Is there a *pure* NE for the IPD?

Let us first consider a two player Iterated Prisoner Dilemma game and that both players use a one-memory strategy. This means that the two players choose vectors $\mathbf{p} = (p_1, \ldots, p_4)$ resp. $\mathbf{q} = (q_1, \ldots, q_4)$, where $p_i, q_i \in [0, 1]$ and $p_1, \ldots, p_4$ resp. $q_1, \ldots, q_4$ determine the probability of cooperation following previous play of the two players of $xy \in (cc, cd, dc, dd)$ (for player X) resp. $xy = (cc, dc, cd, dd)$ (for player Y) (c=cooperation and d=defection). Define the payoff vectors

$$\mathbf{S}_X = (R, S, T, P) \text{ and } \mathbf{S}_Y = (R, T, S, P).$$

Here R=reward, S=sucker, T=temptation , P=punishment, and a common choice is $(T, R, P, S) = (5, 3, 1, 0)$. Here $2R > T + S > 2P$ so cooperation pays in the long run. So in this case

$$\mathbf{S}_X = (3, 0, 5, 1) \text{ and } \mathbf{S}_Y = (3, 5, 0, 1).$$

This means that we have a Markov chain with transition matrix

$$M = \begin{pmatrix} p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_3)(1-q_2) \\ p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4) \end{pmatrix}.$$

Also assume for the moment that M is transitive and so has a unique stationary eigenvector $\mathbf{v}$, i.e., $\mathbf{v}$ is the left eigenvector of M corresponding to eigenvalue 1. Note that $\mathbf{v}$ is unique if $\mathbf{p}, \mathbf{q} \in (0, 1)^4$ because in that case the Markov chain is transitive. In fact, it is even transitive if $p_i, q_j \in (0, 1)$ for all $i, j = 1, 2, 3$.

In Press-Dyson's paper it is shown that if we define

$$D(\mathbf{p}, \mathbf{q}, f) = \det \begin{pmatrix} p_1q_1 - 1 & p_1 - 1 & q_1 - 1 & f_1 \\ p_2q_3 & p_2 - 1 & q_3 & f_2 \\ p_3q_2 & p_3 & q_2 - 1 & f_3 \\ p_4q_4 & p_4 & q_4 & f_4 \end{pmatrix}$$

and assume that the asymptotic distribution vector $\mathbf{v}$ is unique, then the asymptotic expected score is

$$s_X = \mathbf{v} \cdot \mathbf{S}_X = \frac{\mathbf{v} \cdot \mathbf{S}_X}{\mathbf{v} \cdot \mathbf{1}} = \frac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_X)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})} \tag{1}$$

$$s_Y = \mathbf{v} \cdot \mathbf{S}_Y = \frac{\mathbf{v} \cdot \mathbf{S}_Y}{\mathbf{v} \cdot \mathbf{1}} = \frac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_Y)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})} \tag{2}$$

Obviously, the above formula can only be correct if $\mathbf{v}$ is unique.

Example 1. Suppose both players do TFT. This corresponds to $p = q = (1, 0, 1, 0)$. Then there are three stationary vectors (left eigenvectors of M corresponding to the eigenvalue 1), namely $(1, 0, 0, 0)$ (always coorporate), $(0, 1, 1, 0)$ (out of sync), $(0, 0, 0, 1)$ (always defect) depending on the initial distribution and this gives S_X can be 3, $5/2$ and 1 (given the above values of T, R, P, S). A calculation shows that if we let $\mathbf{p} \in (0, 1)^4, \mathbf{q} \in (0, 1)^4$ both tend to $(1, 0, 1, 0)$ then $s_X = s_Y = 2.25$. In Lemma 2 below it is shown that this limit exists and is unique.

As mentioned, $\mathbf{v}$ is unique if $p, q \in (0, 1)^4$.

Question 1. Is it true that the matrix M has a unique stationary vector $\mathbf{v}$ if and only if $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) \neq 0$?

Lemma 1. $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) \leq 0$ and $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) < 0$ if $\mathbf{p}, \mathbf{q} \in (0, 1)^4$.

Proof. That $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) \leq 0$ is proved in Wang-Lin's paper. We repeat their argument for completeness. $D(\mathbf{p}, \mathbf{q}, \mathbf{1})$ depends linearly on each p_i and each q_j separately. So the extrema are attained when $p_i, q_j \in \{0, 1\}$. A simple check of the $2^8 = 256$ possibilities shows that $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) \leq 0$ when all $p_i \in \{0, 1\}, q_j \in \{0, 1\}$. This gives the result. □

Even though the expressions for s_X and s_Y are invalid when $\mathbf{v}$ is unique, one has the following

Lemma 2. The functions $\dfrac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_X)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$ and $\dfrac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_Y)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$ on $(0, 1)^4 \times (0, 1)^4$ extend continuously to $[0, 1]^4 \times [0, 1]^4$. In particular, the above expressions for s_X, s_Y extends continuously to $[0, 1]^4 \times [0, 1]^4$ (but may take different values from $\mathbf{v} \cdot \mathbf{S}_X$ and $\mathbf{v} \cdot \mathbf{S}_Y$ when $\mathbf{v}$ is not unique).

