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Notes, Comments, and Letters to the Editor

Contests with limited resources

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Abstract

Many interesting phenomena (electoral competition, R&D races, lobbying) are instances of multiple simultaneous contests with unconditional commitment of limited resources. Specifically, the following game is analyzed. Two players compete in a number of simultaneous contests. The players have limited resources (budgets) and must decide how to allocate these to the different contests. In each contest the player who expends more resources than his adversary wins a corresponding prize. Mixed-strategy equilibria are characterized in the case of identical values and budgets and the connections with the classical Blotto game are analyzed.

JEL classification: C72; D63; D72

Keywords: Contests; Budget restrictions; Blotto game; All-pay auctions

1. Introduction

Many important economic, social, political, and biological phenomena are instances of multiple simultaneous contests with unconditional (on winning) commitment of limited resources.

As a first example, consider the following model of electoral competition: two candidates for an office try to woo voters in different groups—identified by ideology, demographics, or location—by targeted campaign spending on, say, media coverage. Voters are influenced only by the spending targeted at them and the candidate who spends more on a particular group wins the votes from that group. Each candidate's total spending across groups is, however, subject to an overall budget constraint. Candidates seek to maximize their expected vote share. What sorts of

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spending patterns will emerge in equilibrium? Will electoral competition create inequality among an otherwise homogeneous electorate? Will candidates appeal to all voters equally or will they cultivate favored minorities, leading to a prevalence of special interest politics?

A second example is that of two firms engaged in R&D competition. Each firm has to decide how to allocate a given budget to a number of research projects, say developing new drugs. The commitment of resources to projects is not conditional on winning the prize—expenditure on R&D is irreversible. If the expenditure of the first firm on the development of a drug X is greater than that of the other firm, then the first firm is more likely to patent drug X before the other firm. The overall payoff of the firm is the total value of patents it receives less its total research expenditure. How should the firms allocate their budgets to the projects? Will the firms concentrate their expenditure only on a few projects or will they try to invest in every potential patent?

Other notable examples of structurally similar situations include multiple-object simultaneous auctions with budget-constrained players, rent-seeking and lobbying, arms races and escalation conflicts.

This paper seeks to provide a theoretical framework, along with the methods of constructing equilibria, suitable for the analysis of a wide variety of economic situations. Specifically, the following class of games is analyzed. Two players compete in a number of simultaneous contests or races. The players have limited resources (budgets) and must decide how to allocate these to the different races. In each race the player who allocates more resources than his adversary wins a corresponding prize of known value. Resources devoted by a player to a race are not recoverable and constitute sunk costs. The key strategic problem the players face is that an increase in resources devoted to one race in an attempt to win leaves fewer resources available for other races. Thus, the presence of budget restrictions creates an indirect ‘substitutability’ of prizes which precludes the existence of pure strategy equilibria. The paper fully characterizes mixed strategy equilibria and studies their properties in the case of identical values of the objects and identical budgets.

1.1. Related literature

A zero-sum version of the game studied here is one of the first analyzed in game theory. Its venerable history dates back to Borel [3,4] who posed the problem in 1921 and solved it in 1938 by characterizing a mixed strategy equilibrium. Discrete versions of Borel’s game, known as Colonel Blotto games [9, p. 455], were used in the 1950s to analyze armed conflicts and battlefield tactics. It is, however, difficult to assess in full the results of this early development. Some contributions survived only as abstracts [15], others as classified military memoranda. [1] Standard Colonel Blotto games involve a finite (discrete) set of pure strategies, usually a subset of the set of natural numbers. In the simplest version, Colonel Blotto has a number of battalions at his disposal and needs to decide how to distribute them across N battlefields. Blotto’s adversary simultaneously makes the same decision. If Blotto outnumbers his enemy at a battlefield he receives payoff v, otherwise his payoff is $-v$. The payoff of the game as a whole is the sum of payoffs across all battlefields.

[1] The latter category is mentioned by Blackett [2] who “exposes as incorrect various solutions to Blotto games which appear in the classified literature.”

