Rigidity for real polynomials

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Abstract

We prove the topological (or combinatorial) rigidity property for real polynomials with all critical points real and non-degenerate, which completes the last step in solving the density of Axiom A conjecture in real one-dimensional dynamics.

Contents

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1 Introduction

1.1 Statement of Results

It is a long standing open problem whether Axiom A (hyperbolic) maps are dense in reasonable families of one-dimensional dynamical systems. In this paper, we prove the following.

Density of Axiom A Theorem. *Let f be a real polynomial of degree $d \geq 2$. Assume that all critical points of f are real and that f has a connected Julia set. Then f can be approximated by hyperbolic real polynomials of degree d with real critical points and connected Julia sets.*

Here we use the topology given by convergence of coefficients. Recall that a polynomial is called *hyperbolic* if all of its critical points are contained in the basin of an attracting cycle or infinity. A polynomial with a connected Julia set cannot have critical points contained in the attracting basin of infinity.

The quadratic case was solved earlier by Graczyk-Swiatek and Lyubich, [10, 20] (see also [38]).

We have required that the polynomial f has a connected Julia set, because such a map has a compact invariant interval in $\mathbb{R}$, and thus is of particular interest from the viewpoint of real one-dimensional dynamics. In fact, our method shows that the theorem is still true without this assumption: given any real polynomial f with all critical points real, we can approximate it by hyperbolic real polynomials with the same degree and with real critical points (which may have disconnected Julia sets).

In a sequel to this paper we shall show that Axiom A maps on the real line are dense in the C^k topology (for $k = 1, 2, \ldots, \infty, \omega$), and discuss connections with the Palis conjecture [34] and connections with previous results [12], [7], [16], [37] and also with [2].

Our proof is through the quasi-symmetric rigidity approach suggested by Sullivan [41].

For any positive integer $d \geq 2$, let $\mathcal{F}_d$ denote the family of polynomials f of degree d which satisfy the following properties:

- the coefficients of f are all real;
- f has only real critical points which are all non-degenerate;

- f does not have any neutral periodic point;
- the Julia set of f is connected.

Rigidity Theorem. *Let f and $\tilde{f}$ be two polynomials in $\mathcal{F}_d$. If they are topologically conjugate as dynamical systems on the real line $\mathbb{R}$, then they are quasiconformally conjugate as dynamical systems on the complex plane $\mathbb{C}$.*

In fact, if $\mathcal{F}'_d$ is the family of real polynomials f of degree d with only real critical points of even order, then the methods in this paper can be used to prove the following:

Rigidity Theorem'. *Let f and $\tilde{f}$ be two polynomials in $\mathcal{F}'_d$. If f and $\tilde{f}$ are topologically conjugate as dynamical systems on the real line $\mathbb{R}$, and corresponding critical points have the same order and parabolic points correspond to parabolic points, then f and $\tilde{f}$ are quasiconformally conjugate as dynamical systems on the complex plane $\mathbb{C}$.*

For real polynomials f and $\tilde{f}$ in $\mathcal{F}_d$ which are topologically conjugate on the real line, it is not difficult to see that they are combinatorially equivalent to each other in the sense of Thurston, i.e., there exist two homeomorphisms $H_i : \mathbb{C} \to \mathbb{C}$ which are homotopic rel $PC(f)$, where $PC(f)$ denote the union of the forward orbit of all critical points of f, such that $\tilde{f} \circ H_1 = H_0 \circ f$. This observation reduces the Rigidity Theorem to the following.

Reduced Rigidity Theorem. *Let f and $\tilde{f}$ be two polynomials in the class $\mathcal{F}_d$. Assume that f and $\tilde{f}$ are topologically conjugate on the real line via a homeomorphism $h : \mathbb{R} \to \mathbb{R}$. Then there is a quasisymmetric homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that for any critical point c of f and any $n \geq 0$, we have*

$$\phi(f^n(c)) = h(f^n(c)).$$

Like the previous successful approach in the quadratic case, we exploit the powerful tool, *Yoccoz puzzle*. Also we require a "complex bounds" theorem to treat infinitely renormalizable maps. The main difference is as follows. In the proof of [10, 20], a crucial point was that quadratic polynomials display decay of geometry: the moduli of certain dynamically defined annuli grow at least linearly fast, which is a special property of quadratic maps. The proof in [38] does not use this property explicitly, but instead a combinatorial bound was adopted, which is also not satisfied by higher degree polynomial. So all these proofs break down even for unimodal polynomials with degenerate critical points. Our approach was inspired by a recent observation of Smania [40], which was motivated by the works of Heinonen and Koskela [13], and Kallunki and Koskela [15]. The key estimate (stated in the *Key Lemma*) is the control of geometry for appropriately chosen puzzle pieces. For example, if c is a non-periodic recurrent critical point of f with a minimal ω-limit set, and if f is not renormalizable at c, our result shows that given any Yoccoz puzzle piece $P \ni c$, there exist a constant $\delta > 0$ and a sequence of combinatorially defined puzzle pieces Q_n, $n = 1, 2, \ldots$, which contain c and are pullbacks of P with the following properties:

- $\mathrm{diam}(Q_n) \to 0$;
- Q_n contains a Euclidean ball of radius $\delta \cdot \mathrm{diam}(Q_n)$;
- there is a topological disk $Q_n' \supset Q_n$ such that $Q_n' - Q_n$ is disjoint from the orbit of c and has modulus at least δ.

In [40], Smania proved that in the non-renormalizable unicritical case this kind of control implies rigidity. To deduce rigidity from puzzle geometry control, we are not going to use this result of Smania directly - even in the non-renormalizable case - but instead we shall use a combination of the well-known Spreading Principle (see Section 5.3) and the QC-Criterion stated in Appendix 1. This Spreading Principle states that if we have a K-qc homeomorphism $h\colon P \to \tilde{P}$ between corresponding puzzle neighbourhoods $P, \tilde{P}$ of the critical sets (of the two maps $f, \tilde{f}$) respecting the standard boundary marking (i.e. agrees on the boundary of these puzzle pieces with what is given by the Bötcher coordinates at infinity), then we can spread this to the whole plane to get a K-qc partial conjugacy. Moreover, together with the QC-Criterion this also gives a method of constructing such K-qc homeomorphisms h, which relies on good control on the shape of puzzle pieces $Q_i \subset P$, $\tilde{Q}_i \subset \tilde{P}$ with deeper depth. This different argument enables us to treat infinitely renormalizable maps as well. In fact, in that case, we have uniform geometric control for a terminating puzzle piece, which implies that we have a partial conjugacy up to the first renormalization level with uniform regularity. Together with the "complex bounds" theorem proved in [37], this implies rigidity for infinitely renormalizable maps, in a similar way as in [10, 20].

In other words, everything boils down to proving the Key Lemma. It is certainly not possible to obtain control on the shape of all critical puzzle pieces in the principal nest. For this reason we introduce a new nest which we will call the *enhanced nest*. In this enhanced nest, bounded geometry and decay in geometry alternate in a more regular way. The *successor* construction we use, is more efficient than first return domains in transporting information about geometry between different scales. In addition we use an 'empty space' construction enabling us to control the nonlinearity of the system.

1.2 Organization of this work

The strategy of the proof is to reduce the proof in steps. In §2 we reduce the Density of Axiom A to the Rigidity Theorem stated above. Then, in §3, we reduce it to the Reduced Rigidity Theorem. This two sections can be read independently from the rest of this paper, which is occupied by the proof of the Reduced Rigidity Theorem.

The idea of the proof of the Reduced Rigidity Theorem is to reduce all difficulties to the Key Lemma.

In §4, we give the precise statement of the Key Lemma on control of puzzle geometry for *a polynomial-like box mapping* which naturally appears as the first return map to a certain open set. In §5, we review a few facts on Yoccoz puzzles. These facts will be necessary to derive our Reduced Rigidity Theorem from the Key Lemma, which is done in the next two sections, §6 and §7.

The remaining sections are occupied by the proof of the Key Lemma. In §8 we construct the enhanced nest, and show how to derive the Key Lemma from lower and upper control of the geometry of the puzzle pieces in this nest. In §9, we analyze the geometry of the real trace of the enhanced nest. These analysis will be crucial in proving the lower and upper geometric control for the puzzle pieces, which will be done in §10 and §11 respectively.

The statement and proof of a QC-Criterion are given in Appendix 1 and some general facts about Poincaré discs are given in Appendix 2.

1.3 General terminologies and notations

Given a topological space X and a connected subset X_0, we use $\mathrm{Comp}_{X_0}(X)$ to denote the connected component of X which contains X_0. Moreover, for $x \in X$, $\mathrm{Comp}_x(X) = \mathrm{Comp}_{\{x\}}(X)$.

For a bounded open interval $I = (a, b) \subset \mathbb{R}$, $\mathbb{C}_I = \mathbb{C} - (\mathbb{R} - I)$. For any $\theta \in (0, \pi)$ we use $D_\theta(I)$ to denote the set of points $z \in \mathbb{C}_I$ such that the angle (measured in the range $[0, \pi]$) between the two segments $[a, z]$ and $[z, b]$ is greater than θ.

We usually consider a *real–symmetric* proper map $f : U \to V$, where each of U and V is a disjoint union of finitely many simply connected domains in $\mathbb{C}$, and $U \subset V$. Here "real–symmetric" means that U and V are symmetric with respect to the real axis, and that f commutes with the complex conjugate. A point at which the first derivative f' vanishes is called a *critical point.* We use $\mathrm{Crit}(f)$ to denote the set of critical points of f. We shall always assume that $f^n(c)$ is well defined for all $c \in \mathrm{Crit}(f)$ and all $n \geq 0$, and use $PC(f)$ to denote the union of the forward orbit of all critical points:

$$PC(f) = \bigcup_{c \in Crit(f)} \bigcup_{n \geq 0} \{f^n(c)\}.$$

As usual $\omega(x)$ is the omega-limit set of x.

An interval I is a *properly periodic interval* of f if there exists $s \geq 1$ such that $I, f(I), \ldots, f^{s-1}(I)$ have pairwise disjoint interiors and such that $f^s(I) \subset I$, $f^s(\partial I) \subset \partial I$. The integer s is the *period* of I. We say that f is *infinitely renormalizable* at a point $x \in U \cap \mathbb{R}$ if there exists a properly periodic interval containing x with an arbitrarily large period.

A *nice open set* P (with respect to f) is a finite union of topological disks in V such that for any $z \in \partial P$ and any $n \in \mathbb{N}$, $f^n(z) \notin P$ as long as $f^n(z)$ is defined. The set P is *strictly nice* if we have $f^n(z) \notin \overline{P}$.

Given a nice open set P, let $D(P) = \{z \in V : \exists k \geq 1, f^k(z) \in P\}$. The first entry map

$$R_P : D(P) \to P$$

is defined as $z \mapsto f^{k(z)}(z)$, where $k(z)$ is the minimal positive integer with $f^{k(z)}(z) \in P$. The restriction $R_P|P$ is called *the first return map* to P. The first landing map

$$L_P : D(P) \cup P \to P$$

is defined as follows: for $z \in P$, $L_P(z) = z$, and for $z \in D(P) \setminus P$, $L_P(z) = R_P(z)$. A component of the domain of the first entry map to P is called an *entry domain.* Similar terminologies apply to *return, landing domain.* For $x \in D(P)$, $\mathcal{L}_x(P)$ denotes the entry domain which contains x. For $x \in D(P) \cup P$, $\hat{\mathcal{L}}_x(P)$ denote the landing domain which contains x. So if $x \in D(P) \setminus P$, $\mathcal{L}_x(P) = \hat{\mathcal{L}}_x(P)$. We also define inductively

$$\mathcal{L}_x^k(P) = \mathcal{L}_x(\mathcal{L}_x^{k-1}(P)).$$

We shall also frequently consider a *nice interval*, which means an open interval $I \subset V \cap \mathbb{R}$ such that for any $x \in \partial I$ and any $n \geq 1$, $f^n(x) \notin I$. The terminologies *strictly nice interval, the first entry (return, landing) map to I* as well as the notations $\mathcal{L}_x(I)$, $\hat{\mathcal{L}}_x(I)$ are defined in a similar way as above.

By a *pullback of a topological disk* $P \subset V$, we mean a component of $f^{-n}(P)$ for some $n \geq 1$, and a *pullback of an interval* $I \subset V \cap \mathbb{R}$ will mean a component of $f^{-n}(I) \cap \mathbb{R}$ (rather than $f^{-n}(I)$) for some $n \geq 1$.

See §4 for the definition of *a polynomial-like box mapping*, *child*, *persistently recurrent*, a set with *bounded geometry* and related objects.

See §9 for the definition of a *chain* and its *intersection multiplicity* and *order*. Also the notions of *scaled neighbourhood* and *δ-well-inside* are defined in that section.

For definitions of quasi-symmetric (qs) and quasi-conformal (qc) maps, see Ahlfors [1].

2 Density of Axiom A follows from the Rigidity Theorem

One of the main reason for us to look for rigidity is that it implies density of Axiom A among certain dynamical systems. Our rigidity theorem implies the following, sometimes called the *real Fatou conjecture.*

Theorem 2.1. *Let f be a real polynomial of degree $d \geq 2$. Assume that all critical points of f are real and that f has a connected Julia set. Then f can be approximated by hyperbolic real polynomials with real critical points and connected Julia sets.*

The rigidity theorem implies the instability of non-hyperbolic maps. As is well-known, in the unicritical case the above theorem then follows easily: if a map f is not stable, then the critical point of some nearby maps g will be periodic, and so g will be hyperbolic. In the multimodal case, the fact that the kneading sequence of nearby maps is different from that of f, does not directly imply that one can find hyperbolic maps close to f. The proof in the multimodal case, given below, is therefore more indirect.

By means of conjugacy by a real affine map, we may assume that the intersection of the filled Julia with $\mathbb{R}$ is equal to $[0, 1]$. Let Pol_d denote the family of all complex polynomials g of degree d such that $g(0) = f(0)$ and

$g(1) = f(1)$. Note that this family is parameterized by an open set in $\mathbb{C}^{d-1}$. Let $\mathrm{Pol}_d^{\mathbb{R}}$ denote the subfamily of Pol_d consisting of maps with real coefficients and let X denote the subfamily of $\mathrm{Pol}_d^{\mathbb{R}}$ consisting of maps g which have only real critical points and connected Julia set (so no escaping critical points). Moreover, let Y denote the subset of X consisting of maps g satisfying the following properties:

- every critical point of g is non-degenerate;
- every critical point and every critical value of g are contained in the open interval $(0, 1)$.

Note that Y is an open set in $\mathrm{Pol}_d^{\mathbb{R}}$.

Lemma 2.1. *$X = \overline{Y}$.*

Proof. This statement follows from Theorem 3.3 of [33]. In fact X is the family of boundary anchored polynomial maps $g : [0, 1] \to [0, 1]$ with a fixed degree and a specified shape which are determined by the degree and the sign of the leading coefficient of f. Recall that given a real polynomial $g \in X$, its *critical value vector* is the sequence $(g(c_1), g(c_2), \cdots , g(c_m))$, where $c_1 \leq c_2 \leq \cdots \leq c_m$ are all critical points of g. That theorem claims that the critical value vector determines the polynomial, and any vector $\mathbf{v} = (v_1, v_2, \ldots , v_m) \in \mathbb{R}^m$ such that these v_i lie in the correct order is the critical value vector of some map in X. In any small neighborhood of the critical value vector of f, we can choose a vector $\mathbf{v} = (v_1, v_2, \cdots , v_m)$ so that $\mathbf{v}$ satisfies the strict admissible condition, i.e., these v_i are pairwise distinct. The polynomial map corresponding to this $\mathbf{v}$ is contained in Y. □

Therefore by a perturbation if necessary we may assume that $f \in Y$. For every $g \in Y$, let $\tau(g)$ be the number of critical points which are contained in the basin of a (hyperbolic) attracting cycle. Note the map $\tau : Y \to \mathbb{N} \cup \{0\}$ is lower semicontinuous. Let

$$Y' = \{g \in Y : \tau(g) \text{ is locally maximal at } g\}.$$

As τ is uniformly bounded from above, Y' is dense in Y. Moreover, from the lower semicontinuity of τ, it is easy to see that τ is constant in a neighborhood of any $g \in Y'$. Thus Y' is open and dense in Y. Note also that every map in Y' does not have a neutral cycle (this is well-known, because one can perturb the map so the neutral cycle becomes hyperbolic attracting, see for example the proof of Theorem VI.1.2 in [8]). Doing a further perturbation if necessary, let us assume that $f \in Y'$. Let $r = \tau(f)$.

Let

$$QC(f) = \{g \in \mathrm{Pol}_d : g \text{ is quasiconformally conjugate to } f\}.$$

By Theorem 1 in [35], f does not support an invariant line field in its Julia set, and thus by Theorem 6.9 of [29], the (complex) dimension of the Teichmüller space of f is at most r. Consequently, $QC(f)$ is covered by countably many embedded complex submanifolds of Pol_d which have (complex) dimension at most r, and hence

$$QC^{\mathbb{R}}(f) = QC(f) \cap \mathrm{Pol}_d^{\mathbb{R}}$$

is covered by countably many embedded real analytic submanifolds M_i of X which have (real) dimension at most r. The same argument applies to any map in Y'.

Let $c_1 < c_2 < \cdots < c_{d-1}$ be the critical points of f, and let Λ denote the set of i such that $c_i \in AB(f)$, where $AB(f)$ is the union of basins of attracting cycles. Let $\mathcal{U}$ be a small ball in Pol_d centered at f. (Recall that Pol_d is canonically identified with an open set $\mathbb{C}^{d-1}$.) Then there exist holomorphic functions

$$c_i : \mathcal{U} \to \mathbb{C}, \ 1 \le i \le d-1,$$

such that $c_i(g)$ are all the critical points of g. By shrinking $\mathcal{U}$ if necessary, we may assume that for any $g \in \mathcal{U} \cap X$, $\quad c_1(g) < c_2(g) < \cdots < c_{d-1}(g)$ and for any $g \in \mathcal{U}$ and for any $i \in \Lambda$, $c_i(g) \in AB(g)$.

For a map $g \in \mathcal{U}$, by a *critical relation* we mean a sequence (n, i, j) of positive integers such that $g^n(c_i(g)) = c_j(g)$. Given any submanifold $\mathcal{S}$ of $\mathcal{U}$ which contains g, we say that the critical relation is *persistent* within $\mathcal{S}$ if for any $h \in \mathcal{S}$, we have $h^n(c_i(h)) = c_j(h)$.

By a further perturbation if necessary, we may assume that there is no critical relation (n, i, j) for f with $i \in \Lambda$. By shrinking $\mathcal{U}$ if necessary, this statement remains true for any $g \in \mathcal{U}$.

Completion of proof of Theorem 2.1. Let us keep the notation and assumption on f as above. We are going to prove that $\mathcal{U} \cap \mathrm{Pol}_d^{\mathbb{R}}$ contains a hyperbolic map. Arguing by contradiction, assume that every map g in $\mathcal{U} \cap \mathrm{Pol}_d^{\mathbb{R}}$ is not hyperbolic. Then $r = \tau(f) < d-1$. Since $QC^{\mathbb{R}}(f)$ is covered by countably many embedded submanifolds of X with dimension at most r, $QC(f)$ is nowhere dense in $\mathcal{U} \cap X = \mathcal{U} \cap Y'$.

For positive integers n, $1 \le i, j \le d-1$, let

$$M_{n,i,j} = \{g \in \mathcal{U} \cap X : g^n(c_i(g)) = c_j(g)\}.$$

Each of these $M_{n,i,j}$ is a subvariety of $\mathcal{U} \cap X$ with dimension at most $d-2$. By assumption $M_{n,i,j} = \emptyset$ for $i \in \Lambda$. We claim that there exists some (n, i, j) such that the dimension of $M_{n,i,j}$ is $d-2$.

To see this we use the following fact, whose proof is easy and left to the reader.

Fact 2.1. *Let m be a positive integer, and let B be a Euclidean ball in $\mathbb{R}^m$. Let M_i, $i = 1, 2, \ldots$ be embedded real analytic submanifolds of B such that $dim(M_i) \le m-2$. Then $B - \bigcup_{i=1}^{\infty} M_i$ is arc-connected.*

If all the $M_{n,i,j}$'s have dimension less than $d-2$, then $\Omega = \mathcal{U} \cap X - \bigcup M_{n,i,j}$ is arc-connected. By the standard kneading theory, [32, 25], it follows that any $g \in \Omega$ is topologically conjugate to f on the real line. By our Rigidity Theorem, $g \in QC(f)$. As Ω is dense in $\mathcal{U} \cap \mathrm{Pol}_d^{\mathbb{R}}$, this is a contradiction.

Therefore, we obtain a real analytic codimension-one embedded submanifold $\mathcal{V}_1$ of $\mathcal{U} \cap X$ which has a persistent critical relation (n, i, j) with $i \notin \Lambda$. Let us now apply the same arguments to the new $(d-2)$-dimensional family $\mathcal{V}_1$. More precisely, if $r = d-2$, then this implies that every map in $\mathcal{V}_1$ is hyperbolic, which is a contradiction. So $r < d-2$. Take any $f_1 \in \mathcal{V}_1$. As the Teichmüller space of f_1 also has (complex) dimension r, $QC(f_1) \cap X$ is nowhere dense in $\mathcal{V}_1$. Proceeding as above, we will find a real analytic embedded submanifold $\mathcal{V}_2$ of $\mathcal{V}_1$ which has dimension $d-3$ and has two distinct persistent critical relations. Repeating this argument we complete the proof. $\square$

3 Derivation of the Rigidity Theorem from the Reduced Rigidity Theorem

Definition 3.1. Let f and $\tilde{f}$ be two polynomials of degree d, $d \ge 2$. We say that they are *Thurston combinatorially equivalent* if there exist homeomorphisms $H_i : \mathbb{C} \to \mathbb{C}$, $i = 0, 1$, such that $\tilde{f} \circ H_1 = H_0 \circ f$, and $H_0 \sim H_1$ rel

$PC(f)$ (i.e., H_0 and H_1 are homotopic rel $PC(f)$). The homeomorphism H_0 is called a *Thurston combinatorial equivalence* between these two polynomials, and H_1 is called a *lift* of H_0 (with respect to f and $\tilde{f}$).

Proposition 3.1. *Let f and $\tilde{f}$ be real polynomials of degree $d \geq 2$ with only non-degenerate real critical points. Assume that they are topologically conjugate on the real axis, and let $h : \mathbb{R} \to \mathbb{R}$ be a conjugacy. Let $H : \mathbb{C} \to \mathbb{C}$ be a real–symmetric homeomorphism which coincides with h on $PC(f)$. Then H is a Thurston combinatorial equivalence between f and $\tilde{f}$.*

Remark 3.1. Let H, H' be two real–symmetric homeomorphisms of the complex plane which coincide on a set $E \subset \mathbb{R}$. Then it is clear that $H \sim H'$ rel E.

Proof. Without loss of generality, let us assume that h is orientation-preserving. Let $c_1 < c_2 < \cdots < c_{d-1}$ and $\tilde{c}_1 < \tilde{c}_2 < \cdots < \tilde{c}_{d-1}$ be the critical points of f and $\tilde{f}$ respectively. It suffices to prove that there exists a real–symmetric homeomorphism $H_1 : \mathbb{C} \to \mathbb{C}$ such that $\tilde{f} \circ H_1 = H \circ f$ and $H_1|\mathbb{R}$ preserves the orientation. Indeed, we will then have $H_1 = H$ on $PC(f)$ automatically, which implies that $H_1 \sim H$ rel $PC(f)$.

Let us add a circle $X = \{\infty e^{i2\pi t} : t \in \mathbb{R}/\mathbb{Z}\}$ to the complex plane. Then $\mathbb{C} \cup X$ is naturally identified with the closed unit disk, and f extends to a continuous map from $\mathbb{C} \cup X$ to itself, which acts on X by the formula $t \mapsto dt$ if the coefficient of the highest term of f is positive, or $t \mapsto dt + 1/2$ otherwise.

Let $T = f^{-1}(\mathbb{R})$, and $T_0 = T - \mathrm{Crit}(f)$. Note that T_0 is a (disconnected) one-dimensional manifold.

Let $x_i = \infty e^{(d-i)\pi/d}$ for each $0 \leq i \leq 2d-1$. Since each component of $\mathbb{C} - T$ is a univalent preimage of one of the half planes, it is obviously unbounded. Therefore there cannot be a closed curve in T_0, and thus each component of T_0 is diffeomorphic to the real line. The ends of these components can only be a critical point or a point x_i. By local behaviour of the critical points, for each critical c_i, there is a component γ_i of T_0 which is contained in the upper half plane and has c_i as one end. Note that the other end of γ_i must be in X, for otherwise, $\mathbb{C} - T$ would have a bounded component. As these curves γ_i, $1 \leq i \leq d-1$ are pairwise disjoint, the end of γ_i at infinity must be x_i. We have proved that the intersection of T with the upper half plane consists of $d-1$ curves γ_i, which connects x_i and c_i. By symmetry, the intersection of T with the lower half plane consists of $d-1$ curves γ_i, $d+1 \leq i \leq 2d-1$ which connects x_i and c_{2d-i}.

Similarly, $\tilde{T} = \tilde{f}^{-1}(\mathbb{R})$ has the same structure as T. Thus we can define a real–symmetric homeomorphism $H_1 : T \to \tilde{T}$ as a lift of the map $H : \mathbb{R} \to \mathbb{R}$. Since each component of $\mathbb{C} - T$ is a univalent preimage of the upper or lower half plane, H_1 extends to a homeomorphism of $\mathbb{C}$, as a lift of $H : \mathbb{C} \to \mathbb{C}$. $\square$

Derivation of the Rigidity Theorem from the Reduced Rigidity Theorem. Let f and $\tilde{f}$ be two real polynomials as in the Rigidity Theorem, and let $h : \mathbb{R} \to \mathbb{R}$ be a homeomorphism such that $\tilde{f} \circ h = h \circ f$. The Reduced Rigidity Theorem implies that we can find a real–symmetric qc map $\Phi : \mathbb{C} \to \mathbb{C}$ such that $\Phi = h$ on $PC(f)$, and such that $\tilde{f} \circ \Phi = \Phi \circ f$ holds on a neighborhood of infinity and also on a neighborhood of each periodic attractor of f. By Proposition 3.1, Φ is again a Thurston combinatorial equivalence between f and $\tilde{f}$. Let $\Phi_0 = \Phi$ and let Φ_n, $n \geq 1$, be the successive lifts. Then all these homeomorphisms Φ_n are quasiconformal with the same maximal dilatation as that of Φ. Note that Φ_n is eventually constant out of the Julia set $J(f)$ of f. Since $J(f)$ is nowhere dense, Φ_n converges to a qc map which is a conjugacy between f and $\tilde{f}$. $\square$

Although our main interest is at real polynomials with real critical points, we shall frequently need to consider a slightly larger class of maps: real polynomials with *real critical values.* This is because compositions of maps in $\mathcal{F}_d$ may have complex critical points but only real critical values. Proposition 3.1 is no longer true if we only require f to have real critical values, and this is the reason why we need to assume that f have only real critical points (rather than real critical values) in our main theorem. It is convenient to introduce the following definition.

