Econ Theory (2012) 51:397–433
DOI 10.1007/s00199-011-0673-z

SYMPOSIUM

The non-constant-sum Colonel Blotto game

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Received: 23 May 2008 / Accepted: 22 September 2011 / Published online: 28 October 2011

Abstract The Colonel Blotto game is a two-player constant-sum game in which each player simultaneously distributes his fixed level of resources across a set of contests. In the traditional formulation of the Colonel Blotto game, the players' resources are "use it or lose it" in the sense that any resources that are not allocated to one of the contests are forfeited. This article examines a non-constant-sum version of the Colonel Blotto game that relaxes this use it or lose it feature. We find that if the level of asymmetry between the players' budgets is below a threshold, then there exists a one-to-one mapping from the unique set of equilibrium univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game. Once the asymmetry of the players' budgets exceeds the threshold, this relationship breaks down and we construct a new equilibrium.

Keywords Colonel Blotto game · All-pay auction · Contests · Mixed strategies · Multi-dimensional contest

JEL Classification C72 · D7

We have benefited from the helpful comments of Dan Kovenock, Wolfgang Leininger, and participants in presentations at the University of Iowa and the CESifo Summer Institute Workshop on Advances in the Theory of Contests and its Applications.

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1 Introduction

Originating with Borel (1921), the Colonel Blotto game is a classic model of budget-constrained resource allocation across multiple simultaneous contests. Borel formulates this problem as a constant-sum game involving two players, A and B, who must each allocate a fixed amount of resources, $X_A = X_B$, over a finite number of contests. Each player must distribute their resources without knowing their opponent's distribution of resources. In each contest, the player who allocates the higher level of resources wins, and each player's payoff across all of the contests is the proportion of the wins across the individual contests.[1]

This simple model was a focal point in the early game theory literature (see, for example, Bellman 1969; Blackett 1954, 1958; Borel and Ville 1938; Gross and Wagner 1950; Shubik and Weber 1981; Tukey 1949). The Colonel Blotto game has also experienced a recent resurgence of interest (see, for example, Golman and Page 2009; Hart 2008; Hortala-Vallve and Llorente-Saguer 2011; Kovenock and Roberson 2010; Kvasov 2007; Laslier 2002; Laslier and Picard 2002; Macdonell and Mastronardi 2011; Roberson 2006, 2008; or Weinstein 2005).[2] One of the main appeals of the Colonel Blotto game is that it provides a unified theoretical framework that is relevant to a diverse set of environments ranging from political campaign resource allocation to military conflict. In these constant-sum applications, each player has a fixed level of resources to allocate across the set of contests, and any unused resources have no value.

There are also a number of closely related applications of multi-dimensional resource allocation such as research and development races, rent-seeking, lobbying, and litigation. However, these applications are non-constant sum in that any resources that are not allocated to one of the contests have value, i.e., the players' resources are not "use it or lose it." Kvasov (2007) introduces a non-constant-sum version of the Colonel Blotto game that relaxes this use it or lose it feature of the original formulation.[3] In the case of symmetric budgets, that article establishes that there exists a one-to-one mapping from the unique set of equilibrium univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game.

In this article, we extend the analysis of the non-constant-sum version of the Colonel Blotto game to allow for asymmetric budget constraints. For all configurations of the asymmetric constant-sum Colonel Blotto game with three or more contests, Roberson (2006) provides: (i) the characterization of the unique equilibrium payoffs,[4] (ii) the characterization of each player's set of equilibrium univariate

[1] This is the plurality objective. An alternative objective [the majority or tournament objective] is for each player to maximize the probability that they win a majority of the contests. For $n > 3$, the solution to the majority game is an open question.

[2] Related experimental work includes Arad and Rubinstein (2009), Avrahami and Kareev (2009), Chowdhury et al. (2011), and Hortala-Vallve and Llorente-Saguer (2010).

[3] See also the related non-constant-sum multi-dimensional contests examined in Snyder (1989), Szentes and Rosenthal (2003a,b), Klumpp and Polborn (2006), Clark and Konrad (2007), and Arbatskaya and Mialon (2010).

[4] The case of $n = 2$, with symmetric and asymmetric forces, is discussed by Gross and Wagner (1950). Moving from $n = 2$ to $n \geq 3$ greatly enlarges the space of feasible n-variate distribution functions, and the equilibrium strategies examined in that article differ dramatically from the case of $n = 2$.

marginal distributions, and (iii) the existence of joint distributions which, in addition to providing the equilibrium univariate marginal distributions, expend the players' respective budgets with probability one. We find that as long as the asymmetry between the players' budgets is below a threshold, there exists a one-to-one mapping from the unique set of equilibrium univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game. Once the asymmetry of the players' budgets exceeds the threshold, this relationship breaks down. For this range, we construct an entirely new equilibrium for the non-constant-sum game. In all parameter configurations for which there exist unique equilibrium univariate marginal distributions, these are characterized. For these parameter configurations, we also characterize the unique equilibrium payoffs and the unique equilibrium total expected expenditures.

The non-constant-sum Colonel Blotto game is essentially a set of n independent all-pay auctions in which two players submit n-tuples of bids subject to budget constraints that hold across the entire set of auctions. Therefore, our results may also be seen as extending the analysis of the single all-pay auction with budget-constrained bidders (see Che and Gale 1998) to allow for budget constraints that apply across a finite set of auctions.

Section 2 presents the model. Section 3 provides a brief comparison of the constant-sum and non-constant-sum formulations of the Colonel Blotto game and provides intuition for the connection between the equilibria in these two games. Section 4 characterizes the equilibrium payoffs and the equilibrium sets of univariate marginal distributions for the asymmetric non-constant-sum version of the Colonel Blotto game. Section 5 concludes.

2 The model

Two players, A and B, simultaneously enter bids in a finite number, $n \geq 2$, of independent all-pay auctions. Each all-pay auction has a common value of v for each player. Each player has a fixed level of available resources (or budget), X_i for $i = A, B$. Let $X_A \leq X_B$, and let $\mathbf{x}_i$ denote the n-tuple of bids $(x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,n})$, one bid for each auction j. If both players enter the same bid in an auction and the common bid is X_A [resp., $X_B - (n-1)X_A$], then it is assumed that player B [resp., A] wins the auction. Otherwise, in the case of a tie, each player wins the auction with equal probability. As long as the asymmetry in the players' budgets is below a threshold [$X_B \leq (n-1)X_A$], any tie-breaking rule that avoids the need to have the stronger player B provide a bid arbitrarily close to, but above, player A's maximal bid yields similar results. Similarly, once $X_B > (n-1)X_A$, any tie-breaking rule that avoids the need to have the weaker player A provide a bid arbitrarily close to, but above, a bid of $X_B - (n-1)X_A$ by player B when player B bids X_A in the $n-1$ other auctions yields similar results.

In each all-pay auction j the payoff to player i for a bid of $x_{i,j}$ is given by

$$\pi_{i,j}\left(x_{i,j}, x_{-i,j}\right) = \begin{cases} v - x_{i,j} & \text{if } x_{i,j} > x_{-i,j} \\ -x_{i,j} & \text{if } x_{i,j} < x_{-i,j} \end{cases}$$

where ties are handled as described above. Each player's payoff across all n all-pay auctions is the sum of the payoffs across the individual auctions.

The bid provided to each all-pay auction must be nonnegative. For player i, the set of feasible bids across the n all-pay auctions is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}^n_+ \middle| \sum_{j=1}^{n} x_{i,j} \leq X_i \right\}.$$

Strategies

Given that each of the individual contests is an all-pay auction, it is not difficult to show that there are no pure-strategy equilibria for this class of games. A mixed strategy, which we term a *distribution of resources*, for player i is an n-variate distribution function $P_i : \mathbb{R}^n_+ \rightarrow [0, 1]$ with support (denoted Supp(P_i)) contained in the set of player i's set of feasible bids $\mathfrak{B}_i$ and with one-dimensional marginal distribution functions $\{F_{i,j}\}_{j=1}^n$, one univariate marginal distribution function for each all-pay auction j. To avoid confusion with the support of the joint distribution, when referring to the support of a given univariate marginal distribution—the smallest closed univariate interval whose complement has probability zero—we will make a slight abuse of terminology and use the term *domain* to denote the support of the given univariate marginal distribution function. The n-tuple of player i's bids across the n all-pay auctions is a random n-tuple drawn from the n-variate distribution function P_i.

The Non-Constant-Sum Colonel Blotto game

The N-C-S Colonel Blotto game, which we label

$$NCB\{X_A, X_B, n, v\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of resources subject to their budget constraints, each all-pay auction is won by the player that provides the higher bid in that auction (where in the case of a tie the tie-breaking rule described above applies), and players receive the sum of their payoffs across the individual all-pay auctions.

3 Relationship between the two formulations

Before proceeding with the equilibrium analysis, it is instructive to provide intuition for the connection between the equilibria in the constant-sum and non-constant-sum formulations of the Colonel Blotto game. The formulation of the constant-sum Colonel Blotto game differs from the non-constant-sum game in that in each contest j the payoff to each player i for a bid of $x_{i,j}$ is given by

$$\pi_{i,j}\left(x_{i,j}, x_{-i,j}\right) = \begin{cases} \frac{1}{n} & \text{if } x_{i,j} > x_{-i,j} \\ 0 & \text{if } x_{i,j} < x_{-i,j} \end{cases}$$

where ties are handled as described above. Note that, in the constant-sum game resources that are not allocated to one of the contests have no value; that is, resources are use it or lose it. Each player's payoff across all n contests is the sum of the payoffs in the individual contests.

The following discussion provides a brief sketch of the relationship between the equilibria in the constant-sum and non-constant-sum formulations of the game. We begin this discussion with the disclaimer that this is not a sketch of the formal proofs of the main results [which are provided in the Appendix]. Instead, our objective for this discussion is simply to provide a few informal insights regarding some necessary conditions for equilibrium in both the constant-sum and non-constant-sum Colonel Blotto games and to highlight the relationship between these sets of necessary conditions. For $n \geq 3$ auctions, the Appendix provides the formal proof of the necessity of these conditions.[5]

Given that player $-i$'s strategy is given by the n-variate distribution function P_{-i} with the set of univariate marginal distribution functions $\{F_{-i,j}\}_{j=1}^{n}$, player i's expected payoff for any n-tuple of bids $\mathbf{x}_i \in \mathbb{R}_{+}^{n}$ is:

$$\pi_i\left(\mathbf{x}_i, \{F_{-i,j}\}_{j=1}^{n}\right) = \sum_{j=1}^{n}\left[vF_{-i,j}\left(x_{i,j}\right) - x_{i,j}\right]. \tag{1}$$

Observe that for a given P_{-i}, each player i's expected payoff depends only on the set of univariate marginal distribution functions $\{F_{-i,j}\}_{j=1}^{n}$ and not the correlation structure, utilized by player $-i$, among the univariate marginals.

Given this feature of the expected payoffs, it is useful to note that any joint distribution may be broken into a set of univariate marginal distribution functions and an n-copula, the function that maps the univariate marginal distribution functions into a joint distribution function.[6] Let $\mathcal{C}_i$ denote the collection of all sets of univariate marginal distribution functions $\{F_{i,j}\}_{j=1}^{n}$ which satisfy the constraint that there exists a mapping from the set of univariate marginal distributions into a joint distribution (an n-copula), C, in which the support of the resulting n-variate distribution function $C(F_{i,1}(x_1), \ldots, F_{i,n}(x_n))$ is contained in $\mathfrak{B}_i$.

Assuming that each of the univariate marginal distributions in player i's strategy is differentiable (possibly discontinuously so) and ignoring the possibility of a tie occurring with strictly positive probability, player i's optimization problem may be written as:

5 In the case of $n = 2$, these conditions are not necessary. See the discussion of the case of $n = 2$ at the conclusion of the next section.

6 See Nelsen (1999) or Schweizer and Sklar (1983) for an introduction to copulas.

$$\max_{\left\{\{F_{i,j}\}_{j=1}^{n}\in\mathcal{C}_i\right\}}\sum_{j=1}^{n}\left[\int_0^{\infty}\left[vF_{-i,j}\left(x_{i,j}\right)-x_{i,j}\right]\mathrm{d}F_{i,j}\right]. \tag{2}$$

Observe that the n-copula enters into the players' optimization problems only as a constraint and not as a strategic variable. That is, player i's optimization problem is invariant to the correlation structure among his own univariate marginal distribution functions subject to the constraint that there exists a mapping from the optimal set of univariate marginal distributions into a joint distribution that satisfies the restriction on the support.

Next, recall that the budget constraint holds with probability one. Therefore, the budget constraint must also hold in expectation, and player i's set of univariate marginal distribution functions satisfy the following constraint,

$$\sum_{j=1}^{n}\left[\int_0^{\infty}x_{i,j}\,\mathrm{d}F_{i,j}\right]\leq X_i. \tag{3}$$

Given that Eq. (3) is a constraint on only the set of univariate marginal distributions functions, it will be useful to include this constraint in player i's optimization problem. Thus, we have that player i's optimization problem from Eq. (2) may now be written as,

$$\max_{\left\{\{F_{i,j}\}_{j=1}^{n}\in\mathcal{C}_i\right\}}\sum_{j=1}^{n}\left[\int_0^{\infty}\left[vF_{-i,j}\left(x_{i,j}\right)-(1+\lambda_i)\,x_{i,j}\right]\mathrm{d}F_{i,j}\right]+\lambda_i X_i. \tag{4}$$

This optimization problem is essentially a variational problem involving the maximization of a collection of functionals with the side constraints that there exists a sufficient n-copula and that each univariate marginal distribution is a weakly increasing function. The n Euler–Lagrange equations provide a set of necessary conditions for equilibrium. For each $j=1,\ldots,n$, the corresponding Euler–Lagrange equation is given by

$$\frac{d}{dx}\left[vF_{-i,j}\left(x_{i,j}\right)-(1+\lambda_i)\,x_{i,j}\right]=0. \tag{5}$$

Rearranging terms slightly, it becomes clear that for each auction j equation (5) is precisely the necessary condition that holds for one isolated all-pay auction without a budget constraint and in which the prize has value $v/(1+\lambda_i)$, henceforth the *implicit value of the prize*. The intuition is that the constraint on the total expenditure across all auctions implicitly imposes an opportunity cost $\lambda_i\geq 0$ of resource expenditure.[7] Therefore, the cost of allocating x_j resources to auction j entails not only the explicit

[7] Note that λ_i takes the value of zero in the event that player i does not benefit from the relaxation of his budget constraint.

cost of the bid but also the implicit opportunity cost from not being able to use those resources in another auction. An increase in the implicit opportunity cost of a bid has the dual interpretation of lowering the implicit value of the prize.

