N-dimensional Blotto Game with Heterogeneous Battlefield Values*

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Abstract

This paper introduces the irregular N-gon solution, a new geometric method for constructing equilibrium distributions in the Colonel Blotto game with heterogeneous battlefield values, generalising known construction methods. Using results on the existence of tangential polygons, it derives necessary and sufficient conditions for the irregular N-gon method to be applied, given the parameters of a Blotto game. The method does particularly well when the battlefield values satisfy some clearly defined regularity conditions. The paper establishes the parallel between these conditions and the constrained integer partitioning problem in combinatorial optimisation. The properties of equilibrium distributions numerically generated using the irregular N-gon method are illustrated. They indicate that the realised allocations, weighted by battlefield value, are less egalitarian and depend more strongly on battlefield values than previously thought. In the context of the U.S. presidential elections the explicit construction of equilibria provides new insights into the relation between the size of a state and the campaign resources spent there by presidential candidates.

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1 Introduction

Budget-constrained multidimensional allocation problems were amongst the very first considered in game theory, starting with Borel and Ville (1938). These problems came to be known as "Colonel Blotto" games, after the interpretation of Gross and Wagner (1950). In the simplest version of the Colonel Blotto game, two generals dispose of one divisible unit of military resources each, and face each other on three battlefields of equal value. A battlefield is captured by the general who allocates more resources there than his opponent. Each general aims to maximise the number of captured battlefields. Blotto games have since been used to model military and system defense, as well as electoral competition and R&D races.

This paper considers specifications of the Blotto game in which the players maximise the expected total value of captured battlefields, and there are $N \geq 3$ *heterogeneous* battlefields. That is, every battlefield has the same value for both players, but that value may vary across battlefields. For instance, in a system network a hub may be of greater value – both to the attacker and the defender – than its spokes. In U.S. presidential elections, the different states carry different numbers of electoral votes.

In these Blotto games, a pure strategy exhausting a player's budget is an N-dimensional allocation vector belonging to the $(N-1)$-dimensional simplex, Δ^{N-1}. A mixed strategy is a probability measure on Δ^{N-1} captured by the N-variate cumulative distribution function $F : \Delta^{N-1} \to [0,1]$. A sufficient condition for Nash equilibrium is that the N one-dimensional marginals of F be uniform over the support $[0, 2v_n/(\sum_{n=1}^{N} v_n)]$, where v_n denotes the value of battlefield n.

The first contribution of this paper is to introduce the *irregular N-gon* method, a new geometric method for constructing Nash equilibrium strategies. The method involves projecting a point uniformly distributed on the surface of a sphere onto a two-dimensional support. It relies on the existence of an N-gon that admits an inscribed circle and whose sides have lengths given by the N battlefield values. I provide necessary and sufficient conditions on the N-vector of battlefield values under which such an N-gon exists, ensuring the irregular N-gon construction method can be applied.

Reordering the elements of a given *set* of battlefield values, $\{v_1, \ldots, v_N\}$, allows us to generate a large number of different *vectors* of battlefield values $(v_{\pi(1)}, \ldots, v_{\pi(N)})$, where π is a permutation of the indices $1, \ldots, N$. Any such vector satisfying the conditions for the existence of a suitable associated N-gon can be used to construct an equilibrium strategy using the irregular N-gon solution. Generically, when the irregular N-gon method can be applied, it yields multiple equilibrium strategies. However, it may be possible that every reordering of the elements of $\{v_1, \ldots, v_N\}$ fails the conditions. The existence of a suitable

N-gon is therefore not guaranteed.

The second contribution of this paper is to establish that a set of battlefield values admits a reordering generating a suitable N-gon if it admits a solution to an associated *constrained integer partitioning problem*. In general, establishing whether an integer partitioning problem admits a solution is computationally cumbersome when N grows large. Based on results in the literature on combinatorial optimisation, I derive a simple heuristic for establishing whether a solution to the appropriate integer partitioning problem exists, and thus whether the irregular N-gon method can be used to construct Nash equilibria of the Blotto game.

The third contribution of this paper is to study the distributional properties of the equilibrium allocation generated using the irregular N-gon method, and compare them with those obtained under the previously known construction method of Gross (1950) and Laslier (2002), which partitions battlefields into three groups. Focusing on the allocations relative to battlefield size – which I interpret as per-capita allocations with in mind the example of the U.S. presidential elections where the size of a battlefield is given by its population – I find the following differences. Under the irregular N-gon method, the variation in the ranking of a battlefield according to realised per-capita allocation increases with the battlefield size. Second, the outcome is less egalitarian ex-post. And most strikingly, the per-capita allocations to smaller battlefields are positively correlated. This is surprising, given that the budget constraint is a force making for negative correlation.

Finally, I apply the irregular N-gon method in the context of U.S. presidential elections, in which two candidates compete over 51 battlefields[1] with differing numbers of electoral votes. I assume that the candidates maximise the expected total number of electoral votes won. Concentrating on the 11 swing states of the 2012 election, I numerically construct the equilibrium distribution of campaign resources across states. There is considerable inequity in realised per-capita allocations, with the most neglected half of the population on average receiving only 25% of campaign resources. Moreover, while smaller states are more likely to be allocated intermediate shares, larger states are more likely to be allocated either the highest or the lowest share of per-capita resources. Finally, the per-capita allocations of campaign resources are positively correlated amongst smaller states.

The general appeal of geometric methods for constructing N-dimensional distributions lies in the relative simplicity with which they describe complicated multi-dimensional objects, by using two-dimensional supports. The drawback is that they may fail to generate the full set of distributions satisfying given restrictions on support and marginal distributions.

[1] There are fifty states, plus the District of Columbia.

The geometric method involving the projection of a sphere onto a disk inscribed in an N-gon is described for the case of $N = 3$ heterogeneous battlefields in Borel and Ville (1938) and Gross and Wagner (1950). This was extended by Laslier and Picard (2002) to the case of $N \geq 3$ homogeneous battlefields. In the case of $N \geq 3$ heterogeneous battlefields considered in this paper, Gross (1950) and Laslier (2002) show that the construction method for $N = 3$ heterogeneous battlefields can be used, after partitioning the N battlefields into 3 subsets. In this construction the allocations to battlefields that belong to a common subset are *perfectly* correlated. This is not the case with the irregular N-gon method, and this paper provides a numerical example illustrating the differences, and particularly the equity and correlation properties, of equilibrium distributions constructed using the irregular N-gon method, when compared with the construction method of Gross (1950) and Laslier (2002). Kvasov (2007) discusses the correlation properties arising in the equilibria of N-dimensional all-pay auctions as a result of the players' budget constraints.

Weinstein (2012) proposes a construction method directly in $\mathbb{R}^N$, for the case of N homogeneous battlefields.[2] The mixed strategy equilibria in first-price sealed-bid auctions for objects with complementarities studied in Szentes and Rosenthal (2003) also involve uniform distributions over the surface of three-dimensional objects and uniform marginals.

Roberson (2006), Roberson and Kvasov (2012) and Kovenock and Roberson (2015) exploit the link between the Blotto game and N-dimensional all-pay auctions. They introduce copulas to map N one-dimensional marginals into N-dimensional distributions.[3] They establish the existence of a copula satisfying the budget constraint, given marginals satisfying the equilibrium conditions for the case of N homogeneous battlefields, and allowing players to have different budgets. Schwartz et al. (2014) and Kovenock and Roberson (2015) show that the results of Roberson (2006) extend to heterogeneous battlefields, if there are at least three battlefields of each possible value.[4] In general, however, the existence of a suitable copula in the case of N heterogeneous battlefields remains open.

The conditions for the existence of an N-gon admitting an inscribed circle are derived in Radić (2002).[5] My paper links the existence of an N-gon suitable for applying the irregular N-gon method to a constrained version of the NP-complete "integer partitioning problem". I use results derived in Borgs et al. (2004) (see also Borgs et al. (2001)) to

[2] Weinstein (2012) also considers a game with N homogeneous battlefields in which one player's budget is r times that of his opponent. That paper derives bounds on the equilibrium payoff and shows that they are tight when r is close to 1.

[3] In addition they treat the case of $N = 2$, as do Macdonell and Mastronardi (2012).

[4] Kovenock and Roberson (2015) allow for battlefield values to differ both across battlefields (heterogeneous values) and across players (asymmetric values).

[5] I had independently derived the results of Section 3.2. I thank an anonymous referee on an earlier version of this paper for bringing Radić's work on tangential polygons to my attention.

develop a heuristic assessment of the reliability of the irregular N-gon method, given any parametrisation of the battlefield values.

The Blotto game has frequently been used to model electoral competition. Laslier and Picard (2002) and Laslier (2002) focus on the distributional properties[6] of equilibria. So do Myerson (1993), which corresponds to a Blotto game with a continuum of homogeneous battlefields, and Sahuguet and Persico (2006), who consider two candidates differing in valence and assume that some form of proportional representation is possible. Brams and Davis (1974) propose modeling the allocation of campaign resources in the U.S. presidential elections as a Blotto game, although it is considered debatable[7] whether presidential candidates indeed maximise the expected plurality in their favour. Laffond et al. (1994) and Laslier (2000) provide arguments supporting this view. An alternative view is that candidates maximise their likelihood of winning a simple majority in the election. In a model where the probability of winning a battlefield is given by a logit contest success function, Snyder (1989) shows that the two modeling assumptions imply different equilibrium behaviour, particularly if the effect of a candidate's campaign spending on votes is allowed to vary across states.[8] Equilibrium existence in Blotto games with the simple-majority objective is established in Duggan (2007) and Barelli et al. (2014).

Brams and Davis (1974) consider a Blotto game in which the probability of a player capturing a battlefield is given by a linear Tullock contest success function. That paper does not study the mixed strategy Nash equilibria of the game, and instead concentrates on the pure strategy symmetric minmax strategy profile. The authors derive the "3/2's rule", which implies that the share of campaign resources allocated to larger states is disproportionately large relative to these states' electoral votes. I show that mixed strategy Nash equilibria of the Blotto game do not support the "3/2's rule", nor the notion that larger states receive disproportionately large shares of resources. However, in the equilibrium I construct, the variance in the ranking of a state according to allocated per-capita share of campaign resources tends to increase with its size. More generally, the largest and smallest per-capita shares are most likely to go to larger states, while intermediate shares are more likely to go to smaller states. In other words, candidates tend to focus on "all or nothing" policies in large swing states.

Discrete versions of the Blotto game in which a player's budget consists of indivisible

[6] De Donder (2000) uses numerical simulations to compare the redistributive properties of different solution concepts in majority voting games.

[7] See Colantoni et al. (1975a), Brams and Davis (1975) and Colantoni et al. (1975b). See also Laslier (2005).

[8] Duffy and Matros (2015) study the Blotto game with heterogeneous battlefield values and a logit contest success function for total expected payoff and majority rule objectives.

units of resources are analysed in Hart (2008), Dziubiński (2013) and Hart (2014). They provide the basis for experimental studies[9] of the game.

Section 2 describes the Blotto game with heterogeneous battlefields and re-states a known result characterising its Nash equilibria. Section 3 defines the irregular N-gon solution. It gives necessary and sufficient conditions (due to Radić) for the construction method to be applied. Proposition 2 in Section 3.3 relates these to the integer partitioning problem. The correlation and equity properties of the resulting distributions are described in Section 3.5. I apply my results to analyse the U.S. presidential election in Section 4.

2 Model and Equilibrium

Two players with identical budgets normalised to B choose how to allocate their resources across $N \geq 3$ battlefields indexed by $n \in \{1, ..., N\}$. For both players, the value of battlefield n is $v_n \in (0, \infty)$, and $B := \sum_{n=1}^{N} v_n$. If $v_n = v_{n'}$ for all $n, n' \in \{1, \ldots, N\}$, then battlefields are homogeneous, otherwise they are heterogeneous. Assume that $v_n < \sum_{k \neq n} v_k$, for all $n = 1, ..., N$.

