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SHORT NOTES

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ON "COLONEL BLOTTO" AND ANALOGOUS GAMES*

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1. Introduction. In a number of different settings, we encounter the problem of determining the minimum with respect to the x_i and the maximum with respect to the y_i of the function

$$R_N = k_1(x_1, y_1) + k_2(x_2, y_2) + \cdots + k_N(x_N, y_N) \tag{1.1}$$

over the region defined by

$$\begin{aligned} x_i &\geqq 0, \qquad \sum_{i=1}^{N} x_i = 1, \\ y_j &\geqq 0, \qquad \sum_{j=1}^{N} y_j = 1. \end{aligned} \tag{1.2}$$

If the function k_i is convex in x_i and concave with respect to y_i, $i = 1, 2, \cdots, N$, then we can assert that

$$\min_{\{x_i\}} \max_{\{y_i\}} R_N = \max_{\{y_i\}} \min_{\{x_i\}} R_N \tag{1.3}$$

and conceive of this optimization problem as an example of a two-person zero-sum game.

The "Colonel Blotto" game [1], [2] is an example of this. In this note, we wish to indicate the applicability of dynamic programming [3], [4] to the analytic and computational solution of problems of this nature.

2. Functional equations. In place of (1.2), let the constraint region be

$$\begin{aligned} x_i &\geqq 0, \qquad \sum_{i=1}^{N} x_i = x, \qquad 0 \leqq x < \infty, \\ y_j &= 0, \qquad \sum_{j=1}^{N} y_j = y, \qquad 0 \leqq y < \infty. \end{aligned} \tag{2.1}$$

Write

$$f_N(x, y) = \min_{\{x_i\}} \max_{\{y_i\}} R_N, \tag{2.2}$$

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where $N = 1, 2, \cdots$. Then an application of the principle of optimality yields the recurrence relation

$$f_N(x, y) = \min_{0 \leqq x_N \leqq x} \max_{0 \leqq y_N \leqq y} [k_N(x_N, y_N) + f_{N-1}(x - x_N, y - y_N)] \tag{2.3}$$
$$= \max_{0 \leqq y_N \leqq y} \min_{0 \leqq x_N \leqq x} [\cdots],$$

$N \geqq 2$, with

$$f_1(x, y) = \min_{0 \leqq x_1 \leqq x} \max_{0 \leqq y_1 \leqq y} [k_1(x_1, y_1)] \tag{2.4}$$
$$= \max_{0 \leqq y_1 \leqq y} \min_{0 \leqq x_1 \leqq x} [\cdots].$$

One way of obtaining (2.3) is to conceive of the game as a multistage process with N stages, where (x_N, y_N) are chosen first, (x_{N-1}, y_{N-1}) are chosen next, and so on. The fundamental min-max theorem permits us to permute the minimum and maximum operations as we wish.

3. Homogeneity. A case of some interest is that where each k_i is homogeneous of degree one, i.e.,

$$k_i(tx, sy) = tsk_i(x, y) \tag{3.1}$$

for $x, y, s, t \geqq 0$. It follows then that

$$f_N(x, y) = g_N xy, \tag{3.2}$$

where g_N is independent of x and y. Then (2.3) becomes

$$g_N = \min_{0 \leqq x_N \leqq 1} \max_{0 \leqq y_N \leqq 1} [k_N(x_N, y_N) + g_{N-1}(1 - x_N)(1 - y_N)], \tag{3.3}$$

a relation which permits an explicit analytic solution in some cases. We shall omit the associated max-min relation.

If we have only

$$k_i(tx, ty) = tk_i(x, y), \tag{3.4}$$

as in [2], then (2.3) can be replaced by a recurrence relation for a function of one variable, say $f_N(1, y)$ or $f_N(x, 1)$. This can lead to analytic and computational simplification. To obtain this, we write

$$f_N(x, y) = xf_N\left(1, \frac{y}{x}\right) \tag{3.5}$$
$$= yf_N\left(\frac{x}{y}, 1\right).$$

Thus, for example, setting $z = y/x$, we obtain

$$f_N(1, z) = \min_{0 \leqq x_N \leqq 1} \max_{0 \leqq y_N \leqq z} \left[k_N(x_N, y_N) + (1 - x_N) f_{N-1}\left(1, \frac{z - y_N}{1 - x_N}\right)\right], \tag{3.6}$$

with a corresponding equation for $f_N(z, 1)$.

4. Computational aspects. A disadvantage of this last recurrence equation as opposed to (2.2) lies in the expanding grid in the z-variable. If $f_N(1, z)$ is desired in the interval $[0, z_0]$, we may have to compute $f_{N-1}(1, z)$ in a very much larger interval due to the presence of the term $(z - y_N)/(1 - x_N)$ on the right-hand side of (3.6).

To avoid this difficulty, we use the relation in (3.6) for $0 \leqq z \leqq 1$ and the corresponding relation for $f_N(z, 1)$ also in $0 \leqq z \leqq 1$. Thus, by calculating the two sequences $\{f_N(1, z)\}$, $\{f_N(z, 1)\}$, we can restrict our attention to the fixed interval $0 \leqq z \leqq 1$; see the use of the same device in [4, p. 239].

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