Chapter 1
Dynamics associated to games (fictitious play) with chaotic behavior

Colin Sparrow and Sebastian van Strien

Abstract In this survey we will discuss some recent results on a certain class of dynamical systems, called *fictitious play* which are associated to game theory. Here we simply aim to show that the dynamics one encounters in these systems is unusually rich and interesting. This paper does not require a background in game theory.

1.1 Introduction

Consider games with two players A and B which both can play, randomised, n strategies. So the **state space** of the players is described by two probability vectors

$$p^A \in \Sigma_A \text{ and } p^B \in \Sigma_B$$

where Σ_A and Σ_B are the space of probability vectors in $\mathbb{R}^n$. By convention, p^A is a row vector and p^B a column vector. We assume that player A has **utility** (i.e. **payoff**) $p^A A p^B$ and player B has payoff $p^A B p^B$ where A and B are $n \times n$ matrices.

At a given moment in time, player A can best improve her utility by choosing the unit vector $\mathscr{B}R_A(p^B)$ which corresponds to the largest component of Ap^B. This is the **best response** of player A to position p^B. Formally,

$$\mathscr{B}R_A(p^B) := \arg\max_{p^A} p^A \, A \, p^B. \tag{1.1}$$

Of course, $\mathscr{B}R_A(p^B)$ can be a whole collection of vectors. However, when A is a non-degenerate matrix, this happens only for p^B in certain hyperplanes. (More precisely, in one of the $n \times (n-1)/2$ hyperplanes in Σ_B corresponding to $p^B \in \Sigma_B$ where Ap^B has two or more equal components.) Outside these hyperplanes, $\mathscr{B}R_A(p^B)$ is a unit basis vector. Similarly denote the best response of player B by $\mathscr{B}R_B(p^A)$.

Maths Department, Unversity of Warwick

A **Nash equilibrium** is a choice of strategies from which no unilateral deviation by an individual player is profitable for that player. That is, (p^A_*, p^B_*) is a Nash equilibrium if

$$p^A_* \in \mathscr{B}R_A(p^B_*) \text{ and } p^B_* \in \mathscr{B}R_B(p^A_*).$$

The Nash equilibrium is **unique**, if A and B are invertible and if, moreover, there exists a purely mixed Nash equilibrium (E^A, E^B) (see, for example, [vSS09, Theorem 1.5]). In this case $E^B \in \Sigma_B$ is the vector so that all components of AE^B are the same (so player A is indifferent to all different strategies). The vector E^A can be found similarly.

In the 1950s Brown [Bro51] proposed **fictitious play** as a way in which players are able to naturally find the Nash equilibrium by flowing according to the following differential equation:

$$\begin{aligned} dp^A/dt &= \mathscr{B}R_A(p^B) - p^A \\ dp^B/dt &= \mathscr{B}R_B(p^A) - p^B \end{aligned} \tag{1.2}$$

where $\mathscr{B}R_A(p^B) \in \Sigma_A$ is the best response of player A to players B position, and similarly for $\mathscr{B}R_B(p^A) \in \Sigma_B$. So each player's tendency is to adjust his or her strategy in a straight line from his/her (current) strategy towards their (current) best response.

There is an interpretation of this game as a mechanism by which the players **learn** from the other player's previous actions and then one often writes

$$\begin{aligned} dp^A/ds &= \left(\mathscr{B}R_A(p^B) - p^A\right)/s \\ dp^B/ds &= \left(\mathscr{B}R_B(p^A) - p^B\right)/s, \end{aligned} \tag{1.3}$$

see for example the monograph [FL98]. The dynamics of this system and the previous are the same up to time-parametrisation $s = e^t$. Since $\mathscr{B}R_A$ and $\mathscr{B}R_B$ are not necessarily single-valued, (1.2) and (1.2) are really differential inclusions, rather than differential equations. Under mild assumptions, see [AC84], these differential inclusions have solutions. In actual fact, as we shall see, in many examples the solutions are even still unique.

