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RESEARCH ARTICLE

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The Colonel Blotto game

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Abstract In the Colonel Blotto game, two players simultaneously distribute forces across n battlefields. Within each battlefield, the player that allocates the higher level of force wins. The payoff of the game is the proportion of wins on the individual battlefields. An equilibrium of the Colonel Blotto game is a pair of n-variate distributions. This paper characterizes the unique equilibrium payoffs for all (symmetric and asymmetric) configurations of the players' aggregate levels of force, characterizes the complete set of equilibrium univariate marginal distributions for most of these configurations, and constructs entirely new and novel equilibrium n-variate distributions.

Keywords Colonel Blotto game · Redistributive politics · All-pay auction

JEL Classification Numbers D7

1 Introduction

The Colonel Blotto game, which originates with Borel (1921), is a constant-sum game involving two players, A and B, and n independent battlefields. A has X_A units of force to distribute among the battlefields, and B has X_B units. Each player must distribute their forces without knowing the opponent's distribution. If A sends x_A^k units and B sends x_B^k units to the kth battlefield, the player who provides the

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higher level of force wins battlefield k. The payoff for the whole game is the proportion of the wins on the individual battlefields. An equilibrium[1] of this game is a pair of n-variate distributions, and the first solutions appear in Borel and Ville (1938),[2] who solve the problem for the case of $n = 3$ and $X_A = X_B$. In a 1950 RAND research memorandum Gross and Wagner extend these solutions to allow for any finite $n \geq 3$, but still require that the players' are symmetric in their aggregate levels of force, $X_A = X_B$.

Although the Colonel Blotto game captured the attention of some of the greatest operations researchers of recent generations (see, for instance, Bellman 1969; Blackett 1954, 1958; Shubik and Weber 1981; Tukey 1949) heretofore, the technical difficulty of this problem has restricted the scope of examination to a simplified discrete Colonel Blotto game and symmetric configurations of the players' aggregate levels of force in the continuous Colonel Blotto game.[3] This paper extends the literature on the continuous Colonel Blotto game by characterizing the unique equilibrium payoffs for all symmetric and asymmetric configurations of the players' aggregate levels of force, characterizing the complete set of equilibrium univariate marginal distributions for most of these configurations, and constructing entirely new and novel equilibrium n-variate distributions.[4]

Gross and Wagner's (1950) generalizations of Borel's two solutions to the Colonel Blotto game with symmetric forces exploit properties of regular n-gons.[5] However, for $n > 3$ the use of regular n-gons severely limits the set of n-tuples from which the support of equilibrium n-variate distributions can be formed. Furthermore, the equilibrium n-variate distributions of the game with asymmetric forces examined in this paper cannot be constructed by distributing mass on the surface of regular n-gons. This paper establishes entirely new and novel solutions which do not use regular n-gons.

Since the appearance of the solutions to the symmetric case, it has been an open question whether uniform univariate marginal distributions are a necessary condition for equilibrium.[6] We show that the answer to this question is yes. To characterize the equilibrium univariate marginal distributions, we utilize n-copulas, the functions that map univariate marginal distributions into joint distributions, to separate the players' best response correspondences into a set of univariate marginal distributions and a mapping of this set into an n-variate distribution.[7] Additionally,

[1] Throughout this paper the term "equilibrium" refers to Nash equilibrium although, since the game is constant sum, these are also optimal strategies.

[2] In Borel's course on probability at the University of Paris (1936–1937) two solutions to this problem were given. These are commonly referred to as the *disk* and *hexagonal* solutions and were published in Borel and Ville (1938).

[3] Gross and Wagner (1950) solve the Colonel Blotto game with asymmetric forces in the special case of $n = 2$.

[4] In particular for $\frac{1}{n-1} \leq \frac{X_A}{X_B} \leq 1$, this paper completely characterizes the equilibrium univariate marginal distributions. For the case $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$, this paper provides an equilibrium and the unique equilibrium payoffs. The remaining case $\frac{X_A}{X_B} \leq \frac{1}{n}$ is trivial.

[5] In particular, the sum of the perpendiculars from any point in a regular n-gon to the sides of the regular n-gon is equal to n times the inradius, and letting s be the side length and r be the inradius, $s = 2r \tan \frac{\pi}{n}$ for all regular n-gons.

[6] See for example Gross and Wagner (1950), Kvasov (2005), and Laslier and Picard (2002) who discuss this issue.

[7] See Nelson (1999) for an introduction to copulas.

both Borel's solutions and Gross and Wagner's (1950) generalizations rely on the connectedness of the support. The equilibrium n-variate distributions examined in this paper do not rely on the connectedness of the support, and this paper highlights the fact that the connectedness (or disconnectedness) of the support is a property of the n-copula.

The Colonel Blotto game is a fundamental model of strategic resource allocation in multiple dimensions. Strategic resource allocation in a single dimension, such as the all-pay auction, has been widely used in economics to model contests such as political campaigns, political lobbying, research and development races, litigation and a number of other applications. Most if not all of these applications have multiple dimension analogs. In addition, the Colonel Blotto game has recently been used to analyze electoral competition over redistribution of a fixed budget (Laslier 2002; Laslier and Picard 2002). In the model of redistributive politics candidates simultaneously announce how they will allocate a budget, if elected, by making binding promises to each voter. Each voter votes for the candidate offering the higher level of utility, and each candidate's payoff is the vote share that they receive. The Colonel Blotto game with asymmetric forces, characterized in this paper, corresponds directly to a model of redistributive politics in which one candidate has a valence advantage.[8] Also related is Kvasov (2005), who examines a non-constant-sum version of the Colonel Blotto game in which the allocation of force is costly. That paper alludes to a connection between the standard (constant-sum) Colonel Blotto game with symmetric forces and a non-constant-sum version of the Colonel Blotto game with symmetric forces. This paper formally establishes the connection between the two games for all symmetric and asymmetric configurations of the players' aggregate levels of force.

Section 2 presents the model. Section 3 completely characterizes the equilibrium univariate marginal distributions of the Colonel Blotto game for most of the parameter space. Using a new method for constructing equilibrium n-variate distributions, Section 4 demonstrates the existence of n-copulas with the necessary properties. Section 4 also provides an equilibrium and the unique equilibrium payoffs for the remaining subset of the parameter space. Section 5 concludes.

2 The model

Players Two players, A and B, simultaneously allocate their forces X_A and X_B, respectively, across a finite number, $n \geq 3$, of homogeneous battlefields.[9] Each battlefield j has a payoff of $\frac{1}{n}$. Each player's payoff is the sum of the payoffs across all of the battlefields or, equivalently, the proportion of the battlefields to which the player sends a higher level of force. Let $X_A \leq X_B$. In the case that the players allocate the same level of force to a battlefield, it is assumed that player B wins that battlefield. The specification of the tie-breaking rule does not affect the results

[8] See for example Sahuguet and Persico (2006) who examine a related model of redistributive politics, based on Myerson's (1993) model of redistributive politics with a continuum of voters, in which one candidate has a valence advantage.

[9] The case of $n = 2$, with symmetric and asymmetric forces, is discussed by Gross and Wagner (1950). Moving from $n = 2$ to $n \geq 3$ greatly enlarges the space of feasible n-variate distribution functions, and the equilibrium strategies examined in this paper differ dramatically from the case of $n = 2$.

as long as $\frac{2}{n}X_B \leq X_A$. However, in the case that $\frac{2}{n}X_B > X_A$, this specification of the tie-breaking rule avoids the need to have player B allocate a level of force arbitrarily close to, but above, player A's maximal allocation of force, X_A. A range of tie-breaking rules yield similar results. The force allocated to each battlefield must be nonnegative. For player i, the set of feasible allocations of force across the n battlefields is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^{n} x^j \leq X_i \right\}.$$

Strategies It is well known that for $\frac{1}{n}X_B < X_A \leq X_B$ there is no pure strategy equilibrium for this class of games.[10] A mixed strategy, which we term a *distribution of force*, for player i is an n-variate distribution function $P_i : \mathbb{R}^n_+ \rightarrow [0, 1]$ with support contained in the set of player i's feasible allocations of force, $\mathfrak{B}_i$, and with one-dimensional marginal distribution functions $\left\{F_i^j\right\}_{j\in\{1,\ldots,n\}}$, one univariate marginal distribution function for each battlefield j. The n-tuple of player i's allocation of forces across the n battlefields is a random n-tuple drawn from the n-variate distribution function P_i with the set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^n$.

The Colonel Blotto game *The Colonel Blotto game*, which we label

$$CB\{X_A, X_B, n\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of force subject to their budget constraints, each battlefield is won by the player that provides the higher allocation of force on that battlefield (where player B wins the battlefield in the case of a tie), and players' payoffs equal the proportion of battles won.

3 Optimal univariate marginal distributions

We begin with the case of $\frac{1}{n-1} \leq \frac{X_A}{X_B} \leq 1$. The remaining case $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$ is addressed in section 4. To completely characterize the equilibrium univariate marginal distribution functions for $\frac{1}{n-1} \leq \frac{X_A}{X_B} \leq 1$, we utilize n-copulas, the functions that map univariate marginal distribution functions into joint distribution functions.

