Existence of unique SRB-measures is typical for unimodal families
Existence unique d'une mesure SRB est typique pour familles unimodaux

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Abstract

We show that for a one-parameter family of unimodal polynomials $\{f_c\}$ with even critical order $\ell \geq 2$, for almost all parameters c, f_c admits a unique SRB-measure, being either absolutely continuous, or supported on the postcritical set. As a byproduct we prove that if f_c has a Cantor attractor, then it is uniquely ergodic on its postcritical set.

Nous montrons que si $\{f_c\}$ est une famille a un paramètre de polynômes unimodaux dont l'ordre $\ell \geq 2$ est pair, alors pour presque toute valeur du paramètre c, f_c admet une unique mesure SRB et soit cette mesure est absolument continue, soit son support est l'ensemble postcritique. Nous montrons aussi si f_c a un attracteur de Cantor, alors f_c est uniquement ergodique.

1 Introduction and Statement of Results

About 10 years ago, Jacob Palis conjectured that "most" dynamical systems have a finite number of metric attractors whose union of basins of attraction has total probability, and that each of these attractors either is a periodic

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orbit or supports a physical measure, i.e., a measure whose set of typical points has positive Lebesgue measure. The topological version of this conjecture was recently proved in the one-dimensional case: within the space of C^∞ one-dimensional maps, hyperbolic maps are dense, see [18] and [19]. This paper deals with 'Lebesgue most' parameters within a family of polynomial maps, and proposes a new strategy for proving a probabilistic version of the above conjecture.

Consider the family $f_c(x) = x^\ell + c$, where ℓ is an even positive integer. Let $\mathcal{M}$ denote the set of parameters c such that f_c has a connected Julia set. Then $\mathcal{M} \cap \mathbb{R}$ consists of the parameters $c \in \mathbb{R}$ for which f_c has a compact invariant interval, consisting of the (real) points not escaping to infinity. An f-invariant measure μ is called *physical* or *SRB* if its *basin*, i.e., the set $B(\mu)$ of points x such that for all continuous functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi d\mu,$$

has positive Lebesgue measure. A probability invariant measure which is absolutely continuous w.r.t. the Lebesgue measure is called an *acip*, and we say that a dynamical system $g\colon X \to X$ is *uniquely ergodic* if there is at most one probability measure on X which is invariant under g. There are many parameters $c \in \mathcal{M} \cap \mathbb{R}$ for which f_c has no physical measure, see [15] and also [26]. Our main theorem states that for Lebesgue almost all $c \in \mathcal{M} \cap \mathbb{R}$ there is a physical measure.

Theorem 1. *For Lebesgue a.e. $c \in \mathcal{M} \cap \mathbb{R}$, $f_c\colon \mathbb{R} \to \mathbb{R}$ has a unique physical measure μ. Moreover, either μ is an acip, or μ is supported on $\omega(0)$ and $f_c|\omega(0)$ is uniquely ergodic.*

The basin of the measure μ from the theorem, in fact, has full Lebesgue measure in the compact interval which is invariant under f_a.

It is well-known, see for example [26], that for *all* parameters, f_c has a unique metric attractor which is either a periodic orbit, or a finite union of intervals, or a Cantor set $\omega(0)$. In the last case, $\omega(0)$ is either of solenoidal type (the infinitely renormalizable case) or a "wild attractor" (which attracts a positive measure set of only first Baire category). We should emphasize that if in the above theorem $\mathrm{supp}(\mu) = \omega(0)$, then this need not imply that $\omega(0)$ is the metric attractor. It could, for example, happen that there is a conservative σ-finite acip $\tilde{\mu}$, such that Lebesgue a.e. x is typical for both μ

and $\tilde{\mu}$; yet these points visit any set A whose closure is disjoint from $\omega(0)$ with frequency 0.

For $\ell = 2$ a stronger result is known: for almost all $c \in \mathcal{M} \cap \mathbb{R}$, either f_c is Collet-Eckmann or f_c has a hyperbolic periodic attractor, see [21, 22, 3]. However, the geometry of orbits for $\ell = 2$ and $\ell > 2$ is completely different (for example, wild attractors exist only if ℓ is sufficiently large). For this reason several crucial steps of the proofs in those papers fail for the case $\ell > 2$. For this reason we use a new approach to this problem in this paper.

Decompose the set $\mathcal{M} \cap \mathbb{R}$ as the union of the following pairwise disjoint sets: $\mathcal{M} \cap \mathbb{R} = \mathcal{A} \cup \mathcal{F} \cup \mathcal{I}$, where

$$\begin{aligned}
\mathcal{A} &= \{c \in \mathcal{M} \cap \mathbb{R} : f_c \text{ has a periodic attractor}\},\\
\mathcal{F} &= \{c \in \mathcal{M} \cap \mathbb{R} \setminus \mathcal{A} : f_c \text{ is at most finitely renormalizable}\},\\
\mathcal{I} &= \{c \in \mathcal{M} \cap \mathbb{R} : f_c \text{ is infinitely renormalizable}\}.
\end{aligned}$$

In the first case, f_c has a SRB-measure supported on the periodic attractor and in the third case, it has a SRB-measure supported on the postcritical set $\omega_c(0)$. So we are only concerned in the second case.

Let us further decompose $\mathcal{F}$ as $\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}^n$, where $\mathcal{F}^n$ denotes the subset of $\mathcal{F}$ consisting of parameters c for which f_c is exactly n times renormalizable. Most of our effort will be put into the case $c \in \mathcal{F}^0$ as the finitely renormalizable case can be reduced to the non-renormalizable case. Let us use $\mathcal{F}^0_r$ to denote the subset of $\mathcal{F}^0$ consisting of parameters c for which f_c has a *recurrent* critical point. It is well known that the set of parameters $\mathcal{F}^0 \setminus \mathcal{F}^0_r$ has Lebesgue measure zero and by a classical result of Misiurewicz, f_c has an acip for any $c \in \mathcal{F}^0 \setminus \mathcal{F}^0_r$, see for example [26].

The case when f_c has a recurrent critical point is much more tricky. So let us say that an open interval I is *nice* if $f^n(\partial I) \cap I = \emptyset$ for all $n \geq 0$. An interval J is a called a *child* of I if it is a unimodal pullback of I, i.e., if there exists an interval J' containing the critical value c and an integer $s \geq 0$ so that $f^{s-1}\colon J' \to I$ is a homeomorphism and $J = f^{-1}(J') \ni 0$. If $c \in \mathcal{F}^0_r$ and there exists a nice interval $I \ni 0$ with infinitely many children, then we say that f_c is *reluctantly recurrent*; otherwise it is called *persistently recurrent.* Let us say that a parameter $c \in \mathcal{F}^0_r$ has *decaying geometry property* if either

- f_c is reluctantly recurrent, or
- f_c is persistently recurrent and there exists a sequence of nice intervals $\Gamma^0 \supset \Gamma^1 \supset \cdots \ni 0$ such that for each $n \geq 0$, Γ^{n+1} is the smallest child of Γ^n, and so that $|\Gamma^{n+1}|/|\Gamma^n| \to 0$ as $n \to \infty$.

Let $\mathcal{DG}$ denote the collection of parameters c for which f_c satisfies the decaying geometry condition. We should note that if $\ell = 2$, $\mathcal{F}_r^0 = \mathcal{DG}$ (and in fact, the decay is at least exponentially fast). To deal with parameters $c \in \mathcal{F}_r^0 \setminus \mathcal{DG}$ we first prove in Sections 2 and 3 the following.

Theorem 2. *If $c \in \mathcal{F}_r^0 \setminus \mathcal{DG}$ then $f_c|\omega_c(0)$ is uniquely ergodic. More precisely, if $f = f_c$ is persistently recurrent the following holds:*

- *If f has* low combinatorial complexity*:*
$$\sum_{n \geq 0} 1/\mathcal{G}_n = \infty \tag{1}$$
(where $\mathcal{G}_0, \mathcal{G}_1, \ldots$ are the positive integers associated to the chain $\Gamma^0 \supset \Gamma^1 \supset \cdots \ni 0$ defined in Section 2), then $f|\omega(0)$ is uniquely ergodic, i.e., there exists a unique f-invariant measure μ supported on $\omega(0)$, and either f has an acip or μ is the unique physical measure for f.

- *Assume that the critical point of f is recurrent, but f does* not *satisfy the decaying geometry property. Then $\liminf \mathcal{G}_n < \infty$ and so in particular f has low combinatorial complexity (1).*

To deal with the set $\mathcal{DG}$, we shall carry out a parameter exclusion argument in spite of the fact that $|\Gamma^{n+1}|/|\Gamma^n|$ need not decay exponentially.

For a subset A of a bounded interval I, and $\gamma \geq 1$,
$$Cap_\gamma(A, I) = \sup_h \frac{|h(A)|}{|h(I)|},$$
where h runs over all γ-quasisymmetric maps from I into $\mathbb{R}$. Moreover, let $\mathcal{DC}$ be the subset of $\mathcal{F}_r^0$ consisting of all the parameters c such that for any $\alpha > 0$ the following summability condition holds:
$$\sum_{n=0}^{\infty} \frac{1}{|Df_c^n(c)|^\alpha} < \infty. \tag{2}$$
By [8], for any $c \in \mathcal{DC}$, f_c has an acip which has decay of correlations faster than any polynomial rate.

Theorem 3. *The set $\mathcal{DC}$ has full Lebesgue measure in $\mathcal{F}_r^0 \cap \mathcal{DG}$. To describe the geometry of the set $\mathcal{DC}$ more precisely, for every $c \in \mathcal{DG}$ and every $\varepsilon > 0$ and $\gamma > 1$, there exists a neighborhood $J \ni c$, such that*

$$Cap_\gamma\left((J \setminus \mathcal{DC}), J\right) < \varepsilon.$$

To prove this theorem we shall follow the idea of [21, 3], which uses complex method in an essential way. The new ingredient here is a different strategy to obtain dilatation control of "pseudo-conjugacies". In quadratic case, such control was deduced from "linear growth of the principal moduli" which does not hold in our case (even for maps satisfying our decaying geometry condition). Instead, we shall prove in the case $c \in \mathcal{DG}$, that there exists a sequence of critical puzzle pieces for which the relative size of the first return domains is arbitrarily 'small', see Theorem 5. This result implies dilatation control for the pseudo-conjugacies by an argument used previously in [16, 32, 29].

Finally, we shall show in Section 3 that the geometry implied by the $\mathcal{DG}$ condition excludes existence of Cantor attractors. Therefore we have

Theorem 4. *If f has a Cantor attractor $\omega(0)$ (of solenoid type, or a "wild attractor"), then $f|\omega(0)$ is uniquely ergodic.*

1.1 Organization of the paper

Clearly Theorem 1 follows from Theorems 2 and 3. In Section 2 we show that if f is persistently recurrent and one has low combinatorial complexity, then $f|\omega(0)$ is uniquely ergodic, see Proposition 1. This is done by showing that certain transition matrices act as contractions on the projective Hilbert metric. In Section 3 we use real bounds to complete the proof of Theorem 2. The proof of Theorem 4 is also given in that section. The remainder of the paper is devoted to the proof of Theorem 3. In Section 4 we review how the combinatorics of Yoccoz puzzles changes with the parameter. In Section 5, we study the geometry of the Yoccoz puzzle for maps f_c with decaying geometry property, and prove Theorem 5. In Section 6, we convert this result to an estimate of the dilatation of pseudo-conjugacies. The proof of Theorem 3 will be given in Section 7.

2 A condition for unique ergodicity in the persistently recurrent case

In this section, let f be an arbitrary C^2 unimodal map with a non-flat critical point. We shall assume that the critical point is recurrent, but not periodic. The goal is to give a sufficient condition for $f|\omega(0)$ to be unique ergodic. So we shall assume that f is not renormalizable; if f is finitely renormalizable we pass to the "deepest" renormalization, whereas for infinitely renormalizable maps, $\omega(0)$ is an attractor and $f|\omega(0)$ is isomorphic to the adding machine (defined by "adding 1 and carry") on the space $\{(x_i)_{i=1}^{\infty} \mid x_1 \in \{0, \dots, p_1 - 1\},\ x_i \in \{0, \dots, \frac{p_i}{p_{i-1}} - 1\} \text{ for } i \geq 2\}$. Here p_i is the period of the i-th periodic interval. Such adding machines are well-known to be uniquely ergodic.

2.1 Construction of the nest of children

Recall that an open interval Γ is called *nice* if $f^n(\partial\Gamma) \cap \Gamma = \emptyset$ for all $n \geq 1$. For any nice interval $\Gamma \ni 0$, let $R_\Gamma : \Gamma \to \Gamma$ be the first return map; it has one central unimodal branch and in general infinitely many non-central branches. Let $\rho(I)$ be the collection of return domains of I that intersect $\omega(0)$. A *child* Γ' of Γ is a neighborhood of 0 such that there exists a neighborhood $\tilde{\Gamma}$ of $c_1 := f(0)$ such that $f^{-1}(\tilde{\Gamma}) = \Gamma'$ and $f^{s-1} : \tilde{\Gamma} \to \Gamma$ is monotone onto for some $s \geq 0$. The children of Γ are again nice, nested neighborhoods of 0. Each nice neighborhood has at least one, and if f is not renormalizable at least two children.

If f is *persistently recurrent* then (by definition) each nice neighborhood Γ of 0 has only finitely many children (cf. [35, 7]). Note that persistent recurrence of f implies that $\omega(0)$ is a minimal Cantor set. *Making this assumption,* let Γ^1 be the smallest child of Γ^0. Continue by induction, Γ^{n+1} being the smallest child of Γ^n. Let s_n be the iterate such that f^{s_n-1} maps a (one-sided) neighborhood $\tilde{\Gamma}^{n+1}$ of $f(\Gamma^{n+1})$ monotonically onto Γ^n.

