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Interpretation of electoral mixed strategies

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Abstract. In this paper is remarked that "mixed" strategies in games of electoral competition do not need to be interpreted as random moves. There are two a priori symmetric parties, and a finite (non spatial) set of alternatives. Parties are allowed to take unclear positions, by campaining on a "platform" that is a mix of several alternatives. Each individual nevertheless identifies a party with a single alternative, the number of individuals who identify a party with a given alternative being proportional to the importance of that alternative in the party's platform.

1 Introduction

The fact that majority rule leads to disequilibrium has been for a long time a recurrent result of formal models of voting (see for instance Ordeshook 1986). To obtain voting models which are not submitted to disequilibrium, possibilities that have been considered are numerous, and among them is the introduction of uncertainty at some point in the model. See for instance Coughlin (1990), Anderson et al. (1994), Roemer (1994), Eichenberger and Serna (1996) or more generally the whole litterature on probabilistic voting. The model presented here belongs to this family but (i) we impose no spatial structure on the set of alternatives, and in particular we do not need any kind of "single dimension" hypothesis and (ii) uncertainty is endogeneous, and arises from the parties' rational behavior.

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Parties are allowed to send to the electorate ambiguous messages, formalized as follows. There is a set X of "alternatives". Elements $x, y \ldots$ of X are *complete and mutually exclusive* descriptions of possible policies, including all relevant issues. Voters have complete preferences over X. But during the campaign, a party does not (or at least is not restricted to) present itself as the candidate of one such single clear-cut ideology. Instead, the party presents itself sometimes as taking position x, sometimes as taking position y, etc... Then we have to model the process of identification, by each voter, of a party with an alternative. We do so in a rather naïve way, supposing that a fraction $p(x)$ of the electorate identifies party A with alternative x, this fraction of the electorate being statistically independent of the preferences and of the identification with the other party. We suppose that, when designing its campaign, a party can choose freely the vector p of the proportions of the electorate that will identify this party with such and such policy. But the party cannot choose *which* individuals will identify it with a given policy. An illustrative way to present the model is to imagine that, during the campain, the party spends a fraction $p(x)$ of its time saying "We are the candidates of policy x". The vector p is called the "platform" of the party.

Once a voter has identified each party with an alternative, she votes according to her true preferences. If she is indifferent between the two alternatives, then it may be supposed that she abstains, that she chooses at random one of the two parties, or that she casts "half of a vote" for each. These behaviors are equivalent for the models considered here, given the objective of the parties. Two objectives are considered. According to the first one, the payoff for a party is the number of votes for this party, minus the number of votes for the other party. According to the second one, the party is only interested in winning the election and the payoff can be written as $+1$ for a win, -1 for a loss and 0 for a tie. We refer to the first competition as competition for the *plurality* and to the second as competition for the *majority*. Even with two parties, it is not obvious that these objectives are equivalent. Indeed, in a model closely related to this one, Laffond et al. (1994) show that parties' behavior may be entirely different, at equilibrium, depending on the formulation of the objective (this has nothing to do with the problem of abstention). Fortunately, this paradox does not appear here: at equilibrium, the two formulations are equivalent. A pair (p, q) of strategies (one platform for each party) is an equilibrium of the *Plurality Platform Game* if and only if (p, q) is an equilibrium of the *Majority Platform Game*.

The main result of the paper is that these games do have equilibria (and, under technical asumptions, the equilibrium is unique). To prove this result, one just has to observe that the Plurality Platform Game is formally equivalent to the *mixed extension* of the classical pure Plurality Game, which can be studied with standard tools of the theory of games in mixed strategies. Therefore no new mathematical result is needed and results established for the Mixed Plurality game (see references in Sect. 3.2) can directly be imported for the Platform game. In the absence of a Condorcet winner, equilibrium platforms do not reduce to a single alternative. This means that it is in the party's interest not to set to zero the variance of the voters' perception (see Hinich and Munger (1989) for an opposite view).

Ambiguity in the parties' positions is a standard theme in Politics (at least since Downs 1957) and a well-documented fact (see, among others, Shepsle 1972; Page 1976; Campbell 1983; Chappell 1994). In the model presented here, ambiguity is a rational behavior for the parties, logically related to the existence of a Condorcet cycle in the voters' profile of preferences. This ambiguity diseapears if and only if the preference profile has a Condorcet winner.

The paper is organized as follows. In Sect. 2, the main notations are given, dealing with individual preferences, plurality and the identification process. Section 3 contains the results. In Sect. 3.1 is observed the equivalence between the Plurality Platform Game and the Plurality Mixed Game. In Sect. 3.2 results about the Plurality Platform Game are deduced from this equivalence, including the existence of equilibrium. In Sect. 3.3 is proved the proposition relating the equilibria of the Plurality and Majority Platform Games. Section 4 is the conclusion. In an annex, we come back to the individual process of identificaton party/alternative and give a more detailled motivation for the main assumption about the result of these independent individual processes. A table is provided to summarize the models.

