# Period Three, Chaos and Fractals

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Let us motivate the so-called *Sarkovskii ordering* on $\mathbb{N}$

$$3 \prec 5 \prec 7 \prec 9 \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \cdots \prec 8 \prec 4 \prec 2 \prec 1.$$

Let $f\colon [0,1] \to [0,1]$ be continuous and consider successive *iterates* $x_{n+1} = f(x_n)$ of a point $x_0 = x \in [0,1]$. So $x_n = f^n(x_0)$ where

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{\text{n times}}.$$

We say that $x$ is *periodic* if $f^n(x) = x$ for some $n \geq 1$. The minimal such number is called the *period* of $x$.

**Theorem (Sarkovskii 1964)**

*If $p \prec q$ and $f$ has a periodic point of (minimal) period $p$ then it also has a periodic point of minimal period $q$.*

In particular, period three implies all periods. We will see that the main ingredient for this is the intermediate value theorem.

# The main lemma

**Lemma**

*Let $J \subset [0,1]$ be an interval.*

1. *Then $f(J)$ is also an interval;*
2. *If $f^n(J) \supset J$ then there exists $x \in J$ with $f^n(x) = x$;*
3. *If $J_0, J_1 \subset [0,1]$ are intervals and $f(J_0) \supset J_1$ then there exists an interval $J \subset J_0$ so that $f(J) = J_1$;*
4. *Let $J_0, J_1, \ldots, J_m$ be intervals in $[0,1]$ so that $f(J_i) \supset J_{i+1}$ for $i = 0, \ldots, m-1$. Then there exists an interval $J \subset J_0$ so that $f^i(J) \subset J_i$ for $i = 0, 1, \ldots, m-1$ and $f^m(J) = J_m$.*

The proof of this lemma relies on the *Intermediate Value Theorem*:
if $g\colon [0,1] \to \mathbb{R}$ is continuous and $g(0)g(1) < 0$ then there exists $x \in (0,1)$ with $g(x) = 0$.

*Your project could include a detailed proof of the above lemma.*

# Proof of Sarkovskii's theorem for the period three case: Step 1

Let $x$ be a point of period three, so the set $x, f(x), f(x), \dots$ consists of three distinct points $a < b < c$.
Let

$$I_1 = [a, b] \text{ and } I_2 = [b, c].$$

Since $f$ permutes points from the set $\{a, b, c\}$ (fixing none of these points), depending on whether the middle point goes to the left or to the right,

$$f(b) = a \text{ and } f(a) = c \text{ and } f(c) = b$$

or

$$f(b) = c \text{ and } f(a) = b \text{ and } f(b) = a.$$

Let us assume the former (the latter case goes the same up to relabelling). Then Statement (1) of the previous lemma implies

$$f(I_1) \supset I_1 \cup I_2 \text{ and } f(I_2) \supset I_1$$

## Step 2: For any $m$ there exists a point with $f^m(x) = x$

Remember that

$$f(I_1) \supset I_1 \cup I_2 \text{ and } f(I_2) \supset I_1$$

Let $m$ be an integer and let us take

$$J_0, \ldots, J_{m-2} \text{ all equal to } I_1,$$

$$J_{m-1} = I_2 \text{ and } J_m = I_1.$$

Since $f(J_i) \supset J_{i+1}$ for each $i = 0, 1, \ldots, m-1$, by Part (4) of the lemma, there exists an interval $J \subset J_0 = I_1$ so that $f^i(J) \subset J_i$ for $i = 0, 1, \ldots, m-1$ and

$$f^m(J) = J_m = I_1 \supset J.$$

The last inclusion and Part (2) of the lemma implies that there exists $x \in J$ so that $f^m(x) = x$.

## Step 3: $x$ has exactly period $m$

Remember that

$$I_1 = [a, b]\ ,\ I_2 = [b, c] \text{ and } f(a) = c, f(b) = a, f(c) = b \tag{1}$$

$$x, f(x), \dots, f^{m-2}(x) \in I_1 \text{ and } f^{m-1}(x) \in I_2. \tag{2}$$

Suppose by contradiction that not all the points $x, f(x), \dots, f^{m-1}(x)$ are distinct. Then $f^i(x) = f^j(x)$ for some $0 \le i < j < m$ and therefore $f^{i+k}(x) = f^{j+k}(x)$ for all $k \ge 0$. So

$$f^{i'}(x) = f^{m-1}(x) \text{ for some } 0 \le i' < m-1.$$

By (2), $f^{i'}(x) = f^{m-1}(x) = b$. Hence, using (1),

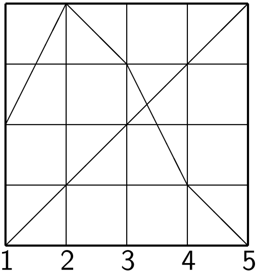
$$f^{i'+1}(x) = f^m(x) = f(b) = a \text{ and } x = f^m(x) = a. \tag{3}$$

But then

$$f(x) = f(a) = c \notin I_1$$

contradicting (2) unless $m = 2$. If $m = 2$ then $i' = 0$ and (3) gives $f(a) = a$, contradicting (1) and that $a, b, c$ are all distinct.