Lemma 1. *If Γ^{n+1} is the smallest child of Γ^n, then for each $J \in \rho(\Gamma^n)$, there exists an iterate $t < s_n$ such that $f^t(\Gamma^{n+1}) \subset J$. In particular, the existence of a smallest child implies that $\#\rho(\Gamma^n) < \infty$.*

Proof. If J is the central domain, then $t = 0$ works. Take $J \in \rho(\Gamma^n)$ non-central, and let t' be minimal such that $f^{t'}(0) \in J$. Then there exists a neighborhood U of c_1 such that $f^{t'-1} : U \to J$ is monotone onto, and iterating some $\beta = \beta(J)$ steps more, U is mapped monotonically onto Γ^n. Therefore $f^{-1}(U)$ is a child of Γ^n. If $t' \geq s_n$, then this child is actually smaller than Γ^{n+1}, a contradiction. $\square$

2.2 Unique ergodicity

Let $I^0 := \Gamma^n$ be any interval in the chain of smallest children. Let I^1 be the central return domain of $R_{\Gamma^n} =: R_0$. This domain is again nice, so it has a central return domain I^2 under the return map $R_1 := R_{I^1} : I^1 \to I^1$. Continue by induction to construct the *principal nest* of Γ^n by defining I^{i+1} as the central return domain of the return map R_i to the previous central domain I^i. Then for some r, $I^r \supsetneq \Gamma^{n+1} \supset I^{r+1}$. For any $y \in \Gamma^n \cap \omega(0)$, the first landing map of y to Γ^{n+1} can be decomposed into return maps R_i. Indeed, write

$$R(z) = \begin{cases} R_0(z) & \text{if } z \in I^0 \setminus I^1; \\ R_{i-1}(z) & \text{if } z \in I^i \setminus I^{i+1} \text{ and } i \geq 1. \end{cases} \tag{3}$$

Let $k = k(y) \geq 0$ be such that $R^k(y)$ is the first landing of y into Γ^{n+1}, and for $0 \leq l \leq k$, write $\alpha_l(y) = i$ if $R^l(y) \in I^i \setminus I^{i+1}$. Define the *combinatorial complexity of* $y \in \Gamma^n$ to be

$$\mathcal{G}_n(y) = \#\{0 \leq l < k \mid \alpha_l(y) \leq \alpha_{l+1}(y)\}.$$

Note that $\mathcal{G}_n(y) \geq 1$, unless $y \in \Gamma^{n+1}$. As $\omega(0)$ is a minimal Cantor set, $k(y)$ is uniformly bounded for $y \in \omega(0) \cap \Gamma^n$. In particular, we have

$$\mathcal{G}_n := \sup_{y \in \omega(0) \cap \Gamma^n} \mathcal{G}_n(y) < \infty.$$

Since we assume that f is not renormalizable, $\mathcal{G}_n \geq 1$ for all n.

Proposition 1 (Non-unique ergodicity implies growing combinatorial complexity). *Let f be a persistently recurrent non-flat C^2 unimodal map such that $\sum_{n \geq 0} \frac{1}{\mathcal{G}_n} = \infty$, then $f|\omega(0)$ is uniquely ergodic.*

Proof. Abbreviate $\rho_n = \rho(\Gamma^n)$. Let y be any point in $\omega(0)$, and $J \in \rho_n$ for some n. Consider the visit frequency interval of y to J:

$$\gamma_n(J) = \left[\liminf_n \frac{1}{n} \#\{i < n \mid f^i(y) \in J\} \ , \ \limsup_n \frac{1}{n} \#\{i < n \mid f^i(y) \in J\} \right]$$

Unique ergodicity implies (and is actually equivalent to) $\gamma_n(J)$ being a point, and independent of y, for each $n \geq 0$ and $J \in \rho_n$.

We can express $\gamma_n(J)$ in terms of the $\gamma_{n+1}(\tilde{J})$'s for $\tilde{J} \in \rho_{n+1}$. Indeed, let A_n be the $\#\rho_n \times \#\rho_{n+1}$ matrix such that the entry $a_{J,\tilde{J}}$ of A_n indicates the number of visits of $\tilde{J}$ to J before $\tilde{J}$ returns to Γ^{n+1}. Then,

$$\gamma_n(J) \subset \frac{1}{N_n} \sum_{\tilde{J} \in \rho_{n+1}} a_{J,\tilde{J}} \gamma_{n+1}(\tilde{J}) := \left\{ \frac{1}{N_n} \sum_{\tilde{J} \in \rho_{n+1}} a_{J,\tilde{J}} z \mid z \in \gamma_{n+1}(\tilde{J}) \right\}$$

for some normalizing constant N_n. Write γ_n for the frequency vector $(\gamma_n(J) \mid J \in \rho_n)^t$. Then composing matrices A_n, we find

$$\gamma_n = \frac{1}{N_{n,m}} A_n \cdot A_{n+1} \cdots A_{m-1} \gamma_m.$$

Write $\mathcal{C}_n$ for the cone $(\mathbb{R}_{\geq 0})^{\#\rho_n}$. Disregarding the normalizing constants $N_{n,m}$, we find that γ_n is independent of y if and only if

$$\ell_n := \cap_{m>n} A_n \cdot A_{n+1} \cdots A_{m-1}(\mathcal{C}_m)$$

is a line, and in that case γ_n is the intersection of ℓ_n and the unit simplex in $\mathcal{C}_n$. Indeed, the visit frequency (and hence the measure) to any $J \in \rho_n$ and any $n \geq 0$ is determined independently of $y \in \omega(0)$. By Kolmogorov's extension theorem, this uniquely determines the measure μ.

Let us have a closer look at the matrices A_n. The first thing to notice is that A_n has strictly positive entries. This is a consequence of Lemma 1, and it is here that we effectively use the fact that Γ^{n+1} is the smallest child of Γ^n. More precisely, if the matrix A_n^+ records all the visits of $\tilde{J}$'s in ρ_{n+1} to J's in ρ_n *before* iterate s_n, then A_n^+ is already strictly positive. Moreover, for each $t < s_n$, $f^t(\Gamma^{n+1})$ intersects at most one return domain $J \in \rho_n$. Thus all columns of A_n^+ are identical. The differences of visits of the respective $\tilde{J}$'s occur only after the iterate s_n, and are recorded in the matrix $A_n^- = A_n - A_n^+$.

Lemma 2. *Each entry of A_n^- can be at most $2\mathcal{G}_n$ times the corresponding entry of A_n^+.*

Proof. Given $x \in \omega(0) \cap \Gamma^{n+1}$, write $y_0 = y = f^{s_n}(x)$ and $y_l = R^l(y)$, where R is as in equation (3). Abbreviate $\alpha_l = \alpha_l(y)$. For $i \geq 1$, $R|I^i = R_{i-1}|I^i$ is the central branch of the return map to I^{i-1}; let t_i be such that $R|I^i = f^{t_i}$.

Claim 1: If $\alpha_l = 0$, i.e., y_l belongs to a non-central domain $J \in \rho(I^0)$, and $R|J = f^t$, then $t \leq s_n$. Moreover, $1 = \#\{0 \leq i < t | f^i(y_l) \in J\} \leq \#\{0 \leq i < s_n | f^i(0) \in J\}$.

Proof: By Lemma 1, there exists $t' > 0$ such that $f^{t'}(0) \in J$, and hence $f^{s_n - t'}(J) \cap I^0 \neq \emptyset$. Therefore $t \leq s_n - t' < s_n$. The second statement of this claim follows because $R|J$ is the first return map to I^0.

Claim 2: Assume that there exist $l < l'$ such that

$$\alpha_l > \alpha_{l+1} > \cdots > \alpha_{l'}$$

then $R^{l'-l}(y_l) = f^t(y_l)$ for some $t \leq s_n$. A fortiori, $\#\{0 \leq i < t | f^i(y_l) \in J\} \leq \#\{0 \leq i < s_n | f^i(0) \in J\}$ for each $J \in \rho(I^0)$.

Proof: Since f is not renormalizable, there exists at least one non-central return domain J of I^0. Therefore there exists a maximal $s'_n < s_n$ and J such, and $f^{s'_n - 1}(\tilde{\Gamma}^{n+1}) = J$, where $\tilde{\Gamma}^{n+1}$ is the one-sided neighborhood of $f(\Gamma^{n+1})$ that maps onto I^0 under f^{s_n}. Since $I^{\alpha_l} \supset \Gamma^{n+1}$, $f^{s'_n}(I^{\alpha_l})$ contains at least one boundary point of J. But the forward orbit of ∂J is disjoint from the open interval I^0, and therefore the return time $t_{\alpha_l} \leq s'_n$. Furthermore, if $f^j(y_l) \in J$ for some $J \in \rho(I^0)$ and $j < t_{\alpha_l}$ while $f^j(\Gamma^{n+1}) \not\subset J$, then $f^j(I^{\alpha_l})$ contains a boundary point of J. This would contradict that $f^{t_{\alpha_l}}(I^{\alpha_l}) \subset I^{\alpha_l - 1}$. Therefore y_l and Γ^{n+1} visit the same return domains along the iterates $0 \leq j < t_{\alpha_l}$. This proves Claim 2 when $l' = l + 1$.

If $l' > l + 1$, then $f^{t_{\alpha_l}}(\Gamma^{n+1}) \subset f^{t_{\alpha_l}}(I^{\alpha_l}) \subset I^{\alpha_{l+1}}$. Hence we can repeat the argument for the iterates $t_{\alpha_l} \leq j < t_{\alpha_l} + t_{\alpha_{l+1}}$, etc.

In fact, the same argument also proves:

Claim 3: Assume that l is such that $0 < \alpha_l \leq \alpha_{l+1}$. Then $R(y_l) = f^t$ for some $t \leq s_n$ and $\#\{0 \leq i < t | f^i(y_l) \in J\} \leq \#\{0 \leq i < s_n | f^i(0) \in J\}$ for each $J \in \rho(I^0)$.

To prove the lemma, take any $x \in \tilde{J} \in \rho(\Gamma^{n+1})$, and decompose $\{0, \ldots, k\}$ into stings $l, l+1, \ldots, l'$ that satisfy the hypotheses of one of the thee above claims. If $\alpha_l \leq \alpha_{l+1}$, then Claim 1 or 3 holds for l, whereas for any maximal string $\alpha_l > \alpha_{l+1} > \cdots > \alpha_{l'}$, Claim 1 or 3 holds for l'. By definition of $\mathcal{G}_n$, there are at most $2\mathcal{G}_n$ such strings, and each such strings, $\#\{0 \leq i < t | f^i(y_l) \in J\} \leq \#\{0 \leq i < s_n | f^i(0) \in J\}$ for each $J \in \rho(\Gamma^n)$. Hence the $J, \tilde{J}$-entry in $A^+_{J,\tilde{J}} \leq 2\mathcal{G}_n A^-_{J,\tilde{J}}$ as asserted. $\square$

To conclude the proof, we will show that the matrices A_n act as contractions in the projective Hilbert metric. Given $v, w \in \mathcal{C}_{n+1}$, define this metric as

$$\Theta(v,w) = \log\left(\frac{\inf\{\mu \mid \mu v - w \in \mathcal{C}_{n+1}\}}{\sup\{\lambda \mid w - \lambda v \in \mathcal{C}_{n+1}\}}\right).$$

Let $A_n : \mathcal{C}_{n+1} \to \mathcal{C}_n$ be a linear map. It is shown in e.g. [4] that $\Theta(A_n v, A_n w) \leq \tanh(D/4)\Theta(v,w)$ for $D = \sup_{v',w' \in \mathcal{C}_{n+1}} \Theta(A_n v', A_n w')$. In particular, A_n is a contraction if A_n maps $\partial\mathcal{C}_{n+1} \setminus \{0\}$ into the interior of $\mathcal{C}_n$. By strict positivity of the A_n, this is true for all n.

Remark 1. A different way of regarding this Hilbert metric is the following, see Figure 1. The lines through 0 and v resp. w span a plane V, which contains the line connecting v and w. Let A and B be the intersections of this line with those coordinate axes that V intersects (A or B could be ∞). The points v, w, A and B bound an arc and divide it into three piece; call the middle piece j and the other pieces j and r. It is not hard to see that the ratio $\frac{\mu}{\lambda}$ equals the cross ratio $\frac{|l \cup j| \cdot |j \cup r|}{|l| \cdot |r|}$. Linear transformations preserve this cross-ratio, and the contraction is due to Schwartz inclusion of the image arc in the cone $\mathcal{C}_n$.

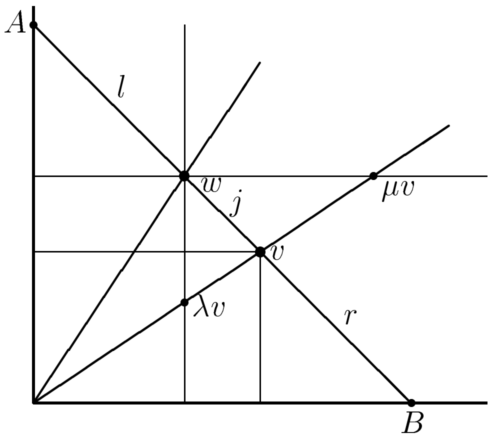

Figure 1: Illustration of the Hilbert metric.

To continue the calculation, in our case each column in A_n^- is at most $2\mathcal{G}_n$ times the corresponding column in A_n^+. Therefore, when comparing two columns a and b in A_n, we always find that $\frac{1}{1+2\mathcal{G}_n} a \leq b \leq (1 + 2\mathcal{G}_n)a$, element-wise. Therefore $\frac{\mu}{\lambda} \leq (1 + 2\mathcal{G}_n)^2$. The contraction factor therefore becomes

$$\begin{aligned}
\tanh(D/4) &= \frac{e^{\frac{1}{2}\log(1+2\mathcal{G}_n)} - e^{-\frac{1}{2}\log(1+2\mathcal{G}_n)}}{e^{\frac{1}{2}\log(1+2\mathcal{G}_n)} + e^{-\frac{1}{2}\log(1+2\mathcal{G}_n)}} \\
&= \frac{\sqrt{1+2\mathcal{G}_n} - \sqrt{\frac{1}{1+2\mathcal{G}_n}}}{\sqrt{1+2\mathcal{G}_n} + \sqrt{\frac{1}{1+2\mathcal{G}_n}}} \\
&= 1 - \frac{2}{(1+2\mathcal{G}_n)(1+\frac{1}{1+2\mathcal{G}_n})} \leq 1 - \frac{1}{1+2\mathcal{G}_n}.
\end{aligned}$$

Therefore ℓ_n is indeed a line if $\prod_{m\geq n}(1-\frac{1}{1+2\mathcal{G}_m}) = 0$, which is equivalent to $\sum_{n\geq 0}\frac{1}{\mathcal{G}_n} = \infty$.

We have shown now that for any n and $J \in \rho(\Gamma^n)$, the visit frequency interval $\gamma_n(J)$ is a point, and independent of the choice of y. Given $x \in \omega(0)$ and $n \geq 0$, there exists a minimal integer $t \geq 0$ such that $f^t(x) \in J \in \rho(\Gamma^n)$. Let $J_n(x)$ denote the pullback of J under f^t to x. Since f is assumed to be C^2 and therefore has no wandering intervals (see [26]), $\cap J_n(x) = \{x\}$. This shows that $f|\omega(0)$ is indeed uniquely ergodic. □

Remark 2. The consecutive visits of the $\tilde{J}$'s in ρ_{n+1} to $\tilde{J}$'s in ρ_n give a direct way to describe $f|\omega(0)$ as a substitution shift based on a chain of substitutions χ_n. The matrices A_n are the associated matrices of the substitutions χ_n, cf. [13, 6]. The proof of unique ergodicity then becomes almost identical to the one given in [6].

Remark 3. The proof of Proposition 1 can be applied to unicritical complex maps as well. In this case, Yoccoz puzzle pieces will take the role of nice intervals, see Section 4. However, since we have no analogue of the "no wandering interval" result from real dynamics, it is not true in all generality that $\cap_n J_n(x) = \{x\}$. Therefore, Proposition 1 can only be used to show that there is a unique invariant probability measure which is measurable with respect to the partition into atoms $\cap_n J_n(x)$, $x \in \omega(0)$.

