Transitive Maps which are not Ergodic with respect to Lebesgue Measure

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Abstract

In this note we shall give examples of rational maps on the Riemann sphere and also of polynomial interval maps which are transitive but not ergodic with respect to Lebesgue measure. In fact, these maps have two disjoint compact attractors whose attractive basins are 'intermingled', each having a positive Lebesgue measure in every open set. In addition, we show that there exists a real bimodal polynomial with Fibonacci dynamics (of the type considered by Branner and Hubbard), whose Julia set is totally disconnected and has positive Lebesgue measure. Finally, we show that there exists a rational map associated to the Newton iteration scheme corresponding to a polynomial whose Julia set has positive Lebesgue measure.

1 Introduction and statement of results

Let N be a manifold and $f\colon N \to N$ be a smooth mapping and define $\omega(x)$ to be the set of accumulation points of the sequence $x, f(x), \ldots$. We say that f is *transitive* if there exists a point x such that $\omega(x) = N$. We say that f is *ergodic with respect to the Lebesgue measure* or simply *ergodic* if each measurable set X for which $f^{-1}(X) = X$ (up to zero Lebesgue measure), has either zero or full Lebesgue measure. In this note we shall show that, even for polynomial interval maps and rational maps on the Riemann sphere, transitivity does not imply ergodicity.

That, in general, transitivity does not imply ergodicity was known before. For example, in the case of interval exchange transformations this implication was shown to be false by M. Keane, see [Ke]. Similarly, it was shown in [Mi2] that the map $\mathbb{C} \ni z \mapsto \exp(z)$ is topologically transitive and in [Ly2], [Ly3] and [Re3] that Lebesgue almost all points are attracted to $\omega(0) \subset \mathbb{R}$ under iterations of the exponential map. So the situation in the exponential map is quite different from our examples because there the attractor is unique but its basin splits up into uncountably many ergodic

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components. (Measure theoretically, it looks more like the basin of an attracting fixed point.) Another example is given in [Kan], see also [AKYZ], there exists a real analytic topologically transitive (non-invertible) map on the annulus $S^1 \times [0,1]$ such that the basin of the boundary circles $X = S^1 \times \{0,1\}$ has full Lebesgue measure. (In fact, the basin of each circle intersects every open set in a set of positive area.) In [AKYZ] two attractors are called *intermingled* if each open set which intersects one basin in a set of positive Lebesgue measure, also intersects the other basin in a set of positive measure. So Kan's example is an example of a map with two intermingled basins. In [AHKY] an example of a quadratic map in two real variables with three intermingled basins is given (for one part of their proof a computer supported estimate is used). In our theorem we will show that this phenomenon even occurs for polynomial maps on $[0,1]$ and rational maps on $\mathbb{C}$:

Theorem 1.1 *Let $N = [0,1]$ or $N = \bar{\mathbb{C}}$. Then there exists a map $f\colon N \to N$ which is polynomial if $N = [0,1]$ and rational if $N = \mathbb{C}$ with the following properties.*

- *f is transitive;*
- *there exist disjoint Cantor sets X_1, X_2 (with zero Lebesgue measure) so that for both $i = 1$ and $i = 2$ the basin*

$$B(X_i) = \{x \; ; \; \omega(x) \subset X_i\}$$

  is dense in N and has positive Lebesgue measure;

- *one can construct f so that $B(X_1) \cup B(X_2)$ has full Lebesgue measure.*

In particular, since the basin $B(X)$ of a set X is completely invariant, f is transitive and not ergodic.

The degree of the maps for which we can prove the above theorem rigorously is extremely large. However, computer experiments indicate that one might already have such maps when the degree is respectively 30 and 64. As a reference for this we refer to [BKNS], [SN] and a forthcoming paper.

We should point out that it was conjectured by R. Mañé (see Conjecture III in [Ma]) that such rational maps cannot exist, see also [He]. Since each point of N is in the closure of both $B(X_1)$ and $B(X_2)$ this situation is called *intermingled.*

It is easy to modify the examples in this theorem: for any integer p there is a map f as above but with disjoint Cantor sets $X_1, \ldots, X_p$ whose basins positive measure and are intermingled.

Of course, the Julia set of the rational map $f\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ from the previous theorem is the entire Riemann sphere. M. Rees [Re2] has shown that within the set of rational maps of degree two, there exists a subset of positive Lebesgue measure consisting of ergodic maps. We should note that typical points of ergodic maps need not have dense forward orbits:

Theorem 1.2 *There exists a rational map $f: \mathbb{C} \to \mathbb{C}$ which is ergodic and such that there exists a Cantor set X of zero Lebesgue measure such that for Lebesgue almost all points z one has $\omega(z) \subset X$.*

The examples in Theorems 1.1 and 1.2 are based on unimodal maps on the interval with a very special dynamics: closest preimages of the critical point occur precisely at the S_n-th backward iterate of c where S_n is the Fibonacci sequence $S_0 = 1$, $S_1 = 2$, $S_{n+1} = S_n + S_{n-1}$ for $n \geq 1$. Therefore such maps are called Fibonacci maps. Related to these unimodal Fibonacci maps, are the bimodal polynomial Fibonacci maps considered by Branner and Hubbard [BH]. These maps have two critical points. One of these escapes to infinity, and the dynamics of the other is essentially the same as that of the unimodal Fibonacci map mentioned above (we shall be more precise about this in Section 4 below). If the non-escaping critical point has order $\ell = 2$ then, as is shown in [BH], the Julia set of the corresponding polynomial is totally disconnected. (For a precise definition of the Fibonacci dynamics, see Section 4.) Using a modulus-area estimate, McMullen was able to prove that the Julia has zero Lebesgue measure if the order ℓ of the non-escaping critical point is equal to two, see [BH]. In [BH], it is asked what happens if $\ell > 2$. If the map is real, the answer is given by the theorem below:

Theorem 1.3 *Let $f: \mathbb{C} \to \mathbb{C}$ be a bimodal Fibonacci map which preserves the real line with one non-escaping critical point of even order ℓ and the other critical point escaping to infinity. Then*

- *the Julia set if totally disconnected when $\ell \geq 2$;*
- *the Julia set has positive Lebesgue measure provided ℓ is sufficiently large.*

Furthermore, we show in the final section that there exists a rational map of the form $f(z) = z - g(z)/g'(z)$ where g is a polynomial such that the Julia set of f has positive Lebesgue measure. Remark that f is the Newton iteration corresponding to g. The polynomial g we shall choose is of the form $g(z) = z^{\ell-1} + (\lambda - 1)z - \lambda$ where ℓ is some large even integer. Numerical experiments indicate that $\ell = 32$ is already sufficiently small.