This survey will describe some results on the dynamics of these games and pose some conjectures and open questions.

1.2 A short introduction into game theory and some simple 2×2 examples

Let us discuss first the simplest (and essentially trivial) case, where both players have only two strategies to choose from, i.e. when Σ_A and Σ_B both correspond to the one-dimensional simplex $\{(p_1, p_2) \in \mathbb{R}^2; p_i \geq 0, p_1 + p_2 = 1\}$. Often instead of writing $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, these two matrices are denoted using the following notation $\begin{pmatrix} (a_1, b_1) & (a_2, b_2) \\ (a_3, b_3) & (a_4, b_4) \end{pmatrix}$. Equivalently, these matrices are encoded in the following way:

Payoff's	Player B chooses left	Player B chooses right
Player A chooses top	(a_1, b_1)	(a_2, b_2)
Player A chooses bottom	(a_3, b_3)	(a_4, b_4)

where the 2nd part of each entry corresponds to the payoff to player B (the column player). As mentioned, $\Sigma_A \times \Sigma_B$ can be thought of as $[0,1] \times [0,1]$ and because of this notation it is traditional (and convenient) to identify the vertical side with the position of player A (the player with the row vector p^A) and the top left corner of $[0,1] \times [0,1]$ with $(1\,0), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Sigma_A \times \Sigma_B$. When we use this identification, payoffs the each of the corners of the square is the corresponding entry of the matrix. (We note that p^A is a row vector, even though p^A represents the position of player A and is displayed on the vertical side of the square.)

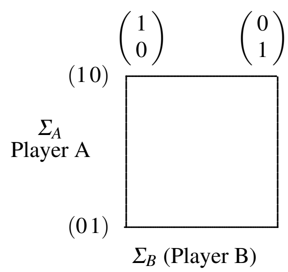

Fig. 1.1 The identification for 2×2 games: the horizontal side corresponds to player B and the top left corner corresponds to the first unit base vector for both players (i.e. the first strategy)

Of course, the evolution described by fictitious play, i.e. the differential inclusion

$$\begin{aligned} dp^A/dt &= \mathscr{B}R_A(p^B) - p^A \\ dp^B/dt &= \mathscr{B}R_B(p^A) - p^B \end{aligned} \tag{1.4}$$

is completely determined by the (multivalued) functions $\mathscr{B}R_A(p^B)$ and $\mathscr{B}R_B(p^A)$. Note that $(\mathscr{B}R_A(p^B), \mathscr{B}R_B(p^A))$ corresponds to one of the corners of $[0,1] \times [0,1]$, except where it is multivalued. So where it is not multivalued, the motion of (1.4) is towards one of the corners. There is a vertical line which determines where $\mathscr{B}R_B(p^A)$ changes, i.e. where the motion (1.4) switches direction (moving towards one of the top corners on one side and to one of the bottom corners on the other side), and a horizontal line where player A switches direction (moving towards one of the left corners on one side and to one of the right corners on the other side). In Figure 1.2 we have drawn a few cases. Case (1) corresponds to a situation where the player A always prefers to move up (except on the segment marked in the figure on the left side, where she is indifferent). There are many matrices which would correspond to this situation, for example when $\begin{pmatrix} (0,-1) & (0,0) \\ (0,0) & (-1,-1) \end{pmatrix}$. Case (2) corresponds to $\begin{pmatrix} (-1,1) & (0,0) \\ (0,0) & (-1,1) \end{pmatrix}$. Here both players have opposite interests (the sum of the payoff's is always zero). Player B is copying A's behavior (because the largest component of $p^A B$ is then equal to the largest component of p^A), whereas player A is doing the opposite to what player B is doing. Finally, Case (3) corresponds to $\begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1,1) \end{pmatrix}$ where both players agree to choose the same strategy. In the prisoner dilemma $\begin{pmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{pmatrix}$ the players always move to the bottom right corner, see Case (4) and for both players the best response is always the 2nd strategy ($\mathscr{B}R_A, \mathscr{B}R_B$ are both constant in this case), even though they both would receive higher payoffs playing the first strategy.