Applying a similar line of reasoning to the constant-sum Colonel Blotto game, it is straightforward to derive the set of necessary conditions for equilibrium given by the n Euler–Lagrange equations for that optimization problem. For each $j = 1, \ldots, n$, the corresponding Euler–Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{n}F_{-i,j}\left(x_{i,j}\right) - \lambda_i x_{i,j}\right] = 0. \tag{6}$$

In this case, we see that for each contest j Eq. (6) is precisely the necessary condition that holds for one isolated all-pay auction without a budget constraint and in which the prize has value $1/(n\lambda_i)$.

As long as there exists a sufficient n-copula, each of the unique equilibrium univariate marginal distribution functions in the two games corresponds directly to the unique equilibrium univariate distribution function in a single two-player all-pay auction with complete information and with each player i's values for the prizes given by $v/(1+\lambda_i)$ and $1/(n\lambda_i)$ respectively (see Hillman and Riley 1989; Baye et al. 1996).[8] Therefore, there exists a one-to-one mapping from the unique set of equilibrium univariate marginal distributions in the non-constant-sum game to those in the constant-sum game as long as there exists a sufficient n-copula.

In general, the constraint on the n-copula is non-binding if for each player the intersection of the hyperplane formed by the n-tuples that exhaust his respective budget and the n-box formed by the domains of each of the univariate marginal distributions for the corresponding all-pay auctions is well behaved. For example, consider the case in which the n-box formed by the domains is $[0, X_A]^n$. If $X_B > (n-1)X_A$, then it is clear that there exist no n-tuples in the intersection of the hyperplane $\{\mathbf{x} \in \mathbb{R}^n_+ \,|\, \sum_{j=1}^n x_j = X_B\}$ and the n-box $[0, X_A]^n$ in which any $x_j = 0$. Thus, the support of player B's distribution of resources cannot be completely contained in his budget-balancing hyperplane and have univariate marginals with domain $[0, X_A]$.

In the constant-sum game, the constraint on the existence of a sufficient n-copula is non-binding as long as $(2/n) < (X_A/X_B) \leq 1$. Within this region, which is illustrated in Panel (i) of Fig. 1, Theorem 2 of Roberson (2006) characterizes the unique equilibrium univariate marginal distribution functions, and Theorem 4 of that article provides the proof of the existence of a sufficient n-copula for this range.

Before tracing out the corresponding region for the non-constant-sum formulation of the game, observe that Panel (i) of Fig. 1 also delineates the regions of the parameter space that correspond to Theorems 3 and 5 of Roberson (2006), labeled regions 3 and 5, respectively. In these regions, in which $(1/n) < (X_A/X_B) \leq (2/n)$, there exists a corresponding parameter region in the non-constant-sum game over which the equilibrium univariate marginal distribution functions in the two games are related. However, this relationship is not necessarily one-one. The issue is that the constraint

8 For generalizations of the all-pay-auction, see Baye et al. (2011) and Siegel (2009). For recent examples of applications, see Levy and Razin (2011) and Sela (2011).

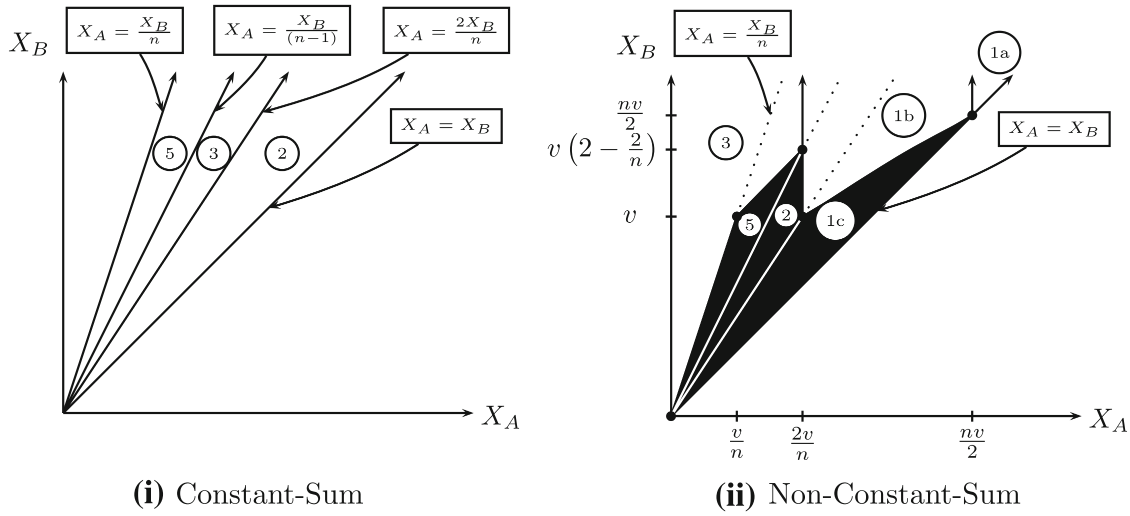

Fig. 1 Parameter space $n \geq 3$

on the existence of a sufficient n-copula comes into play, and the sets of univariate marginal distributions must be adjusted accordingly. In the two games, these adjustments may vary.

For the constant-sum game's remaining parameter configuration $X_A \leq (X_B/n)$, the players are at the extreme end of the asymmetry spectrum. Over this parameter region, the stronger player (B) has a sufficient level of resources to win each of the n contests with certainty, and, due to the use it or lose it feature of the constant-sum formulation, that game becomes trivial. In this region, there is no relationship between the two games. Due to the relaxation of the use it or lose it feature, the non-constant-sum game is never trivial, and in this range, we construct entirely new equilibrium distributions of resources for the non-constant-sum game.

We now introduce what we term the *modified budgets* for the non-constant-sum game with $n \geq 3$. In the expressions for the modified budgets, we define the sets Tk for $k = 1, 2, 3, 5$ to denote the portion of the parameter space that is covered by the corresponding theorem number $k(= 1, 2, 3, 5)$ in the following section.[9] These regions are delineated as follows.

$$\text{T1: } \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| \left(\tfrac{2}{n}\right) \min\{v, X_B\} < X_A \leq X_B\right\}$$

$$\text{T2: } \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_B/(n-1) \leq X_A \leq \left(\tfrac{2}{n}\right) \min\{v, X_B\} \text{ or } X_A = \tfrac{2v}{n} \text{ and } X_B > v\left(2 - \tfrac{2}{n}\right)\right\}$$

$$\text{T3: } \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_A < \left(\tfrac{2v}{n}\right) \text{ and } X_A \leq \max\left\{\frac{X_B - \frac{2v}{n}}{n-2}, \frac{X_B}{n}\right\}\right\}$$

$$\text{T5: } \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| \max\left\{\frac{X_B - \frac{2v}{n}}{n-2}, \frac{X_B}{n}\right\} < X_A < \frac{X_B}{n-1}\right\}$$

[9] Theorem 4 establishes the existence of a pair of n-variate joint distributions that satisfy the conditions specified in Theorem 3.

Recall that the floor function $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Player A's modified budget is given by

$$M_{X_A}(X_A, X_B) = \begin{cases} \min\left\{X_A, \frac{nv}{2}\right\} & \text{if } (X_A, X_B) \in \text{T1} \\ X_A & \text{if } (X_A, X_B) \in \text{T2} \\ \frac{n(X_A)^2}{2v} & \text{if } (X_A, X_B) \in \text{T3} \\ X_A - \frac{\left(1-\frac{nX_A}{2v}\right)\left(nX_A - X_B\right)}{\left\lfloor \frac{X_A}{X_B-(n-1)X_A} \right\rfloor + 1} & \text{if } (X_A, X_B) \in \text{T5} \end{cases}$$

and player B's modified budget is given by

$$M_{X_B}(X_A, X_B) = \begin{cases} \min\left\{X_B, \frac{nv}{2}, \left(\frac{nvX_A}{2}\right)^{1/2}\right\} & \text{if } (X_A, X_B) \in \text{T1} \\ \min\left\{X_B, v\left(2 - \frac{2}{n}\right)\right\} & \text{if } (X_A, X_B) \in \text{T2} \\ n\left(X_A - \frac{X_A^2}{2v}\right) & \text{if } (X_A, X_B) \in \text{T3} \\ \frac{nX_A(nX_B-(n-1)^2X_A)}{2v} + \left(1 - \frac{n(X_B-(n-2)X_A)}{2v}\right) & \\ \quad \times \frac{\left(\left\lfloor \frac{X_A}{X_B-(n-1)X_A} \right\rfloor + 2\right)X_A}{\left\lfloor \frac{X_A}{X_B-(n-1)X_A} \right\rfloor + 1} & \text{if } (X_A, X_B) \in \text{T5} \end{cases}$$

It will be useful to define the set of n-tuples that exhaust the modified budgets M_{X_A} and M_{X_B}. Let $\overline{\mathfrak{B}}_i$ denote this set, defined as

$$\overline{\mathfrak{B}}_i = \left\{ \mathbf{x} \in \mathbb{R}^n_+ \middle| \sum_{j=1}^{n} x_{i,j} = M_{X_i}(X_A, X_B) \right\},$$

and note that $\overline{\mathfrak{B}}_i \subset \mathfrak{B}_i$.

The players' modified budgets, which are illustrated in (X_A, X_B)-space as the shaded regions in Panel (ii) of Fig. 1, are the equilibrium total expected expenditures for each of the equilibria examined in the following section [i.e., for player i, $M_{X_i} = \sum_j E_{F_{i,j}}(x_{i,j})$]. As shown in the Appendix [see Lemma 2], in the T1 and T2 parameter regions with $X_A \neq (2v/n)$, these equilibrium total expected expenditures are unique. In the remaining parameter regions, there exist other payoff non-equivalent equilibria.

Note that given a pair of resource levels X_A and X_B which satisfy $(X_A, X_B) \in \text{T1}$, there are three possible cases: (a) neither player uses all of their available resources [i.e., $M_{X_A} = nv/2$ and $M_{X_B} = nv/2$], (b) only (the weaker) player A uses all of his available resources [i.e., $M_{X_A} = X_A$ and $M_{X_B} = (nvX_A/2)^{1/2}$], and (c) both players A and B use all of their available resources [i.e., $M_{X_A} = X_A$ and $M_{X_B} = X_B$]. The regions corresponding to each of these cases appear in Panel (ii) of Fig. 1 as 1a, 1b, and 1c, respectively. Given that in the constant-sum game resources are use it or lose it, such considerations do not arise in that game.

It is important to observe that when X_A and X_B satisfy the condition that $X_A \in ((2/n)\min\{v, X_B\}, X_B]$ [i.e., regions 1a, 1b, and 1c of Panel (ii) of Fig. 1], the modified budgets satisfy the corresponding condition that $(2/n) < (M_{X_A}/M_{X_B}) \leq 1$. As we will show, there exists a one-to-one correspondence between the sets of equilibrium univariate marginal distribution functions that arise in this region and those that arise in the constant-sum game for the region $(2/n) < (X_A/X_B) \leq 1$. This characterization is formally stated in Theorem 1 of the next section.

Similarly, for X_A and X_B that lie in regions 2 and 5 [which correspond to Theorems 2 and 5] of Panel (ii) of Fig. 1, the modified budgets satisfy the condition that $(1/n) < (M_{X_A}/M_{X_B}) \leq (2/n)$. In these regions, the sets of equilibrium univariate marginal distribution functions are related to those arising in the constant-sum game for the parameter range $(1/n) < (X_A/X_B) \leq (2/n)$. But as mentioned before, this relationship is not necessarily one-one.

For all budget configurations (X_A, X_B) that lie in region 3 of panel (b), we construct an entirely new set of equilibrium distributions of resources [see Theorem 3]. Note that this region covers not only the portion of the parameter space which corresponds to the trivial region of the constant-sum game [i.e., $X_A \leq (X_B/n)$], but also a portion of the parameter space in which the constant-sum game is non-trivial. Again, this breakdown in the relationship between the equilibria in the two games occurs in sufficiently asymmetric regions of the parameter space because of the discrepancy in the value of unused resources in the two formulations.

To summarize, whereas there is a one-to-one relationship between the unique equilibrium sets of univariate marginal distribution functions in the constant-sum and non-constant-sum versions of the game—when the asymmetry between the players' budgets is below a threshold—this relationship is nonlinear with respect to the players' budgets but is linear with respect to the players' modified budgets.

4 Equilibrium distributions of resources

The following Theorems examine the equilibrium distributions of resources for all parameter configurations of the non-constant-sum Colonel Blotto game with $n \geq 3$ auctions. This section concludes with the case of $n = 2$ auctions. In the Theorem 1 parameter range, we characterize each player's unique set of equilibrium univariate marginal distributions. In the Theorem 2 parameter range with $X_A \neq (2v/n)$, we characterize the unique set of equilibrium univariate marginal distributions for player A and provide an equilibrium distribution of resources for player B. Over this range, player B does not have a unique set of equilibrium univariate marginal distribution functions.[10] In the Theorems 3 and 5 parameter ranges, we provide an equilibrium distribution of resources for each player. Over this range, neither player has a unique set of univariate marginal distribution functions.[11] For the Theorems 1 and 2 parameter ranges with $X_A \neq (2v/n)$, the equilibrium expected payoffs and the equilibrium total expected expenditures are unique [see Lemma 2 in the Appendix].

[10] An alternative set of equilibrium univariate marginal distribution functions is provided in the discussion following Lemma 7 in the Appendix.

[11] For information on the non-uniqueness of the univariate marginals over the Theorem 5 [3] parameter range, see the discussion preceding Theorem 5 [at the conclusion of the Appendix].

Three or more auctions

For the game $NCB\{X_A, X_B, n, v\}$ with $n \geq 3$, Theorem 1 examines all parameter configurations that lie in the 1a, 1b, and 1c regions of panel (ii) of Fig. 1. Recall that in these regions the resulting modified budgets satisfy the condition $(2/n) < (M_{X_A}/M_{X_B}) \leq 1$.