The construction of all these examples is based on the results in [BKNS] and [SN]. In [SN] it was shown that each real Fibonacci polynomial of the form $z \mapsto z^\ell + c_1$ with ℓ sufficiently large has a Julia set of positive Lebesgue measure. Note that in that case, the Julia set is connected (because this polynomial has critical points which iterate to infinity).

We wish to thank M. Herman for telling us about references [He] and [Ma]. We also would like to thank J. Milnor, J. Barnes and the referee for some useful discussions and comments.

2 A transitive polynomial interval map which is not ergodic

In this section we will prove that there exists a polynomial interval map which is transitive and not ergodic. The example is a simple modification of the example from [BKNS]. This example deals with unimodal interval maps with the so-called Fibonacci dynamics. Since the precise description does not play a role in this paper, it suffices for the moment to say that there exists a certain number $s \in (\sqrt{2}, 2)$ for which the tent map $F\colon [0,1] \to [0,1]$ with slope $\pm s$ and with $F(0) = F(1) = 0$ has the required dynamics, i.e., the required kneading invariant. We shall need the following

Theorem 2.1 *(See [BKNS])*
There exists ℓ_0 and a tent map $F\colon [0,1] \to [0,1]$ with slope $\pm s$ with $s \in (\sqrt{2}, 2)$ with the following properties.
1. The set $\omega_F(c)$ is a Cantor set and $F\colon \omega_F(c) \to \omega_F(c)$ is minimal, where $c = 1/2$ is the turning point of F.
2. Let $f\colon [0,1] \to [0,1]$ be any unimodal C^2 map which is conjugate to F and such that its critical point c is of even order $\ell \geq \ell_0$. Define $\Lambda = \omega_f(c)$. Then $B(\Lambda)$ has positive Lebesgue measure.

Here we say that a C^2 map has a critical point c of order ℓ if there exists a C^2 coordinate system in which f near c takes the form $f(x) = c_1 - x^\ell$.

We shall apply the above theorem. More precisely, we shall modify the map f from that theorem so that it will become transitive. First we state an abstract lemma which is applicable to the situation of the previous theorem.

Lemma 2.1 *Suppose that $B(\Lambda)$ has positive measure and let U be a neighbourhood of the set Λ. Then the set*

$$Z(U) = \{x \in B(\Lambda)\ ;\ f^n(x) \in U \text{ for all } n \in \mathbb{N}\}$$

has positive measure. Moreover, one can find a set X of positive measure so that $f^i(X) \cap X = \emptyset$ for each $i > 0$ (this means that the map is not conservative).

Proof: Define $Z_N = \{x \in B(\Lambda)\ ;\ f^n(x) \in U \text{ for all } n \geq N\}$. One has that $Z_N \subset Z_{N+1}$ and $\cup_{N \geq 0} Z_N = B(\Lambda)$. Since $B(\Lambda)$ has positive Lebesgue measure, Z_N has positive Lebesgue measure for sufficiently large N. Hence $Z(U) = Z_0 \supset f^N(Z_N)$ has positive Lebesgue measure. Take U so small that the measure of U is less than the measure of $B(\Lambda)$. Then the measure of $Z_0 = Z(U)$ is also less than that of $B(\Lambda)$. It follows that there exists an integer n so that the measure of $Z_{n+1} \setminus Z_n$ is positive. The set $X = Z_{n+1} \setminus Z_n$ has the required properties. □

Let $F\colon [0,1] \to [0,1]$ be the Fibonacci tent-map from the theorem above with $F(0) = F(1)$. Let p be the orientation reversing fixed point of F. Since $F\colon \omega_F(1/2) \to \omega_F(1/2)$

is a minimal, the set $\omega_F(1/2)$ does not contain p. Hence one can choose $p' > p$ such that $[p, p']$ does not contain any point of the closed set $\omega_F(1/2)$. Choose $p < x_1 < x_2 < p'$ and define G to be the piecewise linear map $[0,1] \to [-1,1]$ such that $G = F$ outside $[p,p']$, $G(x_1) = -1$, $G(x_2) = 1$, and such that G is affine on each of the intervals (p, x_1), (x_1, x_2) and (x_2, p'), see the figure. Next extend G to a map $G\colon [-1,1] \to [-1,1]$ by the requirement that $G(-x) = -G(x)$. We can do all this so that the absolute value of the slope of G is everywhere at least $s \in (\sqrt{2}, 2)$. Note that we had chosen p' so that $[p,p'] \cap \omega_F(1/2) = \emptyset$ and therefore $G^i(1/2) = F^i(1/2)$ for $i \geq 0$ and $G(1/2) > p'$. Let us define $\hat{F}\colon [-1,1] \to [-1,1]$ by $\hat{F}(x) = F(x)$ when $x \geq 0$ and $\hat{F}(x) = -F(|x|)$ when $x < 0$.

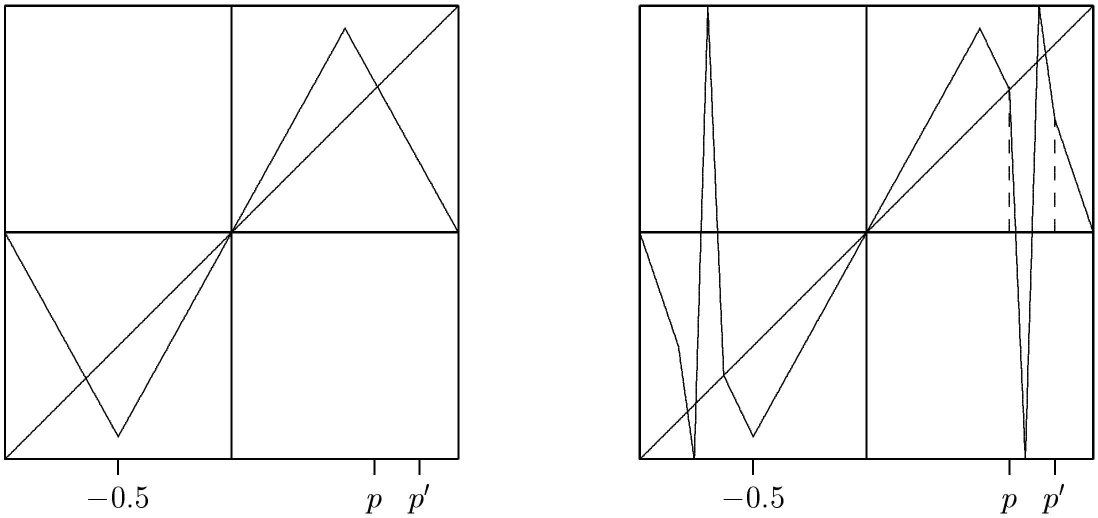

Figure 1: The maps $\hat{F}$ and G.