Proposition 1 does generalize to the real multimodal case; for the definition of nice intervals and its children in the multimodal setting, we refer to [18].

2.3 SRB-measures

For this subsection, we allow f to be a multimodal interval map with non-flat critical points, with a finite set Crit of non-flat critical points. Assume also that f has only repelling periodic points. Such maps have no wandering intervals (cf. [26]). According to [5, 34], the Lebesgue measure has finitely many ergodic components, and the number of ergodic components is bounded by the number of critical points. For each ergodic component, the set of "typical points" E has positive Lebesgue measure, and satisfies exactly one of the following properties:

1. There exists $\varepsilon > 0$ such that for any $x \in E$

$$\limsup_{n\to\infty} \frac{1}{n} \#\{0 \le i < n \mid f^i(x) \notin B(\omega(\mathrm{Crit}), \varepsilon)\} > 0. \tag{4}$$

   In this case, there is an acip with E as set of typical points, see Proposition 2.

2. For all $\varepsilon > 0$ such that for any $x \in E$

$$\lim_{n\to\infty} \frac{1}{n} \#\{0 \le i < n \mid f^i(x) \in B(\omega(\mathrm{Crit}), \varepsilon)\} = 1. \tag{5}$$

   In this case, any possible physical measure is supported in $\omega(\mathrm{Crit})$.

In the next two propositions, we prove that a physical measure μ is either absolutely continuous, or supported on $\omega(\mathrm{Crit})$. In the unimodal setting, Hofbauer and Keller [15] proved the stronger statement that μ is either absolutely continuous, or contained in the convex hull of the weak accumulation points of $\frac{1}{n}\sum_n \delta_{f^i(0)}$, the averages of Dirac measures along the critical orbit. We will prove the weaker statement for multimodal maps without the use of Markov extension arguments.

Proposition 2. *Let f be a C^3 multimodal map with only repelling periodic points, having an ergodic component such that (4) holds for its set E of typical points. Then f has an acip μ, and supp(μ) is a finite union of intervals.*

Proof. Take $\varepsilon > 0$ such that (4) holds. Since f has no wandering intervals, $\cup_n f^{-n}(\mathrm{Crit})$ is dense. For some large $M > 0$, take N so large that $P :=$

$\cup_{n\le N} f^{-n}(\mathrm{Crit})$ is an $\frac{\varepsilon}{2M}$-spanning set. For at least one component J of $I \setminus P$, $\mu(J) > 0$ with $J \cap B(\omega(\mathrm{Crit}); \varepsilon) = \emptyset$ we have

$$\limsup_n \frac{1}{n} \#\{0 \le i < n \mid f^i(x) \in J\} =: \eta > 0$$

for m-a.e. $x \in E$. By construction of P, $f^n(\partial J) \cap J = \emptyset$ for all $n \ge 0$. Therefore, the first return map $F : J \to J$ has only monotone onto branches and each branch $F_i \colon J_i \to J$ can be extended to a diffeomorphism $\hat{F}_i \colon \hat{J}_i \to \hat{J}$ where $\hat{J}$ is a M-neighborhood $\hat{J}$ of J. Because we are assuming that all periodic points are repelling, by Theorem C.2 in [34], F_i has bounded distortion. (The argument for this goes as follows: take a neighborhood U of the critical point, let $F_i = f^{n_i}$ and let $s_i < n_i$ be the last visit of J_i to U. By Kozlovski's theorem [17] (or its multimodal version in [34]), f^{s_i+1} has negative Schwarzian, and by Mañé, $f^{n_i - s_i - 1}$ has bounded distortion. Combined this gives the required statement.) In particular, $|DF(x)|$ is uniformly bounded away from 1. By a telescoping argument, we can derive that the distortion of all branches of all iterates of F are bounded uniformly as well.

Let $J_0 \subset J$ be the set of points on which F^k is defined for all k. Then J_0 is forward invariant under F and $m(J_0) > 0$. The Folklore Theorem [23] gives an F-invariant absolutely continuous probability measure, say ν, such that $\nu(J_0) = 1$, and $\frac{d\nu}{dm}$ is bounded and bounded away from 0. From this it easily follows that $\mathrm{supp}(\nu) = \overline{J_0}$.

For $x \in J_0$, define the return time $\tau(x) > 0$ such that $F(x) = f^{\tau(x)}(x)$, and let $\tau_N(x) = \min\{N, \tau(x)\}$. Then $\tau_N \in L^1(\nu)$, and by Birkhoff's Ergodic Theorem, ν-a.e. $x \in J_0$ satisfies

$$\int \tau_N \; d\nu = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau_N(F^i(x)) \le \liminf_n \frac{1}{n} \sum_{i=0}^{n-1} \tau(F^i(x)) = \frac{1}{\eta} < \infty.$$

This shows that $\eta_0 := \int \tau d\nu < \infty$. Therefore we can pullback ν to obtain an absolutely continuous f-invariant probability

$$\mu(A) = \frac{1}{\eta_0} \sum_k \sum_{i=0}^{k-1} \nu(f^{-i}(A) \cap \{\tau = k\}).$$

The support of μ is the forward orbit of $J \subset \overline{E}$. Since E contains no non-repelling periodic orbit and there are no wandering intervals, $\mathrm{supp}(\mu)$ is a finite union of compact intervals. $\square$

Since we proved Proposition 1 only for unimodal maps, we will state the next result in this case. The multimodal version holds just as well.

Proposition 3. *Suppose that a C^2 non-renormalizable unimodal map f has small combinatorial complexity, i.e., $\sum_n 1/\mathcal{G}_n < \infty$. Suppose also that (5) holds for m-a.e. $x \in [f^2(0), f(0)]$. Then f has a unique physical measure supported on $\omega(0)$.*

Proof. Condition (5) only implies that for Lebesgue a.e. x, any accumulation point of Cesaro means of Dirac measures $\sum_{i=0}^{n-1} \delta_{f^i(x)}$ is an invariant measure supported on $\omega(0)$. But by Proposition 1, $f|\omega(0)$ is uniquely ergodic. Therefore the invariant measure on $\omega(0)$ is physical. □

Remark 4. For C^2 non-flat multimodal maps with all periodic points repelling, compact forward invariant sets that are disjoint from Crit, are hyperbolic and have 0 Lebesgue measure. Therefore each physical measure contains at least one critical point in its support. It follows from [34, Theorem E] that any critical point interior to the support of an acip cannot be in the support of another physical measure. For singular physical measures, this is not true; it is possible, for example, to construct a bimodal map on $[0, 1]$ with two Cantor attractors, such that the basins of both attractors are dense in $[0, 1]$.

3 No decaying geometry implies low combinatorial complexity

Throughout this section we consider a map $f = f_c$. For any interval I, let αI denote the interval of length $\alpha|I|$ that is concentric with I.

Let I be a nice interval. Let us denote the first entry domain to I containing x by $\mathcal{L}_x(I)$. The interval I is called δ*-nice*, if for each $x \in I \cap \omega(0)$ we have $(1 + 2\delta)\mathcal{L}_x(I) \subset I$.

Lemma 3. *There exists $\delta > 0$ such that if $I \ni 0$ is a nice interval with a* non-central return *(i.e., with $R_I(0) \notin \mathcal{L}_0(I)$) then*

$$(1 + 2\delta)\mathcal{L}_0^2(I) \subset \mathcal{L}_0(I).$$

Moreover, for each $\varepsilon > 0$ there exists $\eta > 0$ such that if $|\mathcal{L}_0(I)| \geq (1 - \eta)|I|$ then

$$|\mathcal{L}_0^2(I)| \leq \varepsilon|\mathcal{L}_0(I)|.$$

Proof. See [24] as well as [34]. $\square$

Lemma 4. *For any N and $\rho > 0$, there exists $\rho' > 0$ such that if J is a pullback of a nice interval I with order bounded by N, and if $(1+2\rho)J \subset I$, then J is a ρ'-nice interval. Moreover, $\rho' \to \infty$ as $\rho \to \infty$.*

Proof. See Lemma 9.7 in [18]. $\square$

Lemma 5. *For any $\rho > 0$, $\delta > 0$ there exists $r > 0$ such that if I is a δ-nice interval and $K_1 \supsetneq K_2 \supsetneq \dots$ are children of I, then $I \supset (1+2\rho)K_i$ for $i \geq r$.*

Proof. For each $i \geq 1$ there exists $s_i \in \mathbb{N}$ such that $f^{s_i - 1}$ maps a one-side neighborhood T_i of $f(K_i)$ onto I. Clearly, $f^{s_i}(K_{i+1})$ is contained in a return domain of I. By the real Koebe principle, T_i contains a definite neighborhood of fK_{i+1} and hence K_i contains a definite neighborhood of K_{i+1}. The lemma follows. $\square$

Lemma 6. *For each $\rho > 0$ and $\delta > 0$, there exists $N = (\rho, \delta)$ with the following property. Let I be a δ-nice interval and let Γ be its smallest child. Let $I := I^0 \supset I^1 \supset I^2 \dots$ the principal nest corresponding to I, i.e., $I^i = \mathcal{L}_0(I^{i-1})$ for $i \geq 1$ and let m be a positive integer such that $R_{I^i}(0) = R_{I^0}(0)$ for $i = 0, \dots, m-1$. If there exists $N' \geq N$ and $z \in \omega(0)$ such that $R_I^j(z) \in (I \setminus I^m)$ for $j = 0, \dots, N'$ and at least N of these points are in $I \setminus I^1$, then $(1+2\rho)\Gamma \subset I$.*

Proof. Let us show that I has at least N children. Write $R := R_I$ and let $n_1 < n_2 < \dots < n_N \leq N'$ be so that $R^{n_i}(z) \in I \setminus I^1$. Since $z, \dots, R^{N'}(z) \notin (I \setminus I^m)$, R^{n_i+1} maps a neighborhood J_i of z diffeomorphically onto I. (Here we use that R maps a component of $I^i \setminus I^{i+1}$, $1 \leq i \leq m-1$, diffeomorphically onto a component of $I^{i-1} \setminus I^i$.) It follows that $K_i := \mathcal{L}_0(J_i)$ is a child of I. Since $J_1 \supsetneq J_2 \supsetneq \dots \supsetneq J_N$ we also have $K_1 \supsetneq K_2 \supsetneq \dots \supsetneq K_N$, i.e., I has at least N children. So if we let r be the integer associated to δ and ρ from Lemma 5 then the conclusion of the lemma holds if $N \geq r$. $\square$

Proposition 4. *Assume that f is non-renormalizable and persistently recurrent. Let $\Gamma^0 \supset \Gamma^1 \supset \dots \ni 0$ be a sequence of nice intervals as in Section 2. For each $\rho > 0$ there exists C so that for any $n \geq 2$, if the combinatorial complexity $\mathcal{G}_n \geq C$, then Γ^{n+2} is ρ-nice.*

Proof. By Lemma 4, there exists $\tau = \tau(\rho) > 0$ such that Γ^{n+2} is ρ-nice if $|\Gamma^{n+2}|/|\Gamma^n| < \tau$ since Γ^{n+2} is a pull back of Γ^n of order 2. Assuming that $|\Gamma^{n+2}|/|\Gamma^n| \geq \tau$, let us show that $\mathcal{G}_n$ cannot be too large.

Let $I^0 = \Gamma^n$ and let I^i be the corresponding principal nest. Let $i(0) = 0$ and $i(1) < i(2) < \cdots$ be all the positive integers such that $R_{i(j)-1}(0) \notin I^{i(j)}$. Choose r so that $I^{r+1} \subset \Gamma^{n+1} \subsetneq I^r$, and let q be maximal with $i(q) \leq r$. By Lemma 3, q is bounded from above by a constant $q(\tau)$.

Claim. There exists $\delta = \delta(\tau) > 0$ such that $I^{i(j)}$ is a δ-nice interval for all $0 \leq j \leq q$.

First let us consider the case $1 \leq j \leq q$. As $I^{i(j)+1} \supset \Gamma^{n+2}$ we have $|I^{i(j)+1}|/|I^{i(j)}| \geq \tau$, which implies by the second statement of Lemma 3 that $|I^{i(j)}|/|I^{i(j)-1}|$ is bounded away from 1. By Lemma 4, there exists $\delta = \delta(\tau) > 0$ such that $I^{i(j)}$ is δ-nice. Now let us consider the case $j = 0$. Again by Lemma 4, it suffices to show that $I^0 = \Gamma^n$ is well inside Γ^{n-2}. To see this, let $\hat{I}^0 = \Gamma^{n-2}$ and for $i \geq 1$, $\hat{I}^i = \mathcal{L}_0(\hat{I}^i)$, and let $m \geq 1$ be minimal such that $R_{\hat{I}^{m-1}}(0) \notin \hat{I}^m$. As $\Gamma^{n-2} \cap \omega(0)$ contains a point outside $\hat{I}^1$, we have $\hat{I}^m \subset \Gamma^{n-1}$, and hence $\hat{I}^{m+1} \supset \Gamma^n$. By the first statement of Lemma 3, it follows that

$$(1 + 2\delta_0)\Gamma^n \subset (1 + 2\delta_0)\hat{I}^{m+1} \subset \hat{I}^m \subset \Gamma^{n-2}.$$

This completes the proof of the claim.

Now let $N = N(\tau^{-1}, \delta)$ be as in Lemma 6. Let us show that $\mathcal{G}_n \leq N^{q+1}$. To this end, let $y \in \omega(0) \cap \Gamma^n$ be such that $\mathcal{G}_n(y) = \mathcal{G}_n \geq C$ and let $s \geq 0$ be minimal such that $f^s(y) \in \Gamma^{n+1}$. Note that if $0 < i < r$ and I^i is central, i.e., $R_{I^{i-1}}(0) \in I^i$, then $R_{I^{i-1}}$ maps $I^i \setminus I^{i+1}$ into $I^{i-1} \setminus I^i$, so in the definition of combinatorial complexity, visits to $I^i \setminus I^{i+1}$ do not contribute to $\mathcal{G}_n$. Therefore

$$\#\{0 \leq k < s : f^k(y) \in \bigcup_{j=0}^{q} I^{i(j)} \setminus I^{i(j)+1}\} \geq \mathcal{G}_n.$$

For $j \geq 0$, let

$$\nu(j) = \#\{0 \leq k < s : f^k(x) \in I^{i(j)} \setminus I^{i(j)+1}\}.$$

Note that $\nu(q+1) = 0$.

Let us show that for any $0 \leq j \leq q$, $\nu(j) \leq (\nu(j+1)+1)(N-1)$. Indeed, otherwise, there exists $0 \leq s' < s$ such that the orbit $\{f^k(y)\}_{k=s'}^{s}$ visits $I^{i(j)} \setminus I^{i(j)+1}$ at least N times before it enters $I^{i(j+1)}$. By Lemma 6, this, together with the claim above, implies that if K_j is the last child of $I^{i(j)}$ then

$(1+2\tau^{-1})K_j \subset I^{i(j)}$. Noticing $I^{i(j)} \supset \Gamma^{n+1}$, we have $K_j \supset \Gamma^{n+2}$. Therefore $(1+2\tau^{-1})\Gamma^{n+2} \subset I^{i(j)} \subset \Gamma^n$, contradicting the hypothesis $|\Gamma^{n+2}|/|\Gamma^n| \geq \tau$.

