Period Three, Chaos and Fractals

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Imperial, May 2012

Three Different Projects

- (A) Real one-dimensional dynamics: builds on analysis from the first year.
- (B) Complex dynamics: ask you to computer experiments and computer coding.
- (C) Complex dynamics: looks ahead at material from the 2nd year, but requires no material except what you already know from the first year (and knowledge about complex numbers - what you already know from secondary school).

Does anybody already know which topic he/she wants to do? They will all be fun!

Next, I will discuss each of the projects in some more detail and will tell you **exactly** what you would need to do by next Wednesday if you choose that topic.

This will make up about 1/3 of the project.

For the week after, I will give guidance, but you will have more freedom how to take it further.

Let us motivate the so-called *Sarkovskii ordering* on $\mathbb{N}$

$$3 \prec 5 \prec 7 \prec 9 \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \cdots \prec 8 \prec 4 \prec 2 \prec 1.$$

Let $f\colon [0,1] \to [0,1]$ be continuous and consider successive *iterates* $x_{n+1} = f(x_n)$ of a point $x_0 = x \in [0,1]$. So $x_n = f^n(x_0)$ where

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{\text{n times}}.$$

We say that x is *periodic* if $f^n(x) = x$ for some $n \geq 1$. The minimal such number is called the *period* of x.

Theorem (Sarkovskii 1964)

If $p \prec q$ and f has a periodic point of (minimal) period p then it also has a periodic point of minimal period q.

In particular, period three implies all periods. We will see that the main ingredient for this is the intermediate value theorem.

The task for this week is to understand

- some background and
- all the ingredients for the proof of this theorem.

- $f(x) = x$: then each point has period one and no point has period > 1. This is a pretty boring map.
- $f(x) = 2x(1-x)$ (draw graph).

Then $f(0) = 0$ and for each $x \in [0, 1/2]$,

$$f(x) = 2x(1-x) \leq 1/2 \text{ and } x < f(x).$$

So when $x \in [0, 1]$ we get $x < f(x) < f^2(x) < \cdots < 1/2$ and hence $f^n(x)$ is a bounded increasing sequence. So it has a limit y. Since $f^n(x) \to y$ we also have that $y = f(y)$. Hence $y = 1/2$ (check).

- $f(x) = 2x(1-x)$

  Similarly for $x \in [1/2, 1]$, $\quad f(x) = 2x(1-x) \leq 2x \leq 1/2$ and so then again $f^n(f(x)) \to 1/2$. In other words, there are no periodic points in $[0, 1]$ except $0, 1/2$.

- $f(x) = 4x(1-x)$ (draw graph). Let us show that this map has a point of period three.

Take $z_0 \in [0, 1/2]$ so that $f(z_0) = 1/2$ (how can you know such a point exists and is unique?).

Then take $z_1 \in [0, z_0]$ so that $f(z_1) = z_0$ (how can you know such a point exists and is unique?).

- So f maps
  1. $[z_1, z_0]$ continuously onto $[z_0, 1/2]$, monotone increasingly
  2. $[z_0, 1/2]$ continuously onto $[1/2, 1]$, monotone increasingly
  3. $[1/2, 1]$ continuously onto $[1/2, 1]$, monotone decreasingly

  So f^3 maps $[z_1, z_0]$ continuously onto $[0, 1]$, monotone decreasingly.

  The next lemma will imply that there exists $x \in [z_1, z_0]$ so that $f^3(x) = x$.

Lemma

Let $J \subset [0,1]$ be an interval and $g\colon J \to [0,1]$ continuous and surjective. Then there exists $x \in J$ so that $g(x) = x$.

Proof: If not, then consider $h(x) = g(x) - x$ and we get $h \neq 0$. By the intermediate value theorem $h(x) > 0$ or $h(x) < 0$ for all $x \in J$.

So $g(x) < x$ for all $x \in J \subset [0,1]$ or $g(x) > x$ for all such points. Let's assume we are in the former case.

Write $J = [a,b]$. Since $J \subset [0,1]$, we have $0 \leq a < b \leq 1$.