Lemma 2.2 *G is topologically transitive.*

Proof: Take an interval U. We want to show that

$$G^n(U) = [-1,1] \text{ for some } n \in \mathbb{N}. \tag{1}$$

First note that if U contains $x \in \{0, -1, 1\}$ then for some n the interval $G^n(U)$ contains a one-sided neighbourhood J of x. In fact, it contains an interval J which is maximal with respect to the property that $G|J$ monotone. Since $G^2(J) \supset [-1,1]$ for each such interval J, we get $G^{n+2}(U) \supset [-1,1]$.

Since G has everywhere slope $\geq s > \sqrt{2}$, if either U or $G(U)$ does not contain turning points of G (or if both these intervals do not contain a turning point) then $|G^2(U)| \geq (s^2/2)|U|$. Since $[-1,1]$ is bounded, this implies that there exists n such that $G^n(U)$ and $G^{n+1}(U)$ both contain a turning point. From the first part of the proof we may assume that these turning point are not $0, \pm x_1, \pm x_2$, because otherwise (1) follows immediately. Hence $G^n(U)$ and $G^{n+1}(U)$ both contain one of the turning points 1/2 or $-1/2$. If one of these intervals contains $1/2$ and the other $-1/2$ then $G^{n+1}(U) \ni 0$ and again (1) follows. So assume that $G^n(U), G^{n+1}(U) \ni 1/2$ (the case these intervals contain $-1/2$ is similar). Hence $G^{n+1}(U)$ contains both $1/2$ and $G(1/2) > 1/2$ and therefore the interval $[x_1, x_2]$. Therefore $G^{n+2}(U) = [-1, 1]$. Thus we have proved (1).

A standard argument implies that there exists a residual subset of $[-1, 1]$ of points x for which $\omega(x) = [-1, 1]$. $\square$

One of the reasons we have chosen G as above, is that $\omega_G(1/2) = \omega_F(1/2) \subset [0, 1]$ and that G and F coincide on these sets. In other words, the G orbit of $1/2$ is coincides with the orbit of $1/2$ under the unimodal map $F: [0, 1] \to [0, 1]$. In the proof of the next theorem we shall use this fact, in order to change a map which is conjugate to G (which is 6-modal) into a map which is conjugate to $\hat{F}: [-1, 1] \to [-1, 1]$ (which is bimodal but actually consists of two unimodal maps glued together). In this way we are able to construct a 6-modal transitive polynomial map $f: [-1, 1] \to [-1, 1]$ which is not ergodic.

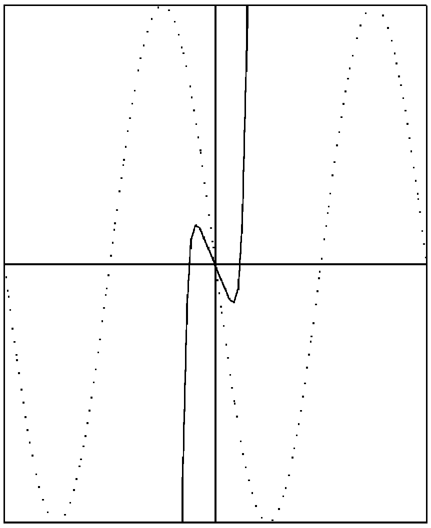

Figure 2: The graph of h is drawn in a doted curve and the graph of g_a in a solid curve.

Theorem 2.2 *There exists ℓ_0 with the following property. For each even $\ell \geq \ell_0$ there exists a polynomial map $f: [-1, 1] \to [-1, 1]$ such that*

- *f is conjugate to G;*
- *the critical points $c_\pm$ corresponding to $\pm 1/2$ are of order ℓ;*
- *the sets $\Lambda_\pm = \omega_f(c_\pm)$ both have a basin of positive Lebesgue measure;*

- *the map f is transitive;*
- *the union of the basins of Λ_- and Λ_+ has full Lebesgue measure.*

Proof: Take a polynomial $h: [-1,1] \to [-1,1]$ which is odd and of degree five with $h(0) = h(\pm 1) = 0$ and with critical points in $\pm u_1, \pm u_2$ with $0 < u_1 < u_2 < 1$ and with $h(u_1) = -1$ and $h(u_2) = 1$, see the figure. Choose ℓ even and fix $u > 0$. Moreover, choose a parameter $a > 0$ and define the polynomial g_a with $g_a(0) = 0$ and with derivative

$$(g_a)'(x) = a \cdot (x^2 - u^2)^{\ell - 1}.$$

g_a is a bimodal map with critical points in $\pm u$. Here we choose $a > 0$ so that $g_a(\pm u) \in (-u_1, u_1)$. Let $z > 0$ be so that $g_a(\pm z) = \pm 1$ and let k_a be the linear map which maps $[-1,1]$ onto $[-z,z]$. Next define $f_a = h \circ g_a \circ k_a$. This map f_a has critical points in $\pm c(a)$ and $\pm x_1(a), \pm x_2(a)$ where $k_a(c) = u$ and $g_a \circ k_a(x_i) = u_i$ and $0 < c < x_1 < x_2 < 1$. Moreover, $f_a(0) = f_a(\pm 1) = 0$, $f(c) \in (0,1)$ and $f_a(x_1), f_a(x_2) \in \{-1, 1\}$. So f_a is a smooth version of the map G from Figure 1. As a increases, the extremal value $g_a(u)$ decreases from 0 to $-u_1$ and therefore $f_a(c)$ increases from 0 to 1. Hence, using the result about full families from Section II.4 of [MS], there exists a parameter a_* (which depends on ℓ) such that the polynomial map $f = f_{a_*}: [-1,1] \to [-1,1]$ is conjugate to the map G. (In fact, f_a is odd and since the orbit of the critical point c and also of $-c$ is 'carried' by a unimodal map and the other critical points are eventually mapped onto fixed points, we could also apply the unimodal result on full families, see for example [CE].) The critical points $\pm c$ of f are of order ℓ. Write $\Lambda_\pm = \omega_f(\pm c)$. Because of the observation made above the statement of this theorem, we can modify $f: [-1,1] \to [-1,1]$ outside a neighbourhood U of $\Lambda_+ \cup \Lambda_-$ to a C^2 map $\tilde{f}: [-1,1] \to [-1,1]$ which consists of 'two disjoint unimodal pieces' (in other words, simply glue in the interval corresponding to $[p, p']$ a monotone piece). Thus we get a map $\tilde{f}$ with $\tilde{f}(x) > 0$ for $x \in (0,1)$ and $\tilde{f}(x) < x$ for $x \in (-1, 0)$. This means that $\tilde{f}: [0,1] \to [0,1]$ and $\tilde{f}: [-1,0] \to [-1,0]$ are unimodal interval maps both with Fibonacci dynamics. By the first lemma above,