Player $i \in \{1, 2\}$ chooses a nonnegative vector of allocations $\boldsymbol{x}^i = (x_1^i, ..., x_N^i)$, where x_n^i denotes the fraction of resources allocated to battlefield n. Player i captures battlefield n if the share x_n^i of resources he allocates to that battlefield exceeds the share x_n^j of resources allocated there by his opponent. Ties are resolved by flipping a fair coin. The sum of a player's resources allocated across all battlefields cannot exceed that player's budget.

A pure strategy for player i is an N-dimensional vector $\boldsymbol{x}^i$ satisfying the budget constraint, $\sum_{n=1}^{N} x_n^i B \leq B$. Let $\mathcal{S}^i$ denote the set of player i's pure strategies:

$$\mathcal{S}^i = \left\{ \boldsymbol{x}^i \in [0, 1]^N : \sum_{n=1}^{N} x_n^i \leq 1 \right\}.$$

An allocation that satisfies the budget constraint with equality belongs to the $(N-1)$-dimensional simplex denoted Δ^{N-1}.

Each player seeks to maximise the aggregate value of captured battlefields. The function $g : \mathcal{S}^i \times \mathcal{S}^j \to \mathbb{R}$ measures the excess aggregate value of battlefields captured by player i if he plays the pure strategy $\boldsymbol{x}^i$ against player j's pure strategy $\boldsymbol{x}^j$:

$$g(\boldsymbol{x}^i, \boldsymbol{x}^j) = \sum_{n=1}^{N} v_n \ sgn(x_n^i - x_n^j), \tag{1}$$

[9] See Avrahami and Kareev (2009), Arad and Rubinstein (2012), Chowdhury et al. (2013), Montero et al. (2016) and Casella et al. (2016).

with $sgn(u) = 1$ if $u > 0$, 0 if $u = 0$ and -1 if $u < 0$.

A mixed strategy for player i is an N-variate joint distribution over $\mathcal{S}^i$. Let $F^i : \mathcal{S}^i \to [0,1]$ denote its cumulative distribution function (c.d.f.), and let $F^{i,(n)} : [0,1] \to [0,1]$ denote the n^{th} one-dimensional marginal of F^i, i.e. the unconditional c.d.f. of x_n^i, player i's allocation to the n^{th} battlefield. Player i's payoff to a mixed strategy is defined as the expectation of $g(\boldsymbol{x}^i, \boldsymbol{x}^j)$ with respect to the strategy profile (F^i, F^j).

The following proposition describes the equilibrium of this game.

Proposition 1. *(Gross (1950), Laslier (2002)) Consider the Colonel Blotto Game with heterogeneous battlefield values.*

(i) This game has no pure-strategy Nash equilibrium.

(ii) Each player's resource constraint binds in equilibrium.

(iii) Let F^* be a probability distribution of $\boldsymbol{x} \in \Delta^{N-1}$ such that each vector coordinate x_n, $n = 1, ..., N$, is uniformly distributed on $[0, 2v_n/B]$. The strategy profile (F^*, F^*) constitutes a symmetric Nash equilibrium.

The first point implies that an equilibrium, if it exists, must be in mixed strategies. The second point implies that the support of any equilibrium strategy is (a subset of) the $(N-1)$-dimensional simplex. Point three states that having univariate marginal distributions that are uniform on $[0, 2v_n/B]$ is a sufficient condition for a mixed strategy with support Δ^{N-1} to constitute a symmetric Nash equilibrium.

3 Constructing Nash equilibrium strategies

This section introduces a new geometric method for constructing a Nash equilibrium strategy as defined in Proposition 1. Let us refer to it as the *irregular N-gon* solution or method, as it relies on the existence of an N-gon that admits an inscribed circle and whose sides have lengths given by the vector $(v_1, \ldots, v_N)$ of battlefield values. We begin by describing the construction method, assuming the existence of such an N-gon. Necessary and sufficient conditions on the vector of battlefield values are then given under which such an N-gon exists and the irregular N-gon construction method can be applied.

Reordering the elements of the set $\{v_1, \ldots, v_N\}$ allows us to generate a large number of different vectors of battlefield values. Any of these vectors satisfying the conditions for the existence of a suitable associated N-gon can be used to construct an equilibrium strategy, using the irregular N-gon solution. However, it may be possible that every reordering of the elements of $\{v_1, \ldots, v_N\}$ fails these conditions. The existence of a suitable N-gon is therefore not guaranteed.

Proposition 2 establishes that a set $\{v_1, \ldots, v_N\}$ admits a reordering generating a suitable N-gon if it admits a solution to an associated constrained integer partitioning problem. In general, establishing whether a given integer partitioning problem admits a solution is computationally cumbersome when N grows large. Based on results in the literature on combinatorial optimisation, I provide a simple heuristic for establishing, given a set $\{v_1, \ldots, v_N\}$ of battlefield values, whether a solution to the appropriate integer partitioning problem exists, and thus whether the irregular N-gon method can be used to construct Nash equilibria of the Blotto game.

Generically, when the irregular N-gon method can be applied, it yields multiple equilibrium strategies. This multiplicity has many causes, which are discussed in Section 3.4. Moreover, further equilibrium strategies may be obtained using other construction methods. I provide a numerical example illustrating the differences, particularly in their correlation properties, of equilibrium distributions constructed using the irregular N-gon method, when compared with the construction method of Gross (1950) and Laslier (2002).

In what follows, the set $\mathcal{V} := \{v_1, \ldots, v_N\}$ of N battlefield values parametrises a Blotto game.[10] Without loss of generality, assume that the indices $n = 1, \ldots, N$ are such that $v_1 \leq v_2 \leq \cdots \leq v_N$. Let π denote a permutation of the indices $1, \ldots, N$ and let $\boldsymbol{w}^\pi = (w_1^\pi, \ldots, w_N^\pi) := (v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(N)})$ denote the vector of battlefield values associated with the permutation π. For notational simplicity, whenever this is unambiguous I will suppress the dependence of $\boldsymbol{w}$ on π. Moreover, in a slight abuse of language, I refer to a vector $\boldsymbol{w}$ as "a permutation" of the vector $\boldsymbol{v} := (v_1, \ldots, v_N)$.

3.1 The irregular N-gon solution

Fix a permutation $\boldsymbol{w}$ or $\boldsymbol{v}$. The following process generates an N-dimensional random vector $\boldsymbol{X} = (X_1, \ldots, X_N)$ in the $(N-1)$-dimensional simplex, Δ^{N-1}, such that for each $n = 1, \ldots, N$, X_n is distributed uniformly over $[0, 2w_n/B]$. Consider a tangential polygon $\boldsymbol{P}$ with vertices $P_1, \ldots, P_N$ (in this order[11]) such that the sides have lengths $|P_{n-1}P_n|$ equal to w_n, for each $n = 1, \ldots, N$. A polygon is *tangential* if there exists a circle $\mathscr{C}$ such that each side of $\boldsymbol{P}$ lies on a tangent line of $\mathscr{C}$. Let O and r denote the center and radius of $\mathscr{C}$. The disk of centre O and radius r is the projection of the sphere $\mathscr{S} \in \mathbb{R}^3$ onto the plane supporting the polygon $\boldsymbol{P}$. Finally, let R be a generic point uniformly distributed on the surface of the sphere $\mathscr{S}$ and let P be the orthogonal projection of R onto the plane supporting $\boldsymbol{P}$.

[10] In a slight abuse of terminology I allow $\mathcal{V}$ to contain multiple elements of the same value. That is, I allow for the possibility that two battlefields have the same value.

[11] All indices are calculated modulo N.

For all n, h_n is the distance of P from the side (P_{n-1}, P_n). The following lemma is a well-known result in geometry.

Lemma 1. *If R is uniformly distributed on the surface of the sphere $\mathscr{S}$, then for each $n = 1, \ldots, N$, $X_n := (h_n w_n)/(rB)$ is uniformly distributed on $[0, 2w_n/B]$.*

Proof. Let $\mathscr{H}_n$ denote the plane that is tangent to the sphere $\mathscr{S}$ and whose projection onto the plane supporting $\boldsymbol{P}$ is the side (P_{n-1}, P_n) of $\boldsymbol{P}$. For all $t \in [0, 2r]$, the *spherical cap of height* t is the region of the sphere $\mathscr{S}$ that lies between the vertical plane $\mathscr{H}_n$ and the vertical plane intersecting $\mathscr{S}$ that is parallel to $\mathscr{H}_n$ and a distance t away from it. Then, for all $t \in [0, 2r]$, $\Pr(h_n < t) = \Pr(R \in \text{cap of height } t)$. Since R is uniformly distributed on the surface of the sphere, this probability equals the surface area of the cap of height t divided by the total surface area of the sphere:

$$\Pr(h_n < t) = \frac{2\pi \int_0^t r \; dx}{2\pi \int_0^{2r} r \; dx} = \frac{t}{2r},$$

and h_n is distributed uniformly on $[0, 2r]$. Consequently, the variable $X_n := (h_n w_n)/(rB)$ is distributed uniformly on $[0, 2w_n/B]$, for each $n = 1, \ldots, N$. □

3.2 Existence conditions

Lemma 1 allows to construct an equilibrium strategy when battlefield values are given by the vector $\boldsymbol{w}$, supposing there exists an associated tangential N-gon $\boldsymbol{P}$ with vertices $P_1, \ldots, P_N$ such that the sides have lengths $|P_{n-1}P_n|$ equal to w_n, for each $n = 1, \ldots, N$. Theorem 1 provides necessary and sufficient conditions on $\boldsymbol{w}$ for the existence of an associated tangential N-gon.

First note that the restriction $w_n < B/2, \; \forall n = 1, \ldots, N$, ensures that a convex N-gon with sides of lengths given by $(w_1, \ldots, w_N)$ exists. However, not every polygon admits an inscribed circle. Given a positive integer κ, we say that a tangential polygon $\boldsymbol{P}$ is κ-tangential if:

$$\sum_{n=1}^{N} \phi_n = 2\kappa\pi,$$

where $\phi_n = \angle P_n O P_{n+1}$ for each $n = 1, \ldots, N$, and O is the centre of the circle inscribed into $\boldsymbol{P}$. Radić (2002) derives the following necessary and sufficient conditions for the existence of a tangential polygon, given a vector $(w_1, \ldots, w_N)$ of lengths.

Theorem 1. *(Radić (2002)) Let $w_1, \ldots, w_N$, $t_1, \ldots, t_N$ be any given lengths (in fact positive numbers) such that*

$$t_n + t_{n+1} = w_n, \quad n = 1, \ldots, N; \tag{2}$$

and let κ be a positive integer such that $\kappa \leq \lfloor \frac{N-1}{2} \rfloor$. Then there exists a κ-tangential polygon whose sides have lengths $w_1, \ldots, w_N$.

Let T_n denote the point of tangency of the side (P_n, P_{n+1}) of the tangential polygon $\boldsymbol{P}$ and its incircle $\mathscr{C}$. The length t_n then equals $|P_n T_n|$, and the theorem above follows from the fact that the distances $|P_n T_n|$ and $|P_n T_{n-1}|$ must be equal.

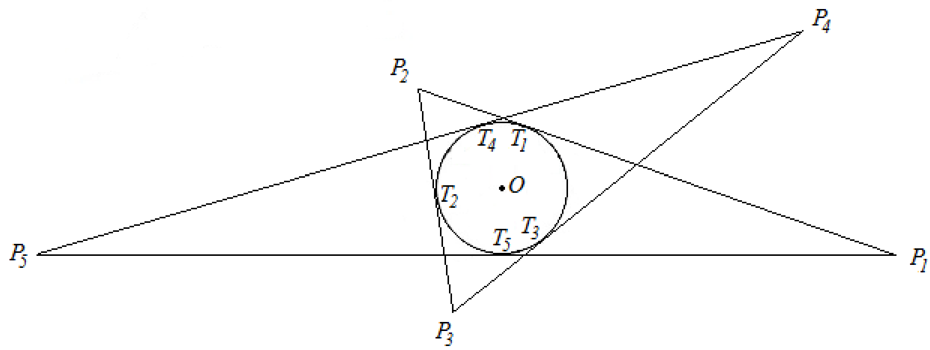

Figure 1: *A 2-tangential polygon for $N = 5$.*

We have the following two corollaries, depending on whether N is odd or even.