Definition 1 *Let I denote the unit interval* $[0, 1]$. *An n-copula is a function C from I^n to I such that*

1. *For all $\boldsymbol{x} \in I^n$, $C(\boldsymbol{x}) = 0$ if at least one coordinate of $\boldsymbol{x}$ is* 0, *and if all coordinates of $\boldsymbol{x}$ are* 1 *except x_k, then $C(\boldsymbol{x}) = x_k$.*

[10] In the case that $\frac{1}{n}X_B \geq X_A$, there, trivially, exists a pure strategy equilibrium, and player B wins all of the battlefields.

2. *For every $\boldsymbol{x}, \boldsymbol{y} \in I^n$ such that $x_k \leq y_k$ for all $k \in \{1, \ldots, n\}$, the C-volume of the n-box $[x_1, y_1] \times \ldots \times [x_n, y_n]$,*

$$V_C\left(\left[\boldsymbol{x}, \boldsymbol{y}\right]\right) = \Delta_{x_n}^{y_n} \Delta_{x_{n-1}}^{y_{n-1}} \cdots \Delta_{x_2}^{y_2} \Delta_{x_1}^{y_1} C\left(\boldsymbol{t}\right)$$

where

$$\begin{aligned}\Delta_{x_k}^{y_k} C\left(\boldsymbol{t}\right) = {} & C\left(t_1, \ldots, t_{k-1}, y_k, t_{k+1}, \ldots, t_n\right) \\ & - C\left(t_1, \ldots, t_{k-1}, x_k, t_{k+1}, \ldots, t_n\right)\end{aligned}$$

is greater than or equal to 0.

Given the definition of an n-copula, we can state the crucial property of n-copulas that we will use.

Theorem 1 (Sklar's Theorem in n-dimensions) *Let H be an n-variate distribution function with univariate marginal distribution functions $F_1, F_2, \ldots, F_n$. Then, there exists an n-copula C such that for all $\boldsymbol{x} \in \mathbb{R}^n$,*

$$H\left(x_1, \ldots, x_n\right) = C\left(F_1\left(x_1\right), \ldots, F_n\left(x_n\right)\right). \tag{1}$$

Conversely, if C is an n-copula and $F_1, F_2, \ldots, F_n$ are univariate distribution functions, then the function H defined by equation (1) is an n-variate distribution function with univariate marginal distribution functions $F_1, F_2, \ldots, F_n$.

The proof of the two-dimensional version of Sklar's theorem is due to Sklar (1959). For a proof of the n-dimensional version see Schweizer and Sklar (1983).

One additional definition that will be used throughout the paper is the support of an n-variate distribution function.

Definition 2 *The support of an n-variate distribution function, H, is the complement of the union of all open sets of $\mathbb{R}^n$ with H-volume zero.*

We now show that the univariate marginal distribution functions and the n-copula are separate components of the players' best response correspondences.

Proposition 1 *In the game $CB\left\{X_A, X_B, n\right\}$, for a given P_{-i}, with the set of univariate marginal distribution functions $\left\{F_{-i}^j\right\}_{j=1}^n$, the Lagrangian of each player i's optimization problem*[11] *can be written as*

$$\max_{\left\{F_i^j\right\}_{j=1}^n} \lambda_i \sum_{j=1}^n \left[\int_0^\infty \left[\frac{1}{n\lambda_i} F_{-i}^j\left(x\right) - x\right] dF_i^j\right] + \lambda_i X_i \tag{2}$$

where the set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^n$ satisfy the constraint that there exists an n-copula, C, such that the support of the n-variate distribution $C\left(F_i^1\left(x^1\right), \ldots, F_i^n\left(x^n\right)\right)$ is contained in $\mathfrak{B}_i$.

[11] This formulation assumes that for all battlefields the players' univariate marginal distributions do not place an atom on the same value. However, it is straightforward to incorporate the tie-breaking rule into the Lagrangian of each player's optimization problem.

Proof In the game $CB\{X_A, X_B, n\}$, for a given P_{-i}, each player i maximizes the sum of the expected payoffs across the individual battlefields

$$\max_{P_i} \sum_{j=1}^{n} \int_0^{\infty} \frac{1}{n} F_{-i}^j(x)\, dF_i^j$$

subject to the constraint that the support of the distribution of force P_i is contained in $\mathfrak{B}_i$.

For a given P_i, let G_i denote the distribution function of $\sum_{j=1}^n x_i^j$ and recall that $G_i(z)$ is the P_i-volume over the region $\left\{\mathbf{x} \in \mathbb{R}_+^n \mid \sum_{j=1}^n x^j \leq z\right\}$. Given that the P_i-volume over the region $\left\{\mathbf{x} \in \mathbb{R}_+^n \mid \sum_{j=1}^n x^j > X_i\right\}$ is 0, it follows that $E_{P_i}\left(\sum_{j=1}^n x_i^j\right) \leq X_i$. Furthermore, $E_{P_i}\left(\sum_{j=1}^n x_i^j\right) = X_i$ if and only if

$$G_i(z) = \begin{cases} 0 & if\ z < X_i \\ 1 & if\ z \geq X_i \end{cases}$$

Recalling that $E_{P_i}\left(\sum_{j=1}^n x_i^j\right) = \sum_{j=1}^n E_{F_i^j}(x)$, it follows that the restriction on the support of the joint distribution, P_i, implicitly places a restriction on the set of univariate marginal distributions. In particular, $\sum_{j=1}^n E_{F_i^j}(x) \leq X_i$ which holds with equality if and only if the budget is spent with probability 1. Finally, from Theorem 1 the n-variate distribution function P_i is equivalent to the set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^n$ combined with an appropriate n-copula, C. The result follows directly. □

Note that from Theorem 1 an n-variate distribution function is equivalent to a set of univariate marginal distribution functions, $\left\{F_i^j\right\}_{j=1}^n$, and an n-copula, C. This in combination with the payoff function of this class of games allows us to separate the players' best response correspondences into the set of univariate marginal distribution functions and n-copula components. Moreover, contrary to the concerns stated by Gross and Wagner (1950), the existence of equilibrium n-variate distribution functions without a connected support is not problematic.[12] Connectedness of the support is a property that arises from the n-copula. Proposition 1 makes no requirement on the connectedness of the resulting n-variate distribution function. In particular, the only requirement on the set of feasible n-copulas is that given a set of optimal univariate marginal distribution functions, $\left\{F_i^j\right\}_{j=1}^n$, the combination of the n-copula and the set of univariate marginal distribution functions must have support contained in $\mathfrak{B}_i$.

[12] For example consider the Colonel Blotto game $CB\{1, 1, 3\}$. It is straightforward to establish that the trivariate distribution with support that uniformly places mass $\frac{1}{2}$ on each of the two following line segments $\left(0, \frac{2}{3}, \frac{1}{3}\right)$ to $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$ and $\left(0, \frac{1}{3}, \frac{2}{3}\right)$ to $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$ is an equilibrium trivariate distribution that has a disconnected support. See Section 4 for details. After this article was accepted for publication, I learned that this equilibrium in the symmetric case was derived independently by Weinstein (2005), who also examines majority Blotto games.

We begin by completely characterizing the set of equilibrium univariate marginal distribution functions for $\frac{1}{n-1} \leq \frac{X_A}{X_B} \leq 1$ and then move on to constructing sufficient n-copulas. Theorem 2 examines the case of $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$ and Theorem 3 examines the case of $\frac{1}{n-1} \leq \frac{X_A}{X_B} < \frac{2}{n}$. The remaining parameter range, $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$, is addressed in Section 4.[13] In this parameter range the equilibrium univariate marginal distributions differ dramatically from those examined in this section.

Theorem 2 *Let X_A, X_B, and $n \geq 3$ satisfy $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$. The unique Nash equilibrium univariate marginal distribution functions of the game $CB\{X_A, X_B, n\}$ are for each player to allocate its forces according to the following univariate distribution functions. For player A*

$$\forall\, j \in \{1, \ldots, n\} \quad F_A^j(x) = \left(1 - \frac{X_A}{X_B}\right) + \frac{x}{\frac{2}{n}X_B}\left(\frac{X_A}{X_B}\right) \quad x \in \left[0, \tfrac{2}{n}X_B\right].$$

Similarly for player B

$$\forall\, j \in \{1, \ldots, n\} \quad F_B^j(x) = \frac{x}{\frac{2}{n}X_B} \quad x \in \left[0, \tfrac{2}{n}X_B\right].$$

The expected payoff for player A is $\frac{X_A}{2X_B}$, and the expected payoff for player B is $1 - \frac{X_A}{2X_B}$.

The formal proof of Theorem 2 is given in Appendix A. However, it is useful to provide some intuition for the uniqueness of the univariate marginal distribution functions.