$$\{x \in B(\Lambda_+)\ ;\ \tilde{f}^n(x) \in U \text{ for all } n \in \mathbb{N}\}$$

has positive Lebesgue measure. Since f agrees with $\tilde{f}$ on U, it follows that $B_f(\Lambda_+)$ has positive Lebesgue measure. Similarly, $B_f(\Lambda_-)$ has positive Lebesgue measure. That f is transitive follows from the second lemma above.

The last statement can be seen as follows. First notice that f is a composition of three polynomials with only real critical points. Hence the Schwarzian derivative of each of these polynomials is negative, see for example Exercise 1.7 on page 285 of [MS]. (This observation also was made in the monograph of Collet and Eckmann, [CE].) Hence the composed function also has negative Schwarzian derivative and therefore each periodic attractor contains in its basin a critical point or a boundary point of the interval. Since the critical points corresponding to $\pm x_1, \pm x_2$ and the boundary points are eventually mapped onto the repelling fixed point 0, and since the critical points $\pm c$

lie both in minimal Cantor sets $\omega_f(\pm c)$, it follows that f has no periodic attractors. In addition, these Cantor sets $\omega_f(\pm c)$ have zero measure (w.r.t. to the one-dimensional) Lebesgue measure), see Theorem 1.3' in [MS]. Therefore the last statement follows from Theorem 2 of [Ly1] which states that each metric attractor of such a map is either equal to the closure of the orbit of a critical point or to a finite union of intervals on which the map is topologically transitive. □

This concludes the proof of Theorem 1.1 in the interval case. We should remark that we could easily modify the above example so that $f(\pm 1) = \pm 1$. We should also remark that we introduced the new map $\tilde{f}$ in the proof above because in [BKNS] only smooth unimodal maps are considered. However, the argument above shows that one can also apply these results to maps which are only partly defined, such as the maps considered in the next section.

Before proving Theorem 1.1 in the rational case, we shall first need some results on polynomial-like mappings.

3 Polynomial-like maps with a Julia set of positive Lebesgue measure

In this section we shall apply the results of [SN] and show that certain polynomial-like mappings have a Julia set of positive Lebesgue measure. Let us first define this notion. Let D^0, D^1, D be topological discs bounded by smooth curves and such that the closures D^i are disjoint and contained in the interior of D. Then

$$R\colon (D^0 \cup D^1) \to D$$

is *ℓ-polynomial-like* if $R|D^1$ is a univalent conformal map onto D and if $R|D^0\colon D^0 \to D$ is a ℓ-fold covering map, i.e., $R|D^0 \to D$ is surjective and the composition of a map of the type $z \mapsto z^\ell$ (up to translation) and a conformal map onto D. The map R is called *unbranched* (using the terminology of McMullen [McM]) if all R-iterates of the critical point of $R|D^0$ are contained in $D^0 \cup D^1$.

So consider a unimodal Fibonacci map f as in the previous section. To keep the notation consistent with [SN] we choose f such that it has a local minimum (one can take for example f of the form $f(z) = z^\ell + c_1$). For simplicity, when $x \in \mathbb{R}$ is not equal to the critical point c then we define $\hat{x} \in \mathbb{R}$ to be the unique point so that $\hat{x} \neq x$ and so that $f(\hat{x}) = f(x)$. To formalize the discussion we need a few properties of the unimodal Fibonacci map; for proofs, see for example [SN]. Consider the Fibonacci sequence S_n defined by $S_0 = 1$, $S_1 = 2$, $S_{n+1} = S_n + S_{n-1}$ for $n \geq 1$. Let u_0 be the orientation reversing fixed point of f. Define $u_n \in \mathbb{R}$ inductively to be the point closest to c so that $f^{S_{n-1}}(u_n) = u_{n-1}$ and so that u_n is on the same side of c as $f^{S_n}(c)$. Hence $u_1 = \hat{u}_0$. One of the properties of a Fibonacci map is that $u_{n+1} \in (u_n, \hat{u}_n)$ and $|u_n| \to 0$ as $n \to \infty$. If we define $U_n = (u_{n-1}, \hat{u}_{n-1})$ then the first return map to U_n has precisely