2 Notations

2.1 Preference profile and plurality game

The model uses two parties and a finite set, V, of voters. We consider a nonempty finite set X, elements of which are called *alternatives*. Alternative will have two roles in the model. They are the objects of preferences for the individuals and they are the objects of the campaign for the parties. A *preference on* X is a complete binary relation on X (transitivity will not be used although it may be assumed). A preference *profile* on X is a vector $\mathbf{R} = (R_v)_{v \in V}$ of preferences on X. We impose no other structure on X than $\mathbf{R}$ and in particular no spatial structure. Nevertheless, the usual euclidean model is not a particular case of our model because we assume that there is only a finite number of alternatives.

Given $\mathbf{R}$ a preference profile on X and x and y two alternatives in X, we call (net) *plurality for* x *against* y the integer:

$$g^{\mathbf{R}}(x, y) = \mathrm{Card}\{v \in V : xR_v y\} - \mathrm{Card}\{v \in V : yR_v x\}.$$

Clearly, the net pluralities define a symmetric, two-player, zero-sum game. We refer to this game as the Plurality Game on X and write, when no confusion can arise, g instead of $g^{\mathbf{R}}$. Variants of this game are the main object of this paper. Players are the two parties, and standard results relate the existence of an equilibrium in the plurality game to the existence of a Condorcet winner in the preference profile. In the game g, the payoffs are the net pluralities, which means that it is better to win with a large margin than with a slim one. It is well known that this consideration diseapears at (pure strategy) equilibrium. Denote $sgn(g)$ the sign of g, that is: $+1$ if $g(x, y) > 0$, -1 if $g(x, y) < 0$ and 0 if $g(x, y) = 0$. Then $sgn(g)$ is again a symmetric, zero-sum game and g and $sgn(g)$ have the same (pure) equilibria, if they have one. Therefore, at pure equilibrium, g and $sgn(g)$ cannot be differentiated.

2.2 Party identification

In the two games of the previous section, the process of identification, by a voter, of a party with an alternative is clear:

- Party A annonces an alternative $a(A)$ and party B annonces $a(B)$.
- Every voter hears both messages, and understands which party proposes which alternative.

As a consequence, voters with identical preferences will always cast identical votes. We now give to the parties the possibility of sending unclear messages: for instance, during the fine talks of its campaign, party A will sometimes appear as taking position $a(A) = x$ and will sometimes appear as taking position $a(A) = x'$. As a result, some voters will identify A with x and some will identify A with x'. Let $p(x)$ be the proportion of the population that identifies A with x. We give to A the possibility to choose, by carefully designing its discourse, the vector p. We call p the *platform* of A. Formally, a platform is an element of the set

$$\Delta(X) = \left\{ p \in [0, 1]^X : \sum_{x \in X} p(x) = 1 \right\}.$$

This set replaces X as the strategy set for players. In order to be able to write the number of votes that will be casted for each party under this new hypothesis, further specification is needed. Let p and q be the platforms choosen by parties A and B.

Assumption 1. For any preference R and any two alternatives x and y, among the individuals whose preference is R, the proportion of those who identify A with x and B with y is $p(x)q(y)$ (and thus does not depend on R).

The signification of this assumption is that the process of identification

- is independent between the two parties
- is independent from the preferences.

It is clear that this assumption is a strong one, and it would be certainly fruitful to replace the assumption by other ones describing specific dependence between party identification and preferences. But in the abstract framework of symmetric parties and no structure on the set of alternatives it is not clear which form of dependence is adequat. Therefore independence is here a natural assumption. An electoral competition game played under Assumption 1 will be called a Platform Game.

3 Results

3.1 Platform Games

We now compute, for each couple of strategies (p, q) chosen by the two parties, their net pluralities, denoted $g(p, q)$. Among the fraction $p(x)q(y)$ of the population who identify A with x and B with y, the number of votes for A minus the number of votes for B is the number of people who prefer x to y minus the number of people who prefer y to x. Thanks to our assumption, this number is the fraction $p(x)q(y)$ of the whole number of people who prefer x to y minus the number of individuals who prefer y to x. Summing over all pairs (x, y) of alternatives, one obtains:

$$g(p, q) = \sum_{x, y \in X} p(x)q(y)g(x, y).$$

Formally, it is possible to consider (p, q) as a probability distribution over the set $X \times X$, then the formula above states that $g(p, q)$ is the mathematical expectation of the random variable $g(x, y)$, we write this as:

$$g(p, q) = E_{p,q}[g(x, y)].$$

Consider now the payoffs. If the payoff for a party is its net plurality g, then we proved that the *Plurality Platform Game* is formally identical to the plurality game played in mixed strategies. We therefore can study this game with tools of the theory of mixed normal-form games. If the parties are only interested in winning the election, that is the payoff function is $sgn(g)$, then the payoff in the *Majority Platform Game* is

$$sgn(g(p, q)) = sgn(E_{p,q}[g(x, y)]) \neq E_{p,q}[sgn(g(x, y))].$$

and is not (in general) the payoff in the mixed majority game. This point leads to greater difficulties in the study of the Majority Platform Game, for instance the payoff function $sgn(g)$ is not a continuous function of its argument (p, q). On the contrary, many results are available for the linear payoff function g. In the sequel, we study the Plurality Platform Game and show how the results in fact also apply to the Majority Platform Game.