Beginning with the characterization of n independent and identical simultaneous two-bidder all-pay auctions with complete information, let F_i^j represent bidder i's distribution of bids for auction j, and v_i^j represent the value of auction j for bidder i. Each bidder i's problem is

$$\max_{\left\{F_i^j\right\}_{j=1}^n} \sum_{j=1}^{n} \int_0^\infty \left[v_i^j F_{-i}^j(x) - x\right] dF_i^j.$$

Since each auction is independent, we can apply the equilibrium characterization of the single all-pay auction with complete information (see Hillman and Riley 1989; Baye, Kovenock, and de Vries 1996). Thus, there exists a unique equilibrium distribution function F_i^j for each auction j. For each auction j and bidder i we have the following three cases:

$$\begin{array}{lll}
if\ v_i^j > v_{-i}^j & F_i^j(x) = \frac{x}{v_{-i}^j} & x \in \left[0, v_{-i}^j\right] \\
if\ v_i^j = v_{-i}^j & F_i^j(x) = \frac{x}{v_i^j} & x \in \left[0, v_i^j\right] \\
if\ v_i^j < v_{-i}^j & F_i^j(x) = \left(\frac{v_{-i}^j - v_i^j}{v_{-i}^j}\right) + \frac{x}{v_{-i}^j} & x \in \left[0, v_i^j\right].
\end{array}$$

[13] The case of $\frac{1}{n}X_B \geq X_A$ is trivial.

Now consider a Colonel Blotto game $CB\{X_A, X_B, n\}$. From equation (2) in Proposition 1, each player's Lagrangian can be written as

$$\max_{\left\{F_i^j\right\}_{j=1}^n} \lambda_i \sum_{j=1}^{n} \left[\int_0^\infty \left[\frac{1}{n\lambda_i} F_{-i}^j(x) - x \right] dF_i^j \right] + \lambda_i X_i$$

subject to the constraint that there exists an n-copula, C, such that the support of the n-variate distribution $C\left(F_i^1\left(x^1\right), \ldots, F_i^n\left(x^n\right)\right)$ is contained in $\mathfrak{B}_i$. Assuming that a sufficient n-copula exists, Appendix A establishes a one-to-one correspondence between the set of equilibrium univariate marginal distribution functions and the equilibrium distribution functions of bids from a unique set of n independent and identical simultaneous two-bidder all-pay auctions. It is important to note the role of the Lagrange multipliers in this correspondence. In particular, the Lagrange multipliers establish a shadow value, $\frac{1}{n\lambda_i}$, for the independent and identical simultaneous all-pay auctions. Appendix A establishes the uniqueness of the Lagrange multipliers.

It is also important to note how the constraint on the set of feasible n-copulas affects the correspondence between the Colonel Blotto game and a unique set of independent and identical simultaneous all-pay auctions. In particular, a potential issue that arises is whether this additional constraint leads to equilibria of the Colonel Blotto game which do not have univariate marginal distributions that correspond to the equilibrium distributions of bids from a set of independent and identical simultaneous all-pay auctions.[14] However, if a sufficient n-copula exists this constraint places no restrictions on the set of potential univariate marginal distribution functions, but rather the set of univariate marginal distributions places a constraint on the set of feasible n-copulas. Section 4, then, establishes the existence of sufficient n-copulas. Thus, the equilibrium univariate marginal distributions of the Colonel Blotto game are equivalent to the equilibrium distributions of bids from a unique set of independent and identical simultaneous all-pay auctions. However, the restriction on the set of feasible n-copulas in the Colonel Blotto game implies that the set of equilibrium n-variate distributions for the game forms a strict subset of the set of all n-variate distribution functions with univariate marginal distributions that coincide with the equilibrium distributions of the corresponding set of all-pay auctions.

The following Theorem addresses the case of $\frac{1}{n-1} \leq \frac{X_A}{X_B} < \frac{2}{n}$.

Theorem 3 *Let X_A, X_B, and $n \geq 3$ satisfy $\frac{1}{n-1} \leq \frac{X_A}{X_B} < \frac{2}{n}$. The unique Nash equilibrium univariate marginal distribution functions of the game $CB\{X_A, X_B, n\}$ are for each player to allocate its forces as follows:*

For player A

$$\forall\, j \in \{1, \ldots, n\} \quad F_A^j(x) = \left(1 - \tfrac{2}{n}\right) + \tfrac{x}{X_A}\left(\tfrac{2}{n}\right) \quad x \in [0, X_A].$$

Similarly for player B

$$\forall\, j \in \{1, \ldots, n\} \quad F_B^j(x) = \begin{cases} \frac{2x\left(X_A - \frac{X_B}{n}\right)}{(X_A)^2} & x \in [0, X_A), \\ 1 & x \geq X_A. \end{cases}$$

[14] Thanks to an anonymous referee for this remark.

The expected payoff for player A is $\frac{2}{n} - \frac{2X_B}{n^2 X_A}$*, and the expected payoff for player B is* $1 - \frac{2}{n} + \frac{2X_B}{n^2 X_A}$.

The formal proof of Theorem 3 is similar to the proof contained in Appendix A for Theorem 2, and is thus omitted. The intuition for this parameter range follows from a one-to-one correspondence with a unique set of n independent and identical simultaneous two-bidder all-pay auctions in which player A has a cap of X_A on bids. The characterization of the all-pay auction in which only one bidder faces a cap on bids follows along lines similar to the all-pay auction with a symmetric cap on bids due to Che and Gale (1998).

4 Existence of sufficient n-copulas

Subject to the constraint that there exist sufficient n-copulas, Theorems 2 and 3 characterize the unique sets of equilibrium univariate marginal distribution functions for $\frac{2}{n} \le \frac{X_A}{X_B} \le 1$ and $\frac{1}{n-1} \le \frac{X_A}{X_B} < \frac{2}{n}$, respectively. There is no known existence result for an n-copula, C, with the necessary property that, given a set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^n$, the support of the n-variate distribution $C\left(F_i^1\left(x^1\right), \ldots, F_i^n\left(x^n\right)\right)$ is contained in $\left\{\mathbf{x} \in \mathbb{R}_+^n \mid \sum_{j=1}^n x^j = X_i\right\}$. However from Theorem 1, it is sufficient to show that for each player there exists an n-variate distribution function that allocates all of that player's forces with probability 1 and that provides the unique sets of equilibrium univariate marginal distribution functions characterized in Theorems 2 and 3. Much of this section is devoted to a proof of the existence of such n-variate distributions. This section concludes by addressing the remaining parameter range, $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$.

Theorem 4 *For each unique set of equilibrium univariate marginal distribution functions,* $\left\{F_i^j\right\}_{j=1}^n$*, characterized in Theorems 2 and 3, there exists an n-copula, C, such that the support of the n-variate distribution function* $C\left(F_i^1\left(x^1\right), \ldots, F_i^n\left(x^n\right)\right)$ *is contained in* $\left\{\boldsymbol{x} \in \mathbb{R}_+^n \mid \sum_{j=1}^n x^j = X_i\right\}$.

The discussion that follows establishes an entirely new and novel way to construct sufficient n-variate distribution functions for the symmetric and most asymmetric configurations of force in the Colonel Blotto game. Recall that the ceiling function $\lceil x \rceil$ gives the smallest integer greater than or equal to x, and that the floor function $\lfloor x \rfloor$ gives the largest integer less than or equal to x. We begin with the case that $\frac{2}{n} \le \frac{X_A}{X_B} \le 1$ as in Theorem 2. This proof is for player A. The proof for player B follows directly as the special case of player A where $\frac{X_A}{X_B} = 1$. The construction of the n-variate distribution function is outlined as follows:

1. Player A selects $n - \left\lceil \frac{nX_A}{X_B} \right\rceil$ of the battlefields, each battlefield chosen with equal probability, and provides zero forces to those battlefields.
2. If $\left\lceil \frac{nX_A}{X_B} \right\rceil - \left\lfloor \frac{nX_A}{X_B} \right\rfloor = 1$, then:
   (a) Player A selects $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$ of the remaining $\left\lceil \frac{nX_A}{X_B} \right\rceil$ battlefields, each of the battlefields chosen with equal probability.

(b) On the randomly selected $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$ battlefields, player A randomizes uniformly on $\left[0, \frac{2}{n}X_B\right]$ on each of these battlefields such that, letting z be the sum of player A's allocations of force on these $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$ battlefields and $G(z)$ be the distribution of z,

$$G(z) = \begin{cases} \frac{(z-(X_A-\frac{2}{n}X_B))\left(1-\left\lceil \frac{nX_A}{X_B} \right\rceil + \frac{nX_A}{X_B}\right)}{\frac{2}{n}X_B} & z \in \left[X_A - \frac{2}{n}X_B, X_A\right), \\ 1 & z \geq X_A. \end{cases}$$

The precise construction generating $G(z)$ is discussed in detail directly following this outline.

(c) Defining the allocation of force on the remaining battlefield as $X_A - z$, it follows directly that the univariate distribution of force on the remaining battlefield places mass $\left\lceil \frac{nX_A}{X_B} \right\rceil - \frac{nX_A}{X_B}$ at 0 and randomizes uniformly on $\left(0, \frac{2}{n}X_B\right]$ with the remaining mass. In addition, for all realizations, $\mathbf{x} \in \mathbb{R}^n_+$, of this strategy $\sum_{j=1}^{n} x^j = X_A$ with probability 1.

(d) There are ${}_nC_{\left\lceil \frac{nX_A}{X_B} \right\rceil} \times {}_{\left\lceil \frac{nX_A}{X_B} \right\rceil}C_1$ ways of dividing n battlefields into disjoint subsets such that $n - \left\lceil \frac{nX_A}{X_B} \right\rceil$ battlefields receive zero forces with probability 1, $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$ battlefields involve randomizations of force as in 2(b) above, and one battlefield involves randomization as in 2(c). The n-variate distribution function formed by placing probability $[{}_nC_{\left\lceil \frac{nX_A}{X_B} \right\rceil} \times {}_{\left\lceil \frac{nX_A}{X_B} \right\rceil}C_1]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $\left(1 - \frac{X_A}{X_B}\right)$ at 0 and randomize uniformly on $\left(0, \frac{2}{n}X_B\right]$.