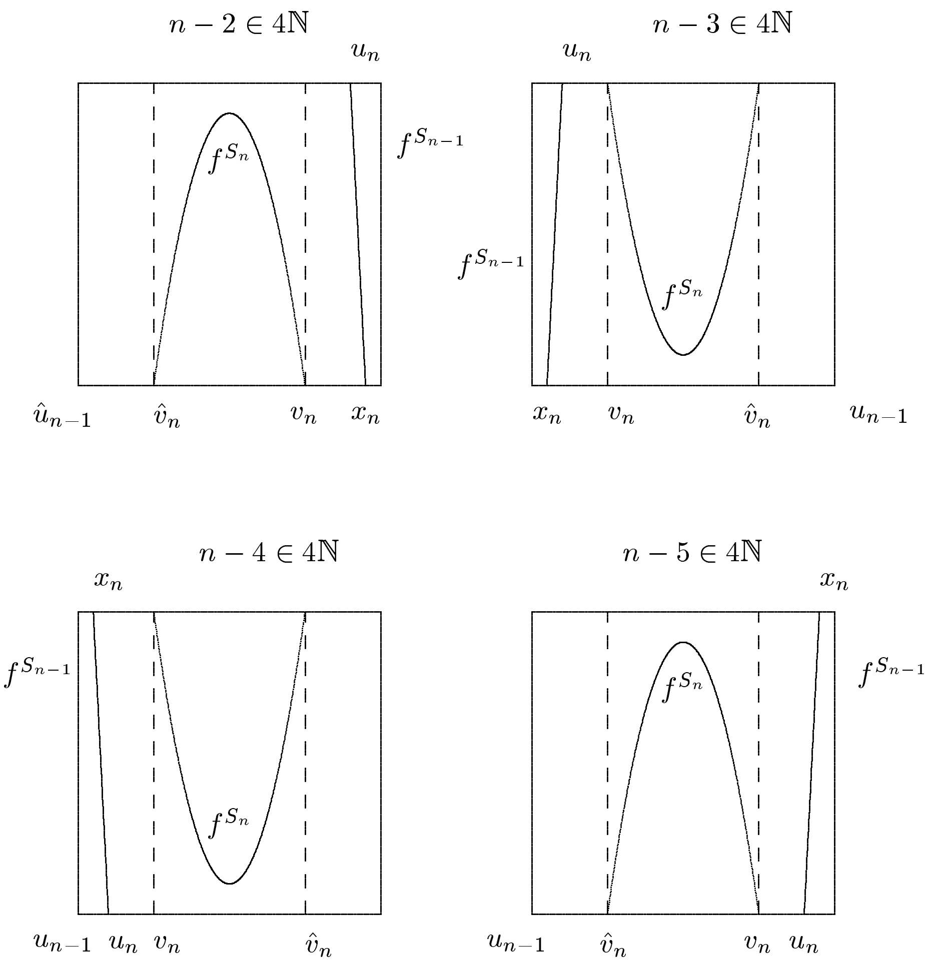

Figure 3: The successive first return maps $R_n: U_n \to U_n$ for $n \geq 2$ (note however that $v_2 = u_2$). Here the components of the domain of R_n which contain points of the orbit of the critical point are $U_n^0 = (v_n, \hat{v}_n$ and $U_n^1 = (x_n, u_n)$.

two domains containing points of $\omega(c)$. These domains are of the form $U_n^0 = (\hat{v}_n, v_n)$ and $U_n^1 = (x_n, u_n)$ and the graph looks as in Figure 3. The return map R coincides with f^{S_n} on U_n^0 and with $f^{S_{n-1}}$ on U_n^1. (This is a one-dimensional version of a polynomial-like mapping as defined above.) We should emphasize that $f^{S_n}(c) \in U_n^1 = (x_n, u_n)$ for each n.

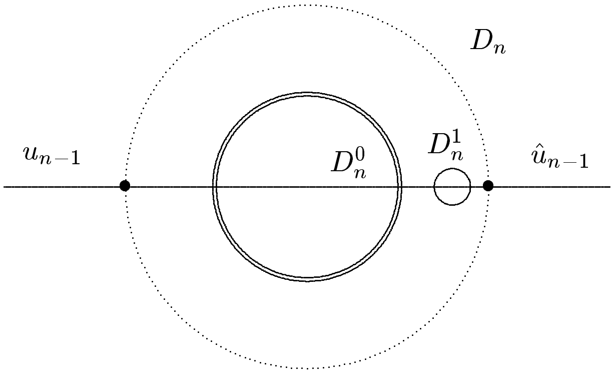

Figure 4: The complex extension of the first return map $R_n: U_n^0 \cup U_n^1 \to U_n$ is a polynomial-like map $R_n: D_n^0 \cup D_n^1 \to D_n$. In the situation drawn $\ell = 4$. The solid (topological) disc D_n^0 is mapped in a ℓ-fold way by f^{S_n} onto the disc D_n with dotted boundary. The smaller (topological) disc D_n^1 is mapped by $f^{S_{n-1}}$ univalently onto D_n. Furthermore, $D_n \cap \mathbb{R} = U_n^0 = (u_{n-1}, \hat{u}_{n-1})$, $D_n^0 \cap \mathbb{R} = U_n^0 = (v_n, \hat{v}_n)$ and $D_n^1 \cap \mathbb{R} = U_n^1 = (x_n, u_n)$. The domain of the first return map to D_n^0 has infinitely many components but all iterates of the critical point under the return map are contained in $U_n^0 \cup U_n^1 \subset D_n^0 \cup D_n^1$.

Theorem 3.1 *There exists ℓ_0 with the following properties. Assume that I^0, I^1 are disjoint open intervals contained in an interval I. Moreover, assume that either*

- *$g: I \to I$ is a real analytic unimodal Fibonacci map with a critical point c of even order ℓ, or,*
- *$g: I^0 \cup I^1 \to I$ is a real analytic map which is conjugate to one of the maps $R: U_n^0 \cup U_n^1 \to U_n$ from Figure 3 and such that the critical point c of g in I^0 has even order ℓ.*

Then

1. *there exist closed topological discs $D^0 \ni c$, D^1 and D in the complex plane $\mathbb{C}$ so that D^0 and D^1 are disjoint and contained in the interior of D;*
2. *there exists $k \in \mathbb{N}$ so that the first return map $\mathcal{R}_g$ of g to D coincides with $g^{S_{k+1}}$ on D^0 and to g^{S_k} on D^1; the map $\mathcal{R}_g: D^0 \cup D^1 \to D$ is an unbranched polynomial-like mapping.*
3. *$\mathcal{R}_g$ is complex analytic and maps D^0 as a ℓ-branched covering map onto D and maps D^1 conformally onto D;*

4. *the set*

$$J(\mathcal{R}_g) = \{z \in (D^0 \cup D^1)\ ;\ \mathcal{R}_g^i(z) \in D^0 \cup D^1 \text{ for all } i \in \mathbb{N}\}$$

has positive Lebesgue measure provided $\ell \geq \ell_0$*;*

5. *in fact, the subset of* $J(\mathcal{R}_g)$ *of points* z *for which* $\mathcal{R}_g^i(z) \to \omega_g(c) \cap D$ *has positive Lebesgue measure.*

In fact, there exists a sequence of such discs $D_n^0, D_n^1 \subset D_n$ *such that the diameter of* D_n *shrinks to zero and so that the modulus of the annulus* $D_n \setminus D_n^0$ *is bounded from below by a number which only depends on* ℓ *(and not on* n *provided* n *is large enough).*

Proof: The proof of this theorem uses several results from [SN]. Let us discuss these ingredients. Choose ℓ_0 as in the Main Theorem of [SN] and fix some even $\ell \geq \ell_0$. Define the sequence u_n as above. As in Section 6 of [SN] one can choose a disc D based on $(u_n, \hat{u}_n)$ and show that its inverse branches are mapped inside themselves provided n is sufficiently large, see Figure 4. (We should note that we use here Lemma VI.5.3 of [MS] instead of the straightforward Schwarz Lemma, just as was done in the addendum to [LS] because g is not defined on all of $\mathbb{C}$.) Thus we obtain a polynomial-like $\mathcal{R}_g\colon (D^0 \cup D^1) \to D$ associated to the real map g. This proves Statements 1 to 3 of the theorem.