It is important to emphasize that in these early models players care only about winning and losing but not about the resources (money, battalions, etc.) they expend in doing so, making the underlying game zero- or constant-sum. However, this assumption may be incongruous with many real-life situations. Candidates may take into account the total campaign spending—raising funds requires effort. Companies definitely do care about minimizing their overall R&D expenditures. And even generals occasionally show concern over casualties.

In an important contribution, Myerson [10] uses a similar zero-sum game to analyze inequality in election campaign spending, focusing on the case with a continuum of homogeneous voters (equivalent in this model to a continuum of identical objects). The infinite-population assumption allows him to treat promises to individual voters as independent and, as a result, to relax budget restrictions: budget constraints must be satisfied only on average. Laslier and Picard [7], building on the work by Borel and Ville and Gross and Wagner [6], analyze two-candidate majority elections with exact budget constraints. They also provide an insightful review of the results related to the construction of equilibrium mixed strategies for two-person zero-sum games with convex set of pure strategies.

While the work referred to above concerns zero- or constant-sum games, there exists some literature on all-pay actions with budget-constrained players [5,8], which is naturally in a non-zero-sum setting. This literature, however, is confined to the one-object case. Multiple-object auctions with budget-constrained bidders are studied by Benoît and Krishna [1], but only when auctions are conducted sequentially. Szentes and Rosenthal [13,14] study the simultaneous first-price sealed-bid auctions when objects are endowed with specific synergies. Even though they impose no budget restrictions, the presence of extreme complementarities precludes the existence of pure strategy equilibria and results in the structure of mixed strategy equilibria somewhat similar to the one of Blotto game.

To summarize, the early literature focuses on the role of budget restrictions but in a zero-sum framework. The all-pay auctions and R&D literature takes into account the costs of competing, but does so, mostly, for one object. In contrast, this paper studies the model that incorporates all these essential elements: multiplicity of objects, simultaneity of contests, unconditional commitment of resources, and, last but not the least, budget restrictions.

2. The model

There are two players with budgets $B_1 = B_2 = B$ and $N > 1$ indivisible objects indexed by $j \in \{1, \ldots, N\}$. The value of object *j* is v_j and is common to either player. All the objects are identical, $v_j = v$ for every *j*. There are no synergies among the objects, the value of any bundle of objects is the sum of the values of individual objects. The game is the one of complete information, all values and budgets are commonly known to all players.

The objects are sold simultaneously by means of (first-price) all-pay auctions. Player *i* submits a nonnegative vector of bids $\mathbf{b}_i = (b_{i1}, \ldots, b_{iN})$, where b_{ij} is a bid by player *i* for object *j*. Player *i* wins object *j* if his bid for that object, b_{ij}, exceeds the bid of the other player, b_{-ij}, and always pays his bid, so the total payment of player *i* is $\sum_j b_{ij}$. Players are budget constrained, the sum of player's bids cannot exceed that player's budget B_i. [2]

[2] It does not matter for the all-pay auction whether the budget constraint is imposed ex ante or ex post because the sum of the bids always equals the actual amount paid. This distinction may be important for the analysis of other auction formats, however.

Formally, a pure strategy of player i is an N-dimensional vector $\mathbf{b}_i$. The set $\mathcal{B}_i$ of all pure strategies of player i is non-empty, compact, convex subset of finite-dimensional Euclidean space

$$\mathcal{B}_i = \left\{ \mathbf{b}_i \in \mathbb{R}_+^N : \sum_j b_{ij} \leqslant B \right\}.$$

The player i's payoff function $\pi_i : \mathcal{B}_i \times \mathcal{B}_{-i} \to \mathbb{R}$ is

$$\pi_i(\mathbf{b}_i, \mathbf{b}_{-i}) = \sum_{j: b_{ij} \geqslant b_{-ij}} v_j - \sum_j b_{ij}.$$

That is, players try to maximize the total value of objects they win less the resources spend on bidding. Note that the payoff function of the corresponding constant-sum (Blotto) game is $\pi_i(\mathbf{b}_i, \mathbf{b}_{-i}) = \sum_{j: b_{ij} \geqslant b_{-ij}} v_j$.