Corollary 1. *(Radić (2002)) Let $N \geq 3$ be an odd number, and let $w_1, \ldots, w_N$ be given lengths. Then for any $\kappa \leq \frac{N-1}{2}$ there exists a κ-tangential polygon whose sides have lengths $w_1, \ldots, w_N$ if and only if*

$$\sum_{k=0}^{N-1} (-1)^{2+k} \; w_{n+k} > 0, \quad n = 1, \ldots, N. \tag{3}$$

This can be derived from the fact that the solution to the system (2) is unique and given by

$$2t_n = \sum_{k=0}^{N-1} (-1)^{2+k} \; w_{n+k}, \quad n = 1, \ldots, N. \tag{4}$$

Corollary 2. *(Radić (2002)) Let $N \geq 4$ be an even number, and let $w_1, \ldots, w_N$ be given lengths. Then for any $\kappa \leq \frac{N-2}{2}$ there exists a κ-tangential polygon whose sides have lengths $w_1, \ldots, w_N$ if and only if the following two conditions hold:*

$$\sum_{k=1}^{N} (-1)^k \; w_k = 0, \tag{5}$$

$$\min\{b_1, b_3, \ldots, b_{N-1}\} > \max\{b_2, b_4, \ldots, b_N\}, \tag{6}$$

where

$$b_k = \sum_{n=1}^{k} (-1)^{n+1}\ w_n, \quad k = 1, \ldots, N.$$

Conditions (4) and (6) determine the location of the tangency points of the inscribed circle with the tangential polygon. When N is odd, given $w_1, \ldots, w_N$, the solution to the system (2) is unique and t_n is given by (4) for each $n = 1, \ldots, N$. When N is even, t_1 belongs to the interval defined in (6) and the remaining t_n, for $n = 2, \ldots, N$, are pinned down by the system (2). (See Appendix 6.1.) Observe that consequently, given $w_1, \ldots, w_N$, and for each $\kappa \leq \frac{N-2}{2}$, if N is even and if (5) and (6) are satisfied, it is possible to build an infinity of κ-tangential polygons, by varying t_1.

Conditions (3) and (5) require that the N-gon with sides given by $w_1, \ldots, w_N$ be balanced in the following sense. When N is even, the summed lengths of odd sides must equal the summed lengths of even sides. When N is odd, choosing any side n and alternately partitioning the remaining sides into two sets, the difference between the summed lengths within each set must not exceed w_n. One interpretation for these conditions is that there must exist a binary partition of the battlefields such that each battlefield in each subset is pivotal.

3.3 The constrained integer partitioning problem

Faced with a Blotto game parametrised by the set $\mathcal{V}$ of battlefield values, will we be able to apply the irregular N-gon method for constructing Nash equilibrium strategies? Equivalently: Does there exist a permutation $\boldsymbol{w}$ of $\boldsymbol{v}$ such that the conditions of Theorem 1 for the existence of a tangential polygon with sides of lengths given by the vector $\boldsymbol{w}$ are satisfied? This section relates this question to the constrained integer partitioning problem.

First, observe that the conditions of Theorem 1 are satisfied by every $\boldsymbol{v} \in \mathbb{R}^N$ when $N = 3$. This corresponds to the case treated in Gross and Wagner (1950). The conditions of Theorem 1 are also satisfied for any $N \geq 3$ in the case of homogeneous battlefields, i.e. when $v_n = v$ for each $n = 1, \ldots, N$. This is treated in Laslier and Picard (2002).[12] For $N \geq 3$, as we depart from the regular N-gon, these conditions become harder to meet, and there exist instances of $\boldsymbol{v}$ for which these conditions always fail.[13]

To find all permutations $\boldsymbol{w}$ of a given $\boldsymbol{v}$ which satisfy Theorem 1, one could just parse, one by one, all possible permutations[14]. This "brute force" method becomes cumbersome

[12] Every triangle admits an inscribed circle, as does every regular N-gon.

[13] For instance, $\boldsymbol{v} = (1, 1, 1, 2)$.

[14] Treating a permutation (e.g. (x, y, z)), its cyclic shifts ((y, z, x) and (z, x, y)), and the order-reversing

when N grows large, as the number $(N-1)!/2$ of possible permutations grows faster than exponentially. In Appendix 6.2 we propose an algorithm that does better than this brute force method. One might still wish to assess, given a vector $\boldsymbol{v}$ of battlefield values, whether a suitable permutation $\boldsymbol{w}$ satisfying Theorem 1 is likely to even exist. I therefore propose the following simple heuristic.

Observe that if, as we have assumed until now, $v_n \in (0, \infty)$ for each $n = 1, \ldots, N$, then, generically, (5) cannot be satisfied. For the remainder of this paper, let us therefore restrict attention to integer-valued battlefields and assume that v_n is a positive integer for all $n = 1, \ldots, N$. This is a reasonable assumption in many applications. For instance, the number of electoral votes in electoral systems is integer-valued. Moreover, the argument also extends to the positive rationals. Since the number of battlefields is finite, we can always re-scale rational values so that they become integers.

In that case, more can be said about when (3) and (5) are likely to be satisfied by some permutation of $\boldsymbol{v}$ by noting that these conditions are related to the *constrained integer partitioning problem*, a well-known problem of combinatorial optimisation, in which a given set of N integers must be partitioned into two subsets of given cardinalities such that the *discrepancy*, the absolute value of the difference between the sum of integers in each subset, is minimised.

Proposition 2. *Fix a set $\mathcal{V} := \{v_1, \ldots, v_N\}$ of N battlefield values.*

(i) For N even, there exists a permutation $\boldsymbol{w}$ of $\boldsymbol{v}$ satisfying the conditions of Theorem 1 if and only if $\mathcal{V}$ admits a solution to the following integer partitioning problem:

***(P1)**: Partition the N elements of $\mathcal{V}$ into two subsets of equal cardinality such that the discrepancy is zero.*

(ii) For N odd, there exists a permutation $\boldsymbol{w}$ of $\boldsymbol{v}$ satisfying the conditions of Theorem 1 only if there exists a solution to the following integer partitioning problem:

***(P2)**: For each $n = 1, \ldots, N$, partition the $N-1$ elements of $\mathcal{V} \setminus \{v_n\}$ into two subsets of equal cardinality such that the discrepancy is less than v_n.*

Proof. (i) The set $\mathcal{V}$ satisfies the integer partitioning problem (P1) if and only if there exists a permutation $\boldsymbol{w}$ of its elements such that:

$$w_1 + w_3 + \cdots + w_{N-1} = w_2 + w_4 + \cdots + w_N.$$

The condition above is equivalent to condition (5).

permutations of each $((z, y, x)$, (x, z, y) and $(y, x, z))$ as identical, the N elements of a set admit $(N-1)!/2$ distinct permutations.

(ii) A permutation $\boldsymbol{w}$ of $\boldsymbol{v}$ satisfies the N conditions in (3) if and only if, for each $n = 1, \ldots, N$,

$$w_n > \sum_{k=1}^{\frac{N-1}{2}} w_{n+2k-1} - \sum_{k=1}^{\frac{N-1}{2}} w_{n+2k}$$

Clearly, a necessary condition is that the set $\mathcal{V}$ satisfies the integer partitioning problem (P2). However, this condition is not sufficient. To satisfy (3) the solutions to the N partitioning problems in (P2) must coincide in a particular way. Appendix 6.3 provides an example of a set $\mathcal{V}$ satisfying the integer partitioning problem (P2), even though there exists no permutation of its elements satisfying condition (3).

$\square$

The problems (P1) and (P2) are computationally hard. The unconstrained partitioning problem is NP-complete, and while some algorithms deliver good approximations of the optimal partition (that is, the partition with the lowest possible discrepancy), the brute force algorithm that compares the discrepancies of all possible partitions is still the best known solution to the problem. Borgs et al. (2004) study the typical behaviour of the optimal partition when the N integers are i.i.d. random variables chosen uniformly from the set $\{1, ..., 2^m\}$ for some integer m. They identify two "phases" for our constrained problems, depending on their computational difficulty.

For N and m tending to infinity while keeping the ratio m/N constant, with probability tending to one there exists a perfect partition, i.e. a partition with discrepancy 0 or 1, when $m/N < 1$. They call this the *perfect phase* of the problem. In that phase, the number[15] of perfect partitions is about $2^{(N-m)}$. For $m/N > 1$, in what the authors call the *hard phase* of the problem, the probability of a perfect partition tends to zero, presumably making computation of the optimal partition more difficult there.

For the purpose of this paper, the results of Borgs et al. (2004) allow the conclusion that conditions (3) and (5) are likely to be more easily satisfied for $m/N < 1$, when there is a relatively large number of battlefields and their values are close, than for $m/N > 1$, when there is a relatively small number of battlefields and their values differ greatly. The limiting case in which $v_n = 1/N$ for all $n = 1, \ldots, N$, and $N \to \infty$ clearly has $m/N \to 0$ and corresponds to the setup in Myerson (1993).

For $m/N > 1$, while it is possible for (3) to be satisfied, it is almost never possible for (5) to be satisfied. We express this in Table 1 by stating that the partitioning problems (P1) and (P2) are more "easily" satisfied when $m < N$ and that we consequently expect to find more permutations $\boldsymbol{w}$ of the vector $\boldsymbol{v}$ of battlefield values that satisfy the conditions

[15] The number of perfect partitions in the perfect phase is lower than in the limiting case where $N \to \infty$ by about twenty percent for a given ratio m/N.

of Corollaries 1 and 2 than for $m > N$, making the irregular N-gon method easier to use. This offers a pragmatic rule of thumb for assessing, given a vector $\boldsymbol{v}$ of battlefield values, how reliable the irregular N-gon method is likely to be.

	$m < N$	$m > N$
(P1)	easy	impossible
(P2)	easy	hard

Table 1: *The heuristic for assessing how reliable our geometric construction method is likely to be, given a vector* $\boldsymbol{v}$ *of* N *i.i.d. battlefield values chosen uniformly from* $\{1, \ldots, 2^m\}$.

The following three examples illustrate the use of this heuristic.[16] Consider the set $\mathcal{V}_1 := \{5, 7, 8, 9, 12\}$ of integers uniformly and independently drawn[17] from the range 1 to 16. Our heuristic based on Borgs et al. (2004) indicates that $\mathcal{V}_1$ should admit a solution to the integer partitioning problem (**P2**), since $N = 5$ and $m = 4$. Indeed, there are six permutations of the elements of $\mathcal{V}_1$ that satisfy the conditions of Theorem 1, out of a possible $(N - 1)!/2 = 12$. They are listed in Section 3.5.

In contrast, consider the sets $\mathcal{V}_2 := \{13, 17, 38, 46, 63\}$ and $\mathcal{V}_3 := \{2, 15, 37, 46, 62\}$ of integers uniformly and independently drawn[18] from the range 1 to 64. In both cases, $m = 6$ (since $2^6 = 64$) and $N = 5$ so that $m/N = 6/5 > 1$. The heuristic based on Borgs et al. (2004) indicates that both sets $\mathcal{V}_2$ and $\mathcal{V}_3$ are in the hard phase: there is only a very small chance of finding even one perfect partition. Even though for N odd, the irregular N-gon method does not require a perfect partition, it requires that there exists at least one permutation $\boldsymbol{w}$ of $\boldsymbol{v}$ that satisfies the constrained integer partitioning problem (P2).

