ONE-PARAMETER FAMILIES OF SMOOTH INTERVAL MAPS: DENSITY OF HYPERBOLICITY AND ROBUST CHAOS

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Abstract. In this note we will discuss the notion of robust chaos, and show that (i) there are natural one-parameter families with robust chaos and (ii) hyperbolicity is dense within generic one-parameter families (and so these families are not robustly chaotic).

1. Statement of Results

In [BYG98] the notion of *robust chaos* was introduced. A family of maps $\{f_t\}_{t\in[0,1]}$ is said to have robust chaos (or to be robustly chaotic) if there exists no parameter $t \in [0,1]$ for which the map f_t has a periodic attractor. Examples of families with robust chaos where given in that paper, but in these families the maps are non-smooth. The authors conjectured that robust chaos does not occur within smooth families of intervals maps $f_t\colon [0,1] \to [0,1]$. Contradicting this conjecture, in [AA01b], [AA01a], [Let01] [ES08] and [Agu09], examples where given of families of smooth one-dimensional maps with robust chaos. Since there is a huge literature on bifurcations of one-parameter families of dynamical systems (starting perhaps with, for example, [NPT83]), we shall clarify the situation in this note.

1.1. Theorem (Robust unimodal families are 'constant')**.**
If $\{f_t\}$ is a smooth unimodal family with robust chaos, then all maps within this family are topologically conjugate.

So the family of robustly chaotic unimodal maps given in the papers cited above are all topologically conjugate to each other. That the family is robustly chaotic is therefore not surprising! For multimodal families this need not be the case:

1.2. Theorem (A family of cubic maps with robust chaos)**.**
There exists a one-parameter family $\{f_t\}$ of smooth multimodal interval maps which is robustly chaotic.

On the other hand, the above example is special: generic one-parameter families are never robustly chaotic. In fact, hyperbolicity is dense within such families:

1.3. Theorem (Hyperbolicity is dense within generic families, and so only exceptional families are robustly chaotic)**.**
Near any one-parameter family of smooth interval maps there exists a one-parameter family $\{f_t\}$ of smooth intervals maps for which

- *the number of critical points of each of the maps f_t is bounded;*
- *the set of parameters t for which*

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– all *critical points of* f_t *are in basins of periodic attractors (such a map* f_t *is called hyperbolic),*
– *critical points of* f_t *are not eventually mapped onto other critical points (such a map* f_t *is said to satisfy the no-cycle condition),*

is open and dense.

In particular, a generic family $\{f_t\}_{t\in[0,1]}$ is *not* robustly chaotic. We should also point out the following two facts , see [dMvS93]: maps f_t which are hyperbolic and satisfy the no-cycle condition, are (i) structurally stable and (ii) Lebesgue almost every x is in the basin of a hyperbolic periodic attractor of f_t.

2. The Proofs

Let us start with the proof of Theorem 1.1. Take a robustly chaotic family $\{f_t\}$ of unimodal maps $f_t\colon [0,1] \to [0,1]$. The itinerary of the critical point c_t of f_t can only change as t varies, if $f_t^n(c_t) = c_t$ for some n. But since $\{f_t\}$ is robustly chaotic, this does not happen. So f_t has the same kneading invariant for each $t \in [0,1]$. Since f_t has no periodic attractors at all, it follows from the non-existence of wandering intervals (see Chapter IV of [dMvS93]) that $f_{t'}$ and f_t are topologically conjugate for all $t, t' \in [0,1]$.

Let us now prove Theorem 1.2 and show that there exists a family of cubic maps with robust chaos and which does not have constant kneading invariant. Consider polynomials $f\colon [0,1] \to [0,1]$ of degree three, so that $f(0) = 0$, $f(1) = 1$ (which implies that $f(x) = ax + bx^2 + (1-a-b)x^3$) and with two critical points $0 < c_1 < c_2 < 1$ so that $0 < c_1 < f(c_2) < f^3(c_2) = f^4(c_2) < c_2 < f^2(c_2) < f(c_1) < 1$. The set of such polynomials corresponds to a real analytic curve in the (a,b) plane (defined by the condition that $f^4(c_2) = f^3(c_2)$). Hence it contains a one-parameter family of maps $\{f_t\}_{t\in[0,1]}$. Since f_t is a polynomial with only real critical points, it has negative Schwarzian (see [dMvS93, Exercise IV.1.7]). Hence by Singer's result, each of its periodic attractors has a critical point in its immediate basin. Since $[f(c_2), 1]$ is mapped into itself, and $f(c_1) \in [f(c_2), 1]$ any periodic attractor of f_t would have to lie in $[f(c_2), 1]$. Since c_2 is the only critical point in $[f(c_1), 1]$, it follows that if f_t has a periodic attractor, then c_2 would have to be in its basin. But since $f^4(c_2) = f^3(c_2)$ is a repelling fixed point, this does not happen. It follows that these maps define a one-parameter family $\{f_t\}$ of smooth bimodal maps which are robustly chaotic. Since $f_t(c_1)$ can vary with t (to be anywhere within the interval $[f_t)^2(c_2), 1]$), the kneading invariant of f_t is not constant. Note that the example is based on the map having a trapping region.