Given a strategy profile $\mathbf{b} = (\mathbf{b}_i, \mathbf{b}_{-i})$ payoff function π_i is continuous in player i's own strategy except when the bids of both players for some object coincide, that is, when there exists j such that $b_{ij} = b_{-ij}$. Such situations are referred to as *ties*. The specifics of the way in which ties are resolved do not affect the results when players have identical budgets, because in such case the equilibrium distributions are atomless and ties occur with probability zero. It is assumed, without loss of generality, that in case of a tie the object is randomly allocated to one of the players and both players have an equal probability of getting the object.

A mixed strategy of player i is an N-variate (joint) distribution function $G_i : \mathcal{B}_i \to [0, 1]$. The corresponding N-variate (probability) density function is $g_i : \mathcal{B}_i \to \mathbb{R}$. The one-dimensional *marginal* distribution functions, $G_{ij} : [0, B_i] \to [0, 1]$, (or, briefly, margins) of G_i are defined by

$$G_{ij}(x) = \int_0^x \left(\int_0^{B_i} \cdots \int_0^{B_i} g_i(\mathbf{z})\, d\mathbf{z}_{-j} \right) dz_j.$$

The function G_{ij} is interpreted as the distribution function of bids of player i for object j. The corresponding marginal density functions are $g_{ij} : [0, B_i] \to \mathbb{R}$.

3. Analysis

The analysis proceeds as follows: first, Proposition 1 establishes that no pure strategy equilibrium exists if more than one object is for sale. Next, Proposition 2 states necessary and sufficient conditions for the existence of mixed strategy equilibria in terms of margins. It asserts that the equilibrium *marginal* distributions are unique, symmetric with respect to objects, and have connected supports. The equilibrium analysis does not, however, provide the full description of the joint distribution, that is, of the equilibrium strategy *per se*. So, as a next step, several methods of constructing a multivariate distribution with given margins are discussed. Finally, Proposition 3 provides a new approach to generating joint distributions (mixed strategies) with given margins when more than three objects are for sale.

3.1. Pure strategy equilibrium: nonexistence

The proof of nonexistence is standard, so only the intuitive argument is provided. On the one hand, the players cannot bid different amounts on the same object because it is always profitable for a winning player to decrease the bid, reducing the amount paid. On the other hand, being tied on all objects also cannot be the equilibrium strategy; a slight increase in the bid on one object leads to winning that object for sure instead of sharing it. The presence of the budget constraints, though, asks for a more careful argument. If, when tied, a player uses up all his budget then such an increase no matter how small leads to the loss of some other object. Still, a profitable deviation exists: outbid the $N-1$ smallest bids of the opponent and place a zero bid on the remaining object. Such a strategy allows a player to win all the prizes except one while spending strictly less than before.

Proposition 1. *No pure strategy equilibrium exists if more than one object is for sale.*

3.2. Mixed strategy equilibria: marginal distributions

The first step in constructing an equilibrium is the analysis of the marginal distributions associated with the joint distribution (mixed strategy). Because there are no synergies among objects, the payoff of player i can be written as

$$\sum_j v \Pr[b_{ij} \geqslant b_{-ij}] - \sum_j b_{ij} = \sum_j [vG_{-ij}(b_{ij}) - b_{ij}].$$

Note that the player's payoff depends only on the marginal distributions G_{-ij} of the other player's mixed strategy G_{-i}. The next Proposition characterizes mixed strategy equilibria in terms of their margins.

Proposition 2. *Let* $m = \min\{v, \frac{2B}{N}\}$ *and* $N > 2$. *A joint distribution function* $G_i : \mathcal{B}_i \to [0, 1]$ *constitutes an equilibrium strategy if and only if all its one-dimensional marginal distribution functions* G_{ij} *are uniform on* $[0, m]$. *All equilibria are payoff equivalent and the payoff to either player is* $\frac{N}{2}(v - m)$.