In the case of $\mathcal{V}_2$, we find two permutations, $(13, 38, 63, 46, 17)$ and $(17, 38, 63, 46, 13)$, that satisfy (P2). Computing all possible cyclical permutations the elements of $\mathcal{V}_3$ we find that none satisfies ($P2$) or the conditions of Theorem 1, and we cannot use the irregular N-gon method to derive a Nash equilibrium strategy in this case, even though such a

[16] Although in all three cases, $N = 5$ and the $(N-1)!/2 = 12$ possible permutations of $\boldsymbol{v}$ can be quickly parsed.

[17] The null hypothesis that the data is distributed according to the uniform distribution on $[0, 16]$ cannot be rejected up to the 61 percent level based on the Kolmogorov-Smirnov test. (The p-value is 0.614616.)

[18] For $\mathcal{V}_2$ the null hypothesis that the data is distributed according to the uniform distribution on $[0, 64]$ cannot be rejected up to the 95 percent level based on the Kolmogorov-Smirnov test. (The p-value is 0.95664.) For $\mathcal{V}_3$ the null hypothesis that the data is distributed according to the uniform distribution on $[0, 64]$ cannot be rejected up to the 98 percent level based on the Kolmogorov-Smirnov test. (The p-value is 0.987942.)

distribution exists.[19]

3.4 Multiplicity of equilibrium strategies

Given a set of battlefield values $\mathcal{V}$, the irregular N-gon method may yield a multiplicity of N-variate distribution functions constituting Nash equilibria of the corresponding Blotto game. There are numerous sources of multiplicity.

1. The irregular N-gon method requires the existence of a tangential polygon with sides of lengths given by $w_1, \ldots, w_N$ for some suitable permutation $\boldsymbol{w}$ of the vector $\boldsymbol{v}$. For any given $\mathcal{V}$ there may be many distinct suitable permutations satisfying the conditions of Theorem 1 for the existence of a κ-tangential polygon.

2. Fix $\mathcal{V}$ and fix a permutation $\boldsymbol{w}$ of $\boldsymbol{v}$ such that $v_{\pi(1)}, \ldots, v_{\pi(N)}$ satisfy the conditions of Theorem 1. When N is even, there exists an infinity of associated lengths $t_1, \ldots, t_N$ satisfying (2), each generating a different κ-tangential polygon. (When N is odd, the lengths $t_1, \ldots, t_N$ are uniquely determined by (4).)

3. For any given $\mathcal{V}$, $w_1, \ldots, w_N$ and $t_1, \ldots, t_N$ satisfying Theorem 1, letting κ vary generates $\left\lfloor \frac{N-1}{2} \right\rfloor$ distinct κ-tangential polygons.

Distinct polygons[20] generate joint distributions with different correlation properties, as discussed in the next section. The irregular N-gon method only generates a unique equilibrium strategy in the case $N = 3$, as $\left\lfloor \frac{N-1}{2} \right\rfloor = 1$ so that κ may only take the value 1, N is odd so that the unique vector $(t_1, \ldots, t_N)$ satisfying (2) is given by (4), and there exists only one relevant permutation of 3 battlefield values.

Finally observe that, since there exist other methods for constructing multivariate distributions satisfying the equilibrium conditions, our method, even when feasible, might not describe the entire set of equilibrium strategies.

3.5 Equity and correlation

The restrictions of Proposition 1 on the support and marginal distributions of equilibrium strategies limits the number of possible interactions between the resource allocations to different battlefields. The fact alone that, in equilibrium, players only use strategies in the $(N-1)$-dimensional simplex, Δ^{N-1}, which does not include the whole of $\times_{n=1}^{N}[0, 2v_n/B]$, precludes some correlation structures for the joint distribution. The particulars of any geometric construction method are liable to limit these interactions further. So far, it has not

[19] It can be constructed using the construction method of Gross (1950) and Laslier (2002).

[20] These may be identical-looking polygons generated by different permutations of the battlefield values.

been possible to fully characterise the set of possible correlations satisfying the restrictions on equilibrium strategies, or to disentangle the effects of the equilibrium restrictions from those of the construction method.

To illustrate this point, consider the case of homogeneous battlefields with $v_i = v$ for each $i = 1, \ldots, N$ and with $N \geq 3$. Given the lengths $t_1, \ldots, t_N$ satisfying (4) or (6) as appropriate, there are $(N-1)!/2$ distinct permutations π of the indices $1, \ldots, N$ such that $(v_{\pi(1)}, \ldots, v_{\pi(N)})$ satisfies Theorem 1. Given any such π, and for $\kappa = 1$, the resources allocated to battlefield $\pi(n)$, $n \in \{1, \ldots, N\}$ under the irregular N-gon method will be (more) positively correlated with the allocation to battlefields on adjacent sides of the N-gon, $\pi(n+1)$ and $\pi(n-1)$, and (more) negatively correlated with the allocation to battlefields on opposite sides of the N-gon $\pi\left(n + \lfloor N/2 \rfloor\right)$ and $\pi\left(n - \lfloor N/2 \rfloor\right)$. (Other correlation properties arise for $\kappa > 1$). To rectify this, Laslier and Picard (2002) propose averaging out over all permutations π such that the corresponding $\boldsymbol{w}^{\pi}$ satisfy Theorem 1. (Although, they only consider 1-tangential polygons with $t_n = v/2$ for each $n = 1, \ldots, N$, ignoring further possible values of κ and, when N is even, of t_n.)

Using a numerical example with heterogeneous battlefields, the remainder of this section illustrates some properties of the irregular N-gon method and compares it with that of Gross (1950) and Laslier (2002), paying particular attention to the equity and correlation properties of the respective equilibrium distributions. (Section 4 illustrates the irregular N-gon method in the context of U.S. presidential elections.) Even though the expected allocation to a given battlefield is uniquely pinned down by the equilibrium condition of Proposition 1, the distribution of allocations depends on the chosen construction method.

I find the following differences. Under the irregular N-gon method the variability in the rank of the per-capita allocation, as measured by the coefficient of variation, tends to increase with the battlefield size. This feature is less pronounced under the construction method of Gross (1950) and Laslier (2002). Second, computing the Lorenz curve and associated Gini coefficient in each case reveals that the irregular N-gon method yields less egalitarian outcomes ex-post. Finally, I find that the per-capita allocations tend to be positively correlated amongst smaller battlefields under the irregular N-gon solution. This is particularly striking, given that the budget constraint is a force making for negative correlation. In contrast, all allocations are negatively correlated when using the method of Gross (1950) and Laslier (2002).

Consider the set of integers $\mathcal{V}_1 := \{5, 7, 8, 9, 12\}$ discussed in the previous section. There are six permutations of the elements of $\mathcal{V}_1$ that satisfy the conditions of Theorem 1, out of a possible twelve. They are:

$$(7) \qquad \begin{array}{c} (7,5,12,9,8),\ (8,5,12,9,7),\ (5,7,12,9,8), \\ (8,7,12,9,5),\ (5,8,12,9,7),\ (7,8,12,9,5). \end{array}$$

Since for any given permutation, the $t_1, \ldots, t_N$ are uniquely determined by (4) when N is odd, each of the permutations above corresponds to two κ-tangential polygons: one for $\kappa = 1$ and the other for $\kappa = 2$. Different values of κ generate equilibrium distributions with different correlation properties. To emphasise this dependence, we let F_κ^* denote the distribution function generated using a κ-tangential N-gon, and F_0^* denote the distribution function obtained by averaging out over all $\kappa \leq \frac{N-2}{2}$.

For each κ, the equilibrium distribution F_κ^* is numerically constructed as follows. For each permutation $\boldsymbol{w}$ in (7) we generate $10,000$ points uniformly distributed on the surface of the sphere $\mathscr{S}$ and for each point compute the realisation of the allocation vector. See Appendix 6.4 for details. For each realisation $\boldsymbol{x}$, the battlefields are then ranked according to, x_n/v_n, their allocated share of resources weighted by the battlefield value. This can be interpreted as ranking battlefields according to the per-capita share of resources allocated to them, if the value of a battlefield is understood to reflect the size of its population.

The two left panels in Table 2 describe the probability that a battlefield with value v_n is ranked k^{th} according to the per-capita share of resources it receives under F_κ^*, for $\kappa = 1$ and for $\kappa = 2$. Each row describes the probabilities with which each battlefield occupies a particular rank. For instance, the battlefield with value v_1 is allocated the smallest share of per-capita resources with probability 14.01 when $\kappa = 1$ and with probability 19.8 when $\kappa = 2$. The two right panels in Table 2 are the correlation matrices of F_κ^*, for $\kappa = 1$ and for $\kappa = 2$. The i, j entry is the correlation, under F_κ^*, of the per-capita allocation X_i/v_i to the battlefield with value v_i, and the per-capita allocation X_j/v_j to the battlefield with value v_j. (The matrix is symmetric.) Figure 2 gives a visual representation of the rank and correlation matrices of Table 2.

When $\kappa = 1$, the smallest and largest per-capita share are both most likely to be allocated to the largest battlefield, and both least likely to be allocated to the smallest battlefield. The smallest battlefield is most likely to be allocated the second smallest or second largest per-capita share. The per-capita shares of the three smallest battlefields are more positively correlated to one another. The per-capita shares of the largest two battlefields are negatively correlated to the per-capita allocations in all other battlefields.

When $\kappa = 2$ the smallest battlefield is most likely to receive the median per-capita share. Compared with the case $\kappa = 1$, more extreme per-capita allocations (rank 1 and 5) become more likely for smaller battlefields (1, 2 and 3) and less likely for larger battlefields (4 and 5). For the latter, the likelihood of receiving intermediate per-capita allocations (rank 2

Rank × Battlefield, $\kappa = 1$

Rank	v_1	v_2	v_3	v_4	v_5
1	14.01	17.75	20.31	20.39	27.54
2	27.13	21.09	17.28	23.87	10.63
3	17.7	22.31	24.9	10.78	24.3
4	27.12	21.06	17.11	24.32	10.39
5	14.03	17.79	20.4	20.64	27.14

Battlefield × Battlefield, $\kappa = 1$

Battlefield	v_1	v_2	v_3	v_4	v_5
v_1	1	0.17	0.11	−0.27	−0.39
v_2	0.17	1	−0.03	−0.26	−0.44
v_3	0.11	−0.03	1	−0.24	−0.52
v_4	−0.27	−0.26	−0.24	1	−0.32
v_5	−0.39	−0.44	−0.52	−0.32	1

Rank × Battlefield, $\kappa = 2$

Rank	v_1	v_2	v_3	v_4	v_5
1	19.8	20.64	21.71	16.28	21.56
2	15.29	17.25	20.54	23.5	23.42
3	29.74	24.2	15.52	21.46	9.09
4	15.44	17.27	20.65	22.98	23.66
5	19.73	20.64	21.58	15.78	22.28

Battlefield × Battlefield, $\kappa = 2$

Battlefield	v_1	v_2	v_3	v_4	v_5
v_1	1	0.04	−0.49	0.49	−0.48
v_2	0.04	1	−0.77	0.24	−0.27
v_3	−0.49	−0.77	1	−0.19	0.13
v_4	0.49	0.24	−0.19	1	−0.97
v_5	−0.48	−0.27	0.13	−0.97	1

Table 2: *Rank and correlation matrices of per-capita allocations X_n/v_n for $\kappa = 1$ and $\kappa = 2$.*