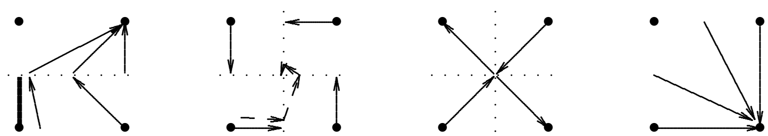

Fig. 1.2 The possible motions in 2×2 games (up to relabeling, and shifting the indifference lines (drawn in dotted lines).

In fact, it is easy to see that the dynamics in any 2×2 game is topologically of one of these types. Therefore, from this point of view, the next interesting case is that of a 3×3 game. (The dynamics of 2×3 games can be essentially reduced to that of a 2×2 game, with a normal direction added, see [vSS09, Theorem 1.5].) In a later section we will review some results on 3×3 games, and see that these are much more complicated than 2×2 games.

1.3 Convergence to Nash equilibria in the zero-sum case

If $B + A = 0$ then we have a so-called **zero-sum** game. It was shown in the 1950s by Robinson [Rob51] that then the differential inclusion (1.4) converges (albeit slowly) to the set of Nash equilibria. This situation corresponds to Case (2) in Figure 1.2.

Of course, matrices A, B for which $A + B \neq 0$, could have the same best responses $\mathscr{B}R_A$ and $\mathscr{B}R_B$ as matrices $\tilde{A}, \tilde{B}$ for which $\tilde{A} + \tilde{B} = 0$. For example, $(A,B) = \begin{pmatrix} (2,-1) & (1,0) \\ (1,0) & (2,-1) \end{pmatrix}$ and $(\tilde{A}, \tilde{B}) = \begin{pmatrix} (1,-1) & (0,0) \\ (0,0) & (1,-1) \end{pmatrix}$ have the same best responses. (Indeed, since p^B is a probability vector, $Ap^B = p^B + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \tilde{A}p^B + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and hence player A has for both games the same best-response; for player B the same holds because his matrix is the same for both games.) Because of this, one calls two matrices A, B zero-sum if and only if they induce the same best reponses as two matrices $\tilde{A}, \tilde{B}$ for which $\tilde{A} + \tilde{B} = 0$.

In the zero-sum case, it is easy to see that the motion (1.4) converges. Indeed, take

$$H(p^A, p^B) = \mathscr{B}R_A(p^B)\,A\,p^B - p^A\,A\,\mathscr{B}R_B(p^A).$$

Note that $\mathscr{B}R_A(p^B)\,A\,p^B \geq p^A\,A\,p^B \geq p^A\,A\,\mathscr{B}R_B(p^A)$. That is, $H \geq 0$ and $H(p^A, p^B) = 0$ iff (p^A, p^B) is a Nash equilibrium. Since $\mathscr{B}R_A$ and $\mathscr{B}R_B$ are piecewise constant, (1.2) implies

$$\begin{aligned}\frac{dH}{dt} &= \mathscr{B}R_A(p^B)\,A\,\frac{dp^B}{dt} - \frac{dp^A}{dt}\,A\,\mathscr{B}R_B(p^A)\\ &= \mathscr{B}R_A(p^B)\,A\,(\mathscr{B}R_B(p^A) - p^B) - (\mathscr{B}R_A(p^B) - p^A)\,A\,\mathscr{B}R_B(p^A) = -H.\end{aligned}$$

It follows that solutions go to the zero-set of H, i.e. to the set of Nash equilibria.