Definition 3.2. Let f and $\tilde{f}$ be polynomials with real coefficients such that all critical values belong to the real line. We say that they are *strongly combinatorially equivalent* if they are Thurston combinatorially equivalent, and there exists a real–symmetric homeomorphism $H : \mathbb{C} \to \mathbb{C}$ such that $\tilde{f} \circ H = H \circ f$ on the real axis.

By Proposition 3.1, if f and $\tilde{f}$ have only real non-degenerate critical points, and they are topologically conjugate on $\mathbb{R}$, then they are strongly combinatorially equivalent.

4 Statement of the Key Lemma

In this section, we give the precise statement of our Key Lemma on puzzle geometry. As we will need universal bounds to treat the infinitely renormalizable case, we shall not state this lemma for a general real polynomial which does not have a satisfactory initial geometry. Instead, we shall first introduce the notion of "polynomial-like box mappings", and state the puzzle geometry for this class of maps. These polynomial-like box mappings appear naturally as first return maps to certain puzzle pieces, see for example Lemma 6.7.

Definition 4.1. Let $b \geq 1$ and $m \geq 0$ be integers. Let V_i, $0 \leq i \leq b-1$, be topological disks with pairwise disjoint closures, and let U_j, $0 \leq j \leq m$, be topological disks with pairwise disjoint closures which are compactly contained in V_0. We say that a holomorphic map

$$f : \left(\bigcup_{j=0}^{m} U_j\right) \cup \left(\bigcup_{i=1}^{b-1} V_i\right) \to \bigcup_{i=0}^{b-1} V_i \tag{1}$$

is a *polynomial-like box mapping* if the following hold:

- for each $1 \leq j \leq m$, there exists $0 \leq i = i(j) \leq b-1$ such that $f : U_j \to V_i$ is a conformal map;
- for U equal to $U_0, V_1, \ldots, V_{b-1}$, there exists $0 \leq i = i(U) \leq b-1$ such that $f : U \to V_i$ is a 2-to-1 branched covering.

The *filled Julia set* of f is defined to be

$$K(f) = \{z \in \mathrm{Dom}(f) : f^n(z) \in \mathrm{Dom}(f) \text{ for any } n \in \mathbb{N}\};$$

and the *Julia set* is $J(f) = \partial K(f)$.

In fact, everything we do will go through in the case where critical points are degenerate of even order. If $b = 1$, then such a map is frequently called *generalized polynomial-like.*

We say that f is *real–symmetric* if each of the topological disks V_i, U_j are symmetric with respect to the real axis, and f commutes with complex conjugation map. Throughout this paper, we shall only consider real–symmetric polynomial-like box mappings. Let $\mathcal{P}_b$ denote the set of real–symmetric polynomial-like box mappings (1) satisfying the following properties:

- the critical points of f are contained in the filled Julia set of f, and they are all non-periodic recurrent with the same ω-limit set;
- each branch of f is contained in the Epstein class, that is, for any interval $J \subset \mathrm{Dom}(f) \cap \mathbb{R}$ which does not contain a critical point of f, then $f^{-1}|f(J)$ extends to a univalent map defined on $\mathbb{C}_{f(J)}$.

Given a polynomial-like box mapping as above, a *puzzle piece* of depth n is a component of $f^{-n}(V_0)$. Let $P_n(x)$ denote the puzzle piece of depth n which contains x. A puzzle piece is called *critical* if it contains a critical point. Given two critical puzzle pieces P, Q, we say that Q is a *child* of P if it is a unimodal pullback of P, i.e., if there exists a positive integer n such that $f^n : Q \to P$ is a double branched covering.

Definition 4.2. We say that f is *persistently recurrent* if each critical puzzle piece has only finitely many children.

We say that f is *renormalizable* at a critical point c, if there is a puzzle piece $P_n(c)$ and a positive integer s such that $f^j(c) \notin P_n(c)$ for all $1 \le j \le s-1$ and $f^s(c) \in P_n(c)$, and the map $f^s : P_{n+s}(c) \to P_n(c)$ is a polynomial-like mapping (in the sense of Douady and Hubbard [9]) with a connected Julia set. In other words, f is renormalizable at c if c returns to all puzzle pieces $P_n(c)$ and the return times are all the same for sufficiently large n. For a map in $\mathcal{P}_b$, since the critical points have all the same ω-limit set, the map is renormalizable at one critical point if and only if it is renormalizable at any critical point. Note that a renormalizable polynomial-like box mapping is persistently recurrent.

Definition 4.3. A critical puzzle piece $P_n(c)$ is called *terminating* if the return time of c to $P_m(c)$ is the same for each $m \ge n$.

We say that f is τ-*extendible* if there are topological disks $V_i' \supset V_i$, $0 \le i \le b-1$ with $\mathrm{mod}(V_0' - \overline{V_0}) \ge \tau$ such that the following hold:

1. for each $1 \le i \le b-1$, if $0 \le k \le b-1$ is so that $f(V_i) = V_k$, then $f|V_i$ extends to a holomorphic 2-to-1 branched covering from V_i' to V_k';
2. for each $0 \le j \le m$, if k is such that $f(U_j) = V_k$, then there exists a topological disk $U_j' \supset U_j$, so that $f|U_j$ extends to a holomorphic map from U_j' to V_k' which is conformal if $j \ne 0$ and a 2-to-1 branched covering if $j = 0$;

3. moreover, $U'_j \subset V_0$, and $(U'_j - U_j) \cap PC(f) = \emptyset$.

We are most interested in real–symmetric polynomial-like box mappings with further properties:

$$D_{\pi-\sigma}(V_0 \cap \mathbb{R}) \subset V_0 \subset D_\sigma(V_0 \cap \mathbb{R}), \tag{2}$$

$$PC(f) \cap V_0 \subset \frac{1}{1+2\tau} V_0 \cap \mathbb{R}, \tag{3}$$

where $\sigma \in (0, \pi/2)$. Let $\mathcal{P}_b^{\tau,\sigma}$ denote the set of τ-extendible maps in $\mathcal{P}_b$ with the properties (2) and (3).

Definition 4.4. We say that a topological disk Ω has *ξ-bounded geometry* if it contains a Euclidean ball of radius $\xi \operatorname{diam}(\Omega)$.

Key Lemma. (Puzzle Geometry Control) *Let $f \in \mathcal{P}_b^{\tau,\sigma}$ be a persistently recurrent polynomial-like box mapping, and let c be a critical point of f. Then there is a constant $\xi = \xi(\tau, \sigma, b) > 0$ with the following properties.*

1. *Assume that f is non-renormalizable. Then for any $\varepsilon > 0$, there is a puzzle piece Y which contains c and a topological disk Y' with $V_0 \supset Y' \supset Y$ such that*
   - $\operatorname{diam}(Y) < \varepsilon$*;*
   - $(Y' - Y) \cap PC(f) = \emptyset$*, and* $\operatorname{mod}(Y' - Y) \geq \xi$*;*
   - *Y has ξ-bounded geometry, i.e., $Y \supset B(c, \xi \operatorname{diam}(Y))$.*
2. *Assume that f is renormalizable. Then there are terminating puzzle pieces $Y' \supset Y \ni c$, such that $Y \supset B(c, \xi \operatorname{diam}(Y))$ and $\operatorname{mod}(Y' - Y) \geq \xi$.*

Furthermore, if $\tilde{f} \in \mathcal{P}_b^{\tau,\sigma}$ is a map which is strongly combinatorially equivalent *to f, then the geometric bounds also apply to* the corresponding puzzle pieces for $\tilde{f}$.

Here, we say that $f : U \to V$ and $\tilde{f} : \tilde{U} \to \tilde{V}$ are *strongly combinatorially equivalent* if there are real–symmetric homeomorphisms $H_i : V \to \tilde{V}$, $i = 1, 2$, such that the following hold:

- $\tilde{f} \circ H_1 = H_0 \circ f$ on U;

- $H_1 = H_0$ on $V \setminus U$;
- $H_1 = H_0$ on $V \cap \mathbb{R}$.

Note that $H_1 \sim H_0$ rel $(V \setminus U) \cup (V \cap \mathbb{R})$. So by lift of homotopy, we can find a sequence of real–symmetric homeomorphisms $H_n : V \to \tilde{V}$, $n \geq 0$, such that $\tilde{f} \circ H_{n+1} = H_n \circ f$, and $H_n \sim H_{n+1}$ on $f^{-n}(V \setminus U)$. In particular, given a puzzle piece P of depth m for f, $\tilde{P} := H_m(P) = H_{m+1}(P) = \cdots$ is a puzzle piece for $\tilde{f}$, which is called the puzzle piece (for $\tilde{f}$) corresponding to P.

5 Yoccoz puzzle

5.1 External angles

Let f be a polynomial with degree greater than 1. Assume that the filled Julia set $K(f)$ is connected. Then by Riemann mapping theorem, there is a unique conformal map

$$B = B_f : \mathbb{C} - K(f) \to \mathbb{C} - \overline{\mathbb{D}}$$

which is tangent to the identity at infinity. The B-preimage $\mathcal{R}_\theta$ of a radical line $\{re^{i\theta} : 1 < r < \infty\}$ is called an *external ray* of angle θ, and the B preimage of the round circle $\{|z| = R\}$ with $R > 1$ is called an *equipotential curve.* Recall that the *Green function* of f is defined as

$$G(z) = G_f(z) = \begin{cases} \log|B(z)| & \text{if } z \in \mathbb{C} - K(f) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.1. *Let f and $\tilde{f}$ be two polynomials of degree $d \geq 2$ with real coefficients and real critical values which are strongly combinatorially equivalent and let H be a strong combinatorial equivalence between them. Assume that neither of these polynomials has a neutral periodic point, and assume that $h = H|\mathbb{R}$ preserves the orientation. Then for any preperiodic point $p \in J(f) \cap \mathbb{R}$ of f, the f-external ray of angle θ lands at p if and only if the $\tilde{f}$-external ray of angle θ lands at $\tilde{p} = H(p)$.*

We first prove that at each periodic point p which is contained in the interior of $K(f) \cap \mathbb{R}$, there are exactly two external rays landing at p.

Lemma 5.1. *Let f be a polynomial of degree ≥ 2. Assume that f has a connected Julia set. For any repelling periodic point p, if γ_i, $1 \leq i \leq n$ are the external rays landing at p, and V is a component of $\mathbb{C} - (\bigcup_{i=1}^n \gamma_i) \cup \{p\}$, then V intersects the orbit of some critical value.*

Proof. It is well known that there exists a positive integer m such that $f^m(\gamma_i) = \gamma_i$ for all i. See [30]. Thus $f^{-m}(V)$ has a component U which is contained in V and has p on its boundary. If V is disjoint from the orbits of the critical values, then $f^m : U \to V$ must be a conformal map, which implies that $U = V$. Let g denote the inverse of $f^m|V$. By the local dynamics at p, for any z which is close to p, we have $g^k(z) \to p$ as $k \to \infty$. So p is a Denjoy-Wolff point of g, that is, $g^k(z) \to p$ holds for any $z \in V$. Since V contains infinitely many points from the Julia set, we know that this is impossible. □

Applying this result to real polynomials, we have

Lemma 5.2. *Let f be a real polynomial with all critical values real. Assume that the Julia set is connected. Then for each repelling periodic point p of f,*

- *if $p \notin \mathbb{R}$, then there exists exactly one external ray landing at p;*
- *if p is contained in the interior of $K(f) \cap \mathbb{R}$, then there exists exactly two external rays landing at p.*

Proof. Let γ_i, $1 \leq i \leq n$ be the external rays landing at p. By the previous lemma, we know that any component V of $\mathbb{C} - \bigcup_{i=1}^n \gamma_i - \{p\}$ intersect the orbit of a critical value. Since all critical values are on the real axis and since f is real, the orbit of any critical value is on the real axis. Thus V intersects the real axis. The statements follow. □

Proof of Proposition 5.1. Let f, $\tilde{f}$ and H be as in Proposition 5.1. For any repelling periodic point $z \in int(K(f) \cap \mathbb{R})$, let $A(z)$ denote the angles of the f-external rays landing at z, let γ_z^+ (γ_z^-, respectively) denote the f-external ray in the upper (lower, respectively) which lands at p, and let $\gamma_z = \gamma_z^+ \cup \gamma_{\bar{z}}^- \cup \{z\}$. For $\tilde{z} = h(z)$, let $\tilde{A}(\tilde{z})$, $\tilde{\gamma}_{\tilde{z}}^+$, $\tilde{\gamma}_{\tilde{z}}^-$, $\tilde{\gamma}_{\tilde{z}}$ be the corresponding objects for $\tilde{f}$.

For a region V bounded by f-external rays, let $\text{ang}(V)$ denote the length of the set of angles of f-external rays which are contained in V. (We consider this set of angles as a subset of $\mathbb{R}/\mathbb{Z}$, endowed with the standard Lebesgue measure.) Note that if V' is a component of $f^{-1}(V)$ then

$$\deg(f)\text{ang}(V') = k\text{ang}(V),$$

where k is the degree of the proper map $f : V' \to V$, and $\deg(f)$ is the degree of $f : \mathbb{C} \to \mathbb{C}$. Similarly we define $\tilde{\mathrm{ang}}(V)$ for regions bounded by $\tilde{f}$-external rays.

Now let p be a repelling periodic point of f which is contained in the interior of $K(f) \cap \mathbb{R}$, and let P be the f-orbit of p. By possibly changing H on $\mathbb{C} - \mathbb{R}$, we may assume that

- for any $z \in P$, $H(\gamma_z^i) = \tilde{\gamma}_{\tilde{z}}^i$ and $\tilde{G}(H(w)) = G(w)$ for any $w \in \gamma_z^i$, where $i \in \{+, -\}$, and G and $\tilde{G}$ are the Green functions of f and $\tilde{f}$ respectively.

Let $H_0 = H$, and for $n \geq 0$, inductively define H_{n+1} to be the lift of H_n (so $H_{n+1}|\mathbb{R} = H|\mathbb{R}$; for the definition of a lift see Section 3). Note that $H_1 = H$ on the set

$$X = (\bigcup_{z \in P} \gamma_z) \cup \mathbb{R},$$

and thus $H \sim H_1$ rel X. Consequently, for each $n \geq 0$, we have

$$H_{n+1} \sim H_n \text{ rel } f^{-n}(X).$$

Let s be the period of p. Then $f^{2s}(\gamma_p^i) = \gamma_p^i$ for $i \in \{+, -\}$. Let U^- (U^+, respectively) denote the component of $\mathbb{C} - \gamma_p$ which contains the left (right, respectively) component of $\mathbb{R} - \{p\}$. As we have noted, $H_{2s} = H$ on γ_p. Let V_i, $i = 1, \ldots, N$ be the components of $\mathbb{C} - f^{-2s}(\gamma_p)$ which are contained in U^+, and let $\tilde{V}_i = H_{2s}(V_i)$. Let k_i denote the degree of the proper map $f^{2s}|V_i$. Note that k_i is also the degree of $\tilde{f}^{2s}|\tilde{V}_i$, and that $f^{2s}(V_i) = U^+$ if and only $\tilde{f}^{2s}(\tilde{V}_i) = \tilde{U}^+$. Let Λ^- and Λ^+ denote the set of i's with $f^{2s}(V_i) = U^-$ and $f^{2s}(V_i) = U^+$ respectively. Note that

$$\begin{aligned}
\deg(f)\mathrm{ang}(U^+) &= \sum_{i=1}^{N} \deg(f)\mathrm{ang}(V_i) \\
&= \sum_{i \in \Lambda^-} k_i \mathrm{ang}(U^-) + \sum_{i \in \Lambda^+} k_i \mathrm{ang}(U^+) \\
&= \sum_{i \in \Lambda^-} k_i + (\sum_{i \in \Lambda^+} k_i - \sum_{i \in \Lambda^-} k_i)\mathrm{ang}(U^+),
\end{aligned}$$

where in the last equality, we used the relation $\mathrm{ang}(U^-) + \mathrm{ang}(U^+) = 1$. Therefore,

$$\mathrm{ang}(U^+) = \frac{\sum_{i \in \Lambda^-} k_i}{\deg(f) + \sum_{i \in \Lambda^-} k_i - \sum_{i \in \Lambda^+} k_i}.$$

The same equality is true for $\tilde{\text{ang}}(\tilde{U}^+)$, and thus $\text{ang}(U^+) = \tilde{\text{ang}}(\tilde{U}^+)$. Therefore the f-external rays landing at p and $\tilde{f}$-external rays landing at $h(p)$ have the same angles.

Now we see that we can choose the homeomorphism H so that it coincides with $B_{\tilde{f}}^{-1} \circ B_f$ on $X' = \{G(z) \geq 1\} \cup \gamma_p^+ \cup \gamma_p^-$. Then $H_n = B_{\tilde{f}}^{-1} \circ B_f$ on $f^{-n}(X')$, which implies the angles of the f-external rays landing at any preimage q of p coincide with that of the $\tilde{f}$-external rays landing at $\tilde{q} = h(q)$. □

5.2 Yoccoz puzzle partition

Given a polynomial with a connected Julia set, Yoccoz introduced the powerful method of cutting the complex plane using external rays and equipotential curves. We are going to review this concept in this section.

Let f be a polynomial with a connected Julia set. To define a Yoccoz puzzle, we specify a forward invariant subset Z of the Julia set and a positive number r. We require that the set Z satisfies the following properties:

1. for each $z \in Z$, there are at least two external rays landing at z;
2. $Z \cap PC(f) = \emptyset$;
3. each periodic point in Z is repelling.

Let Γ_0 be the union of the equipotential curve $\{G(z) = r\}$, the external rays landing on Z and the set Z. We call a bounded component of $\mathbb{C} - \Gamma_0$ a *puzzle piece of depth* 0 (with respect to (Z, r)). Similarly, for each $n \in \mathbb{N}$, a bounded component of $\mathbb{C} - f^{-n}(\Gamma)$ is called a *puzzle piece of depth* n (with respect to (Z, r)).

Let $\mathcal{Y}_n$ denote the family of puzzle pieces of depth n, and let $\mathcal{Y} = \bigcup_{n=0}^{\infty} \mathcal{Y}_n$. A puzzle piece P is a *nice open set* in the sense that $f^k(\partial P) \cap P = \emptyset$ for any $k \geq 1$. Any two puzzle pieces P, Q are either disjoint, or nested, i.e., one is contained in the other.

Fact 5.1. *If $U \supset Crit(f)$ is a union of puzzle pieces, then*

$$E(U) = \{z \in K(f) : f^n(z) \notin U \textit{ for all } n \in \mathbb{N}\}$$

is a nowhere dense compact set with zero measure.

Proof. Since U is open, the set $E(U)$ is certainly closed and thus compact. To show the other statements, we may assume that all components of U are puzzle pieces of the same depth. Using the "thickening" technique, one shows that the set $E(U)$ is expanding, i.e., there is a conformal metric ρ, defined on a neighborhood of $E(U)$ such that for some $C > 0$ and $\lambda > 1$, $\|Df^n(z)\|_\rho \geq C\lambda^n$ holds for any $z \in E(U)$ and $n \in \mathbb{N}$. It follows that $E(U)$ is nowhere dense and has zero Lebesgue measure. For details, see [31]. $\square$

Now let us consider two strongly combinatorially equivalent polynomials f and $\tilde{f}$ which have real coefficients and real critical values and do not have neutral periodic points. Let homeomorphism $H : \mathbb{C} \to \mathbb{C}$ be a strongly combinatorial equivalence between f and $\tilde{f}$. Without loss of generality, let us assume that $h = H|\mathbb{R}$ is orientation-preserving.

Definition 5.1. A f-forward invariant set Z is called *admissible (with respect to f)* if it is a finite set contained in the interior of $K(f) \cap \mathbb{R}$ and disjoint from $PC(f)$.

Given an f-admissible set Z and any $r > 0$, let us construct a Yoccoz puzzle $\mathcal{Y}$ for f. Note that $\tilde{Z}$ is an $\tilde{f}$-admissible set, so we can construct a Yoccoz puzzle $\tilde{\mathcal{Y}}$ for the map $\tilde{f}$ using the set $\tilde{Z} = h(Z)$ and the same r. Re-choosing H if necessary, we may assume that it coincides with $B_{\tilde{f}}^{-1} \circ B_f$ on $\{G(z) \geq r\}$ as well as on $\Gamma_0 - J(f)$. Let $H_0 = H$, and for each $n \geq 1$ inductively define H_n to be the lift of H_{n-1} so that $H_n = h$ on $\mathbb{R}$. Set $X = \Gamma_0 \cup (K(f) \cap \mathbb{R})$. Then $H_{n+1} \sim H_n$ on $f^{-n}(X)$. In particular, for any puzzle piece $P \in \mathcal{Y}_n$, $H_n(P)$ is a puzzle piece in $\tilde{\mathcal{Y}}_n$, and $H_n = B_{\tilde{f}}^{-1} \circ B_f$ on $(\partial P - J(f))$. Let us denote $\tilde{P} = H_n(P)$. Note that $H_{n+i}(P) = H_n(P) = \tilde{P}$ for any $n, i \geq 0$.

Definition 5.2. Let P be a puzzle piece in $\mathcal{Y}_n$, and let $\tilde{P}$ be the corresponding puzzle piece in $\tilde{\mathcal{Y}}_n$. We say that a homeomorphism $\phi : P \to \tilde{P}$ *respects the standard boundary marking* if ϕ extends continuously to ∂P, and $\phi|\partial P = H_n|\partial P$.

Lemma 5.3. *For every puzzle piece P, there exists a qc map $\phi : P \to \tilde{P}$ which respects the standard boundary marking.*

Proof. For each $z \in \bigcup_{n=0}^{\infty} f^{-n}(Z)$, let $\mathcal{R}_z$ be the union of the f-external rays landing at z, and let $\tilde{\mathcal{R}}_{\tilde{z}}$ be the union of the $\tilde{f}$-external rays landing at $\tilde{z}$. A neighborhood Ω of z is called *f-transversal* if it is a Jordan disk bounded by a smooth curve which intersects each ray in $\mathcal{R}_z$ transversally at a single point, and $G(\partial\Omega\cap\mathcal{R}_z)$ consists of one point. An $\tilde{f}$-transversal neighborhood of $\tilde{z}$ is defined in an analogous way. Clearly, for any $\varepsilon > 0$ and any $z \in \bigcup_{n=0}^{\infty} f^{-n}(Z)$, there exists an f-transversal ($\tilde{f}$-transversal, respectively) neighborhood Ω of z ($\tilde{\Omega}$ of $\tilde{z}$, respectively) which has diameter less than ε. Moreover for a given ε, there exists $\eta > 0$ such that for any $0 < \rho < \eta$ we can find such neighborhoods with the property that $G(\Omega\cap\mathcal{R}_z) = \tilde{G}(\tilde{\Omega}\cap\tilde{\mathcal{R}}_{\tilde{z}}) = \rho$.

Claim. For any $z \in \bigcup_{n=0}^{\infty} f^{-n}(Z)$, there exists $\varepsilon > 0$ such that the following holds. Let Ω be an f-transversal neighborhood of z which is contained in $B(z, \varepsilon)$, and let $\tilde{\Omega}$ be an $\tilde{f}$-transversal neighborhood of $\tilde{z}$ which is contained in $B(\tilde{z}, \varepsilon)$. Assume that $G(\partial\Omega\cap\mathcal{R}_z) = \tilde{G}(\partial\tilde{\Omega}\cap\tilde{\mathcal{R}}_{\tilde{z}})$. Then there exists a qc homeomorphism $\phi : \Omega \to \tilde{\Omega}$ such that

- $B_{\tilde{f}}^{-1} \circ B_f$ on $\Omega\cap\mathcal{R}_z$,
- $\phi : \partial\Omega \to \partial\tilde{\Omega}$ is a diffeomorphism.

First notice that we may assume that z is a periodic point of f, as z is f-preperiodic and the orbit of z is disjoint from $\mathrm{Crit}(f)$. Let s be a positive integer such that f^s leaves each ray in $\mathcal{R}_z$ invariant. Since $|(f^s)'(z)| > 1$, if ε is sufficiently small, $f^s|B(z, \varepsilon)$ is a conformal map onto its image, which contains $B(z, \varepsilon)$ compactly. Similarly, this statement holds for the corresponding objects with tilde. Let g denote the inverse of the map $f^s|B(z, \varepsilon)$, and let $\tilde{g}$ be defined in an analogous way. Then there exists a positive integer N such that $g^N(\Omega) \subset\subset \Omega$ and $\tilde{g}^N(\tilde{\Omega}) \subset\subset \tilde{\Omega}$. Let $A = \Omega - g^N(\Omega)$ and $\tilde{A} = \tilde{\Omega} - \tilde{g}^N(\tilde{\Omega})$. Note that $g^N(\partial\Omega)$ intersects each ray in $\mathcal{R}_z$ transversally at a single point, and the analogy for the corresponding objects with tilde is also true. So we can find a diffeomorphism $\phi_0 : A \to \tilde{A}$ such that

- $\phi_0 = B_{\tilde{f}}^{-1} \circ B_f$ on $A\cap\mathcal{R}_z$;
- $\phi_0 \circ g = \tilde{g} \circ \phi_0$ on $\partial\Omega$.

For any $k \geq 1$, we inductively define a diffeomorphism $\phi_k : g^{kN}(A) \to \tilde{g}^{kN}(\tilde{A})$ using the formula

$$\phi_k \circ g^N = \tilde{g}^N \circ \phi_{k-1}.$$

As $\phi_k = \phi_{k-1}$ on $g^{kN}(\partial\Omega)$ we can glue these diffeomorphisms together to get a diffeomorphism

$$\phi : \Omega - \{z\} \to \tilde{\Omega} - \{z\},$$

with $\phi = B_{\tilde{f}}^{-1} \circ B_f$ on $\Omega \cap \mathcal{R}_z$. As quasiconformal maps these ϕ_k have the same maximal dilatation, so ϕ is quasiconformal and it extends naturally to a qc map from Ω to $\tilde{\Omega}$. This proves the claim.