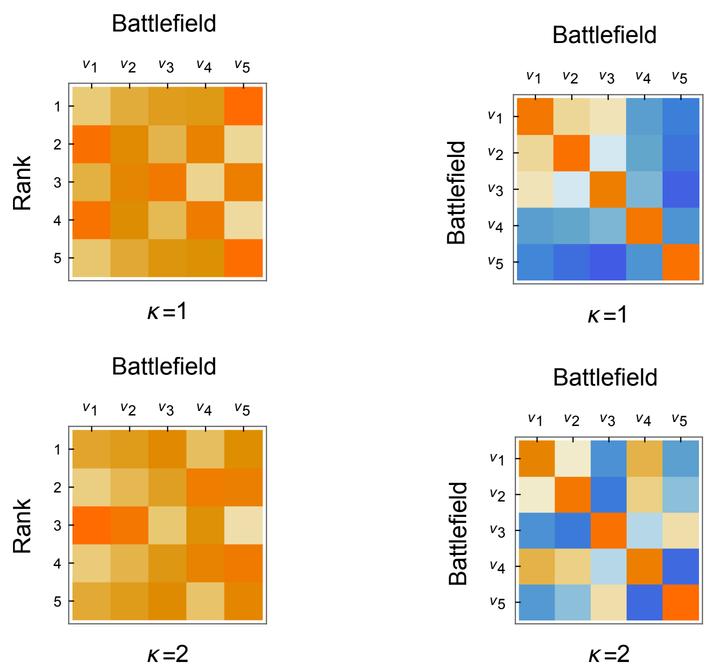

Figure 2: *Rank and correlation matrices of per-capita allocations X_n/v_n for $\kappa = 1$ and $\kappa = 2$. (In this an other visual representations of matrices, blue tones indicate negative values and orange tones indicate positive values. Darker colours indicate a larger absolute value.)*

and 4) is is now higher. Correlations are less systematic than for $\kappa = 1$.[21]

Averaging over $\kappa = 1$ and $\kappa = 2$, we obtain the rank and correlation matrices of F_0^* described in Table 3 and illustrated in Figure 3. In addition, we report the coefficient of variation c_V for the distribution of the rank of per-capita allocations obtained by each battlefield. We make an observation that remains true for other parametrisations of the battlefield values $\mathcal{V}$ (see Section 4): the largest and smallest per-capita shares are most likely to go to larger battlefields, while intermediate shares are more likely to go to smaller battlefields. Moreover, the variability of the allocation rank, as measured by the coefficient of variation, tends to increase with the battlefield size.

Rank \ Battlefield	v_1	v_2	v_3	v_4	v_5
1	16.90	19.19	21.01	18.33	24.55
2	21.21	19.17	18.91	23.68	17.02
3	23.72	23.25	20.21	16.12	16.69
4	21.28	19.16	18.88	23.65	17.02
5	16.88	19.21	20.99	18.21	24.71
c_V	0.444	0.462	0.478	0.464	0.506

Battlefield \ Battlefield	v_1	v_2	v_3	v_4	v_5
v_1	1	0.11	−0.19	0.11	−0.43
v_2	0.11	1	−0.40	−0.01	−0.35
v_3	−0.19	−0.40	1.	−0.21	−0.20
v_4	0.11	−0.01	−0.21	1	−0.64
v_5	−0.43	−0.35	−0.20	−0.64	1

Table 3: *Rank and correlation matrices of per-capita allocations X_n/v_n, averaged over $\kappa = 1$ and $\kappa = 2$. The coefficients of variation of the rank distribution for each battlefield are also reported.*

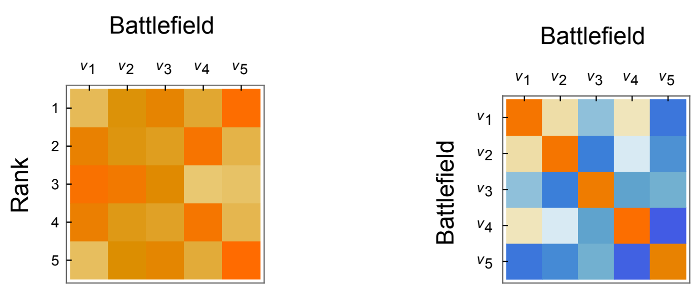

Figure 3: *Rank and correlation matrices of per-capita allocations X_n/v_n, averaged over $\kappa = 1$ and $\kappa = 2$.*

I compare the irregular N-gon method with that of Gross (1950) and Laslier (2002). Their method requires partitioning the battlefields into 3 subsets, $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$, and building a triangle with sides of lengths given by the sum of battlefield values in each subset. Because every triangle admits an incircle, the projection method described in Section 3.1 can always be applied to generate a three-dimensional random vector of allocations,

[21] Appendix 6.9 illustrates the effects of varying κ on the rank and correlation matrices in the U.S. presidential election game.

(Y_1, Y_2, Y_3). If battlefield $n \in \{1, \ldots, N\}$ belongs to subset $\mathcal{N}_k$, $k \in \{1,2,3\}$, then the share of resources allocated to battlefield n is given by $X_n = Y_k v_n / (\sum_{n \in \mathcal{N}_k} v_n)$, and the random vector $(X_n)_{n=1}^N \in \Delta^{N-1}$ satisfies the equilibrium conditions of Proposition 1.

It is immediate that, under this construction method, the allocations to battlefields that belong to a common subset are *perfectly* correlated, for any given partition of battlefields $\{\{\mathcal{N}_1\}, \{\mathcal{N}_2\}, \{\mathcal{N}_3\}\}$. However, there are many possible ways of partitioning N battlefields into 3 non-empty subsets[22], subject to the constraint that the sum of battlefields values within any subset does not exceed the sum of remaining battlefields values. This last condition ensures that it is possible to build a triangle with sides of lengths given by the sum of battlefield values in each subset. In our example, given $\mathcal{V}_1$, there are 12 such partitions:

$$\begin{aligned} &\{\{5\},\{7,12\},\{8,9\}\},\ \{\{7\},\{5,9\},\{8,12\}\},\ \{\{7\},\{5,12\},\{8,9\}\},\\ &\{\{8\},\{5,9\},\{7,12\}\},\ \{\{8\},\{5,12\},\{7,9\}\},\ \{\{9\},\{5,7\},\{8,12\}\},\\ &\{\{9\},\{5,8\},\{7,12\}\},\ \{\{9\},\{5,12\},\{7,8\}\},\ \{\{12\},\{5,7\},\{8,9\}\},\\ &\{\{12\},\{5,8\},\{7,9\}\},\ \{\{12\},\{5,9\},\{7,8\}\},\ \{\{5,7,8\},\{9\},\{12\}\}. \end{aligned} \tag{8}$$

Constructing the equilibrium distribution for each of the 12 partitions above, then considering their average, I obtain the rank and correlation matrices described in Table 4 and illustrated in Figure 4.

	Battlefield				
Rank	v_1	v_2	v_3	v_4	v_5
1	32.82	34.92	35.84	31.76	37.10
2	34.10	30.19	28.48	36.48	25.90
3	33.09	34.90	35.68	31.76	37.00
c_V	0.405	0.418	0.423	0.398	0.431

Battlefield	v_1	v_2	v_3	v_4	v_5
v_1	1	−0.15	−0.14	−0.00	−0.23
v_2	−0.15	1	−0.27	−0.14	−0.24
v_3	−0.14	−0.27	1	−0.04	−0.42
v_4	−0.00	−0.14	−0.04	1	−0.64
v_5	−0.23	−0.24	−0.42	−0.64	1

Table 4: *Rank and correlation matrices of per-capita allocations X_n/v_n under the construction method of Gross (1950) and Laslier (2002). The coefficients of variation of the rank distribution for each battlefield are also reported.*

Even though the per-capita allocations X_n/v_n within each subset $\mathcal{N}_k$ are perfectly correlated given a particular partition of battlefields, when averaging out over partitions the per-capita allocation to any battlefield is negatively correlated to that in all other battlefields. Moreover, the probability for any battlefield of obtaining the k^{th} largest per-capita

[22] In general there are $\left\{ {N \atop k} \right\}$ ways to partition N objects into k non-empty subsets, where $\left\{ {N \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{a}{b}j^n$ denotes the Stirling number of the second kind.

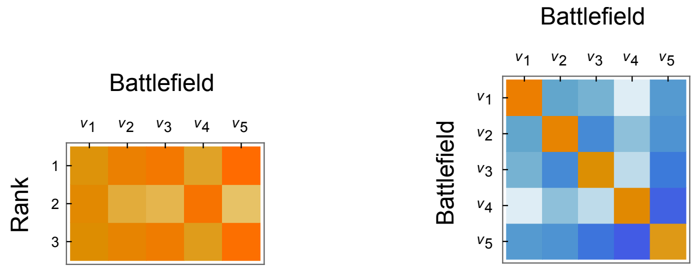

Figure 4: *Rank and correlation matrices of per-capita allocations X_n/v_n under the construction method of Gross (1950) and Laslier (2002).*

allocation, is closer to 1/3 for every $k \in \{1,2,3\}$ when using the Gross (1950) and Laslier (2002) construction method, than it is to 1/5 for every $k \in \{1,2,3,4,5\}$ when using the irregular N-gon method. (See Table 3 and Figure 3). According to the coefficients of variation, under the irregular N-gon method the allocation to larger battlefields tends to be more variable than the allocation to smaller ones (See also Table 6). This relation between rank variability and battlefield size is less pronounced under the construction method of Gross (1950) and Laslier (2002): for each battlefield the coefficient of variation is larger under the former construction method than under the latter.

Furthermore, computing the Lorenz curve and associated Gini coefficient in each case reveals that the irregular N-gon method yields less egalitarian outcomes ex-post. The Lorenz curve describes the cumulative distribution of per-capita allocation shares. The Gini coefficient associated with a Lorenz curve takes values between 0 and 1 and measures the departure of the Lorenz curve from the 45° line (perfect equity). Larger values of the Gini coefficient indicate a less equitable allocation of resources.

Here, the Lorenz curve is numerically constructed as follows. For each generated realisation of the allocation vector we cumulate the per-capita allocation shares ranked from the poorest to the wealthiest. Averaging out over all permutations in (7) and over $\kappa = 1, 2$ generates the Lorenz curve for the irregular N-gon method. Averaging out over all permutations in (8) generates the Lorenz curve for the construction method of Gross (1950) and Laslier (2002).

These Lorenz curves are depicted in Figure 5. The realised allocation of resources is clearly inequitable, with the poorest twenty units of battlefield value (corresponding to the poorest half) receiving less than 26% of resources in both cases. The irregular N-gon method implies greater inequality, as the associated Lorenz curve first-order stochastically dominates the Lorenz curve generated using the construction method of Gross (1950) and Laslier (2002). The null hypothesis that the two Lorenz curves are the same is rejected at the 0.1 percent significance level based on a two-sample Kolmogorov-Smirnov test.[23]

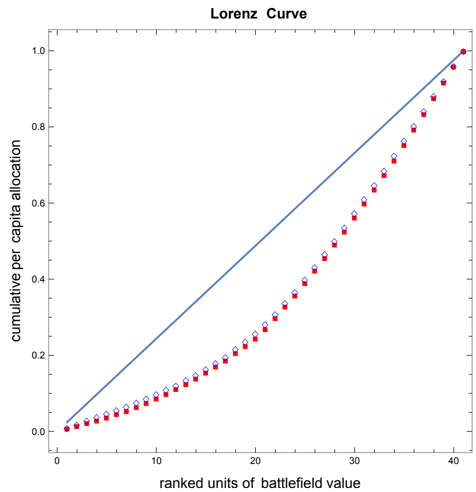

Figure 5: *Lorenz curves generated using the irregular N-gon method (full squares) and that of Gross (1950) and Laslier (2002) (empty diamonds). The associated Gini coefficients are* 0.288 *and* 0.270 *respectively.*

4 Application: U.S. presidential elections

One compelling application of this model is the election of U.S. presidents by the Electoral College. During primaries, two candidates, one Democrat, the other Republican, are chosen to represent their party in the general election, which is then held simultaneously in all fifty U.S. states and the District of Columbia. Each state is allocated a number of electoral votes proportional to its population.[24] A candidate gains all electoral votes of a given state[25] if he receives a plurality of the votes cast in that state. To win the election, a candidate must win more than half the electoral votes.

[23] The Kolmogorov-Smirnov test statistic is 0.0127 while the critical value at the 0,1% significance level is 0.0080 for $12 \times 10,000$ observations.

[24] These numbers vary over time. The distribution of electoral votes across U.S. states for the most recent four decades is given in Appendix 6.5.