Proof. See Appendix. $\square$

It is instructive to compare mixed strategy equilibria described in Proposition 2 with mixed strategy equilibria of the corresponding zero-sum game [4]. The uniformity of the margins furnishes sufficient condition for a joint distribution to be an equilibrium mixed strategy both in zero- and non-zero-sum games. In addition, the uniformity of the margins is also necessary condition for an equilibrium in non-zero-sum games. This result hinges on the connectedness of the (closures of) supports of the margins, costly bidding precludes the existence of gaps in the supports. In zero-sum case, it is known [7] that if the supports are connected then the uniformity also becomes necessary. However, in Blotto game the connectedness itself remains very plausible but unproven conjecture.

Costly bidding also leads to a different upper bound on the magnitude of bids. In zero-sum games a bid for any object is, with probability one, in $[0, \frac{2B}{N}]$ while in non-zero-sum game a player's bid for any object cannot exceed that player's valuation of the object and is, consequently, in $[0, m]$. Intuition can be formulated in terms of the opportunity cost of money. In the zero-sum game, the opportunity cost of money is calculated comparing the different objects and the bidding of the other player. With costly bidding, the opportunity cost of money has to be compared, in addition, with the constant value of not using it in any contest.

3.3. Mixed strategy equilibria: joint distributions

The characterization of the equilibrium strategy in terms of marginal distributions is not an exhaustive one. Several interesting and important properties—such as the uniqueness and the correlation—cannot be studied without the knowledge of the joint distribution.

The problem of constructing a multivariate distribution function from its one-dimensional margins given support restrictions is an example of ill-posed inverse problem. It does not admit the complete characterization; it is not possible, in general, to determine whether a particular problem has the unique solution or any at all. In the game here, restrictions on the support of the joint distribution emerge because the players are budget constrained and are allowed to use strategies only in the set $\mathcal{B}_i$, which does not include all of $C(N) = [0, m]^N$ (or, in the zero-sum case, all of $C_0(N) = [0, \frac{2B}{N}]^N$). Were it otherwise, it would be possible to construct a joint distribution with any given correlation properties.

The case of only two objects for sale, $N = 2$, admits the complete characterization when budgets are small, $B \leqslant v$. The unique[3] bivariate distribution function with uniform margins and the support included in $\mathcal{B}_i$ is

$$G_2(\mathbf{x}) = \frac{1}{B} \max\{0, x_1 + x_2 - B\}.$$

It places positive uniform weight only on the budget line $x_1 + x_2 = B$ and the bids for two objects have perfect negative correlation. The solution is no longer unique when budgets are large, $v < B \leqslant 2v$. Moreover, it is possible to have zero or positive correlation between bids even for budgets that are strictly smaller than $2v$. Fig. 1 depicts three solutions for $B = \frac{3}{2}v$, the density is uniform on the heavy lines and is zero everywhere else. Note that none of these solutions can be transformed to produce a solution of the zero-sum game.

These simple examples illustrate the differences and similarities between zero- and non-zero-sum cases implied by the different caps on the bids in games with arbitrary N. The support of any zero-sum solution is included in the set $\mathcal{B}_i \cap C_0(N)$, while the support of any non-zero-sum solutions is included in the set $\mathcal{B}_i \cap C(N)$. The sets $C_0(N)$ and $C(N)$ are the same when the budgets are relatively small, $B \leqslant \frac{Nv}{2}$, and in such case, the solutions to both games coincide. When, however, the budgets are relatively large, $\frac{Nv}{2} < B \leqslant Nv$, there is more flexibility in constructing joint distributions in non-zero-sum case, as examples in Fig. 1 demonstrate. Intuitively, as the budget increases it becomes less 'restrictive' in non-zero-sum games and 'substitutability' among the objects diminishes, while in zero-sum games the budget is always binding. A player with the larger budget is partially able to treat the objects as independent. Consider for example the equilibrium strategy illustrated in Fig. 1(c). In this equilibrium a player makes, with the positive probability, arbitrarily small bids on both objects, effectively not spending any resources at all.