Now let $P \in \mathcal{Y}$ be a puzzle piece. Take a small constant $\varepsilon > 0$. For any $z \in \partial P \cap K(f)$, we choose an f-transversal neighborhood $\Omega_z \subset B(z, \varepsilon)$ for z and an $\tilde{f}$-transversal neighborhood $\tilde{\Omega} \subset B(\tilde{z}, \varepsilon)$ for $\tilde{z}$, so that $G(\Omega \cap \mathcal{R}_z) = \tilde{G}(\tilde{\Omega} \cap \mathcal{R}_{\tilde{z}})$. Then by the claim above, we have a qc map $\phi_z : \Omega_z \to \tilde{\Omega}_{\tilde{z}}$ which is smooth on $\partial\Omega_z$ and coincides with $B_{\tilde{f}}^{-1} \circ B_f$ on $\Omega_z \cap \mathcal{R}_z$. Since $P - \Omega$ is a Jordan disk whose boundary consists of finitely many smooth curves with transversal intersections, and so is $\tilde{P} - \tilde{\Omega}$, we can find a qc map ψ from $P - \Omega$ to $\tilde{P} - \tilde{\Omega}$ so that $\psi = \phi_z$ on $\partial\Omega_z$ for each $z \in \partial P \cap J(f)$ and so that $\psi = B_{\tilde{f}}^{-1} \circ B_f$ on $(P - \Omega) \cap \mathcal{R}_z$. Gluing these qc maps ϕ_z and ψ together, we obtain a qc map $\phi : P \to \tilde{P}$ with standard boundary marking. $\square$

Remark 5.1. If the puzzle piece is symmetric with respect to $\mathbb{R}$, then we can choose the map $\phi : P \to \tilde{P}$ to be symmetric to $\mathbb{R}$ as well. See [1].

5.3 Spreading principle

The next proposition shows that we can spread a qc map between the critical puzzle pieces with standard boundary making to the whole complex plane, which is a key ingredient (although well-known to many people). For an outline on how we shall use this proposition see below Proposition 6.1.

Spreading Principle. *Let $U \supset Crit(f)$ be a nice open set consisting of puzzle pieces in $\mathcal{Y}$. Let $\phi : U \to \tilde{U}$ be a K-qc map which respects the standard boundary marking. Then there exists a K-qc map $\Phi : \mathbb{C} \to \mathbb{C}$ such that the following hold:*

1. *$\Phi = \phi$ on U, and*

2. *for each $z \notin U$, we have*
$$\tilde{f} \circ \Phi(z) = \Phi \circ f(z),$$

3. *$\bar{\partial}\Phi = 0$ on $\mathbb{C} - D(U)$, where $D(U)$ denotes the domain of the first landing map under f to U;*

4. *for each puzzle piece $P \in \mathcal{Y}$ which is not contained in $D(U)$, $\Phi(P) = \tilde{P}$ and $\Phi : P \to \tilde{P}$ respects the standard boundary marking.*

Proof. For each puzzle piece P, we choose an arbitrary qc map $\phi_P : P \to \tilde{P}$ with the standard boundary marking. Let $K' \geq K$ be an upper bound for the dilatation of the qc maps ϕ_P, where P runs over all puzzle pieces of depth 0, and all critical puzzle pieces which are not contained in U.

For a puzzle piece $P \in \mathcal{Y}_n$, let $k = k(P) \leq n$ be the minimal non-negative integer such that $f^k(P)$ is a critical puzzle piece or has depth 0, and let $\tau(P) = f^k(P)$. Then $f^k : P \to \tau(P)$ is a conformal map, and so is $\tilde{f}^k : \tilde{P} \to \widetilde{\tau(P)}$. Given a qc map $q : \tau(P) \to \widetilde{\tau(P)}$, we can define a qc map $p : P \to \tilde{P}$ by the formula $\tilde{f}^k \circ p = q \circ f^k$. Note that the maps p and q have the same maximal dilatation, and that if q respects the standard boundary marking, then so does p.

Let W_0 be the domain bounded by the equipotential curves $\{G(z) = r\}$ which we used to construct the puzzle $\mathcal{Y}$. Let Y_0 be the union of all puzzle pieces in $\mathcal{Y}_0$. For $n \geq 0$, inductively define Y_{n+1} to be the subset of Y_n consisting of puzzle pieces P of depth $n+1$ so that P is not contained in $D(U)$. Note that each puzzle piece in $Y_n - Y_{n+1}$ of depth $n+1$ is a component of $D(U)$.

We define Φ_0 to be the qc map which coincides with $B_{\tilde{f}}^{-1} \circ B_f$ on $\mathbb{C} - W_0$, and with ϕ_P for each component of Y_0. For each $n \geq 0$, assume that Φ_n is defined, then we define Φ_{n+1} so that

- $\Phi_{n+1} = \Phi_n$ on $\mathbb{C} - Y_n$,
- for each component P of Y_n, $\Phi_{n+1} = B_{\tilde{f}}^{-1} \circ B_f$ on $P - \bigcup_{Q \in \mathcal{Y}_{n+1}} Q$, and for each component $Q \in \mathcal{Y}_{n+1}$ which is contained in P, if $Q \not\subset Y_{n+1}$, then Φ_{n+1} is the pullback of ϕ, and otherwise it is the pullback of $\phi_{\tau(P)}$.

For each $n \geq 0$, Φ_n is a K'-qc map. Note that Φ_n is eventually constant on $\mathbb{C} - \bigcap_n Y_n$. Since $\bigcap Y_n = E(U)$ is a nowhere dense set, Φ_n converges to a qc map Φ. The properties (1), (2) and (4) follow directly from the construction, and (3) follows from the fact that $E(U)$ has measure zero. $\square$

6 Reduction to the infinitely renormalizable case

In this and the next section, we shall prove the Reduced Main Theorem by assuming the Key Lemma. The idea is to construct K-qc maps between the corresponding critical puzzle pieces with standard boundary marking so that we can apply the Spreading Principle from Section 5.3. To do this we shall need control on the geometry of these puzzle pieces and apply the Key Lemma.

Of course, the puzzle pieces around a renormalizable critical point need not to have a uniformly bounded geometry since they converge to the small Julia set. Infinitely renormalizable critical points are particularly problematic since they are renormalizable with respect to any Yoccoz puzzle. We shall leave this problem to the next section, and assume the following proposition for the moment.

Proposition 6.1. *Let f and $\tilde{f}$ be two polynomials in $\mathcal{F}_d$, $d \geq 2$, which are topologically conjugate on $\mathbb{R}$. Let c be a critical point of f at which f is infinitely renormalizable and let $\tilde{c}$ be the corresponding critical point of $\tilde{f}$. Then there exists a quasisymmetric homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that*

$$\phi(f^n(c)) = \tilde{f}^n(\tilde{c})$$

for any $n \geq 0$.

The goal of this section is to derive the Reduced Rigidity Theorem from the Key Lemma and the above proposition.

Throughout this section, f and $\tilde{f}$ are polynomials in $\mathcal{F}_d$, $d \geq 2$, which are topologically conjugate on the real line, and $h : \mathbb{R} \to \mathbb{R}$ is a topological conjugacy which is quasisymmetric in each component of $AB(f) \cap \mathbb{R}$, where $AB(f)$ denotes the union of basins of attracting cycles of f. Without loss of generality, let us assume that h is monotone increasing.

We shall first construct an appropriate Yoccoz puzzle $\mathcal{Y}$ for f (and the corresponding one $\tilde{\mathcal{Y}}$ for $\tilde{f}$) so that every critical point which is renormalizable with respect to this Yoccoz puzzle either has very tame behaviour or is infinitely renormalizable. This is done in §6.1. This enables us to find qc standard correspondence between the corresponding puzzle pieces around (combinatorially) eventually-renormalizable critical points with bounded maximal dilatation by applying Proposition 6.1. This is done in §6.2. In §6.3, we analyze the geometry of puzzle pieces around all other critical points. We show that we can find an arbitrarily small combinatorially defined puzzle neighborhood W of these critical points with uniformly bounded geometry such that the first entry map to W has good extendibility. To deal with persistently recurrent critical points, we shall assume the Key Lemma. Finally, in

§6.4, we show how the Reduced Rigidity Theorem follows from the puzzle geometry control by applying the Spreading Principle from Section 5.3 and the QC-Criterion from Appendix 1.

6.1 A real partition

As we have seen, the construction of a Yoccoz puzzle involves the choice of a finite forward invariant set Z. In this subsection, we shall specify our choice of this set. Recall that a f-forward invariant set Z is called admissible (with respect to f) if it is a finite set contained in the interior of $K(f) \cap \mathbb{R}$ and disjoint from $PC(f)$. As there are exactly two external rays which are symmetric with respect to $\mathbb{R}$ landing at z, a Yoccoz puzzle for f can be constructed using this set Z and $r = 1$.

Definition 6.1. Let c be a critical point of f and let Z be an admissible set for f. For every $n \geq 0$, let $Q_n^Z(c)$ denote the component of $\mathbb{R} - f^{-n}(Z)$ which contains c. We say that f *is Z-recurrent at c* if for any $n \geq 0$, there exists some $k \geq 1$ such that $f^k(c) \in Q_n^Z(c)$. We say that f is *Z-renormalizable at c*, or *c is Z-renormalizable* if there exists a positive integer s, such that $f^s(c) \in Q_n^Z(c)$ for any $n \geq 0$, and the minimal positive integer s with this property is called the *Z-renormalization period* of c.

For a Z-renormalizable critical point c, we define

$$A^Z(c) = \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{s-1} Q_n^Z(f^i(c)) \cap \mathrm{Crit}(f),$$

where s stands for the Z-renormalization period of c. Note that any critical point $c' \in A^Z(c)$ is also Z-renormalizable with period s, and that $A^Z(c) = A^Z(c')$.

Fact 6.1. *Let Z be an admissible set. Then for each $c \in Crit(f)$, if f is Z-recurrent but not Z-renormalizable at c, then $|Q_n^Z(c)| \to 0$ as $n \to \infty$.*

Proof. Let $I_n = Q_n^Z(c)$, and let s_n be the return time of c to I_n. Then s_n is defined for every $n \geq 0$ and $s_n \to \infty$ as $n \to \infty$. If $|I_n|$ does not tend to zero as n tends to infinity, then there is a one-side neighborhood J of c which is contained in $\bigcap I_n$. Note that $\{f^i(J)\}_{i=0}^{\infty}$ are pairwise disjoint since so are $f^i(I_n)$, $0 \leq i \leq s_n - 1$ for every $n \geq 0$. Since f does not have a wandering interval, it follows that c is contained in the attracting basin of an attracting cycle O. As c enters $\bigcap I_n$ infinitely many times, the periodic orbit O intersects $\bigcap I_n$, which implies that the return time of c to I_n are the same for all sufficiently large n, a contradiction. $\square$

Let $\mathrm{Crit}_t(f)$ denote the set of critical points c which are contained in the attracting basin of f or have a finite orbit. A polynomial f is called *trivial* if $\mathrm{Crit}_t(f) = \mathrm{Crit}(f)$. In the following we shall assume that f is non-trivial, because otherwise the Reduced Rigidity Theorem is obvious.

Lemma 6.1. *Assume that f is non-trivial. Then there exists an admissible set Z such that if c is a Z-recurrent critical point, then either of the following holds:*

1. *$c \in Crit_t(f)$, f is Z-renormalizable at c, and $A^Z(c) \subset Crit_t(f)$;*
2. *f is recurrent and not Z-renormalizable at c, and $|Q_n^Z(c)| \to 0$ as $n \to \infty$;*
3. *f is infinitely renormalizable at c, and $A^Z(c) = \omega(c) \cap Crit(f)$.*

Moreover, in the second case, $\partial Q_0^Z(c) \cap Per(f) = \emptyset$.

Proof. First of all, since f is non-trivial, it has infinitely many periodic points, and thus we have a repelling periodic orbit X_0 which is admissible. For any $c \in \mathrm{Crit}(f) - \mathrm{Crit}_t(f)$ such that f is renormalizable but not infinitely renormalizable at c, let $J = J_c$ be the smallest properly periodic interval which contains c, and let s be the period of J. Since $f^s|J$ has a critical point c which has an infinite forward orbit and is not contained in the attracting basin of a periodic attractor, $f^s|J$ has infinitely many periodic points. Thus, we can find a repelling periodic orbit $X_1(c)$ which is admissible with respect to f and intersects J. Let X_1 be the union of these $X_1(c)$'s and let $X = X_0 \cup X_1$. By logic, if f is X-renormalizable at c, then either $c \in \mathrm{Crit}_t(f)$ or f is infinitely renormalizable. Note that the last statement remains true if we replace X by any larger admissible set of f. By Fact 6.1, if f is X-recurrent but not X-renormalizable at c, we have $|Q_n^X(c)| \to 0$ as $n \to \infty$.

Let us now consider a critical point c at which f is infinitely renormalizable. It is well known that c is approximated by periodic points of f from both sides. Thus for any $\varepsilon > 0$, we can find an admissible periodic orbits Y_c such that $|Q_0^{Y_c}(c)| < \varepsilon$. By the no wandering interval theorem, for any $\delta > 0$, there exists $\varepsilon > 0$ such that the length of any pullback of $(c-\varepsilon, c+\varepsilon)$ is less than δ. As every point in $A^{Y_c}(c)$ is contained in a pullback of $Q_0^{Y_c}(c)$, it follows that $A^{Y_c}(c) \subset \omega(c)$ provided that we have chosen $\varepsilon > 0$ sufficiently small. Let Y be the union of such Y_c's. Then for every critical point c at which f is infinitely renormalizable, we have $A^Y(c) \subset A^{Y_c}(c) \subset \omega(c)$.

Now let $Z = X \cup Y$. Then for every Z-recurrent critical point c, either of the three possibilities listed in the lemma happens. For the last statement to hold, we simply replace Z with $f^{-n}(Z) \cap \mathbb{R}$ for an appropriately large n. $\square$

Let us fix an admissible set Z as above, and construct Yoccoz puzzle $\mathcal{Y}$, $\tilde{\mathcal{Y}}$ for f and $\tilde{f}$, using Z and $\tilde{Z}$ respectively. For any point x and any integer $n \geq 0$, we use $P_n(x)$ to denote the puzzle piece in $\mathcal{Y}_n$ which contains x if there is such one. The notation $\tilde{P}_n(x)$ is defined in analogous way. Note that for any $x \in \mathbb{R} - \bigcup_{n=0}^{\infty} f^{-n}(Z)$, and any $n \geq 0$, the real trace of $P_n(x)$ equals to $Q_n^Z(x)$. We shall also fix a sequence of combinatorial equivalences H_n as in the previous section.

Let $\mathrm{Crit}_{rn}(f)$ denote the set of Z-renormalizable critical points of f, and $\mathrm{Crit}_{er}(f)$ the set of *post Z-renormalizable* critical points, i.e., critical points c for which there exists an integer $k \geq 0$ and a critical point $c' \in \mathrm{Crit}_{rn}(f)$ such that $f^k(c) \in \bigcap_n P_n(c')$. So $\mathrm{Crit}_{er}(f) \supset \mathrm{Crit}_{rn}(f)$. Similarly we define $\mathrm{Crit}_{rn}(\tilde{f})$ and $\mathrm{Crit}_{er}(\tilde{f})$.

6.2 Correspondence between puzzle pieces containing post-renormalizable critical points

Lemma 6.2. *There exists a constant $K > 1$ such that for each $c \in \mathrm{Crit}_{er}(f)$ and any $n \geq 0$, there exists a real–symmetric K-qc map $\phi : P_n(c) \to \tilde{P}_n(\tilde{c})$ which respects the standard boundary marking, and matches h on $\bigcap_n P_n(c) \cap \mathbb{R}$.*

Proof. It suffices to prove the lemma in the case that f is Z-renormalizable at c. Let s be the minimal positive integer such that $f^s(c) \in P_n(c)$ for any $n \geq 0$. For $0 \leq i \leq s$, let $I_i = \bigcap_{n=0}^{\infty} P_n(f^i(c)) \cap \mathbb{R}$. Then $\{I_i\}_{i=0}^s$ is a cycle of properly periodic intervals. Let N be a sufficiently large positive integer such that $P_N(f^i(c)) - I_i$ does not contain any critical point for every $0 \leq i \leq s-1$. Let $U_i = P_{N+s-i}(f^i(c))$ and let

$$F : \bigcup_{i=0}^{s-1} U_i \to \bigcup_{i=1}^{s} U_i$$

be the restriction of f. Let $\tilde{F}$ be the corresponding map for $\tilde{f}$.
Claim. There exists a real–symmetric qc map

$$\Phi : \bigcup_{i=0}^{s} U_i \to \bigcup_{i=0}^{s} \tilde{U}_i$$

which is a conjugacy between F and $\tilde{F}$ such that

- $\Phi(\partial U_i) = H_{N+s}|\partial U_i$ for every $0 \le i \le s$;
- $\Phi = h$ on $\bigcup_{i=0}^{s-1} I_i$.

Let us prove the claim. By a combinatorial equivalence between F and $\tilde{F}$ we mean a homeomorphism $\varphi_0 : \bigcup_{i=0}^{s} U_i \to \bigcup_{i=0}^{s} \tilde{U}_i$ such that there exists a homeomorphism $\varphi_1 : \bigcup_{i=0}^{s} U_i \to \bigcup_{i=0}^{s} \tilde{U}_i$ with the following properties:

- $\tilde{F} \circ \varphi_1 = \varphi_0 \circ F$ holds on $\bigcup_{i=0}^{s-1} U_i$;
- $\varphi_0 = \varphi_1$ on $PC(F) \cup (U_s - U_0)$.

Note that H_{N+s} is a combinatorial equivalence between F and $\tilde{F}$.

Let $J \subset \mathbb{R}$ be a small neighborhood of the periodic attractors of F. In the following we are going to find a real–symmetric qc combinatorial equivalence Φ_0 between F and $\tilde{F}$ such that Φ_0 coincides with H_{N+s} on $\bigcup_{i=0}^{s} \partial U_i$ and with h on J. Once we find this map Φ_0, the desired qc conjugacy Φ can be constructed by a similar argument as what we used to derive the Rigidity Theorem from the Reduced Rigidity Theorem. The details are left to the reader.

Let us prove the existence of Φ_0. To this end, we first apply Lemma 5.3 and the Spreading Principle from Section 5.3 to find a real–symmetric qc map $\Psi : \mathbb{C} \to \mathbb{C}$ so that for every $0 \le i \le s$, $\Psi(U_i) = \tilde{U}_i$ and $\Psi|\partial U_i = H_{N+s}|\partial U_i$. More precisely, let U to be the union of critical puzzle pieces of f with depth $N+s$. By Lemma 5.3, we can find a real–symmetric qc map $\psi : U \to \tilde{U}$ which respects the standard boundary marking for each component of U. Applying the Spreading Principle from Section 5.3 we find the map Ψ. Next we notice that for each $0 \le i \le s-1$, there is a qs map $\phi_i : I_i \to \tilde{I}_i$ which coincides with h on $I_i \cap PC(F)$ as well as J. Indeed, if $c \in \mathrm{Crit}_t(f)$, then every critical point of F either is contained in the attracting basin of a periodic attractor or has a finite forward orbit, and thus such a ϕ_i obviously exists; if f is infinitely renormalizable at c, then this is guaranteed by Proposition 6.1. Finally note that $PC(F) \cap U_i$ is compactly contained in U_i, $0 \le i \le s-1$, and thus we can find a real–symmetric qc homeomorphism from $\bigcup_{i=0}^{s} U_i$ onto $\bigcup_{i=0}^{s} \tilde{U}_i$ which coincides with Ψ on $\bigcup_{i=0}^{s} \partial U_i$, and with ϕ_i for every $0 \le i \le s-1$. This map is the desired Φ_0. The proof of the claim is completed.

Let K_0 be the maximal dilatation of Φ. Then for any $k \ge 0$, Φ provides a real–symmetric qc homeomorphism from $P_{N+ks}(c)$ onto $\tilde{P}_{N+ks}(\tilde{c})$ respecting the standard boundary marking. Changing N to be $N+j$, $j = 1, 2, \ldots, s-1$, and repeating the above argument, we complete the proof of this lemma. $\square$

6.3 Geometry of the puzzle pieces around other critical points

Definition 6.2. Let A be a subset of $\mathrm{Crit}(f)$, and let V be a nice open set which contains A. We say that V is a *puzzle neighborhood* of A if each component of V is a puzzle piece intersecting A.

Let $\delta > 0$ and $N \in \mathbb{N}$. Let $V' \supset V$ be an open set consisting of pairwise disjoint topological disks and let B be any subset of $\mathrm{Crit}(f)$. For every $a \in A$, Let V_a and V'_a denote the components of V and V' containing a respectively. We say that the first landing map R (under f) to V is *N-extendible to V' with respect to B* if the following holds: if $f^s : U \to V_a$ is a branch of the first landing map R, and if $U' = \mathrm{Comp}_U(f^{-s}(V'_a))$, then

$$\#\{0 \le j \le s-1 : \big(f^j(U') - f^j(U)\big) \cap B \ne \emptyset\} \le N.$$

We say that the first landing map R is *(δ, N)-extendible with respect to B* if there exists a topological disk $V'_a \supset V_a$ for every $a \in A$ such that $\mathrm{mod}(V'_a - V_a) \ge \delta$ and such that R is N-extendible to $\bigcup_a V'_a$ with respect to B.

Recall that a Jordan disk Ω in $\mathbb{C}$ has η-bounded geometry if it contains a Euclidean ball of radius $\eta \,\mathrm{diam}(\Omega)$. The goal of this subsection is to prove the following.

Proposition 6.2. *There exist a positive constant δ and a positive integer N such that the following holds. For every $\varepsilon > 0$, there is a puzzle neighborhood W of $\mathrm{Crit}(f) \setminus \mathrm{Crit}_{er}(f)$ with the following properties:*

1. *every component of W has diameter $< \varepsilon$;*

2. *every component of W has δ-bounded geometry;*

3. *the first landing map under f to W is (δ, N)-extendible with respect to $Crit(f) \setminus Crit_{er}(f)$.*

Moreover, these statements remain true if we replace the objects for f with the corresponding ones for $\tilde{f}$.

Before we prove this proposition let us state the following consequence which will be convenient for us.

Corollary 6.3. *For any integer $n \geq 0$ there exists a puzzle neighborhood W of $Crit(f) \setminus Crit_{er}(f)$ such that for every landing domain U to W, the following hold.*

- *both of U and $\tilde{U}$ have η-bounded geometry;*
- *U is contained in a puzzle piece $P \in \mathcal{Y}_n$, and moreover,*

$$\text{mod}(P - U) \geq \eta, \ \text{mod}(\tilde{P} - \tilde{U}) \geq \eta,$$

where $\eta > 0$ is a constant independent of n.

Proof. First note that there exists an integer n_0 such that for any critical point $c \in \text{Crit}(f) \setminus \text{Crit}_{er}(f)$ and $c' \in \text{Crit}_{er}(f)$, and for any $k \geq 0$, we have $f^k(c') \notin P_{n_0}(c)$. We may assume that $n \geq n_0$.

Let W be a puzzle neighborhood of $\text{Crit}(f) \setminus \text{Crit}_{er}(f)$ with properties specified in the previous proposition such that for every $c \in \text{Crit}(f) \setminus \text{Crit}_{er}(f)$ we have $\text{mod}(P_n(c) - W_c) \geq 1$ and $\text{mod}(\tilde{P}_n(\tilde{c}) - \tilde{W}_{\tilde{c}}) \geq 1$, where W_c (respectively $\tilde{W}_{\tilde{c}}$) is the component of W (respectively $\tilde{W}$) which contains c (respectively $\tilde{c}$). Then for every $c \in \text{Crit}(f) \setminus \text{Crit}_{er}(f)$, we have a topological disk W'_c with $W_c \subset W'_c \subset P_n(c)$, such that $\text{mod}(W'_c - W_c) \geq \delta'$ and such that the first landing map to W under f is N-extendible to $W' = \bigcup_c W'_c$ with respect to $\text{Crit}(f) \setminus \text{Crit}_{er}(f)$. As the forward orbits of critical points in $\text{Crit}_{er}(f)$ are disjoint from W', the first landing map to W is 0-extendible with respect to $\text{Crit}_{er}(f)$.

Now let U be a landing domain to W and let $f^s : U \to W_c$ be the first landing map to W. Let $U' = \text{Comp}_U(f^{-s} W'_c)$. Then $f^s : U' \to W'_c$ has a uniformly bounded degree. Thus U has a bounded geometry, and $\text{mod}(U' - U)$ is bounded away from zero. As $W'_c \subset P_n(c)$, U' is contained in a puzzle piece $P \in \mathcal{Y}_n$. This proves the statements about the landing domains to W. The proof for the objects marked with tilde is similar. $\square$

Here we use the following fact which will be used repeatedly throughout the paper.

Fact 6.2. *Let $\phi\colon U' \to V'$ be a proper map between topological discs of degree N. Let $V \subset V'$ be a topological disc such that* $\mathrm{mod}(V' - V) \geq \eta$ *and let U be a component of $\phi^{-1}(V)$. Then* $\mathrm{mod}(U' - U) \geq \eta/N$ *and if V has δ–bounded geometry then U has $\delta'(\delta, \eta, N)$–bounded geometry.*

The rest of this subsection will be occupied by the proof of Proposition 6.2. To prove this proposition, we shall first introduce a partial order and an equivalence relation on the critical set $\mathrm{Crit}(f)$ (and also $\mathrm{Crit}(\tilde{f})$). Then we construct an arbitrarily small puzzle neighborhood of every equivalence class with bounded geometry and good extendibility. Finally we show how to get a puzzle neighborhood of the whole set $\mathrm{Crit}(f) \setminus \mathrm{Crit}_{er}(f)$.

Let us begin with two preparatory lemmas.