Theorem 1 *Let X_A, X_B, v, and $n \geq 3$ satisfy $(2/n)\min\{v, X_B\} < X_A \leq X_B$ (equivalently $(2/n) < (M_{X_A}/M_{X_B}) \leq 1$). The n-variate distribution functions P_A^* and P_B^* constitute a Nash equilibrium of the game $NCB\{X_A, X_B, n, v\}$ if and only if they satisfy the following two conditions:* (1) *For each player i,* $\mathrm{Supp}(P_i^*) \subset \mathfrak{B}_i$ *and* (2) *P_i^*, $i = A, B$, provides the corresponding unique set of univariate marginal distribution functions $\{F_{i,j}^*\}_{j=1}^n$ outlined below.*

$$\forall\ j \in \{1, \ldots, n\}\ F_{A,j}^*\left(x_j\right) = \left(1 - \frac{M_{X_A}}{M_{X_B}}\right) + \frac{x_j}{(2/n)M_{X_B}}\left(\frac{M_{X_A}}{M_{X_B}}\right) \quad \text{for } x_j \in \left[0, \frac{2}{n}M_{X_B}\right].$$

$$\forall\ j \in \{1, \ldots, n\}\ F_{B,j}^*\left(x_j\right) = \frac{x_j}{(2/n)M_{X_B}} \quad \text{for } x_j \in \left[0, \frac{2}{n}M_{X_B}\right].$$

The unique equilibrium expected payoff for player A is $(nvM_{X_A}/2M_{X_B}) - M_{X_A}$, and the unique equilibrium expected payoff for player B is $nv\left(1 - (M_{X_A}/2M_{X_B})\right) - M_{X_B}$. The unique equilibrium total expected expenditure for player A is $M_{X_A}(X_A, X_B) = \min\{X_A, (nv/2)\}$, and the unique equilibrium total expected expenditure for player B is $M_{X_B}(X_A, X_B) = \min\{X_B, (nv/2), (nvX_A/2)^{1/2}\}$.

The existence of a pair of n-variate distribution functions that satisfy conditions (1) and (2) of Theorem 1 is provided in Roberson (2006). In particular, Theorem 4 of Roberson (2006) establishes the existence of n-variate distribution functions for which $\mathrm{Supp}(P_i^*) \subset \mathfrak{B}_i$ and that provide the necessary sets of univariate marginal distribution functions given in Theorem 1. The proof of the uniqueness of the equilibrium sets of univariate marginal distribution functions, equilibrium payoffs, and equilibrium total expected expenditures is given in the Appendix.

Although it is straightforward to show that any pair of n-variate distribution functions that satisfies conditions (1) and (2) of Theorem 1 forms an equilibrium, it is useful to provide the intuition for this result. We begin with the expected payoffs for each player. Let P_B^* denote a feasible n-variate distribution function for player B with the univariate marginal distributions $\{F_{B,j}^*\}_{j=1}^n$ given in Theorem 1. If player B is using P_B^*, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A \bigcap [0, (2/n)M_{X_B}]^n$ [i.e., one bid for each of the n all-pay auctions such that $\sum_j x_{A,j} \leq X_A$ and $x_{A,j} \in \left[0, (2/n)M_{X_B}\right]$ for each auction j], is

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = \sum_{j=1}^{n}\left[vF_{B,j}^*\left(x_{A,j}\right) - x_{A,j}\right].$$

Recall that for all j, $F^*_{B,j}\left(x_j\right) = (x_j)/((2/n)M_{X_B})$ for $x_j \in \left[0, (2/n)M_{X_B}\right]$. Simplifying yields

$$\pi_A\left(\mathbf{x}_A, P^*_B\right) = \left(\frac{nv}{2M_{X_B}} - 1\right)\sum_{j=1}^{n} x_{A,j}. \tag{7}$$

Similarly, the expected payoff π_B to player B from any n-tuple of bids $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, (2/n)M_{X_B}]^n$—when player A uses a feasible n-variate distribution P^*_A with the univariate marginal distributions $\{F^*_{A,j}\}_{j=1}^n$ given in Theorem 1—follows directly,

$$\pi_B\left(\mathbf{x}_B, P^*_A\right) = nv\left(1 - \frac{M_{X_A}}{M_{X_B}}\right) + \left(\frac{nvM_{X_A}}{2M^2_{X_B}} - 1\right)\sum_{j=1}^{n} x_{B,j}. \tag{8}$$

Observe that neither player can bid below 0 and that bidding above $(2/n)M_{X_B}$ is suboptimal. Thus, for the Theorem 1, parameter range equations (7) and (8) provide the maximal payoffs (for player A and player B, respectively) for any feasible n-tuple of bids across the n all-pay auctions.

Recall that there are three possible cases: (a) neither player uses all of his available resources, (b) only (the weaker) player A uses all of his available resources, and (c) both players A and B use all of their available resources. These three regions are shown graphically in panel (ii) of Fig. 1 as regions 1a, 1b, and 1c, respectively. Suppose that we are in case (a) in which neither player uses all of his available resources. Case (a) corresponds to the situation in which the total value of the n auctions nv is low enough relative to the players' budgets that neither player has incentive to commit all of his resources. In the Theorem 1 parameter range, player A's modified budget is given by $M_{X_A} = \min\{X_A, nv/2\}$. If player A does not use all of his budget, then it must be that $X_A > (nv/2)$ and so $M_{X_A} = (nv/2)$. Similarly from player B's modified budget in the Theorem 1 range $[M_{X_B} = \min\{X_B, nv/2, (nvX_A/2)^{1/2}\}]$, it follows that if player A (the weaker player) is not using all of his budget then $M_{X_B} = (nv/2)$. Because $M_{X_A} = M_{X_B} = (nv/2)$, the expected payoffs given in (7) and (8) are $\pi_A\left(\mathbf{x}_A, P^*_B\right) = 0$ and $\pi_B\left(\mathbf{x}_B, P^*_A\right) = 0$, respectively. Observe that in case (a) neither player has incentive to change their total resource expenditure, $\sum_j x_{i,j}$, across the n all-pay auctions. That is, because $M_{X_A} = M_{X_B} = (nv/2)$ and the opponent is using the equilibrium strategy, the expected payoff to player i, given in Eqs. (7) and (8), is zero for all $\mathbf{x}_i \in [0, v]^n$ regardless of player i's total expenditure, $\sum_j x_{i,j}$, in the n all-pay auctions.

Now suppose that we are in case (b) in which only player A uses all of his budget. Case (b) corresponds to the situation in which the total value of the n all-pay auctions nv is high enough that the weaker player optimally commits all of his resources, but not so high that the stronger player must also commit all of his resources to the n all-pay auctions. From the preceding discussion, it follows that $X_A \leq (nv/2)$ and thus $M_{X_A} = X_A$. If player B is not using all of his budget then from $M_{X_B} = \min\{X_B, nv/2, (nvX_A/2)^{1/2}\}$, it must be that $X_B > (nvX_A/2)^{1/2}$ and so $M_{X_B} =$

$(nvX_A/2)^{1/2}$. Inserting M_{X_A} and M_{X_B} into Eqs. (7) and (8) and simplifying yields

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = \left(\left(\frac{nv}{2X_A}\right)^{1/2} - 1\right)\sum_{j=1}^{n} x_{A,j} \tag{9}$$

and

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = nv\left(1 - \left(\frac{2X_A}{nv}\right)^{1/2}\right). \tag{10}$$

Recall that in case (b) $X_A \leq (nv/2)$ and so $((nv/2X_A)^{1/2} - 1) \geq 0$. From Eq. (9), we see that player A is indifferent with regard to which all-pay auctions to commit resources, but has incentive to increase his total resource expenditure across the n all-pay auctions [i.e., $\sum_j x_{A,j}$]. However in case (b), player A's equilibrium distribution of resources P_A^* expends his budget with probability one [i.e., at each point $b_A \in \text{Supp}(P_A^*)$, $\sum_j x_{A,j} = X_A$].[12] From Eq. (10), we see that, when $M_{X_A} = X_A$ and $M_{X_B} = (nvX_A/2)^{1/2}$ are inserted into player B's expected payoff given in Eq. (8), player B's expected payoff is the same for all n-tuples $\mathbf{x}_B \in (0, (2nvX_A)^{1/2}]^n$. That is player B's expected payoff is independent of his total expenditure $\sum_j x_{B,j}$ [so long as $\mathbf{x}_B \in (0, 2(nvX_A/2)^{1/2}]^n$], and so player B does not have incentive to change his total resource expenditure across the n all-pay auctions.

Finally, suppose that we are in case (c) in which each player is at his respective budget constraint. Case (c) corresponds to the situation in which the total value of the n all-pay auctions nv is high enough that both players optimally commit all of their resources to the n all-pay auctions. Thus, $M_{X_A} = X_A$ and $M_{X_B} = X_B$. From Eqs. (7) and (8), it follows that

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = \left(\frac{nv}{2X_B} - 1\right)\sum_{j=1}^{n} x_{A,j} \tag{11}$$

and

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = nv\left(1 - \frac{X_A}{X_B}\right) + \left(\frac{nvX_A}{2X_B^2} - 1\right)\sum_{j=1}^{n} x_{B,j}. \tag{12}$$

In case (c), $X_A < (nv/2)$ and $X_B < (nvX_A/2)^{1/2} < (nv/2)$. Observe in Eq. (11) that $((nv/2X_B) - 1) > 0$ and, thus, player A has incentive to increase his total resource expenditure across the n all-pay auctions, but in his equilibrium distribution of resources P_A^* he is already at his budget constraint with probability one [i.e., at each point $\mathbf{x}_A \in \text{Supp}(P_A^*)$, $\sum_j x_{A,j} = X_A$]. Similarly, in Eq. (12) $((nvX_A/2X_B^2) - 1) > 0$

[12] Recall that Roberson (2006) establishes the existence of n-variate distribution functions for which $\text{Supp}(P_i^*) \subset \overline{\mathfrak{B}}_i$, and that in this case $M_{X_A} = X_A$. It follows directly that player A expends his budget with probability one.

and, thus, player B has incentive to increase his total resource expenditure across the n all-pay auctions, but in his equilibrium distribution of resources P_B^*, he is already at his budget constraint with probability one [i.e., at each point $\mathrm{x}_B \in \mathrm{Supp}(P_B^*)$, $\sum_j x_{B,j} = X_B$].

Because Roberson (2006) demonstrates the existence of a pair of n-variate distributions $\{P_{A,}^*, P_{A,}^*\}$ in which $\mathrm{Supp}(P_i^*) \subset \overline{\mathfrak{B}}_i$ for $i = A, B$ and that provides the univariate marginal distributions specified in condition (2) of Theorem 1, it follows from the arguments given above that such a pair of n-variate distribution functions constitute an equilibrium in all three cases (a), (b), and (c). The proof of the uniqueness of the univariate marginal distributions is given in the Appendix.

Once $(M_{X_A}/M_{X_B}) = (2/n)$ both the uniqueness of player B's equilibrium univariate marginal distributions and the relationship with the two-player all-pay auction with complete information fail to hold. The reason for this breakdown is that once $X_B/(n-1) \leq X_A \leq (2/n)\min\{v, X_B\}$, or equivalently $(1/(n-1)) \leq (M_{X_A}/M_{X_B}) \leq (2/n)$, it is possible for player B's equilibrium univariate marginals to have atoms that lie strictly within the interior and at the upper bound of the domain and player B's equilibrium total expected expenditure is not unique.[13] In Theorem 2, we provide the unique equilibrium univariate marginal distributions for player A and provide an equilibrium set of univariate marginal distributions for player B.

Theorem 2 *Let X_A, X_B, v, and $n \geq 3$ satisfy $X_B/(n-1) \leq X_A \leq (2/n)\min\{v, X_B\}$ or $X_A = (2v/n)$ and $X_B > v(2-(2/n))$ [equivalently $1/(n-1) \leq (M_{X_A}/M_{X_B}) \leq (2/n)$]. The n-variate distribution function P_A^* is a Nash equilibrium strategy for player A in the game $NCB\{X_A, X_B, n, v\}$ if and only if it satisfies the following two conditions: (1) $\mathrm{Supp}(P_A^*) \subset \mathfrak{B}_A$ and (2) P_A^* provides the corresponding set of univariate marginal distribution functions $\{F_{A,j}^*\}_{j=1}^n$ outlined below.*

$$\forall\, j \in \{1, \ldots, n\} \quad F_{A,j}^*\left(x_j\right) = \left(1 - \frac{2}{n}\right) + \frac{x_j}{X_A}\left(\frac{2}{n}\right) \quad \text{for } x_j \in [0, X_A].$$

Sufficient conditions for P_B^* to be a Nash equilibrium strategy include: $\mathrm{Supp}(P_B^*) \subset \mathfrak{B}_B$ and that P_B^* provides the corresponding set of univariate marginal distribution functions $\{F_{B,j}^*\}_{j=1}^n$ outlined below.

$$\forall\, j \in \{1, \ldots, n\} \quad F_{B,j}^*\left(x_j\right) = \begin{cases} \frac{2x_j\left(X_A - \frac{M_{X_B}}{n}\right)}{(X_A)^2} & \text{for } x_j \in [0, X_A) \\ 1 & \text{for } x_j \geq X_A \end{cases}.$$

In equilibria satisfying these conditions on P_A^* and P_B^*, the expected payoff for player A is $2v(1-(M_{X_B}/nX_A)) - X_A$, the expected payoff for player B is $nv - 2v(1-(M_{X_B}/nX_A)) - M_{X_B}$, the total expected expenditure for player A is $M_{X_A}(X_A, X_B) = X_A$, and the total expected expenditure for player B is $M_{X_B}(X_A, X_B) = \min\{X_B, v(2-(2/n))\}$.

[13] See the discussion at the conclusion of the Appendix.

If $X_A \neq (2v/n)$, then the equilibrium expected payoffs and total expected expenditures are unique. In the event that $X_A = (2v/n)$ player B's equilibrium total expected expenditure is not unique. As a direct consequence player A's equilibrium expected payoff is not unique when $X_A = (2v/n)$.

The existence of a pair of n-variate distribution functions that satisfy Theorem 2's necessary and sufficient condition for player A and sufficient condition for player B is provided in Theorem 4 of Roberson (2006). Following Lemma 7 in the Appendix, the Appendix concludes with a sketch, for $X_A \neq (2v/n)$, of the proof of uniqueness for the equilibrium payoffs, the equilibrium total expected expenditures, and player A's univariate marginal distributions. If $X_A = (2v/n)$, then there exist equilibria in which player B uses strategies P_B in which $\sum_j E_{F_{B,j}}(x_{B,j}) \neq M_{X_B}(X_A, X_B)$, where since $(X_A, X_B) \in T2$, $M_{X_B}(X_A, X_B) = \min\{X_B, v(2-(2/n))\}$. In fact, there exists a continuum of equilibria in which P_B satisfies a modified form of the sufficient conditions given in Theorem 2. The modification to the sufficient conditions for P_B^* is that the term M_{X_B} in the univariate marginal distributions given above may be replaced by any value in the set $[v, \min\{X_B, v(2-(2/n))\}]$. In this case, it is clear that the equilibrium payoffs are not unique. Player B's set of equilibrium univariate marginal distributions is, also, not unique, and an alternative set of equilibrium univariate marginal distributions for player B is given in the discussion, following Lemma 7, at the conclusion of the Appendix.