3. If $\left\lceil \frac{nX_A}{X_B} \right\rceil - \left\lfloor \frac{nX_A}{X_B} \right\rfloor = 0$, then:

(a) On the remaining $\frac{nX_A}{X_B}$ battlefields, player A randomizes uniformly on $\left[0, \frac{2}{n}X_B\right]$ on each of these battlefields such that, letting z be the sum of player A's allocations of force on these battlefields and $G(z)$ be the distribution of z,

$$G(z) = \begin{cases} 0 & z < X_A, \\ 1 & z \geq X_A. \end{cases}$$

The precise construction generating $G(z)$ is discussed in detail directly following this outline.

(b) There are ${}_nC_{\frac{nX_A}{X_B}}$ ways of dividing the n battlefields into disjoint subsets such that $n - \frac{nX_A}{X_B}$ battlefields receive zero forces with probability 1 and $\frac{nX_A}{X_B}$ battlefields involve randomizations of force as in 3(a). The n-variate distribution function formed by placing probability $[{}_nC_{\frac{nX_A}{X_B}}]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $\left(1 - \frac{X_A}{X_B}\right)$ at 0 and randomize uniformly on $\left(0, \frac{2}{n}X_B\right]$.

The pivotal steps in this construction are points 2 (b) and 3 (a), and we will now show that there exist such multivariate distribution functions. Beginning with the case that $\frac{2}{n} \leq \frac{X_A}{X_B} \leq \frac{3}{n}$, from points 2 and 3 player A allocates force to at least two and not more than three battlefields, which we label battlefields 1, 2, and 3. Let x_i denote the allocation of force to battlefield $i \in \{1, 2, 3\}$, $z = x_2 + x_3$, and $x_1 = X_A - z$. Consider the support of a bivariate distribution function, F, for x_2 and x_3 which uniformly places mass $\left(X_A/\frac{2}{n}X_B\right) - 1$ on each of the two following line segments:

$$\begin{aligned}
&\left(\tfrac{2}{n}X_B, X_A - \tfrac{2}{n}X_B\right) \text{ to } \left(X_A - \tfrac{2}{n}X_B, 0\right), \\
&\left(X_A - \tfrac{2}{n}X_B, \tfrac{2}{n}X_B\right) \text{ to } \left(0, X_A - \tfrac{2}{n}X_B\right),
\end{aligned}$$

and uniformly places the remaining mass, $3 - (nX_A/X_B)$, on the line segment

$$\left(\tfrac{2}{n}X_B, X_A - \tfrac{2}{n}X_B\right) \text{ to } \left(X_A - \tfrac{2}{n}X_B, \tfrac{2}{n}X_B\right).$$

This support is shown in Figure 1.

In the expression for this bivariate distribution function we will use the following notation:

$$\begin{aligned}
&\text{R1: } \left\{(x_2, x_3) \in \left[0, \tfrac{2}{n}X_B\right]^2 | x_2 > \frac{\frac{4}{n}X_B - X_A}{X_A - \frac{2}{n}X_B} x_3 + X_A - \tfrac{2}{n}X_B\right\} \\
&\text{R2: } \left\{(x_2, x_3) \in \left[0, \tfrac{2}{n}X_B\right]^2 | x_3 > \frac{\frac{4}{n}X_B - X_A}{X_A - \frac{2}{n}X_B} x_2 + X_A - \tfrac{2}{n}X_B\right\} \\
&\text{R3: } \left\{(x_2, x_3) \in \left[0, \tfrac{2}{n}X_B\right]^2 | x_2 + x_3 > X_A\right\} \\
&\text{R4: } \left\{(x_2, x_3) \in \left[0, \tfrac{2}{n}X_B\right]^2 | (x_2, x_3) \notin \text{R1} \cup \text{R2} \cup \text{R3}\right\}
\end{aligned}$$

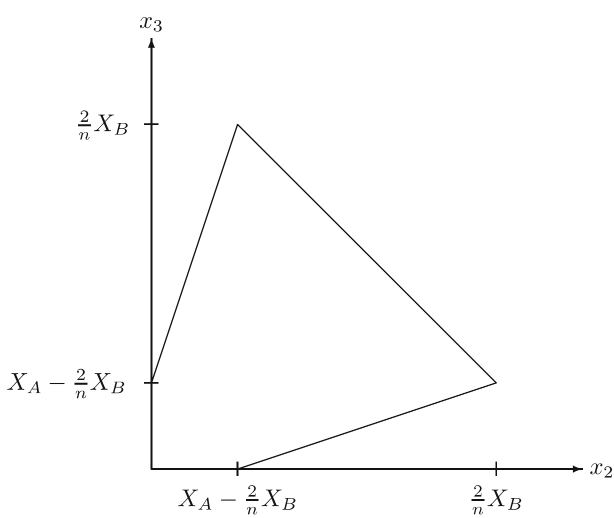

Fig. 1 Support of the bivariate distribution F

The bivariate distribution function for x_2, x_3 is given by

$$F(x_2, x_3) = \begin{cases} \frac{x_3}{\frac{2}{n}X_B} & (x_2, x_3) \in \mathrm{R1} \\ \frac{x_2}{\frac{2}{n}X_B} & (x_2, x_3) \in \mathrm{R2} \\ \frac{x_2+x_3}{\frac{2}{n}X_B} - 1 & (x_2, x_3) \in \mathrm{R3} \\ \frac{\max\{x_2 - X_A + \frac{2}{n}X_B, 0\} + \max\{x_3 - X_A + \frac{2}{n}X_B, 0\}}{\frac{2}{n}X_B\left[\frac{\frac{4}{n}X_B - X_A}{X_A - \frac{2}{n}X_B}\right]} & (x_2, x_3) \in \mathrm{R4} \end{cases}$$

The univariate marginal distributions are given by $F\left(x_2, \frac{2}{n}X_B\right) = \frac{x_2}{\frac{2}{n}X_B}$ and $F\left(\frac{2}{n}X_B, x_3\right) = \frac{x_3}{\frac{2}{n}X_B}$. Thus, F provides the necessary univariate marginal distributions for battlefields 2 and 3.

If $\frac{2}{n} = \frac{X_A}{X_B}$, then player A randomizes on only 2 battlefields and the support of this bivariate distribution function F collapses to the line segment $\left(\frac{2}{n}X_B, 0\right)$ to $\left(0, \frac{2}{n}X_B\right)$, i.e. the support is $\left\{(x_1, x_2) \in \mathbb{R}^2_+ | x_1 + x_2 = X_A\right\}$.[15]

If $\frac{2}{n} < \frac{X_A}{X_B} < \frac{3}{n}$, then, from the support of the bivariate distribution function F, it follows that

$$G(z) = \begin{cases} \left(\frac{z - \left(X_A - \frac{2}{n}X_B\right)}{\frac{2}{n}X_B}\right)\left(\frac{nX_A}{X_B} - 2\right) & z \in \left[X_A - \frac{2}{n}X_B, X_A\right) \\ 1 & z \geq X_A \end{cases}.$$

Since $x_1 \equiv X_A - x_2 - x_3$, the univariate marginal distribution for battlefield 1 places an atom of size $3 - \frac{nX_A}{X_B}$ at 0 and randomizes uniformly on $\left(0, \frac{2}{n}X_B\right]$ with the remaining mass, and for all realizations of (x_1, x_2, x_3), $x_1 + x_2 + x_3 = X_A$ with probability 1. Equivalently, the combination of $x_1 = X_A - z$ with the bivariate distribution F for x_2 and x_3 defines a trivariate distribution function, F', with support that uniformly places mass $\left(X_A / \frac{2}{n}X_B\right) - 1$ on each of the two following line segments:

$$\left(0, \tfrac{2}{n}X_B, X_A - \tfrac{2}{n}X_B\right) \text{ to } \left(\tfrac{2}{n}X_B, X_A - \tfrac{2}{n}X_B, 0\right),$$
$$\left(0, X_A - \tfrac{2}{n}X_B, \tfrac{2}{n}X_B\right) \text{ to } \left(\tfrac{2}{n}X_B, 0, X_A - \tfrac{2}{n}X_B\right),$$

and uniformly places the remaining mass, $3 - (nX_A/X_B)$, on the line segment

$$\left(0, \tfrac{2}{n}X_B, X_A - \tfrac{2}{n}X_B\right) \text{ to } \left(0, X_A - \tfrac{2}{n}X_B, \tfrac{2}{n}X_B\right).$$

The projections of this support onto the x_2, x_3-, x_2, x_1-, and x_3, x_1-planes are given in Figure 2.

[15] It should be pointed out that in the case that $\frac{2}{n} = \frac{X_A}{X_B}$, the bivariate distribution function F is exactly the Fréchet–Hoeffding lower bound two-copula,

$$W = \max\{F(x_1) + F(x_2) - 1, 0\}$$

combined with $F(x_i) = \frac{x_i}{\frac{2}{n}X_B}$ for $x_i \in \left[0, \frac{2}{n}X_B\right]$ and $i = 1, 2$.