Let us show that there exists a continuous map $f\colon [1,5] \to [1,5]$ with period 5 and no period 3. To do this take $f$ piecewise affine with $f(1) = 3, f(3) = 4, f(4) = 2, f(2) = 5, f(5) = 1$.




By inspection $f^3[1,2] = [2,5], f^3[2,3] = [3,5], f^3[4,5] = [1,4]$ and there are no fixed points of $f^3$ on these intervals. $f^3$ maps $[3,4]$ monotonically decreasing onto $[1,5]$: it is a composition of monotone decreasing maps $[3,4] \to [2,4] \to [2,5] \to [1,5]$. So the fixed point of $f^3$ on $[3,4]$ agrees with the fixed point of $f$ on this interval.

## Suggestions for project A (on Sarkovskii's theorem

- Adapt the above proof to show that the existence of a point of period 5 implies the existence of periodic points of any period (except 3). For a complete proof see for example
  1. R.L. Devaney, *An introduction to chaotic dynamical systems* or
  2. W. de Melo and S. van Strien, *One dimensional dynamics*, Section II.1. You will be able to download this book from the module webpage).
- Show that the map $f(x) = 4x(1-x)$ has a periodic point of any period. Do this by drawing a graph of the map $f(x) = 4x(1-x)$ and 'constructing' a point $x \in (0, 1/2)$ so that $x < f(x) < \cdots < f^{m-2}(x) < 1/2 < f^{m-1}(x)$ and $f^m(x) = x$.
- Show that the only periodic points of the map $f(x) = 3x(1-x)$ are the two fixed points 0 and $2/3$. (Hint: analyse the graph of $f^2$.)
- More suggestions will be posted on the webpage of this module.

**Theorem**

*A polynomial $f : \mathbb{C} \to \mathbb{C}$ of deg $> 1$ has $\infty$ many periodic points.*

Before proving this, we remark (without proof) that all but finitely many of these periodic points $p$ have the property:

$$\text{if } f^n(p) = p \text{ then } |(f^n)'(p)| > 1.$$

Such periodic points are called *repelling* because by the Mean Value Theorem

$$|f^n(x) - p| = |f^n(x) - f^n(p)| = (f^n)'(\xi)|x - p| > |x - p|.$$

We denote the set of repelling periodic points by $Per_{rep}(f)$.

## Theorem

*A polynomial $f : \mathbb{C} \to \mathbb{C}$ of deg $> 1$ has $\infty$ many periodic points.*

Prerequisite for the proof: *Fundamental Theorem of Algebra* (which is taught in year two). Let $f$ be a polynomial $f$ of *degree $d$*,

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0 \text{ with } a_d \neq 0.$$

Then

$$f(z) = a(z - z_1)^{d_1} \cdot \cdots \cdot (z - z_k)^{d_k}$$

where $z_1, \ldots, z_k$ are all distinct and $d_1 + \cdots + d_k = d$. The positive integer $d_i$ is called the *multiplicity* of the zero $w_i$. In particular we get:

$$\deg(f) = \text{ sum of multi. at zeroes of } f.$$

**Proof of the existence of $\infty$ many periodic points**. Assume by contradiction that

$$p_1, \dots, p_k \text{ are the only periodic pts (with periods } n_1, \dots, n_k). \quad (4)$$

- Let $N = n_1 \cdots n_k$. For each $i = 1, \dots, k$, $N$ is a multiple of $n_i$ and so $f^N(p_i) = p_i$.
- By an explicit calculation the multiplicity $m$ of $f^N(z) - z$ and $f^{2N}(z) - z$ at each $p_i$ is the same. Indeed, w.l.o.g. we can assume $p_i = 0$ and $f^N(z) = z + az^m + h.o.t.$ and so $f^{2N}(z) = (z + az^m + \dots) + a(z + az^m + \dots)^m = z + 2az^m + \dots$.
- By (4) the only zeroes of $f^{2N}(x) - x$ are $p_1, \dots, p_k$. Hence

$$\sum \text{mult. at zeroes of } f^N(z) - z = \sum \text{mult. of zeroes of } f^{2N}(z) - z.$$

- This and the Fund. Thm of Algebra gives the middle equality in:

$$deg(f^N(z)) = deg(f^N(z) - z) = deg(f^{2N}(z) - z) = deg(f^{2N}(z)).$$

- But this is impossible since $deg(f^{2N}) > deg(f^N)$.

