gamma^\diamond} \right)^2 \left( \frac{\epsilon^2}{(1-\epsilon)} \right) \right). \end{aligned} \quad (28)$$

For each  $\gamma^\diamond \in (2\epsilon, 1)$  that is a solution to Eq. (15), it follows from Eqs. (24), (27) and (28) that

$$\frac{1}{n\lambda^1} \geq \frac{1}{n\lambda_B^\diamond|_{\gamma^\diamond}}. \quad (29)$$

From the inequality in Eq. (29), it follows that the Corollary 2 condition for case  $(\diamond)$  with  $\gamma^\diamond \in (2\epsilon, 1)$  (i.e.,  $\frac{1}{n\lambda_B^\diamond} > 1$ ) cannot be satisfied unless the Corollary 2 condition for  $\gamma^* = 1$  is also satisfied, i.e., for  $\gamma^\diamond \in (2\epsilon, 1)$  it becomes more difficult to satisfy the Corollary 2 condition as  $\gamma^*$  moves away from the value of 1.

Next, for case  $(\diamond)$  and any  $\gamma^\diamond \in \left[\frac{\epsilon}{1-\epsilon}, 2\epsilon\right]$  that is a solution to Eq. (15), the Corollary 2 condition, which follows from Eqs. (27) and (28), is given by,

$$1 < \frac{2\epsilon}{n\lambda_A^\diamond |_{\gamma^\diamond}} = \frac{2\epsilon}{(\gamma^\diamond)^2 \left(\frac{n_A}{2n}\right) + \frac{n_D}{2n} \left(\epsilon + (\gamma^\diamond)^2 \left(\frac{\epsilon^2}{(1-\epsilon)}\right)\right)} \quad (30)$$

Because the right-hand side of Eq. (30) is decreasing in  $\gamma^\diamond$  it follows that,

$$\frac{2\epsilon}{n\lambda_A^\diamond |_{\gamma^\diamond}} \leq \frac{1}{\frac{n_A}{2n} + \frac{n_D}{2n} \left(\epsilon + \left(\frac{1}{2\epsilon}\right)^2 \left(\frac{\epsilon^2}{(1-\epsilon)}\right)\right)} \leq \frac{1}{n\lambda^1}.$$

Thus, for any  $\gamma^\diamond \in \left[\frac{\epsilon}{1-\epsilon}, 2\epsilon\right]$  that is a solution to Eq. (15), the Corollary 2 condition for case  $(\diamond)$  cannot be satisfied unless the Corollary 2 condition for  $\gamma^* = 1$  is also satisfied. This completes the proof of case  $(\diamond)$  of part (2) of Proposition 3. To complete the proof of Proposition 3, note that the proof for case  $(\blacklozenge)$  of part (2) of Proposition 3 follows from a symmetric argument.

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