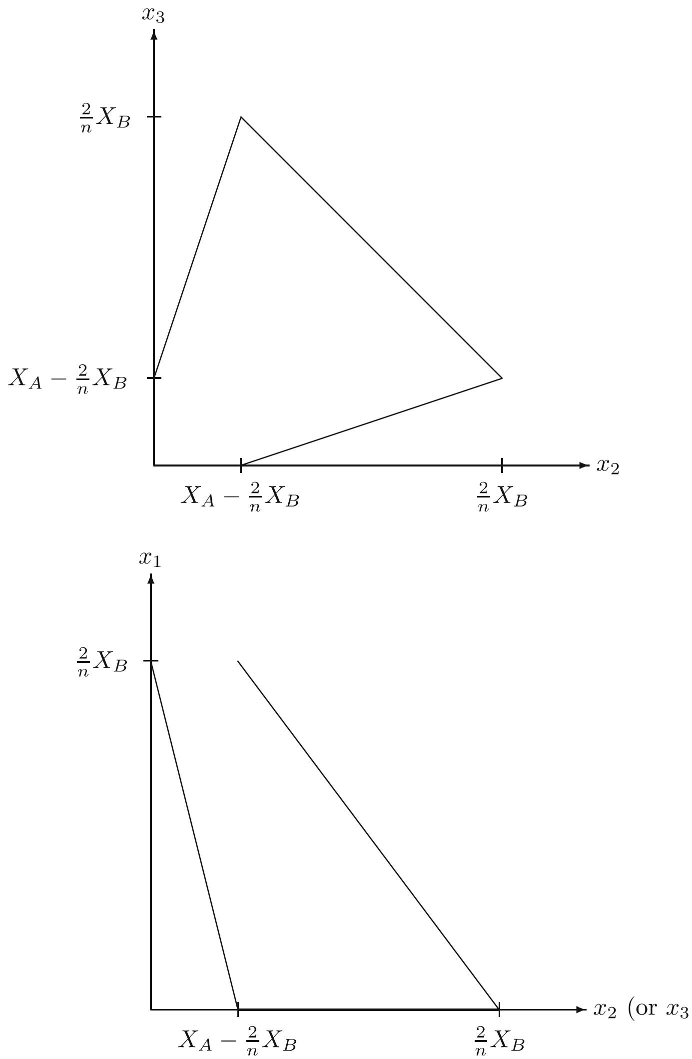

Fig. 2 Projections of the support of the trivariate distribution F' onto the x_2, x_3-, x_2, x_1-, and x_3, x_1- planes

If $\frac{X_A}{X_B} = \frac{3}{n}$, then player A randomizes on three battlefields according to the trivariate distribution function F' which has support that for $\frac{X_A}{X_B} = \frac{3}{n}$ uniformly places mass $\frac{1}{2}$ on each of the two following line segments:

$$\left(0, \tfrac{2}{n}X_B, \tfrac{1}{n}X_B\right) \text{ to } \left(\tfrac{2}{n}X_B, \tfrac{1}{n}X_B, 0\right),$$
$$\left(0, \tfrac{1}{n}X_B, \tfrac{2}{n}X_B\right) \text{ to } \left(\tfrac{2}{n}X_B, 0, \tfrac{1}{n}X_B\right).$$

From the preceding discussion it is clear that each of the three univariate marginal distribution functions randomizes uniformly on $\left[0, \frac{2}{n}X_B\right]$ and that for all realizations of (x_1, x_2, x_3), $x_1 + x_2 + x_3 = X_A$ with probability 1. Furthermore, it is also clear that this support is not a connected set.

Similarly, for $\frac{3}{n} < \frac{X_A}{X_B} \leq \frac{4}{n}$ player A allocates force to at least three and not more than four battlefields. In this case, let $z = x_2 + x_3 + x_4$ and $x_1 = X_A - z$. Consider the support of the trivariate distribution function, F, for x_2, x_3, and x_4 which uniformly places mass $2 - \left(X_A/\frac{2}{n}X_B\right)$ on each of the two following line segments:

$$\left(0, \tfrac{2}{n}X_B, X_A - \tfrac{2}{n}X_B\right) \text{ to } \left(\tfrac{2}{n}X_B, X_A - \tfrac{2}{n}X_B, 0\right),$$
$$\left(0, X_A - \tfrac{2}{n}X_B, \tfrac{2}{n}X_B\right) \text{ to } \left(\tfrac{2}{n}X_B, 0, X_A - \tfrac{2}{n}X_B\right),$$

and uniformly places mass $\left(X_A/\frac{2}{n}X_B\right) - (3/2)$ on each of the two following line segments:

$$\left(0, 0, X_A - \tfrac{2}{n}X_B\right) \text{ to } \left(\tfrac{2}{n}X_B, X_A - \tfrac{2}{n}X_B, 0\right),$$
$$\left(0, X_A - \tfrac{2}{n}X_B, 0\right) \text{ to } \left(\tfrac{2}{n}X_B, 0, X_A - \tfrac{2}{n}X_B\right).$$

This support is shown in Figure 3.

Given this support, it is straightforward to establish that each of the three univariate marginal distribution functions randomizes uniformly on $\left[0, \frac{2}{n}X_B\right]$. In addition, this trivariate distribution function has the property that the distribution of z places an atom of size $4 - \frac{nX_A}{X_B}$ at X_A and randomizes uniformly on $\left[X_A - \frac{2}{n}X_B, X_A\right)$ with the remaining mass. Since at every point on the support $x_1 + x_2 + x_3 + x_4 = X_A$, it follows directly that the univariate marginal distribution on battlefield 1 places an atom of size $4 - \frac{nX_A}{X_B}$ at 0 and randomizes uniformly on $\left(0, \frac{2}{n}X_B\right]$ with the remaining mass.

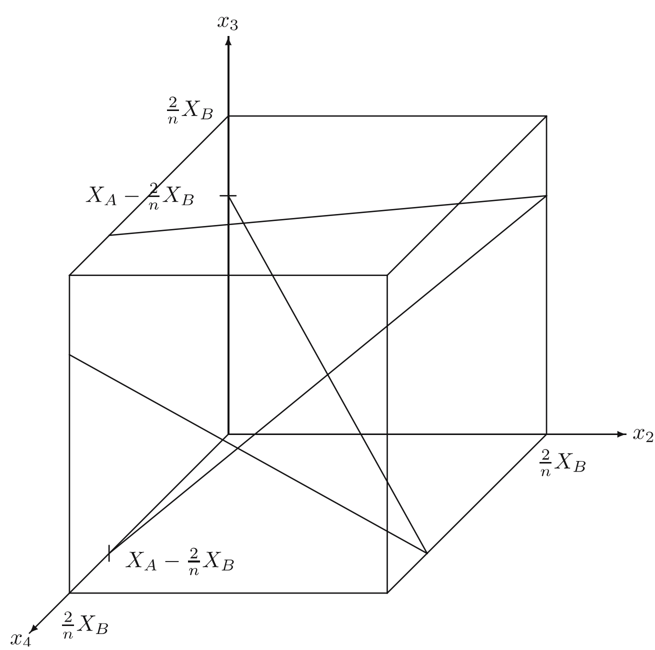

Fig. 3 Support of the trivariate distribution F

Since we can always use independent combinations of the bivariate and trivariate distributions used to establish that points 2 (b) and 3 (a) hold for $\frac{2}{n} \le \frac{X_A}{X_B} \le \frac{3}{n}$, the remaining cases, $\frac{4}{n} < \frac{X_A}{X_B} \le 1$, follow directly. For example, in the case that $\frac{4}{n} < \frac{X_A}{X_B} \le \frac{5}{n}$ it is clear that $\frac{2}{n} < \frac{X_A}{X_B} - \frac{2}{n} \le \frac{3}{n}$. Thus, player A can independently use the construction in 3(a) of the outline for $\frac{X_A}{X_B} = \frac{2}{n}$ and the construction in 2(b) of the outline for $\frac{2}{n} < \frac{X_A}{X_B} \le \frac{3}{n}$. In this case, player A randomly selects $n - 5$ battlefields which each receive zero force and breaking the remaining five battlefields into a set of two battlefields and a set of three battlefields, independently randomizes on these two disjoint subsets as described above. Since the bivariate and trivariate distribution functions are independent it is straightforward to show that the support across all five battlefields is contained in $\left\{\mathbf{x} \in \mathbb{R}^5_+ \mid \sum_{i=1}^5 x_i = X_A\right\}$. In general, for all $\frac{4}{n} < \frac{X_A}{X_B} \le 1$ there exist combinations of independent bi- and trivariate distribution functions to establish that points 2 (b) and 3 (a) hold.