Let $f(z) = z^\ell + c_1$ be the Fibonacci polynomial and let $\mathcal{R}_f\colon (\tilde{D}^0 \cup \tilde{D}^1) \to \tilde{D}$ be a corresponding polynomial-like (first return) map. According to Section 6 of [SN] there exists a quasiconformal homeomorphism h conjugating the polynomial-like mappings

$$\mathcal{R}_g\colon (D^0 \cup D^1) \to D \text{ and } \mathcal{R}_f\colon (\tilde{D}^0 \cup \tilde{D}^1) \to \tilde{D}$$

and which maps c_g to c_f. Hence h maps $\Lambda_g = \omega_g(c) \cap D$ to $\Lambda_f = \omega_f(c) \cap D$.

The set of points which is attracted to the set Λ_f has positive Lebesgue measure according to the Main Theorem of [SN]. Since a quasiconformal homeomorphism is absolutely continuous, the set of points which is attracted to the set Λ_g has therefore also positive Lebesgue measure. This concludes the proof of Statements 4 and 5 of the theorem. □

4 A polynomial map with a Julia set which is a Cantor set of positive Lebesgue measure

In [BH], Branner and Hubbard considered a bimodal polynomial map $g\colon \mathbb{C} \to \mathbb{C}$ which has one escaping and one non-escaping critical point with so-called Fibonacci dynamics. One way to define this dynamics is as follows. Let u_0, u_1 and u_2 be as in Figure 3 on the left. Furthermore, define $\tilde{S}_0 = 1$, $\tilde{S}_1 = 1$ and $\tilde{S}_{n+1} = \tilde{S}_n + \tilde{S}_{n-1}$ (then $\tilde{S}_n = S_{n-1}$

for $n \geq 1$). Now consider the first return map to $[u_0, u_1]$. The first return map is as in Figure 5. The left branch of this return map of g does not contain any points from the orbit of the non-escaping critical point and can be 'forgotten'. We say that the bimodal map g has *Fibonacci dynamics* if this first return map of g (with the left branch removed) is conjugate to the first return map of the unimodal map f to U_2 (corresponding to the top left situation in Figure 3 where $n = 2$ and therefore $v_2 = x_2$). In particular, the map g has no periodic attractors.

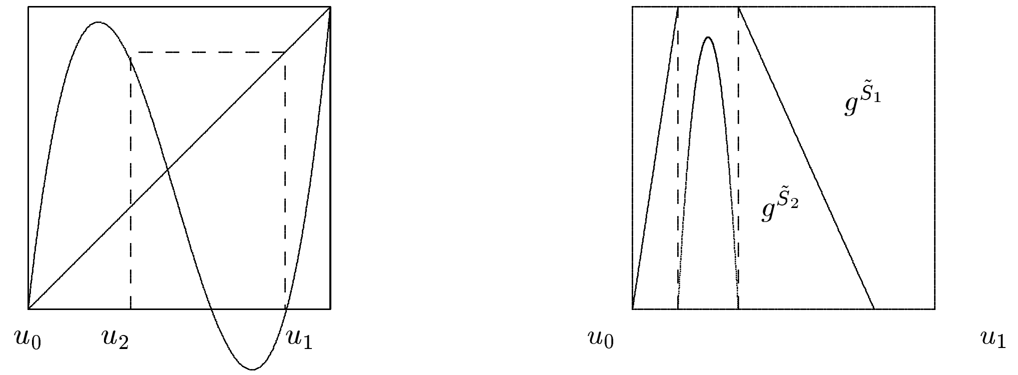

Figure 5: On the left the bimodal Fibonacci map g and on the right the first return map of g to $[u_0, u_1]$ (here $\tilde{S}_1 = 1$ and $\tilde{S}_2 = 2$).

From the Theorem of the previous section it follows that one can associate to this map a polynomial-like mapping $\mathcal{R}$ whose associated filled Julia set $J(\mathcal{R})$ has positive Lebesgue measure. Since g has no periodic attractors (or neutral periodic orbits), the Julia set of g consists of the points not iterating to infinity. Hence the Julia set of g contains the set $J(\mathcal{R})$. So g has a Julia set of positive Lebesgue measure. To prove $J(g)$ is totally disconnected one can use the arguments that are used in the treatment of the unimodal Fibonacci case in [LS].

5 A transitive rational non-ergodic map

In this section we construct a rational map on $\bar{\mathbb{C}}$ which is transitive but not ergodic. To construct the map, we proceed as follows. Define

$$h(z) = 100 \frac{z}{z^2 + 100}.$$

This rational function has two critical points $z = \pm 10$ and one has $h(z) \to 0$ as $|z| \to \infty$. Furthermore, choose ℓ even, let $a > 0$ and define the polynomial g_a so that $g_a(0) = 0$ with derivative

$$(g_a)'(z) = a \cdot (1 - z^2)^{\ell - 1}.$$

Finally, define the rational function $f_a \colon \mathbb{C} \to \mathbb{C}$ by

$$f_a(z) = h \circ g_a(z).$$

The rational function maps $\mathbb{R}$ into itself and this real function has a graph as drawn below.

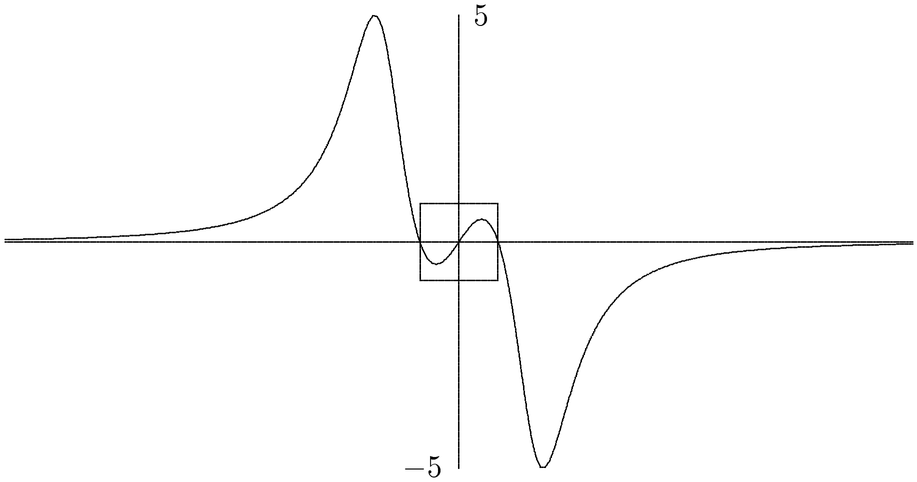

Figure 6: The graph of $f_a = h \circ g_a$ for $a = 6$ and $\ell = 2$ and the invariant box $[-x_\ell, x_\ell] \times [-x_\ell, x_\ell]$.