# The basin at infinity

Define the *basin of infinity* $B(\infty) = \{x; f^n(x) \to \infty \text{ as } n \to \infty\}$.

**Lemma**

*Let $f$ be a polynomial. Then there exists $K > 0$ so that if $|z| \geq K$ then $z \in B(\infty)$.*

Proof when $f(z) = z^2 + c$. Take $K = \max(3, |c|)$ and $z \in \mathbb{C}$ with $|z| \geq K$. Then

$$|f(z)| = |z^2 + c| \geq |z||z| - |c| \geq K|z| - |c| \geq 3|z| - |z| = 2|z|$$

and so

$$|f^n(z)| \geq 2^n|z| \to \infty \text{ as } n \to \infty.$$

## The Julia set of a polynomial

There are several equivalent definitions of the Julia set $J(f)$ of a polynomial $f$. We will give two. Project B will explore how to show their equivalence, and how to effectively compute the Julia set.

Let $A \subset \mathbb{C}$ and define the *boundary* $\partial A$ (resp. the *closure* $\overline{A}$) of $A$ as the set of points $x$ so that within each ball $B_r(x)$ around $x$ there are points from $A$ and also points from the complement of $A$ (resp. so that $B_r(x) \cap A \neq \emptyset$). Let $\#A$ be the *cardinality* of a set.

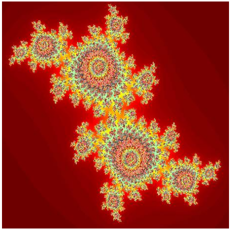
**Theorem (Definition of Julia set)**

$$J(f) := \partial B(\infty) = \overline{Per_{rep}(f)}.$$

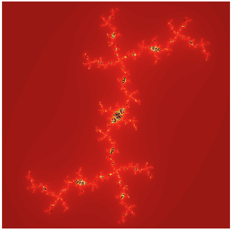
**Theorem (Useful way of computing/drawing a Julia set)**

*Take $p$ so that $\#f^{-2}(p) \geq 3$. Then for each $\delta > 0$ there exists $n$ so that each point $x \in J(f)$ is at most $\delta$ apart from some $z \in f^{-n}(p)$.*

Below are some examples of Julia sets of $f(z) = z^2 + c$ for various choices of $c$. It turns out that whether or not $|f^n(0)| \to \infty$ determines whether $J(f)$ is 'connected'.



$f^n(0) \to \infty$



Is 0 in the basin of a periodic attractor?

0 is in basin of attracting fixed point

The previous theorem can be proved using:

**Theorem (Montel)**

*Let $p_n, q_n$ be a sequence polynomials and $D$ a disc in $\mathbb{C}$ so that $q_n \neq 0$ for all $z \in D$ and all $n$. Define $R_n := p_n/q_n$. If none of the sets $D, R_1(D), R_2(D), \dots$ contains $0$ or $1$ then there exists $M < \infty$ so that $|R_n(z)| \leq M$ and $|R_n'(z)| \leq M$ for all $n$ and all $z \in D$.*

Let us show how this implies:

**Theorem (Useful way of computing/drawing a Julia set)**

*Take $p$ so that $\#f^{-2}(p) \geq 3$. Then for each $\delta > 0$ there exists $n$ so that for each point $w \in J(f) = \partial B(\infty)$ there exists $z \in f^{-n}(p)$ with $|z - w| < \delta$.*

**Step 1.** Take $r_1, r_2, r_3$ distinct with $f^2(r_1) = f^2(r_2) = f^2(r_2) = p$. (Check that there are plenty of $p$'s $\#f^{-2}(p) \geq 3$ when $f$ has degree $\geq 2$).