To sketch the proof that a pair of n-variate distributions that satisfy the conditions of Theorem 2 form an equilibrium, let P_B^* denote a feasible n-variate distribution for player B with the univariate marginal distributions $\{F_{B,j}^*\}_{j=1}^n$ given in Theorem 2. If player B is using P_B^*, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A$, is

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = \left(\frac{2v\left(X_A - (M_{X_B}/n)\right)}{X_A^2} - 1\right)\sum_{j=1}^{n} x_{A,j}. \tag{13}$$

Note that $(2v/X_A^2)(X_A - (M_{X_B}/n)) - 1 \geq 0$ is equivalent to $M_{X_B} \leq (n - (nX_A/2v))X_A$. If $X_A < (2v/n)$, it follows from Eq. (13) that player A has incentive to choose n-tuples $\mathbf{x}_A \in [0, X_A]^n$ such that $\sum_j x_{A,j} = X_A$. When $X_A = (2v/n)$, player A's expected payoff from any n-tuple $\mathbf{x}_A \in [0, X_A]^n$ is zero.

Similarly, the expected payoff π_B to player B from any n-tuple of bids across the n all-pay auctions $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, X_A]^n$, when player A uses a feasible n-variate distribution P_A^* with the univariate marginal distributions $\{F_{A,j}^*\}_{j=1}^n$ given in Theroem 2, is

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = nv\left(1 - \frac{2}{n}\right) + \left(\frac{2v}{nX_A} - 1\right)\sum_{j=1}^{n} x_{B,j}. \tag{14}$$

Because $X_A \leq (2v/n)$ it follows that $(2v/nX_A) - 1 \geq 0$. If $X_A < (2v/n)$, player B has incentive to choose n-tuples $\mathbf{x}_B \in (0, X_A]^n$ such that $\sum_j x_{B,j} = X_B$. If

$X_A = (2v/n)$, then any n-tuple $\mathbf{x}_B \in (0, X_A]^n$ provides player B with an expected payoff of $nv(1 - (2/n))$.

Because Roberson (2006) demonstrates the existence of a pair of n-variate distributions that result in the univariate marginal distributions given in Theorem 2 and that satisfy the respective budget restrictions with probability one [i.e., for $i = A, B$ at each point $b_i \in \text{Supp}(P_i^*)$, $\sum_j x_{i,j} = M_{X_i}$], it follows from the arguments given above that such a pair of n-variate distribution functions constitutes an equilibrium. A sketch of the proof of the uniqueness of player A's univariate marginal distributions is given in the discussion, following Lemma 7, at the conclusion of the Appendix.

The following Theorem constructs entirely new equilibrium distributions of resources for the highly asymmetric portion of the parameter space in which the relationship between the constant-sum and non-constant-sum versions of the game breaks down.

Theorem 3 *Let X_A, X_B, v, and $n \geq 3$ satisfy $X_A < (2v/n)$ and $X_A \leq \max\{(X_B - (2v/n))/(n-2), X_B/n\}$. The n-variate distribution functions P_A^* and P_B^* constitute a Nash equilibrium of the game $NCB\{X_A, X_B, n, v\}$ if they satisfy the following two conditions:* (1) *For each player i,* $\text{Supp}(P_i^*) \subset \mathfrak{B}_i$ *and* (2) *$P_i^*, i = A, B$, provides the corresponding set of univariate marginal distribution functions $\{F_{i,j}^*\}_{j=1}^n$ outlined below.*

$$\forall\, j \in \{1, \ldots, n\} \quad F_{A,j}\left(x_j\right) = \left(1 - \frac{X_A}{v}\right) + \frac{x_j}{v} \quad \text{for } x_j \in [0, X_A].$$

$$\forall\, j \in \{1, \ldots, n\} \quad F_{B,j}\left(x_j\right) = \begin{cases} \frac{x_j}{v} & \text{for } x_j \in [0, X_A) \\ 1 & \text{for } x_j \geq X_A \end{cases}$$

In equilibria satisfying these conditions on P_A^* and P_B^*, the expected payoff for player A is 0, *the expected payoff for player B is $nv(1 - (X_A/v))$, the total expected expenditure for player A is $(X_A)^2(n/2v)$, and the total expected expenditure for player B is $n(X_A - (X_A)^2/2v)$.*

We begin with a sketch of the proof that a pair of n-variate distribution functions that satisfy the conditions of Theorem 3 form an equilibrium and then move on to the proof of existence of such a pair of n-variate distribution functions.

To see that these two sets of univariate marginal distributions form an equilibrium in the Theorem 3 parameter region, let P_B^* denote a feasible n-variate distribution for player B with the univariate marginal distributions $\{F_{B,j}^*\}_{j=1}^n$ given in Theorem 3. If player B is using P_B^*, then player A's expected payoff π_A, when player A chooses any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A$ is

$$\pi_A\left(\mathbf{x}_A, P_B^*\right) = 0. \tag{15}$$

From Eq. (15), player A does not have incentive to increase or decrease his total expenditure in the n all-pay auctions.

Similarly, the expected payoff π_B to player B from any n-tuple of bids across the n all-pay auctions $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, X_A]^n$, when player A uses a feasible n-variate distribution P_A^* with the univariate marginal distributions $\{F_{A,j}^*\}_{j=1}^n$ given in Theorem 3, is

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = nv\left(1 - \frac{X_A}{v}\right). \tag{16}$$

Thus, player B also has the same expected payoff for each $\mathbf{x}_B \in (0, X_A]^n$ and therefore has no incentive to increase or decrease his total expenditure in the n all-pay auctions.

Assuming that there exists a pair of n-variate distribution functions that satisfy conditions (1) and (2) of Theorem 3, it follows from the arguments given above that such a pair of n-variate distribution functions constitute an equilibrium. We now establish the existence of sufficient n-variate distributions for the Theorem 3 parameter range.

Theorem 4 *For each set of equilibrium univariate marginal distribution functions, $\{F_{i,j}\}_{j=1}^n$, given in Theorem 3, there exists an n-copula, C, such that the support of the n-variate distribution function $C(F_{i,1}(x_1), \ldots, F_{i,n}(x_n))$ is contained in $\mathfrak{B}_i$.*

We begin with the proof for player A. The construction of a sufficient n-variate distribution function for player A and $X_A \geq (v/n)$ is outlined as follows [recall that in the Theorem 3 parameter region $X_A < (2v/n)$]. The remaining case that $X_A < (v/n)$ is addressed directly following this case.

1. Player A selects $n - 2$ of the all-pay auctions, each all-pay auction chosen with equal probability, and bids zero in each of those all-pay auctions.
2. On the remaining 2 all-pay auctions, player A randomizes uniformly on three line segments: (i) $\{(x_1, x_2) \in \mathbb{R}_+^2 | \, x_1 + x_2 = 2X_A - (2v/n)\}$, (ii) $\{(x_1, x_2) | \, x_1 = 0 \text{ and } 2X_A - (2v/n) \leq x_2 \leq X_A\}$, and (iii) $\{(x_1, x_2) | \, x_2 = 0 \text{ and } 2X_A - (2v/n) \leq x_1 \leq X_A\}$. This support is shown in panel (i) of Fig. 2, and this randomization is discussed in greater detail directly following this outline.
3. There are ${}_nC_2$ ways of dividing the n all-pay auctions into disjoint subsets such that $n - 2$ all-pay auctions receive bids of zero with probability 1 and 2 all-pay auctions involve randomizations of resources as in point 2. The n-variate distribution function formed by placing probability $[{}_nC_2]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each has a mass point of $(1 - (X_A/v))$ at 0 and randomizes uniformly on $(0, X_A]$ with the remaining mass.

The pivotal step in this construction is point 2. Let x_i denote the allocation of resources to all-pay auction $i \in \{1, 2\}$. Consider the support of a bivariate distribution function, G_A, for x_1 and x_2 which uniformly places mass $1 - (nX_A/2v)$ on each of the following two line segments:

$$\left\{(x_1, x_2) | \, x_1 = 0 \text{ and } 2X_A - \frac{2v}{n} \leq x_2 \leq X_A\right\}$$
$$\left\{(x_1, x_2) | \, x_2 = 0 \text{ and } 2X_A - \frac{2v}{n} \leq x_1 \leq X_A\right\}.$$

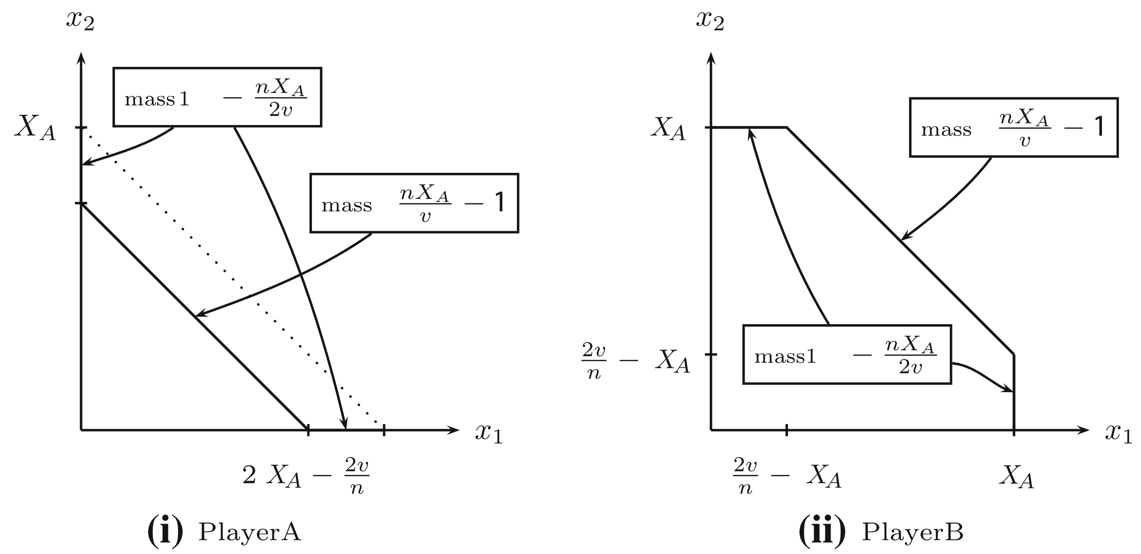

Fig. 2 Supports of players' bivariate distributions in Theorem 3 parameter range

and uniformly places the remaining mass, $(nX_A/v) - 1$, on the line segment

$$\left\{(x_1, x_2) \in \mathbb{R}^2_+ | \, x_1 + x_2 = 2X_A - \tfrac{2v}{n}\right\}.$$

This support is shown in panel (i) of Fig. 2.

In the expression for this bivariate distribution function, we will use the following notation.

R1: $\left\{(x_1, x_2) \in \left[0, 2X_A - \frac{2v}{n}\right]^2\right\}$
R2: $\left\{(x_1, x_2) \in \left(2X_A - \frac{2v}{n}, X_A\right] \times \left[0, 2X_A - \frac{2v}{n}\right]\right\}$
R3: $\left\{(x_1, x_2) \in \left[0, 2X_A - \frac{2v}{n}\right] \times \left(2X_A - \frac{2v}{n}, X_A\right]\right\}$
R4: $\left\{(x_1, x_2) \in \left(2X_A - \frac{2v}{n}, X_A\right]^2\right\}$

The bivariate distribution function for x_1, x_2 is given by

$$G_A(x_1, x_2) = \begin{cases} \left(\frac{n}{2v}\right) \max\left\{x_1 + x_2 - 2X_A + \frac{2}{vn}, 0\right\} & \text{if } (x_1, x_2) \in \text{R1} \\ \left(1 - \frac{nX_A}{v}\right) + \frac{nx_1}{2v} + \frac{nx_2}{2v} & \text{if } (x_1, x_2) \in \text{R2} \cup \text{R3} \cup \text{R4} \end{cases}$$

The univariate marginal distributions are given by $G_A(x_1, X_A) = (1 - (nX_A/2v)) + (nx_1/2v)$ and $G_A(X_A, x_2) = (1 - (nX_A/2v)) + (nx_2/2v)$. To see that G_A provides the necessary univariate marginal distributions, observe that given the randomization outlined above player A allocates zero resources to each all-pay auction j with probability $((n-2)/n) + (2/n)(1 - (nX_A/2v)) = (1 - (X_A/v))$, and randomizes uniformly over the interval $(0, X_A]$ with the remaining mass.

If $X_A < (v/n)$, then player A allocates zero resources to $n-1$ of the all-pay auctions and provides a random level of resources in the one remaining all-pay auction. In this one remaining all-pay auction player A has a mass point of $(1-(nX_A/v))$ at 0 and randomizes uniformly over the interval $[0, X_A]$ with the remaining mass.

The proof for player B is similar. The construction of a sufficient n-variate distribution function for player B and $X_A \geq (v/n)$ is outlined as follows. In the Theorem 3 parameter region $X_B \geq \min\{nX_A, (n-2)X_A + (2v/n)\}$. If $X_A \geq (v/n)$ then $X_B \geq (n-2)X_A + (2v/n)$. The remaining case in which $X_A < (v/n)$ and $X_B \geq nX_A$ is addressed directly following this case.

1. Player B selects $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability, and bids X_A in each of those all-pay auctions.
2. On the remaining 2 all-pay auctions, player B randomizes uniformly on three line segments: (i) $\{(x_1, x_2) \in \mathbb{R}^2_+ | \, x_1 + x_2 = (2v/n)\}$, (ii) $\{(x_1, x_2) | \, x_1 = X_A \text{ and } 0 \leq x_2 \leq (2v/n) - X_A\}$, and (iii) $\{(x_1, x_2) | \, x_2 = X_A \text{ and } 0 \leq x_1 \leq (2v/n) - X_A\}$. This support is shown in Panel (ii) of Fig. 2, and this randomization is discussed in greater detail directly following this outline.
3. There are ${}_nC_2$ ways of dividing the n all-pay auctions into disjoint subsets such that $n-2$ all-pay auctions receive X_A with probability 1 and 2 all-pay auctions involve randomizations of force as in point 2. The n-variate distribution function formed by placing probability $[{}_nC_2]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each has a mass point of $(1-(X_A/v))$ at X_A and randomizes uniformly on $[0, X_A)$ with the remaining mass.

The pivotal step in this construction is again point 2. Let x_i denote the allocation to all-pay auction $i \in \{1, 2\}$. Consider the support of a bivariate distribution function, G_B, for x_1 and x_2 which uniformly places mass $1-(nX_A/2v)$ on each of the two following line segments

$$\{(x_1, x_2) | \, x_1 = X_A \text{ and } 0 \leq x_2 \leq \tfrac{2v}{n} - X_A\}$$
$$\{(x_1, x_2) | \, x_2 = X_A \text{ and } 0 \leq x_1 \leq \tfrac{2v}{n} - X_A\}.$$

and uniformly places the remaining mass, $(nX_A/v) - 1$, on the line segment

$$\{(x_1, x_2) | \, x_1 + x_2 = \tfrac{2v}{n}\}$$

This support is shown in Panel (ii) of Fig. 2.

The bivariate distribution function for x_1, x_2 is given by

$$G_B(x_1, x_2) = \begin{cases} \left(\frac{n}{2v}\right) \max\left\{x_1 + x_2 - \frac{2}{vn}, 0\right\} & \text{if } (x_1, x_2) \in [0, X_A)^2 \\ \frac{nx_1}{2v} & \text{if } x_2 = X_A, \; x_1 \in [0, X_A) \\ \frac{nx_2}{2v} & \text{if } x_1 = X_A, \; x_2 \in [0, X_A) \\ 1 & \text{if } x_1, x_2 \geq X_A \end{cases}$$

Following from the arguments given above for player A, it follows that G_B provides the necessary univariate marginal distributions for all-pay auctions 1 and 2.

If $X_A < (v/n)$ and $X_B \geq nX_A$, then player B allocates X_A to $n-1$ of the all-pay auctions and provides a random level of resources in the one remaining all-pay auction. In this one remaining all-pay auction player B has a mass point of $(1-(nX_A/v))$ at X_A and randomizes uniformly over the interval $[0, X_A)$ with the remaining mass.

This completes the proof of the existence of sufficient n-variate distributions for the Theorem 3 parameter range.

In the remaining region in which $\max\{(X_B - \frac{2v}{n})/(n-2), X_B/n\} < X_A < X_B/(n-1)$, as in the corresponding constant-sum parameter range, both players have atoms in the interior of the domains of their univariate marginal distribution functions. It should be noted that in this region, the results are sensitive to the specification of the tie-breaking rule.

Let Δ denote the amount of resources available to player B if player B has bid X_A in $n-1$ of the auctions:

$$\Delta = X_B - (n-1)X_A.$$

Recalling that the floor function $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, define k as

$$k = \left\lfloor \frac{X_A}{X_B - (n-1)X_A} \right\rfloor = \left\lfloor \frac{X_A}{\Delta} \right\rfloor.$$

In this region of the parameter space, $(n-1)X_A < X_B < nX_A$ and so $k \geq 1$. It will also be helpful to note that $X_A/(k+1) < \Delta \leq X_A/k$.

In this region of the parameter space, the equilibrium univariate marginal distributions are not unique.

Theorem 5 *Let X_A, X_B, v, and $n \geq 3$ satisfy* $\max\{(X_B - (2v/n))/(n-2), X_B/n\} < X_A < X_B/(n-1)$. *The n-variate distribution functions P_A^* and P_B^* constitute a Nash equilibrium of the game $NCB\{X_A, X_B, n, v\}$ if they satisfy the following two conditions:* (1) *For each player i,* $\mathrm{Supp}(P_i^*) \subset \mathfrak{B}_i$ *and* (2) *P_i^*, $i = A, B$, provides the corresponding set of univariate marginal distribution functions $\{F_{i,j}^*\}_{j=1}^n$ outlined below, $\forall\, j \in \{1, \ldots, n\}$*

$$F_{B,j}^*(x) = \begin{cases} \frac{x}{v} & \text{if } x \in \left[0, \frac{X_A}{k+1}\right) \\ \frac{\left(\frac{2}{n} - \frac{\Delta + X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\frac{X_A}{k+1}, \frac{2X_A}{k+1}\right) \\ \vdots & \vdots \\ \frac{i\left(\frac{2}{n} - \frac{\Delta + X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\frac{iX_A}{k+1}, \frac{(i+1)X_A}{k+1}\right) \\ \vdots & \vdots \\ \frac{k\left(\frac{2}{n} - \frac{\Delta + X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\frac{kX_A}{k+1}, X_A\right) \\ 1 & \text{if } x \geq X_A \end{cases}.$$

If $k \geq 2$, then $\forall\ j \in \{1, \ldots, n\}$

$$F^*_{A,j}(x) = \begin{cases} 1 - \frac{2}{n} + \frac{\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in [0, \Delta) \\ 1 - \frac{2}{n} + \frac{2\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\Delta, \Delta + \frac{X_A - \Delta}{k-1}\right) \\ \vdots & \vdots \\ 1 - \frac{2}{n} + \frac{(i+1)\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\Delta + \left(i-1\right)\left(\frac{X_A - \Delta}{k-1}\right), \Delta + i\left(\frac{X_A - \Delta}{k-1}\right)\right) \\ \vdots & \vdots \\ 1 - \frac{2}{n} + \frac{k\left(\frac{2}{n} - \frac{X_A}{v}\right)}{k+1} + \frac{x}{v} & \text{if } x \in \left[\Delta + \left(k-2\right)\left(\frac{X_A - \Delta}{k-1}\right), X_A\right) \\ 1 & \text{if } x > X_A \end{cases}.$$

If $k = 1$, then $\forall\ j \in \{1, \ldots, n\}$

$$F^*_{A,j}(x) = \begin{cases} 1 - \frac{2}{n} + \frac{\left(\frac{2}{n} - \frac{X_A}{v}\right)}{2} + \frac{x}{v} & \text{if } x \in [0, \Delta) \\ 1 - \frac{2}{n} + \frac{1.5\left(\frac{2}{n} - \frac{X_A}{v}\right)}{2} + \frac{x}{v} & \text{if } x \in [\Delta, X_A) \\ 1 & \text{if } x > X_A \end{cases}.$$

In equilibria satisfying these conditions on P^*_A and P^*_B, the expected payoff for player A is $[(2vk/n) - k(\Delta + X_A)]/(k+1)$, the expected payoff for player B is $(n-1)(v - X_A) + v[1 - (2/n) + [(2/n) - (X_A/v)]/(k+1)]$, the total expected expenditure for player A is $X_A - (1 - nX_A/2v)(X_A - \Delta)/(k+1)$, and the total expected expenditure for player B is $nX_A(nX_B - (n-1)^2 X_A)/2v + (1 - n(\Delta + X_A)/2v)(k+2)X_A/(k+1)$.

We begin with a sketch of the proof that a pair of n-variate distribution functions that satisfy the conditions of Theorem 5 form an equilibrium and then move on to the proof of existence of such a pair of n-variate distribution functions. We will focus primarily on the case that $k \geq 2$ and conclude with the case that $k = 1$.

We first show that player A cannot increase his payoff by deviating to a feasible strategy and then examine the case of player B. Let P^*_B denote a feasible n-variate distribution function for player B with the univariate marginal distribution functions $\{F^*_{B,j}\}$ given in Theorem 5. We begin with the case that player A chooses an n-tuple of bids, $\mathbf{x}_A$, in which ties arise with probability zero. That is, $\mathbf{x}_A \in \mathfrak{B}_A$ such that for all $j = 1, \ldots, n$ and $i = 1, \ldots, k+1$, $x_{A,j} \neq iX_A/(k+1)$. If player B is using P^*_B, then player A's expected payoff π_A, when player A chooses any such n-tuple of bids is

$$\pi_A\left(\mathbf{x}_A, P^*_B\right) = \sum_{j=1}^{n}\left[vF^*_{B,j}\left(x_{A,j}\right) - x_{A,j}\right]. \tag{17}$$

To simplify the following discussion, for each $j = 1, \ldots, n$ let player B's univariate marginal distributions be written as

$$F^*_{B,j}\left(x_{A,j}\right) = \gamma_B(x_{A,j}) + \frac{x_{A,j}}{v}, \tag{18}$$

where, because we are focusing on the case that ties occur with probability zero, the term $\gamma_B(x_{A,j})$ is the sum of the mass on all atoms that lie strictly below $x_{A,j}$ and is given by the expression for $F^*_{B,j}$ in the statement of Theorem 5. Note that for each of player B's univariate marginal distribution functions each atom that lies strictly in the interior of the domain has the same mass, $[(2/n) - ((\Delta + X_A)/v)]/(k+1)$. Thus, the term $\gamma_B(x_{A,j})$ is equal to the number of atoms that lie below $x_{A,j}$ multiplied by the mass on each atom. Inserting Eq. (18) into Eq. (17) and simplifying, player A's expected payoff is given by

$$\pi_A\left(\mathbf{x}_A, P^*_B\right) = v\sum_{j=1}^{n}\gamma_B(x_{A,j}), \tag{19}$$

which is equal to the value of the prize multiplied by both the number of player B's atoms that player A outbids and by the mass on each atom, $[(2/n)-((\Delta+X_A)/v)]/(k+1)$.

Next note that in Theorem 5's set of univariate marginal distribution functions for player B, $\{F^*_{B,j}\}_{j=1}^n$, the step size between each atom is $X_A/(k+1)$, and the first atom occurs at $X_A/(k+1)$. There are a total of $k+1$ atoms in each of player B's univariate marginal distributions. Recall that the rule for breaking ties at a common bid of X_A in an auction is that player B wins the auction. In the event that player A bids X_A in auction j, then—because the $(k+1)$th atom is at X_A—player A outbids exactly k of player B's atoms. Suppose that in auction j player A's bid $x_{A,j}$ strictly outbids $\theta \leq k$ of player B's atoms. Because the step size between atoms is $X_A/(k+1)$, strictly outbidding $\theta \leq k$ of player B's atoms requires that $x_{A,j} > \theta X_A/(k+1)$. Then because $\mathbf{x}_A \in \mathfrak{B}_A$ and $x_{A,j} > \theta X_A/(k+1)$, it follows that $\sum_{j'\neq j} x_{A,j'} < (k+1-\theta)(X_A/(k+1))$. This implies that, in auctions other than j, the maximal number of player B's atoms that player A can feasibly outbid with his remaining budget is $k-\theta$. Equivalently, across all auctions, player A can outbid at most k of player B's atoms. Thus, player A's expected payoff for an any n-tuple of bids $\mathbf{x}_A \in \mathfrak{B}_A$ such that for all $j = 1, \ldots, n$ and $i = 1, \ldots, k+1$, $x_{A,j} \neq iX_A/(k+1)$ is less than or equal to his equilibrium expected payoff,

$$\pi_A\left(\mathbf{x}_A, P^*_B\right) = v\sum_{j=1}^{n}\gamma_B(x_{A,j}) \leq \frac{vk\left(\frac{2}{n} - \frac{\Delta+X_A}{v}\right)}{k+1}. \tag{20}$$

To conclude the proof that player A has no feasible payoff increasing deviations, we now address the case that, with positive probability, a tie occurs in one or more auctions. Recall that if a tie occurs and the common bid is neither X_A nor $X_B-(n-1)X_A$, then each player wins the auction with equal probability. It follows that if exactly two ties occur, then player A's expected payoff is less than or equal to his equilibrium expected payoff given in Eq. (20). However, if more than two ties occur, then player A's expected payoff is strictly less than his equilibrium expected payoff.

Using a similar argument for player B, it can be shown that the maximal number of player A's atoms that player B can outbid is $(n-1)(k+1)+1$. One difference in this case is that in each auction player A's atom at zero has more mass than each of the other atoms, but each of the other atoms has the same mass. Observe that in the Theorem 5 equilibrium univariate marginal distributions for player B, player B's bid is almost surely strictly positive. Therefore, player B outbids player A's atom at zero in each of the auctions.

There are three cases to consider: (i) player B bids X_A in $(n-1)$ of the auctions, (ii) player B bids X_A in $(n-2)$ of the auctions, and (iii) player B bids X_A in less than $(n-2)$ of the auctions. Beginning with case (i), recall that $\Delta = X_B - (n-1)X_A$. If player B bids X_A in $(n-1)$ of the auctions and in the remaining auction j bids $x_{B,j} \in (0, \Delta]$, then, because player A has $k+1$ atoms in each univariate marginal, player B outbids $(n-1)(k+1)+1$ of A's atoms and the expected payoff for player B is

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = \Big(n-1\Big)\Big(v - X_A\Big) + v\Big(1 - \frac{2}{n} + \frac{\frac{2}{n} - \frac{X_A}{v}}{k+1}\Big). \tag{21}$$

Thus, there are no feasible payoff increasing deviations in which player B bids X_A in $(n-1)$ of the auctions.

For cases (ii) and (iii), we begin with the restriction that player B chooses an arbitrary n-tuple of bids $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, X_A]^n$ such that for each auction j and $i = 1, \ldots, k-2$, $x_{B,j} \neq \Delta + i[(X_A - \Delta)/(k-1)]$. That is, player B chooses an n-tuple of bids in which for each auction j with $x_{B,j} \in (\Delta, X_A)$ a tie occurs with probability zero. Recall that in any auction j in which player B bids $x_{B,j} = X_A$, the tie-breaking rule implies that player B wins that auction with probability one. Conversely, if player B bids $x_{B,j} = \Delta$ in auction j and a tie occurs, then player A wins that auction.

If player A is using P_A^* and player B chooses an n-tuple of bids of the form assumed, then player B's expected payoff may be written as

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = v \sum_{j=1}^{n} \gamma_A\left(x_{B,j}\right), \tag{22}$$

where $\gamma_A(x_{B,j})$ is the sum of the mass on all atoms that lie strictly below $x_{B,j}$ if $x_{B,j} \in (0, X_A)$, and is the sum of the mass on all atoms, $1 - (X_A/v)$, if $x_{B,j} = X_A$.

In case (ii), if player B chooses any n-tuple of bids $\mathbf{x}_B \in \mathfrak{B}_B \bigcap (0, X_A]^n$ such that a bid of X_A is made in all but two auctions, denoted j' and j'', then player B's budget constraint implies that $x_{B,j'} + x_{B,j''} \leq \Delta + X_A$. In this case, player B's expected payoff is

$$\pi_B\left(\mathbf{x}_B, P_A^*\right) = \Big(n-2\Big)\Big(v - X_A\Big) + v\gamma_A(x_{B,j'}) + v\gamma_A(x_{B,j''}), \tag{23}$$

and for any feasible pair of bids $x_{B,j'}$ and $x_{B,j''}$ in $(0, X_A]^2$ such that $x_{B,j'} + x_{B,j''} = \Delta + X_A$ and ties occur with probability zero in auctions j' and j'' player B outbids $k+2$ of A's atoms in auctions j' and j'', which yields the equilibrium expected payoff. Thus, there are no feasible payoff increasing deviations of this form in which player B bids X_A in $(n-2)$ of the auctions

For case (iii), observe that because player A has an atom at X_A in each of his univariate marginal distributions, player B cannot increase his payoff by deviating to feasible n-tuples that bid X_A in less than $(n-2)$ auctions. This follows from the tie-breaking rule at bids of X_A and the following two facts. First, for each of player A's univariate marginal distribution functions, each atom that lies strictly in the interior of the domain has the same mass, $[(2/n) - (X_A/v)]/(k+1)$. Second, player A has atoms at 0 and at Δ (where $X_A/(k+1) < \Delta \leq X_A/k$), but the step size between the remaining atoms is $(X_A - \Delta)/(k-1) > \Delta$. This completes the sketch of the proof that player B has no feasible payoff increasing deviations when player B chooses an n-tuple of bids in which for each auction j with $x_{B,j} \in (\Delta, X_A)$ a tie occurs with probability zero. If player B chooses an n-tuple of bids in which there exists at least one auction j with $x_{B,j} \in (\Delta, X_A)$ such that a tie occurs with positive probability (i.e., for some $i = 1, \ldots, k-2$, $x_{B,j} = \Delta + i[(X_A - \Delta)/(k-1)]$), then the arguments given for player A in the case that one or more ties occurs with strictly positive probability rule out the possibility of this being a payoff increasing deviation for player B.