It follows that $\nu(j) \leq N^{q-j+1} - N^{q-j}$ for all $0 \leq j \leq q$. So $\mathcal{G}_n \leq \sum_{i=0}^{q} \nu(j) \leq N^{q+1}$. $\square$

Proof of Theorem 2. The first part of Theorem 2 follows from Propositions 1 and 3, and the second part from the previous proposition. $\square$

Proof of Theorem 4. Assuming that f is not uniquely ergodic on $\omega(0)$, let us show that f has no Cantor attractor. For the reason explained at the beginning of Section 2, we may assume that f is non-renormalizable. Moreover, we may assume that f is persistently recurrent as this is a necessary condition for the existence of Cantor attractors, see [20, 7].

By Proposition 1, the combinatorial complexity $\mathcal{G}_n$ tends to ∞ as $n \to \infty$. Furthermore, Proposition 4 states that for n sufficiently large, Γ^n is ρ-nice, and hence $(1+2\rho)\mathcal{L}_0(\Gamma^n) \subset \Gamma^n$ for ρ large.

To prove non-existence of Cantor-attractors, we will use a by now standard random walk argument on an induced map, see e.g. [9]. Let us first define the inducing scheme: Let $R_n : \Gamma^n \setminus \mathcal{L}_0(\Gamma^n) \to \Gamma^n$ be the first return to Γ^n. Recall that for each $n \geq 1$, there exists s_n such that $f^{s_n - 1}$ maps a one-sided neighborhood of $f(\Gamma^n)$ monotonically onto Γ^{n-1}. Let j be minimal such that $R_{n-1}^j \circ f^{s_n}|\Gamma_n$ has a branch whose image intersects $\mathcal{L}_0(\Gamma^{n-1})$ but is properly contained in Γ^{n-1}. This branch is part of the central branch of f^{s_n+t} for some $t \geq 0$. Let V_n be the maximal neighborhood of 0 such that $f^{s_n+t}(\partial V) \subset \partial\mathcal{L}_0(\Gamma^{n-1})$ and $f^{s_n+t-1}|f(V)$ is monotone. Then $V_n \supset \Gamma^{n+1}$, because otherwise $f^{s_n+t}(\Gamma^{n+1})$ contains a boundary point of $\mathcal{L}_0(\Gamma^{n-1})$. Moreover, the central branch of f^{s_n+t} covers a boundary point of Γ^{n-1}. Because $\mathcal{L}_0(\Gamma^{n-1})$ lies deep inside Γ^{n-1}, V_n lies deep inside Γ^n.

Now we define R. For $x \in V_n \setminus \Gamma^{n+1}$, let $R(x)$ be the first return map to Γ^{n+1}. Hence $R|V_n \setminus \Gamma^{n+1}$ has (countably many) branches onto Γ^{n+1}.

For $x \in \Gamma^n \setminus V_n$, let $R(x) = R_{n-1}^{j(x)} \circ f^{s_n}(x)$ where $j(x) \geq 0$ is minimal such that there is a neighborhood U_x such that $R_{n-1}^{j(x)} \circ f^{s_n}$ maps U_x monotonically onto Γ^{n-1}. Obviously, $j(y)$ and U_y are the same for all $y \in U_x$.

Using this definition for all n, we find that R is defined Lebesgue a.e., and it is a Markov induced map preserving the partition generated by the intervals Γ^n.

To describe the random walk, let $\alpha_k = n$ if $R^k(x) \in \Gamma^n \setminus \Gamma^{n+1}$. The α_k can be considered as random variable which satisfy the conditional probabilities

$$\frac{m(\alpha_k = n \text{ and } \alpha_{k+1} = n-1)}{m(\alpha_k = n)} \geq 1 - \mathcal{O}(\rho'),$$

and

$$\frac{m(\alpha_k = n \text{ and } \alpha_{k+1} = n+r)}{m(\alpha_k = n)} \leq \mathcal{O}(|\Gamma^{n+r}|/|\Gamma^n|),$$

which decreases at least exponentially fast in r. Therefore, provided C and hence ρ' are sufficiently large, the drift of the random walk is

$$\mathbb{E}(\alpha_{k+1}|\alpha_k = n) = \sum_{r \geq -1} r \ \frac{m(\alpha_k = n \text{ and } \alpha_{k+1} = n+r)}{m(\alpha_k = n)} \leq -\frac{1}{2},$$

for n sufficiently large. A similar computation shows that the variance is bounded as well. Hence we can apply the random walk argument from [9] to conclude that $\liminf \alpha_k < \infty$ for Lebesgue a.e. x, excluding the existence of a Cantor attractor. $\square$

4 Yoccoz puzzle

Let us consider the family $f_c = z^\ell + c$ parametrized by $c \in \mathbb{C}$. By definition, the filled Julia set K_c of f_c is the completion of the open set

$$A_c(\infty) = \{z \in \mathbb{C} : f_c^n(z) \to \infty \text{ as } n \to \infty\},$$

which is the attracting basin of infinity. The Green function

$$G_c : \mathbb{C} \to \mathbb{R}_+ = \{t \geq 0\}, z \mapsto \lim_n \frac{1}{\ell^n} \log^+ |f_c^n(z)|,$$

is a subharmonic function vanishing exactly on the filled Julia set K_c. The classical Böttcher Theorem provides us a unique conformal representation

$$B_c : \{z : G_c(z) > G_c(0)\} \to \{z||z| > r_c\}, \text{ where } \log r_c = G_c(0),$$

which satisfies $B_c'(\infty) = 1$ and $B_c \circ f_c = (B_c)^\ell$.

The Green function is equal to $\log|B_c|$ on the domain of B_c. The level curve $\{G_c(z) = r\}, r > 0$ is called the *equipotential curve of level* r, and denoted by $E_c(r)$. The *external ray of angle* $t \in \mathbb{R}/\mathbb{Z}$ is the gradient curve of G_c stemming from infinity with the angle t (measured via the Böttcher coordinate B_c), and denoted by $R_c(t)$. When c is contained in the *Multibrot set*

$$\mathcal{M} = \{c \in \mathbb{C} : K_c \text{ is connected}\},$$

the map B_c is defined in the whole complement $A_c(\infty)$ of the filled Julia set K_c, and so $R_c(t) = B_c^{-1}(\{re^{2\pi it} : r > 1\})$. In this case, any external ray $R_c(t)$ with t rational has a well defined landing point $\lim_{r\to 1^+} B_c^{-1}(re^{2\pi it})$ which is contained in the Julia set ∂K_c; vice versa, a repelling or parabolic point is the common landing point of finitely many external rays with rational angle. When K_c is disconnected, provided that $\arg B_c(c) \neq \ell^k t$ for all $k \geq 1$, the external ray $R_c(t)$ is still a smooth curve joining infinity and ∂K_c, so each point in $R_c(t)$ has a well define potential.

For every $c \in \mathbb{C}$, the domain of B_c contains the critical value c of f_c so that $B_c(c)$ is well defined. By [11], the set $\mathcal{M}$ is connected and the map $\Phi(c) = B_c(c)$ defines a conformal map from $\mathbb{C} \setminus \mathcal{M}$ onto $\mathbb{C} \setminus \overline{\mathbb{D}}$. As in the dynamical plane, the *parameter (external) ray* of angle $t \in \mathbb{R}/\mathbb{Z}$ is the set

$$\mathcal{R}(t) = \Phi^{-1}(\{re^{2\pi it} : r > 1\}),$$

and the *equipotential* of level $r > 0$ is the closed curve

$$\mathcal{E}(r) = \{c \in \mathbb{C} \setminus \mathcal{M} : \log|\Phi(c)| = r\}.$$

Let $\mathcal{H}$ denote the component of the interior of $\mathcal{M}$ which contains 0. This is the region where f_c has an attracting fixed point. For $c_0 \in (\mathcal{M} \setminus \mathcal{H}) \cap \mathbb{R}$, f_{c_0} has an orientation fixed point α_{c_0} in $\mathbb{R}$. There exist exactly two external rays $R_{c_0}(t^-)$, $R_{c_0}(t^+)$ landing at α_{c_0}, see Lemma 5.2 in [18]. These two external rays are symmetric to each other with respect to the real axis, and permuted by f_c:

$$\ell t^- = t^+, \ell t^- = t^+, \quad \text{mod } 1.$$

Arguing as in Theorem 2.1 in [27], the corresponding dynamical rays $\mathcal{R}(t^-)$ and $\mathcal{R}(t^+)$ land at a common point $\gamma \in \mathbb{R}$. The configuration $\mathcal{R}(t^-) \cup \mathcal{R}(t^+) \cup \{\gamma\}$ cuts the parameter plane into two connected components, and we use $\mathcal{W}$ to denote the one which does not contain 0 (the 1/2-wake). The set

$\mathcal{W}$ consists of all c for which f_c has a repelling fixed point α_c at which the external rays $R_c(t^+)$ and $R_c(t^-)$ land. In particular,

$$\mathcal{W} \supset (\mathcal{M} \setminus \mathcal{H}) \cap \mathbb{R} \ni c_0.$$

4.1 Yoccoz puzzle

Now let us recall the definition of Yoccoz puzzle for $c \in \mathcal{W}$. Let $X_c^n = \{z \in \mathbb{C} : G_c(z) < 1/\ell^n\}$. By definition, the *Yoccoz puzzle* of f_c is the following sequence of graphs:

$$S_c^0 = \partial X_c^0 \cup \left(X_c^0 \cap \bigcup_{t \in \{t^+, t^-\}} \overline{R_c(t)} \right)$$
$$S_c^n = f_c^{-n} S_c^0,\ n = 1, 2, \ldots .$$

A component of $X_c^n \setminus S_c^n = f_c^{-n}(X_c^0 \setminus S_c^0)$ will be called *a puzzle piece of depth n*. A puzzle piece of depth n which contains a point z will be denoted by $P_c^n(z)$.

Definition 1. Let $m > n \geq 0$ be integers. We say that $P_c^m(0)$ is a *child* of $P_c^n(0)$ if $f^{m-n-1} : P_c^{m-1}(c) \to P_c^n(0)$ is a conformal map.

Lemma 7. *Assume that $c \in \mathcal{W} \cap \mathbb{R}$ is such that f_c is non-renormalizable. Then $P_c^2(c) \Subset P_c^1(c)$.*

Proof. Otherwise, $P_2(c)$ contains $-\alpha_c$ in its closure. As $c \in \mathbb{R}$, this implies that $P_2(0) \cap \mathbb{R}$ is a periodic interval of period 2, contradicting that f_c is non-renormalizable. $\square$

4.2 First return maps

Consider a map $f = f_c$ with $c \in \mathcal{W}$. Let V be a puzzle piece which contains 0. Let $D(V) = \{z \in \mathbb{C} : \exists k \geq 1 \text{ such that } f^k(z) \in V\}$. The first return map g_V is defined as follows: for each $z \in D(V) \cap V$, if $k \geq 1$ is the return time of z to V, i.e., the minimal $k \geq 1$ such that $f^k(z) \in V$, then $g_V(z) = f^k(z)$. It is well-known that the return time is constant on each component P of $D(V) \cap V$ and that $g_V|P$ is conformal if $P \not\ni 0$ and ℓ-to-1 otherwise. If $0 \in D(V)$, and V is *strictly nice: $f^k(\partial V) \cap \overline{V} = \emptyset$* for all $k \geq 1$, then the first return map g_V is an R-map as defined below.

Definition 2. Let $V, U_j, j = 0, 1, \ldots$ be Jordan disks such that such that the $\overline{U_j}$ are pairwise disjoint and contained in U. A holomorphic map $g : \bigcup_{j=0}^{\infty} U_j \to U$ is called an *R-map* (where "R" stands for "return") if the following hold:

- $g : U_0 \to V$ is an ℓ-to-1 proper map with a unique critical point at 0,
- for all $i \geq 1$, $g : U_j \to V$ is conformal.

The renormalization $\mathcal{L}g$ is, by definition, the first return map of g to U_0, which is again an R-map provided that $g^k(0) \in U_0$ for some $k \geq 1$.

The following is a lemma which we shall need later.

For an R-map $g : \bigcup_i U_i \to V$ define

$$\mathrm{mod}(g) = \mathrm{mod}(V \setminus \overline{U}_0),\ \ \mathrm{mod}'(g) = \inf\{\mathrm{mod}(V \setminus \overline{U_i}), i \geq 1\}.$$

Lemma 8. *Let $g : \bigcup U_i \to V$ be an R-map. Let W be a return domain to U_0 (under g) such that $\mathcal{L}g|W = g^s|W$. Then*

$$\mathrm{mod}(U_0 \setminus W) \geq \frac{1}{\ell}((s-1)\mathrm{mod}'(g) + \mathrm{mod}(g)). \tag{6}$$

Proof. For each $1 \leq j \leq s$, let i_j be such that $U_{i_j} \ni g^j(W)$. Then $i_j \neq 0$ for all $1 \leq j \leq s-1$ and $i_s = 0$. Let Q_j be the component of $g^{-j}(V)$ containing $g(W)$ for $j = 0, 1, \ldots, s$. Then $W = g^{-1}(Q_s)$. For any $j \leq s-1$, $g^j : (Q_j, Q_{j+1}) \to (V, U_{i_{j+1}})$ is a conformal map. So

$$\begin{aligned}\mathrm{mod}(V \setminus Q_s) &\geq \sum_{j=0}^{s-1} \mathrm{mod}(Q_j \setminus Q_{j+1}) \\ &= \sum_{j=0}^{s-1} \mathrm{mod}(V \setminus U_{i_{j+1}}) \geq (s-1)\ell\mathrm{mod}'(g) + \mathrm{mod}(g).\end{aligned}$$

Since $\mathrm{mod}(U_0 \setminus W) \geq \mathrm{mod}(V \setminus Q_s)/\ell$, the lemma follows. □

4.3 Holomorphic motion

Definition 3. A holomorphic motion of a set $X \in \mathbb{C}$ over a complex manifold D is a map

$$\mathbf{h} : D \times X \to D \times \mathbb{C},\ \ (\lambda, z) \mapsto (\lambda, h_\lambda(z)),$$

which satisfies the following properties:

- for any $\lambda \in D$, $h_\lambda : X \to \mathbb{C}$ is injective;
- for any $z \in X$, $\lambda \mapsto h_\lambda(z)$ is holomorphic;
- $h_* = id_X$ for some $* \in D$.

We shall also say that $\mathbf{h}$ is a holomorphic motion of X over $(D, *)$.