Proof. $D(\mathbf{p}, \mathbf{q}, \mathbf{1})$ depends linearly on each p_i and each q_j separately, so each zero of $D(\mathbf{p}, \mathbf{q}, \mathbf{1})$ is a product of simple zeros p_i, q_j. On the other hand $\dfrac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_X)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$ and $\dfrac{D(\mathbf{p}, \mathbf{q}, \mathbf{S}_Y)}{D(\mathbf{p}, \mathbf{q}, \mathbf{1})}$ do not have poles, so this means that the poles of these expressions are removable. QQQQ Is this argument correct???? QQQQ □

Remark 1. There are quite a few, but perhaps only something like 40, choices for pairs of vectors $\mathbf{p}, \mathbf{q} \in \{0, 1\}^4$ for which $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) = 0$. However, there are also zeros of D of the following form: $\mathbf{q} = (1, 1, 0, 0)$ then $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) = 0$ when either $p_1 = p_3$ or $p_2 = p_4$.

Lemma 3. Varying p_i in the interval $[0,1]$ and letting the other variables remain constant, the expression (1),(2) has a maximum either at $p_i = 0$ or at $p_i = 1$.

Proof. Since the fractions in (1),(2) give the average payoff, these fractions remain bounded even if $D(\mathbf{p},\mathbf{q},\mathbf{1}) \to 0$. Therefore a cancelation in these fractions occurs when $D(\mathbf{p},\mathbf{q},\mathbf{1}) = 0$. Hence the expression $\dfrac{D(\mathbf{p},\mathbf{q},\mathbf{S}_Y)}{D(\mathbf{p},\mathbf{q},\mathbf{1})}$ is a Moebius transformation with bounded values for $\mathbf{p} = (p_1,\dots,p_4), \mathbf{q} = (q_1,\dots,q_4) \in [0,1]^4$. The lemma follows. $\square$

In the lemma below we compute $BR_p(q)$ (using the computer):

Lemma 4. Based on the continuous extension of s_X, s_Y one has:

$$BR_p(q) = \begin{cases} \{(0,0,0,0),(0,0,1,0),(1,0,0,0),(1,0,1,0)\} & q=(0,0,0,0) \\ \{(0,0,0,0),(1,0,0,0)\} & q=(0,0,0,1) \\ \{(0,0,1,1),(1,0,1,1)\} & q=(0,0,1,0) \\ \{(0,0,0,0),(0,1,0,0),(1,0,0,0)\} & q=(0,0,1,1) \\ \{(0,0,0,0),(1,0,0,0)\} & q=(0,1,0,0) \\ \{(0,0,0,0),(0,1,0,0),(1,0,0,0)\} & q=(0,1,0,1) \\ \{(0,0,0,1),(1,0,0,1)\} & q=(0,1,1,0) \\ \{(0,0,0,0),(0,1,0,0),(1,0,0,0),(1,0,0,1)\} & q=(0,1,1,1) \\ \{(0,0,0,0),(0,0,1,0),(1,0,0,0),(1,0,0,1),(1,0,1,0),(1,0,1,1),(1,1,0,0),(1,1,0,1),(1,1,1,0),(1,1,1,1)\} & q=(1,0,0,0) \\ \{(0,0,0,0),(0,0,0,1),(1,0,0,0),(1,0,0,1),(1,0,1,1)\} & q=(1,0,0,1) \\ \{(1,1,0,1),(1,1,1,0),(1,1,1,1)\} & q=(1,0,1,0) \\ \{(0,0,0,0),(0,0,0,1),(0,1,0,0),(0,1,0,1),(1,0,0,0),(1,0,0,1),(1,1,0,0),(1,1,0,1),(1,1,1,0),(1,1,1,1)\} & q=(1,0,1,1) \\ \{(0,0,0,0)\} & q=(1,1,0,0) \\ \{(0,0,0,0),(0,0,0,1),(0,1,0,0),(0,1,0,1)\} & q=(1,1,0,1) \\ \{(0,0,0,1),(0,1,0,1)\} & q=(1,1,1,0) \\ \{(0,0,0,0),(0,0,0,1),(0,1,0,0),(0,1,0,1)\} & q=(1,1,1,1) \end{cases}$$

Note that only $p \in BR_p(q), q \in BR_q(p)$ with $p = q$ if and only if $(p,q) = (0,0,0,0)$ and QQQ

$$A(BR_p(q)) = \begin{cases} 1 & q = (0,0,0,0) \\ 3 & q = (0,0,0,1) \\ 2.5 & q = (0,0,1,0) \\ 3 & q = (0,0,1,1) \\ 2.333 & q = (0,1,0,0) \\ 5 & q = (0,1,0,1) \\ 5 & q = (0,1,1,0) \\ 5 & q = (0,1,1,1) \\ 1 & q = (1,0,0,0) \\ 3 & q = (1,0,0,1) \\ 3 & q = (1,0,1,0) TFT \\ 3 & q = (1,0,1,1) \\ 3 & q = (1,1,0,0) \\ 5 & q = (1,1,0,1) \\ 5 & q = (1,1,1,0) \\ 5 & q = (1,1,1,1) \end{cases}$$

Question 2. Is there a pure Nash equilibrium for this infinite dimensional game, with payoffs determined by (1),(2) where we take the limit values. (Of course one should check when $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) = 0$.)

Question 3. Work out precisely what happens when $D(\mathbf{p}, \mathbf{q}, \mathbf{1}) = 0$. I guess s_X, s_Y are then multivalued (depend on the initial choice of play).

Remark 2. $\mathbf{p}, \mathbf{q} \in [0,1]^4$ can be viewed as pure actions. Mixed actions would correspond to probability measures in this space.