[25] Two states, Maine and Nebraska, allocate all but two of their electoral votes by Congressional district, so that a candidate needs to win a plurality of votes in a Congressional district to win that district's electoral vote. The remaining two electoral votes (corresponding to the number of Senators in each state) are allocated to the candidate winning a plurality of the vote in the state. So, in these two states, the electoral votes may be split. I am grateful to an anonymous referee for this observation. Since neither state was considered a swing state in the 2012 election, this singularity does not affect the results of this section.

Modelling the U.S. presidential elections as a Blotto game allows us to address the following question: does the Electoral College system intrinsically lead to inequity in the realised per-capita allocation of campaign resources across states? Does the per-capita allocation depend on a state's size?

Modelling the U.S. presidential elections as a Blotto game requires the following five assumptions: (i) there are two presidential candidates (ii) who face identical budgets, (iii) if a candidates allocates more campaign resources than his opponent to a state, he receives a majority of votes in that state (iv) no state is ex-ante biased in favour of either candidate and (v) each candidate wishes to maximise the expected total number of electoral votes in his favour, rather than the probability of winning the election.

Even though third party candidates have won electoral votes in the past, they have seldom campaigned in all 51 states. Nevertheless, their presence is considered to have had a significant effect on the outcome of close elections.[26] For the purpose of this numerical exercise, I abstract from third party candidates.

Assumption (ii) is necessarily satisfied if we understand the campaign resource to mean time spent campaigning in a state. What if we think of money as the resource? Since the introduction of the public funding program[27] in 1976, candidates may accept public funds for their campaign, in which case they must abide by pre-specified spending limits. If both candidates are publicly funded, they face identical budget constrains[28]. However, in the last two presidential elections, at least one of the major party candidates rejected public funding. Data on the last four presidential elections (see Appendix 6.6) shows that, in practice, the campaign expenditures of candidates can greatly differ.

The positive relationship between campaigning effort and votes is well documented, whether campaigning effort is understood to be time spent campaigning in a state (Herr (2008)) or financial campaign expenditures in that state (Chapman and Palda (1984)). However, Levitt (1994) rejects this claim in the context of U.S. House elections.

Assumptions (iv) and (v) may seem more problematic. For every past presidential elections, all candidates were confident enough about securing the votes of their "safe" states and concentrated their campaigning efforts mostly on "swing" states. In my numerical exercise I will therefore restrict attention to the "swing" states of the 2012 presidential election. Those are listen in Table 5 together with their electoral votes in that year.

[26] Perot in 1992 and Nader in 2000 are often cited as the most recent cases of potential "spoiler candidates". Lee (2012) argues that third party candidates are an integral part of the two-party system, even when they fail to win a significant percentage of votes.

[27] See the Federal Election Commission brochure for Public Funding of Presidential Elections at `http://www.fec.gov/pages/brochures/pubfund.shtml`.

[28] In the 2004 presidential election, both candidates rejected primary matching funds, but accepted public fund in the general election.

CO (9)	MI (16)	NC (15)	VA (13)
FL (29)	NV (6)	OH (18)	WI (10)
IA (6)	NH (4)	PA (20)	

Table 5: *Electoral votes in the 2012 swing states.*

As to the candidates' aim, in general one might be more inclined to assume that candidates maximise the probability of their winning the election. However, because presidential elections coincide with Senate and House of Representatives elections, maximising the expected plurality of electoral votes in their favour is at least a candidate's secondary objective (as opposed to maximising the expected plurality of the popular vote, or the probability of winning a majority of state electors).

4.1 Equilibrium in the U.S. presidential election game

Under assumptions (i)-(v) I model the 2012 presidential election as the following Colonel Blotto game: two candidates with identical budgets normalised to 1 choose how to allocate their campaign funds across $N = 11$ "swing" states indexed by $n \in \{1, ..., N\}$. The value of state n is v_n and corresponds to the number of electoral votes attached to that state in the 2012 election (Table 5). Let $\mathcal{V} = \{4, 6, 6, 9, 10, 13, 15, 16, 18, 20, 29\}$ denote the set of state values. Candidate i's plurality, the number of electoral votes won minus the number of electoral votes lost, is measured by the function $g_i : \mathcal{S}_i \times \mathcal{S}_i \to \mathbb{R}$ defined in (1). This matches the setup of Section 2, and we can use the results of Section 3 to numerically construct a symmetric equilibrium strategy of this game.

Assume[29] that the values in $\mathcal{V}$ are drawn uniformly from $[0, 32]$. In the context of Section 3.3 we have $m = 5$ (since $2^5 = 32$) and $N = 11$ so that $m/N < 1$. This indicates that the problem is in the perfect phase of Borgs et al. (2004) and our partitioning problem should be "easy" to solve.[30] Using the algorithm described in Appendix 6.2, I derive the set $\mathcal{G}^*(\mathcal{V})$ of all distinct permutations $\boldsymbol{w}$ of $\boldsymbol{v}$ satisfying Theorem 1. There are 4896 such permutations, about 0.27% of the $(N-1)!/2 = 1\ 814\ 400$ possible permutations of $\boldsymbol{v}$.

[29] The null hypothesis that the values in $\mathcal{V}$ are distributed according to the uniform distribution on $[0, 32]$ cannot be rejected up to the 28 percent significance level based on the Kolmogorov-Smirnov test. (The p-value is 0.28.)

[30] Had we considered the entire fifty U.S. states and the District of Columbia (ignoring the singularities of Nebraska and Maine), the resulting combinatorial problem would also have been in the perfect phase (since $N = 51$, $m = 6$, and $m/N = 6/51 < 1$). Indeed, a solution can readily be found by tâtonnement, as shown in Appendix 6.7, which illustrates one possible permutation of the 51 battlefields satisfying the conditions of Theorem 1.

Thus, for each κ, there exist 4896 distinct tangential polygons associated with $\mathcal{V}$.[31] Each tangential polygon generates its own joint distribution over Δ^{10}, each with its own correlation properties. I construct each joint distribution numerically as follows: for each permutation in $\mathcal{G}^*(\mathcal{V})$ I generate 1000 points uniformly distributed on the surface of the sphere $\mathscr{S}$ and for each point compute the realisation of the allocation vector. See Appendix 6.4 for details.

Following Laslier and Picard (2002), given κ, I obtain the joint distribution F^*_κ by averaging out over all permutations $\boldsymbol{w}$ in $\mathcal{G}^*(\mathcal{V})$. Each κ generates distinct equity and correlation properties. I show that varying κ does not significantly affect the Lorenz curve associated with the distribution F^*_κ. We can therefore average out over all $\kappa \leq \lfloor \frac{N-1}{2} \rfloor$ to obtain the equilibrium strategy F^*_0.

4.2 Lorenz curve

Let us investigate whether the per-capita allocation of campaign resources depends on a state's size, and how equitable the realised allocations are. Brams and Davis (1974) argue that larger states receive a disproportionately larger share in equilibrium. This is clearly excluded by Proposition 1. Define the per-capita allocation in state n to be x_n/v_n, splitting the resources allocated in equilibrium to state n equally among its electoral votes. In equilibrium, since the allocation x_n to state n is distributed uniformly over $[0, 2v_n/B]$, we have $\mathbb{E}[x_n/v_n] = 1/B$ for all n, implying ex-ante equity.

In this sense, the Blotto game does not does not support the claim that larger states will systematically receive the largest per-capita share of campaign resources. Instead it predicts that ex-ante, campaign resources will be distributed equitably. However, I show that this is not the case ex-post.

For each $\kappa \leq \lfloor \frac{N-1}{2} \rfloor$, I construct the Lorenz curve describing the cumulative distribution of per-capita allocation shares. For each permutation $w \in \mathcal{G}^*(\mathcal{V})$, I generate $1,000$ realisations of the equilibrium allocation vector $\boldsymbol{x}$ and rank electoral votes from the poorest to the wealthiest, according to the per-capita share of campaign resources allocated to them. Cumulating the allocation shares and averaging out over all permutations $\boldsymbol{w} \in \mathcal{G}^*(\mathcal{V})$ generates the Lorenz curve, for a given κ. These are illustrated in Appendix 6.8.

The Gini coefficients vary significantly with κ, and consistent with this, some Lorenz curves first-order stochastically dominate others given our sample data (See Appendix 6.8). Averaging out over all $\kappa \leq \lfloor \frac{N-1}{2} \rfloor$, gives the Lorenz curve generated by the equilibrium

[31] Observe that because N is odd, given κ and a permutation $\boldsymbol{w} := (w_1, \ldots, w_{11}) \in \mathcal{G}^*(\mathcal{V})$, there exists a unique associated vector $\boldsymbol{t} := (t_1, \ldots, t_{11})$ given by (4) such that $(\boldsymbol{t}, \boldsymbol{w})$ satisfies Theorem 1. Therefore there exists a unique tangential polygon associated with each pair $(\kappa, \boldsymbol{w})$, $\kappa \leq \lfloor \frac{N-1}{2} \rfloor$, $\boldsymbol{w} \in \mathcal{G}^*(\mathcal{V})$.

distribution F_0^*. It is depicted in Figure 6. Its Gini coefficient is 0.304. Clearly, the realised allocation of resources is inequitable, with the most neglected half of the population receiving less than 25% of campaign resources.

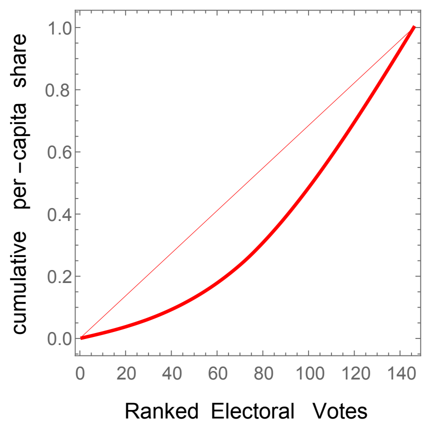

Figure 6: *The Lorenz curve generated by the equilibrium distribution* F_0^*.

4.3 Equity and correlation

When constructing the equilibrium distribution F_0^*, for each generated data point I rank the states according to the size of the per-capita share of resources obtained. The (k, n) entry of Table 6 gives the probability that the n^{th} smallest state in terms of electoral votes receives the k^{th} smallest per-capita share of campaign resources under F_0^*. For instance, the smallest per-capita share of resources (rank 1) is allocated to the smallest state (NH) with 7.34% probability and to the largest state (FL) with 11.35% probability.

The results described in Table 6 and illustrated in Figure 7 show that the Blotto game does not predict that larger states will systematically receive the largest per-capita share ex-post. In fact, the largest state in our sample is as likely to receive the smallest share, as the largest one. In line with results from Section 3.5, the variability in the ranking of a state, as measured by the coefficient of variation, tends to increase with its size. The largest and smallest per-capita shares are most likely to go to larger states, while intermediate shares are more likely to go to smaller states.