Optimal λ-lemma. *(***Slodkowski** *[33]) Let $D \subset \mathbb{C}$ be a topological disk and let $c_0 \in D$. Given any holomorphic motion $\mathbf{h}$ of a set $X \subset \mathbb{C}$ over (D, c_0), there exists a holomorphic motion $\widetilde{\mathbf{h}}$ of $\mathbb{C}$ over (D, c_0) such that $\widetilde{\mathbf{h}}|D \times X = \mathbf{h}$. Moreover, $\tilde{h}_c$ is a $K(r)$-qc map, where r is the hyperbolic distance between c and c_0 in D and $\lim_{r\to 0} K(r) = 1$.*

We shall use the terminology *tube* for a holomorphic motion $\mathbf{h}$ of a Jordan curve γ over a Jordan disk D. We say that the tube is *proper* if $\mathbf{h}$ extends to a homeomorphism from $\overline{D} \times \gamma$ onto its image. A holomorphic motion $\mathbf{h}$ of a closed Jordan disk $\overline{V}$ over another Jordan disk D will be called a *filled tube*. A filled tube is called *proper* if the restriction $\mathbf{h}|D \times \partial V$ is.

Given a filled tube $\mathbf{h} : D \times \overline{V} \to D \times \mathbb{C}$, a holomorphic map $\varphi : D \to \mathbb{C}$ will be called a *diagonal of* $\mathbf{h}$ if the following hold:

- $\varphi(c) \in h_c(V)$ for all $c \in D$,
- φ has a continuous extension to $\overline{D}$, and
- $c \mapsto h_c^{-1} \circ \varphi(c)$ defines a homeomorphism from ∂D onto ∂V.

By the Argument Principle, for each $z \in V$, the equation $h_c(z) = \varphi(c)$ has a unique solution in D. See [21].

Lemma 9. *There exists $M > 0$ with the following property. Let D be a Jordan disk. Let $V \Supset U$ be Jordan disks with $\mathrm{mod}(V \setminus \overline{U}) > 2M$. Let $\mathbf{h} : D \times \overline{V} \to D \times \mathbb{C}$ be a proper filled tube and let $\varphi : D \to \mathbb{C}$ be a diagonal of $\mathbf{h}$. Assume that for each $c \in D$, there exists a 2-qc map $\hat{h}_c : V - \overline{U} \to h_c(V - \overline{U})$ which coincides with h_c on $\partial V \cup \partial U$. Then $D' = \{c \in D : h_c^{-1}(\varphi(c)) \in U\}$ is a topological disk, and*

$$\mathrm{mod}(D \setminus \overline{D'}) \geq \frac{1}{2}\mathrm{mod}(V \setminus U) - M.$$

Proof. See Section 4.3 in [21]. $\square$

4.4 Parapuzzle

Let us define the *Yoccoz parapuzzle* as follows. Let $\mathcal{X}^n = \{c \in \mathbb{C} \setminus \mathcal{M} : \log|\Phi(c)| < 1/\ell^n\}$ and $T_n = \{t \in \mathbb{R}/\mathbb{Z} : \ell^n t \in \{t^+, t^-\}\}$. Define

$$\mathcal{S}^n = \partial\mathcal{X}^n \cup \left(\bigcup_{t \in T_n} \overline{\mathcal{R}(t)} \right).$$

A component of $\mathcal{X}^n \setminus \mathcal{S}^n$ is called *a parapuzzle of depth* n and denoted by $\mathcal{P}_n(c)$ if it contains c.

The following lemma describes how the combinatorics of Yoccoz puzzle changes with the parameter.

Lemma 10. *Let $c_0 \in \mathcal{F}^0_r$. Then for any $n \geq 2$, there exists a holomorphic motion*

$$\mathbf{p}_n : \mathcal{P}_n(c_0) \times \mathbb{C} \to \mathcal{P}_n(c_0) \times \mathbb{C}, \ \ (c, z) \mapsto (c, h_{n,c}(z))$$

such that for each $c \in \mathcal{P}_n(c_0)$, the following hold:

1. *for each $0 \leq i \leq n$, $S^i_c = p_{n,c}(S^i_{c_0})$;*
2. *for each $z \notin X^n_c$, $p_{n,c}(z) = B_c^{-1} \circ B_{c_0}(z)$;*
3. *for all $1 \leq i \leq n$ and all $z \in S^i_{c_0}$, $f_c \circ p_{n,c}(z) = p_{n,c} \circ f_{c_0}(z)$.*

Moreover, the restriction $\mathbf{p}_n | \mathcal{P}_n(c_0) \times \overline{P^n_{c_0}(c_0)}$ is proper filled tube which has the identity map as a diagonal.

Sketch of proof. We shall only give a sketch of proof here. For the details we refer to Section 2 in [30]. Although only quadratic polynomials are considered there, the proof works through in the general unicritical case.

We take $\mathbf{p}_n$ to be the restriction of holomorphic motion H_{n-1} constructed in Lemma 2.5 of [30] to $\mathcal{P}_n(c_0) \times \mathbb{C}$. Assuming $n \geq 2$, let us show that $\mathbf{p}_n | \mathcal{P}_n(c_0) \times \overline{P^n_{c_0}(c_0)}$ is a proper tube. For $n = 2$, by Lemma 7, we have $P^2_c(c) \Subset P^1_c(c)$, which implies that $\mathcal{P}_2(c_0) \Subset \mathcal{P}_1(c_0)$ by Lemma 2.8 in [30]. For $n > 2$ one proceeds by induction. The fact that the identity map is a diagonal to the filled tube follows from Lemma 2.6 in [30]. ☐

Remark 5. Clearly, the map $p_{n,c}$ is holomorphic outside $X^n_{c_0}$. For any $z \in S^n_{c_0} \setminus K_{c_0}$, $p_{n,c}(z) \in S^c_n \setminus K_c$ and $B_c \circ p_{n,c}(z) = B_{c_0}(z)$.

Remark 6. As $t^+ = -t^- \mod 1$, the set $\mathcal{S}^n$ is real-symmetric. Consequently, any parapuzzle piece which intersects $\mathbb{R}$ is real-symmetric.

5 Properties of the Julia sets

Given a topological disk Ω and a set A, define

$$\lambda(A|\Omega) = \sup_{\varphi} \frac{m(\varphi(A \cap \Omega))}{m(\varphi(\Omega))},$$

where φ runs over all conformal maps from Ω into $\mathbb{C}$ and m denotes the planar Lebesgue measure.

Definition 4. Let V be a topological disk, and let U_i, $i = 0, 1, \ldots$ be pairwise disjoint topological disks contained in V. We say that the family $\{U_i\}$ is ε-*absolutely-small in* V if $\lambda(\bigcup_i U_i|V) < \varepsilon$, and for each i, the diameter of U_i in the hyperbolic metric of V is less than ε.

The main result of this section is the following:

Theorem 5. *Consider a map $f = f_c$ with $c \in \mathcal{DG}$. Then for any $\varepsilon > 0$, there exists a critical puzzle piece Y such that the collection of the components of the domain of the first return map to Y is ε-absolutely-small in Y.*

5.1 Extendibility

For a puzzle piece Y, let $D(Y)$ denote the set of all points z for which there exist $k = k(z) \geq 1$ with $f^k(z) \in Y$, let $E(Y) = D(Y) \cup Y$, and let $g_Y : D(Y) \cap Y \to Y$ denote the first return map to Y.

We shall say that a Jordan disk $\hat{Y} \supset Y$ is an *extension domain* of g_Y, if for each component U of $D(Y) \cap Y$, there exists a Jordan disk $\hat{U}$ with $Y \supset \hat{U} \supset U$ such that $f^{s-1} : f(\hat{U}) \to \hat{Y}$ is a conformal map, where s denotes the return time of U to V, i.e., $g_V|U = f^s|U$. We say that g_Y is C-*extendible* if there exists an extension domain $\hat{Y}$ with $\text{mod}(\hat{Y} \setminus \overline{Y}) \geq C$.

A critical puzzle Y is called C-*nice* if for each return domain U to Y we have $\text{mod}(Y \setminus U) \geq C$. Remark that if g_Y is C-extendible, then Y is C/ℓ-nice:

$$\text{mod}(Y \setminus \overline{U}) \geq \text{mod}(\hat{U} - \overline{U}) \geq \text{mod}(\hat{Y} \setminus \overline{Y})/\ell \geq C/\ell.$$

The following lemma will be convenient for us to find extension domains.

Lemma 11. *Let $\hat{Y} \supset Y$ be puzzle pieces such that $f^k(\partial Y) \cap \hat{Y} = \emptyset$ for all $k \geq 1$.*

- *If $Y \ni 0$, then $\hat{Y}$ is an extension domain of g_Y.*
- *If $\hat{Z}$ is a critical puzzle piece such that $f^{s-1} : f(\hat{Z}) \to \hat{Y}$ is a conformal map for some $s \in \mathbb{N}$, and $f^s(0) \in Y$, then $\hat{Z}$ is an extension domain of g_Z, where $Z = Comp_0(f^{-s}Y)$.*

Proof. Let U be a return domain to Y and let r be the return time. For each $0 \le i \le r$ let Q_i denote the component of $f^{i-r}(\hat{Y})$ which contains $f^i(U)$. For each $0 \le i < r$, $Q_i \cap \partial Y = \emptyset$ for otherwise there exists $z \in \partial Y$ with $f^{r-i}(z) \in \hat{Y}$. This shows that $Q_i \subset Y$ if $0 \in Q_i$. In particular, $Q_0 \subset Y$. Moreover, this implies that $Q_i \not\ni 0$ for all $0 < i < r$. In fact, otherwise, we would have $f^i(U) \subset Q_i \subset Y$, contradicting the fact that r is the return time of U to Y. This proves that $\hat{Y}$ is an extension domain of g_Y. For the second statement, one checks that $f^k(\partial Z) \cap \hat{Z} = \emptyset$ for all $k \ge 1$ and then applies the first statement of the lemma. □

5.2 A recursive argument

To prove Theorem 5 let us start with a slightly more general situation.

Lemma 12. *For any $\varepsilon > 0$ there exists $C > 0$ such that if Y is a critical puzzle piece and if the first return map g_Y is C-extendible, then*

$$1 - \lambda(E(Y^1)|Y) \ge \frac{m(Y \setminus D(Y))}{m(Y \setminus D(Y)) + \varepsilon m(Y)}(1 - \frac{\varepsilon}{4}), \tag{7}$$

where Y^1 is the critical return domain to Y. Moreover, if Y' is a child of Y, then

$$1 - \lambda(D(Y')|Y') \ge \frac{(1-\varepsilon)m(Y \setminus D(Y))}{m(Y \setminus D(Y)) + \varepsilon m(Y)} \ge \frac{1 - \lambda(D(Y)|Y)}{1 - \lambda(D(Y)|Y) + \varepsilon}(1 - \varepsilon). \tag{8}$$

Proof. Let us use $B_Y(r)$ to denote the hyperbolic ball in Y with center 0 and radius r. Let $\delta > 0$ be a small constant so that

$$\lambda(B_Y(2\delta)|Y) \le \frac{\varepsilon}{4}.$$

Define $U_0 = Y \setminus D(Y)$, define V_0 to be the union of all components P of $Y \cap D(Y)$ with $P \cap B_Y(\delta) = \emptyset$, and define W_0 to be the union of all other component of $D(Y) \cap Y$. Moreover inductively define U_i, V_i, W_i for all $i \geq 1$ as follows:

$$U_i = \{z \in V_{i-1} : g_Y^i(z) \in U_0\};$$
$$V_i = \{z \in V_{i-1} : g_Y^i(z) \in V_0\};$$
$$W_i = \{z \in V_{i-1} : g_Y^i(z) \in W_0\}.$$

By definition of C-extendibility, there exits a topological disk $\hat{Y} \supset Y$ with $\mathrm{mod}(\hat{Y} \setminus \overline{Y}) \geq C$ and satisfying the following: for each component P of $D(Y) \cap Y$, there exists a topological disk $\hat{P}$ with $P \subset \hat{P} \subset Y$ and such that $f^{s-1} : f\hat{P} \to \hat{Y}$ is a conformal map, where s denotes the return time of P into Y. Take γ to be the core-curve of the annulus $\hat{Y} \setminus \overline{Y}$, i.e., γ is the Jordan curve in $\hat{Y} \setminus \overline{Y}$ which separate $\hat{Y} \setminus \overline{Y}$ into two annuli with modulus $\mathrm{mod}(\hat{Y} \setminus \overline{Y})/2$. Let $\tilde{Y}$ be the domain bounded by γ and define $\tilde{P} = \mathrm{Comp}_P(f^{-s}\tilde{Y})$. Then $\mathrm{mod}(Y \setminus \overline{\tilde{P}}) \geq \mathrm{mod}(\hat{Y} \setminus \overline{\tilde{Y}})/(2\ell) \geq C/2\ell$. If C is sufficiently large, then this implies that if $P \subset V_0$ then $0 \notin \tilde{P}$. It follows that for any $i \geq 1$ and any component A of V_{i-1}, $R_Y^i|A$ extends to a conformal map onto $\tilde{Y}$. By the Koebe distortion theorem, the distortion $\mathrm{Dist}(R_Y^i|A)$ is small. Note also that $W_0 \subset B_Y(2\delta)$. Thus

$$\frac{m(A \cap U_i)}{m(A \cap W_i)} \geq \frac{1}{2}\frac{m(U_0)}{m(W_0)} = \frac{1}{2}\frac{m(U_0)}{m(Y)}\frac{m(Y)}{m(W_0)} \geq \frac{2}{\varepsilon}\frac{m(U_0)}{m(Y)}.$$

Since $E(Y^1) \cap Y \subset \bigcup_i W_i$, this implies that for each component P of V_0,

$$\frac{m(P \setminus E(Y^1)}{m(P \cap E(Y^1))} \geq \frac{2}{\varepsilon}\frac{m(U_0)}{m(Y)}. \tag{9}$$

Let us estimate $\lambda(Y \setminus E(Y^1)|Y)$. Let φ be a conformal map from Y into $\mathbb{C}$. Then

$$\frac{m(\varphi(Y \setminus E(Y^1)))}{m(\varphi Y)} \geq \frac{m(\varphi U_0)}{m(\varphi Y)} + \sum_{P \in \mathcal{V}_0} \frac{m(\varphi(P \setminus E(Y^1)))}{m(\varphi(P))}\frac{m(\varphi(P))}{m(\varphi Y)},$$

where $\mathcal{V}_0$ denote the collection of the components of V_0. As $\mathrm{mod}(Y \setminus \overline{P}) \geq C/\ell$, $\mathrm{Dist}(\varphi|P) << 1$ provided that C is sufficiently large. By (9), this implies

$$\frac{m(\varphi(P \setminus E(Y^1)))}{m(\varphi(P))} \geq \frac{m(U_0)}{m(U_0) + \varepsilon m(Y)},$$

and hence

$$\begin{aligned}
\frac{m(\varphi(Y\setminus E(Y^1)))}{m(\varphi Y)} &\geq \frac{m(U_0)}{m(U_0)+\varepsilon m(Y)}\left(\frac{m(\varphi U_0)}{m(\varphi Y)}+\sum_{P\in\mathcal{V}_0}\frac{m(\varphi(P))}{m(\varphi Y)}\right)\\
&= \frac{m(U_0)}{m(U_0)+\varepsilon m(Y)}\left(1-\frac{m(\varphi(W_0))}{m(\varphi(Y))}\right)\\
&\geq \frac{m(U_0)}{m(U_0)+\varepsilon m(Y)}\left(1-\lambda(B_Y(2\delta)|Y)\right)\\
&\geq \frac{m(U_0)}{m(U_0)+\varepsilon m(Y)}(1-\frac{\varepsilon}{4}).
\end{aligned}$$

This proves (7).