There are other examples for which it is shown that the game converges (for example in $2 \times n$ games see [Ber05], and games with some other special properties, see for example [Hah08], [MS96], [Ber07], [MR91], [Kri92]. However, for all those other cases the Nash equilibrium is on the boundary of the state space $\Sigma_A \times \Sigma_B$, and usually in those cases the Nash equilibrium is not unique and the flow does not have unique attractor. Therefore, following [Hof95], we would like to pose the following:

Conjecture 1. Assume that all solutions of (1.2) converge to a unique equilibrium. Then (1.2) is associated to a zero-sum game.

We would like to mention here that we have shown in [vS09] that for any zero sum game (with some non-degeneracy conditions), the motion (1.4) can be viewed as the product of the *Hamiltonian* motion $\frac{d\bar{p}}{dt} = \frac{\partial H}{\partial \bar{q}}, \frac{d\bar{q}}{dt} = -\frac{\partial H}{\partial \bar{p}}$ on $\Sigma_A \times \Sigma_B$ associated to the (Hamilton) function H and a motion towards the Nash equilibrium.

More precisely, because of the non-degeneracy conditions, the game has a unique Nash equilibrium $E = (E^A, E^B) \in \Sigma := \Sigma_A \times \Sigma_B$ and $H^{-1}(1)$ is the boundary of a ball around E. Moreover, there exists a continuous map $\pi\colon \Sigma \setminus \{E\} \to H^{-1}(1) \cap \Sigma$ so that $\pi(p^A, p^B) = \lambda(p^A, p^B)\cdot(E^A, E^B) + (1 - \lambda(p^A, p^B))\cdot(p^A, p^B)$ for some scalar $\lambda(x) > 0$ and $\pi(x) \in H^{-1}(1)$. (Take $\lambda(x) = 1 - 1/H(p^A, p^B)$.) So

$$(p^A, p^B) \mapsto (\pi(p^A, p^B), \lambda(p^A, p^B)) \in (H^{-1}(1) \cap \Sigma) \times \mathbb{R}^+$$

can be viewed as (higher dimensional) spherical coordinates around E. The dynamics (1.3) in these spherical coordinates $(\bar{p}^A, \bar{p}^B) = \pi(p^A, p^B), \lambda = \lambda(p^A, p^B)$ becomes

$$\begin{aligned}&\frac{d\bar{p}}{dt} = \frac{\partial H}{\partial \bar{q}}, \frac{d\bar{q}}{dt} = -\frac{\partial H}{\partial \bar{p}} \quad \textbf{on} \quad \Sigma_A \times \Sigma_B,\\ &\frac{d\lambda}{dt} = -\lambda.\end{aligned} \tag{1.5}$$

Of course, the Hamiltonian is not smooth. It is continuous and piecewise affine, and locally the flow is just a translation flow. However, as is shown in [vS09], the associated Hamiltonian flow is *unique and continuous*.

1.4 A family of (not necessarily non-zero) sum games containing Shapley's example displaying a periodic orbit

In the case of non-zero sum games, one certainly does not always convergence. Indeed, there is a famous example due to Shapley [Sha64] from the 1960s which shows that in general the evolution does NOT converge to a Nash equilibrium of the game. The Shapley example exhibits periodic behavior.

Indeed, take the family of 3×3 games

$$A_\beta = \begin{pmatrix} 1 & 0 & \beta \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix} \quad B_\beta = \begin{pmatrix} -\beta & 1 & 0 \\ 0 & -\beta & 1 \\ 1 & 0 & -\beta \end{pmatrix}, \tag{1.6}$$

which depend on a parameter $\beta \in \mathbb{R}$.

This family of examples was chosen in [SvSH08] because it contains Shapley's example (when $\beta = 0$) for which he had shown the existence of a periodic attractor. For $\beta = \sigma$, where $\sigma := (\sqrt{5}-1)/2 \approx 0.618$ is the golden mean, the game is equivalent to a zero-sume game (rescaling B to $\tilde{B} = \sigma(B-1)$ gives $A+\tilde{B} = 0$), so then play always converges to the interior equilibrium E^A, E^B as we have seen in the previous section. So varying $\beta \in [0,1)$ should reveal how this periodic orbit disappears. In fact, it reveals a lot more interesting behavior!