We now examine the case that $\frac{1}{n-1} \le \frac{X_A}{X_B} < \frac{2}{n}$ as in Theorem 3. The existence of a sufficient n-variate distribution for player A in this parameter range is a special case of the Theorem 2 parameter range when $X_A = \frac{2}{n} X_B$. This proof is for player B. The construction of the n-variate distribution function is outlined as follows:

1. Player B selects $\left\lfloor \frac{2X_B}{X_A} \right\rfloor - n$ of the battlefields, each battlefield chosen with equal probability, and provides a force of X_A to each of those battlefields.
2. If $\left\lceil \frac{2X_B}{X_A} \right\rceil - \left\lfloor \frac{2X_B}{X_A} \right\rfloor = 1$, then:
   (a) Player B selects $2n - \left\lceil \frac{2X_B}{X_A} \right\rceil$ of the remaining $2n - \left\lfloor \frac{2X_B}{X_A} \right\rfloor$ battlefields, each of the battlefields chosen with equal probability.
   (b) On the randomly selected $2n - \left\lceil \frac{2X_B}{X_A} \right\rceil$ battlefields, player B randomizes uniformly on $[0, X_A]$ on each of the battlefields such that, letting z be the sum of player B's allocations of force on all $n - 1$ of the battlefields addressed in 2(a) and 2(b) and $G(z)$ be the distribution of z,
   $$G(z) = 1 + \left(\frac{z - X_B}{X_A}\right)\left(1 - \frac{2X_B}{X_A} + \left\lfloor \frac{2X_B}{X_A} \right\rfloor\right)$$
   for $z \in [X_B - X_A, X_B]$. The precise construction of $G(z)$ is given in detail directly following this outline.
   (c) Defining the allocation of force on the remaining battlefield as $X_B - z$, it follows directly that the univariate distribution of force on the remaining battlefield places mass $\frac{2X_B}{X_A} - \left\lfloor \frac{2X_B}{X_A} \right\rfloor$ at X_A and randomizes uniformly on $[0, X_A)$ with the remaining mass. In addition for all realizations, $\mathbf{x} \in \mathbb{R}^n_+$, of this strategy $\sum_{j=1}^n x^j = X_B$ with probability 1.
   (d) There are ${}_nC_{\left(2n - \left\lfloor \frac{2X_B}{X_A} \right\rfloor\right)} \times {}_{\left(2n - \left\lfloor \frac{2X_B}{X_A} \right\rfloor\right)}C_1$ ways of dividing n battlefields into disjoint subsets such that $\left\lfloor \frac{2X_B}{X_A} \right\rfloor - n$ battlefields receive X_A forces with probability 1, $2n - \left\lceil \frac{2X_B}{X_A} \right\rceil$ battlefields involve randomizations of force as in 2(b), and one battlefield involves randomization as in 2(c). The n-variate distribution function formed by placing probability $[{}_nC_{\left(2n - \left\lfloor \frac{2X_B}{X_A} \right\rfloor\right)} \times {}_{\left(2n - \left\lfloor \frac{2X_B}{X_A} \right\rfloor\right)}$

$C_1]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $\frac{\frac{2}{n}X_B}{X_A} - 1$ at X_A and randomize uniformly on $[0, X_A)$.

3. If $\left\lceil \frac{2X_B}{X_A} \right\rceil - \left\lfloor \frac{2X_B}{X_A} \right\rfloor = 0$, then:
   (a) On the remaining $2n - \frac{2X_B}{X_A}$ battlefields, player B randomizes uniformly on $[0, X_A]$ on each of these battlefields such that, letting z be the sum of player B's allocation of force on all of the battlefields and $G(z)$ be the distribution of z,

$$G(z) = \begin{cases} 0 & z < X_B, \\ 1 & z \geq X_B. \end{cases}$$

   The precise construction of $G(z)$ is given in detail directly following this outline.
   (b) There are ${}_nC_{\left(2n - \frac{2X_B}{X_A}\right)}$ ways of dividing the n battlefields into disjoint subsets such that $\frac{2X_B}{X_A} - n$ battlefields receive X_A forces with probability 1 and $2n - \frac{2X_B}{X_A}$ battlefields involve randomizations of force as in 3(a). The n-variate distribution function formed by placing probability $[{}_nC_{\left(2n - \frac{2X_B}{X_A}\right)}]^{-1}$ on each of these n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $\left(\frac{\frac{2}{n}X_B}{X_A} - 1\right)$ at X_A and randomize uniformly on $[0, X_A)$.

The pivotal steps in this construction are, again, points 2 (b) and 3 (a), and we will now show that there exist such multivariate distribution functions. In fact these multivariate distributions are quite similar to those used for the Theorem 2 parameter range. We will, thus, only provide the supports of the bivariate and trivariate distributions that establish that points 2 (b) and 3 (a) hold. Beginning with the case that $n - 3 \leq \frac{2X_B}{X_A} - n \leq n - 2$ (or equivalently $\frac{1}{n-1} \leq \frac{X_A}{X_B} \leq \frac{1}{n-\frac{3}{2}}$), from points 2 and 3 player B allocates a force of X_A to at least $n - 3$ and not more than $n - 2$ battlefields. Given that $n - 3$ battlefields have received a force of X_A, for the three remaining battlefields let x_i denote the allocation of force to battlefield $i \in \{1, 2, 3\}$. Consider the support of a trivariate distribution function for x_1, x_2, x_3 which uniformly places mass $n - 1 - (X_B/X_A)$ on each of the two following line segments:

$$(0, X_A, X_B - X_A(n-2)) \text{ to } (X_A, X_B - X_A(n-2), 0),$$
$$(0, X_B - X_A(n-2), X_A) \text{ to } (X_A, 0, X_B - X_A(n-2))$$

and uniformly places the remaining mass, $(2X_B/X_A) - 2n + 3$, on the line segment

$$(X_A, 0, X_B - X_A(n-2)) \text{ to } (X_A, X_B - X_A(n-2), 0).$$

This support is shown in Figure 4.

Given this support, it is straightforward to establish that the univariate marginal distribution functions on battlefields 2 and 3 randomize uniformly on $[0, X_A]$ and that the univariate marginal distribution function for battlefield 1 places an atom of size $\frac{2X_B}{X_A} - 2n + 3$ at X_A and randomizes uniformly on $[0, X_A)$ with the remaining mass.

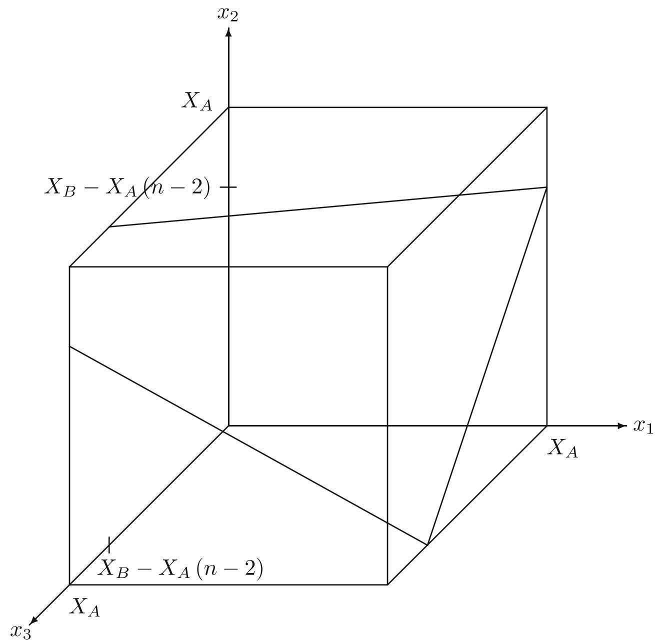

Fig. 4 Support of the trivariate distribution F

Similarly, for $n-4 \leq \frac{2X_B}{X_A} - n < n-3$ (or equivalently $\frac{1}{n-\frac{3}{2}} < \frac{X_A}{X_B} \leq \frac{1}{n-2}$) player B allocates a force of X_A to at least $n-4$ and not more than $n-3$ battlefields. Given that $n-4$ battlefields have received a force of X_A, for the four remaining battlefields let x_i denote the allocation of force to battlefield $i \in \{1, 2, 3, 4\}$, $z' = x_2 + x_3 + x_4$ and $x_1 = X_B - z' - X_A(n-4)$. Consider the support of a trivariate distribution function for x_2, x_3, x_4 which uniformly places mass $2 + (X_B/X_A) - n$ on each of the two following line segments:

$$(0, X_A, X_B - X_A(n-2)) \text{ to } (X_A, X_B - X_A(n-2), 0),$$

$$(0, X_B - X_A(n-2), X_A) \text{ to } (X_A, 0, X_B - X_A(n-2))$$

and uniformly places mass $n - (X_B/X_A) - (3/2)$ on each of the two following line segments:

$$(0, X_A, X_B - X_A(n-2)) \text{ to } (X_A, X_B - X_A(n-2), X_A),$$

$$(0, X_B - X_A(n-2), X_A) \text{ to } (X_A, X_A, X_B - X_A(n-2)).$$

This support is shown in Figure 5.