Lemma 5.1 *The map f_a has the following properties.*

- *All critical* **values** *of f_a are real: they are $\pm f_a(1)$ and $\pm 5 = h(\pm 10)$. The critical points are ± 1 (degenerate of order ℓ) and the points $z \in \mathbb{C}$ for which $g_a(z) = \pm 10$.*
- *$f_a : \mathbb{R} \to \mathbb{R}$ is an odd function.*
- *For each $\ell \in 2\mathbb{N}$, there exists a parameter $a(\ell)$ and a point $x_\ell > 1$ so that $f = f_{a(\ell)} : [0, x_\ell] \to [0, x_\ell]$ is a unimodal map with Fibonacci dynamics (in particular, $f(0) = f(x_\ell)$).*
- *$f(\pm 5) \in (-1, 1) \subset (-x_\ell, x_\ell)$ for ℓ large.*

Proof: The first property follows from the chain rule. The second property is obvious. For the third property we want to prove that f_a contains a 'full unimodal family'. To do this first note that $0 \leq g_a'(x) \leq a$ when $0 \leq x \leq 1$. Hence $g_a(1) \leq a$. Moreover, $g_a'(x) \leq -a2^{\ell-1}$ when $x \geq \sqrt{3}$ and $g_a'(x) \leq 0$ when $x \geq 1$. Therefore

$$g_a(x) \leq g_a(1) - (x - \sqrt{3})a2^{\ell-1} \leq a[1 - (x - \sqrt{3})2^{\ell-1}]$$

for $x \geq \sqrt{3}$. In particular, $g_a(2) < 0$ when $\ell \geq 4$. It follows that there exists $x_\ell \in (1, 2)$ so that $f_a(x_\ell) = 0$ (and $f_a : [0, x_\ell] \to [0, \infty)$). Now restricted to $\mathbb{R}^+$ the function g_a takes its maximum at 1, and $g_a(1)$ is a positive multiple of a. Since h is monotone increasing on $[0, 10]$ and since $h(10) = 5 > 2 > z_\ell$ it follows that there exists $b(\ell) > 0$ so that $[0, b(\ell)] \ni a \mapsto f_a(1/2) \in [0, x_\ell]$ is a homeomorphism. It follows that $f_a : [0, x_\ell] \to [0, x_\ell]$,

$a \in (0, b(\ell)]$, is a full family of unimodal map, see for example Section II.4 of [MS]. In particular, there exists a parameter $a(\ell) \in (0, b(\ell))$ for which this map has Fibonacci dynamics. Finally, note that for the Fibonacci map $f_{a(\ell)}$ one has $f_{a(\ell)}(1) \geq 1$. Since $h'(x) \leq 1$ for $x \in \mathbb{R}$ and $0 \leq g'_a(x) \leq a$ when $0 \leq x \leq 1$ this implies that $a(\ell) \geq 1$. Now

$$g_a(5) \leq a[1 - (5 - \sqrt{3})2^{\ell-1}] \leq a(1 - 2^\ell) < 0.$$

Hence $f(5) = h(g_{a(\ell)}(5)) \geq h(a(1-2^\ell)) \geq h(1-2^\ell) \in (-1, 0)$ for ℓ sufficiently large. □

For simplicity write f for the map $f_{a(\ell)}$ from the previous lemma.

Corollary 5.1 *Provided ℓ is sufficiently large the rational map $f: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ has no periodic attractors. In fact, the Fatou set of f is empty.*

Proof: It is well-known that each periodic attractor (or parabolic point) has a critical point in its immediate basin (see for example p. 194 of [Bea]). Now, each critical value lies on the real line and, by the last property in the previous lemma is eventually mapped inside $(-1, 1) \subset [-x_\ell, x_\ell]$ provided ℓ is large enough. So possible periodic attractors (resp. parabolic periodic points) have to lie in $[-x_\ell, x_\ell]$. Now the only critical points in this interval are $\pm c$ and by construction $\omega(\pm c)$ are both (minimal) Cantor sets. To prove the second assertion, we use the classification theorem of Sullivan. This theorem tells us that each component of the complement of the Julia set is eventually periodic and either contains a periodic attractor, or a neutral periodic point or is a Siegel disc or a Hermann ring. As we just showed the first two possibilities cannot occur. Moreover, see p. 194 of [Bea], the boundaries of Siegel discs and Hermann rings are contained in the orbit closures of the critical points of f (and hence in $[-x_\ell, x_\ell]$). Hence a Siegel disc or a Hermann ring cannot exist either. □

Theorem 5.1 *The rational map $f: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is transitive and the basin of each of the sets $\Lambda_\pm = \omega(\pm c)$ has positive Lebesgue measure.*

Proof: By the previous corollary the Julia set of f is the entire Riemann sphere. Hence for each point $x \in \bar{\mathbb{C}}$ there exists arbitrarily small neighbourhoods U such that $f^n(U) \supset \bar{\mathbb{C}}$ for some n. It is easy to show from this that f is transitive.

That the basin of $\Lambda_\pm$ has positive Lebesgue measure follows from the second alternative of Theorem 3.1 above. □

Because the basins of $\Lambda_\pm$ are disjoint, completely invariant and both have positive Lebesgue measure, the map constructed is not ergodic. To show that the basins of $\Lambda_\pm$ together have full measure we need the following extension of Theorem 3.9 of [McM].

Proposition 5.1 *If f is a rational map of degree greater than one, then either*

- *f is transitive and ergodic or*
- *for Lebesgue almost all points $z \in J(f)$ one has that the spherical distance*

$$\lim_{n\to\infty} d(f^n x, P_+(f)) \to 0.$$

Here

$$P_+(f) = \bigcup_{c\in C_+(f)} \omega(c),$$

$$C_+(f) = \{c \in C(f) \text{ such that } c \in \omega(c_2) \text{ for some } c_2 \in C(f)\}$$

and $C(f)$ is the set of critical points of the rational map f.

Proof: The proof of the proposition is a simple extension of the proof of Theorem 3.9 of [McM]. The idea of the proof is that if B is a ball which is disjoint from P_+ then pullback's of B do at most meet a bounded number of critical points of f. The reason for this is i) that the hyperbolic diameter of B decreases as one takes preimages and ii) if B' is a sufficiently small disc containing a critical point $c \notin C_+(f)$ then preimages of B' do not contain critical points at all. All this implies that the reasoning of the proof of Theorem 3.9 of [McM] applies. □

We should emphasize that there are many cases when $P_+(f)$ is empty: for example there exists a neighbourhood U of the set of critical points such that for each critical point c one gets $f^n(c) \notin U$ for each $n > 0$. In this case the above theorem implies that the map is ergodic.