In the case of 3×3 games, Σ_A, Σ_B are both the set of probability vectors in $\mathbb{R}^3$ and so they are both a triangular simplex. So the state space is the product of two such triangular simpleces (i.e. topologically a ball in $\mathbb{R}^4$). Shapley's periodic orbit is drawn Figure 1.3. The periodic orbit, which lives in $\Sigma_A \times \Sigma_B$, spirals in a clockwise fashion when projected on each of the triangles Σ_A and Σ_B (where the corners are labelled as in Figure 1.6).

For the family of games (1.6), when $\beta = 0$, the sets where the players are indifferent to two or more strategies are marked in dotted lines in Figure 1.4. In [SvSH08] it was proven that for $\beta \in (0,\sigma)$ where $\sigma = (\sqrt{5}-1)/2$ there still exists a periodic attractor.

Throughout the remaing part of this survey we will only consider games coming from this family (1.6).

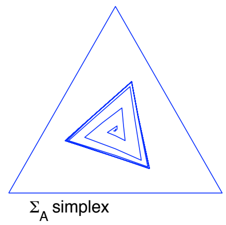

Fig. 1.3 Shapley's periodic orbit.

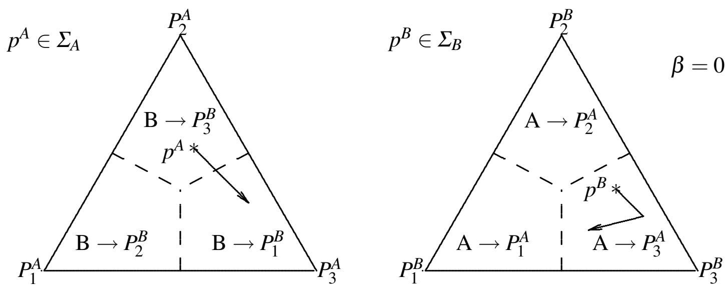

Fig. 1.4 The preferences of the players when $\beta = 0$.

1.5 Analysis of stationary points of flow: stable sets of the stationary point are extremely complicated

In this section we will give a more detailed description of some results about the dynamics associated to the above family of games and show that the situation is rather different than that of smooth dynamical systems.

Let $E^A = (E^B)^T = (1/3, 1/3, 1/3)$ and $E = (E^A, E^B)$. At this point, the players are indifferent between all three strategies (i.e. $\mathscr{B}R_A(E^B) = \Sigma_A$ and $\mathscr{B}R_B(E^A) = \Sigma_B$). So the right hand side of (1.2) includes the zero vectors, and E can be thought of as a stationary point of (1.2). (In fact, E is the only point in $\Sigma_A \times \Sigma_B$ where the right hand side of (1.2) can be zero.)

It seems reasonable to expect that we should be able to determine the local behavior near the Nash equilibrium E. It turns out that this question is rather subtle.

Theorem 1 (Continuity of flow).

- *For* $\beta \in (0,1)$*, the differential inclusion (1.2) has a unique continuous flow* outside E.
- *For* $\beta < (-1,0]$*, the differential inclusion (1.2) has a flow which is not continuous (in many places).*

This theorem is proved in [vSS09] and, as it turns out, when $\beta \neq \sigma$, orbits which start in E can choose to remain there or can leave this set. So E is not genuinely a stationary point.