One pair of equilibrium n-variate distribution functions that satisfy the conditions of Theorem 5 is described as follows. The support of each player's n-variate joint distribution function consists of an absolutely continuous distribution over a set of line segments in $\mathbb{R}_n^+$ combined with a set of atoms on n-tuples. Mass is distributed among the atoms and line segments in such a way that the opponent is indifferent among all feasible pure strategies and the mass sums to one. More precisely, player A randomly bids 0 in $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability $(n-2)/n$, and randomizes according to a symmetric bivariate distribution function in the remaining 2 auctions. Conversely, player B randomly bids X_A in $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability $(n-2)/n$, and randomizes according to a symmetric bivariate distribution function in the remaining 2 auctions. The supports of these bivariate distributions are illustrated in Fig. 3. To avoid confusion between atoms in the supports of the respective bivariate distribution functions described below and atoms in the resulting univariate marginal distributions, 2-tuples in the support of a bivariate distribution which receive positive mass are referred to as bivariate atoms. Similarly, points in the support of a univariate marginal distribution that receive positive mass are referred to as univariate atoms.

We now specify the conditional bivariate marginal distribution that player A utilizes in the 2 randomly chosen all-pay auctions which do not receive a bid of 0. Player A's conditional bivariate marginal distribution in these 2 auctions has $k+1$ bivariate atoms,[14] each bivariate atom receiving the same weight, $(1 - (nX_A)/(2v))/(k+1)$.

[14] Observe that at each of the "bivariate atoms" described here player A allocates 0 resources to the other $n-2$ auctions. Thus, each of these bivariate atoms is actually an atom on an n-tuple.

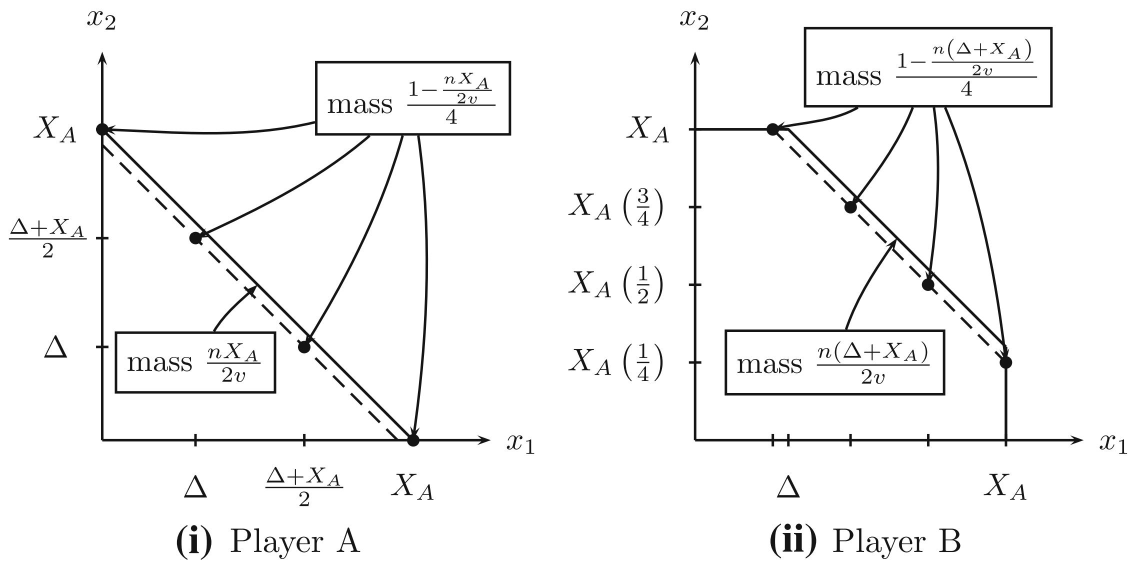

Fig. 3 Supports of players' bivariate distributions in Theorem 5 parameter range ($k = 3$)

These bivariate atoms are located at the points $(0, X_A)$, $(X_A, 0)$, and

$$\left(\Delta + \left(k - 1 - i\right)\left(\frac{X_A - \Delta}{k - 1}\right), \Delta + \left(i - 1\right)\left(\frac{X_A - \Delta}{k - 1}\right)\right), \quad i = 1, \ldots, k - 1. \tag{24}$$

Player A uniformly distributes the remaining mass of $(nX_A)/(2v)$ along the line segment $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_1 + x_2 = X_A\}$. To see that this construction provides the necessary univariate marginal distributions, observe that in the randomization outlined above player A allocates zero resources to each all-pay auction j with probability $(n - 2)/n + (2/n)[1 - (nX_A/2v)]/(k + 1) = 1 - (2/n) + [2/(n(k + 1))] - [X_A/(v(k + 1))]$, randomizes uniformly over the interval $(0, X_A]$ with probability $(2/n)(nX_A)/(2v) = X_A/v$, and has the specified univariate atoms with the remaining probability.

Moving on to player B's conditional bivariate marginal distribution, consider a bivariate distribution function with $k + 1$ bivariate atoms, each bivariate atom receiving the same weight, $[1 - (n/2v)(\Delta + X_A)]/(k + 1)$. These bivariate atoms are located at the points

$$\left(\frac{(k + 1 - i)X_A}{(k + 1)}, \frac{(1 + i)X_A}{(k + 1)}\right), \quad i = 0, \ldots, k. \tag{25}$$

Player B uniformly distributes the remaining mass of $n(\Delta + X_A)/(2v)$ along the three line segments $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_1 + x_2 = X_B - (n - 2)X_A\}$, $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_1 = X_A \text{ and } 0 \leq x_2 \leq \Delta\}$, and $\{(x_1, x_2) \in \mathbb{R}^2_+ |\ x_2 = X_A \text{ and } 0 \leq x_1 \leq \Delta\}$. To see that this construction provides the necessary univariate marginal distributions, observe that in the randomization outlined above player B allocates X_A resources to each all-pay auction j with probability $((n - 2)/n) + (\Delta/v) + [(2/n) - ((\Delta + X_A)/v))]/(k + 1)$, randomizes uniformly over the interval $[0, X_A)$ with probability $(2/n)(nX_A)/(2v) = X_A/v$, and has the specified univariate atoms with the remaining probability.

It is important to note that for this pair of equilibrium n-variate distribution functions, neither player exhausts his budget with probability one. However, as shown below, each n-tuple in the support yields the equilibrium expected payoff for the corresponding player, and as shown above, neither player has a, feasible, payoff increasing deviation.

Recall that each player i's expected payoff [Eqs. (19) and (22)] is proportional, subject to the tie-breaking rule, to the number of player $-i$'s univariate atoms that he outbids. As is indicated in the equilibrium payoffs given in Theorem 5, player A outbids k of player B's univariate atoms, and player B outbids $(n-2)(k+1)+k+2$ of player A's univariate atoms. Next, note the following inequality

$$\frac{(k-i)X_A}{k+1} < \Delta + \left(k-1-i\right)\left(\frac{X_A - \Delta}{k-1}\right) < \frac{(k+1-i)X_A}{k+1}, \tag{26}$$

which holds for all $i = 1, \ldots, k-1$. This inequality follows directly from the relationship between Δ, k, and X_A. In particular, $X_A/(k+1) < \Delta \leq X_A/k$. The inequality in Eq. (26) shows that when player A bids $\Delta + (k-1-i)((X_A - \Delta)/(k-1))$ in an auction he outbids $k-i$ of player B's univariate atoms in that auction. Conversely, as Eq. (26) holds for all $i = 1, \ldots, k-1$, it also shows that when player B bids $(k+1-i)X_A/(k+1)$ in an auction he outbids $k+1-i$ of player A's univariate atoms in that auction. From the locations of each player's bivariate atoms given in Eqs. (24) and (25), it follows that for each player i each of his bivariate atoms outbids the same number of player $-i$'s univariate atoms as is indicated in his equilibrium expected payoff [k atoms for player A and $(n-2)(k+1)+k+2$ for player B]. This completes the proof of Theorem 5 for $k \geq 2$.

We now address the case of $k = 1$. Just as with $k \geq 2$, the equilibrium univariate marginal distributions are not unique. For $k = 1$, the construction specified above for player A fails to be feasible given his budget constraint. In this case, player A's univariate marginals are modified, but for player B, the construction specified above, but with $k = 1$, still applies.

The sketch of the proof that a pair of n-variate distribution functions, which satisfy the conditions of Theorem 5 with $k = 1$, form an equilibrium follows along the same lines as for $k \geq 2$. For the proof of existence of such an n-variate distribution function for player A, consider the following construction.

Player A randomly allocates 0 resources to $n-2$ of the all-pay auctions, each all-pay auction chosen with equal probability, $(n-2)/n$. On the remaining 2 all-pay auctions player A utilizes a bivariate distribution function with 4 bivariate atoms, each bivariate atom receiving the same weight, $(1-(nX_A)/(2v))/4$. Player A's bivariate atoms on these two remaining all-pay auctions are located at the points $(0, X_A)$, $(X_A, 0)$, $(0, \Delta)$, and $(\Delta, 0)$. Player A uniformly distributes the remaining mass of $(nX_A)/(2v)$ along the line segment $\{(x_1, x_2) \in \mathbb{R}^2_+ | \, x_1 + x_2 = X_A\}$. To see that this construction provides the necessary univariate marginal distributions, observe that in the randomization outlined above player A allocates zero resources to each all-pay auction j with probability $(n-2)/n + (2/n)[1-(nX_A/2v)]/2 = 1-(2/n)+(1/n)-[X_A/(2v)]$, randomizes uniformly over the interval $(0, X_A]$ with probability $(2/n)(nX_A)/(2v) = X_A/v$, and has the specified univariate atoms with the remaining probability.

As before, two of player A's atoms do not exhaust player A's budget. However, each of these bivariate atoms clearly outbids one of player B's univariate atoms and results in the equilibrium expected payoff for player A.

Two auctions

Before outlining the case of two auctions, it is important to note that for $n = 2$ the equilibrium univariate marginal distributions are non-unique for all parameter regions.[15] However, as is shown in the Appendix, in the Theorems 1 and 2 parameter ranges with $X_A \neq (2v/n)$, the equilibrium payoffs and total expenditures are unique.

Recall that in both panels of Fig. 1, the parameter space is partitioned by the four rays: (a) $X_A = X_B/n$, (b) $X_A = X_B/(n-1)$, (c) $X_A = 2X_B/n$, and (d) $X_A = X_B$. In the case that $n = 2$, the last three of these collapse to the single ray $X_A = X_B$, and the first of these becomes $X_A = X_B/2$. The following partition of the parameter space, for $n = 2$, provides the portions of the parameter space in which the theorems in the preceding subsection provide sufficient, but not necessary, conditions for equilibrium.

$$\text{T1*:} \ \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| v < X_A \leq X_B\right\}$$
$$\text{T2*:} \ \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_B = X_A \leq v \ \text{ or } \ X_A = v \ \text{ and } \ X_B > v\right\}$$
$$\text{T3a*:} \ \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_B \geq v \ \text{ and } \ \tfrac{X_B}{2} < X_A < v\right\}$$
$$\text{T3b*:} \ \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_A < v \ \text{ and } \ X_A \leq \tfrac{X_B}{2}\right\}$$
$$\text{T5*:} \ \left\{(X_A, X_B) \in \mathbb{R}^2_+ \middle| X_B < v \ \text{ and } \ \tfrac{X_B}{2} < X_A < v\right\}$$

These regions and the resulting modified budgets are illustrated in Fig. 4 below.

Recall that in the constructions provided for the Theorems 3 and 5 parameter regions, each player allocated a specified bid to $(n-2)$ of the all-pay auctions [for player A, this was a bid of 0, and for player B, this was a bid of X_A]. When $n = 2$, $(n-2)/n = 0$ and the constructions for both of those regions simply become the bivariate distributions that were specified for the remaining two auctions. It is straightforward to show that in the Theorems 1 and 2 regions the Fréchet-Hoeffding lower bound 2-copula combined with the univariate marginals specified in Theorems 1 and 2, which for player $i = A, B$ are given by

$$P_i^*\left(b_{i,1}, b_{i,2}\right) = \max\left\{F_{i,1}^*\left(b_{i,1}\right) + F_{i,2}^*\left(b_{i,2}\right) - 1, 0\right\},$$

results in a pair of bivariate distribution functions for which $\text{Supp}(P_i^*) \subset \mathfrak{B}_i$ and that provide an equilibrium pair of univariate marginal distribution functions.

[15] With $n = 2$ each player's pair of univariate marginals need not be independent of the identity of the auction. For example, the location of and/or the mass placed on atoms need not be symmetric across auctions. For further information, see Macdonell and Mastronardi (2011).

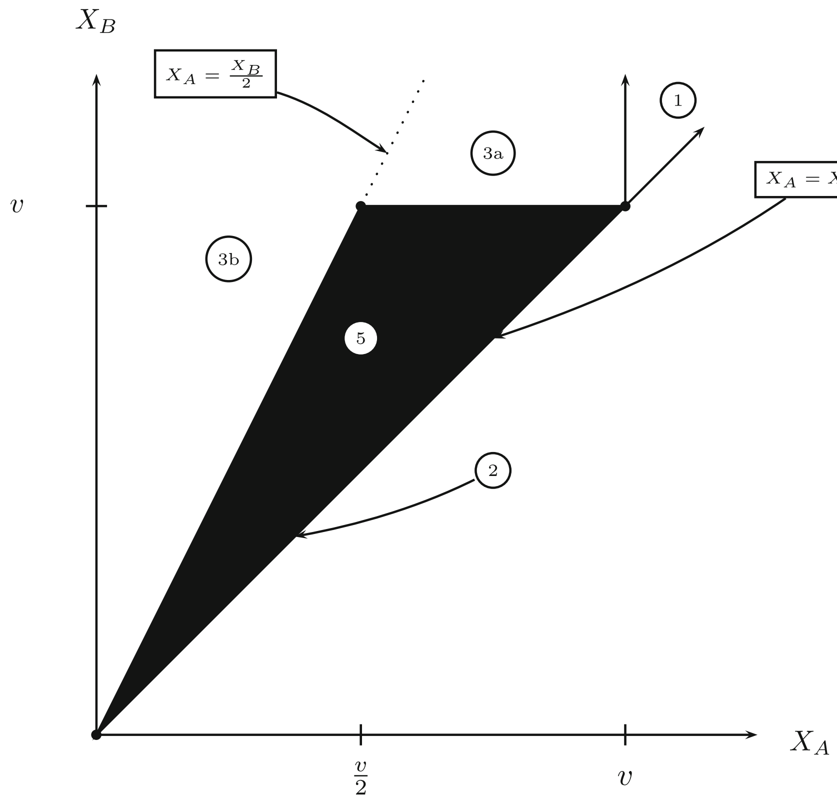

Fig. 4 Parameter space $n = 2$

5 Conclusion

Kvasov (2007) introduces a non-constant-sum version of the Colonel Blotto game that relaxes the "use it or lose it" feature of the traditional constant-sum formulation of the game. In the case of symmetric budgets, that article establishes that there exists a one-to-one mapping from the set of unique univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game. As the analysis of the non-constant-sum version of the Colonel Blotto game is extended to allow for asymmetric budget constraints, we find that—as long as the level of asymmetry between the players' budgets is below a threshold—there still exists a one-to-one mapping from the unique set of equilibrium univariate marginal distribution functions in the constant-sum game to those in the non-constant-sum game. The classic Colonel Blotto game provides an important benchmark in the study of the logic of strategic multi-dimensional conflict, and, as our results show, the nature of the incentives in such conflicts remains largely unchanged when the use it or lose it feature of the constant-sum game is relaxed.

Appendix

For the Theorem 1 parameter range with $n \geq 3$ (denoted as T1), this Appendix characterizes each player's unique: equilibrium univariate marginal distribution functions, equilibrium payoffs, and equilibrium total expected expenditures. We also show that the uniqueness of the equilibrium payoffs and equilibrium total expected expenditures extends to the case of $n = 2$. The proof for the Theorem 2 parameter range with $X_A \neq (2v/n)$ follows along similar lines, and we conclude with a sketch of that proof.

For $(X_A, X_B) \in T1$, the proof of the uniqueness of the univariate marginal distributions involves formally showing that, as the Euler–Lagrange equations given in Eq. (5) of Sect. 3 suggest, there exists a one-to-one correspondence between the equilibrium univariate marginal distributions in the Non-Constant-Sum Colonel Blotto game and the equilibrium distributions of bids from a unique set of two-bidder independent and identical simultaneous all-pay auctions. The uniqueness of the equilibrium univariate marginal distributions follows from the characterization of the all-pay auction by Hillman and Riley (1989) and Baye et al. (1996).

In the case of the standard constant-sum formulation of the Colonel Blotto game, the proof of the uniqueness of the equilibrium marginal distributions (Roberson 2006) utilizes the fact that in a two-player constant-sum game with multiple equilibria, all equilibria are interchangeable. In Lemmas 1–3, we show that for the Theorem 1 parameter range, this interchangeability of equilibria property also applies to the Non-Constant-Sum Colonel Blotto game. Given this result on the interchangeability of equilibria, the rest of the proof follows along lines similar to Roberson (2006).

In the discussion that follows, we will utilize the following notational conventions. Given an n-variate distribution function P_i with $\mathrm{Supp}(P_i) \subset \mathcal{B}_i$ and the set of univariate marginal distribution functions $\{F_{i,j}\}_{j=1}^n$, let M_{X_i} denote the total expected expenditure across the entire set of auctions, that is $M_{X_i} \equiv \sum_{j=1}^n E_{F_{i,j}}(x_{i,j})$. Also, let $\bar{s}_{i,j}$ and $\underline{s}_{i,j}$ denote the upper and lower bounds of player i's distribution of resources for all-pay auction j.

We begin the proof of the interchangeability of equilibria in the Non-Constant-Sum Colonel Blotto game by showing that if the pair of the players' resources $(X_A, X_B) \in T1$ [i.e., $(2/n)\min\{v, X_B\} < X_A \leq X_B$], then in any equilibrium the total expected expenditures (M_{X_A}, M_{X_B}) are uniquely determined by (X_A, X_B) and equal to those given in Theorem 1. The proof of this result is done in two steps. First, Lemma 1 shows that if $(X_A, X_B) \in T1$, then in any equilibrium $\{P_A, P_B\}$ the pair of equilibrium total expected expenditures (M_{X_A}, M_{X_B}) must lie in the set of equilibrium total expected expenditures for Theorem 1 as illustrated by the shaded region 1c in panel (ii) of Fig. 1 and delineated by the conditions: (i) $(2/n)M_{X_B} \leq M_{X_A} \leq M_{X_B}$, (ii) $M_{X_i} \leq (nv/2)$ for $i = A, B$, and (iii) if $M_{X_A} > (2v/n)$ then $M_{X_B} \leq (nvM_{X_A}/2)^{1/2}$. Then, Lemma 2 shows that in the Theorem 1 parameter region the equilibrium total expected expenditures are uniquely determined by the pair of the players' resources (X_A, X_B).

Lemma 1 *If* $(X_A, X_B) \in \mathrm{T1}$, *then in any equilibrium* $\{P_A, P_B\}$ *the pair of total expected expenditures* (M_{X_A}, M_{X_B}) *are contained in the region delineated by: (i)* $(2/n)M_{X_B} \leq M_{X_A} \leq M_{X_B}$, *(ii)* $M_{X_i} \leq (nv/2)$ *for* $i = A, B$, *and (iii) if* $M_{X_A} > (2v/n)$ *then* $M_{X_B} \leq (nvM_{X_A}/2)^{1/2}$.

Proof First, note that the total value at stake in the auctions is nv. Let α_i denote the fraction of the total value of the auctions that player i expects to win in this equilibrium,

$$\alpha_i = \frac{1}{n} E_{P_i}\left[\sum_{j=1}^{n} F_{-i,j}(x_{i,j})\right] = 1 - \frac{1}{n} E_{P_{-i}}\left[\sum_{j=1}^{n} F_{i,j}(x_{-i,j})\right] \tag{27}$$

where the first [second] expectation is taken with respect to player i's joint distribution P_i [player $-i$'s joint distribution P_{-i}] and the second equality follows from $\alpha_A + \alpha_B = 1$. It is instructive to note that the α_i term is precisely player i's expected payoff in the corresponding constant-sum Colonel Blotto game with budget constraints given by the expected expenditures (M_{X_A}, M_{X_B}). Player i's expected payoff may be written as:

$$\pi_i(P_i, P_{-i}) = nv\alpha_i - M_{X_i}. \tag{28}$$

First, we show that there exist no equilibria in which $M_{X_A} + M_{X_B} > nv$. This proof is by contradiction. Suppose that $(X_A, X_B) \in T1$, and that there exists an equilibrium $\{P_A, P_B\}$ in which $M_{X_A} + M_{X_B} > nv$. From Eq. (28), it follows that the sum of the players' expected payoffs is

$$\pi_A(P_A, P_B) + \pi_B(P_B, P_A) = nv - M_{X_A} - M_{X_B}. \tag{29}$$

Because in any equilibrium each player must have a nonnegative expected payoff, it follows that the sum of the players' expected payoffs must also be nonnegative. Thus, from Eq. (29), there exist no equilibria in which $M_{X_A} + M_{X_B} > nv$, a contradiction to the assumption that there exists such an equilibrium.

Focusing now on equilibria in which $M_{X_A} + M_{X_B} \leq nv$, for the T1 region, there are two remaining cases to consider:[16] (i) $M_{X_A} > M_{X_B}$, and (ii) $M_{X_A} \leq M_{X_B}$, $M_{X_A} > (2v/n)$, and $M_{X_B} > (nvM_{X_A}/2)^{1/2}$.

We begin with case (i). By way of contradiction, suppose that there exists an equilibrium $\{P_A, P_B\}$ in which $M_{X_A} > M_{X_B}$. Because $X_B \geq X_A$, player B can always duplicate player A's strategy and earn an expected payoff of at least $(nv/2) - M_{X_A}$. That is

$$\pi_B(P_B, P_A) \geq \frac{nv}{2} - M_{X_A} \tag{30}$$

From Eqs. (28) and (30), it follows that $\alpha_B \geq (1/2) - (M_{X_A} - M_{X_B})/nv$. Because $\alpha_A + \alpha_B = 1$ it follows that

$$\pi_A(P_A, P_B) \leq \frac{nv}{2} - M_{X_B} \tag{31}$$

We will now use the upper bound on player A's expected payoff from the strategy profile $\{P_A, P_B\}$, given in Eq. (31), to show that there exists a profitable deviation, $\widetilde{P}_A$, for player A. From Roberson (2006) [see the comments following Theorem 1

[16] Observe that cases (i) and (ii) together with $M_{X_A} + M_{X_B} > nv$ correspond to the non-shaded portions of the T1 region in panel (ii) of Fig. 1.

in this article], we know that there exists a joint distribution function $\widetilde{P}_A$ that satisfies the three following properties: Supp$(\widetilde{P}_A) \subset \mathfrak{B}_A$, the total equilibrium expected expenditures are given by $\widetilde{M}_{X_A} = \min\{X_A, (nvM_{X_B}/2)^{1/2}\}$, and the set of univariate marginal distributions are given by

$$\forall\, j \in \{1, \ldots, n\} \;\; \widetilde{F}^*_{A,j}(x) = \frac{x}{(2/n)\widetilde{M}_{X_A}} \quad \text{for } x \in \left[0, \tfrac{2}{n}\widetilde{M}_{X_A}\right].$$

Player A's expected payoff from the feasible deviation $\widetilde{P}_A$ is

$$\begin{aligned}\pi_A(\widetilde{P}_A, P_B) &= nv\left(1 - E_{P_B}\left(\sum_{j=1}^{n} \widetilde{F}^*_{A,j}\left(x_{B,j}\right)\right)\right) - \widetilde{M}_{X_A} \\ &\geq nv\left(1 - \frac{M_{X_B}}{2\widetilde{M}_{X_A}}\right) - \widetilde{M}_{X_A}.\end{aligned} \tag{32}$$

If Supp$(P_B) \subset [0, \frac{2}{n}\widetilde{M}_{X_A}]^n$, then Eq. (32) holds with equality.

Recall that in the equilibrium $\{P_A, P_B\}$ Eq. (31) provides an upper bound on player A's expected payoff. However, $\widetilde{P}_A$ is a feasible payoff increasing deviation from P_A. That is, because $M_{X_A} + M_{X_B} \leq nv$ and by assumption $M_{X_A} > M_{X_B}$, it follows that $M_{X_B} < (nv/2)$. Thus, $M_{X_B} < \widetilde{M}_{X_A} < (nv/2)$, and it follows from Eqs. (31) and (32) that $\pi_A(\widetilde{P}_A, P_B) > \pi_A(P_A, P_B)$. A contradiction to the assumption that there exists an equilibrium $\{P_A, P_B\}$ in which $M_{X_A} > M_{X_B}$.

The proof of case (ii) follows along a similar line as the proof for case (i). By way of contradiction, suppose that there exists an equilibrium $\{P_A, P_B\}$ in which $M_{X_A} \leq M_{X_B}$, $M_{X_A} > (2v/n)$, and $M_{X_B} > (nvM_{X_A}/2)^{1/2}$. Parallel to the lower bound of player B's expected payoff in case (i) given in Eq. (30), in case (ii) player A can establish a lower bound on his expected payoff. As with the upper bound of player A's expected payoff in case (i) given in Eq. (31), in case (ii) the upper bound on player B's expected payoff is given by $nv(1 - \alpha_A) - M_{X_B}$. It can then be shown that there exists a profitable deviation for player B, a contradiction to the assumption that such an equilibrium exists. This completes the proof of Lemma 1. □

Lemma 2 *If* $(X_A, X_B) \in$ T1*, then in any equilibrium* $\{P_A, P_B\}$*, the pair of total expected expenditures* (M_{X_A}, M_{X_B}) *is equal to the pair of equilibrium total expected expenditures uniquely determined by* (X_A, X_B) *in Theorem 1. Furthermore, the equilibrium expected payoffs are also uniquely determined by* (X_A, X_B).

Proof By way of contradiction suppose that for some $(X_A, X_B) \in$ T1, there exists an equilibrium $\{P_A, P_B\}$ with a pair of total expected expenditures (M_{X_A}, M_{X_B}) that satisfies Lemma 1 [i.e., (M_{X_A}, M_{X_B}) is contained in the set of equilibrium total expected expenditures for Theorem 1] but in which the pair (M_{X_A}, M_{X_B}) differs from the pair of total expected expenditures uniquely determined by (X_A, X_B) in Theorem 1.

The outline of the proof is as follows. First, we show how feasible and total-expected-expenditure invariant deviations from $\{P_A, P_B\}$ may be used to determine the payoffs in the original equilibrium $\{P_A, P_B\}$. Then, we show that because the pair

(M_{X_A}, M_{X_B}) differs from the pair of total expected expenditures uniquely determined by (X_A, X_B) in Theorem 1, at least one player i has a strictly payoff increasing deviation—in which player i's total-expected-expenditure differs from M_{X_i}—from the assumed equilibrium $\{P_A, P_B\}$.