If a zero-sum solution has connected supports of all its one-dimensional margins then it is always possible to transform it into a non-zero-sum solution by scaling it by $\frac{Nm}{2B}$. The resulting joint distribution always satisfies the budget constraint because the budgets are the same in both games and the distribution is scaled *down*.

Lemma 1. *Any equilibrium of a zero-sum game, such that the supports of all its one-dimensional margins are connected, is also an equilibrium of a non-zero sum game given an appropriate affine transformation (scaling).*

[3] The proof of uniqueness can be found in [11].

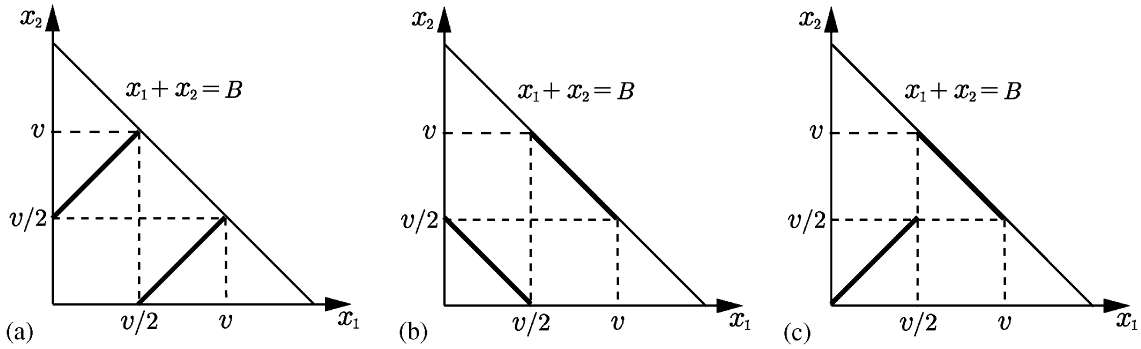


Fig. 1. (a) $Corr = -0.5$; (b) $Corr = 0.5$; and (c) $Corr = 0.75$.

On the other hand, only some non-zero-sum solutions can be transformed into zero-sum solutions. The non-zero-sum solutions need to be scaled up because $m \leqslant \frac{2B}{N}$, so the scaling coefficient is $\frac{2B}{Nm}$. After the scaling the support of a solution must be included in $\mathcal{B}_i \cap C_0(N)$, thus the support of the non-zero-sum solution must be included in the inverse image (under the scaling) of the budget set. Note that this condition is vacuously satisfied when budgets are small, $m = \frac{2B}{N}$.

Lemma 2. *Any equilibrium of a non-zero-sum game with the support included in the set*

$$\left\{\sum_i x_i \leqslant \frac{Nm}{2};\ x_i \geqslant 0 \textit{ for } i = 1, \dots, N\right\}$$

is also an equilibrium of a zero-sum game given the appropriate affine transformation (*scaling*).

For zero-sum game and $N = 3$, two equilibrium joint distributions, called the disk and hexagonal solutions, were constructed independently by Borel [4] and by Gross and Wagner [6]. Both solutions have uniform margins with connected supports and involve randomization over the set

$$\mathrm{Hex}(B) = \left\{\sum_{i=1}^{3} x_i = B;\ 0 \leqslant x_i \leqslant \frac{2B}{3} \text{ for } i = 1, 2, 3\right\}.$$

Thus, (by Lemma 1) these solutions can also be scaled to produce non-zero-sum solutions. When budgets are large, $\frac{3}{2}v < B \leqslant 3v$, additional families of solutions, which cannot be obtained from zero-sum solutions by scaling, can be constructed using ideas similar to those in Fig. 1.

The Borel construction generalizes to give a solution for the N-object game [6], the support of which is a two-dimensional set, regardless of N. As a result, bids for different objects are heavily correlated and the knowledge of any two bids allows to calculate the rest of the strategy. The next Proposition describes a novel way of constructing joint distributions with uniform margins in the presence of support restrictions. In contrast to generalized disk solution it produces a variety of solutions with different levels of correlation.