As previously observed, using κ-tangential polygons with different values of κ leads to different distributions F_κ^* with different correlation properties. In Appendix 6.9, Figure 11 illustrates the correlation matrices of F_κ^*, and Figure 12 represent the rank distributions, when κ takes the values 1 to 5. Averaging out over all κ, gives F_0^* whose correlation matrix is illustrated in Figure 8. The per-capita allocations of campaign resources are positively

Rank	NH(4)	IA(6)	NV(6)	CO(9)	WI(10)	VA(13)	NC(15)	MI(16)	OH(18)	PA(20)	FL(29)
1	7.34	8.17	8.17	8.61	8.73	9.46	9.32	9.29	9.8	9.77	11.35
2	8.41	8.35	8.35	8.93	8.95	9.23	9.49	9.63	9.46	9.78	9.41
3	9.48	9.22	9.22	9.09	9.24	9.18	8.83	8.78	8.92	8.75	9.29
4	9.32	9.18	9.18	9.07	9.09	8.78	9.05	8.98	8.99	9.37	8.99
5	9.41	9.58	9.58	9.45	9.12	9.09	8.99	8.90	8.88	8.87	8.14
6	11.8	10.84	10.84	9.53	9.72	8.53	8.47	8.73	8.11	7.84	5.59
7	9.48	9.61	9.61	9.43	9.10	9.09	9.02	8.89	8.82	8.81	8.13
8	9.35	9.20	9.20	9.11	9.07	8.75	9.08	9.02	8.96	9.3	8.95
9	9.54	9.24	9.24	9.14	9.25	9.17	8.84	8.81	8.88	8.67	9.22
10	8.49	8.39	8.39	8.97	8.98	9.24	9.54	9.66	9.42	9.42	9.51
11	7.38	8.21	8.21	8.66	8.75	9.48	9.37	9.31	9.76	9.44	11.45
c_V	0.500	0.509	0.509	0.519	0.522	0.533	0.532	0.532	0.538	0.540	0.560

Table 6: *Distributions of state rankings under F_0^*, according to per-capita share of resources obtained by that state, and associated coefficients of variation.*

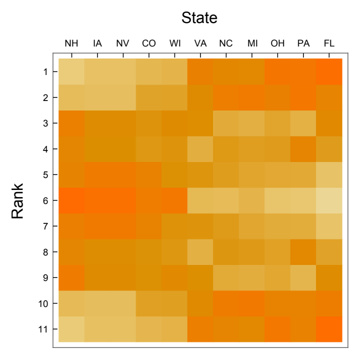

Figure 7: *Plot of the data in Table 6. Darker cells indicate higher values.*

correlated amongst smaller states (NH-CO). For the remaining states (WI-FL), the larger the state, the more negatively its allocation is correlated with those of other states.

5 Conclusion

I introduce the irregular N-gon solution, a new geometric method for generating instances of the N-variate distributions that constitute equilibria of the Blotto game with heterogeneous battlefield values. I relate my method to the constrained integer partitioning problem and propose a heuristic for assessing its reliability, given a set of parameters. I illustrate the irregular N-gon solution in a numerical example, and contrast it with the solution generated following Gross (1950) and Laslier (2002). Finally, I apply the irregular

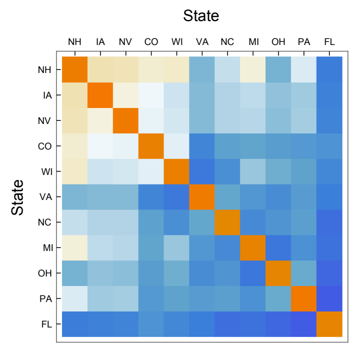

Figure 8: *Plot of the correlation matrix of F_0^*. Blue tones indicate negative values and orange tones indicate positive values. Darker colours indicate a larger absolute value.*

N-gon method to a game based on the U.S. presidential elections. Concentrating on the 11 swing states of the 2012 election, I study the equity of campaign resource allocation across states. I find that there is substantial inequality in the realised allocation of resources. The variation in the ranking of a state according to realised per-capita allocation tends to increase with its size. The per-capita allocations to smaller states are positively correlated. The largest and smallest per-capita shares are most likely to go to larger states, while intermediate shares are more likely to go to smaller states.

6 Appendix

6.1 Bounds on t_n when N is even.

For N even, it follows from (2) that

$$0 < t_n < v_n, \quad n = 1, \ldots, N. \tag{9}$$

Using (2) to express each t_n in terms of t_1:

$$t_n = \begin{cases} -t_1 + b_{n-1} & n = 2, 4, \ldots, N, \\ t_1 - b_{n-1} & n = 1, 3, \ldots, N-1, \end{cases} \tag{10}$$

where $b_n = \sum_{j=1}^{n} (-1)^{j+1} v_j$ for $n = 1, \ldots, N$ was defined in Corollary 2 and we use the convention $b_0 \equiv 0$. Using (10) in (9) we obtain

$$t_n = \begin{cases} b_{n-1} > t_1 > b_{n-1} - v_n, & n = 2, 4, \ldots, N, \\ b_{n-1} < t_1 < b_{n-1} + v_n, & n = 1, 3, \ldots, N-1. \end{cases}$$

Since $b_n = b_{n-1} - v_n$ when n is even, and $b_n = b_{n-1} + v_n$ when n is odd, the above becomes

$$t_n = \begin{cases} b_{n-1} > t_1 > b_n, & n = 2, 4, \ldots, N, \\ b_{n-1} < t_1 < b_n, & n = 1, 3, \ldots, N-1. \end{cases}$$

The above system of inequalities imposes $N/2$ upper bounds, and $N/2$ lower bounds on t_1.

Condition (6) is necessary and sufficient for all N bounds to hold, and the interval in (6) defines the set of admissible values for t_1. For a given t_1, equation (10) uniquely defines t_n for all remaining $n = 2, \ldots, N$.

6.2 Algorithm

The following algorithm is used to build the set $\mathcal{G}^*(\mathcal{V})$ of all permutations of the set $\mathcal{V}$ of battlefield values satisfying Theorem 1, treating a permutation, its cyclic shifts, and the order-reversing permutations of all of them as identical.[32]. Fix $\mathbf{v} := (v_1, \dots, v_N)$ and for each permutation π of $1, \dots, N$ let $\mathbf{w}^\pi := (w_1^\pi, \dots, w_N^\pi) = (v_{\pi(1)}, \dots, v_{\pi(N)})$ denote the corresponding permutation of $\mathbf{v}$. The existence of a tangential polygon with sides given by $\mathbf{w}^\pi$ requires condition (2) to hold. This implies that for each $n = 1, \dots, N$, we have $0 \leq t_n \leq w_{n-1}^\pi$ and $0 \leq t_{n+1} \leq w_{n+1}^\pi$ and therefore

$$w_n^\pi \leq w_{n+1}^\pi + w_{n-1}^\pi \quad \forall n = 1, \dots, N.$$

Observing that this condition is harder to satisfy the larger w_n^π, we use the following algorithm to construct $\mathcal{G}^*(\mathcal{V})$.

1. Let $\mathcal{G}_0(\mathcal{V})$ denote the set of all possible permutations w^π of $\mathbf{v}$, treating a permutation, its cyclic shifts, and the order-reversing permutations of all of them as identical[33]. For each corresponding π, let $\tilde{n}^\pi := \arg\max_{n \in \{1,\dots,N\}} w_n^\pi$ denote the index of the highest-valued coordinate of $\mathbf{w}^\pi$.

2. Construct the set $\mathcal{G}_1(\mathcal{V}) \subseteq \mathcal{G}_0(\mathcal{V})$ of cyclical permutations satisfying

$$w_{\tilde{n}^\pi}^\pi \leq w_{\tilde{n}^\pi - 1}^\pi + w_{\tilde{n}^\pi + 1}^\pi.$$

3. For $k = 2, \dots, (N-1)/2$, construct the set $\mathcal{G}_k(\mathcal{V}) \subseteq \mathcal{G}_{k-1}(\mathcal{V})$ of cyclical permutations satisfying

$$\begin{cases} w_{\tilde{n}^\pi - k + 1}^\pi \leq w_{\tilde{n}^\pi - k}^\pi + w_{\tilde{n}^\pi - k + 2}^\pi, \\ w_{\tilde{n}^\pi + k - 1}^\pi \leq w_{\tilde{n}^\pi + k}^\pi + w_{\tilde{n}^\pi + k - 2}^\pi. \end{cases}$$

4. Eliminate from $\mathcal{G}_{\frac{N-1}{2}}(\mathcal{V})$ all permutations that do not satisfy (3).

The surviving permutations constitute the set $\mathcal{G}^*(\mathcal{V})$.

Because the algorithm above exploits the conditions of Theorem 1, it does better at deriving the set $\mathcal{G}^*(\mathcal{V})$ than the brute force method of testing all possible permutations of the elements of $\mathcal{V}$.

[32] See footnote 14

[33] Therefore, $|\mathcal{G}_0(\mathcal{V})| = (N-1)!/2$.

6.3 Counter-example

Consider the set $\mathcal{V} = \{1, 2, 3, 109, 110, 111, 325\}$ of battlefield valuations. The set $\mathcal{G}_1$, defined in Appendix 6.2, is empty, since $325 > 110 + 111$. Therefore, there exists no permutation of the elements of $\mathcal{V}$ satisfying condition (2).

However, $\mathcal{V}$ satisfies the N integer partitioning problems in Proposition 2 (ii):

v_n	partition of $\mathcal{V} \setminus \{v_n\}$ into two subsets of equal cardinality	discrepancy
1	$\big\{\{109, 110, 111\}, \{2, 3, 325\}\big\}$	0
2	$\big\{\{109, 110, 111\}, \{1, 3, 325\}\big\}$	1
3	$\big\{\{109, 110, 111\}, \{1, 2, 325\}\big\}$	2
109	$\big\{\{3, 110, 111\}, \{1, 2, 325\}\big\}$	104
110	$\big\{\{3, 109, 111\}, \{1, 2, 325\}\big\}$	105
111	$\big\{\{3, 109, 110\}, \{1, 2, 325\}\big\}$	106
325	$\big\{\{3, 109, 111\}, \{1, 2, 110\}\big\}$	110

6.4 Numerically constructing the equilibrium distribution

This section describes the method used for numerically computing the allocation of resources under F^*_κ. Fix a given permutation $\boldsymbol{w}$ of the vector of battlefield values $\boldsymbol{v}$ such that $\boldsymbol{w} \in \mathcal{G}^*(\mathcal{V})$ and fix a $\kappa \le \lfloor \frac{N-1}{2} \rfloor$. Using Mathematica, we generate 10,000 points randomly distributed on the surface of a sphere. Each data point is then used to compute one realisation $\boldsymbol{x}$ of the N-dimensional allocation vector. Recall that indices are defined modulo N. We use the notation of Section 3.1.

Let ρ_n denote the distance $|OP_n|$ and let α_n denote the angle $\widehat{T_{n-1}OP_n}$ which by construction equals the angle $\widehat{P_nOT_n}$. The distance $|OT_n|$ is given by the radius r_κ of the incircle of the tangential N-gon $\boldsymbol{P}$. (In this section this radius is indexed by κ to emphasise the dependence.)

Consider the right triangle $OT_{n-1}P_n$, where $\widehat{T_{n-1}OP_n} = \alpha_n$ and where $|OT_{n-1}| = r_\kappa$, $|P_nT_{n-1}| = t_n$ and the hypotenuse is $|OP_n| = \rho_n$. We have that

$$\sin \alpha_n = \frac{t_n}{\sqrt{r_\kappa^2 + t_n^2}}.$$

Since α_n is one of the non-right apices of $OT_{n-1}P_{n-1}$, we have that $-\pi/2 < \alpha_n < \pi/2$ so that $\arcsin(\sin \alpha_n)$ is defined[34] and

$$\alpha_n(r_\kappa) = \arcsin\left(\frac{t_n}{\sqrt{r_\kappa^2 + t_n^2}}\right).$$

Since arcsin is strictly increasing over its domain, $\alpha_n(r)$ is strictly decreasing in r_κ.

Theorem 2 and Corollary 3 in Radić (2002) give the radius r_κ of the incirle of a κ-tangential polygon as an implicit function of the lengths $t_1, \ldots, t_N$. Based on that we obtain expressions for α_n and ρ_n, $n = 1, \ldots, N$.

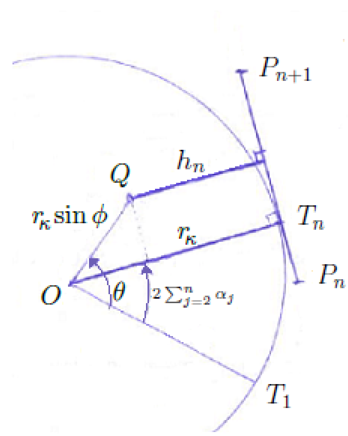

Figure 9: *Projecting Q onto (P_n, P_{n+1}).*

Let $\theta \in [0, 2\pi]$ and $\phi \in [-\pi/2, \pi/2]$ denote the polar coordinates of R, the point uniformly distributed over the surface of the sphere $\mathscr{S} = (O, r_\kappa)$. Let Q denote the projection of R onto the disk (O, r_κ) inscribed in the N-gon $\boldsymbol{P}$. Let h_n denote the orthogonal distance of Q from the line (P_n, P_{n+1}).