Lemma 6.3. *Let A be a subset of Crit(f), and let $V' \supset V$ be two puzzle neighborhoods of A. For each $a \in A$, let V_a denote the component of V which contains a, and let V'_a denote that of V'. Assume that*

$$f^k(\partial V_a) \cap V'_a = \emptyset \quad \textit{for all } k \geq 1. \tag{4}$$

Under these circumstances, if $f^s : U \to V_a$ is a branch of the first landing map to V, then for every $c \in A$, we have

$$\#\{0 \leq i \leq s-1 : \; c \in \mathrm{Comp}_{f^i(U)}(f^{-(s-i)}(V'_a))\} \leq 1.$$

Proof. Let $U' = \mathrm{Comp}_U(f^{-s}(V'_a))$. For every $0 \leq i \leq s$, let $U'_i = f^i(U')$. Arguing by contradiction, assume that there exist $0 \leq i_1 < i_2 < s$ such that $c \in U'_{i_j}$, $j = 1, 2$. Then U'_{i_2} is contained in the domain of the first entry map to V'_a, and U'_{i_1} is contained in the domain of the first entry map to U'_{i_2}. Since V' is a nice open set, $U'_{i_2} \subset V'_c$. From (4), it follows that $U'_{i_1} \subset V_c$. In particular, $f^{i_1}(U) \subset V_c$, which contradicts the hypothesis that $f^s : U \to V_a$ is a branch of the first landing map to V. □

Lemma 6.4. *Let $c \in$ Crit(f) $\setminus$ Crit$_{er}$(f). Assume that for some constant $\eta > 0$, there exists an arbitrarily large positive integer n such that $P_n(c)$ has η-bounded geometry. Then*

$$\mathrm{diam}(P_n(c)) \to 0 \quad \textit{as } n \to \infty.$$

Proof. Arguing by contradiction, assume that $\mathrm{diam}(P_n(c))$ is bounded away from zero. Then we will find a Euclidean ball $B(z,\varepsilon)$ which is contained in $P_n(c)$ for all $n \geq 0$. As

$$f^n(B(z,\varepsilon)) \subset f^n(P_n(c)) = P_0(f^n(c)) \subset \{G(z) \leq 1\},$$

$B(z,\varepsilon)$ is contained in the interior of the filled Julia set, and hence it is contained in the attracting basin of a periodic attractor. It follows that $c \in \mathrm{Crit}_{er}(f)$, a contradiction. $\square$

Now let us define a partial order $<_Z$ on $\mathrm{Crit}(f)$ as follows: for any $c, c' \in \mathrm{Crit}(f)$, $c \leq_Z c'$ if $c = c'$ or the forward orbit of $f(c')$ intersects the puzzle piece $P_n(c)$ for all $n \geq 0$. This partial order induces an equivalence relation $\sim_Z$ on $\mathrm{Crit}(f)$ in the natural way: $c \sim_Z c'$ if $c \leq_Z c'$ and vice versa. For every $c \in \mathrm{Crit}(f)$, let $[c]_Z$ denote the equivalence class of c, and let

$$Back(c) = \{c' \in \mathrm{Crit}(f) : c \leq_Z c'\}; \tag{5}$$

$$Forw(c) = \{c' \in \mathrm{Crit}(f) : c' \leq_Z c\}. \tag{6}$$

If $c, c' \notin \mathrm{Crit}_{er}(f)$ then $c \leq_Z c'$ iff $c \in \omega(c')$ or $c' = c$. Similarly, we define the corresponding objects for $\tilde{f}$. Note that $c \leq_Z c'$ if and only if $\tilde{c} \leq_Z \tilde{c}'$.

Definition 6.3. Let c be a Z-recurrent critical point of f. Let $n \geq 0$ and $k \in \mathbb{N}$, and let c' be a critical point such that $c' \sim_Z c$. We say that $P_{n+k}(c')$ is *a child* of $P_n(c)$ if $f^k(c') \in P_n(c)$ and $f^{k-1} : P_{n+k-1}(c') \to P_n(c)$ is a conformal map. We say that f is *Z-persistently recurrent* at c if for any $n \geq 0$ and any $c' \sim_Z c$, $P_n(c')$ has only finitely many children. Otherwise, we say that f is *Z-reluctantly recurrent* at c.

A Z-persistently recurrent critical c is minimal in the order $\leq_Z$, i.e., if $c' \leq_Z c$, then $c \sim_Z c'$. Note also that a Z-renormalizable critical point is Z-persistently recurrent.

Lemma 6.5. *Let c be a Z-reluctantly recurrent critical point of f. Then there exists a constant C, and for every $n \geq 0$, there exists an arbitrarily large positive integer m such that $f^m(c) \in P_n(c)$ and such that the degree of the map $f^m : P_{m+n}(c) \to P_n(c)$ is bounded from above by C.*

Proof. By definition, there exist $n_0 \geq 0$ and $c' \sim_Z c$ such that $P_{n_0}(c')$ has infinitely many children. So for any $k_0 \geq 0$, there exists a positive integer

$k \geq k_0$ and a critical point $c'' \sim_Z c$ for which $f^k(c'') \in P_{n_0}(c')$ and the degree of the map $f^k : P_{n_0+k}(c'') \to P_{n_0}(c')$ is bounded from above by $\deg(f) = d$. Note that c is contained in the domain of the first landing map to $P_{n_0+k}(c'')$, and let r be the landing time of c to $P_{n_0+k}(c'')$. As a first landing map, $f^r : P_{n_0+k+r}(c) \to P_{n_0+k}(c'')$ has a uniformly bounded degree. Assume for the moment $n \geq n_0$. Let s be the landing time of $f^{k+r}(c)$ to $P_n(c)$. Again the degree of the map $f^s : P_{n+s}(f^{k+r}(c)) \to P_n(c)$ is uniformly bounded from above. As $P_{n+s}(f^{k+r}(c)) \subset P_{n_0}(c')$, the proper map $f^{k+r+s} : P_{n+k+r+s}(c) \to P_n(c)$ can be written as the composition of three proper maps with uniformly bounded degree, and thus its degree is uniformly bounded from above. This proves the lemma in the case $n \geq n_0$. For the case $n < n_0$, we observe that there exists $n' > n_0$ such that $f^{n'-n}(c) \in P_n(c)$. $\square$

Lemma 6.6. (Puzzle geometry in the reluctantly recurrent case) *Let c be a Z-reluctantly recurrent critical point. Then there exists a positive constant η with the following properties. For any $\varepsilon > 0$, there are puzzle neighborhoods $W' \supset W$ of $Back(c)$ such that*

1. *each component of W has η-bounded geometry;*
2. *for each $p \in Back(c)$, $\operatorname{diam}(W'_p) \leq \varepsilon$ and $\mod (W'_p - W_p) \geq \eta$; and*
3. *$f^k(\partial W_p) \cap W'_p = \emptyset$ for each $p \in Back(c)$ and each $k \geq 1$,*

where W_p and W'_p denote the component of W and W' containing p respectively. Moreover, these statements remain true if we replace f with $\tilde{f}$, and replace p, c, W, W' with the corresponding objects for $\tilde{f}$.

Proof. The last assertion will follow from the proof. So let us only prove the assertion for objects without tilde.

Let $n_0 \in \mathbb{N}$ be a large positive integer such that for every $p \in \mathrm{Crit}(f) \setminus Back(c)$, the orbit of p is disjoint from $P_{n_0}(c)$. Let $V_c \ni c$ be a puzzle piece of depth $\geq n_0$, and let $\mathcal{U}$ be a family of (countably many) pairwise disjoint puzzle pieces U which are compactly contained in V so that

- $\{U : U \in \mathcal{U}\}$ is a covering of the domain of the first return map to V_c, and
- $f^k(\partial U) \cap V_c = \emptyset$ for all $k \geq 1$.

Such a pair will be called *good*. For any $\delta > 0$, a good pair $(V_c, \mathcal{U})$ will be called δ*-good* if for each $U \in \mathcal{U}$, U has δ-bounded geometry and $\mathrm{mod}(V_c - U) \geq \delta$.

Let $(V_c, \mathcal{U})$ be a good pair. For each $p \in Back(c) \setminus \{c\}$, let $V_p = \mathcal{L}_p(V_c)$ and let $t_p \in \mathbb{N}$ be the entry time of V_p to V_c. Then $f^{t_p}(p)$ is contained in a puzzle piece U which belongs to $\mathcal{U}$, and let $W_p = \mathrm{Comp}_p(f^{-t_p}(U))$. Let W_c denote the puzzle piece in $\mathcal{U}$ which contains c. Let us consider the first entry map R to $W := \bigcup_{p \in Back(c)} W_p$. Let $f^s : P \to W_p$ be a branch of R, and let $P' = \mathrm{Comp}_P(f^{-s} V_p)$. We claim that

1. the proper map $f^s : P' \to V_p$ has a uniformly bounded degree;
2. if $P \subset W_c$, then $P' \subset W_c$.

To prove the former statement, we first notice that every critical point in $\bigcup_{i=0}^{s-1} f^i(P')$ must be contained in $Back(c)$, by the choice of n_0. As $f^k(\partial W_p) \cap V_p = \emptyset$ for every $p \in Back(c)$ and $k \in \mathbb{N}$, it follows from Lemma 6.3 that every $p \in Back(c)$ can only be contained in one of these topological disks $f^i(P')$, $1 \leq i \leq s-1$. Thus the degree of $f^s|P'$ is uniformly bounded from above. To show the latter statement, note that P' is contained in the domain of the first entry map to V_c, while W_c contains all return domains to V_c which intersect it. The proof of the claim is completed.

In particular, if $(V_c, \mathcal{U})$ is a δ-good pair, then for some $\delta' > 0$ we have

- W_p has δ'-bounded geometry;
- $\mathrm{mod}(V_p - W_p) \geq \delta'$;
- $f^k(\partial W_p) \cap V_p = \emptyset$ for all $k \geq 1$.

So it suffices to prove that for some $\delta > 0$ and any $n \in \mathbb{N}$, we can find a δ-good pair $(V_c, \mathcal{U})$ so that the depth of V_c is larger than n. Note that by Lemma 6.4, the existence of such pairs implies that the diameter of $P_n(c)$ tends to zero as $n \to \infty$.

To this end, we first notice that it suffices to find one pair. Indeed, if $(V_c, \mathcal{U})$ is a δ-good pair, and if $\hat{V}_c$ is a pullback of V_c with order N which contains c, then the pair $(\hat{V}_c, \hat{\mathcal{U}})$ is a δ'-good pair, where $\hat{\mathcal{U}}$ denotes the corresponding pullback of $\mathcal{U}$, and $\delta' > 0$ is a constant depending only on δ and N. As f is reluctantly recurrent at c, we have infinitely many pullbacks of

V_c containing c and with uniformly bounded order, and thus the statement follows.

It remains to prove the existence of such a pair. As we are assuming that $P_0(c) \cap \mathbb{R}$ does not contain a periodic point in its boundary, it follows that $P_0(c)$ is strictly nice, and thus so is any pullback of this puzzle piece. Therefore there exists a positive integer $n_1 \geq n_0$ such that $P_{n_1}(c)$ is strictly nice. Let $V_c = P_{n_1}(c)$, and let $\mathcal{U}$ be the family of all return domains to V_c. Then $(V_c, \mathcal{U})$ is a good pair. Define W_p, $p \in Back(c)$ as above, and let $\mathcal{W}$ be the family of entry domains to $\bigcup_p W_p$ which are contained in W_c. From the claim above we know that for any $P \in \mathcal{W}$ there is a topological disc $P' \supset P$ such that $f^s \colon P' \to V_p$ has bounded degree and $P' \subset W_c$. Since there are finitely many domains V_p, the pair $(W_c, \mathcal{W})$ is δ-good where δ depends on the geometry of W_p, $\mathrm{mod}(V_p \setminus W_p)$ and the number of critical points. This completes the proof. $\square$

Let us now construct puzzle neighborhoods for a Z-persistently recurrent critical point $c \in \mathrm{Crit}(f) \setminus \mathrm{Crit}_{er}(f)$.

Lemma 6.7. *Let $c \in Crit(f) \setminus Crit_{er}(f)$ be a Z-persistently recurrent critical point. Then there exists a positive constant $\delta > 0$ such that for any $\varepsilon > 0$, there exists a puzzle neighborhood W of $[c]_Z$ with the following properties:*

- *for each $p \in Back(c)$, $\mathrm{diam}(W_p) < \varepsilon$, where $W_p = \mathrm{Comp}_p(W)$;*
- *each component of W has δ-bounded geometry;*
- *the first landing map (under f) to W is (δ, N)-extendible with respect to $[c]_Z$.*

Moreover, the statements remain true if we replace the objects for f with the corresponding ones for $\tilde{f}$.

Before we prove this lemma, let us describe a procedure to produce a polynomial-like box mapping from a strictly nice puzzle piece $V = P_n(c)$ with a sufficiently large depth n. For every $p \in [c]_Z$, let V_p be the landing domain to V which contains p. (So $V_c = V$.) Note that when n is sufficiently large, $V_p \cap V_{p'} = \emptyset$ for any $p, p' \in [c]_Z$ with $p \neq p'$. Let us label these puzzle pieces V_p as $V_0 = V_c, V_1, \ldots, V_{b-1}$, where $b = \#[c]_Z$. Let $U_0 \ni c, U_1, \ldots, U_m$ be all the entry domains to $\bigcup V_p$ which intersect $orb(c) \cap V_c$. As c is Z-persistently recurrent, the number of U_i's is finite. Since V is strictly nice, these U_i are compactly contained in V_c and have pairwise disjoint closure. Let

$$F : \left(\bigcup_{i=0}^{m} U_i\right) \cup \left(\bigcup_{i=1}^{b-1} V_i\right) \to \bigcup_{i=0}^{b-1} V_i$$

be the (appropriate restriction of the) first entry map to $\bigcup_{i=0}^{b-1} V_i$ under f. Then it is easy to check that F belongs to the class $\mathcal{P}_b$. We shall call this map the *polynomial-like box mapping associated to* $V = P_n(c)$.

Proof. We are going to derive this lemma from the Key Lemma. First let us prove a simple fact about the shape of puzzle pieces which intersect the real line.

Fact 6.3. *Let P be a puzzle piece (in $\mathcal{Y}$ or $\tilde{\mathcal{Y}}$) which intersects $\mathbb{R}$. Then there exists $\sigma \in (0, \pi/2)$ such that*

$$D_{\pi-\sigma}(P \cap \mathbb{R}) \subset P \subset D_\sigma(P \cap \mathbb{R}).$$

Proof. This is not hard so show using a linearization and compactness argument. Let us prove it as follows. Without loss of generality, let us assume that $P \in \mathcal{Y}$. Let z_0 be an endpoint of $P \cap \mathbb{R}$ which is contained in the Julia set of f, and let γ be an external rays landing at z_0. It suffices to prove that for any one-side neighborhood K of z_0, there exists a constant $\sigma \in (0, \pi/2)$, such that $\gamma \cap \{G(z) \le 1\} \subset D_\sigma(K)$. To this end, first notice that we may assume that z is a periodic point of f. Then there exists a positive integer k such that $f^k(\gamma) = \gamma$. Let $\gamma_i = \gamma \cap \{1/d^{k(i+1)} \le G(z) \le 1/d^{ki}\}$ for any $i \ge 0$. Take σ to be a small constant so that $\gamma_0 \subset D_\sigma(K)$, where d is the degree of the map f. We may assume that K is small so that $f^k : K \to f^k(K)$ is a diffeomorphism and $f^k(K) \supset K$. Let $K_0 = K$ and inductively define K_i, $i \ge 1$, to be the interval which is contained in K_{i-1} such that $f^k(K_i) = K_{i-1}$. Note that $\gamma_i \subset \mathrm{Comp}_{K_i}(f^{-k}\gamma_{i-1}) \subset \mathrm{Comp}_{K_i}(f^{-ki}\gamma_0)$. By the Schwarz lemma, it follows that $\gamma_i \subset D_\sigma(K_i) \subset D_\sigma(K)$ for any $i \ge 0$. Therefore $\gamma \subset D_\sigma(K)$. $\square$

Let us continue the proof of Lemma 6.7. Let us choose a strictly nice puzzle piece $V = P_n(c)$ with a sufficiently large depth n and let F be the polynomial-like mapping associated to V. As the first landing map to V has only finitely many domains intersecting $\bigcup_{p \in [c]_Z} orb_f(p)$, and the closure of every such a domain is disjoint from ∂V, there exists $\tau > 0$ such that

$$\left((1+2\tau)(V \cap \mathbb{R}) - \frac{1}{1+2\tau}(V \cap \mathbb{R})\right) \cap \left(\bigcup_{p \in [c]_Z} orb_f(p)\right) = \emptyset.$$

It follows that F is the class $\mathcal{P}_b^{\tau,\sigma}$ for appropriately chosen constants τ, σ. As $c \notin \mathrm{Crit}_{rn}(f)$, this polynomial-like box mapping F is non-renormalizable. By the Key Lemma, there exists a constant $\delta > 0$ such that for every $\varepsilon > 0$ there exists a puzzle piece Y for F (which is also a puzzle piece in $\mathcal{Y}$) which contains c and satisfies the following.

- $\mathrm{diam}(Y) < \varepsilon$,
- Y has δ-bounded geometry;
- there exists a topological disk $Y' \supset Y$ such that $(Y' - Y) \cap orb_f(c) = \emptyset$ and $\mathrm{mod}(Y' - Y) \geq \delta$.

Let $W = \bigcup_{p \in [c]_Z} \hat{\mathcal{L}}_p(Y)$. Then W is a puzzle neighborhood of $[c]_Z$ such that every component of W has a uniformly bounded geometry, and such that the first landing map is uniformly extendible with respect to $[c]_Z$. This proves this lemma for f. To prove the corresponding statements for $\tilde{f}$, just repeat the above argument. $\square$

Lemma 6.8. *Let c be a Z-non-recurrent critical point of f which is contained in Crit$(f) \setminus$ Crit$_{er}(f)$. Then there is a constant $\eta > 0$ and for every $\varepsilon > 0$ there exists a puzzle piece $W \ni c$ (in $\mathcal{Y}$) such that* $\mathrm{diam}(W) < \varepsilon$ *and such that W has η-bounded geometry. Moreover, the statement remains true if we replace W by $\tilde{W}$.*

Proof. Once again, the last assertion will follow from the proof, and so we shall only prove the lemma for f.

Recall that $Forw(c) = \{c' \in \mathrm{Crit}(f) : c' \leq_Z c\}$. If $Forw(c)$ contains a reluctantly recurrent critical point, then this lemma follows from the previous lemma and Lemma 6.3. From now on we assume that $Forw(c)$ does not contain a reluctantly critical point, and distinguish a few cases.

Case 1. $Forw(c) = \emptyset$. In this case, there exists an integer $n_0 \geq 0$ such that for any $k \geq 1$, $P_{n_0}(f^k(c))$ does not contain a critical point. Then for any $n \geq 1$, $f^{n-1} : P_{n+n_0-1}(f(c)) \to P_{n_0}(f^n(c))$ is a conformal map with uniformly bounded distortion. It follows that $P_{n+n_0-1}(f(c))$ and hence $P_{n+n_0}(c)$ has uniformly bounded geometry.

Case 2. $Forw(c) \neq \emptyset$ does not contain any Z-recurrent critical point. Let c' be a minimal element in $Forw(c)$. As $Forw(c') \subset Forw(c)$, it follows that $Forw(c') = \emptyset$. By Case 1, there exists $\eta > 0$ such that $P_n(c')$ has η-bounded geometry. By Lemma 6.4, this implies that $\mathrm{diam}(P_n(c')) \to 0$ as $n \to \infty$.

Let n_1 be a large positive integer so that for any critical points $c_1, c_2 \in \mathrm{Crit}(f)$ with $c_1 \not\leq_Z c_2$, then $f^k(c_2) \notin P_{n_1}(c_1)$ for all $k \geq 1$. For any $n \geq n_1$, let s_n be the entry time of c to $P_n(c')$. Consider the map $f^{s_n} : P_{n_1+s_n}(c) \to P_{n_1}(c')$. Since $Forw(c)$ does not contain any Z-recurrent critical point, this map has a uniformly bounded degree. It follows that $P_{s_n+n}(c)$ has η'-bounded geometry for a constant $\eta' > 0$ which depends only on η.
Case 3. Assume that $Forw(c)$ contains a persistently recurrent critical point $p \in \mathrm{Crit}(f) \setminus \mathrm{Crit}_{er}(f)$, By Lemma 6.7, there exists a positive constant $\delta > 0$ and a sequence of puzzle pieces $P_{n_i}(p)$ such that the following hold:

- $P_{n_i}(p)$ has δ-bounded geometry;
- there exists a topological disk $\Omega_i \supset P_{n_i}(P)$ such that $\Omega_i \setminus \overline{P}_{n_i}(P)$ is an annulus disjoint from $orb(p)$ and with modulus at least δ.

By replacing Ω_i with a slightly smaller topological disk, we may assume that $\mathrm{diam}(\Omega_i) \to 0$ as $i \to \infty$. Let s_i be the first entry time of c to $P_{n_i}(p)$, and let $W_i = \mathcal{L}_c(P_{n_i}(p))$, $W_i' = \mathrm{Comp}_c\, f^{-s_i}(\Omega_i)$. It suffices to show that $f^{s_i} : W_i' \to \Omega_i$ has a uniformly bounded degree.

To prove this, we may assume that Ω_i is contained in $P_n(p)$ for a large n. For any $q \in \mathrm{Crit}(f)$, let $\nu_q = \#\{0 \leq j \leq s_i - 1 : f^j(U_i') \ni q\}$. Let us show that $\nu_q \leq 1$ for every critical point q. Arguing by contradiction, assume that $\nu_q \geq 2$ for some q. Then $q \in Back(p) \cap Forw(c)$ and q must be a Z-recurrent critical point. Since $Forw(c)$ does not contain a Z-reluctantly recurrent critical point, q is Z-persistently recurrent. From $q \in Back(p)$, it follows that $q \in [p]_Z$. Since $f^{s_i} : W_i \to P_{n_i}(p)$ is a first entry, there can be at most one j with $q \in f^j(W_i')$. So there must be some j such that $q \in f^j(W_i' \setminus W_i)$, which implies that $orb(q)$ intersects $\Omega_i - P_{n_i}(p)$. This is absurd.
Case 4. $Forw(c)$ contains a point $p \in \mathrm{Crit}_{er}(f)$. We may assume that $p \in \mathrm{Crit}_{rn}(f)$.

Let s be the minimal positive integer such that $f^s(p) \in P_n(p)$ for any $n \geq 0$. Let N be a positive integer such that $f^s|P_N$ has all its critical points in $\bigcap_n P_n(p)$. Let $I(p) = \bigcap_n P_n(p) \cap \mathbb{R}$. Note that $(P_n(p) - P_{n+s}(p)) \cap \mathbb{R} \neq \emptyset$ for all $n \geq 0$, for otherwise, $P_n(p) \cap \mathbb{R} = I(p)$, which would imply that $c \in \mathrm{Crit}_{er}(f)$.

For each $k \geq 1$, let r_k be the first entry time of c to $P_{N+ks}(p)$. Note that there are infinitely many k's such that $f^{r_k}(c) \in P_{N+ks}(p) \setminus P_{N+(k+1)s}(p)$. Let us consider such a k. Let

$$V_k = P_{N+s}(f^{r_k+ks}(c)),\ U_k = P_{N+(k+1)s}(f^{r_k}(c)).$$

By our choice of N, $f^{ks} : U_k \to V_k$ is a conformal map. Moreover, this map can be extended to a conformal map onto $\mathbb{C}_K$, where K is the component of $(P_{N-s}(p) - P_{N+2s}(p)) \cap \mathbb{R}$ which contains $V_k \cap \mathbb{R}$, and hence its distortion is bounded by a constant $C > 1$ independent of k, by the Koebe distortion theorem. As k varies, there are only finitely many possibility of V_k. Therefore, for some positive constant δ the following hold:

- the puzzle pieces U_k has δ-bounded geometry;
- $\mathrm{mod}(P_{N+(k-1)s}(p) - U_k) \geq \delta$.

We want to show that the puzzle pieces $W_k = P_{N+(k+1)s+r_k}(c)$ has δ'-bounded geometry for some $\delta' > 0$. It suffices to prove that the degree of the proper map

$$f^{r_k} : P_{N+(k-1)s+r_k}(c) \to P_{N+(k-1)s}(p)$$

is bounded from above by a constant.

For each $q \in \mathrm{Crit}(f)$, let ν_q be the number of i's, $1 \leq i \leq r_k - 1$, such that $q \in P_{N+(k-1)s+r_k-i}(f^i(x))$. We shall prove that $\nu_q \leq 2$, which thus completes the proof. We first notice that if $\nu_q \neq 0$, then $q \in Forw(c) \cap Back(p)$ provided that k is sufficiently large. As we are assuming that every Z-recurrent critical point in $Forw(c)$ is contained in $\mathrm{Crit}_{rn}(f)$, q is either Z-non-recurrent or is contained in $[p]$. If q is Z-non-recurrent, then $\nu_q = 1$ if k is sufficiently large. So assume that $q \in [p]$. Let $\mathbf{V} = \bigcup_{q\in[p]} \hat{\mathcal{L}}_q(P_{N+ks}(p))$, and $\mathbf{V}' = \bigcup_{q\in[p]} \hat{\mathcal{L}}_q(P_{N+(k-1)s}(p))$. Then both $\mathbf{V}$ and $\mathbf{V}'$ are puzzle neighborhoods of $[p]$. It is clear that $f^m(\partial\hat{\mathcal{L}}_q(P_{N+ks}(p))) \cap \hat{\mathcal{L}}_q(P_{N+(k-1)s}(p)) = \emptyset$ for any $q \in [p]$ and any $m \geq 1$. Let r'_k be the first entry time of c to $\mathbf{V}$, then by Lemma 6.3,

$$\#\{1 \leq i < r'_k : q \in P_{N+(k-1)s+r_k-i}(f^i(c))\} \leq 1.$$

Note that $f^{r_k-r'_k}|P_{N+(k-1)s+r_k-r'_k}(f^{r'_k}(c))$ is a branch of the first entry map to $\mathbf{V}'$, and thus

$$\#\{r'_k \leq i < r_k : q \in P_{N+(k-1)s+r_k-i}(f^i(c))\} \leq 1.$$

This proves that $\nu_q \leq 2$. The proof of the lemma is completed. $\square$

Let A be subset of $\mathrm{Crit}(f)$. Let us say that A is *(δ, N) -well controlled* if for any $\varepsilon > 0$, we can find a puzzle neighborhood W of A (so $\tilde{W}$ is a puzzle neighborhood of $\tilde{A}$) such that the following holds:

- each component of W (respectively $\tilde{W}$) has diameter less than ε;
- each component of W (respectively $\tilde{W}$) has δ-bounded geometry;
- the first landing map to W (respectively $\tilde{W}$) under f (respectively $\tilde{f}$) is (δ, N)-extendible with respect to A (respectively $\tilde{A}$).

Summarizing Lemmas 6.6, 6.7, 6.8, we have proved the following:

Proposition 6.4. *Let c be a critical point of f which is not contained in $Crit_{er}(f)$. Then there exists $\delta > 0$, $N \in \mathbb{N}$ such that $[c]_Z$ is (δ, N)-well controlled.*

Lemma 6.9. *Let A, B be disjoint subsets of $Crit(f) \setminus Crit_{er}(f)$, such that for every $c \in A$ and $c' \in B$, we have $c \not\leq_Z c'$. If both of A and B are (δ, N)-well controlled, then $A \cup B$ is $(\delta/2, N)$-well controlled.*

Proof. Let W_A and W_B be puzzle neighborhoods of A and B respectively. Assume that the minimal depth of components of W_B (which are puzzle pieces) is not less than the maximal depth of that of W_A. Then $W_A \cup W_B$ is a puzzle neighborhood of $A \cup B$. To see this, we notice that for puzzle pieces P, Q, if there is some $k \in \mathbb{N}$ such that $f^k(\partial P) \cap Q \neq \emptyset$, then k is not greater than the depth of P, and $f^k(P) \subset Q$.