Given a game Γ with $N > 3$ objects and budgets B consider the two auxiliary games: the game Γ_L with $L > 1$ objects and budgets $B_L = \frac{L}{N}B$ and the game Γ_{N-L} with $N - L > 1$ objects and budgets $B_{N-L} = \frac{N-L}{N}B$. Suppose G_L is the equilibrium distribution function of game Γ_L and G_{N-L} is the equilibrium distribution function of game Γ_{N-L}. Define distribution function

$G = G_L \times G_{N-L}$ as follows: a player bids on the group of L objects according to G_L and bids on the rest of objects $(N - L)$ according to G_{N-L}.

Proposition 3 (*Decomposition*). *Suppose $N > 3$ and $N - 1 > L > 1$. If G_L and G_{N-L} are equilibrium distributions of Γ_L and Γ_{N-L} then $G = G_L \times G_{N-L}$ is an equilibrium distribution function of Γ.*

Proof. It suffices to verify that the joint distribution function G has uniform one-dimensional margins on $[0, m]$ where $m = \max\{v, \frac{2B}{N}\}$. Since G_L is the equilibrium distribution function of the game Γ_L then, according to Proposition 2, G_L has uniform margins on $[0, m_L]$ where $m_L = \max\{v, \frac{2B_L}{L}\}$. By construction, $\frac{2B_L}{L} = \frac{2B}{N}$ and, thus, $m_L = m$. By the same token, G_{N-L} also has uniform margins on $[0, m]$. □

The number of solutions given by the Proposition 3 grows very quickly and already for $N = 10$ equals 12,288.[4]

The first immediate consequence of the Proposition 3 is that a solution for the game with $N > 3$ objects may be constructed from the solutions to the games with $N = 2$ and $N = 3$ objects. In other words, equilibria of games G_2 and G_3 are the smallest 'building blocks' needed to generate mixed strategy equilibria in games with arbitrary N.

Corollary 1. *For any $N > 1$ there exists an equilibrium joint distribution G constructed using only G_2's and G_3's.*

The correlation properties of the multivariate distribution G when $N > 3$ are addressed by the next corollary.

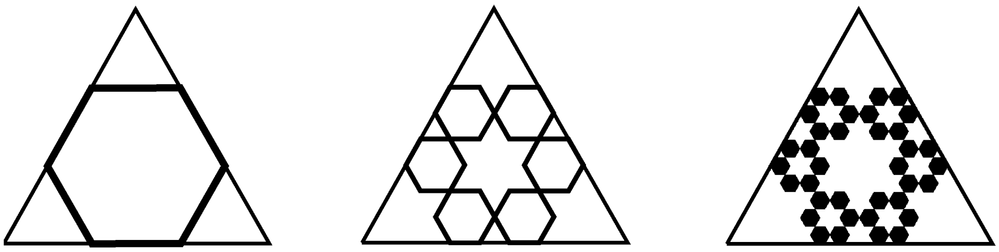
Corollary 2. *For any $N > 3$ there exists an equilibrium joint distribution G such that the bids for $\lfloor \frac{N}{2} \rfloor$ objects are not correlated. Moreover, the bids for at most three objects are correlated.*

Intuitively one might think that the bids on different objects should be negatively correlated, bidding more on one object leaves fewer resources available for the others. Corollary 2 reveals that such intuition is somewhat flawed. Already for $N = 4$ there exist equilibria in which bids for some pairs of objects are independent.

The results above can also be applied to the analysis of inequality in the models of electoral competition [10,7]. The levels of inequality (as measured, for example, by Gini index) depend on the correlation properties of the joint distribution and differ for different solutions. They range from egalitarian ones, when the joint distribution is constructed using only G_2's, to very unequal ones, when the heavily correlated generalized disk solution is used. No unambiguous answer exists unless some restrictions are imposed on the equilibrium joint distribution.