Since our example is not ergodic, it follows that almost all points iterate towards $P_+(f)$. In our example $P_+(f)$ is equal to $\Lambda_- \cup \Lambda_+$. This concludes the construction of the rational example from Theorem 1.1.

6 An ergodic transitive rational map with an invariant Cantor set which attracts a set a full measure

Let $g_a(z) = z^\ell + a$ (with $a \in \mathbb{R}$) and consider the family

$$f_a(z) = h \circ g_a(z)$$

where h is as in the previous section. In the same way as before one can choose a so that there exists a real symmetric interval I such that f_a restricted to I is a unimodal map with Fibonacci dynamics. The map f_a has 0 critical points in 0 and in some points $z \in \mathbb{C}$ so that $f_a(z) = \pm 5$. Choosing ℓ sufficiently large, $f_a \circ f_a(\pm 5) \in I$. Denote by Λ the closure of the orbit of the critical point 0. Using the result of Section 2, the basin

$B(\Lambda)$ has positive Lebesgue measure. To show that f is ergodic we need the following fact.

Fact: Consider a forward invariant set X of positive Lebesgue measure. Let us show that 0 is a Lebesgue density point of f. Take a Lebesgue density point p of X. Since $P_+(f) = \omega(0)$ (and 0 is recurrent), we can take – according to Proposition 5.1 – this point p so that its forward iterates enter any neighbourhood of 0. Take a sequence of neighbourhoods $D_n^0 \subset D_n$ of 0 as in Theorem 3.1 such that the diameter of D_n shrinks to zero. Let $i(n)$ be minimal so that $f^{i(n)}(p) \in D_n^0$. Since D_n^0 is a component of the first return map to D_n it follows that the branch of $f^{-i(n)}$ which takes $f^{i(n)}(p)$ back to p has a univalent extension to D_n. Since the modulus of $D_n \setminus D_n^0$ is uniformly bounded in n (it is not uniformly bounded in ℓ) it follows by Koebe that the restriction of this branch of $f^{-i(n)}$ to D_n^0 has a distortion which is uniformly bounded in n. Since the diameter of D_n shrinks to zero, also the diameter of $f^{-(n)}(D_n)$ shrinks to zero. Since p is a Lebesgue density point of X and X is forward invariant, it follows that 0 is also a density point of f.

Because of the above fact, one cannot have two disjoint forward invariant sets of positive Lebesgue measure. It follows that f is ergodic.

7 A Newton iteration map with a Julia set of positive Lebesgue measure

Consider a polynomial g and the corresponding Newton iteration map

$$f(x) = x - \frac{g(x)}{g'(x)}.$$

The zeroes of g are superattracting fixed points of f. It is also known that if g is a real polynomial with only real zeroes, then f has a Julia set of zero Lebesgue measure and Lebesgue almost all points outside the Julia set iterate towards zeroes of the polynomial g, see [Barna]. On the other hand, there exist real polynomials of the form $g(z) = z^{\ell-1} + (\lambda - 1)z - \lambda$ with ℓ even such that f has attracting periodic orbits of period $n > 2$, see for example [CGS]. Let us show

Theorem 7.1 *There exists ℓ_0 such that for each $\ell \in 2\mathbb{N}$ larger than ℓ_0 there exists $\lambda \in (0,1)$ so that the Newton map associated to $g(z) = z^{\ell-1} + (\lambda-1)z - \lambda$ has a Julia set of positive Lebesgue measure.*

Proof: Let us fix ℓ and write g_λ to emphasize the dependence of g on the parameter λ. First notice that the map $\mathbb{R} \ni x \mapsto g_1(x)$ has a unique critical point at $x = 0$ which is an inflection point (the order of the critical point is odd) and $g_1(0) < 0$. Furthermore, $g_0(-1) > 0$. Moreover, $g'(x) = \ell x^{\ell-2} + (\lambda - 1)$ has two real zeroes when $\lambda \in (0,1)$. From all this it follows that there exists $\delta > 0$ such that for each $\lambda \in (\delta, 1)$ the derivative $g'_\lambda(x)$ has two zeroes $\pm z_\lambda$ with $z_\lambda \in (0,1)$ and so that $g_\lambda(x) < 0$ for $x < 1$. In addition,

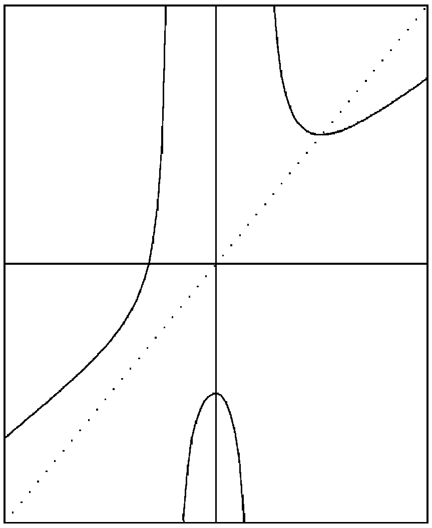

Figure 7: The graph of the Newton iteration associated to $g_\lambda = x^{\ell-1} + (\lambda-1)x - \lambda$ with $\lambda \in (\delta, 1)$.

for each $\lambda \in (\delta, 1)$ the function $[-1,1] \ni x \mapsto g_\lambda(x)$ has a unique maximum depending continously on λ; this maximum takes all values between -1 to 0.

From all this it follows that $f_\lambda(x)$ has a graph as shown in Figure 7. The vertical asymptotics correspond to $x = \pm z_\lambda$. Let R_ℓ be the first return map to $[-z_\lambda, z_\lambda]$. It consists of a number monotone branches and possibly also a central unimodal piece (the last possibility arrises if 0 eventually maps back inside $[-z_\lambda, z_\lambda]$). It is easy see that, as λ varies in $[\delta, 1]$, the family of central unimodal maps contains a full unimodal family. (Just consider how a fundamental domain of g_λ in $(-\infty, -z_\lambda)$ depends on λ compared to the speed in which $g_\lambda(0)$ depends on λ.) In particular, there exist (a countable number of) parameters such that the central unimodal map has Fibonacci dynamics. So using the result of Section 3 it follows that the associated Julia set of f has positive Lebesgue measure. The Julia set is nowhere dense, because 1 (a zero of g) is a superattracting fixed point of f. $\square$

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