Let m be a large positive integer such that for every $b \in B$, the forward orbit of b is disjoint from $\bigcup_{a \in A} P_m(a)$.

Let $\{W_A^n\}, \{W_B^n\}$, $n = 1, 2, \ldots$ of puzzle neighborhoods of A and B respectively, such that

1. every component of W_A^n (respectively W_B^n) has δ-bounded geometry;
2. the first landing map to W_A^n (respectively W_B^n) is (δ, N)-extendible with respect to A (respectively B);
3. the maximal diameter of the components of W_A^n goes to zero as $n \to \infty$, and so does that of W_B^n.

We claim that for every n sufficiently large, there is a positive integer n' such that $W_A^n \cup W_B^{n'}$ is a puzzle neighborhood of $A \cup B$ and the first landing map to $W_A^n \cup W_B^{n'}$ is $(\delta/2, N)$-extendible with respect to $A \cup B$.

For each $a \in A$, let W_a^n denote the component of W_A^n which contains a. The notation W_b^n for $b \in B$ is defined similarly. By definition, for every W_a^n, there is a topological disk $\hat{W}_a^n$ with

$$\operatorname{mod}(\hat{W}_a^n - W_a^n) \geq \delta, \tag{7}$$

such that if $f^s : U \to W_a^n$ is a branch of the first landing map to W_A^n, then

$$\#\{0 \leq j \leq s-1 : f^j(\hat{U}) \cap A \neq \emptyset\} \leq N,$$

where $\hat{U}$ is the component of $f^{-s}(\hat{W}_a^n)$ which contains U. By replacing $\hat{W}_a^n$ with a smaller topological disk we may assume that $\operatorname{diam}(\hat{W}_a^n) \to 0$ as $n \to \infty$. (The right hand side of (7) becomes $\delta/2$.) Provided that n is sufficiently large, we can assume that $\hat{W}_a^n \subset P_m(a)$. Then for any $0 \leq j \leq s-1$, $f^j(\hat{U})$ cannot intersect B. Similarly, for each $b \in B$ and every $n' \in \mathbb{N}$, there exists a topological disk $\hat{W}_b^{n'}$ such that $\operatorname{mod}(\hat{W}_b^{n'} - W_b^{n'}) > \delta/2$ and such that if $f^s : U \to W_b^{n'}$ is a branch of the first landing map to $W_B^{n'}$, then

$$\#\{0 \leq j \leq s-1 : f^j(\hat{U}) \cap B \neq \emptyset\} \leq N.$$

Moreover, $\operatorname{diam}(\hat{W}_b^{n'}) \to 0$ as $n \to \infty$. Thus $\hat{W}_b^{n'} \subset P_n(b)$ provided that n' is large enough. Now let us consider the set $W_{A\cup B}^n := W_A^n \cup W_B^{n'}$. By the remark at the beginning of this proof, this is a puzzle neighborhood of $A \cup B$. Let us prove that the first landing map to $W_A^n \cup W_B^{n'}$ is $(\delta/2, N)$-extendible with respect to $A \cup B$.

To this end, let U be a component of the domain of the first landing map to $W_{A\cup B}^n$, and let s be the landing time. If $f^s(U) = W_a^n$ for some $a \in A$, then noticing that $f^s : U \to W_a^n$ is also a branch of the first landing map to W_A^n, we have $\#\{0 \leq j \leq s-1 : f^j(\hat{U}) \cap A \neq \emptyset\} \leq N$, where $\hat{U} = \operatorname{Comp}_U(f^{-s}(\hat{W}_a^n))$. As $\hat{W}_a^n \subset P_m(a)$ is disjoint from $\bigcup_{b\in B} orb(b)$, $f^j(\hat{U}) \cap B = \emptyset$ for all $0 \leq j \leq s-1$, and hence the number of j's with $f^j(\hat{U}) \cap (A \cup B) \neq \emptyset$ is at most N. Now assume that $f^s(U) = W_b^{n'}$ for some $b \in B$. Then since $f^s : U \to W_b^{n'}$ is a branch of the first landing map to $W_B^{n'}$, the number of j's with $f^j(\hat{U}) \cap B \neq \emptyset$ is at most N, where $\hat{U} = \operatorname{Comp}_U f^{-s}\hat{W}_b^{n'}$. Since $\hat{W}_b^{n'} \subset P_n(b)$, $f^j(\hat{U}) \cap A = \emptyset$ for all $0 \leq j \leq s-1$, for otherwise $f^j(\hat{U})$ is contained in $P_{n+j}(a)$ for some $a \in A$, which contradicts the hypothesis that $f^s|U$ is a branch of the first landing map to $W_{A\cup B}^n$. This completes the proof. $\square$

Now we can complete the proof of Proposition 6.2.

Proof of Proposition 6.2. We first decompose the set $\mathrm{Crit}(f)$ as follows. Let $\mathrm{Crit}_0(f)$ be the set of critical points which are largest in the partial order $\leq_Z$, that is, a critical point c belongs to $\mathrm{Crit}_0(f)$ if and only if for every $c' \in \mathrm{Crit}(f)$, $c \leq_Z c'$ implies that $c' \sim_Z c$. For every $k \geq 0$, assume that $\mathrm{Crit}_k(f)$ is defined, then $\mathrm{Crit}_{k+1}(f)$ is defined to be the set of critical points which are largest in $\mathrm{Crit}(f) \setminus \mathrm{Crit}_k(f)$ in the partial order $\leq_Z$. Clearly, there is a non-negative integer m such that $\cup_{k \leq m} \mathrm{Crit}_k(f) = \mathrm{Crit}(f)$. Let $\mathrm{Crit}'(f) = \mathrm{Crit}(f) \setminus \mathrm{Crit}_{er}(f)$ and let $\mathrm{Crit}'_k(f) = \mathrm{Crit}_k(f) \cap \mathrm{Crit}'(f)$.

By Lemma 6.9, it suffices to show that for some $\delta > 0$, $N \in \mathbb{N}$, $\mathrm{Crit}'_k(f)$ is (δ, N)-well controlled for every $0 \leq k \leq m$. By Proposition 6.4, for every $c \in \mathrm{Crit}'(f)$, the equivalence class $[c]_Z$ is uniformly well controlled. For every $0 \leq k \leq m$, $\mathrm{Crit}'_k(f)$ is a finite union of equivalence classes which are not comparable to each other in the partial order $\leq_Z$, and thus again by Lemma 6.9, we see that $\mathrm{Crit}'_k(f)$ is uniformly well controlled. The proof is completed. □

6.4 Proof of the Reduced Rigidity Theorem from rigidity in the infinitely renormalizable case

Proof of the Reduced Rigidity Theorem. Let us assume Proposition 6.1, which will be proved in the next section. In Lemma 6.2, we have proved that there is a constant $K > 1$ such that for every $c \in \mathrm{Crit}_{er}(f)$, and any $n \geq 0$, we have a real–symmetric K-qc map $\phi : P_n(c) \to \tilde{P}_n(\tilde{c})$ which respects the standard boundary marking, and is equal to h on the $\bigcap_n P_n(c) \cap \mathbb{R}$. In the following, we are going to prove that

Claim. *For every critical point $c \in Crit(f) \setminus Crit_{er}(f)$, and every $n \geq 0$, there is a real–symmetric K-qc map $\phi : P_n(c) \to \tilde{P}_n(\tilde{c})$ which respects the standard boundary marking, where K is a constant independent of n.*

The Reduced Main Theorem follows from this claim by the Spreading Principle from Section 5.3. Indeed, provided that the claim is true, we can then construct a real–symmetric K-qc map $\Phi_n : \mathbb{C} \to \mathbb{C}$ such that $\tilde{f} \circ \Phi_n = \Phi_n \circ f$ holds on $\mathbb{C} - \bigcup_{c \in Crit(f)} P_n(c)$. By passing to a subsequence, Φ_n converges to a quasiconformal map whose real trace coincides with h. Thus h is qs.

It remains to prove the claim. Let us fix an integer $n \geq 0$, and choose a puzzle neighborhood W of $\mathrm{Crit}(f)\backslash\mathrm{Crit}_{er}(f)$ as in Corollary 6.3. Let k be the maximal depth of components of W and let $V = W \cup \left(\bigcup_{c\in Crit_{er}(f)} P_k(c)\right)$. Then V is a nice open set containing $\mathrm{Crit}(f)$.

We first take an arbitrary real–symmetric qc map $\phi : V \to \tilde{V}$ which respects the standard boundary marking such that $\phi|P_k(c)$ is as in Lemma 6.2 for $c \in \mathrm{Crit}_{er}(f)$. Thus the maximal dilatation of ϕ is bounded by K on these components. On the other components, we do not have a bound on the maximal dilatation at this moment. However, by the Spreading Principle from Section 5.3, we have for any critical point c a qc homeomorphism $\Phi : P_n(c) \to \tilde{P}_n(\tilde{c})$ with the following properties.

- Φ respects the standard boundary marking,
- $\bar{\partial}\Phi = 0$, a.e., on $P_n(c)\backslash D(V)$, where $D(V)$ is the domain of first landing map (under f) to V;
- on each component U of $D(V)$, $\Phi(U) = \tilde{U}$ and Φ is the pullback of ϕ (i.e., $\Phi|U$ is the appropriate branch of $\tilde{f}^{-t} \circ \phi \circ f^t$ where t is the landing time of U to V).

The QC-Criterion in Appendix 1, proves that $\Phi : \partial P_n(c) \to \partial\tilde{P}_n(\tilde{c})$ extends to a qc map between these two puzzle pieces with a bound on its maximal dilatation. More precisely, let $X = D'(V) \cap P_n(c)$, where $D'(V)$ denotes the union of the components U of the first landing map R to V such that $R(U) \subset W$. Then the dilatation of Φ is bounded by K outside X. Note that any component P of X is also a landing domain to W, and thus both of P and $\tilde{P}$ has η-bounded geometry, and $\mathrm{mod}(P_n(c) - P) \geq \eta$, $\mathrm{mod}(\tilde{P}_n(\tilde{c}) - \tilde{P}) \geq \eta$, where $\eta > 0$ is as in Corollary 6.3. The proof is completed. $\square$

7 Rigidity in the infinitely renormalizable case (assuming the Key Lemma)

In this section, using a complex bounds theorem and the Key Lemma, we prove

Theorem 7.1. *Let f and $\tilde{f}$ be two polynomials in $\mathcal{F}_d$, $d \geq 2$, which are topologically conjugate on $\mathbb{R}$. Let c be a critical point of f at which f is* *infinitely renormalizable and let $\tilde{c}$ be the corresponding critical point of $\tilde{f}$. Then there exists a quasisymmetric homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that*

$$\phi(f^n(c)) = \tilde{f}^n(\tilde{c})$$

for any $n \geq 0$.

We shall use a similar strategy as in [10]. The main step is to construct qs maps, with extra regularity, between corresponding properly periodic intervals for f and $\tilde{f}$ which also send maximal properly periodic intervals at the next renormalization level of f to the corresponding ones of $\tilde{f}$. See Lemma 7.1. To do this we shall need the Complex Bounds theorem [37] and apply apply the Key Lemma to get appropriate control of the geometry of certain puzzle pieces.

7.1 Properties of deep renormalizations

Let $[c]$ be the subset of $\mathrm{Crit}(f)$ consisting of critical points c' with the property $\omega_f(c) = \omega_f(c') \ni c, c'$. Let $b = \#[c]$. Let $h : K(f) \cap \mathbb{R} \to K(\tilde{f}) \cap \mathbb{R}$ be a topological conjugacy between f and $\tilde{f}$. We shall continue to mark with tilde objects of $\tilde{f}$.

Let $s_1 < s_2 < \ldots$ be all positive integers such that f has a properly periodic interval with period s_n which contains c. Let I_n be the maximal (closed) properly periodic interval which contains c and has period s_n, and for each $0 \leq i \leq s_n$, let $I_n^i = \mathrm{Comp}_{f^i(c)}(f^{-(s_n - i)}(I_n)) \cap \mathbb{R}$. Obviously except possibly the first a few n's, we have $\partial I_n \cap PC(f) = \emptyset$. In this case, f maps the interior of I_n^i to that of I_n^{i+1}, $0 \leq i \leq s_n - 1$.

Fact 7.1. *There exists a positive integer $N = N(f)$ such that if $n \geq N$, then the following hold.*

1. *There exist $0 = i_0 < i_1 < \ldots < i_{b-1} < s_n$ such that $I_n^{i_j}$ contains a critical point in $[c]$ and for any other $0 \leq i \leq s_n$, I_n^i is disjoint from Crit(f). Moreover, $f^{i_{j+1} - i_j}(I_n^{i_j})$ contains the critical point in $I_n^{i_{j+1}}$.*

2. *Let $J_n = \bigcup_{j=0}^{b-1} I_n^{i_j}$, where i_j's are as above. Then the distortion of the first landing map to J_n under f, restricted to $\bigcup_{i=0}^{s_n - 1} I_n^i$ is bounded from above by a constant C which depends only on $b = \#[c]$.*

3. *For any* $0 \le i \le s_n$ *and* $0 \le i' \le s_{n+1}$*, if* $I^{i'}_{n+1} \subset I^i_n$*, then*

$$(1+2\tau)I^{i'}_{n+1} \subset I^i_n,$$

where $\tau > 0$ *is a constant depending only on* b*.*

4. *The derivative of the map* $f^{s_n} : I_n \to I_n$ *is bounded from above by a constant* C *which depends only on* b*.*

5. *The multipliers of the periodic points of* $f^{s_n} : I_n \to I_n$ *are bounded from below by a constant* $\rho = \rho(f) > 1$*.*

These facts are well known: (1) is a consequence of non-existence of wandering intervals, (2-4) follow from the real bounds (see [6], [25] and [37]), and (5) follows from Theorem B in [26] or Theorem V.B in [25].

Therefore, for every $n \ge N$, I_n is an interval bounded by a fixed point β_n of f^{s_n} and its symmetric point with respect to c. Moreover for every $0 \le i \le s_n - 1$, $f^i(\beta_n)$ is the only fixed point of f^{s_n} in ∂I^i_n. Let $\lambda_n = (f^{s_n})'(\beta_n)$. By the fact above, we know that λ_n is bounded away from both infinity and 1 by constants independent of n. Similarly we define $\tilde{\lambda}_n = (\tilde{f}^{s_n})'(\tilde{\beta}_n)$. By choosing N larger if necessary, $\tilde{\lambda}_n$ is also bounded away from both infinity and 1.

Definition 7.1. Let $C > 1$ be a constant. For every $n \ge N$, and every $0 \le i \le s_n - 1$, let $\mathcal{A}_{n,i}(C)$ denote the family of orientation-preserving homeomorphisms $\phi : I^i_n \to \tilde{I}^i_n$ which satisfy the following:

- ϕ is C-qs;
- for any $x \in I^i_n$ and $a \in \partial I^i_n$,

$$\frac{1}{C}\left(\frac{|x-a|}{|I^i_n|}\right)^{\log\tilde{\lambda}_n/\log\lambda_n} \le \frac{|\phi(x)-\phi(a)|}{|\tilde{I}^i_n|} \le C\left(\frac{|x-a|}{|I^i_n|}\right)^{\log\tilde{\lambda}_n/\log\lambda_n}.$$

Moreover, let $\mathcal{B}_{n,i}(C)$ denote the set of homeomorphisms $\phi : I^i_n \to \tilde{I}^i_n$ such that

- $\phi \in \mathcal{A}_{n,i}(C)$;
- for any $0 \le j \le s_{n+1} - 1$, if $I^j_{n+1} \subset I^i_n$, then $\phi(I^j_{n+1}) = \tilde{I}^j_{n+1}$, and $\phi|I^j_{n+1} \in \mathcal{A}_{n+1,j}(C)$.

Lemma 7.1. *There exists a constant $C > 1$ such that for any $n \geq N$ and $0 \leq i \leq s_n - 1$, the family $\mathcal{B}_{n,i}(C)$ is non-empty.*

The partial conjugacies provided by this lemma will be glued together to supply a qs conjugacy between the critical orbits. This will be done in §7.5.

Definition 7.2. We say that the n-th renormalization of f (at c) is of *intersection type*, if there exists $1 \leq j \leq s_n - 1$, such that $\partial I_{n+1} \cap \partial I_{n+1}^j \neq \emptyset$.

As we shall see below, Lemma 7.1 is easy to prove in the case that the n-th renormalization is of intersection type. The remaining case is much more complicated. By means of complex methods, we shall prove

Proposition 7.2. *Let $n \geq N$ and assume that the n-th renormalization is not of intersection type. Then for every constant $C > 1$, there exists a constant $C' > 1$ (independent of n, i) such that the following holds. Assume that for every $0 \leq j \leq s_{n+1} - 1$, an orientation-preserving C-qs homeomorphism $p_j : I_{n+1}^j \to \tilde{I}_{n+1}^j$ is given. Then for every $0 \leq i \leq s_n - 1$ such that I_n^i contains a critical point of f, there exists a homeomorphism $\psi_i : I_n^i \to \tilde{I}_n^i$ which is in the class $\mathcal{A}_{n,i}(C')$ such that $\psi_i = p_j$ whenever both sides are defined.*

Proof of Lemma 7.1. Note that by Fact 7.1 (2), we only need to prove $\mathcal{B}_{n,i}(C)$ is non-empty for some constant $C > 1$ in the case that I_n^i contains a critical point. Suppose that the n-th renormalization of f is of intersection type. Then $s_{n+1} = 2s_n$, and so I_{n+1}^i and $I_{n+1}^{s_n+i}$ are the only intervals in the cycle $\{I_{n+1}^j\}_{j=0}^{s_{n+1}}$ which are contained in I_n^i. It is well known that the configuration $(I_n^i; I_{n+1}^i, I_{n+1}^{s_n+i})$ has a uniformly bounded geometry, i.e., the length of each of components of $I_n^i - \partial I_{n+1}^i \cup \partial I_{n+1}^{s_n+i}$ is comparable to that of I_n^i. Similarly the configuration $(\tilde{I}_n^i; \tilde{I}_{n+1}^i, \tilde{I}_{n+1}^{s_n+i})$ also has a uniformly bounded geometry. Thus the lemma holds for an appropriate choice of C in this case. Now assume that the n-th renormalization is not of intersection type. In this case this lemma follows from Proposition 7.2. To see this, we first observe that there exists a constant C_1 such that $\mathcal{A}_{n+1,j}(C_1)$ is non-empty, $0 \leq j \leq s_{n+1}$, since λ_{n+1} and $\tilde{\lambda}_{n+1}$ are uniformly bounded away from both infinity and one. Taking p_j to be a map in $\mathcal{A}_{n+1,j}(C_1)$ in Proposition 7.2, the map ψ_i given there is in the class $\mathcal{B}_{n,i}(C)$ for some $C > 1$. $\square$

To prove Proposition 7.2, we shall first use the complex bounds theorem to reduce it to a problem between certain real polynomials (in the class $\mathcal{T}_b$ defined below), see Lemma 7.3. To prove Lemma 7.3, we shall first construct a Yoccoz puzzle and apply the Key Lemma to get a uniform geometric control of a terminating puzzle piece, see Lemma 7.4. Then we apply the Spreading Principle from Section 5.3 and the QC-Criterion in Appendix 1.

7.2 Compositions of real quadratic polynomials

Definition 7.3. Let $\mathcal{Q}$ denote the family of real quadratic polynomials $q_t : z \mapsto t(1-z^2)-1$, $1 \le t \le 2$. For $b \in \mathbb{N}$, let $\mathcal{Q}_b$ be the family of polynomials F which can be expressed as the composition of b real quadratic polynomials, $F = q_{t_{b-1}} \circ q_{t_{b-2}} \circ \cdots \circ q_{t_0}$.

As we shall see in the next subsection, maps in $\mathcal{Q}_b$ are models for renormalizations with sufficiently large periods. Note that a map F in $\mathcal{Q}_b$ has a connected Julia set $J(F)$, with $J(F) \cap \mathbb{R} = [-1,1]$, and that -1 is a fixed point of F. Moreover, the diameter of $J(F)$ is bounded from above by a constant which depends only on b. In fact $J(F)$ is contained in the closed unit disk $\overline{\mathbb{D}}$.

To each map $F = q_{t_{b-1}} \circ q_{t_{b-2}} \circ \cdots \circ q_{t_0}$, we can associate an *extended map* $\mathbf{F} : \mathbb{C} \times \mathbb{Z}_b \to \mathbb{C} \times \mathbb{Z}_b$ defined by $\mathbf{F}(z,i) = (q_{t_i}(z), i+1)$. Let $\mathcal{T}_b$ denote the subfamily of $\mathcal{Q}_b$ consisting of maps F with the following property: the critical points $c_i = (0,i)$ of the extended map $\mathbf{F}$ are all non-periodic and recurrent, and have the same ω-limit set which is a minimal set. Note that a map in $\mathcal{T}_b$ does not have a periodic attractor.

Fact 7.2. *Fix a positive integer b. For any k there exist constants $\delta_k > 0$ and $\rho_k > 1$ such that for any $F \in \mathcal{T}_b$, the following hold.*

1. *If c is a critical point of F^k, then $|F^k(c) - c| \ge \delta_k$;*
2. *If $x \in (-1,1)$ is a fixed point of F^k, then*

$$|(F^k)'(x)| \ge \rho_k.$$

Proof. To prove the first statement, we argue by contradiction. Assume that this statement is wrong. Then there exists a sequence of maps $F_n \in \mathcal{T}_b$, $n = 1, 2, \ldots$ such that F_n has a critical point c_n with $|F_n^k(c_n) - c_n| \to 0$ as $n \to \infty$. By passing to a subsequence we may assume that F_n converges to a map $F \in \mathcal{Q}_b$ and that c_n converges to a point c which is in $\mathrm{Crit}(F)$. Then $F^k(c) = c$. So c is an attracting periodic point of F, which implies that F_n

has an attracting periodic point provided that n is sufficiently large. This contradicts the fact $F_n \in \mathcal{T}_b$.

Now let us prove the second statement. So let $F \in \mathcal{T}_b$ and let $x \in (-1, 1)$ be a fixed point of F^k. Let $T \ni x$ be the maximal interval such that $F^{2k}|T$ is monotone on T, and let L, R be the components of $T \setminus \{x\}$. Since F does not have a periodic attractor, $F^{2k}(L) \supset L$ and $F^{2k}(R) \supset R$. Note that both endpoints of T are critical points of F^{2k} and thus by the first statement of this lemma, it follows that $|F^{2k}(L)|/|L|$ and $|F^{2k}(R)|/|R|$ are both bounded away from 1. Since F has negative Schwarzian, this implies that $(F^{2k})'(x)$ and hence $|(F^k)'(x)|$ is bounded away from 1. $\square$

Fact 7.3. *There exists a constant C with the following property. Let $F \in \mathcal{T}_b$ and let x_0 be a fixed point of F which is contained in $(-1, 1)$. Then there is a well defined sequence $x_0 > x_1 > x_2 > \cdots$ such that*

- *x_1 is the point in $F^{-1}(x_0) \cap \mathbb{R}$ closest to -1;*
- *for every $n \geq 2$, $F(x_n) = x_{n-1}$*

and

$$\frac{1}{C}\Lambda^{-n} \leq |x_n - (-1)| \leq C\Lambda^{-n}, \tag{8}$$

where $\Lambda = F'(-1)$.

Proof. Let z_0 be the critical point of F which is in $\mathbb{R}$ and closest to -1. It is easy to see that $F([-1, 1]) = F([-1, z_0])$, and therefore $-1 < x_1 < z_0$. Since F does not have a periodic attractor, $F((-1, z_0)) \supset (-1, z_0)$. The existence of the sequence x_n follows. Moreover, we can find a sequence $z_0 > z_1 > z_2 > \ldots > -1$ such that $F(z_n) = z_{n-1}$ and $x_n \in (z_n, z_{n-1})$ for all $n \geq 1$. Obviously it suffices to show that $|z_n - (-1)| \asymp \Lambda^{-n}$ to get (8). By the previous fact, $|F(z_0) - z_0|$ is bounded from below by a positive constant δ depending only on b. Note that the diffeomorphism $F^n : (-1, z_n) \to (-1, z_0)$ extends to a diffeomorphism onto $(-\infty, F(z_0))$ which contains the δ-neighborhood of $(-1, z_0)$. As F has negative Schwarzian, it follows that the distortion of $F^n|(-1, z_n)$ is uniformly bounded from above. It follows that $|z_n - (-1)| \asymp \Lambda^{-n}$. $\square$

7.3 Complex bounds

Now let us state the complex bounds theorem which was proved in [37].

Theorem 7.3. (Complex bounds) *Let $g : [0,1] \to [0,1]$ be a real analytic interval map with only non-degenerate critical points. Let $c \in (0,1)$ be a critical point of g at which g is infinitely renormalizable and let b be the number of critical points of g which are contained in $\omega(c)$. Let $I \ni c$ be a properly periodic interval of g, and let s be its period. If $|I|$ is sufficiently small, then $g^s : I \to I$ extends to a holomorphic mapping $G : \Omega \to \Omega'$ of degree 2^b such that*

$$\mathrm{mod}(\Omega' \setminus \overline{\Omega}) \geq \mu,$$

where $\mu > 0$ is a constant depending only on b.

G is often called a DH polynomial-like mapping. By Douady and Hubbard straightening theorem [9], this polynomial-like mapping $G : \Omega \to \Omega'$ is conjugate to a polynomial F of degree 2^b near their filled Julia set by a K-qc map ϕ, where $K = K(\mu)$ is a constant. In fact, as this map $G : \Omega \to \Omega'$ is a composition of b real–symmetric double branched covering, the polynomial F belongs to the class $\mathcal{T}_b$. Moreover, the conjugacy ϕ can be chosen to be real–symmetric as well.

Applying this argument to the case $g = f$ and $I = I_n$, we obtain the following corollary.

Corollary 7.4. *There exists a constant $K > 1$ such that the following holds. Let n be a large positive integer. Then there exist a real polynomial $F \in \mathcal{T}_b$, and a K-qs map $\phi : I_n \to [-1,1]$ such that $f^{s_n}|I_n$ is topologically conjugate to $F|[-1,1]$ via ϕ.*

Similarly we obtain a real polynomial $\tilde{F} \in \mathcal{T}_b$, and a K-qs map $\tilde{\phi} : \tilde{I}_n \to [-1,1]$ such that $\tilde{f}^{s_n}|\tilde{I}_n$ is topologically conjugate to $\tilde{F}|[-1,1]$ via ϕ.