It is also worth mentioning that in two Borel solutions for $N = 3$ and their extensions for $N > 3$ the supports of the joint distributions are connected sets and the joint distribution functions itself are absolutely continuous. These properties, however, are just the artifacts of the particular solutions and are not shared by all equilibrium mixed strategies! The connectedness of the support of the

[4] The total number of different solutions is $2^N(p(N) - p(N-1))$, where $p(N)$ is the partition function: the number of ways of writing the integer N as a sum of positive integers.



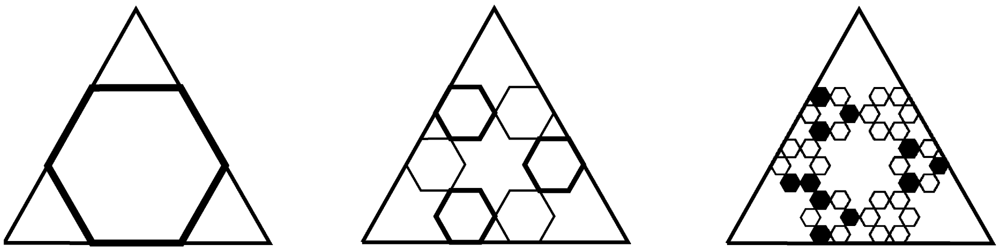


Fig. 2. (a) Fractal solution of dimension $\frac{\ln 6}{\ln 3}$; (b) Fractal solution of dimension 1.

multivariate distribution is not implied by the connectedness of the supports of one-dimensional margins. [5]

Developing the insight in [6], it is possible to construct families of joint distributions with supports of Lebesgue measure less than ε for any $\varepsilon > 0$. Fig. 2(a) illustrates the construction for $N = 3$. At the first step, 6 regular hexagons are inscribed into the original one, defined on Hex. At the kth step, 6 regular hexagons are inscribed into each hexagon generated on the $(k-1)$th step. The solution of kth generation involves choosing, with equal probability, one out of 6^k hexagons and randomizing over it according to some solution for $N = 3$. Fig. 2(b) depicts the first three generations of a solution with even less 'dense' support (at the kth step only 3^k hexagons are used). These 'fractal' solutions possess interesting features. Even though objects are identical ex ante they are never treated symmetrically ex post. In particular, in all realizations there are bids smaller than $2B/9$, there are bids larger than $4B/9$, but there are no bids in the interval $[2B/9, 4B/9]$. In addition, the 'egalitarian' bids in the neighborhood of $(B/3, B/3, B/3)$ are made with probability zero.

The last remark concerns somewhat novel nature of the multiplicity of equilibria in this game. Multiple equilibria emerge because the joint distribution is not determined uniquely by its margins. Interestingly, the uniqueness of margins also implies interchangeability of equilibria in this game, which is not usual for the non-zero-sum games.

Appendix

Proof of Proposition 2. *Necessity*. Let $P = (x_1, \ldots, x_N)$ be an arbitrary point in the closure of the support of an equilibrium mixed strategy of player i. Then there exists $\varepsilon > 0$ such that for

[5] Weinstein [16] constructs another interesting family of one-dimensional joint distributions for any $N \geqslant 3$.

every j the marginal equilibrium distribution function G_{-ij} of player $-i$ is strictly increasing and continuous in the interval $(x_j - \varepsilon, x_j + \varepsilon)$. The proof is by contradiction.

Suppose there exists $j = j^*$ such that G_{-ij} is constant in the interval $(x_{j^*} - \varepsilon, x_{j^*} + \varepsilon)$. Consider a point $P' = (x_1, \ldots, x_{j^*} - \varepsilon, \ldots, x_N)$. The difference in payoffs when moving from P to P' is given by

$$\sum_j \left(dx_j \int_{H_j} g_{-i}\, d\sigma \right) - \frac{\varepsilon}{2},$$

where H_j is the hyperplane $x_j = \text{const}$, and σ is the Lebesgue measure. By construction, $dx_j = x_j - x'_j = 0$ for every $j \neq j^*$ and the hyperplane H_{j^*} does not intersect the support of an equilibrium joint distribution. Thus, the difference is negative and the strategy $G_{-ij}(x)$ is not optimal.