Let us finally prove Theorem 1.3. Take a one-parameter $\{f_t\}_{t\in[0,1]}$ family of real polynomial interval maps of degree d. By taking d large enough, we can take this family arbitrarily close to the original family of interval maps (in any topology). Let P be the space of all real polynomial interval maps of degree d. By [KSvS07b] (which is based on [KSvS07a]) each map $g \in P$ can be approximated by a map $\hat{g}$ for which all critical points are in basins of periodic attractors. Hence we can identify P with $\mathbb{R}^n$, $\{f_t\}$ with a curve $c\colon [0,1] \to \mathbb{R}^n$ and the set of maps in P for which all critical points are in basins of periodic attractors with an open and dense subset X of $\mathbb{R}^n$. Maps which fail the no-cycle condition correspond to maps for which an iterate of a critical point lands on another critical point; the corresponding parameters lie on analytic codimension-one varieties. So we can and will assume that X corresponds to hyperbolic maps for which the no-cycle conditions holds. Hence Theorem 1.3 follows from

2.1. Lemma.
Let $c\colon [0,1] \to \mathbb{R}^n$ *be a curve, and let* X *be an open and dense subset of* $\mathbb{R}^n$. *Then there exist a set* $A \subset \mathbb{R}^n$ *which is dense (in fact of 2nd Baire category) so that for each* $\alpha \in A$, $F_\alpha := \{t \in [0,1]; c(t) + \alpha \in X\}$ *is open and dense.*

Proof. Since the curve c is continuous and X is open, F_α is open for each $\alpha \in \mathbb{R}^n$. To prove that F_α is dense, take $\delta > 0$ and define the set A_δ of $\alpha \in \mathbb{R}^n$ so that for each $t \in [0,1]$ there exists t' with $|t - t'| < \delta$ and so that $t' \in F_\alpha$.

Let us show that A_δ is dense. Assume by contradiction it is not dense. Then there exists an open set U of $\alpha \in \mathbb{R}^n$ for which there exists $t_\alpha \in [0,1]$ so that for each $t \in [0,1]$ with $|t - t_\alpha| < \delta$ one has $t \notin F_\alpha$. So if we take $n > 1/\delta$ then for each $\alpha \in U$ there exists $k \in \{0, 1, \ldots, n\}$ so that $k/n \notin F_\alpha$, i.e. $c(k/n) + \alpha \notin X$. Let U_k be the set of $\alpha \in U$ so that $c(k/n) + \alpha \notin X$. Note that $U_0 \cup \cdots \cup U_n = U$. It follows that the closure of at least one of the sets U_{k_0} contains an open set (otherwise $U - \overline{U}_i$ is dense in U for each i, and so $\bigcap_{i=0,\ldots,n}(U - \overline{U}_i) = U - \bigcup_{i=0,\ldots,n} \overline{U}_i$ is dense in U, a contradiction). It follows that there exists a subset $U'_{k_0} \subset U_{k_0}$ so that $\overline{U'_{k_0}}$ contains an open set. Since $U'_{k_0} \subset U_{k_0}$, for each $\alpha \in U'_{k_0}$ one has $c(k_0/n) + \alpha \notin X$. But since X is open then for each $\alpha \in \overline{U'_{k_0}}$ one has $c(k_0/n) + \alpha \notin X$. But this contradicts the assumption that X is open and dense. Thus we have shown that A_δ is dense for each $\delta > 0$.

Since A_δ is also open, it follows by the Baire property that $A := \cap_{\delta > 0} A_\delta$ is dense. By construction, for each $\alpha \in A$, we have that F_α is dense. ■

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