Hence $g(x) < x \leq b \leq 1$ for all $x \in [a,b]$. So there exists no $x \in J$ so that $g(x) = 1$.
This contradicts that g is supposed to be surjective.

Prove the following lemma, and if you get stuck ask questions next week.

Lemma

Let $f\colon [0,1] \to [0,1]$ *be continuous and* $J \subset [0,1]$ *be an interval.*

1. *Then* $f(J)$ *is also an interval;*
2. *If* $f^n(J) \supset J$ *then there exists* $x \in J$ *with* $f^n(x) = x$*;*
3. *If* $J_0, J_1 \subset [0,1]$ *are intervals and* $f(J_0) \supset J_1$ *then there exists an interval* $J \subset J_0$ *so that* $f(J) = J_1$*;*
4. *Let* $J_0, J_1, \ldots, J_m$ *be intervals in* $[0,1]$ *so that* $f(J_i) \supset J_{i+1}$ *for* $i = 0, \ldots, m-1$*. Then there exists an interval* $J \subset J_0$ *so that* $f^i(J) \subset J_i$ *for* $i = 0, 1, \ldots, m-1$ *and* $f^m(J) = J_m$*.*

The proof of this lemma relies on the
Intermediate Value Theorem:
if $g\colon [0,1] \to \mathbb{R}$ is continuous and $g(0)g(1) < 0$ then there exists $x \in (0,1)$ with $g(x) = 0$.

Your project could include a detailed proof of the above lemma.

Look through the first page of the proof of Sarkovskii's theorem, and ask questions about it if you do not follow the proof.

We will discuss this more in detail next week.

Take a polynomial $f\colon \mathbb{C} \to \mathbb{C}$, say $f(z) = z^2 + c$. So you take a complex number $z \in \mathbb{C}$, compute $z \cdot z$ and then add c. Then repeat this again and again.

This week you will take the **Julia set** $J(f)$ of $f(z) = z^2 + c$ when $c \neq 0$ to be defined in the following way:

$$J(f) = \partial B(\infty) = \partial\{z; |f^n(z)| \to \infty \text{ as } n \to \infty\}.$$

Next week you then will also consider another method and begin to understand the Julia set and the Mandelbrot set even better.

Project B: Computing Julia sets

Let's try to compute/draw the Julia set

$$J(f) = \partial B(\infty) = \partial\{z; |f^n(z)| \to \infty \text{ as } n \to \infty\}.$$

Take $R = \max(|c|, 3)$. Then for each $z \in \mathbb{C}$ with $|z| > R$,

$$|f(z)| = |z^2 + c| \geq |z||z| - |c| \geq R|z| - |c| \geq 2|z|,$$

because $R - 2 \geq 1$ and $|z| \geq R \geq |c|$. Hence when $|c| \leq 10$ and $|z| > 10$ then $|f^n(z)| \to \infty$ and

$$B(\infty) \supset \{z; |z| \geq 10\}$$

To draw points which points are **not** in $B(\infty)$, take $|c| < 10$, pick some integer, say 30, and compute for each $z \in \mathbb{C}$ with $|z| < 10$ (or rather each pixel point) the minimal value $n = n(z) \leq 30$ so that $|f^n(z)| \geq 10$ and plot this point z on the computer screen with a colour which depend on $n(z)$.

```
c = 0.36 + i*0.1;
x = -2:0.02:2; y = -2:0.02:2;
[x0, y0] = meshgrid(x,y);
n = zeros(size(x0));
for k = 1:length(y),
  for m = 1:length(x),
    z = x0(k,m) + i*y0(k,m);
    for itr = 1:30,
      z = z.^2 + c;
      if abs(z) > 10,
        n(k,m) = itr; break,
      end,
    end,
  end,
end,
plotcolour(x0,y0,n);
pixgrid(size(x0));
```

What do you need to do this week:

- Get this code to work on matlab and do some experiments (see lecture)
- Understand why the regions with the same colour split in several areas if $|f(0)| = |c| \geq 10$. Hint: consider the set

$$U := \{z; |z^2 + c| \leq 10\} = \{z; z^2 \in B(-c; 10)\}$$

  where $B(-c; 10)$ is the ball with radius 10 and centre $-c = -f(0)$. If $B(-c; 10)$ does not contain 0 then this U consists of two pieces. Now consider this argument when $|f^2(0)| = |f(c)| \geq 10$.