Suppose that $G_{-ij}(x)$ is discontinuous at P for all j. Define

$$\alpha(j) = \limsup_{x_j} G_{-ij}(x) - \liminf_{x_j} G_{-ij}(x)$$

and let $j^* \in \arg\min_j \alpha(j)$. Consider a point $P' = (x_1 + \frac{\varepsilon}{N-1}, \ldots, x_{j*} - \varepsilon, \ldots, x_N + \frac{\varepsilon}{N-1})$. The difference in payoffs when moving from P to P' is given by

$$\sum_j \left(dx_j \int_{H_j} g_{-i}\, d\sigma \right) - \frac{v}{2} \left(\sum_{j \neq j^*} \alpha(j) - \alpha(j^*) \right)$$

and is negative for a small enough ε, contrary to the assumption that G_{-ij} is an equilibrium mixed strategy.

Suppose that G_{-ij} is discontinuous at P only for some j. Denote by J the set of all such j and consider point P' such that $x'_j = x_j + \frac{\varepsilon}{|J|}$ for all $j \in J$ and $x'_j = x_j - \varepsilon$ for some $j \notin J$. Again, the difference in payoffs when moving from P to P' is given by

$$\sum_j \left(dx_j \int_{H_j} g_{-i}\, d\sigma \right) - \frac{v}{2} \sum_{j \in J} \alpha(j)$$

and is negative for a small enough ε, contrary to the assumption that $G_{-ij}(x)$ is an equilibrium mixed strategy. In both cases the existence of a small enough ε is guaranteed by the fact that the function G_{-ij} can have at most a countable number of discontinuities.

Thus, it follows that g_{-i} is positive for any j in the neighborhood of x_j. Consider a point $P' \neq P$ in the neighborhood of P, such that $\sum_j dx_j = 0$. The difference in payoffs when moving from P to P', given by

$$\sum_j \left(dx_j \int_{H_j} g_{-i}\, d\sigma \right),$$

must be zero, implying that $g_{-ij}(x) = \int_{H_j} g_{-i}\, d\sigma$ does not depend on j.

Next, it is shown that g_{-ij} is also independent of x. If P and P' are two points in the support of the joint distribution with a coordinate in common, say $x_j = x'_j$, then $g_{-ij}(x_j) = g_{-ij}(x'_j)$. Since the supports of margins are connected, $g_{-ij}(x_j) = g_{-ij}(x'_j)$ for every pair of points.

Player i's expected payoff from bidding $\mathbf{b}_i$ is increasing in $\sum_j b_{ij}$ if $g_{-ij} < \frac{1}{v}$.

As a result, in such case player i uses up all his budget. Since the average bid on any object cannot exceed $\frac{B}{N}$ the maximal bid on any object cannot exceed $\frac{2B}{N}$. Thus, if $v > \frac{2B}{N}$ then G_{-ij} is uniform on $[0, \frac{2B}{N}]$. If $g_{-ij} = \frac{1}{v}$ then any feasible strategy $\mathbf{b}_i$ gives the same payoff. Moreover, the maximal bid on any object cannot exceed v. Thus, if $v < \frac{2B}{N}$ then G_{-ij} is uniform on $[0, v]$. Denoting $m = \min\{v, \frac{2B}{N}\}$ the payoff to a player in either case can be written as $\frac{N}{2}(v - m)$.

Sufficiency. The proof of sufficiency is similar to the one for the zero-sum games [12,7].

Remark. Note that Proposition 2 is formulated for $N > 2$. The case of two objects for sale, $N = 2$, must be treated separately. The main difference is that only the symmetry between objects $g_{i1}(x_{i1}) = g_{i2}(x_{i2})$ but not necessarily the uniformity is implied. Any marginal density such that $g(x) = g(m - x)$ constitutes an equilibrium. □

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