We should remark that the maps F and $\tilde{F}$ are strongly combinatorially equivalent (See Definition 3.2), and moreover the extended maps $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are topologically conjugate on the reals, that is, they are topologically conjugate as dynamical systems on $\mathbb{R} \times \mathbb{Z}_b$.

Lemma 7.2. *There exists a constant C which does not depend on n, such that for every $x \in int(I_n)$ and $a \in \partial I_n$ we have*

$$\frac{1}{C}\left(\frac{|x-a|}{|I_n|}\right)^{\log\Lambda/\log\lambda_n} \leq \frac{|\phi(x)-\phi(a)|}{2} \leq C\left(\frac{|x-a|}{|I_n|}\right)^{\log\Lambda/\log\lambda_n},$$

where $\Lambda = F'(-1)$. Moreover, the analogous statement for $\tilde{\phi}$ holds as well.

Proof. In the proof of Fact 7.3, we have defined a sequence $z_0 > z_1 > z_2 \ldots$ in $K(F) \cap \mathbb{R}$ such that z_0 is the left-most critical point of $F|\mathbb{R}$, and $F(z_k) = z_{k-1}$, and proved that $|(-1) - z_k| \asymp \Lambda^{-k}$. A similar argument shows that $|\beta_n - \phi^{-1}(z_k)| \asymp \lambda_n^{-k}$, where β_n is the boundary point of I_n as defined before. This implies this lemma for ϕ. The proof of the analogous statement for $\tilde{\phi}$ is similar. □

Instead of I_n, we can repeat the above argument to a properly periodic interval I_n^i containing a critical point. So Proposition 7.2 is reduced to the following lemma.

Lemma 7.3. *Let F and $\tilde{F}$ be two polynomials in $\mathcal{T}_b$ so that the extended maps $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are topologically conjugate on the reals. Assume that F is renormalizable, and the first renormalization of F is not of intersection type. Then for every $C > 1$, there exists $C' > 1$ which is independent of F and $\tilde{F}$ such that the following holds. Assume that for every maximal properly periodic interval J of F an orientation-preserving C-qs homeomorphism $p_J : J \to \tilde{J}$ is given, then there exists a C'-qs homeomorphism $\Gamma : [-1,1] \to [-1,1]$ such that $\Gamma = p_J$ on J. Moreover, there exists a constant C'' which depends only on b such that for any $x \in (-1,1)$ and $a \in \{-1,1\}$, we have*

$$\frac{1}{C''}\left(\frac{|x-a|}{2}\right)^{\log\tilde{\Lambda}/\log\Lambda} \le \frac{|\Gamma(x)-\Gamma(a)|}{2} \le C''\left(\frac{|x-a|}{2}\right)^{\log\tilde{\Lambda}/\log\Lambda}, \tag{9}$$

where $\Lambda = F'(-1)$ and $\tilde{\Lambda} = \tilde{F}'(-1)$.

7.4 Puzzle geometry control

We will now prove Lemma 7.3. So we have two polynomials $F = q_{t_{b-1}} \circ \cdots q_{t_0}$ and $\tilde{F} = q_{\tilde{t}_{b-1}} \circ \cdots \circ q_{\tilde{t}_0}$ in the class $\mathcal{T}_b$. These two maps are topologically conjugate on the real line via a homeomorphism $H : [-1,1] \to [-1,1]$ which extends to a combinatorial equivalence between them.

Let us take an orientation-reversing fixed point α of F, for example, the one which is contained in $(-1,1)$ and furthest from the origin. Let $M_0 = (\alpha, -\alpha)$, and for $n \ge 1$ define inductively M_n to be the component of the domain of the first return map (under F) to M_{n-1} which has α in its boundary. Note that the return time of M_n to M_{n-1} is always 2. If $M_1 = M_0$, then M_0 is a properly periodic interval, and so the first renormalization of F

is of intersection type. Since we are assuming that the first renormalization of F is not of intersection type, we have $M_1 \subsetneq M_0$.

Lemma 7.4. (Puzzle geometry control) *Let F and $\tilde{F}$ be as in Lemma 7.3. Assume that the first renormalization of F is not of intersection type. Let J_0 be the maximal properly periodic interval for f which contains 0. Then there exists two puzzle pieces $P' \supset P$ for F which contain J_0 and are contained in the same Yoccoz puzzle $\mathcal{Y}$ for F such that*

- *the first return time of 0 to P' is equal to the first renormalization period s;*
- *$P' - P$ is disjoint from the postcritical set $PC(F)$;*
- $\operatorname{mod}(P' - P) \geq \eta$*;*
- *P has η-bounded geometry,*

where $\eta > 0$ is a constant independent of F. Moreover, if we replace the puzzle pieces P' and P with the corresponding objects for $\tilde{F}$, the statements remain true.

Proof. As we have not yet fixed a Yoccoz puzzle partition, an element of an *arbitrary* Yoccoz puzzle for F will be called an *artificial puzzle piece* for F. Let us say that a real–symmetric Jordan disk Ω has θ-bounded shape, $0 < \theta < \pi/2$, if $D_{\pi-\theta}(\Omega \cap \mathbb{R}) \subset \Omega \subset D_\theta(\Omega \cap \mathbb{R})$. We shall prove that there exist two artificial puzzle pieces $V' \supset V$ which both contain J_0 such that the following hold.

1. V has θ-bounded shape,
2. $V' - V$ is disjoint from the $PC(F)$ and its modulus is at least τ,
3. $F^k(\partial V) \cap V' = \emptyset$ for all $k \geq 1$,
4. $PC(F) \cap V \subset (1 + 2\tau)^{-1}(V \cap \mathbb{R})$,

where $\tau > 0$, $\theta \in (0, \pi/2)$ are constants depending only on b.

Let us first show that this statement implies the existence of P and P' claimed by this lemma. To this end we notice that $V \times \{0\}$ is a nice domain for the extended map $\mathbf{F}$, and that the complex box mapping G associated to $V_0 \times \{0\}$ for $\mathbf{F}$ falls into the class $\mathcal{P}^{\tau,\theta}$. More precisely, let $V_i \subset \mathbb{C} \times \{i\}$, $0 \leq$

$i \le b-1$, be the landing domain to $V \times \{0\}$ under $\mathbf{F}$ which contains $c_i = (0, i)$, and let $U_0 \ni c_0, U_1, \ldots, U_m$ be the landing domains to $\bigcup_i V_i$ under $\mathbf{F}$ which are contained in V_0 and intersect $PC(\mathbf{F})$. Then by definition, the polynomial-like box mapping G is the first entry map from $\left(\bigcup_{j=0}^m U_j\right) \cup \left(\bigcup_{i=1}^{b-1} V_i\right)$ to $\bigcup_{i=0}^{b-1} V_i$. By properties (1-4) of V and V', it follows that $G \in \mathcal{P}_b^{\tau,\sigma}$. This map G is renormalizable, and $J_0 \times \{0\}$ is a properly periodic interval for G. Applying the Key Lemma completes the proof.

To prove the existence of V and V', let us first assume that $F^2 : M_1 \to M_0$ is monotone. Then, $F^2 : -M_1 \to M_0$ is an orientation reserving diffeomorphism, and so this map has a unique fixed point, which we denote by γ. Let $Y' \ni 0$ be the F-puzzle piece bounded by the external rays through α and $-\alpha$, and the equipotential curve $\{G(z) = 2\}$, where G is the Green function of F, and let $Y \ni 0$ denote the puzzle piece bounded by the external rays through γ and $-\gamma$, and the equipotential curve $\{G(z) = 1\}$.

Fact 7.4. *There exists $\tau > 0$ and $\sigma \in (0, \pi/2)$ such that*

- *Y has θ-bounded shape;*
- $\mathrm{mod}(Y' - Y) \ge \tau$.

Proof. First notice that the length of each component of $I - \{\alpha, -\alpha, \gamma, -\gamma\}$ is uniformly bounded away from zero. By Fact 7.2 the multipliers of α and γ are both uniformly bounded from above and away from 1. Note that $Y \subset\subset Y'$ and a compactness argument shows that $\mathrm{mod}(Y' - Y)$ is bounded away from zero. Now let $T \ni \gamma$ be the maximal open interval such that $F^4|T$ is monotone. Let L be the component of $T - \{\gamma\}$ which is contained in $(\gamma, 0)$ and let R be the other one. As we have shown, $|L|$ and $|R|$ are both bounded away from zero. Again by a compactness argument, we show that for some constant $\theta > 0$, $\partial Y \cap \{G(z) \ge 1/2^{4b}\}$ is contained in $D_\theta(L) \cap D_\theta(R)$, and disjoint from $D_{\pi-\theta}((\gamma, -\gamma))$. Note that $F^4(L) \supset L$ and $F^4(R) \supset R$. By the Schwarz lemma, it follows that the external rays landing at γ is contained in $D_\theta(L) \cap D_\theta(R)$. By symmetry, the external rays landing at $-\gamma$ is contained in $D_\theta(-L) \cap D_\theta(-R)$. Therefore,

$$Y \subset D_\theta(L) \cup D_\theta(-L) \subset D_\theta((\gamma, -\gamma)),$$

and

$$\partial Y \cap D_{\pi-\theta}((\gamma, -\gamma)).$$

This proves the first statement. $\square$

Let us continue the proof of Lemma 7.4. The artificial puzzle piece Y has a bounded shape, but $PC(F)$ may come close to its boundary. In the following, we shall pull-back this topological disk to find the desired V. Let us say that an interval $K \times \{i\} \subset \mathbb{R} \times \{i\}$, $i \in \mathbb{Z}_b$ is an $\mathbf{F}$-pullback of $M_0 \times \{0\}$ of *depth* k if it is a component of $\mathbf{F}^{-k}(M_0 \times \{0\}) \cap (\mathbb{R} \times \{i\})$. (So $i + k = 0 \mod b$.) Let m be the maximal positive integer such that $M_0 \times \{0\}$ has a unimodal $\mathbf{F}$-pullback of depth m. Let $K \times \{i\} \subset \mathbb{R} \times \{i\}$ be this pullback. Then for each $(x, i) \in PC(\mathbf{F}) \cap K \times \{i\}$, we must have $\mathbf{F}^m((x, i)) \in (M_0 - (M_1 \cup (-M_1))) \times \{0\}$. To see this, arguing by contradiction, assume that $\mathbf{F}^m((x, i)) \in Q \times \{0\}$, where Q is a component of $M_1 \cup (-M_1)$. Then

$$K' \times \{i\} = \mathrm{Comp}_{(x,i)}(\mathbf{F}^{-m}(Q \times \{0\})) \cap (\mathbb{R} \times \{i\})$$

is either a unimodal or a monotone pullback of $Q \times \{0\}$ according to $K' \ni 0$ or not. In the former case, $K' \times \{i\}$ would be a unimodal pullback of $M_0 \times \{0\}$ of depth $m + 2$, which contradicts the maximality of m. In the latter case, since $K' \times \{i\}$ intersects $PC(F)$, it has a unimodal pullback which would become a unimodal pullback of $M_0 \times \{0\}$ with a higher depth, again a contradiction.

Let $W = \mathrm{Comp}_{c_i}(\mathbf{F}^{-m}(Y \times \{0\}))$ and $W' = \mathrm{Comp}_{c_i}(\mathbf{F}^{-m}(Y' \times \{0\}))$, where $c_i = (0, i)$. Then $W' - W$ is disjoint from the postcritical set of $\mathbf{F}$. Since

$$\mathbf{F}^s : (W', W) \to (Y' \times \{0\}, Y \times \{0\})$$

is a double branched covering,

$$\mathrm{mod}(W' - W) = \mathrm{mod}(Y' - Y)/2 \geq \tau/2.$$

Moreover, W has uniformly bounded geometry. To see this one first applies the Koebe distortion theorem to see that $\mathbf{F}(W)$ has uniformly bounded geometry and then applies Lemmas 13.2 and 13.3. Now we define $V \subset \mathbb{C}$ to be the topological disk such that $V \times \{0\}$ is the component of the domain of the first landing map onto W under $\mathbf{F}$, and V' the corresponding one for W'. Then V and V' are puzzle pieces of F, and they contain J_0. Let us check that these puzzle pieces satisfy properties (1-4) stated at the beginning of the proof. Properties 1, 2 and 4 come from the corresponding statements for (W', W) which we have proved (we need to redefine the constants), and so we only need to check the third one.

To this end, we first notice that it is sufficient to show that $\mathbf{F}^k(\partial V \times \{0\}) \cap (V' \times \{0\}) = \emptyset$ for all $k \geq 1$. Arguing by contradiction, assume that this is not true, and let k be the minimal positive integer such that there exists $z \in \partial V$ with $\mathbf{F}^k((z,0)) \in V' \times \{0\}$. Let r be the first landing time of $c_0 = (0,0)$ to W under $\mathbf{F}$. Since $W' - W$ is disjoint from $PC(F)$, this is also the first landing time of c_0 to W' under $\mathbf{F}$. Thus $\mathbf{F}^j(V' \times \{0\})$, $0 \leq j \leq r$ are pairwise disjoint, which implies that $k > r$. Next we notice that

$$F^n(\partial Y) \cap V' \subset F^n(\partial Y) \cap Y = \emptyset$$

for any $n \geq 0$, and thus $k < r + m$. Consequently, $\mathbf{F}^{k-m}(W')$ intersects V' since both of them contains $\mathbf{F}^k(z)$. As these sets are pullbacks of a nice domain Y', it follows that $V' \subset \mathbf{F}^{k-m}(W')$, In particular, $\mathbf{F}^{k-m}(W')$ contains a critical point, which contradicts the fact that $\mathbf{F}^k : W' \to Y'$ is a double branched covering.

Now let us assume that $F^2|M_1$ is not monotone. Then $F^2|M_2$ is monotone. To see this let p denote the critical point of $F^2|M_1$ which is closest to α. Then $F^2(M_1) = (F^2(p), \alpha)$ which implies that $F^2(p) \notin M_1$ for otherwise M_1 would become a properly periodic interval of F whose orbit does not contain 0, which is impossible. Next let $1 \leq i \leq 2b-1$ be minimal such that $q_{t_{i-1}} \circ q_{t_{i-2}} \circ \cdots q_{t_0}(M_1)$ contains 0. (In other word, i is minimal such that $\mathbf{F}^i|M_1 \times \{0\}$ contains a critical point of $\mathbf{F}$.) Write $\hat{M}_0 = q_{t_{i-1}} \circ \cdots \circ q_{t_0}(M_1)$ and $\hat{M}_1 = q_{t_{i-1}} \circ \cdots \circ q_{t_0}(M_2)$. Then $\hat{M}_1$ is a return domain to $\hat{M}_0$ under the map $\hat{F} = q_{t_{i+b-1}} \circ q_{t_{i+b-2}} \circ q_{t_i}$, and this first return map is monotone. Let $\hat{J}_0$ be the maximal properly periodic interval of $\hat{F}$ which contains 0. Repeating the previous argument, replacing F with $\hat{F}$, we see that that there exist two puzzle pieces $\hat{P}' \supset \hat{P}$ which contains $\hat{J}_0$, and satisfies the properties claimed in this lemma. Note that $\hat{J}_0 \times \{i\}$ and $J_0 \times \{0\}$ are both maximal properly periodic intervals of $\mathbf{F}$, and so they are contained in the same cycle of properly periodic intervals for $\mathbf{F}$ since the critical points of $\mathbf{F}$ all have the same ω-limit set. Let P (respectively P') be the Jordan domains such that $P \times \{0\}$ (respectively $P' \times \{0\}$) is the component of the first entry map to $\hat{P}$ (respectively $\hat{P}'$) under $\mathbf{F}$ which contains $c_0 = (0,0)$. These puzzle pieces are what we look for.

As the whole argument is combinatorial, the last assertion of this lemma follows. □

Proof of Lemma 7.3. We keep the notation introduced before the statement of Lemma 7.4. Let $P' \supset P$ be the puzzle pieces as in Lemma 7.4. Let $\mathcal{Y}$ be a Yoccoz puzzle for F which has P and P' as pieces, and let $\tilde{\mathcal{Y}}$ be the corresponding puzzle for $\tilde{F}$.

Let $P'' = \mathcal{L}_0(P')$, that is, the component of the domain of the first return map to P' under F which contains 0. From now on we shall assume that $P \subset P''$, and $\mathrm{mod}(P'' - P)$ is bounded away from zero. (Otherwise we simply replace P with $\mathcal{L}_0(P)$ in the following argument).

For each critical point c of F, let P_c (respectively P'_c, P''_c) be the component of the domain of the first landing map L_P to P (respectively P', P'') under F. Since we are assuming that $\omega(c) \ni 0$, these puzzle pieces exist. As $P' - P$ is disjoint from $PC(F)$, the first landing time of c to P coincides with that to P', and $(P'_c - P_c) \cap PC(F) = \emptyset$. Since $L_P : P'_c \to P'$ has a uniformly bounded degree, and since P has η-bounded geometry, there exists a constant $\delta > 0$ such that

- $\mathrm{mod}(P''_c - P_c) \geq \delta$,
- the puzzle pieces P_c have δ-bounded geometry.

Let D (respectively D', D'') denote the domain of the first landing map to $\bigcup_c P_c$ (respectively $\bigcup_c P'_c$, $\bigcup_c P''_c$) under F. Then for a similar reason, redefining the constant $\delta > 0$ if necessary, the following holds: for any component U of D,

- U has δ-bounded geometry,
- if $U'' = \mathrm{Comp}_U(D'')$, then $\mathrm{mod}(U'' - U) \geq \delta$.

Step 1. Let us prove that there exists a constant $K > 1$, and for every $c \in \mathrm{Crit}(F)$, there exists a K-qc homeomorphism $\psi_c : P''_c \to \tilde{P}''_c$ which respects the standard boundary marking. To this end, we first take for every $c \in \mathrm{Crit}(F)$ an arbitrary real–symmetric qc map $\phi_c : P_c \to \tilde{P}_c$ respecting the standard boundary marking. Applying the Spreading Principle from Section 5.3, we obtain a real–symmetric qc map $\Phi : \mathbb{C} \to \mathbb{C}$ such that

- for every puzzle piece $U \in \mathcal{Y}$ which is not contained in the domain D, we have $\Phi(U) = \tilde{U}$ and $\Phi : U \to \tilde{U}$ respects the standard boundary marking,
- $\bar{\partial}\Phi = 0$ a.e., on $\mathbb{C} - D$.

In particular, for every $c \in \mathrm{Crit}(F)$, $\Phi(P''_c) = \tilde{P}''_c$ and $\Phi|P''_c$ respects the standard boundary marking.

Now let us fix a critical point c, and let $X = D \cap P_c''$. By what we have proved above, each component U of X has δ-bounded geometry, and $\mathrm{mod}(U'' - U) \geq \delta$, where $U'' = \mathrm{Comp}_U(D'')$. Note that $U'' \subset P_c''$, and thus $\mathrm{mod}(P_c'' - U) \geq \delta$. Similarly the analogous statements for $\Phi(U)$ are true as well. By the QC-Criterion from Appendix 1, there exists a K-qc homeomorphism $\psi_c : P_c'' \to \tilde{P}_c''$ with the same boundary marking as $\Phi|\partial P_c''$ which is standard, where $K = K(\delta) > 1$ is a constant.

Step 2. Let s be the first renormalization period of F. Then $F^s : P'' \to P'$ is a DH polynomial-like mapping with a connected Julia set $\mathcal{J}$. We claim that there exists a K-qc map $u : P' - \mathcal{J} \to \tilde{P}' - \tilde{\mathcal{J}}$, where $K > 1$ is as above.

To see this, we first apply the Spreading Principle from Section 5.3 once again, using the maps ψ_c, and obtain a real–symmetric K-qc map $\Psi : P' \to \bigcup_c \tilde{P}'$ respecting the standard boundary marking, Moreover, $\Psi|P'' = \psi_0|P''$, and so $\Psi : P' \to \tilde{P}'$ also respects the standard boundary marking. As $P' - \mathcal{J} = \bigcup_{n=0}^{\infty} G^{-n}(P' - P'')$, where $G = F^s|P''$, we can define, for each $n \geq 0$, a real–symmetric K-qc map $u_n : G^{-n}(P' - P'') \to \tilde{G}^{-n}(\tilde{P}' - \tilde{P}'')$ using the formula $\tilde{G}^n \circ u_n = \Psi \circ G^n$. The qc maps match continuous on their common domains, and we can glue together to obtain the desired map u.

Step 3. Let $J_0 = \mathcal{J} \cap \mathbb{R}$, which is the maximal properly periodic interval of F which contains the critical point 0. Let us show that $\mathrm{diam}(\mathcal{J})/|J_0|$ is bounded from above by a constant. In fact, as $\mathrm{mod}(P' - \mathcal{J}) \geq \mathrm{mod}(P' - P)$ is bounded away from 0, by Lemma 2.4 in [21], there exist two topological disks $V \subset\subset W$ which contain $\mathcal{J}$ such that $F^s : V \to W$ is a DH polynomial-like mapping which has $\mathcal{J}$ as its Julia set and such that $\mathrm{mod}(W - V)$ is bounded away from zero. By Douady-Hubbard straightening theorem, there exists a real–symmetric K-qc map ξ and a polynomial $G \in \mathcal{T}_b$ such that $F^s : V \to W$ is conjugate to G near their Julia sets via ξ, where K is a constant. So $\xi(J_0) = [-1, 1]$ and $\xi(\mathcal{J})$ is the Julia set of G. As the diameter of G is uniformly bounded from above, the statement follows.

Step 4. Now let us prove that there exists a real–symmetric K-qc map $\gamma = \gamma_{P'} : P' \to \tilde{P}'$ which respects the standard boundary marking and such that $\gamma|J_0 = p|J_0$ is as in the assumption of this lemma. We have seen above that there exists a real–symmetric K-qc map $\Psi : P' \to \tilde{P}'$, and so this statement will follow if we prove the following

1. $\mathrm{mod}(P' - J_0)$ and $\mathrm{mod}(\tilde{P} - \tilde{J}_0)$ are both uniformly bounded away from zero;

2. $\operatorname{mod}(\tilde{P} - \tilde{J}_0)/\operatorname{mod}(P' - J_0)$ is uniformly bounded away from both infinity and zero.

Notice that $J_0 \subset P$ and $\tilde{J}_0 \subset \tilde{P}$, and so (1) follows. By Step 3, we see that $\operatorname{mod}(P' - J_0) \asymp \operatorname{mod}(P' - \mathcal{J})$ and $\operatorname{mod}(\tilde{P}' - \tilde{J}_0) \asymp \operatorname{mod}(\tilde{P}' - \tilde{\mathcal{J}})$. By Step 2, $\operatorname{mod}(\tilde{P}' - \tilde{\mathcal{J}}) \asymp \operatorname{mod}(P' - \mathcal{J})$. Thus (2) holds.

Step 5. Similarly we show that for every component Q of D', there exists a K-qc map $\gamma_Q : Q \to \tilde{Q}$ which respects the standard boundary marking. Moreover, if Q contains a maximal properly periodic interval J (of F) we have that $\gamma_Q|J = p|J$ is as in the assumption of this lemma.

Step 6. We can now complete the proof. Recall that D' is the domain of the first landing map to P' under F. Since $\omega(c) \ni 0$ for every $c \in \mathrm{Crit}(F)$, D' contains the critical set of F. For every component Q of D', we have proved that there exists a qc map $\gamma_Q : Q \to \tilde{Q}$ with the standard boundary marking and with a bound on its dilatation. Of course when Q is real–symmetric, we can take γ_Q to be real–symmetric as well. Moreover if Q contains a maximal properly periodic interval J of F, we can choose γ_Q such that it coincides with p_J. Applying the Spreading Principle from Section 5.3 (taking U be the union of all components of D' which contain a critical point or a properly periodic interval), we obtain a real–symmetric K-qc map $\Gamma : \mathbb{C} \to \mathbb{C}$ which coincides with p_J on each J. Observe that there exists a fixed point α of F such that Γ coincides with the topological conjugacy h on $\bigcup_{n=0}^{\infty} F^{-n}(\alpha) \cap \mathbb{R}$. Applying Fact 7.3, we see that (9) holds for an appropriate choice of the constant C''. □

7.5 Gluing

So far we have proved Lemma 7.1. So for every $n \geq N$ and $0 \leq i \leq s_n - 1$ we have an orientation preserving homeomorphism $\phi_{n,i} : I_n^i \to \tilde{I}_n^i$ which belongs to the class $\mathcal{B}_{n,i}(C)$, where C is a constant. Next we are going to glue these $\phi_{n,i}$'s together to get a qs conjugacy between $orb_f(c)$ and $orb_{\tilde{f}}(\tilde{c})$.

Conclusion of the proof of Theorem 7.1. Let $\phi_{n,i}$ be as above. Define

$$\phi : \bigcup_{i=0}^{s_N - 1} I_N^i \to \bigcup_{i=0}^{s_N - 1} \tilde{I}_N^i$$

to be the unique homeomorphism such that for any $n \geq N$ and $0 \leq i \leq s_n - 1$, $\phi = \phi_{n,i}$ on $I_n^i - \bigcup_{j=0}^{s_{n+1}-1} I_{n+1}^j$. Note that ϕ maps I_n^i to $\tilde{I}_n^i$ for each (n,i). Since

$\max_i |I_n^i|$ and $\max_i |\tilde{I}_n^i|$ shrinks to zero, ϕ forms a conjugacy between $orb_f(c)$ and $orb_{\tilde{f}}(\tilde{c})$.

Let us prove that ϕ is qs, by which we mean that the restriction of ϕ to each component of its domain is qs. To this end, let $u < v < w$ be three points in some I_N^i such that $v - u = w - v$. We need to estimate $(\phi(v) - \phi(u))/(\phi(w) - \phi(v))$. This estimate will follows from the claims below and will be done at the end of the proof.

Claim 1. For any n, i, and any $a \in \partial I_n^i$ and $b \in int(I_n^i)$, we have

$$\frac{1}{C_1}|\phi_{n,i}(a) - \phi_{n,i}(b)| \leq |\phi(a) - \phi(b)| \leq C_1|\phi_{n,i}(a) - \phi_{n,i}(b)|,$$

where $C_1 > 1$ is a constant independent of n, i.

Indeed, by construction $\phi(a) = \phi_{n,i}(a)$. If $b \notin \bigcup_j I_{n+1}^j$, then we also have $\phi(b) = \phi_{n,i}(b)$, and thus the inequality holds. Let us assume that $b \in U = I_{n+1}^j$ for some j, and let x be the intersection of (a, b) with ∂U. Also, let y be the other endpoint of U. Note that

$$\begin{aligned} |\phi_{n,i}(a) - \phi_{n,i}(x)| = |\phi(a) - \phi(x)| \qquad & \leq |\phi(a) - \phi(b)| \\ \leq |\phi(a) - \phi(y)| \qquad & = |\phi_{n,i}(a) - \phi_{n,i}(y)|. \end{aligned}$$

By Fact 7.4, $|a - x|/|x - y|$ is bounded away from zero uniformly. Since $\phi_{n,i}$ is C-qs, it follows

$$|\phi_{n,i}(a) - \phi_{n,i}(x)| \asymp |\phi_{n,i}(a) - \phi_{n,i}(y)|,$$

which completes the proof of this claim.