Given this support, it is straightforward to establish that each of the three univariate marginal distribution functions randomizes uniformly on $[0, X_A]$. In addition, this trivariate distribution function has the property that the distribution of z' places an atom of size $4 + \frac{2X_B}{X_A} - 2n$ on $X_B - X_A(n-3)$ and randomizes uniformly on $((X_B - X_A(n-3), X_B - X_A(n-4)]$ with the remaining mass. Since at every point on the support $x_1 + x_2 + x_3 + x_4 = X_B - (n-4)X_A$, it follows directly that the univariate marginal distribution on battlefield 1 places an atom of size

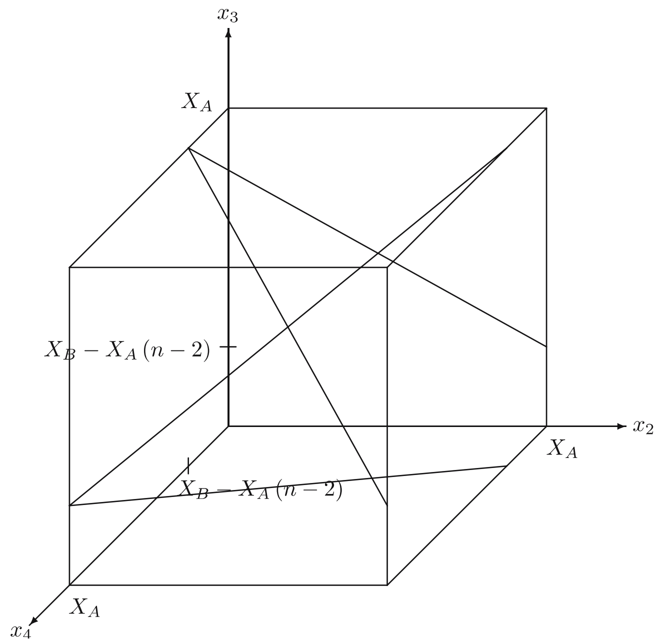

Fig. 5 Support of the trivariate distribution F

$4 + \frac{2X_B}{X_A} - 2n$ at X_A and randomizes uniformly on $[0, X_A)$ with the remaining mass.

Since we can always use independent combinations of the bivariate and trivariate distributions used to establish that points 2 (b) and 3 (a) hold for $n - 4 \leq \frac{2X_B}{X_A} - n \leq n - 2$, the remaining cases, $0 \leq \frac{2X_B}{X_A} - n < n - 4$, follow directly.

In the remaining case that $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$, Theorem 3 would provide the unique set of equilibrium univariate marginal distributions if a sufficient n-copula were to exist for each player. However, such an n-copula does not exist for player B, and thus, the Lagrange multipliers in the players' optimization problems may equal zero. This follows from the fact that for player B the univariate marginal distributions in Theorem 3 require an atom of size $\frac{2X_B}{nX_A} - 1$ at X_A and randomize uniformly over the interval $[0, X_A)$ with the remaining mass. However, for $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$ the intersection of the set $\left\{\mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^n x^j = X_B\right\}$ with the n-box $[0, X_A]^n$ contains no n-tuples in which one battlefield receives 0 forces. That is, if player B uses all of its forces it can allocate X_A forces to $n - 1$ battlefields and the force allocated to the remaining battlefield must be $X_B - X_A\,(n - 1)$ which is greater than 0 since $\frac{X_A}{X_B} < \frac{1}{n-1}$. Thus, for $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$ it is not possible for player B to allocate all of its forces with probability 1 and use the univariate marginal distributions given by Theorem 3.

For the case that $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$ one equilibrium is given by an extension of the case of $n = 2$ with asymmetric forces discussed by Gross and Wagner (1950). This equilibrium is not unique. However, since the Colonel Blotto game is a constant-sum game the equilibrium payoffs are unique.

Theorem 5 *Define* $k = \left\lceil \frac{X_A}{X_B - X_A(n-1)} \right\rceil$. *Let* X_A, X_B, *and* n *satisfy* $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$ *(i.e.* $2 \leq k < \infty$*). A Nash equilibrium of the game* $CB\{X_A, X_B, n\}$ *is for each player to allocate its forces according to the following n-variate distributions:*

Player A randomly allocates 0 *forces to* $n-2$ *of the battlefields, each battlefield chosen with equal probability,* $\frac{n-2}{n}$. *On the remaining two battlefields player A utilizes a bivariate distribution function with k mass points, each mass point receiving the same weight,* $\frac{1}{k}$. *Player A's mass points on these two remaining battlefields are located at the points*

$$\left((k-1-i)\frac{X_A}{k-1}, i\frac{X_A}{k-1}\right), \quad i = 0, \ldots, k-1.$$

Player B randomly allocates X_A *forces to* $n-2$ *battlefields, each battlefield chosen with equal probability,* $\frac{n-2}{n}$. *On the remaining two battlefields player B utilizes a bivariate distribution function with k mass points, each mass point receiving the same weight,* $\frac{1}{k}$. *Player B's mass points on the two remaining battlefields are located at*

$$\left(X_A - i\frac{nX_A - X_B}{k-1}, X_A - (k-1-i)\frac{nX_A - X_B}{k-1}\right), \quad i = 0, \ldots, k-1.$$

The unique expected payoff for player A is $\frac{2k-2}{kn^2}$, *and the unique expected payoff for player B is* $1 - \frac{2k-2}{kn^2}$.

The proof of Theorem 5 is given in Appendix B. In the case that $n = 2$, Theorem 5 coincides with Gross and Wagner's (1950) equilibrium for the Colonel Blotto game with asymmetric forces and $n = 2$. It is also important to note that in the limit as $\frac{X_A}{X_B}$ approaches $\frac{1}{n-1}$ from below this set of equilibrium univariate marginal distributions converges to the unique set of equilibrium univariate marginal distributions given by Theorem 3 and the players' payoffs converge to those in Theorem 3.

5 Conclusion

The Colonel Blotto game is a fundamental model of strategic resource allocation in multiple dimensions. This paper extends the literature on the Colonel Blotto game in several important ways. In particular, the technical difficulty of the Colonel Blotto game has, heretofore, restricted the focus to the case of symmetric configurations of the players' aggregate levels of force. This paper extends the literature on the Colonel Blotto game by characterizing the unique equilibrium payoffs for all symmetric and asymmetric configurations of the players' aggregate levels of force and characterizing the complete set of equilibrium univariate marginal distributions for most of these configurations.

Gross and Wagner's (1950) generalizations of Borel's two solutions to the Colonel Blotto game with symmetric forces exploit properties of regular n-gons. However, the equilibrium n-variate distributions of the Colonel Blotto game with asymmetric forces cannot be constructed from distributing mass on the surface of regular n-gons, and this paper establishes entirely new and novel solutions which do not use regular n-gons. Furthermore, unlike both Borel's solutions and Gross and Wagner's generalizations, the equilibrium n-variate distributions examined in this paper do not rely on the connectedness of the support.

Appendix A

The proof of Theorem 2, which is contained in the following lemmas, establishes that there exists a one-to-one correspondence between the equilibrium univariate marginal distributions of the Colonel Blotto game and the equilibrium distributions of bids from a unique set of two-bidder independent and identical simultaneous all-pay auctions. The uniqueness of the equilibrium univariate marginal distributions then follows from the characterization of the all-pay auction by Hillman and Riley (1989) and Baye et al. (1996). In the discussion that follows, $\bar{s}_i^j$ and $\underline{s}_i^j$ are the upper and lower bounds of candidate i's distribution of force for battlefield j and $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$.

Lemma 1 *For each* $i \in \{A, B\}$ *and for* $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$, $\lambda_i > 0$.

Proof The fact that the univariate marginal distributions provided in Theorem 2 and the corresponding n-variate distributions constructed in Theorem 4 form an equilibrium is easily verified. It is also easily verified that in this equilibrium $\lambda_i > 0$ for each player i. By way of contradiction, suppose that there exists an equilibrium in which player $-i$ does not use their entire budget, X_{-i}, with probability 1. For any parameter configuration such that $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$, a feasible strategy for player i is the joint distribution constructed in Theorem 4 that corresponds to the univariate marginal distributions, given in Theorem 2, for this parameter configuration. In this case, player $-i$'s expected payoff would be strictly less than the expected payoff given in Theorem 2. This is a contradiction since player $-i$ can obtain the expected payoff given in Theorem 2 by using the feasible strategy of the joint distribution constructed in Theorem 4 that corresponds to the univariate marginal distributions, given in Theorem 2, for this parameter configuration. □

The next four lemmas follow along the lines of the proofs in Baye et al. (1996).

Lemma 2 *For each* $j \in \{1, \ldots, n\}$, $\bar{s}_{-i}^j = \bar{s}_i^j = \bar{s}^j$.

Lemma 3 *In any equilibrium* $\left\{F_i^j, F_{-i}^j\right\}_{j\in\{1,\ldots,n\}}$, *no* F_i^j *can place an atom in the half open interval* $\left(0, \bar{s}^j\right]$.

Lemma 4 *For each* $j \in \{1, \ldots, n\}$ *and for each* $i \in \{A, B\}$, $\frac{1}{n\lambda_i} F_{-i}^j(x) - x$ *is constant* $\forall\, x \in \left(0, \bar{s}^j\right]$.

Lemma 5 $\forall\, j \in \{1, \ldots, n\}$, $F_B^j(0) = 0$ *and, thus,* $\frac{1}{n\lambda_A} F_B^j(x) - x = 0\ \forall\, x \in \left[0, \bar{s}^j\right]$.

The following lemma characterizes the relationship between λ_A and λ_B.

Lemma 6 *In equilibrium* $\lambda_A = \lambda_B \frac{X_B}{X_A}$.