Now let Y' be a child of Y and let s be such that $f^s(Y') = Y$. As $Y' \subset Y^1$, we have $\lambda(E(Y')|Y) \leq \lambda(E(Y^1)|Y)$. Let $Q_0 \ni 0, Q_1, Q_2, \ldots$ be the components of $f^{-s}(D(Y)) \cap Y'$, and let $\mathcal{I} = \{i \geq 0 : i = 0 \text{ or } f^s(Q_i) = Y^1\}$. Then

$$\#\mathcal{I} \leq \ell + 1.$$

As $\mathrm{mod}(Y' \setminus Q_i) \geq \mathrm{mod}(Y \setminus f^s(Q_i))/\ell \geq C/\ell^2$ for all i, it follows that

$$\lambda(\bigcup_{i\in\mathcal{I}} P_i|Y') \leq \frac{\varepsilon}{2},$$

provided that C is sufficiently large. Let φ be any conformal map into $\mathbb{C}$, and let $U_0' = f^{-s}(U_0) \cap Y'$. For any $i \notin \mathcal{I}$, $R_Y \circ f^s$ maps Q_i conformally onto Y and maps $Q_i \cap D(Y')$ onto $Y \cap E(Y')$, so

$$\begin{aligned}
\frac{m(\varphi(Q_i\setminus D(Y')))}{m(\varphi(Q_i))} &\geq 1-\lambda(E(Y')|Y)\\
&\geq \frac{m(U_0)}{m(U_0)+\varepsilon m(Y)}(1-\frac{\varepsilon}{4}).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{m(\varphi(Y' \setminus D(Y')))}{m(\varphi(Y'))} &= \frac{m(\varphi(U_0'))}{m(\varphi(Y'))} + \sum_{i \notin \mathcal{I}} \frac{m(\varphi(Q_i \setminus D(Y')))}{m(\varphi(Q_i))} \frac{m(\varphi(Q_i))}{m(\varphi(Y'))} \\
&\geq \frac{m(U_0)}{m(U_0) + \varepsilon m(Y)} (1 - \frac{\varepsilon}{4}) \left(1 - \sum_{i \in \mathcal{I}} \frac{m(\varphi(Q_i))}{m(\varphi(Y'))} \right) \\
&\geq \frac{m(U_0)}{m(U_0) + \varepsilon m(Y)} (1 - \frac{\varepsilon}{4}) \left(1 - \frac{\varepsilon}{2} \right) \\
&\geq \frac{m(U_0)}{m(U_0) + \varepsilon m(Y)} (1 - \varepsilon) \\
&\geq \frac{1 - \lambda(D(Y)|Y)}{1 - \lambda(D(Y)|Y)} (1 - \varepsilon).
\end{aligned}$$

□

Remark 7. Note that the first part of (8) implies that (provided that g_Y is C-extendible with a large C), $1 - \lambda(D(Y')|Y') > 0$. This follows from the simple observation that $Y - D(Y)$ has a non-empty interior.

5.3 Proof of Theorem 5

Proposition 5. *Assume that $c \in \mathcal{DG}$. Then there exists a sequence of critical puzzle pieces*

$$Y_1 \Supset Y_2 \Supset Y_3 \Supset \cdots$$

and a sequence of numbers $C_n \to \infty$ as $n \to \infty$ such that the following hold:

- *for each n, Y_{n+1} is a child of Y_n;*
- *the first return map to Y_n is C_n-extendible.*

Proof. We shall distinguish two cases.
Case 1. f_c is reluctantly recurrent.

Step 1. Let $N \in \mathbb{N}$ be such that $P^N(0)$ has infinitely many children. Then for all $n \geq N$, $P^n(0)$ has infinitely many children. In fact, if $P^{N+s}(0)$ is a child of $P^N(0)$, and if $k \geq 0$ is minimal such that $f^{s+k}(0) \in P^n(0)$, then $P^{n+k+s}(0)$ is a child of $P^n(0)$.

Step 2. Let V be a critical puzzle piece of depth $\geq N$, and let U be its central return domain. We claim that there exists an arbitrarily large $s \in \mathbb{N}$, such that $f^s(0) \in U$ and $W = \mathrm{Comp}_0(f^{-s}V)$ is a child of V.

To see this, fix a positive integer M. There exists $s_1 > M$ such that $W_1 = \mathrm{Comp}_0(f^{-s_1}V)$ is a child of V. As 0 is recurrent, there exists a minimal $m \in \mathbb{N} \cup \{0\}$ such that $g_V^m(f^{s_1}(0)) \in U$, where g_V denotes the first return map to V. By minimality of m, there exists a neighborhood Q of $f^{s_1}(0)$ such that g_V^m maps Q conformally onto V. Let $W := \mathrm{Comp}_0(f^{-s_1}Q)$. Then clearly W is a child of V with transition time $s \geq s_1 > M$ and $f^s(0) \in U$.

Step 3. Let U, V be as in Step 2. Assume that $U \Subset V$. Let us show that for every $C > 0$, any child W of V with a sufficiently large transition time is C-nice.

Let $s_1 < s_2 < \cdots$ be all the positive integers such that $f^{s_n}(0) \in U$ and such that $W_n' = \mathrm{Comp}_0(f^{-s_n}V)$ is a child of V, $n \geq 1$. Then $W_n := \mathrm{Comp}_0(f^{-s_n}U)$ is a child of U. Let $W_0' = V$ and $W_0 = U$. Note that $W_n \supset W_{n+1}'$ for all n. For all $n \geq 1$, since $f^{s_n} : W_n' \setminus \overline{W}_n \to V \setminus \overline{U}$ is a covering map of degree ℓ,

$$\mathrm{mod}(W_n' \setminus \overline{W}_n) = \mu := \mathrm{mod}(V \setminus \overline{U})/\ell > 0.$$

To complete this step, let us show that if W is a child of V such that $W \subset W_{n-1}$, then W is $n\mu/\ell$-nice.

To this end, let $s \in N$ be such that $f^s(W) = V$. Let P be a return domain to W and let r be the return time. Clearly, $r \geq s$. If $r = s$, then $f^s(P) = W$, so

$$\mathrm{mod}(W \setminus P) \geq \frac{\mathrm{mod}(V \setminus \overline{W})}{\ell} \geq \ell^{-1} \sum_{i=0}^{n-1} \mathrm{mod}(W_i' \setminus \overline{W}_i) \geq n\mu/\ell.$$

If $r > s$, then $f^s(P)$ is a landing domain to W. For $0 \leq i \leq n-1$, let Q_i', Q_i denote the landing domain to W_i' and W_i respectively. Then $\mathrm{mod}(Q_i' \setminus Q_i) \geq \mathrm{mod}(W_i' \setminus W_i) \geq \mu$. Since $f^s(P) \subset Q_{n-1}$, it follows that $\mathrm{mod}(V \setminus f^s(P)) \geq n\mu$ and hence

$$\mathrm{mod}(W \setminus P) \geq \mathrm{mod}(V \setminus f^s(P))/\ell \geq n\mu/\ell.$$

Step 4. Let us now complete the proof of Theorem 5 in the reluctantly recurrent case.

Let us first prove that there exists a 1-nice critical puzzle piece Y_1. Take a critical puzzle piece V of depth $\geq N$, such that its central return domain U is compactly contained in V. Such a puzzle piece exists: one can take V to be a critical pull back of $P^3(0)$. By Step 3, V has a 1-nice child which is Y_1.

Once Y_{2n-1} is defined, let Y_{2n} its the central return domain. By Step 2 and Step 3, there exists $s_n \in \mathbb{N}$, such that $f^{s_n}(0) \in Y_{2n}$, and $W'_n = \mathrm{Comp}_0(f^{-s_n} Y_{2n-1})$ is a child of Y_{2n-1} and $W_n = \mathrm{Comp}_0(f^{-s_n} Y_{2n})$ is $(n+1)$-nice a child of Y_{2n}. Define $Y^{2n+1} = W_n$. Note that by Lemma 11, W'_n is an extension domain of the first return map to Y_{2n+1}. It is easy to see that so defined $Y_n, n \geq 1$ satisfies all the requirement in this proposition.

Case 2. f_c is persistently recurrent and there exists a chain of nice intervals $\Gamma^0 \supset \Gamma^1 \supset \cdots \ni 0$ such that Γ^{n+1} is the smallest child of Γ^n and so that $|\Gamma^{n+1}|/|\Gamma^n| \to 0$ as $n \to \infty$. Now let us consider the enhanced nest of puzzle pieces $\mathbf{I}_n \supset \mathbf{K}_n \supset \mathbf{L}_n \supset \mathbf{I}_{n+1} \supset \ldots$ defined in Section 8 of [18] and let I_n, L_n, K_n be their real traces. This construction is based on the fact that to each critical puzzle piece $\mathbf{I}$ one can associate an integer ν so that if we define

$$\mathcal{A}(\mathbf{I}) := \mathrm{Comp}_0(f^{-\nu}(\mathcal{L}_{f^\nu(0)}(\mathbf{I}))) \ \subset \ \mathcal{B}(\mathbf{I}) := \mathrm{Comp}_0(f^{-\nu}(\mathbf{I}))$$

then $f^\nu \colon \mathcal{B}(\mathbf{I}) \to \mathbf{I}$ has degree bounded by some universal constant and $\mathcal{B}(\mathbf{I}) - \mathcal{A}(\mathbf{I})$ is disjoint from the critical set. (In fact, in the unicritical case one can choose ν so that $\mathcal{L}_{f^\nu(0)}(\mathbf{I}) = \mathcal{L}_0(\mathbf{I})$.) If we denote the smallest child of $\mathbf{I}$ by $\Gamma(\mathbf{I})$ then the enhanced nest is inductively defined by $\mathbf{L}_n = \mathcal{A}(\mathbf{I}_n), \mathbf{K}_n = \mathcal{B}(\mathbf{I}_n), \mathbf{I}_{n+1} = \Gamma^T(\mathbf{L}_n)$ where T is a fixed integer chosen in Section 8.1 of [18]. By this construction, there exists some fixed T so that $\mathbf{I}_{n+1}$ is a descendant of $\mathbf{I}_n$ of generation $\leq T'$ with T' fixed. Hence there exists a sequence of puzzle pieces $Y_1 \Supset Y_2 \Supset Y_3 \Supset \ldots$ such that for each n, Y_{n+1} is a child of Y_n and so that the puzzle pieces from the enhanced nest all appear in the sequence $Y_1, Y_2, \ldots$. By the Key Lemma stated in Section 4 in [18], there exists $\eta = \eta(\ell) > 0$ such that for all n sufficiently large, $\mathbf{I}_n$ has η-bounded geometry: $B(0, \eta \mathrm{diam}(\mathbf{I}_n)) \subset \mathbf{I}_n$. Moreover, there exists $\xi > 0$ and a neighborhood $\mathbf{I}'_n$ of $\mathbf{I}_n$ so that $\mathbf{I}'_n \cap \omega(0) \subset \mathbf{I}_n$ and $\bmod(\mathbf{I}'_n \setminus \mathbf{I}_n) \geq \xi$ for each $n \geq 0$. It follows that all Y_i have η' bounded geometry for all i large, see [18].

By construction, for any n, there are at least two nice intervals Γ^i and Γ^{i+1} between I_n and I_{n+1}. It follows that $|I_{n+1}|/|I_n|$ tends to zero. Hence, by Proposition 8.1 in [18], $\sup_{x \in \omega(0) \cap I_n} |\mathcal{L}_x(I_n)|/|I_n| \to 0$ and by the bounded geometry $\mathrm{mod}(\mathbf{I}_n - \mathcal{L}_{f^{\nu_n}(0)}(\mathbf{I}_n)) \to \infty$. Since $f^k(\partial \mathcal{L}_{f^{\nu_n}(0)} \mathbf{I}_n)) \cap \mathrm{int}(\mathbf{I}_n) = \emptyset$ for all $k \geq 1$, we can apply the second part of Lemma 11 (possibly repeatedly if $\mathcal{B}(\mathbf{I}_n)$ is not a child, but a grandchild $\mathbf{I}_n$), and obtain that $\mathcal{B}(\mathbf{I}_n)$ is a C_n-extension domain of the first return map to $\mathbf{L}_n = \mathcal{A}(\mathbf{I}_n)$ with $C_n \to \infty$. Since $\mathcal{B}(\mathbf{I}_n) \setminus \mathbf{L}_n$ is disjoint from the critical set, we can repeatedly apply the second part of Lemma 11 to the children (and their children) of $\mathbf{L}_n$. Since we only need to repeat this at most T' times until we get to $\mathbf{L}_{n+1}$, this implies the C_i'-extendibility of the first return maps to each of the puzzle pieces Y_i with $C_i' \to \infty$. $\square$

Proof of Theorem 5. Let $Y_n, n \geq 1$ be as in the above proposition, and let $\mu_n = 1 - \lambda(D(Y_n)|Y_n)$. By Remark 7, there exists n_0 such that for all $n \geq n_0$, $\mu_n > 0$. By Lemma 12, for any $\varepsilon > 0$,

$$\mu_{n+1} \geq \frac{\mu_n}{\mu_n + \varepsilon}(1 - \varepsilon),$$

holds for all n sufficiently large, which implies that

$$\liminf_{n \to \infty} \mu_n \geq 1 - 2\varepsilon.$$

Therefore, $\lim_n \mu_n = 1$. $\square$

6 Pseudo-conjugacy

Definition 5. Let $g : \bigcup_i U_i \to V$ and $\tilde{g} : \bigcup_i \tilde{U}_i \to \tilde{V}$ be R-maps. A qc map $\varphi : \mathbb{C} \to \mathbb{C}$ is called *a pseudo-conjugacy* between them if φ maps V onto $\tilde{V}$, U_i onto $\tilde{U}_i$, and respects the boundary dynamics: for each $z \in \partial U_i$, $\varphi \circ g(z) = \tilde{g} \circ \varphi(z)$.

Proposition 6. *Let g and $\tilde{g}$ be R-maps, and let φ be a pseudo-conjugacy between them which is conformal a.e. outside the domain of g. There exists a universal constant $\varepsilon_0 > 0$ such that provided that $\{U_i\}$ is ε_0-absolutely-small in V, there exists a qc pseudo-conjugacy ψ such that $\psi = \varphi$ on $\mathbb{C} \setminus \bigcup_i U_i$; and such that ψ is 2-qc on $\mathbb{C} \setminus \overline{U_0}$.*

For the proof we need the following lemma.