Claim 2. For any interval $(a, b) \subset I_n^i$ with $[a, b] \not\subset \bigcup_j int(I_{n+1}^j)$, we have

$$\frac{1}{C_2}|\phi_{n,i}(a) - \phi_{n,i}(b)| \leq |\phi(a) - \phi(b)| \leq C_2|\phi_{n,i}(a) - \phi_{n,i}(b)|,$$

where $C_2 > 1$ is a constant independent of (n, i).

The boundary of $\bigcup_j I_{n+1}^j$ divides (a, b) into finitely many intervals T_k. We may assume that (a, b) is contained in a component K of $I_n^i - \partial \bigcup_j I_{n+1}^j$. If K is not a component of $\bigcup_j I_{n+1}^j$, then $\phi(a) = \phi_{n,i}(a)$ and $\phi(b) = \phi_{n,i}(b)$ by construction, and so the inequality holds. Thus we assume that K is component of $\bigcup_j I_{n+1}^j$. Since $[a, b] \not\subset int(K)$, either $a \in \partial K$ or $b \in \partial K$. By the previous claim,

$$|\phi(a) - \phi(b)| \asymp |\phi_{n+1,j}(a) - \phi_{n+1,j}(b)|.$$

Since both of $\phi_{n+1,j}$ and $\phi_{n,i}|I^j_{n+1}$ belong to the class $\mathcal{A}_{n+1,j}(C)$, the last term is comparable to $|\phi_{n,i}(a) - \phi_{n,i}(b)|$, which proves this claim.

Claim 3. For any $\varepsilon > 0$ there is an $\varepsilon' > 0$ such that for any interval $(a, b) \subset I^i_n$, if $|a - b| \geq \varepsilon |I^i_n|$, then $|\phi(a) - \phi(b)| \geq \varepsilon' |\tilde{I}^i_n|$.

Let $m \geq 0$ be maximal such that $(a, b) \subset \bigcup_j I^j_{n+m}$. For each $0 \leq k \leq m$, let $0 \leq j_k \leq s_{n+k} - 1$ be such that $(a, b) \subset I^{j_k}_{n+k}$. By Claim 2, $|\phi(a) - \phi(b)| \asymp |\phi_{n+m,j_m}(a) - \phi_{n+m,j_m}(b)|$. Since $|I^{j_m}_{n+m}| \geq |a - b| \geq \varepsilon |I^{j_0}_n|$, all the intervals $I^{j_k}_{n+k}$, $0 \leq k \leq m$, are comparable to (a, b). By Fact 7.1, $|I^{j_{k+1}}_{n+k+1}|/|I^{j_k}_{n+k}|$ is uniformly bounded away from 1 for each $0 \leq k \leq m-1$. Thus m is bounded in terms of ε. Since ϕ_{n+m} is C-qs, $|\phi_{n+m,j_m}(a) - \phi_{n+m,j_m}(b)| \asymp |\phi_{n+m,j_m}(I^{j_m}_{n+m})| = \tilde{I}^{j_m}_{n+m}$. Since ϕ_{n+k,j_k}, $0 \leq k \leq m-1$ are all C-qs, $|\tilde{I}_{n+m,j_m}| \asymp |\tilde{I}_{n,i}|$. This proves this claim.

Let us now prove that $A = |\phi(u) - \phi(v)|/|\phi(w) - \phi(v)|$ is bounded away from both infinity and zero. Let $M \geq 0$ be maximal such that u, v, w are contained in I^i_M for some $0 \leq i \leq s_M - 1$. If neither $[u, v]$ nor $[v, w]$ is contained in $\bigcup_j \int(I^j_{M+1})$, then by Claim 2, $|\phi(u) - \phi(v)| \asymp |\phi_{M,i}(u) - \phi_{M,i}(v)|$, and $|\phi(v) - \phi(w)| \asymp |\phi_{M,i}(v) - \phi_{M,i}(w)|$, and hence A is bounded from both above and from zero. So without loss of generality, let us assume $[u, v]$ is compactly contained in I^j_{M+1} for some j. By the maximality of M, $[v, w]$ is not contained in this interval, and thus we still have $|\phi(v) - \phi(w)| \asymp |\phi_{M,i}(v) - \phi_{M,i}(w)|$. So it suffices to prove that $|\phi(u) - \phi(v)| \asymp |\phi_{M,i}(u) - \phi_{M,i}(v)|$. Let $v' \in (v, w) \cap \partial I^j_{M+1}$ and we shall distinguish two cases.

Case 1. $|v' - v|/|u - v|$ is very small. By Claim 2,

$$\frac{|\phi(v) - \phi(w)|}{|\phi(v') - \phi(w)|} \asymp \frac{|\phi_{M,i}(v) - \phi_{M,i}(w)|}{|\phi_{M,i}(v') - \phi_{M,i}(w)|},$$

which is bounded from both above and below since $\phi_{M,i}$ is C-qs. Similarly, $|\phi(u) - \phi(v')| \asymp |\phi_{M,i}(u) - \phi_{M,i}(v')|$ and $|\phi(v) - \phi(v')| \asymp |\phi_{M,i}(v) - \phi_{M,i}(v')|$. Since $|v' - v|/|u - v'|$ is very small, it follows that $|\phi(v) - \phi(v')|/|\phi(u) - \phi(v')|$ is very small, and thus

$$\begin{aligned} |\phi(u) - \phi(v)| &= |\phi(u) - \phi(v')| - |\phi(v) - \phi(v')| \\ &\asymp |\phi_{M,i}(u) - \phi_{M,i}(v')| \\ &\asymp |\phi_{M,i}(u) - \phi_{M,i}(v)|. \end{aligned}$$

Case 2. $|v - v'|/|u - v|$ is bounded away from zero. In this case, by Claim 2 and the C-quasisymmetric property of $\phi_{M,i}$, it follows that

$$|\phi(v) - \phi(v')| \asymp |\phi(v') - \phi(w)| \asymp |\phi(v) - \phi(w)|,$$

and so it suffices to bound $A' = |\phi(u) - \phi(v)|/|\phi(v) - \phi(v')|$. If $[u, v]$ is not contained in a component of $\bigcup_j I^j_{M+2}$, then again by claim 2, $A' \asymp |\phi_{M+1,j}(u) - \phi_{M+1,j}(v)|/|\phi_{M+1,j}(v) - \phi_{M+1,j}(v')|$ is bounded. Assume that $[u, v]$ is contained in $I^{j'}_{M+2}$ for some $0 \le j' \le s_{M+2} - 1$. Note that (v, v') contains a component T of $I^j_{M+1} - I^{j'}_{M+2}$ and thus $|u - v| \ge |v - v'|$ is a definite proportion of $|I^{j'}_{M+2}|$. By Claim 3, $|\phi(u) - \phi(v)| \asymp \phi(I^{j'}_{M+2}) = \tilde{I}^{j'}_{M+2}$. Note that $|v - v'| \asymp |T|$, and thus $|\phi_{M+1,j}(v) - \phi_{M+1,j}(v')| \asymp |\phi_{N+1,j}(T)|$. By Claim 1, $|\phi(v) - \phi(v')| \asymp |\phi_{N+1,j}(T)|$. Since $|T| \asymp |I^{j'}_{M+2}|$ in our case, A' is bounded. $\square$

8 Proof of the Key Lemma from Upper and Lower Bounds

8.1 Construction of the enhanced nest

Let us fix a map f which is in the class $\mathcal{P}^{\tau,\sigma}_b$ as in (1) and let c_0 denote the critical point in V_0. Throughout § 8-11, we assume that f is *persistently recurrent.*

Recall that a puzzle piece is a component of $f^{-n}(V_0)$ for some $n \ge 0$. A puzzle piece $\mathbf{I}$ is *strictly nice* in the sense of Martens: for any $x \in \partial\mathbf{I}$ and any $n \ge 1$, $f^n(x) \notin \bar{\mathbf{I}}$ (if $f^n(x)$ is defined). Therefore any component of the domain of the first return map to $\mathbf{I}$ is compactly contained in $\mathbf{I}$.

Note that a puzzle piece is symmetric with respect to $\mathbb{R}$ if its intersection with $\mathbb{R}$ is non-empty. We shall use the following convention: *a puzzle piece is denoted by a bold letter and its real trace is denoted by the corresponding roman letter.* We are going to construct a sequence of puzzle pieces

$$\mathbf{I}_0 \supset \mathbf{L}_0 \supset \mathbf{K}_0 \supset \mathbf{I}_1 \supset \mathbf{L}_1 \supset \dots \tag{10}$$

around c_0, called the *enhanced nest* for the map f. The enhanced nest will be the main objects for us to study, and we shall see that it contains the puzzle pieces with properties specified in the Key Lemma.

Lemma 8.1. *Let $\mathbf{I} \ni c$ be a puzzle piece. Then there is a positive integer ν with $f^\nu(c) \in \mathbf{I}$ such that the following holds. Let $\mathbf{U}_0 = \mathrm{Comp}_c(f^{-\nu}(\mathbf{I}))$ and $\mathbf{U}_j = f^j(U)$ for $0 \le j \le \nu$. Then*

1. $\#\{0 \le j \le \nu - 1 : \mathbf{U}_j \cap \mathit{Crit}(f) \ne \emptyset\} \le b^2$;

2. $\mathbf{U}_0 \cap PC(f) \subset \mathrm{Comp}_c\left(f^{-\nu}(\mathcal{L}_{f^\nu(c)}(\mathbf{I}))\right)$.

The proof of this lemma will be given at the end of this subsection.

For each puzzle piece $\mathbf{I} \ni c$, let $\nu = \nu(\mathbf{I})$ be the smallest positive integer with the properties specified by Lemma 8.1. We define

$$\mathcal{A}(\mathbf{I}) = \mathrm{Comp}_c(f^{-\nu}(\mathcal{L}_{f^\nu(c)}(\mathbf{I}))),$$
$$\mathcal{B}(\mathbf{I}) = \mathrm{Comp}_c(f^{-\nu}(\mathbf{I})).$$

Definition 8.1. Given a puzzle piece $\mathbf{P} \ni c$, by a *successor* of $\mathbf{P}$, we mean a puzzle piece of the form $\hat{\mathcal{L}}_c(\mathbf{Q})$, where $\mathbf{Q}$ is a child of $\hat{\mathcal{L}}_{c'}(\mathbf{P})$ for some $c' \in Crit(f)$.

Since f is persistently recurrent, each critical puzzle piece $\mathbf{P}$ has a smallest successor, which we denote by $\Gamma(\mathbf{P})$. Remark that if $\mathbf{Q}$ is an entry domain to $\mathbf{P}$ intersecting $PC(f)$, then $\hat{\mathcal{L}}_c(\mathbf{Q})$ is an successor of $\mathbf{P}$ by definition, and thus $\hat{\mathcal{L}}_c(\mathbf{Q}) \supset \Gamma(\mathbf{P})$.

Now we can define the enhanced nest (10) as follows: $\mathbf{I}_0 = V_0$ and for each $n \geq 0$,

$$\begin{aligned}
\mathbf{L}_n &= \mathcal{A}(\mathbf{I}_n),\\
\mathbf{M}_{n,0} &= \mathbf{K}_n = \mathcal{B}(\mathbf{L}_n),\\
\mathbf{M}_{n,j+1} &= \Gamma(\mathbf{M}_{n,j}) \text{ for } 0 \leq j \leq T-1,\\
\mathbf{I}_{n+1} &= \mathbf{M}_{n,T} = \Gamma^T \mathcal{B}\mathcal{A}(\mathbf{I}_n),
\end{aligned}$$

where $T = 5b$. This choice is made because of Lemma 8.2 (it is not optimal).

We define $\chi = \chi(f) := \infty$ if f is non-renormalizable, and otherwise define it to be the minimal non-negative integer such that $\mathbf{I}_\chi$ is terminating, i.e., the return time of c to $\mathbf{I}_\chi$ is equal to the first renormalization period of f.

Proof of Lemma 8.1. First, let $b = 1$. In this case we can take $\mathbf{U}_0$ to be the smallest successor of $\mathbf{I}$ (it exists because f is persistently recurrent) and let ν be the positive integer with $f^\nu(\mathbf{U}_0) = \mathbf{I}$. The first assertion of the lemma is obvious and the second one follows from the minimality of $\mathbf{U}_0$.

Now let us deal with the general case. For simplicity of notation let us assume that the critical point in $\mathbf{I}$ is c_0. We claim that for each $c \in \mathrm{Crit}(f)$ there exist two puzzle pieces $\mathbf{P}'_c \subset\subset \mathbf{P}_c$ containing c with the following properties:

1. each $\mathbf{P}_c$ is a pull back of $\mathbf{I}$ of order $\leq b^2 - b$;

2. for each $c \in \mathrm{Crit}(f)$, and any $z \in (\mathbf{P}_c - \mathbf{P}'_c) \cap PC(f)$, there exist a positive integer r, a puzzle piece $\mathbf{V}$ containing z and a critical point $\hat{c}$ such that $f^r : \mathbf{V} \to \mathbf{P}_{\hat{c}}$ is a conformal map.

Before we prove the claim let us show how it implies the lemma. Let s be the maximal positive integer such that one of the $\mathbf{P}_c$ has a child of depth s, i.e., s is maximal such that there exist a critical puzzle piece $\mathbf{Q}$ and a critical point c such that $f^s : \mathbf{Q} \to \mathbf{P}_c$ is a double branch covering. Let us prove that for each $x \in \mathbf{Q} \cap PC(f)$ we have $f^s(x) \in \mathbf{P}'_c$.

Arguing by contradiction, assume that this is false. Then by the second property above, there exist a puzzle piece $\mathbf{V} \ni f^s(x)$, a positive integer r and a critical point $\hat{c}$ such that $f^r : \mathbf{V} \to \mathbf{P}_{\hat{c}}$ is conformal. Let $\mathbf{W} = \mathrm{Comp}_x(f^{-s}\mathbf{V})$. If $\mathbf{W} \ni c$, then $\mathbf{W}$ is a child of $\mathbf{P}_{\hat{c}}$ of depth $r+s$, contradicting the maximality of s. So $\mathbf{W} \not\ni c$ and thus $f^{r+s} : \mathbf{W} \to \mathbf{P}_{\hat{c}}$ is conformal. Since $\mathbf{W} \cap PC(f) \neq \emptyset$, we can find a critical puzzle piece $\mathbf{W}'$ and a positive integer t such that $f^t : \mathbf{W}' \to \mathbf{W}$ is a double branched covering. Then $\mathbf{W}'$ is a child of $\mathbf{P}_{\hat{c}}$ of depth $r + s + t$, again contradicting the maximality of s.

Now let $\mathbf{U}_0 = \hat{\mathcal{L}}_{c_0}(\mathbf{Q})$ and let ν be the positive integer such that $f^\nu(\mathbf{U}_0) = \mathbf{I}$. Then it is easy to check that ν satisfies all the properties required in the lemma.

It remains to prove the claim. Let $\mathbf{T}_0 := \mathbf{I}$ and $\mathbf{J}_0 := \mathcal{L}_{c_0}(\mathbf{I})$. First assume that for every critical point $c \neq c_0$, $R_{\mathbf{I}}(c) \in \mathbf{J}_0$. In this case we can take $\mathbf{P}_c = \hat{\mathcal{L}}_c$ and $\mathbf{P}'_c = \hat{\mathcal{L}}_c(\mathbf{J}_0)$. In fact, for every $c \neq c_0$, $R_{\mathbf{I}} : \mathcal{L}_c(\mathbf{I}) \to \mathbf{I}$ has all its critical values in $\mathbf{J}_0$ and thus $R_{\mathbf{I}} : \mathbf{P}_c - \mathbf{P}'_c \to \mathbf{I} - \mathbf{J}_0$ is an (unbranched) covering. Now let us suppose that there is a critical point c_1, $c_1 \neq c_0$ such that $R_{\mathbf{I}}(c_1) \notin \mathbf{J}_0$. Let $\mathbf{T}_1 = \mathbf{J}_0 \cup \mathrm{Comp}_{c_1}(R_{\mathbf{I}}^{-1}(\mathcal{L}_{R_{\mathbf{I}}(c_1)}(\mathbf{I})))$. The domain $\mathbf{T}_1$ is strictly nice and, thus, any critical return domain of $\mathbf{T}_1$ is compactly contained in $\mathbf{T}_1$. Both of the puzzle pieces from $\mathbf{T}_1$ are pullbacks of $\mathbf{T}$ of order bounded by b.

Let $\mathbf{J}_1 = \mathcal{L}_{c_0}(\mathbf{T}_1) \cup \mathcal{L}_{c_1}(\mathbf{T}_1)$. The proof is completed again unless there is a critical point c_2, $c_2 \neq c_0, c_1$, such that $R_{\mathbf{T}_1}(c_2) \notin \mathbf{J}_1$. In the latter case $\mathbf{T}_2 = \mathbf{J}_1 \cup \mathrm{Comp}_{c_2}(R_{\mathbf{T}_1}^{-1}(\mathcal{L}_{R_{\mathbf{T}_1}(c_2)}(\mathbf{T}_1)))$ is again a strictly nice set and all the puzzle pieces from $\mathbf{T}_2$ are pullbacks of $\mathbf{I}$ of order bounded by $b + (b-1)$.

We can carry on in this way until we get the following situation:

- there is a collection $\mathbf{T}_m$, $m < b$, of puzzle pieces around some critical points and this collection is strictly nice;
- for any other critical point c', $R_{\mathbf{T}_m}(c') \in \mathbf{J}_m$, where $\mathbf{J}_m = \bigcup_{c \in \mathbf{T}_m} \mathcal{L}_c(\mathbf{T}_m)$;

- any puzzle piece of $\mathbf{T}_m$ is a pullback of $\mathbf{I}$ of order bounded by $b+(b-1)+\cdots+(b-m+1)\le b^2-b$.

In this case we just take $\mathbf{P}_c=\hat{\mathcal{L}}_c(\mathbf{T}_m)$ and $\mathbf{P}'_c=\hat{\mathcal{L}}_c(\mathbf{J}_m)$ for every $c\in\ Crit(f)$ to complete the proof. $\square$

8.2 Properties of the enhanced nest

We first state a proposition on the geometry of the real traces of the puzzle pieces in the enhanced nest. This result shows that the real geometry is under a good control and is the origin for our further analysis on the geometry of those puzzle pieces.

Definition 8.2. A nice interval I is called *ρ-nice*, if for each $x\in I\cap PC(f)$ we have

$$(1+2\rho)\mathcal{L}_x(I)\subset I.$$

Moreover, for any $\rho>0$, let $\mathcal{T}_\rho$ denote the family of all ρ-nice interval I with the property

$$\big((1+2\rho)I-(1+2\rho)^{-1}I\big)\cap PC(f)=\emptyset.$$

Proposition 8.1. (Real geometry for the enhanced nest) *Assume $f\in\mathcal{P}_b^{\tau,\sigma}$.*

1. *There exists $\rho=\rho(\tau,b)>0$ such that for each $0\le n\le\chi$ and for any $Z\in\{I_n,K_n,L_n:0\le n\le\chi\}$, $Z\in\mathcal{T}_\rho$.*

2. *For any $C>0$ there exists an $\varepsilon>0$ such that for any $0\le n\le\chi$, if there is $x\in PC(f)\cap I_n$ with $|\mathcal{L}_x(I_n)|\le\varepsilon|I_n|$, then $|I_n|\ge C|I_{n+1}|$ and I_{n+1} is a C-nice interval.*

3. *For any $C>0$, there exists $C'>0$ such that if $|I_n|/|I_{n+1}|>C'$, then $I_{n+2}\in\mathcal{T}_C$.*

This proposition will be proved in Section 9.

We shall also need the following combinatorial information later. For each $n\ge 0$, let s_n,t_n be the positive integers such that $f^{s_n}(\mathbf{L}_n)=\mathbf{I}_n$ and $f^{t_n}(\mathbf{K}_n)=\mathbf{L}_n$. Moreover, for each $1\le j\le T-1$, let $q_{n,j}$ be such that $f^{q_{n,1}}(\mathbf{M}_{n,1})=\mathbf{K}_n$ and $f^{q_{n,j+1}}(\mathbf{M}_{n,j+1})=\mathbf{M}_{n,j}$. Moreover, let

$$p_n=s_n+t_n+q_{n,1}+q_{n,2}+\cdots+q_{n,T}. \tag{11}$$

So $f^{p_n}(\mathbf{I}_{n+1}) = \mathbf{I}_n$. For any nice interval J containing c_0, let $r(J)$ denote the minimal return time from J to itself and let $\hat{r}(J)$ be the entry time of $x \in PC(f)$ into J.

Lemma 8.2. (Transition and return time relation) *Let $T = 5b$. Then for any $0 \leq n \leq \chi - 2$, the following hold.*

- $b^2 r(L_n) \geq s_n \geq r(I_n)$;
- $b^2 r(K_n) \geq t_n \geq r(L_n)$;
- $r(M_{n,i}) \geq q_{n,i} \geq 2r(M_{n,i-1})$, $i = 1, \ldots, T$;
- $\hat{r}(I_n) \leq q_{n,1} \leq \frac{1}{2} q_{n,2} \leq \cdots \leq \frac{1}{2^{T-1}} r(I_{n+1})$;
- $3\, r(I_{n+1}) \geq p_n$;
- $p_{n+1} \geq 2p_n$.

Proof. Consider the chain $\{G_j\}_{j=0}^{s_n}$ with $G_{s_n} = I_n$ and $G_0 = L_n$. Let $0 = j_0 < j_1 < j_2 < \ldots < j_\nu = s_n$ be all the integers such that $G_{j_i} \cap I_n \neq \emptyset$. Note that

$$L_n \subset G_{j_1} \subset G_{j_2} \subset \ldots \subset G_{j_\nu},$$

and hence, by Lemma 8.1, $\nu \leq b^2$. It is clear that $j_{i+1} - j_i \leq r(L_n)$ for $0 \leq i \leq \nu - 1$. Thus

$$s_n = \sum_{i=0}^{\nu-1} (j_{i+1} - j_1) \leq \nu r(L_n) \leq b^2 r(L_n).$$

As s_n is clearly not smaller than the return time of c to I_n, $s_n \geq r(I_n)$. This proves the first inequality. The second one can be done in a very similar way.

As $n \leq \chi - 2$, all these intervals $M_{n,i-1}$, $1 \leq i \leq T$ are non-terminating, and thus $R_{M_{n,i-1}}(M_{n,i}) \cap M_{n,i} = \emptyset$, which implies that $q_{n,i} \geq 2r(M_{n,i-1})$. Furthermore, we observe that if $\{G_j\}_{j=0}^{q_{n,i}}$ is the chain with $G_{q_{n,i}} = M_{n,i-1}$ and $G_0 = M_{n,i}$, then $G_j \not\ni c$, and hence $G_j \cap M_{n,i} = \emptyset$ for all $0 < j < q_{n,i}$. Therefore $r(M_{n,i}) \geq q_{n,i}$. This proves the third inequality.

Note that for any nice interval J containing c_0, if q is so that $f^q(\Gamma(\mathbf{J})) = \mathbf{J}$, then $q \geq \hat{r}(J)$. Because K_n is a subset of I_n and $M_{n,1}$ is the smallest successor of $M_{n,0}$, $\hat{r}(I_n) \leq \hat{r}(K_n) \leq q_{n,1}$.

The last two inequalities follow from the first three by direct computation, using the fact that $r(I) \leq r(I')$ for any symmetric nice intervals $I \supset I'$. (Here we use the choice for T.) $\square$

8.3 Proof of the Key Lemma (assuming upper and lower bounds)

Assume as before that f is a persistently recurrent polynomial-like box mapping in the class $\mathcal{P}_b^{\tau,\sigma}$. The Key Lemma will follow from the following two propositions.

Proposition 8.2 (Lower Bounds). *There exists a constant $\eta = \eta(\tau, \sigma, b) > 0$ such that for each $0 \le n \le \chi$ we have*

$$B(c_0, \eta|I_n|) \subset \mathbf{I}_n.$$

The proof of this proposition will be given in Section 10.

Proposition 8.3 (Upper Bounds). *There exists a constant $C = C(\tau, \sigma, b) > 1$ such that for any $0 \le n \le \chi$, the following hold:*

- $$\mathrm{diam}(\mathbf{I}_n) \le C|I_n|;$$
- *there exists a topological disk $\Omega \supset \mathbf{I}_n$ such that $\Omega - \mathbf{I}_n$ is disjoint from $PC(f)$ and*
$$\mathit{mod}(\Omega - \mathbf{I}_n) \ge \frac{1}{C}.$$

These upper bounds will be proved in Section 11.

Proof of the Key Lemma. The first statement follows immediately from the two propositions above: we just take Y to be $\mathbf{I}_n$ for a sufficiently big n. As f is non-renormalizable, $\chi = \infty$, so $\mathrm{diam}(\mathbf{I}_n) \asymp |I_n|$ is small when n is large.

Let us prove the second statement. So assume that f is renormalizable. Let $Y' = \mathbf{I}_\chi$. Let η, C, ρ be the constants as in Propositions 8.2, 8.3, 8.1 respectively. Let N be a positive integer such that $(1 + 2\rho)^N \ge C/\eta$. If $\chi < N$, then consider the map $g = f^{p_0+p_1+\cdots+p_{\chi-1}} : \mathbf{I}_\chi \to \mathbf{I}_0$, and let $Y = \mathrm{Comp}_{c_0}(g^1(U))$, where U is the component of $\mathrm{Dom}(f)$ which contains $g(c_0)$. It is easy to check that the second statement holds for an appropriate constant ξ in this case (the degree of $g\colon \mathbf{I}_\chi \to \mathbf{I}_0$ is bounded when χ is bounded). Assume $\chi \ge N$. By Proposition 8.1, we have

$$|I_{\chi-N}| \ge (1 + 2\rho)^N |I_\chi|.$$

By Propositions 8.2 and 8.3, it follows that

$$\mathbf{I}_\chi \subset D_*(\frac{1}{2} I_{\chi-N}).$$

Consider the map

$$g = f^{p_{\chi-1}+p_{\chi-2}+\cdots+p_{\chi-N}} : \mathbf{I}_\chi \to \mathbf{I}_{\chi-N},$$

and let $Y = \mathrm{Comp}_{c_0}(g^{-1}(U))$, where $U = \mathcal{L}_{g(c_0)}(\mathbf{I}_\chi)$. Notice that $R_{\mathbf{I}_\chi} \circ g : Y \to \mathbf{I}_\chi$ is a proper map with bounded degree, it follows that Y has ξ-bounded geometry for an appropriately chosen ξ. Moreover, as $\mathrm{mod}(\mathbf{I}_{\chi-N} - \overline{U})$ is bounded away from zero, so is $\mathrm{mod}(Y' - Y)$.