3.2 Plurality is the goal

The most important result is that the Plurality Platform Game has equilibria. This result is simply the basic min-max theorem (von Neumann 1928) about the existence of optimal strategies for zero-sum games played in mixed strategies. Fishburn (1984) uses this result to define probabilistic social choice rules. Moreover, since the game is symmetric, attention can be restricted to symmetric equilibria of the form (p, p). An alternative is called *essential* if it is played with positive probability at some equilibrium. If (p, p) is an equilibrium of the game, then $g(x, p) = 0$ if x is essential and, by the Equalizer theorem, $g(x, p) < 0$ if x is not essential. The set of essential strategies defines a Social Choice Correspondence, that is a mapping from the set of profiles on X to the set of non-empty subsets of X. Normative properties of this choice correspondence are mentionned by Dutta and Laslier (1999) and Laslier (2000), drawing on the theory of two-player zero sum games. In general the equilibrium is not unique, but a sufficient condition is given by Laffond et al. (1997): the equilibrium is unique if all the numbers $g(x, y)$, for $x \neq y$, are odd integers. This condition holds if all the individual preferences are strict and there is an odd number of individuals. In case of unicity, both parties propose the same platform at equilibrium, and the set of essential alternatives is called the Plurality Bipartisan set (Laffond et al. 1994).

A second result is that the set of essential alternatives reduces to one alternative if and only if this alternative is a Condorcet winner of the preference profile. This means that if the "pure" game has an equilibrium, then the parties have no strategic interest in choosing sophisticated platforms.

3.3 Majority is the goal

As we mentioned earlier, the Majority Platform Game cannot be interpreted as the mixed extension of the (pure) majority game. The mixed extension of the pure majority game is studied in Laffond et al. (1993) and an example is given in Laffond et al. (1994) showing that the mixed extensions of the majority and plurality games can have very different equilibria. But remark that such is not the case for the platform games: The Plurality Platform Game g and the Majority Platform Game $f = sgn(g)$ have the same equilibria. This is so because a platform game is in fact a *pure strategy* game. If a proof is needed, just notice that $g(p', q) \leq 0$ for all p' if and only if $f(p', q) \leq 0$ for all p', the conclusion follows immediately.

As a consequence, it appears that, at equilibrium, the two assumptions that parties seek for the largest possible number of votes (the plurality, or "size" of the majority) or that they simply wish to win the election cannot be distinguished. Note that the same result holds not only for g and $sgn(g)$ but for any monotonic transformation $\phi(g)$ of the plurality g such that $\phi(g) = 0$ if and only if $g = 0$; it is not even needed that the two parties have the same objective. This is a nice feature of the "platform" model, in contrast with the strange behavior of the "mixed strategies" models. (An additional nice feature is of course that the platform model dispenses with the controversial supposition that political parties choose at random their programs.) Note, however, that the equivalence between the plurality and majority assumptions is only proved at equilibrium. But the games are different. If some other solution concept is used then it is possible that the two games may be differentiated. For instance, say that strategy p *covers* strategy p' if the payoff for p against p' is positive and the payoff for p against q is, for any q, greater than or equal to the payoff for p' against q. Then if p covers p' for g then p also covers p' for $sgn(g)$, but the converse may be false.

4 Conclusion

In this paper we have studied a non-spatial model of two-party electoral competition whose main features are: (a) The two parties are ex-ante symmetric. (b) Parties are allowed to take ambiguous positions, by campaining on a "platform", that is a mix of possibly several alternatives. (c) Each individual nevertheless identifies a party with a single alternative. (d) This process of identification is independent from one voter to another and is also independent from the voter's preference. The parameters of this process (the platform) are the strategic variables of action for the parties. The resulting game between the two parties is not subject to the usual drawback of many comparable electoral games, namely the absence of equilibrium. Even in the case where the voters' preference profile does not have a Condorcet winner, a "platform game" always has an equilibrium, and this equilibrium is often unique.