Lemma 13. *There exists an $\varepsilon_1 > 0$ with the following property. Let $\varphi : \Omega \to \tilde{\Omega}$ be a K-qc map between Jordan disks and let $A \subset \Omega$ be a measurable set with $\lambda(A|\Omega) \leq \varepsilon_1$. Assume that φ is conformal a.e. outside A. Then there exists a* $\max(K/4, 2)$*-qc map $\hat{\varphi}$ such that $\hat{\varphi} = \varphi$ on $\partial\Omega$.*

Proof. Without loss of generality we may assume that $\Omega = \tilde{\Omega} = \mathbb{D}$. Moreover, we may assume that $K \leq 8$, because otherwise φ can be written as the decomposition of two qc maps $\varphi_2 \circ \varphi_1$, such that φ_1 is 8-qc and conformal a.e. outside A, and φ_2 is $K/8$-qc.

Assuming that ε_1 is small, let us prove that $\varphi|\partial\mathbb{D}$ extends to a 2-qc map from $\mathbb{D}$ onto itself. By classical quasiconformal mapping theory, it suffices to show that if a, b, c, d are consecutive distinct points in $\partial\mathbb{D}$ with

$$Cr(a, b, c, d) := \frac{d-a}{c-a}\frac{c-b}{d-b} = \frac{1}{2},$$

then $Cr(h(a), h(b), h(c), h(d))$ is close to $1/2$. Let us consider Möbius transformations σ, τ such that $\sigma(a, b, c) = \tau(h(a), h(b), h(c)) = (1, -i, -1)$, and let $\tilde{\varphi} = \tau \circ \varphi \circ \sigma^{-1}$. Notice that $\sigma(d) = -i$ and $\tau(\varphi(d)) = \tilde{\varphi}(-i)$. It suffices to show that $\tilde{\varphi}(-i)$ is close to $-i$. Note that $\tilde{\varphi}$ is 8-qc and conformal a.e. outside $\tilde{A} = \sigma(A)$. As

$$\frac{m(\tilde{A})}{m(\mathbb{D})} \leq \lambda(A|\mathbb{D}) < \varepsilon_1,$$

the desired estimate follows from the formula for the solution of Beltrami equations. See Chapter 5 of [1]. $\square$

Proof of Proposition 6. Let $\mathcal{Q}$ be the collection of all qc maps θ which coincide with φ on $\mathbb{C} \setminus V$, and let $K_0 = \inf\{K(\theta) : \theta \in \mathcal{Q}\}$, where $K(\theta)$ denotes the maximal dilatation of θ. For each $K \geq 1$, all K-qc maps in $\mathcal{Q}$ form a compact family, so there exists $\theta_0 \in \mathcal{Q}$ which is K_0-qc.

Define $\psi : \mathbb{C} \to \mathbb{C}$ to be the map such that $\psi = \varphi$ on $\mathbb{C} \setminus \bigcup_{i\neq 0} U_i$, and such that $\tilde{g} \circ \psi = \theta_0 \circ g$ holds on $\bigcup_{i\neq 0} U_i$. Then ψ is a qc map. In fact, for each $k \in \mathbb{N}$ there exists a homeomorphism $\psi_k : \mathbb{C} \to C$ such that $\psi_k = \psi$ in $\bigcup_{i=1}^{k} U_i$ and $\psi_k = \varphi$ otherwise. By Lemma 2 in [12], for each k, ψ_k is qc with $K(\psi_k) \leq \max(K(\varphi), K_0)$, thus $\psi = \lim_k \psi_k$ is qc. Note that ψ is conformal a.e. outside $\bigcup_i U_i$ because it coincides with φ in that region. Moreover, ψ is K_0-qc on $\bigcup_{i\neq 0} U_i$.

Now let us apply Lemma 13 to show that there exists a map $\theta_1 \in \mathcal{Q}$ which is $\max(K_0/2, 2)$-qc. Let $\gamma \subset V \setminus U_0$ be the Jordan curve which separates $V \setminus \overline{U_0}$ into two annuli with modulus $\mathrm{mod}(V \setminus \overline{U_0})/2$ and let A_0 be the Jordan disk bounded by γ. Then provided that $\varepsilon_0 < \varepsilon_1/2$ is small enough, $\mathrm{mod}(V \setminus A_0)$ is large, so that $\lambda(A_0|V) < \varepsilon_1/2$. Let $A_1 = \bigcup_{i\neq 0} U_i$, $A = A_0 \cup A_1$. Then $\lambda(A|V) < \varepsilon_1$. Moreover, there exists a $2K_0$-qc map $\chi : A_0 \to \theta(A_0)$ with $\chi = \psi$ on ∂A_0. Extend χ to be a qc map from V to $\tilde{V}$ by setting $\chi = \psi$ on $V \setminus A_0$. Then χ is a $2K_0$-qc map which is conformal a.e. outside A. The existence of θ_1 is then guaranteed by Lemma 13.

By the minimality of K_0, we have $K_0 \le \max(K_0/2, 2)$, i.e., $K_0 \le 2$. Thus ψ constructed above satisfies all the requirements. $\square$

7 R-families

7.1 Construction of R-families

To transfer information from the dynamical plane to the parameter plane, we shall use the techniques introduced in [21, 3]. We shall need the notion of R-family.

Definition 6. Let D be a Jordan disk and let $c_0 \in D$. An *R-family* over (D, c_0) is a family $\mathbf{g}$ of R-maps

$$g_c : \bigcup_{i=0}^{\infty} U_{i,c} \to V_c, \ c \in D$$

with the following properties:

- $(c, z) \mapsto (c, g_c(z))$ is holomorphic in both variables c and z;
- there exists a holomorphic motion $\mathbf{h}$ of $\mathbb{C}$ over (D, c_0) such that for each $c \in D$, h_c is a pseudo-conjugacy between g_{c_0} and g_c;
- the filled tube $\mathbf{h}|D \times \overline{V_{c_0}}$ is proper, and the map $c \mapsto g_c(0)$ is a diagonal of this filled tube.

We shall say that $\mathbf{h}$ is an *equipment* of $\mathbf{g}$ and that $(\mathbf{g}, \mathbf{h})$ is an *equipped R-family.*

Let us say that an R-family is *well-controlled* if for each $c \in D$, there exists a qc map $\psi_c : \mathbb{C} \to \mathbb{C}$ such that $\psi_c = h_c$ on $\partial V_{c_0} \cup (\bigcup_i \partial U_{i,c_0})$ (so ψ_c is a pseudo-conjugacy between g_{c_0} and g_c), and such that ψ_c is 2-qc outside U_{0,c_0}.

The following proposition tells us how to obtain an R-family.

Proposition 7. *Let $c_0 \in \mathcal{W}$ and let $n \in \mathbb{N}$ be such that there exists a minimal $s_0 \in \mathbb{N}$ with $f_{c_0}^{s_0}(0) \in P_{c_0}^n(0)$, and such that $f_{c_0}^k(\partial P_{c_0}^n(0)) \cap \overline{P_{c_0}^n(0)} = \emptyset$ for all $k \ge 1$. Then for each $c \in \mathcal{P}_{n+s_0-1}(c_0)$, the first return map g_c to $P_c^n(0)$ under f_c is an R-map, and*

$$g_c, \ c \in \mathcal{P}_{n+s_0-1}$$

is an R-family. Moreover, this family has an equipment

$$\mathbf{h}:\mathcal{P}_{n+s_0-1}(c_0)\times\mathbb{C}\to\mathcal{P}_{n+s_0-1}(c_0)\times\mathbb{C}$$

such that $\mathbf{h}(c,\cdot)$ *is conformal a.e. on* $\mathbb{C}\setminus dom(g_{c_0})$.

Proof. Let $\mathbf{p}_n:\mathcal{P}_n\times\mathbb{C}\to\mathcal{P}_n\times\mathbb{C}$ be the holomorphic motion as in Lemma 10. Let $Y_1,Y_2,\dots,Y_N$ be all the off-critical puzzle pieces of depth n for f_{c_0}, and let $Y_{i,c}=p_{n,c}(Y_i)$. For any word $\mathbf{i}=i_0i_1\cdots i_{k-1}\in\{1,2,\dots,N\}^k$, $k\geq 1$, denote $|\mathbf{i}|=k$ and define

$$\begin{aligned}Y_{\mathbf{i},c}&=\{z\in Y_{i_0,c}:f_c^j(z)\in W_{i_j,c},j=0,1,\dots,k-1\}\\ W_{\mathbf{i},c}&=\{z\in Y_{\mathbf{i},c}:f_c^k(z)\in P_c^n(0)\}.\end{aligned}$$

For each $\mathbf{i}\in\{1,2,\dots,N\}^k$ and each $c\in\mathcal{P}_n$, there exists a unique qc map $\varphi_{\mathbf{i},c}:Y_{\mathbf{i},c_0}\to Y_{\mathbf{i},c}$ such that $f_c^k\circ\varphi_{\mathbf{i},c}=p_{n,c}\circ f_{c_0}^k$, which maps $Y_{\mathbf{i}j,c_0}$ onto $Y_{\mathbf{i}j,c}$ for every $j\in\mathbb{N}$ and $W_{\mathbf{i},c_0}$ onto $W_{\mathbf{i},c}$. Clearly, $\varphi_{\mathbf{i},c}$ is conformal a.e. on $Y_{\mathbf{i},c_0}\setminus(W_{\mathbf{i},c_0}\cup\bigcup_{j=1}^{\infty}Y_{\mathbf{i},j})$. Note that

$$Q_c:=\bigcap_k\bigcup_{|\mathbf{i}|=k}Y_{\mathbf{i},c}=\{z\in\mathbb{C}:f^k(z)\notin P_c^n(0)\text{ for all }k\geq 0\},$$

is a hyperbolic set, and thus has zero measure. Define

$$\varphi_c(z)=\begin{cases}\varphi_{\mathbf{i},c}(z) & \text{if } z\in Y_{\mathbf{i},c_0}-\bigcup_{j=1}^{\infty}Y_{\mathbf{i},j}\\ p_{n,c}(z) & \text{if } G_{c_0}(z)\geq 1/\ell^n.\end{cases}$$

Then $\Phi(c,z)=(c,\varphi_c(z))$ defines a holomorphic motion of the set $\mathbb{C}\setminus Q_{c_0}$ over $\mathcal{P}_n(c_0)$. By the Optimal λ-lemma, it extends to a holomorphic motion of $\mathbb{C}$ over $\mathcal{P}_n(c_0)$, again denoted by Φ. Since Q_{c_0} has zero planar measure, $\varphi_c:\mathbb{C}\to\mathbb{C}$ is conformal a.e. outside $\bigcup_{\mathbf{i}}W_{\mathbf{i},c_0}$. Note that for all $0\leq k\leq n+s_0-1$, $\varphi_c|P_{c_0}^k(c_0)=p_{k,c}|P_{c_0}^k(c_0)$. In particular, the identity map is a diagonal of the filled tube $\Phi|\mathcal{P}_n(c_0)\times\overline{P_{c_0}^n(c_0)}$.

Let $\mathbf{i}_0,\mathbf{i}_1,\dots$ be the set of all indexes such that $W_{\mathbf{i}_j,c_0}\subset P_{n-1}^{c_0}(c_0)$, so organized that $W_{\mathbf{i}_0,c_0}\ni c_0$. Then $U_{j,c}:=f_c^{-1}(W_{\mathbf{i}_j,c})$ are the components of the domain of g_c, and $g_c|U_{j,c}=f_c^{|\mathbf{i}_j|+1}|U_{j,c}$. By assumption, for all j, $U_{j,c_0}\Subset P_{c_0}^n(0)$, which implies that $U_{j,c}\Subset P_c^n(0)$ for all $c\in\mathcal{P}_n$.

Clearly, $\mathcal{P}_{n+s_0-1}(c_0)=\{c\in\mathcal{P}_n:c\in W_{\mathbf{i}_0,c}\}$. For $c\in\mathcal{P}_{n+s_0-1}$, the first return map g_c is an R-map. Finally, define a holomorphic motion $\widetilde{\Phi}$ of $\mathbb{C}$ over $\mathcal{P}_{n+s_0-1}$ such that $\widetilde{\varphi}(c,z)=\varphi_c(z)$ if $z\notin W_{\mathbf{i}_0,c_0}$ and such that $\widetilde{\varphi}(c,c_0)=(c,c)$. By pulling back $\widetilde{\Phi}$ we obtain a holomorphic motion $\mathbf{h}$ of $\mathbb{C}$ over $\mathcal{P}_{n+s_0-1}$ with the desired properties. □

Let us say that an R-family $\mathbf{g}$ is *standard* if it can be obtained as in the proposition. Thus any standard R-family is based over a parapuzzle piece $\mathcal{P}_m(c_0)$, and it has an equipment $\mathbf{h}$ so that h_c is conformal a.e. outside the domain of g_{c_0}.