Theorem 2 (Stable manifold of equilibrium is extremely complicated). *Consider $\beta \in (0, \sigma)$ and let ϕ_t be the flow of the fictitious play (1.2). Then the stable manifold*

$$W^s(E) = \{x \in E; \phi_t(x) \to E \text{ as } t \to \infty\}$$

of E is extremely complicated:

- *There exists a countable infinite number of polygons in $\Sigma \setminus E$ so that the cone with apex E over all these polygons is contained in the stable manifold of E.*
- *There exists an attracting periodic orbit γ in $\Sigma \setminus E$ (the continuation of Shapley's periodic orbit) and which also attracts points arbitrarily close to E.*

So the (local) stable manifold of the Nash equilibrium E is definitely not a full neighbourhood of E, but does contain a countable union of codimension-one sets. We believe that the stable manifold of E is equal to this set:

Conjecture 2. The stable manifold of E is a union of codimension-one sets (cones over certain polygons with apex E).

Question 1. Is the (local) stable manifold of E a closed set?

Usually, in smooth dynamical systems a stable manifold of a singular point is a manifold. Here the situation is rather more complicated, regardless whether the above conjecture is true or not.

The stable manifold of the attracting periodic orbit γ contains a neighbourhood of γ and points arbitrarily close to E, but definitely not a countable union of cones with apex E. So we would like to ask the following:

Question 2. Determine the global topology of the stable manifold of γ.

1.6 Bifurcations of periodic orbits of this family

Let us concentrate on some simple periodic orbits of the fictitious play (1.3) associated to the matrices (1.6). One way of describing an orbit is by a symbolic sequence, indicating the sequence of corners of $\Sigma_A \times \Sigma_B$ the solution is successively heading for. Indeed, note that the best response of A to any $p^B \neq E^B$ is either an integer $i \in \{1,2,3\}$ or a mixed strategy set $\bar{i}$ where $\bar{i} := \{1,2,3\} \setminus \{i\}$ corresponding to where player A is indifferent between two strategies but will **not** play i. Similarly for B. Hence one can associate to any orbit $(p^A(t), p^B(t))$ outside E, a sequence of times $t_0 := 0 < t_1 < t_2 < \dots$ and a sequence of best-response strategies $(i_0, j_0), (i_1, j_i), (i_2, j_2), \dots$ where

$$(i_n, j_n) = (BR^A(p^B(t), BR^B(p^A(t)))) \text{ for } t \in (t_n, t_{n+1})$$

with i_n and j_n equal to 1, 2, 3, $\bar{1}$, $\bar{2}$ or $\bar{3}$ for each $n = 0, 1, 2, \ldots$ and so that

$$(i_n, j_n) \neq (i_{n+1}, j_{n+1}) \text{ for all } n \geq 0$$

(i.e. the players really do switch strategy at time t_n).

So $(\bar{1},\bar{1}),(\bar{1},\bar{2}),(\bar{2},\bar{2}),(\bar{2},\bar{3}),(\bar{3},\bar{3}),(\bar{3},\bar{1})$ means that during the first leg of the orbit both players initially do *not* play strategy 1, and so the leg of the orbit lies in the set where the players are indifferent to strategy 2 and 3. During the 2nd leg of the orbit player A is still indifferent between 2 and 3, and player B between 1 and 3. Such an orbit lies on the dashed lines indicated in Figure 1.4 for the case when $\beta = 0$. For $\beta \in (0,1)$ the corresponding dashed lines will be tilted clockwise.

Shapley's orbit is of the following type $(1,2)$, $(2,2)$, $(2,3)$, $(3,3)$, $(3,1)$, $(1,1)$. So this periodic orbit, heads successively in six directions, and indeed is a hexagon. In the theorem below we describe the periodic orbits which form hexagons.

Theorem 3 (The existence and stability of simple periodic orbits). *There exists $\tau \in (\sigma, 1)$ (here $\tau \approx 0.915$ is a root of some polynomial of degree 6) with the following property.*

- *For $\beta \in (0, \sigma)$ the clockwise periodic Shapley orbit, which has symbolic sequence $(1,2)$, $(2,2)$, $(2,3)$, $(3,3)$, $(3,1)$, $(1,1)$, exists and is (locally) attracting.*
- *For $\beta \in (\sigma, 1)$ there exists another periodic orbit. We call this the anti-Shapley orbit, because it goes anticlockwise around the triangles and has symbolic sequence $(1,3),(1,2),(3,2),(3,1)$, $(2,1),(2,3)$. This orbit is of saddle-type when $\beta \in (\sigma, \tau)$ and attracting when $\beta \in (\tau, 1)$.*
- *For $\beta \in (\sigma, 1)$ there exists a third periodic orbit, called Γ, where both players choose mixed strategies $(\bar{1},\bar{1}),(\bar{1},\bar{2}),(\bar{2},\bar{2}),(\bar{2},\bar{3}),(\bar{3},\bar{3}),(\bar{3},\bar{1})$.*
- *This sequence of strategies corresponds to a fully-invariant set $C(\Gamma)$ (so an orbit starting in this set remains in this set, and an orbit starting outside this set remains outside this set); this fully invariant set exists for each $\beta \in (0,1)$ and contains a periodic orbit when $\beta \in (\sigma, 1)$.*

There are no other periodic orbits with a symbolic sequence of length at most six.

We would like to state the following:

Conjecture 3. There are no periodic orbits other than the Shapley orbit when $\beta \in (0, \sigma)$.

Conjecture 4. There are no periodic orbits attracting periodic orbits when $\beta \in (\sigma, \tau)$.

The bifurcation which occurs when $\beta = \sigma$ is somewhat reminiscent of that of a Hopf bifurcation, except that one has immediately complicated dynamics right after the bifurcation.

Theorem 4 (The bifurcation at $\beta = \sigma$). *At the bifurcation $\beta = \sigma$ the following happens:*

- *As $\beta \uparrow \sigma$, the Shapley orbit shrinks to E;*
- *When $\beta = \sigma$ the Nash equilibrium E is a global attractor;*
- *When $\beta > \sigma$ there exists infinitely many periodic orbits.*
- *When $\beta \downarrow \sigma$ all periodic orbits, including the anti-Shapley orbit and Γ shrink to E.*

At $\beta = \tau$ the anti-Shapley periodic orbit undergoes a non-generic periodic doubling bifurcation: at $\beta = \tau$ there exists a whole continuum of periodic orbits.

1.7 Random walk behavior

The dynamics is indeed much more complicated than one normally encounters.

Theorem 5 (The Hamiltonian flow acts like a 'random walk': an example in $\mathbb{R}^4$). *There exists a periodic orbit Γ (described in Theorem 3) with the following property: If one takes the first return map F to a section Z transversal to Γ (through some point $x \in \Gamma$), then for each $k \in \mathbb{N}$*

- *there exists a sequence of periodic points $x_n \in Z$ of exactly period k of the first return to Z accumulating to x;*
- *the first return map F to Z has infinite topological entropy.*
- *The dynamics acts as a random-walk. More precisely, there exist annuli A_n in Z (around $\Gamma \cap Z$ so that $\cup A_n \cup \{x\}$ is a neighbourhood of x in Z) shrinking geometrically to $\Gamma \cap Z$, so that for each sequence $n(i) \geq 0$ with $|n(i+1) - n(i)| \leq 1$ there exists a point $z \in Z$ so that $F^i(z) \in A_{n(i)}$ for all $i \geq 0$.*

One obvious consequence of the random walking described in the theorem, is the following unusual behavior. Take $\varepsilon > 0$ small and define the local and unstable stable set corresponding to rate τ as

$$W_\varepsilon^{s,\tau}(\Gamma) := \{x; \mathrm{dist}(\phi_t(x), \Gamma) \leq \varepsilon \text{ for all } t \geq 0 \text{ and}$$
$$\lim_{t\to\infty} \frac{1}{|t|} \log(\mathrm{dist}(\phi_t(x), \Gamma)) \to \tau\}$$

$$W_\varepsilon^{u,\tau}(\Gamma) := \{x; \mathrm{dist}(\phi_t(x), \Gamma) \leq \varepsilon \text{ for all } t \leq 0 \text{ and}$$
$$\lim_{t\to-\infty} \frac{1}{|t|} \log(\mathrm{dist}(\phi_t(x), \Gamma)) \to \tau\}.$$