Beginning with the first step, because (M_{X_A}, M_{X_B}) satisfies Lemma 1, we know from Roberson (2006) that there exists a joint distribution function P_A^* which satisfies the two following properties: $\text{Supp}(P_A^*) \subset \overline{\mathfrak{B}}_A$ and the set of univariate marginal distributions are given by

$$\forall\, j \in \{1, \ldots, n\} \;\; F_{A,j}^*(x) = \left(1 - \frac{M_{X_A}}{M_{X_B}}\right) + \frac{x}{(2/n)M_{X_B}} \left(\frac{M_{X_A}}{M_{X_B}}\right) \quad \text{for } x \in \left[0, \tfrac{2}{n} M_{X_B}\right]$$

Observe that the feasible deviation P_A^* has a total expected expenditure of M_{X_A}. Such a feasible deviation ensures that

$$\alpha_A \geq \frac{M_{X_A}}{2M_{X_B}} \text{ and } \alpha_B \leq 1 - \frac{M_{X_A}}{2M_{X_B}}. \tag{33}$$

Similarly, there exists a feasible deviation P_B^* with $\text{Supp}(P_B^*) \subset \overline{\mathfrak{B}}_B$ and the set of univariate marginal distributions:

$$\forall\, j \in \{1, \ldots, n\} \;\; F_{B,j}^*(x) = \frac{x}{(2/n)M_{X_B}} \quad \text{for } x \in \left[0, \tfrac{2}{n} M_{X_B}\right]$$

Note that P_B^* is a feasible deviation which is invariant with respect to the total expected expenditure M_{X_B}. Such a strategy ensures that

$$\alpha_A \leq \frac{M_{X_A}}{2M_{X_B}} \text{ and } \alpha_B \geq 1 - \frac{M_{X_A}}{2M_{X_B}}. \tag{34}$$

From Eqs. (33) and (34), it follows that the original equilibrium strategy profile $\{P_A, P_B\}$ yields the respective total expected fractions of contests won

$$\alpha_A = \frac{M_{X_A}}{2M_{X_B}} \text{ and } \alpha_B = 1 - \frac{M_{X_A}}{2M_{X_B}}. \tag{35}$$

Inserting Eq. (35) back into Eq. (28), the players' expected payoffs from the original equilibrium strategy profile $\{P_A, P_B\}$ are

$$\pi_A(P_A, P_B) = \frac{nvM_{X_A}}{2M_{X_B}} - M_{X_A} \text{ and } \pi_B(P_B, P_A) = nv\big(1 - \frac{M_{X_A}}{2M_{X_B}}\big) - M_{X_B}. \tag{36}$$

We now show that because the pair of total expected expenditures (M_{X_A}, M_{X_B}) in the original equilibrium strategy profile $\{P_A, P_B\}$ differ from the pair of total expected expenditures uniquely determined by (X_A, X_B) in Theorem 1 (denoted $M_{X_i}^*$ for

$i = A, B$), at least one player has a strictly payoff increasing deviation from the assumed equilibrium $\{P_A, P_B\}$.

By assumption $\{P_A, P_B\}$ is an equilibrium, and thus neither player i can increase his expected payoff by deviating to a feasible strategy with a different total expected expenditure M_{X_i}. Recall that in Theorem 1 player A's equilibrium total expected expenditure is $M^*_{X_A} = \min\{X_A, (nv/2)\}$. By way of contradiction assume that $M_{X_A} \neq M^*_{X_A}$. If $M_{X_B} = (nv/2)$, then because (M_{X_A}, M_{X_B}) satisfies Lemma 1 it must be the case that $M_{X_A} = (nv/2)$. A contradiction to the assumption that $M_{X_A} \neq M^*_{X_A}$. We now examine the remaining case that $M_{X_B} < (nv/2)$. Because (M_{X_A}, M_{X_B}) satisfies Lemma 1 and $M_{X_A} \neq M^*_{X_A}$, it follows that either: (a) $X_A \geq (nv/2)$ and $M_{X_A} < (nv/2)$ or (b) $X_A < (nv/2)$ and $M_{X_A} < X_A$. Following along similar lines to the feasible deviations outlined above, from Roberson (2006), there exists a joint distribution function $\widetilde{P}_A$ that satisfies the property that $\text{Supp}(\widetilde{P}_A) \subset \mathcal{B}_A$, has a total expected expenditure $\widetilde{M}_{X_A}$ such that $M_{X_A} < \widetilde{M}_{X_A} \leq M^*_{X_A}$, and ensures that $\widetilde{\alpha}_A \geq (\widetilde{M}_{X_A}/2M_{X_B})$. Thus, it follows from Eq. (36) that in both cases, player A has a strictly payoff increasing deviation. A contradiction to the assumption that $\{P_A, P_B\}$ is an equilibrium.

A similar argument shows that if $M_{X_B} \neq M^*_{X_B}$, then at least one player has a feasible strictly payoff increasing deviation. To summarize, we have shown that if (X_A, X_B) lies in the T1 parameter range and $\{P_A, P_B\}$ is an equilibrium with the pair of total expected expenditures (M_{X_A}, M_{X_B}), then $M_{X_A} = \min\{X_A, (nv/2)\}$ and $M_{X_B} = \min\{X_B, (nv/2), (nvX_A/2)^{1/2}\}$.

Given the uniqueness of the equilibrium total expected expenditures, the uniqueness of the equilibrium payoffs follows directly. □

Lemma 3 *If $(X_A, X_B) \in$ T1, then any equilibrium $\{P_A, P_B\}$ is interchangeable with any equilibrium $\{P^*_A, P^*_B\}$ which satisfies the conditions of Theorem 1.*

Proof Suppose that—in addition to an equilibrium $\{P^*_A, P^*_B\}$ which satisfies the conditions in Theorem 1—there exists an equilibrium $\{P_A, P_B\}$ that violates condition (2) of Theorem 1 [i.e., the condition on the sets of univariate marginal distributions]. From Lemma 2, all equilibria have the same expected expenditures (M_{X_A}, M_{X_B}) and the same expected payoffs. From Eq. (28), it follows that there is a unique equilibrium pair (α^*_A, α^*_B).

If $\{P_A, P_B\}$, with (M_{X_A}, M_{X_B}), is an equilibrium, then it must be the case that neither player has a feasible payoff increasing deviation. Without loss of generality, suppose that player A deviates to the strategy P^*_A which satisfies the conditions in Theorem 1. Because this is a feasible deviation which is invariant to the expected expenditure M_{X_A} and player A's expected payoff π_A does not increase, it follows that α_A does not increase. As $\alpha_A + \alpha_B = 1$, this implies directly that α_B does not decrease.

Conversely, because $\{P^*_A, P^*_B\}$ is an equilibrium neither player has a feasible payoff increasing deviation. Thus, if player B deviates from P^*_B to P_B, player B's expected payoff π_B does not increase. Then, because the deviation P_B is invariant to the expected expenditure M_{X_B}, it follows that α_B must not increase under this deviation. Because $\alpha_A + \alpha_B = 1$ and α_B does not increase, it must be the case that α_A does not decrease.

Because, when player B chooses P_B and player A choose P^*_A, both α_A and α_B neither increase nor decrease we can conclude that they stay at the unique values (α^*_A, α^*_B), and that the players' expected payoffs remain at the unique levels specified by Lemma 2. Furthermore, neither player has a feasible payoff increasing deviation. We have thus shown that if player B chooses P_B and player A chooses P_A^*, then $\{P_A^*, P_B\}$ forms an equilibrium which satisfies Lemmas 1 and 2. By a symmetric argument, it follows that $\{P_A, P_B^*\}$ also forms an equilibrium which satisfies Lemmas 1 and 2. Thus, any equilibrium $\{P_A, P_B\}$ is interchangeable with any equilibrium $\{P_A^*, P_B^*\}$ which satisfies the conditions in Theorem 1. □

Because of Lemma 3's result on the interchangeability of equilibria, arguments along the lines of the proofs in Baye et al. (1996) can be used to establish the next three lemmas.

Lemma 4 *If* $(X_A, X_B) \in \mathrm{T1}$*, then in any equilibrium* $\{P_A, P_B\}, \bar{s}_{i,j} = \bar{s} = (2/n)M_{X_B}$ *and* $\underline{s}_{i,j} = \underline{s} = 0$ *for each* $i \in \{A, B\}$ *and* $j \in \{1, \ldots, n\}$.

Lemma 5 *If* $(X_A, X_B) \in \mathrm{T1}$*, then in any equilibrium* $\{P_A, P_B\}$ *no* $F_{i,j}$ *can place an atom in the half-open interval* $(0, (2/n)M_{X_B}]$

Lemma 6 *If* $(X_A, X_B) \in \mathrm{T1}$*, then in any equilibrium* $\{P_A, P_B\}$ *there exists, for* $i = A, B$*, a* $\lambda_i \geq 0$ *such that* $\forall j = 1, \ldots, n, vF_{-i,j}(x) - (1 + \lambda_i)x$ *is constant* $\forall x \in (0, (2/n)M_{X_B}]$.

Note that the conditions stated in Lemma 6 are equivalent to the Euler–Lagrange equations given in Eq. (5) of Section 3. We now complete the proof of the uniqueness of the univariate marginals.

Lemma 7 *If* $(X_A, X_B) \in \mathrm{T1}$*, then in any equilibrium* $\{P_A, P_B\}, \lambda_A = -1 + ((nv)/(2M_{X_B}))$ *and* $\lambda_B = -1 + ((nvM_{X_A})/(2M_{X_B}^2))$*. Therefore, each set of univariate marginal distribution functions* $\{F_{i,j}\}_{j=1}^n$*,* $i = A, B$ *satisfies the conditions in Theorem 1.*

Proof By definition $M_{X_i} = \sum_{j=1}^n \int_0^{\bar{s}} x\, \mathrm{d}F_{i,j}(x)$. From Lemma 4, $\bar{s} = (2/n)M_{X_B}$ and the lower bound for each univariate marginal distribution is 0. From Lemma 6, $dF_{i,j}(x) = ((1 + \lambda_{-i})/v)dx$.

For player B,

$$M_{X_B} = \frac{(1 + \lambda_A)}{v} \sum_{j=1}^{n} \int_0^{(2/n)M_{X_B}} x\, \mathrm{d}x. \tag{37}$$

Solving Eq. (37) for λ_A, uniquely yields $\lambda_A = -1 + ((nv)/(2M_{X_B}))$. From Lemmas 5 and 6, it follows that for each auction j, $F_{B,j}(x) = F_{B,j}(0) + x(n/2M_{X_B})$ for $x \in [0, (2/n)M_{X_B}]$. Then, because $F_{B,j}((2/n)M_{X_B}) = 1$ it follows that $F_{B,j}(0) = 0$.

For player A,

$$M_{X_A} = \frac{(1 + \lambda_B)}{v} \sum_{j=1}^{n} \int_0^{(2/n)M_{X_B}} x\, \mathrm{d}x. \tag{38}$$

Solving Eq. (38) for λ_B, uniquely yields $\lambda_B = -1 + ((nvM_{X_A})/(2M^2_{X_B}))$. From Lemmas 5 and 6, it follows that for each auction j, $F_{A,j}(x) = F_{A,j}(0) + x((nM_{X_A})/(2M^2_{X_B}))$ for $x \in [0, (2/n)M_{X_B}]$. Then, because $F_{A,j}((2/n)M_{X_B}) = 1$ it follows that $F_{A,j}(0) = 1 - M_{X_A}/M_{X_B}$.

In the Theorem 1 range, there were three cases: (a) neither player uses all of his resources, (b) only the weaker player (A) uses all of his resources, and (c) both players use all of their resources. In case (a), it follows that $\lambda_A = \lambda_B = 0$. Otherwise, at least one player i would have an incentive to increase his expenditure up toward X_i. Similarly, in case (b), it follows that $\lambda_B = 0$, and $\lambda_A \geq 0$, and in case (c) $\lambda_B \geq 0$, and $\lambda_A \geq 0$. Returning to the definition of M_{X_i} and the expressions for each $F_{i,j}$ given above, it follows that in the Theorem 1 parameter range, $M_{X_A} = \min\{X_A, (nv/2)\}$ and $M_{X_B} = \min\{X_B, (nvM_{X_A}/2)^{1/2}\}$. □

We conclude the Appendix with a brief discussion of how these results extend to the case of $n = 2$ and the Theorem 2 parameter range with $X_A \neq (2v/n)$ and $n \geq 3$. Note that Lemma 1 holds for all $n \geq 2$ and can be extended to cover all parameter configurations. Similarly, Lemma 2 holds for all $n \geq 2$, but the lemma can only be extended to the case in which the player's resources (X_A, X_B) lie in the Theorem 2 parameter range with $X_A \neq (2v/n)$. If $M_{X_A} = (2v/n)$ then M_{X_B} can take any value in the interval $[v(2-(2/n)), X_B]$. That is any feasible pair of strategies $\{P_A, P_B\}$ with $M_{X_A} = (2v/n)$ and $M_{X_B} \in [v(2-(2/n)), X_B]$ and which provide the corresponding sets of univariate marginal distributions stated in Theorem 2 is an equilibrium. Similar issues regarding the nonuniqueness of the players' equilibrium total expected expenditures arise in the Theorems 3 and 5 parameter ranges.

In the Theorem 2 parameter range with $X_A \neq (2v/n)$, Lemma 4 applies to both players but $\bar{s} = X_A$. For this parameter range, Lemma 5 only applies to player A. The issue is that when (X_A, X_B) is in the Theorem 2 parameter range, interchangeability of equilibrium strategies is no longer sufficient to rule out mass points in player B's univariate marginal distributions. In particular, because $\bar{s} = X_A$ each $F_{B,j}$ can now place an atom at $\bar{s} = X_A$. Furthermore, mass points may exist in the interior of the domain of player B's univariate marginals. Consider an equilibrium $\{P_A, P_B\}$, with $M_{X_A} = X_A$ and $M_{X_B} = X_B = (n/2)X_A$, in which player A uses a strategy consistent with Theorem 2 and player B uses the strategy formed by player B bidding $(X_A/2)$ in each auction with probability $(1-(2/n))$ and with probability $(2/n)$ player B utilizing a strategy consistent with Theorem 2. This is a feasible strategy for player B [which satisfies Lemma 2], and in this strategy player B's univariate marginals are given by:

$$\forall\ j \in \{1, \ldots, n\}\ \ F_{B,j}\left(x_j\right) = \begin{cases} \frac{2x_j}{nX_A} & \text{for } x_j \in \left[0, \frac{X_A}{2}\right) \\ 1 - \frac{2}{n} + \frac{2x_j}{nX_A} & \text{for } x_j \in \left[\frac{X_A}{2}, X_A\right] \end{cases}.$$

As long as player A uses all of his available resources X_A and bids above $(X_A/2)$ in a single auction—as is the case if player A is using a strategy consistent with Theorem 2—this yields the unique equilibrium expected payoff $[v - X_A]$ for player A.[17]

[17] Note that if player A bids $(X_A/2)$ in two auctions, then the tie-breaking rule applies and player A's expected payoff is equal to the unique equilibrium payoff.

Furthermore, it is straightforward to show that there are no profitable deviations for player A, and thus, such a pair of joint distributions forms an equilibrium. The issue here is that at those points in the support of player A's equilibrium strategy where ties occur with positive probability: (i) player A is at his budget constraint and (ii) ties occur in at most two auctions. In order for player B to create such a situation, it must be the case that $X_B \geq (n/2)X_A$, and so this issue does not arise in the Theorem 1 range.

Because the extension of Lemma 5 to the Theorem 2 parameter range applies to only player A's set of univariate marginal distributions, it clearly follows that the extensions of Lemmas 6 and 7 also only apply to player A's set of univariate marginal distributions.

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