[34] The function arcsin is defined as follows: For $-1 \le x \le 1$ and $-\pi/2 < y < \pi/2$, $x = \sin y \Leftrightarrow y = \arcsin x$.

We orient the two dimensional plane around the origin O by convening that $T_1 = r_\kappa e^{i0}$. We then have that

$$P_n = \rho_n \ e^{i\left(2\sum_{j=2}^{n} \alpha_j - \alpha_n\right)},$$

and

$$T_n = r_\kappa \ e^{i2\sum_{j=2}^{n} \alpha_j}.$$

Moreover, since

$$Q = r_\kappa \ \sin\phi \ e^{i\theta},$$

we obtain that

$$h_n = r_\kappa - r_\kappa \sin\phi \ \cos\left(\theta - 2\sum_{j=2}^{n} \alpha_j\right),$$

and have shown in Section 3.1 that $X_n = (h_n w_n)/(r_\kappa B) \sim U[0, 2w_n/B]$.

6.5 Distribution of electoral votes (Source: www.fec.gov)

State	1981- 90	1991- 00	2001- 10	2011- 20	State	1981- 90	1991- 00	2001- 10	2011- 20
Alabama	9	9	9	9	Missouri	11	11	11	10
Alaska	3	3	3	3	Montana	4	3	3	3
Arizona	7	8	10	11	Nebraska	5	5	5	5
Arkansas	6	6	6	6	Nevada	4	4	5	6
California	47	54	55	55	New Hampshire	4	4	4	4
Colorado	8	8	9	9	New Jersey	16	15	15	14
Connecticut	8	8	7	7	New Mexico	5	5	5	5
Delaware	3	3	3	3	New York	36	33	31	29
D.C	3	3	3	3	North Carolina	13	14	15	15
Florida	21	25	27	29	North Dakota	3	3	3	3
Georgia	12	13	15	16	Ohio	23	21	20	18
Hawaii	4	4	4	4	Oklahoma	8	8	7	7
Idaho	4	4	4	4	Oregon	7	7	7	7
Illinois	24	22	21	20	Pennsylvania	25	23	21	20
Indiana	12	12	11	11	Rhode Island	4	4	4	4
Iowa	8	7	7	6	South Carolina	8	8	8	9
Kansas	7	6	6	6	South Dakota	3	3	3	3
Kentucky	9	8	8	8	Tennessee	11	11	11	11
Louisiana	10	9	9	8	Texas	29	32	34	38
Maine	4	4	4	4	Utah	5	5	5	6
Maryland	10	10	10	10	Vermont	3	3	3	3
Massachusetts	13	12	12	11	Virginia	12	13	13	13
Michigan	20	18	17	16	Washington	10	11	11	12
Minnesota	10	10	10	10	West Virginia	6	5	5	5
Mississippi	7	7	6	6	Wisconsin	11	11	10	10
					Wyoming	3	3	3	3

6.6 Presidential campaign finance summaries

(Data: Federal Election Commission, Presidential Campaign Finance Summaries, `http://www.fec.gov/press/bkgnd/pres_cf/pres_cf_Even.shtml`)

		Disbursements prior to Super Tuesday$^{(i)}$	Total disbursements
2012	Obama* (D)	$66,121,822†	$469,930,646
		$78,712,495‡	
	Romney (R)	$55,745,321†	$298,158,415
		$68,107,847‡	
2008	Obama (D)	$115,689,084†	$488,331,269
		$158,579,005‡	
	Mc Cain (R) (II)	$49,650,185†	$207,523,221
		$58,432,608‡	
2004	Kerry (D) (II)	$30,119,415†	$243,294,897
		$37,867,817‡	
	Bush* (R) (II)	$38,901,223†	$286,628,893
		$46,724,159‡	
2000	Gore (D) (I) (II)	$25,703,131†	$77,863,579
		$34,047,289‡	
	Bush (R) (II)	$47,964,764†	$136,651,579
		$60,724,475‡	

Notation: (D) Democrat, (R) Republican. (I) indicates that the candidate accepted primary matching funds; and (II) that he accepted government funds in the general election. $*$ indicates an incumbent, † indicates disbursements up to and including the month of January, ‡ indicates disbursements up to and including the month of February.

(i) "Super Tuesday" took place on the following dates: 6th Mar. 2012, 5th Feb. 2008, 3th Feb 2004, 7th Mar. 2000. In 2004 many sates held their primaries on 2nd Mar., dubbed "Mini Tuesday".

6.7 One permutation of U.S. states satisfying Theorem 1 (2008 Electoral College).

For clarity, w_n, t_n and t_{n+1} are multiplied by 538. (There are 538 electoral votes in total.)

n	w_n	t_n	t_{n+1}	n	w_n	t_n	t_{n+1}
1	31	2	29	26	3	1	2
2	8	6	2	27	3	2	1
3	9	3	6	28	3	1	2
4	10	7	3	29	4	3	1
5	11	4	7	30	4	1	3
6	17	13	4	31	4	3	1
7	20	7	13	32	4	1	3
8	21	14	7	33	5	4	1
9	27	13	14	34	5	1	4
10	21	8	13	35	5	4	1
11	15	7	8	36	6	2	4
12	15	8	7	37	6	4	2
13	15	7	8	38	7	3	4
14	10	3	7	39	7	4	3
15	7	4	3	40	8	4	4
16	7	3	4	41	9	5	4
17	6	3	3	42	9	4	5
18	5	2	3	43	10	6	4
19	5	3	2	44	10	4	6
20	4	1	3	45	11	7	4
21	3	2	1	46	11	4	7
22	3	1	2	47	11	7	4
23	3	2	1	48	12	5	7
24	3	1	2	49	13	8	5
25	3	2	1	50	34	26	8
-	-	-	-	51	55	29	26

6.8 Lorenz curves and Gini coefficients

	$\kappa = 1$	$\kappa = 2$	$\kappa = 3$	$\kappa = 4$	$\kappa = 5$
Gini coefficient:	0.310	0.309	0.305	0.297	0.300

Table A: *Gini Coefficients for* $\kappa = 1, \ldots, 5$.

	$\kappa' = 5$	$\kappa' = 3$	$\kappa' = 2$	$\kappa' = 1$
$\kappa = 4$	YES (0.0032)	NO (0.0068)	NO (0.0117)	YES (0.0111)
$\kappa = 5$		NO (0.0053)	NO (0.0096)	NO (0.0090)
$\kappa = 3$			NO (0.0050)	YES (0.0052)
$\kappa = 2$				NO (0.0028)

Table B: *The first entry in each cell specifies whether the Lorenz curve* L_κ *(row) first-order stochastically dominates the Lorenz curve* $L_{\kappa'}$ *(column) given our sample data, for* $\kappa, \kappa' \in \{1, \ldots, 5\}$. *The entry in brackets specifies the Kolmogorov-Smirnov test statistic obtained in a two-sample Kolmogorov-Smirnov test of the hypothesis that* L_κ *and* $L_{\kappa'}$ *are identical cumulative distributions. The critical value at the* 0.1% *significance level is* 0.00125 *for* 1000×4896 *observations. All reported test statistics exceed this value, and in each case the null hypothesis is strongly rejected by our sample data. The curves are listed from most to least equitable according to the Gini coefficients presented in Table A.*

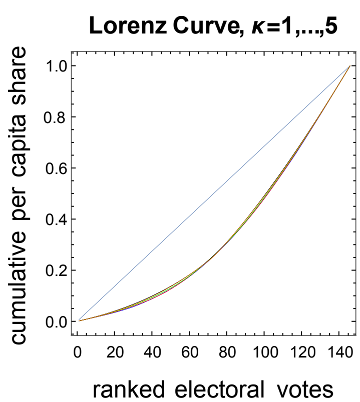

Figure 10: *The Lorenz curves for* $\kappa = 1$ *in blue,* $\kappa = 2$ *in red,* $\kappa = 3$ *in green,* $\kappa = 4$ *in black,* $\kappa = 5$ *in orange.*

6.9 Rank and Correlation properties of F^*_κ

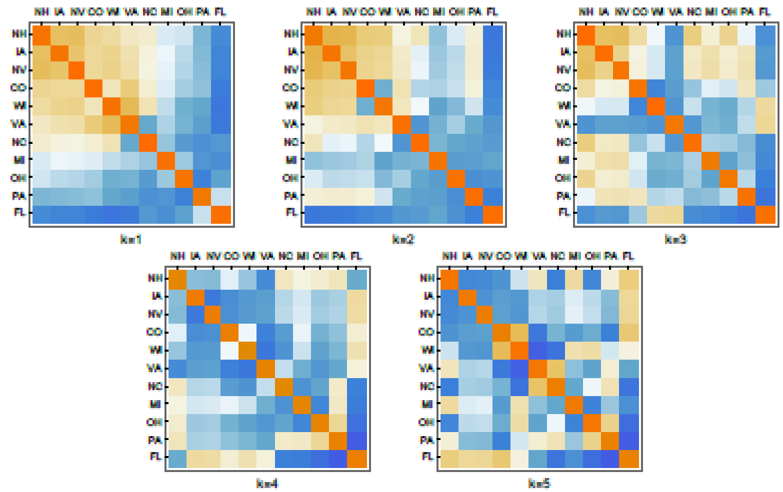

Figure 11: *Plots of the correlation matrices of* F^*_κ*, for* $\kappa = 1, \ldots, 5$.

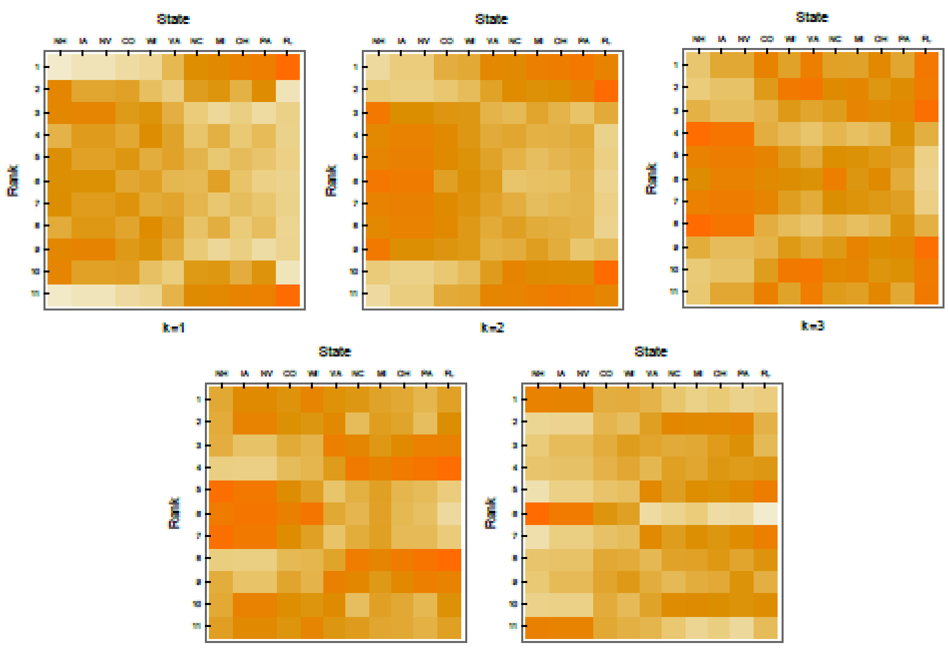

Figure 12: *Plots of rank distributions under* F^*_κ *for* $\kappa = 1, \ldots, 5$.

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