7.2 Renormalization of R-families

Let D be a Jordan disk, and let us consider an R-family

$$\mathbf{g} = \{g_c : \bigcup_i U_{i,c} \to V_c,\ c \in D\}. \tag{10}$$

We shall use holomorphic motion to relate some sets in the dynamical plane with some sets in the parameter plane. More precisely, for each word $\mathbf{i} = i_0 i_1 \ldots i_{k-1}$ of non-zero integers define

$$\begin{aligned} D_{\mathbf{i}} &= \{c \in D : g_c^j(g_c(0)) \in U_{i_j,c} \text{ for } j = 0, 1, \ldots, k-1\}; \\ D'_{\mathbf{i}} &= \{c \in D_{\mathbf{i}} : g_c^k(g_c(0)) \in U_{0,c}\}, \end{aligned}$$

and for each $c \in D$ define

$$\begin{aligned} U_{\mathbf{i},c} &= \{z \in V_c : g_c^j(z) \in U_{i_j,c} \text{ for } j = 0, 1, \ldots, k-1\}; \\ W_{\mathbf{i},c} &= \{z \in U_{\mathbf{i},c} : g_c^k(z) \in U_{0,c}\}. \end{aligned}$$

Lemma 14. *For each $\boldsymbol{i}_0$, the renormalizations $\mathcal{L}g_c, c \in D'_{\mathbf{i}_0}$ form an R-family.*

Proof. Let $\mathbf{h} : D \times \mathbb{C} \to D \times \mathbb{C}$ be an equipment for the family $\mathbf{g} := \{g_c\}_{c \in D}$ so that $h_{c_0} = id_{\mathbb{C}}$ for some $c_0 \in D'_{\mathbf{i}_0}$. Arguing as in the proof of Proposition 7, we construct a holomorphic motion

$$\Phi : D \times \mathbb{C} \to D \times \mathbb{C},\ (c, z) \mapsto (c, \varphi_c(z))$$

which is again an equipment of $\mathbf{g}$, and maps $W_{\mathbf{i},c_0}$ onto $W_{\mathbf{i},c}$. Next define a holomorphic motion $\widetilde{\Phi}|D'_{\mathbf{i}_0} \times \mathbb{C}$ so that $\tilde{\varphi}_c(z) = \varphi(c, z)$ if $z \notin W_{\mathbf{i}_0}$ and $\tilde{\varphi}_c(c_0) = c$. Finally pull back this $\widetilde{\Phi}$ we obtain a holomorphic motion which equips $\mathcal{L}g_c\, c \in D'_{\mathbf{i}_0}$ to an R-family. $\square$

For an R-family as in (10) we define

$$\operatorname{mod}(\mathbf{g}) = \inf_{c \in D} \operatorname{mod}(g_c) = \inf_{c \in D} \operatorname{mod}(V_c \setminus \overline{U_{0,c}}).$$

Lemma 15. *Assume that* $\mathrm{mod}(\mathbf{g})$ *is sufficiently large and that* $\mathbf{g}$ *is a well-controlled R-family. Then for each* $\mathbf{i}_0$,

$$\mathrm{mod}(D_{\mathbf{i}_0} \setminus \overline{D}'_{\mathbf{i}_0}) \geq \frac{1}{2}\mathrm{mod}(\mathbf{g}) - M, \tag{11}$$

where $M > 0$ *is a universal constant. Moreover,* $\mathcal{L}\mathbf{g} = \{\mathcal{L}g_c, c \in D'_{\mathbf{i}_0}\}$ *is again a well controlled R-family.*

Proof. Let $\mathbf{h}$ and Φ be as in the proof of the previous lemma. Let $k = |\mathbf{i}_0|$. For each $c \in D$, φ_c maps $U_{\mathbf{i}_0,c_0}$ and $W_{\mathbf{i}_0,c_0}$ onto $U_{\mathbf{i}_0,c}$ and $W_{\mathbf{i}_0,c}$ respectively. Moreover, $g_c^k \circ \varphi_c = h_c \circ g_{c_0}^k$ holds on $\partial U_{\mathbf{i}_0,c_0} \cup \partial W_{\mathbf{i}_0,c_0}$. By the assumption that $\mathbf{g}$ is a well controlled family, for each c there exists a qc map $\hat{h}_c$ such that $\hat{h}_c = h_c$ on $\partial V_{c_0} \cup \partial U_{0,c_0}$, and such that $\hat{h}_c$ is 2-qc outside U_{0,c_0}. It follows that there exists a qc map $\hat{\varphi}_c$ which coincides with φ_c on the boundary of the annulus $U_{\mathbf{i},c_0} \setminus W_{\mathbf{i}_0,c_0}$ and is 2-qc in this annulus. The estimate (11) follows by Lemma 9.

When $\mathrm{mod}(\mathbf{g})$ is sufficiently large, $\mathrm{mod}(D \setminus D'_{\mathbf{i}_0}) \geq \mathrm{mod}(D_{\mathbf{i}_0} \setminus D'_{\mathbf{i}_0})$ is large, so by the Optimal λ-lemma, φ_c is 2-qc for all $c \in D'_{\mathbf{i}_0}$. Therefore $\tilde{\varphi}_c$ is 2-qc outside $W_{\mathbf{i}_0,c_0}$. As an equipment of $\mathcal{L}\mathbf{g}$ is obtained by pull back the holomorphic motion $\widetilde{\Phi}$, it follows that $\mathcal{L}\mathbf{g}$ is well-controlled. □

Remark 8. It is clear from the argument above that if $\mathbf{g}$ is a standard R-family, then for any $\mathbf{i}$, $D_{\mathbf{i}}$, $D'_{\mathbf{i}}$ are parapuzzle pieces, and the family $\mathcal{L}\mathbf{g}$ is again a standard family.

Before stating the next proposition, let us first give a fact on the capacity.

Lemma 16. *Let* $\Omega \Supset \Omega'$ *be real-symmetric Jordan disks, and let* $J \supset J'$ *be their real traces. Assume that* $\mathrm{mod}(\Omega \setminus \overline{\Omega'})$ *is sufficiently large. Then for each* $\gamma \geq 1$ *there exists* $\eta = \eta(\gamma)$ *such that*

$$Cap_\gamma(J', J) \leq \exp\left(-\eta\, \mathrm{mod}(\Omega \setminus \overline{\Omega'})\right).$$

Proof. It is well-known that provided that $\mathrm{mod}(\Omega \setminus \overline{\Omega'})$ is large enough, for any $z_0 \in \Omega'$ there exists a round annuli $A = \{r < |z - z_0| < R\} \subset \Omega \setminus \overline{\Omega'}$ with $\mathrm{mod}(A) \geq \mathrm{mod}(\Omega \setminus \overline{\Omega'}) - M$, where M is a universal constant. Let us take $z_0 \in \Omega' \cap \mathbb{R}$, $T = (z_0 - R, z_0 + R)$, $T' = (z_0 - r, z_0 + r)$. Then clearly, $J' \subset T' \subset T \subset J$, so $Cap_\gamma(J', J) \leq Cap_\gamma(T', T)$. For each γ-qs map h from T into $\mathbb{R}$, clearly $|hT'|/|hT|$ is bounded from above by a power of r/R. The lemma follows. □

Proposition 8. *Let $c_0 \in \mathcal{F}^0$ and let $\mathbf{g}$ be a standard R-family over $D = \mathcal{P}_m(c_0)$. Assume that the R-family $\mathbf{g} = \{g_c\}$ is well controlled, and that* $\text{mod}(\mathbf{g})$ *is sufficiently large. Then for any $\gamma \geq 1$ there exists $\eta > 0$ such that*

$$Cap_\gamma(\tilde{D} \cap \mathbb{R}, D \cap \mathbb{R}) \leq \exp\left(-\eta \,\text{mod}(\mathbf{g})\right), \; \textit{where } \tilde{D} = \bigcup_{|\boldsymbol{i}| \leq 4\ell} D'_{\boldsymbol{i}}.$$

Proof. Let $J_{\mathbf{i}} = D_{\mathbf{i}} \cap \mathbb{R}$ and $J'_{\mathbf{i}} = D'_{\mathbf{i}} \cap \mathbb{R}$. By Lemma 9, provided that $\text{mod}(\mathbf{g})$ is large enough, for any word $\mathbf{i}$ we have

$$\text{mod}(D_{\mathbf{i}} \setminus \overline{D}'_{\mathbf{i}}) \geq \text{mod}(\mathbf{g})/2 - M > \text{mod}(\mathbf{g})/3.$$

By Lemma 16, this implies that

$$Cap(J'_{\mathbf{i}}, J_{\mathbf{i}}) \leq \exp(-\frac{\eta}{3}\text{mod}(\mathbf{g})).$$

For any $k \geq 0$, the $J_{\mathbf{i}}$'s with $|\mathbf{i}| = k$ are pairwise disjoint, thus

$$Cap(\bigcup_{|\mathbf{i}|=k} J'_{\mathbf{i}}, D \cap \mathbb{R}) \leq \sup_{|\mathbf{i}|=k} Cap(J'_{\mathbf{i}}, J_{\mathbf{i}}) \leq \exp\left(-\frac{\eta}{3}\text{mod}(\mathbf{g})\right).$$

Therefore

$$Cap(\tilde{D} \cap \mathbb{R}, D \cap \mathbb{R}) \leq (4\ell + 1) \exp\left(-\frac{\eta}{3}\text{mod}(\mathbf{g})\right).$$

Redefining the constant η completes the proof.

□

7.3 Proof of Theorem 3

The proof of Theorem 3 is based on the following lemmas.

Lemma 17. *Let $c_0 \in \mathcal{DG}$. Then for any $C > 0$ there exists a standard R-family $\mathbf{g}$ over some parapuzzle piece $\mathcal{P}_m(c_0)$ such that $\mathbf{g}$ is well controlled and*

$$\text{mod}(\mathbf{g}) \geq 2\ell C, \; \text{mod}'(\mathbf{g}) \geq C.$$

Proof. Let $\varepsilon > 0$ be a small number. By Theorem 5, there exists an arbitrarily large $n \in \mathbb{N}$ such that the domain of the first return map to $P^n_{c_0}(0)$ under f_{c_0} is ε-absolutely small in $P^n_{c_0}(0)$. By Proposition 7, there is a parapuzzle piece $\mathcal{P}_m(c_0)$ such that $\mathbf{g} = \{g_c\}_{c \in \mathcal{P}_m(c_0)}$ forms a standard R-family, where g_c denotes the first return map to $P_c^n(0)$ under f_c. Provided that ε was chosen sufficiently small, by Proposition 6, this family is well controlled and thus $\mathrm{mod}(\mathbf{g}) \geq \mathrm{mod}(g_{c_0})/2$ is large. By Lemma 15, there is a smaller parapuzzle piece $\mathcal{P}_{m'}(c_0)$ (with $m' > m$) such that $\mathcal{L}g_c, c \in \mathcal{P}_{m'}(c_0)$ forms another standard well-controlled R-family $\hat{\mathbf{g}}$. Moreover, by Lemma 8, $\mathrm{mod}(\hat{\mathbf{g}})$ and $\mathrm{mod}'(\hat{\mathbf{g}})$ are both large. $\square$

Recall that $\mathcal{DC}$ is the subset of $\mathcal{F}_0$ consisting of all the parameters c for which the summability condition (2) holds for all $\alpha > 0$. In the following we shall use the following criterion:

Lemma 18. *Let $c \in \mathcal{F}$. Then $c \in \mathcal{DC}$ if one of the following holds:*

1. *$c \notin \mathcal{F}_r^0$;*
2. *for f_c, there exists a nice interval $I \ni 0$ with the following property: if we define $I^0 = I$ and define I^{k+1} to be the central return domain to I^k, then $|I^{i+1}|/I^i|$ decreases to 0 at least exponentially fast.*

Proof. In the first case, the map has no periodic attractor and the critical point is non-recurrent. It is well known that f_c satisfies the Collet-Eckmann condition: $|Df_c^n(c)|$ is exponentially big in n, which implies that $c \in \mathcal{DC}$. In the second case, the result was proved in [25]. $\square$

Lemma 19. *For any $\delta > 0$ and $\gamma \geq 1$, there exists $C > 0$ with the following property. Let $\mathbf{g}$ be a well-controlled standard R-family over a parapuzzle $\mathcal{P}_m(c_0)$ with $c_0 \in \mathbb{R}$ such that $\mathrm{mod}(\mathbf{g}) \geq 2\ell C$ and $\mathrm{mod}'(\mathbf{g}) \geq C$. Let $T = \mathcal{P}_m(c_0) \cap \mathbb{R}$.*

$$Cap_\gamma(T \setminus \mathcal{DC}, T) \leq \delta.$$

Proof. The strategy is to construct a sequence of open sets

$$\Omega^{(0)} = \mathcal{P}_m(c_0) \supset \Omega^{(1)} \supset \Omega^{(2)} \supset \cdots$$

with the following properties:

- for each k, $\Omega^{(k)}$ is a disjoint union of parapuzzle pieces $\Omega^{(k,j)}$ which intersect $\mathbb{R}$;

- for each (k,j), there exists a standard R-family $\mathbf{g}^{(k,j)}$ over $\Omega^{(k,j)}$ which is well-controlled and
$$\mathrm{mod}(\mathbf{g}^{(k,j)}) \geq 2^{k+1}\ell C, \mathrm{mod}'(\mathbf{g}^{(k,j)}) \geq 2^k C; \tag{12}$$
- for each component $\mathcal{P}$ of $\Omega^{(k)}$, we have
$$\mathrm{Cap}_\gamma((\mathcal{P}\cap\mathbb{R}) \setminus (\Omega^{(k+1)} \cup \mathcal{DC}), \mathcal{P}\cap\mathbb{R}) \leq 2^{-k-1}\delta. \tag{13}$$

The existence of these $\Omega^{(k)}$ completes the proof. In fact, the equation (13) implies that
$$\mathrm{Cap}_\gamma((T \setminus \bigcap_k \Omega^{(k)}) \setminus \mathcal{DC}, T) \leq \delta.$$
Moreover, by Lemma 18, the modulus estimate (12) shows that for any $c \in T \cap \bigcap_k \Omega^{(k)}$, $c \in \mathcal{DC}$.

Let us construct these sets by induction. The choice of $\Omega^{(0)}$ satisfies the requirement by assumption. Assume now that $\Omega^{(k)}$ is constructed. Take a component $\mathcal{P}$ of $\Omega^{(k)}$, and let $\hat{\mathbf{g}}$ be the R-family over $\mathcal{P}$ which is given by the induction assumption. For each word $\mathbf{i}$ of positive integers, define $\mathcal{P}_\mathbf{i}$ and $\mathcal{P}'_\mathbf{i}$ as in the previous subsection. The set $\Omega^{(k+1)}$ is defined to be the union of all sets of the form $\mathcal{P}'_\mathbf{i}$ with $|\mathbf{i}| > 4\ell$ which intersect $\mathbb{R}$. This is clearly a disjoint union of parapuzzle pieces intersecting $\mathbb{R}$. By Lemma 15, for each $\mathcal{P}_\mathbf{i}$, $\mathcal{L}\hat{g}_c, c \in \mathcal{P}_\mathbf{i}$ form a well-equipped R-family. Applying Lemma 8 to $\hat{g}_c$, we obtain
$$\begin{aligned}\mathrm{mod}(\mathcal{L}\hat{g}_c) &\geq \frac{(|\mathbf{i}|-1)\mathrm{mod}'(\hat{g}_c) + \mathrm{mod}(\hat{g}_c)}{\ell} \\ &\geq \frac{(4\ell-1)\mathrm{mod}'(\hat{g}_c) + \mathrm{mod}(\hat{g}_c)}{\ell} \geq 2^{k+2}\ell C,\end{aligned}$$
and
$$\mathrm{mod}'(\mathcal{L}\hat{g}_c) \geq \frac{\mathrm{mod}(\hat{g}_c)}{\ell} \geq 2^{k+1}C.$$
By Proposition 8, for each $\mathcal{P}$,
$$\begin{aligned}\mathrm{Cap}_\gamma(\bigcup_{|\mathbf{i}|\leq 4\ell} \mathcal{P}_\mathbf{i} \cap \mathbb{R}, \mathcal{P}\cap\mathbb{R}) &\leq \exp\left(-\eta\mathrm{mod}(\hat{\mathbf{g}})\right) \\ &\leq \exp\left(-2^k\ell\eta C\right) \leq 2^{-k-1}\delta,\end{aligned}$$

provided that C is sufficiently large. Note that $(\mathcal{P} \setminus \bigcup_{\mathbf{i}} \mathcal{P}_{\mathbf{i}}) \cap \mathbb{R} \subset \mathcal{F}^0 - \mathcal{F}_r^0$, so by Lemma 18,

$$\mathcal{P} \setminus (\Omega^{(k+1)} \cup \mathcal{DC}) \subset \bigcup_{|\mathbf{i}| \leq 4\ell} \mathcal{P}_{\mathbf{i}}.$$

This completes the construction and thus the proof of the lemma. □

We finish with

Proof of Theorem 3. Combining Lemmas 17 and 19, we obtain the theorem. □

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