Then the above system has for each $\varepsilon > 0$ and **each** $\tau \geq 0$ close enough to zero, that both $W_\varepsilon^{s,\tau}$ and $W_\varepsilon^{s,\tau}$ are non-empty in any neighbourhood of Γ.

The reason why one has such strange dynamics is that the first return map $P\colon Z \to Z$ near to Γ has a very special form. If we identify Z with $\mathbb{R}^2$ and $\Gamma \cap Z$ with $0 \in \mathbb{R}^2$ (by projecting using the projection π introduced in Section 1.3, then P is essentially a composition of maps of the form

$$P(x) = A \circ R_{1/||x||}(x).$$

Here $||(x_1, x_2)|| = |x_1| + |x_2|$ is the l_1 norm on $\mathbb{R}^2$, R_t is a rotation through angle t leaving the 'circles' in the l_1 norm invariant (i.e. $||R_t(x)|| = ||x||$) and A is a matrix of the form $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $0 < \lambda_1 < 1 < \lambda_2$.

1.8 Robustness

The above results do not require the matrices A and B to be of a special form, and hold for games corresponding to an open set of matrices:

Theorem 6 (Robustness). *For each $\beta \in (0,1)$ with $\beta \neq \sigma$, there exists $\varepsilon > 0$ so that for all 3×3 matrices A and B with*

$$||A - A_\beta||, ||B - B_\beta|| < \varepsilon$$

the previous theorems also hold.

1.9 Connection with other results on non-smooth dynamical systems

The transition map of (1.3) between hyperplanes are piecewise projective maps. In fact, as we show in [vS09], taking the appropriate induced flow, we get that the transition map is piecewise a translation. This connects this paper with an exciting body of work on piecewise isometries (with papers by R. Adler, P.Ashwin, M. Boshernitzan, A. Goetz, B. Kichens, T. Nowicki, A. Quas, C. Tresser and many others). Most of these paper deal with piecewise continuous maps, while the maps we encounter are continuous. Another loose connection of our work is to that of the huge and very active field of translation flows (associated to interval exchange transformations, translation surfaces and Teichmüller flows) (with recent papers by A. Avila, Y. Cheung, A. Eskin, G. Forni, P. Hubert, H. Masur, C. McMulen, M. Viana, J-C. Yoccoz, A. Zorich and many many others). But of course our flow does not act on a surface with a hyperbolic metric, and so this connection seems also rather remote. Finally, there is a growing literature on bifurcations on nonsmooth dynamical systems, mainly motivated by mechanical systems with 'dry friction', 'sliding', 'impact' and so on. As the number of workers in this field is enormous, we just refer to the recent survey of M. di Binardo et al [dBBC$^+$08] and the monograph by M. Kunze [Kun00]. Of course our paper is very much related to this work, although the motivation and the result seem to be of a different nature from what can be found in those papers.

1.10 Conclusion

We have seen that there is a lot of complicated behavior associated to fictitious play.

Of course an economist can say: people behave rationally and if they do not converge then they will notice this. So periodic behaviour and chaos is a mathematical curiosity. Perhaps this is not so clear though.

But this reminds us how chemists and other scientists have changed their approach. They used to be only interested in stationary processes. But now they realize that non-stationary processes are often more efficient, and certainly that they occur in a wide-range of important situations.

Moreover, perhaps it is to be expected that learning behaviour does not converge to equilibria except in somewhat exceptional cases, such as zero-sum games?

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