Dynamics of games
Project 1: permanent replicator dynamics

In this project you are allowed to use books and the hand-outs, but it is important that you write up your project by yourself and that you can explain what you have written in detail during the short oral.

Attached to this this project description are

- pages 146-149, pages 149-151 and pages 162, 44-45 and 79 of the book by Hofbauer and Sigmund, Evolutionary Games and Population Dynamics
- Y. Sato , E. Akiyama and J. P. Crutchfield, *Stability and diversity in collective adaptation*, Physica D 210 (2005) 21–57, `https://www.sciencedirect.com/science/article/pii/S0167278905002708`.

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1. Submit your solutions of the selected exercises from the lecture notes (which are the same for all projects).

2. Prove the following theorem.

   Theorem 1. Assume that P is continuous, $P > 0$ on the interior of Δ and for x in the interior of Δ define
   $$\Psi(x) := \frac{\dot{P}(x)}{P(x)}$$
   Assume that Ψ extends continuously to the boundary and that $x \in \partial\Delta$,
   $$\int_0^T \Psi(x(t))\,dt > 0 \text{ for some } T > 0.$$
   Then the corresponding dynamical system is permanent. More precisely, show that there exists $\delta > 0$ so that for each x in the interior Δ there exists $T > 0$ so that for $t \geq T$, we have $x_i(t) \geq \delta$ for all $t \geq T$ and $i = 1, \ldots, n$.

3. Show Theorem 1 implies that the hypercycle system from section 1.5 in the lecture notes is permanent for all $n \geq 5$. You may assume that $k_i = 1$ for all i.

   (Note that in the notes the function $P(x) = x_1 \cdots x_{n-1}$ was chosen, and that
   $$\Psi(x) = \frac{\dot{P}(x)}{P(x)} = \sum_{i=1}^{n} \frac{\dot{x}_i}{x_i} = 1 - n \sum_{j=1}^{n} x_j x_{j-1}.$$
   Note that for $n \geq 5$ it is NOT true that $\Psi(x) > 0$ for all x close to the boundary of Δ (but $P > 0$ near e_i). What does this imply for $t \mapsto P(x(t))$ for a solution of $x(t)$ of the replicator dynamics.

4. Write Matlab code or python code in order to produce solutions of the following two replicator dynamics for single population games:

$$A_0 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & . & . & . & k_1 \\ k_2 & 0 & 0 & . & . & . & 0 \\ 0 & k_3 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & k_n & 0 \end{pmatrix}$$

with $k_i = 1$ and $n = 5$. Compute the NE's for these systems. Analyse the ω-limit of orbits for these systems. Discuss in detail in what way you think the system associated to A_1 has 'chaotic' solutions for $n \geq 5$ and non-chaotic solutions for $n \leq 4$. Do the solutions associated to A_1 for $n \geq 5$ tend to the NE? Also consider the two player game

$$A = \begin{pmatrix} \epsilon_x & -1 & 1 \\ 1 & \epsilon_x & -1 \\ -1 & 1 & \epsilon_x \end{pmatrix} \text{ and } B = \begin{pmatrix} \epsilon_y & -1 & 1 \\ 1 & \epsilon_y & -1 \\ -1 & 1 & \epsilon_y \end{pmatrix}$$

where we use the 1st notation. Show the output for a few values of $\epsilon_x.\epsilon_y \in (-1, 1)$ replicating the figures from the paper of Sato et al.

5. [**Mastery question for 4th year and MSc students**] In Sections 2.1 and 2.2 of the paper of Sato et al, reinforcement adaptation is introduced. Discuss this model focussing on

   (a) its relationship to replicator dynamics (see Section 2.4 of that paper).

   (b) the dynamics of the examples discussed in Section 4 of that paper. Compute the NE's for these examples. Do solutions tend to these Nash equilibria? Compute the average of payoff over time along orbits, and compare this with the payoff players would receive it they chose to play their NE solution at every step.

   You are not required to cover all aspects of this paper (for example, you can skip the discussion about the Hamiltonian structure and Section 3 of this paper.