**Step 2.** Take $w \in \partial B(\infty)$ and a disc $D \ni w$. Since $w \in \partial B(\infty)$, there exists $z \in D$ so that $|f^n(z)| \to \infty$. Taking $S_n := (2/\delta)(f^n - r_1)$ we get $S_n(z) \to \infty$. Hence, by Montel, there exists $n$ and $z \in D$ so that $S_n(z)$ is equal to 0 or 1. If $S_n(z) = 0$ then $f^n(z) = r_1$ and so $f^{n+2}(z) = f^2(r_1) = p$ and we are done. So we can assume that there exist $z \in D$ and $n$ so that $S_n(z) = 1$ which means $f^n(z) = r_1 + \delta/2$. (We will use this in Step 4.)

**Step 3.** Define

$$R_n(z) := \frac{(f^n(z) - r_2)}{(f^n(z) - r_1)} \frac{(p - r_1)}{(p - r_2)}.$$

It follows from Montel that either

- there exists $z \in D$ with $R_n(z) \in \{0, 1, \infty\}$ or
- there exists $M < \infty$ so that $|R_n(z)| \leq M$ for all $n$ and all $z \in D$.

In the former case, $f^n(z) = r_2$, $f^n(z) = r_1$ or $f^n(z) = p$ (check this!) Hence $f^{n+2}(z) = p$ and we are done.

**Step 4.** In the latter case, for each $z \in D$ and each $n$ the norm of $R_n(z) = \frac{(f^n(z)-r_2)}{(f^n(z)-r_1)}\frac{(p-r_1)}{(p-r_2)}$ is at most $M$ and so there exists $M_1$ with $\frac{|f^n(z)-r_2|}{|f^n(z)-r_1|} \leq M_1$ for all $n$ and all $z \in D$. Hence there exists $\delta > 0$ so that $|f^n(z) - r_1| \geq \delta$ for all $z \in D$. But this contradicts that there exists $z$ and $n$ so that $f^n(z) = r_1 + \delta/2$.

*Suggestions for project B (on Julia sets):*

1. Draw the Julia set of $z^2 + c$ for a number of choices of $c$, either in Matlab or in Maple (or in any other computer language of your choice). You can do this by using the algorithm suggested by the previous theorem. You may also use some software freely available on the web. It is absolutely fine if you concentrate your project on this part, but then you need to add to your presentation the code you have written and give a lively description of the computer experiments you made (with lots of pictures)!!
2. Determine by hand the Julia set of the function $f(z) = z^2$.
3. Use the previous theorem to show that the Julia set of $z^2 - 2$ is contained in $[-2, 2]$. (Hint: use the previous theorem, $\#f^{-2}(0) \geq 3$ and that $f^{-1}([-2, 2]) \subset [-2, 2]$. In actual fact, one can prove that the Julia set of this map is equal to $[-2, 2]$.

4. Let $p$ be a repelling periodic point. Show that $p \in \partial B(\infty)$. (Hint: if $p \notin \partial B(\infty)$ then (since $p$ is periodic) there exists a disc $D$ around $p$ so that no point in $D$ is in $B(\infty)$. Then use:

**Lemma**

*Let $f$ be a polynomial. Then there exists $K > 1$ so that if $|z| \geq K$ then $z \in B(\infty)$.*

So there exists $K > 1$ so that $|f^n(z)| \leq K$ for all $n$ and all $z \in D$. Now define $R_n(z) = 3K - f^n(z)$. So $R_n$ does not take the values 0 and 1 on $D$. Hence, by Montel, there exists $A < \infty$ so that $|(R_n)'(z)| \leq A$ for all $z \in D$ and all $n$. But this is impossible because $(R_n)'(p) = (f^n)'(p) \to \infty$ because $p$ is a **repelling** periodic point. Here you need to use the chain rule.)

5. Let $w \in \partial B(\infty)$. Show that for each $\delta > 0$ there exists a periodic point $p$ so that $|p - w| < \delta$. (Hint: see below.)

For (5) you need to use a slightly more general version of the theorem of Montel where one can take $p_n$ and $q_n$ of the form $f^n(z) - \phi_1(z)$ and where $\phi_i$ are inverse functions of $f$. You will also need the following fact (which you can use without proof): If $D$ is a disc containing $w$ and $f^n(D) \supset D$ then there exists $p \in D$ with $f^n(p) = p$.
You will be given some extra help on this during the project meetings.

There are many popular books on chaos and fractals.
H.O. Peitgen and P.H. Richter, *The Beauty of fractals*, Springer Verlag, Berlin 1986). (With lots of pictures but with not so much maths.)
R.L. Devaney, *An introduction to chaotic dynamical systems*, Addison-Wesley, 1986. (A very nice introduction to dynamical systems.)
R.L. Devaney wrote several popularising texts. For school kids: