THE THEORY OF PLAY AND INTEGRAL EQUATIONS WITH SKEW SYMMETRIC KERNELS[1]

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LET US consider a game in which the winnings depend both on chance and the skill of the players. Let us confine ourselves to the case of two players, A and B, and a game symmetric in the sense that if A and B adopt the same method of play their chances are equal. One may propose to investigate whether it is possible to determine a method of play better than all others; i.e., one that gives the player who adopts it a superiority over every player who does not adopt it. Let us first define what we should understand by a method of play. It is a code that determines for every possible circumstance (supposed finite in number) exactly what the person should do. In most ordinary games, the number of methods is extremely large, but nonetheless always finite. If the player A adopts the method C_i and B the method C_k, the calculus of probability permits the calculation of the probability of A winning, which we will call a, and that of B, which will be $b = 1 - a$. We will set

$$\begin{cases} a = \tfrac{1}{2} + \alpha_{ik}, \\ b = \tfrac{1}{2} + \alpha_{ki}. \end{cases} \tag{1}$$

The numbers α_{ik} and α_{ki}, contained between $-\tfrac{1}{2}$ and $+\tfrac{1}{2}$, satisfy the relation

$$\alpha_{ik} + \alpha_{ki} = 0. \tag{2}$$

The symmetry of the game is expressed by the relations

$$\alpha_{ii} = 0. \tag{3}$$

We will say that a manner of playing C_i is bad, if α_{ih} is negative or zero for every h. We will exclude the bad manners of playing. After this exclusion there may be other manners of playing that have become bad. These are those manners C_j such that α_{jk} is negative or zero for every manner C_k not already excluded as bad. We continue this process until no further bad manner of playing remains. It may then turn out that there is an indifferent manner of playing C_0, such that α_{0k} is zero for all k; we will provisionally leave this case aside. The manners of playing C_h that subsist are then such that α_{hk} is positive for at least one value of k. If there existed a manner of playing C_h such that α_{hk} is always positive or zero, that manner of playing would be the best. In case

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that best manner does not exist, one may wonder if it is not possible, lacking a code chosen once and for all, to play in an advantageous manner by varying his play. If one wants to formulate a precise rule for varying the play, with only features of the game entering the rule, and not psychological observations on the player to whom one is opposed; that rule will necessarily be equivalent to a statement such as the following. The probability that, at a given moment of play, A adopts the code C_k to determine his conduct at that moment is p_k. The analogous probability for B will be denoted by q_k, and, denoting by n the number of codes that remain, one has

$$\sum_1^n p_k = 1, \qquad \sum_1^n q_k = 1. \tag{4}$$

The probability that A wins is, taking account of (1), (2), (3), and (4),

$$\sum_1^n \sum_1^n (1/2 + \alpha_{ik}) p_i\, q_k = 1/2 + \alpha;$$

where

$$\alpha = \sum_1^n \sum_1^n \alpha_{ik}\, p_i\, q_k = \sum_{i=1}^{i=n} \sum_{k=1}^{k=i-1} \alpha_{ik}(p_i\, q_k - p_k\, q_i). \tag{5}$$

In the particular case that $n = 3$, this formula becomes

$$\alpha = \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ \alpha_{23} & \alpha_{31} & \alpha_{12} \end{vmatrix}. \tag{6}$$

If, as we are supposing, none of the three manners of playing C_1, C_2, C_3 are bad, one sees immediately that none of the three is better than the others. The three numbers α_{23}, α_{31}, α_{12} are therefore of the same sign. It is easy to find positive numbers p_1, p_2, p_3 satisfying the relations (4) and such that α is zero whatever the numbers q_1, q_2, q_3 may be. It is therefore possible to adopt a manner of playing which enables one to compete with even chances against every player. This manner of playing consists in drawing at random, before taking any decision, under conditions that attribute probabilities p_1, p_2, p_3, to the codes C_1, C_2, C_3 respectively. But it is easy to see that, once n exceeds 7, this circumstance will occur only for particular values of the α_{ik}. In general, whatever the p's may be, it will be possible to choose the q's in (5) in such a manner that α has any sign determined in advance. Since this is the situation, whatever variety is introduced by A into his play, once this variety is defined, it will be enough for B to know it in order that he may vary his play in such a manner as to have an advantage over A. The reciprocal is also true, whence we should conclude that the calculation of probabilities can serve only to facilitate elimination of bad manners of playing and the calculation of α_{ik} ; for the rest, the art of play depends on psychology and not on mathematics.

It is easy to extend the preceding considerations to the case where the manners of playing form an infinite continuum, if it is desired to embrace at once the continuous and the discontinuous, the relation (4) must be replaced by relations such as the following:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\phi(x, y) = 1, \qquad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\phi_1(x_1, y_1) = 1, \tag{7}$$

the increasing functions ϕ, and ϕ_1 depending, for example, on two variables and the integrals being defined in the sense of Stieltjes. These functions define the manners of playing of the players A and B. The probability of winning is defined by a skew symmetric function $f(x, y, x_1, y_1)$; i.e., the relation (2) is replaced by

$$f(x, y, x_1, y_1) = -f(x_1, y_1, x, y). \tag{8}$$

The value of α is then given by the Stieltjes integral

$$\alpha = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y, x_1, y_1)\, d\phi(x, y)\, d\phi_1(x_1, y_1). \tag{9}$$

Numerous problems about such a game can thus be reduced to the study of integral equations with a skew symmetric kernel. This kernel depends on the conventions of the game, while the diverse forms of the integral equations depend on the problem posed.

Among games for which the manners of playing form a doubly infinite continuum, one of the simplest is the following: A and B each choose three positive numbers the sum of which is equal to 1.

$$x + y + z = 1, \qquad x_1 + y_1 + z_1 = 1; \tag{10}$$

and each player arranges the numbers he has chosen in a determined order. A wins, if two of the numbers chosen by him are superior to the corresponding numbers of B; i.e., if

$$(x_1 - x)(y_1 - y)(z_1 - z) > 0, \tag{11}$$

and loses in the contrary case. There is a tie, if the inequality in (11) becomes equality. One can naturally generalize in many ways, replacing (10) and (11) by other relations.

A much simplified form of this game, interesting to study as an illustration of the preceding discussion, consists in supposing the numbers x, y, z, x_1, y_1, z_1 to be positive integers satisfying the relations

$$\begin{aligned} x + y + z &= 7, \\ x_1 + y_1 + z_1 &= 7. \end{aligned} \tag{12}$$

Gain and loss still depend on the sign of the product (11). The number 7 is the smallest integer for which the game does not have simple manners of playing superior to all others. Taking sufficiently great integers one will obtain cases where no complex manner of play can avoid loss against an adversary who knows the manner of playing and takes it into account.

The problems of probability and analysis that one might raise concerning the art of war or of economic and financial speculation are not without analogy to the problems concerning games, but they generally have a much higher degree of complexity. For their practical solution, the mathematical mind must be aided by the strategic mind.[2] The only advice the mathematician could give, in the absence of all psychological information, to a player A whose adversary B seeks to utilize the preceding remarks is that he should so vary his plans that the probabilities attributed by an outside observer to his different manners of playing shall never be defined.

The function $\phi(x, y)$ must then vary at each instant, and vary without following *any law at all.* One may well doubt if it is possible to indicate an effective and sure means of carrying out such counsel. It seems that, to follow it to the letter, a complete incoherence of mind would be needed, combined, of course, with the intelligence necessary to eliminate those methods we have called bad.

[2] Translator's note: There is an allusion here, familiar to French readers, to a contrast drawn by Pascal between the mathematical mind (esprit de géométrie) and the strategic mind (esprit de finesse).