Each one of the above hypothesis can be challenged, but their status are somehow different. The idea (c) that each individual *in fine* identifies a party with a single alternative is the most basic because it is directly related to the the definition of an "alternative" as the object of the individual's preference. The idea (b) that parties campaign on "platforms" rather than single alternatives is the main idea of this paper, and certainly a key ingredient for the existence of equilibrium. It can be argued that voters will be more confident to a party which takes a clear position than to a party whose discourse can be interpreted in one way or another (Bartels 1986; Austen-Smith 1987). And it is a fact that parties do pretend to have clear-cut positions. But it is another fact that campaign speaches contain contradictory statements, and it is this fact which is here taken into account. The idea (a) that the two parties are symmetric certainly does not fit with reality. At least for historical reasons, the main parties in all countries are not identical. In our model, symmetric parties lead to the existence of symmetric equilibria. Dropping the symmetry assumption would lead to models with non-symmetric equilibria. Suppose for instance that only a convex subset Δ_A (resp. Δ_B) of the set Δ of platforms is availlable for party A (resp B). Then the point made in this paper is still valid as to the interpretation of "mixed" strategies. The constrained zero-sum game still has a non-empty convex set of interchangeable equilibria. Last, the idea (d) that the individual identifications are independent has a rather technical flavor, it would be interesting to know whether it is really the case. In particular, it would be interesting to propose alternative hypothesis modelling correlations between what an individual thinks (her preference relation) and what she understands of a party fuzzy speach (the platform).

Annex

A statistical model for party identification

Assumption 1 is stated in terms of fraction of the electorate. Here we explicitely consider the statistical model which yields this assumption at the limit for an infinite population. To do so, we start from an initial population $V = V^1$ with $k = \mathrm{Card}(V)$ and replicate it N times into a population V^N with Nk individuals. Elements $v \in V$ of the initial population are called *types*. The type of an individual i is denoted $v(i)$. Two individuals of the same type v have the same preference R_v. Two platforms $p, q \in \Delta(X)$ are given.

Each individual $i \in V^N$ identifies party A with an alternative $a(i)$ and party B with an alternative $b(i)$ and votes for A if $a(i)P_v b(i)$, where $v = v(i)$ is her type and P_v is the strict preference relation for her type. If $b(i)P_v a(i)$ then i votes for B. If i is indifferent between the two alternatives, then we can either suppose that i abstains or that she chooses one of the two parties by tossing a coin, this is of no real importance because we only need to derive the difference between the number of votes in favor of the two parties. We suppose that the $2Nk$ variables $a(i), b(i)$ for $i \in V^N$ are independent random variables, the law of $a(i)$ being p and the law of $b(i)$ being q.

Let $\varepsilon(i, p, q)$ be equal to $+1$ if i votes for A, -1 if she votes for B and 0 if she abstains. The net plurality is the random variable:

$$\gamma^N(p, q) = \sum_{i \in V^N} \varepsilon(i, p, q).$$

Summing type by type,

$$\gamma^N(p, q) = \sum_{v \in V} \sum_{i : v(i) = v} \varepsilon(i, p, q),$$

and the inner term $\sum_{i:v(i)=v} \varepsilon(i, p, q)$ is the sum of N independent variables with the same law. The variable $\varepsilon(i, p, q)$ is equal to $+1$ with probability $\sum_{xP_v y} p(x)q(y)$, to -1 with probability $\sum_{yP_v x} p(x)q(y)$, and to 0 in the other cases. The expectation of $\varepsilon(i, p, q)$ is thus:

$$m(v) = \sum_{xP_v y} p(x)q(y) - \sum_{yP_v x} p(x)q(y)$$

and its standard deviation is some number $\sigma(v)$ which does not need to be explicited. For N large enough the average of these N identical variables can thus be approximated by a normal variable with mean $m(v)$ and standard deviation $\sigma(v)/\sqrt{N}$.

Considering the various types, the variable $(1/N)\gamma^N(p, q)$ is approximately normal, with mean $\sum_{v \in V} m(v) = g(p, q)$ and standard deviation $(1/\sqrt{N}) \sum_{v \in V} \sigma(v)$. If N is large, randomness vanishes and one finds the model used in the preceding sections. Clearly, the derivation of g proposed here is independent of the description of the objectives of the parties, be them the plurality g itself or the majority $sgn(g)$.

Summary table

Four models of electoral competition have been described with the same set $\Delta(X)$ of strategies, that differ with respect to the the objectives of the parties

Table 1 Four games of electoral competition

	Plurality	Majority
Independent identifications	Plurality Platform Game	Majority Platform Game
Global identification	Plurality Mixed Game	Majority Mixed Game

(plurality or majority) and with respect to the process of identification by the voters of a party with an alternative. "Global identification" means that all voters hear the same (random) alternative.

The Plurality Platform Game and the Plurality Mixed Game are formaly identical (they have the same payoff functions), but the other games are different. Moreover, the Majority Platform Game has the same equilibria as the Plurality Games.

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