Proof By way of contradiction, suppose that $\lambda_A \neq \lambda_B \frac{X_B}{X_A}$. For Theorem 2's parameter range, in any equilibrium each player allocates all of their forces with certainty and in expectation, thus

$$X_B \sum_{j=1}^{n} \int_0^{\bar{s}^j} x dF_A^j(x) = X_A \sum_{j=1}^{n} \int_0^{\bar{s}^j} x dF_B^j(x) . \tag{3}$$

But, from Lemmas 3, 4, and 5, it follows that

$$dF_A^j(x) = n\lambda_B dx \tag{4}$$

for all $x \in \left(0, \bar{s}^j\right]$, and

$$dF_B^j(x) = n\lambda_A dx \tag{5}$$

for all $x \in \left[0, \bar{s}^j\right]$. Substituting equations (4) and (5) in equation (3), we have

$$\lambda_B X_B \sum_{j=1}^{n} \int_0^{\bar{s}^j} nx\, dx = \lambda_A X_A \sum_{j=1}^{n} \int_0^{\bar{s}^j} nx\, dx$$

which is a contradiction since

$$\sum_{j=1}^{n} \int_0^{\bar{s}^j} nx\, dx = \sum_{j=1}^{n} \int_0^{\bar{s}^j} nx\, dx$$

but $\lambda_A \neq \lambda_B \frac{X_B}{X_A}$. □

The following lemma establishes the value of $\bar{s}^j$.

Lemma 7 $\bar{s}^j = \frac{1}{n\lambda_A}$.

Proof From Lemmas 4 and 5, we know that for each player i and any battlefield j

$$\frac{1}{n\lambda_i} F_{-i}^j(x) - x$$

is constant $\forall x \in \left(0, \bar{s}^j\right]$. It then follows that player i would never use a strategy that provides offers in $\left(\frac{1}{n\lambda_i}, \infty\right)$ since an offer of zero strictly dominates such a strategy. Noting that $\frac{1}{n\lambda_A} \leq \frac{1}{n\lambda_B}$, we have that $\bar{s}^j \leq \frac{1}{n\lambda_A}$ and that $\forall x \in \left(0, \bar{s}^j\right]$

$$\frac{1}{n\lambda_i} F_{-i}^j(x) - x \geq \frac{1}{n\lambda_i} - \bar{s}^j.$$

By way of contradiction, assume that $\bar{s}^j < \frac{1}{n\lambda_A}$ then by allocating a level of force to battlefield j that is greater than $\bar{s}^j$ by an arbitrarily small amount, player A can earn arbitrarily close to $\frac{1}{n\lambda_A} - \bar{s}^j > 0$ on battlefield j, which contradicts Lemma 5. □

The following lemma establishes that there exists a unique pair λ_A, λ_B that satisfies the budget constraint.

Lemma 8 *There exists a unique value for λ_A, and thus for λ_B. $\lambda_A = \frac{1}{2X_B}$ and thus $\lambda_B = \frac{X_A}{2X_B^2}$.*

Proof The budget constraint determines the unique pair λ_A, λ_B. Thus, λ_A solves

$$n \int_0^{\frac{1}{n\lambda_A}} xn\lambda_A dx = X_B.$$

Solving for λ_A we have that

$$\lambda_A = \frac{1}{2X_B}.$$

It follows directly from Lemma 6 that $\lambda_B = \frac{X_A}{2X_B^2}$. □

This completes the proof of Theorem 2.

Appendix B

The proof of Theorem 5, stated below, establishes the existence of an equilibrium in the game $CB\{X_A, X_B, n\}$ for X_A, X_B, and n such that $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$. In the discussion that follows, recall that $k = \left\lceil \frac{X_A}{X_B - X_A(n-1)} \right\rceil$, and thus, $2 \leq k < \infty$.

First, the strategies in the statement of Theorem 5 are feasible since for player A

$$(k-1-i)\frac{X_A}{k-1} + i\frac{X_A}{k-1} = X_A,$$

and for player B

$$X_A(n-2) + X_A - i\frac{nX_A - X_B}{k-1} + X_A - (k-1-i)\frac{nX_A - X_B}{k-1} = X_B$$

for all $i = 0, \dots, k-1$.

Second, each player is indifferent between each point in the support of their strategy. For this equilibrium the univariate marginal distributions for player A and $\forall\ j \in \{1, \dots, n\}$ are

$$F_A^j(x) = \begin{cases} \frac{kn-2k+2}{kn} & x \in \left[0, \frac{X_A}{k-1}\right), \\ \frac{kn-2k+4}{kn} & x \in \left[\frac{X_A}{k-1}, 2\frac{X_A}{k-1}\right), \\ \vdots & \vdots \\ \frac{kn-2k+2(i+1)}{kn} & x \in \left[i\frac{X_A}{k-1}, (i+1)\frac{X_A}{k-1}\right), \\ \vdots & \vdots \\ \frac{kn-2}{kn} & x \in \left[\frac{(k-2)X_A}{k-1}, X_A\right), \\ 1 & x \geq X_A. \end{cases}$$

Similarly for player B and $\forall\ j \in \{1, \ldots, n\}$

$$F_B^j(x) = \begin{cases} 0 & x \in [0, X_B - X_A(n-1)), \\ \frac{2}{kn} & x \in \left[X_B - X_A(n-1), X_A - \frac{(k-2)(nX_A - X_B)}{k-1}\right), \\ \vdots & \vdots \\ \frac{2(i+1)}{kn} & x \in \left[X_A - \frac{(k-1-i)(nX_A-X_B)}{k-1}, X_A - \frac{(k-2-i)(nX_A-X_B)}{k-1}\right), \\ \vdots & \vdots \\ \frac{2(k-1)}{kn} & x \in \left[X_A - \frac{nX_A-X_B}{k-1}, X_A\right), \\ 1 & x \geq X_A. \end{cases}$$

In addition note that for $i = 1, \ldots, k-1$,[16]

$$X_A - (k-i)\frac{nX_A - X_B}{k-1} < i\frac{X_A}{k-1} \leq X_A - (k-1-i)\frac{nX_A - X_B}{k-1}.$$

Thus, given that player B is following the equilibrium strategy, player A's allocation of the level of force $i\frac{X_A}{k-1}$ to a battlefield yields the expected payoff $\frac{2i}{kn^2}$ for each $i = 0, \ldots, k-1$. Similarly, player A's remaining force $(k-1-i)\frac{X_A}{k-1}$ has an expected payoff of $\frac{2(k-1-i)}{kn^2}$. Thus, for each $i = 0, \ldots, k-1$ player A's allocation of force

$$\left((k-1-i)\frac{X_A}{k-1}, i\frac{X_A}{k-1}\right)$$

has an expected payoff of $\frac{2k-2}{kn^2}$. The argument for player B is symmetric.

Third, neither player can increase their payoff by deviating to another feasible strategy. Given that player B is following the equilibrium strategy, the payoff to player A for any allocation of force in which no battlefield is allocated a level of force above $X_B - X_A(n-1)$ is zero. Similarly, if, for some $i = 1, \ldots, k-2$,[17] player A allocates a level of force of $X_A - (k-1-i)\frac{nX_A-X_B}{k-1} + \epsilon$ to a battlefield the expected payoff on that battlefield is $\frac{2(i+1)}{kn^2}$. Player A's remaining forces are $(k-1-i)\frac{nX_A-X_B}{k-1} - \epsilon$ and

$$(k-1-i)\frac{nX_A - X_B}{k-1} - \epsilon \leq X_A - (i+1)\frac{nX_A - X_B}{k-1}$$

since, from the definition of k, $nX_A - X_B \leq X_A\frac{k-1}{k}$. If player A allocates all of its remaining force to a single battlefield the maximum expected payoff on that battlefield is $\frac{2(k-i-2)}{kn^2}$. Thus, for player A any feasible allocation of force in which only one or two battlefields receive a strictly positive level of force has a maximum

[16] For the remaining case that $i = 0$, $0 < X_B - X_A(n-1)$.

[17] For the remaining case that $i = k-1$, player A's payoff from allocating all X_A forces to a given battlefield is the same as if player A allocates $X_A - \frac{nX_A-X_B}{k-1} + \epsilon$ to the battlefield. This follows from the tie-breaking rule and the fact that in this case player A's remaining forces are $\frac{nX_A-X_B}{k-1} - \epsilon$, and $\frac{nX_A-X_B}{k-1} - \epsilon < X_B - X_A(n-1)$, for all admissible k and $\epsilon > 0$, so that the payoff from player A's remaining forces is 0.

expected payoff of $\frac{2k-2}{kn^2}$. In addition, since the step size between each mass point in player B's equilibrium strategy is $\frac{nX_A - X_B}{k-1}$, player B's minimal allocation of force is $X_B - X_A\,(n-1) \geq \frac{nX_A - X_B}{k-1}$, and each mass point has the same weight, player A cannot achieve a higher expected payoff from dividing these remaining forces among more than one battlefield. Thus, given that player B is following the equilibrium strategy, the expected payoff to player A for an arbitrary strategy $\mathbf{x} \in \mathfrak{B}_A$ is

$$\frac{1}{n}\sum_{j=1}^{n} F_B^j\left(x_j\right) \leq \frac{2k-2}{kn^2}.$$

The argument for player B is symmetric.

This completes the proof of Theorem 5. □

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