Project C: Some theorems s on polynomials acting on the complex plane

This week you are going to analyse the following theorem.

Theorem

A polynomial $f : \mathbb{C} \to \mathbb{C}$ *of deg* > 1 *has* ∞ *many periodic points.*

Take $f(z) = z^2$. By the theorem, this is supposed to have infinitely many periodic points. Where are they?

If $|z| > 1$ then $|f(z)| = |z|^2 > 1$ and more generally $|f^n(z)| = |z|^{2^n}$ and so $|f^n(z)| \to \infty$ as $n \to \infty$.

If $|z| < 1$ then $|f(z)| = |z|^2 < 1$ and more generally $|f^n(z)| = |z|^{2^n}$ and so $|f^n(z)| \to \infty$ as $n \to \infty$.

So periodic points have to be on the unit circle $\{z; |z| = 1\}$ (or equal to the fixed point 0).

If $|z| = 1$ then $z = e^{i\phi}$ with $\phi \in [0, 2\pi)$. Hence $f(z) = e^{2i\phi}$ and $f^n(z) = e^{2^n i\phi}$.

So $z = f^n(z)$ is equivalent to $\phi = 2^n i\phi \mathrm{mod} 2\pi$. That is, $(2^n - 1)\phi \in 0 \mathrm{mod} 2\pi$.

So $z = f^n(z)$ is equivalent to $(2^n - 1)\phi \in 0 \text{mod} 2\pi$ when $z = e^{i\phi}$.

$n = 1$ then $\phi = 0$.

$n = 2$ then $\phi = 0, \pi/3, 2\pi/3$. Which are the periodic points of period two?

$n = 3$ then $\phi = 0, \pi/7, 2\pi/7, 3\pi/7, 4\pi/7, 5\pi/7, 6\pi/7$. Which are the periodic points of period three? What are the orbits

$$0 \to 0$$

$$\pi/7 \to 2\pi/7 \to 4\pi/7 \to \pi/7$$

and

$$3\pi/7 \to 6\pi/7 \to 5\pi/7 \to 3\pi/7.$$

How many periodic points does the map $f(z) = z^2 - 2$ have? Note that f maps $[-2, 2]$ onto $[-2, 2]$. So we can apply Sharkovskii's theorem.

From what we did in the beginning of this session, f has a periodic point of period three. So there are infinitely many periodic points in $[-2, 2]$.

In fact, the Julia set of this map is equal $[-2, 2]$. So when $z \in [-2, 2]$ then it is **not** the case that $|f^n(z)| \to \infty$. It follows that $[-2, 2] \subset J(f)$.

Let us now show that $J(f) = [-2, 2]$.

Proof that $J(f) \subset [-2, 2]$. To prove this, we shall use the result (we discussed last time):

Theorem

Take p so that $\# f^{-2}(p) \geq 3$. Then for each $\delta > 0$ there exists n so that for each point $w \in J(f) = \partial B(\infty)$ there exists $z \in f^{-n}(p)$ with $|z - w| < \delta$.

Note that $f(z) \in [-2, 2]$ then $z \in [-2, 2]$. In other words, if $z \in [-2, 2]$ then $f^{-1}(z) \subset [-2, 2]$. Repeating this gives $f^{-n}(z) \subset [-2, 2]$.

Moreover, $\# f^{-2}(0) \geq 3$ (see graph).

By the above theorem, for each $\delta > 0$ there exists n so that the Julia set $J(f)$ lies δ close to the set $f^{-n}(0) \subset [-2, 2]$.

So $J(f) \subset [-2, 2]$.

Project C: Final task for this week

Read and understand the proof of the following

Theorem

A polynomial $f : \mathbb{C} \to \mathbb{C}$ *of deg* > 1 *has* ∞ *many periodic points.*