As the whole construction is topological, the last statement follows. □

9 Real bounds

Let us start with some definitions. A sequence of intervals $\{G_j\}_{j=0}^s$ is called a *chain* if G_j is a component of $f^{-1}(G_{j+1}) \cap \mathbb{R}$. The *intersection multiplicity* of a chain is the maximal number of intervals in the chain with a non-empty intersection. The *order* of a chain is the number of intervals in the chain containing a critical point. If I is a real interval of the form $(a-b, a+b)$ and $\lambda > 0$ then we define $\lambda I = (a - \lambda b, a + \lambda b)$. By definition $(1+2\delta)I$ is called the δ-scaled neighbourhood of I. We say that I is δ-well-inside J if $J \supset (1+2\delta)I$.

Throughout Sections 9, 10 and 11 all constants depend on the class $\mathcal{P}_b^{\tau,\sigma}$, and all nice intervals involved are the intersection of puzzle pieces with the real line.

The goal of this section is to prove Proposition 8.1. To do this we shall use the following well known fact frequently.

Fact 9.1. *Let $\{G_j\}_{j=0}^s$ and $\{G'_j\}_{j=0}^s$ be chains such that $G_j \subset G'_j$ for all $0 \le j \le s$. For any $N \in \mathbb{N}$ and any $\rho > 0$ there exists $\rho' > 0$ such that the following holds. Assume that the order of $\{G'_j\}_{j=0}^s$ is at most N and that $(1+2\rho)G_s \subset G'_s$. Then $(1+2\rho')G_0 \subset G'_0$. Moreover, for a fixed N, $\rho' \to \infty$ as $\rho \to \infty$.*

Proof. See [25]. Alternatively it follows easily from the fact that $f^s : G'_0 \to G'_s$ extends to a branched covering $F : U \to \mathbb{C}_{G'_s}$ with degree bounded from above by 2^N. □

We shall also use the following results which have been known previously.

Lemma 9.1. *There exists $\delta > 0$ such that if I is a nice interval around a critical point c and $R_I(c) \notin \mathcal{L}_c(I)$, then*

$$(1+2\delta)\mathcal{L}_c^2(I) \subset \mathcal{L}_c(I).$$

Proof. See Theorem A in [42]. □

Lemma 9.2. *For any $\delta > 0$ there is $\epsilon > 0$ such that if I is a nice interval, J is any subinterval of I such that $(1+2\delta)J \subset I$, x is a point, $k \geq 1$ and $f^k(x) \in J$, then*

$$(1+2\epsilon)\operatorname{Comp}_x(f^{-k}(J) \cap \mathbb{R}) \subset \mathcal{L}_x(I).$$

Proof. See Theorem B in [42]. □

Lemma 9.3. *For any $C > 0$ and $d > 0$ there is $C' > 0$ such that if I is a nice set, $J = \mathcal{L}_x^d(I)$ and $(1+2C')J \subset I$, then*

$$(1+2C)\mathcal{L}_y(J) \subset \mathcal{L}_y(I)$$

for any y which is contained in the domain of the first entry map to J.

Proof. See Proposition 4.1 in [37]. □

Lemma 9.4. *There exists a constant $\rho_0 > 0$ with the following property. Let I be a nice interval containing a critical point c, and $I^1 = \mathcal{L}_c(I)$. Let s be the return time of c into I. Then either of the following holds:*

1. *$(1+2\rho_0)I^1 \subset I$;*
2. *the chain $\{G_j\}_{j=0}^s$ with $G_s = (1+2\rho_0)I$ and $G_0 \ni c$ has intersection multiplicity bounded from above by a constant $N = N(b)$.*

Proof. See Lemma 2 in [42]. □

Definition 9.1. A sequence of nice intervals containing a critical point c

$$I^0 \supset I^1 \supset I^2 \supset \ldots \supset I^m$$

is a *central cascade* around c if $I^{i+1} = \mathcal{L}_c(I^i)$ for all $i = 0, \ldots, m-1$ and the return times of c to $I^0, \ldots, I^{m-1}$ are all the same.

Lemma 9.5. *For any $\delta > 0$, there exist $\kappa > 0$ and $C > 0$ with the following properties. Let us consider a central cascade $I := I^0 \supset I^1 \supset I^2 \supset \ldots \supset I^m$ with $m \geq 2$. Let s be the return time of I^1 to I. Assume that $|I^2| \geq \delta|I^0|$. Then for any critical point z of the map $R_I|I^2$ we have*

$$|f^s(z) - z| \geq \kappa|I^0| \text{ and } |(f^s)'(x)| \leq C \text{ for all } x \in I^2.$$

Proof. Let $\rho_0 > 0$ be as in Lemma 9.4. Then $f^s : I_2 \to I^1$ extends to a branched covering $F : \Omega \to \Omega' := \mathbb{C}_{(1+2\rho_0)I^1}$ with degree uniformly bounded from above. To see this it suffices to show that the order ν of the chain $\{G_j\}_{j=0}^s$ with $G_s = (1 + 2\rho_0)I^1$ and $G_0 \supset I^2$ is uniformly bounded. If $I^0 \supset (1 + 2\rho_0)I^1$, then $\nu \leq b$, and otherwise $\nu \leq N(b)$ by Lemma 9.4.

Let $\Omega_0' = D_{\pi-\sigma}((1+2\rho_0)I^1)$ and let $\Omega_0 = F^{-1}(\Omega_0')$. Note that $\mathrm{mod}(\Omega_0' \setminus I^1)$ and therefore $\mathrm{mod}(\Omega_0 - I^2)$ is bounded away from zero. So there exists a constant κ_1 such that Ω_0 contains $X := \bigcup_{x \in I^2} B(x, \kappa_1|I^2|)$. By Cauchy's formula and since $|I^2| \geq \delta|I^0|$, it follows that $|F'|$ is bounded from above on $\bigcup_{x \in I^2} B(x, \kappa_1/2|I^2|)$. In particular $|(f^s)'|$ is bounded from above on I^2. Moreover there exists $\kappa_2 > 0$ such that for each $z \in \mathrm{Crit}(R_I|I^2)$ and for each $w \in B(z, \kappa_2|I^2|)$, $|(f^s)'(w)| \leq 1/2$. If $|f^s(z) - z|/|I^2| << 1$ then $f^s(B(z, \kappa_2|I^2|)) \subset\subset B(z, \kappa_2|\kappa_2|I^2||)$ which implies that f^s has an attracting fixed point, a contradiction. $\square$

Lemma 9.6. *For any $C > 0$, there exists $C' > 0$ such that if $I \supset J$ are nice intervals around a critical point c and $(1 + 2C')J \subset I$, then for any x, we have*

$$(1 + 2C)\mathcal{L}_x(J) \subset \mathcal{L}_x(I). \tag{12}$$

Proof. Let $I^0 := I$ and $I^n = \mathcal{L}_c(I^{n-1})$ for all $n \geq 1$. Let $m(0) = 0$ and let $m(1) < m(2) < \cdots$ be all the positive integers such that $R_{I^{m(i)-1}}(c) \notin I^{m(i)}$. Let k be the maximal integer such that $J \subset I^{m(k)}$. By Lemma 9.1, for any $1 \leq i \leq k-1$, $I^{m(i)}$ contains a definite neighborhood of $I^{m(i)+1}$. By Lemma 9.2, for any x, $\mathcal{L}_x(I^{m(i)})$ contains a definite neighborhood of $\mathcal{L}_x(I^{m(i)+1})$. As $\mathcal{L}_x(J) \subset \mathcal{L}_x\mathcal{L}_c(I^{m(k-1)+1})$ and $\mathcal{L}_x(I^{m(1)}) \subset \mathcal{L}_x(I)$, (12) follows if k is sufficiently large.

So assume that k is uniformly bounded. By Lemma 9.3, it suffices to find two nice intervals $J \subset J' \subset I' \subset I$ such that $J' \supset \mathcal{L}_c(I')$ and $|I'|/|J'|$ is sufficiently large. Because k is bounded, it enough to consider the case $k = 0$, i.e., $J \supsetneq I^{m(1)}$. The existence of I', J' then follows from the previous lemma: when $|I|/|J|$ is sufficiently large, then either $|I|/|I^1|$, $|I^1|/|I^2|$ or $|I^{m(1)-1}|/|J|$ is large. This completes the proof. $\square$

Lemma 9.7. *Let c be a critical point and let $I \supset J \ni c$ be nice intervals. Assume that J is a pullback of I with order bounded by N. Fix an N. Then for any $\rho > 0$, there exists $\rho' > 0$ such that if $(1+2\rho)J \subset I$, then J is a ρ'-nice interval. Moreover, $\rho' \to \infty$ as $\rho \to \infty$.*

Proof. Let $\{G_j\}_{j=0}^{s}$ be the chain with $G_s = I$ and $G_0 = J$. Let us assume for the moment that

$$G_j \cap J = \emptyset \text{ for all } 0 < j < s. \tag{13}$$

Then for any $x \in J$, either $f^s(x) \in J$ or $f^s(\mathcal{L}_x(J)) \subset \mathcal{L}_{f^s(x)}(J)$. Applying Lemmas 9.2 and 9.6 and Fact 9.1, it follows that there exists a constant ρ' with $\rho' \to \infty$ as $\rho \to \infty$, and such that $(1+2\rho')\mathcal{L}_x(J) \subset J$. This proves the lemma under the assumption (13).

Now assume that (13) fails, and let $s' < s$ be the maximal positive integer such that $I' = G_{s'}$ intersects J. Note that $G_{s'} \supset J \ni c$. Thus there exists $\rho_1 = \rho_1(\rho) > 0$ with $\rho_1 \to \infty$ as $\rho \to \infty$, and such that either $I \supset (1+2\rho_1)I'$ or $I' \supset (1+2\rho_1)J$. In the former case, by what we have proved, I' is a ρ_1'-nice interval and in particular $(1+2\rho_1')J \subset I'$. Note that the order of the chain $\{G_j\}_{j=0}^{s'}$ is at most $N-1$, and thus the lemma follows by induction on N. $\square$

Lemma 9.8. *There exists a constant $\delta > 0$ such that if $\mathbf{I}$ is a non-terminating critical puzzle piece, then*

$$(1+2\delta)\Gamma(\Gamma(I)) \subset I.$$

Proof. Let m be the minimal positive integer such that $R_{I^{m-1}}(c) \notin I^m$, where c is the critical point in $\mathbf{I}$. Since $\mathbf{I}$ is non-terminating, there exists a return domain $\mathbf{J}$ to $\mathbf{I}$ other than the central one $\mathbf{I}^1 = \mathcal{L}_c(\mathbf{I})$ which intersects the postcritical set $PC(f)$. As $\mathbf{P} = \mathcal{L}_c(\mathbf{J})$ is a successor of $\mathbf{I}$, $\Gamma(\mathbf{I}) \subset \mathbf{P} \subset \mathbf{I}^m$. Therefore, $\Gamma\Gamma(\mathbf{I}) \subset \mathbf{I}^{m+1}$ and the statement follows from Lemma 9.1. $\square$

Lemma 9.9. *For any $\rho > 0$ there exists $\rho' > 0$ with $\rho' \to \infty$ as $\rho \to \infty$, such that if I is a ρ-nice interval containing a critical point c then $(1+2\rho')\mathcal{A}(I) - \mathcal{A}(I)$ and $\mathcal{B}(I) - (1+2\rho')^{-1}\mathcal{B}(I)$ are both disjoint form $PC(f)$.*

Proof. By definition, $\mathcal{B}(I) - \mathcal{A}(I)$ is disjoint from $PC(f)$. Moreover, there exists a positive integer ν such that $f^\nu : \mathcal{B}(\mathbf{I}) \to \mathbf{I}$ is a branched covering with a bounded degree and such that $f^\nu(\mathcal{A}(I))$ is contained in a return domain to I. The statement follows. $\square$

Proof of Proposition 8.1. 1. First of all, by Fact 9.1, for every $N \in \mathbb{N}$, and any $\rho > 0$ there exists $\rho' > 0$ such that the following holds. Let I be a nice interval and let J be a pull back of I with order $\leq N$. Then

- if I is ρ-nice, then J is ρ'-nice;
- if $((1+2\rho)I - I) \cap PC(f) = \emptyset$, $((1+2\rho')J - J) \cap PC(f) = \emptyset$; and
- if $(I - (1+2\rho)^{-1}I) \cap PC(f) = \emptyset$, then $(J - (1+2\rho')^{-1}J) \cap PC(f) = \emptyset$.

Moreover for a fixed N, $\rho' \to \infty$ as $\rho \to \infty$.

By this observation and by Lemma 9.9, it suffices to prove that there exists a constant $\rho > 0$ such that I_n is ρ-nice for all $0 \leq n \leq \chi - 2$. Since $I_{n+1} = \Gamma^T(K_n)$ and K_n is non-terminating, it follows from Lemma 9.8 that $|K_n|/|I_{n+1}|$ is bounded away from 1. By Lemma 9.7, I_{n+1} is a ρ-nice interval for an appropriately chosen constant $\rho > 0$. By taking $\rho > 0$ smaller, we may assume that I_0 is also ρ-nice. This completes the proof of the first statement of this proposition.

2. By Lemma 9.7, it suffices to prove that $|I_n|/|I_{n+1}|$ is sufficiently large when ε is sufficiently small. Let $x \in I_n \cap PC(f)$, and assume that the length of $J = \mathcal{L}_x(I_n)$ is small compared to that of I_n. If $J \ni c$, then $J \supset I_{n+1}$ and thus $|I_n|/|I_{n+1}|$ is large. Assume that $J \not\ni c$. By the first statement of this lemma, $I_n \cap PC(f) \subset (1+2\rho)^{-1}I_n$, so J is deep inside I_n. By Lemma 9.3, $\mathcal{L}_c(J)$ is deep inside I_n. Let $J' = \mathcal{L}_x(K_n)$. Then $J' \subset J$, and thus $\mathcal{L}_c(J') \subset \mathcal{L}_c(J)$. Since $\mathcal{L}_c(J') \supset \Gamma(K_n) \supset I_{n+1}$, it follows that $|I_n|/|I_{n+1}|$ is large.

3. By Lemma 9.7, for any $C' > 0$, there exists $C'' > 0$ such that if $|I_n|/|I_{n+1}| \geq C''$ then I_{n+1}, L_{n+1}, K_{n+1} are all C-nice. As $I_{n+2} = \Gamma^T \mathcal{B}\mathcal{A}(I_{n+1})$, applying Lemma 9.9, we see that for any $C > 0$ there exists $C' > 0$ such that if I_{n+1} is C'-nice then $I_{n+2} \in \mathcal{T}_C$. $\square$

10 Lower bounds for the enhanced nest

As before, let $f \in \mathcal{P}_b^{\tau,\sigma}$ be persistently recurrent. The goal of this section is to prove

Proposition 10.1. *There exists a constant $\eta = \eta(\tau, \sigma, b) > 0$ such that for each $0 \leq n \leq \chi$ we have*

$$B(c_0, \eta|I_n|) \subset \mathbf{I}_n.$$

Denote

$$\eta_n = \inf_{x \in PC(f) \cap I_n} \sup_{\{r>0 : B(x,r) \subset \mathbf{I}_n\}} \frac{r}{|I_n|}.$$

Lemma 10.1. *1. There exists a constant $\delta > 0$ such that for all $0 \le n < \chi$*

$$\eta_{n+1} > \delta \eta_n.$$

2. There exist $\kappa > 0$, $\varepsilon > 0$ such that if $|I_{n+1}|/|I_n| \le \varepsilon$, then

$$\eta_{n+2} \ge \min(\kappa, 2\eta_{n+1}).$$

Proof. Let $V = D_{\pi-\sigma}(I_n)$ and let $U = \mathrm{Comp}_c(f^{-p_n}(V))$, where p_n is as in (11). Then $f^{p_n} : U \to V$ is a proper map with a uniformly bounded degree. By Proposition 8.1, $I_n \in \mathcal{T}_\rho$, where $\rho > 0$ is a constant. By the Koebe distortion theorem and by Lemmas 13.2 and 13.3, we conclude that

$$D_{\pi-\theta}(I_{n+1}) \subset U \subset D_\theta(I_{n+1}),$$

where $\theta \in (0, \pi/2)$ is a constant. By a limit argument, this implies the following

1. There exist constant $C > 1$ and $\kappa_1 > 0$ such that

$$|(f^{p_n})'(z)| \le C \frac{|I_n|}{|I_{n+1}|} \tag{14}$$

   for $z \in \mathbb{C}$ with $d(z, PC(f) \cap I_{n+1}) < \kappa_1$.

2. There exists $\kappa_2 > 0$ such that for any $z \in B(c_0, 2\kappa_2 |I_{n+1}|)$,

$$|(f^{p_n})'(z)| \le \frac{1}{2} \frac{|I_n|}{|I_{n+1}|}. \tag{15}$$

Now the first statement follows immediately from (14). To show the second, note that by the third term of Proposition 8.1, there exists an ε such that $|I_{n+1}|/|I_n| \le \varepsilon$ implies that $I_{n+2} \cap PC(f) \subset B(0, \kappa_2 |I_{n+2}|)|$ and apply (15). $\square$

Lemma 10.2. *There exist a positive integer k_0 and a constant $\gamma > 0$ such that for all $0 \le n \le \chi$ and for all $x \in PC(f) \cap I_n$ we have*

$$B(x, \gamma |\mathrm{Comp}_x \mathrm{Dom}(R_{I_n}^{k_0})|) \subset \mathbf{I}_n.$$

Proof. Denote $N = p_0+\cdots+p_{n-1}$. Notice that $f^N(\mathbf{I}_n) = \mathbf{I}_0$. From Lemma 8.2 we know that $2p_{n-1} \geq N$ and $r(I_n) \geq N/6$.

Let $x \in PC(f) \cap I_n$ and let $W = \hat{\mathcal{L}}_{f^N(x)}(I_n)$. Let

$$U := \mathrm{Comp}_x(f^{-N}(W)) \cap \mathbb{R}.$$

Then $U = \mathrm{Comp}_x(\mathrm{Dom}(R_{I_n}^{k_0}))$ for some k_0. Since $r(I_n) \geq N/6$ we have $k_0 \leq 6$. This also implies that the pullback $f^N : U \to W$ has order bounded by $6b$.

By Proposition 8.1, $(1+2\rho)I_n - (1+2\rho)^{-1}I_n$ is disjoint from $PC(f)$. The interval W is a pullback of I_n of universally bounded order, so there is a universal constant $\rho' > 0$ such that $(1+2\rho')W \setminus (1+2\rho')^{-1}W \cap PC(f) = \emptyset$. As $D_{\pi-\sigma}(W) \subset D_{\pi-\sigma}(I_0) \subset \mathbf{I}_0$, by the Koebe distortion theorem and Lemmas 13.2 and 13.3, it follows that $\mathbf{I}_n \supset \mathrm{Comp}_x(f^{-N}(W)) \supset D_{\theta'}(U)$, where $\theta' \in (\pi/2, \pi)$ is a constant depending only on θ, ρ' and N. The proof is completed. $\square$

Proof of Proposition 10.1. If η_n is very small, then due to Lemma 10.2 there exists a domain U of $R_{I_n}^{k_0}$ intersecting $PC(f)$ and such that $|U|/|I_n|$ is very small. This implies that there exists a return domain J to I_n intersecting $PC(f)$ such that $|J|/|I_n|$ is small. By the second term of Proposition 8.1, it follows that each return domain to I_{n+1} is deep inside I_{n+1}. In particular, $|I_{n+2}|/|I_{n+1}|$ is small. By the second statement of Lemma 10.1, $\eta_{n+2} \geq \min(\kappa, 2\eta_{n+1})$. By the first statement of that lemma, it follows that for some constant $\kappa' > 0$ we have $\eta_{n+2} \geq \min(\kappa', 2\eta_{n+1})$. As η_0, η_1 are bounded away from zero, the proposition follows. $\square$

11 Upper bounds for the enhanced nest

Consider a persistently recurrent map f from the class $\mathcal{P}_b^{\tau,\sigma}$ (defined in Section 4) and let c_0 be the critical point in V_0. Our aim in this section is to prove Proposition 8.3, i.e., an upper bound for certain puzzle-pieces. For the construction and properties of the enhanced nest $\mathbf{I}_n$ we refer to Subsections 8.1 and 8.2. Let $I_n = \mathbf{I}_n \cap \mathbb{R}$. The first goal of this section is the prove the following result.

Theorem 11.1. *There exists $\theta > 0$ and n_0 so that for all $n \geq n_0$, $\mathbf{I}_n \subset D_\theta(I_{n-n_0})$.*

The proof of this theorem uses real bounds for the enhanced nest (Proposition 8.1), the real bounds from Section 9, and an analysis of what happens when we pull-back Poincaré disks with slits through critical points many times. Several of the basic results we will use can be found in Appendix 2, so it is probably a good idea to read Appendix 2 before reading this section.

The proof is somewhat related to the proof in [37]; the main difference is that we deal with the bounded and unbounded geometry situations simultaneously.

Throughout this section we will assume that all nice intervals are intersections of puzzle pieces with the real line.

11.1 Pulling-back domains along a chain

The main purpose of this subsection is to prove Proposition 11.2 and Proposition 11.3 below.

Let us first state some preparatory lemmas. Throughout we shall fix the class $\mathcal{P}_b^{\tau,\sigma}$, see Section 4. So all constants (even 'universal constants') *do* depend on this class.

We remind the reader that $(1+2\delta)I$ is defined to be the δ-scaled neighborhood of I, see Section 9.

In order to see what happens when you pull-back a disc of the form $D_\theta(I)$ along some chain, we first deal with central cascades.

Let I be a nice interval containing a critical point c and define the *principal nest* $I^k = \mathcal{L}_c(I^{k-1})$, $k \geq 1$, where $I^0 = I$.

Lemma 11.1. *For each $\delta > 0$ there exist $\delta' > 0$ and $\lambda \in (0,1)$ with the following property. Consider $I = I^0 \supset I^1 \supset \cdots \supset I^{\hat{m}}$ where $\hat{m}$ is the minimal integer such that $R_I|I^1$ has some critical value which is not in $I^{\hat{m}}$. Let r be so that $R_I|I^1 = f^r$. Assume that $\{G_j\}_{j=0}^{pr}$ is a disjoint chain with $G_{ir} \subset I^1$ for $0 \leq i \leq p-1$ and $G_{pr} \subset I$ a nice interval (i.e. the intersection of a puzzle-piece with the real line). Moreover, assume that $\{\hat{G}_j\}_{j=0}^{pr}$ is a chain with $G_{pr} \subset \hat{G}_{pr} \subset (1+\delta)\hat{G}_{pr} \subset I$ and $\hat{G}_0 \supset G_0$. Define*

$$V = D_\theta(\hat{G}_{pr}) \cap \mathbb{C}_{G_{pr}} \text{ and } U = \operatorname{Comp}_{G_0} f^{-pr} V.$$

Then for each $z \in U$ there exists an interval K such that either

$$z \in D_{\lambda\theta}(K) \text{ and } G_0 \subset K \subset (1+\delta')K \subset I \tag{16}$$

or *there exists* $0 \le p' \le p$ *with*

$$f^{p'r}(z) \in D_{\lambda\theta}(K) \text{ and } G_{p'r} \subset K \subset (1+\delta')K \subset I^{\hat{m}}. \tag{17}$$

Proof. Let us define E_1, E_2 to be the maximal open subintervals of $I^{\hat{m}}$ containing a boundary point of $I^{\hat{m}}$ in their closures and on which branches of f^r are diffeomorphic. Because f^r is a composition of folding maps, $f^r(E_1) = f^r(E_2) = f^s(I^{\hat{m}})$. Label these so that $f^r|E_1$ is monotone increasing, and define $X = I^{\hat{m}} \setminus (E_1 \cup E_2)$.

First of all note that this lemma follows from Lemma 13.6 if $p < 10$ or $\hat{m} = 1$: if $p < 10$, then (16) holds and if $\hat{m} = 1$ then (17) holds for $p' = p - \hat{m}$. Let us assume that $p \ge 10$ and $\hat{m} \ge 2$.

Claim 1. There exists a (universal) constant $\kappa_1 > 0$ such that if $|I^2|/|I^0| \le \kappa_1$, then the lemma holds. Indeed, by Lemma 13.6, $f^{(p-2)r}(z) \in D_{\lambda\theta}(I^2)$. Note that $|I^2|/|I^0|$ is small implies that $|I^2|/|I^1|$ is small. Again by Lemma 13.6 for any $x \in I^2$, $\mathrm{Comp}_x\, f^{-r}(D_{\lambda\theta}(0.5I^1)) \subset D_{\lambda'\theta}(I^2) \subset D_{\lambda\theta}(0.5I^1)$. Hence (16) holds if $|I^2|/|I^0|$ is small.

So let us also assume $|I^2|/|I^0| \ge \kappa_1$. By Lemma 9.5, there exist constants $\kappa_2 > 0$ and $C > 1$ such that

- for any critical point c of $f^r|I^2$, $d(f^r(c), c) \ge \kappa_2|I^0|$;
- $|(f^r)'(x)| \le C$ for any $\in I^2$.

In particular, $|I^{\hat{m}-1}|/|I^0| \ge \kappa_2$. Moreover, there exists $\kappa_3 > 0$ such that

- $d(X, \partial I^{\hat{m}-1}) \ge \kappa_3|I^0|$;
- either $f^r(I^{\hat{m}}) \cap I^{\hat{m}} = \emptyset$ or $|E_i| \ge \kappa_3|I^0|$, $i = 1, 2$.

Now let us make another claim.

Claim 2. There exists a constant $\gamma \in (0,1)$ such that either (16) or (17) holds with $\delta' = \gamma$, or there exist $0 \le p_* < p$, $0 \le i \le \hat{m} - 1$ and an interval $G^*_{p_*r}$ with $G_{p_*r} \subset G^*_{p_*r} \subset (1+2\gamma)G^*_{p_*r} \subset I^i$, $U_{p_*r} \subset D_{\lambda_1\theta}(G^*_{p_*r})$ and $|I^{i+1}|/|I^i| \le \gamma$.

Proof of Claim 2. By Lemma 9.2, there exists $\delta_1 > 0$ depending on δ such that for any $0 \le i \le p-1$, $(1+2\delta_1)\hat{G}_{ir} \subset I^1$. If $\hat{G}_{ir}$ does not contain a critical point of $R_I|I^1$ for all $i = 0, 1, \ldots, p-1$, then by Schwarz we have $z \in D_\theta(\hat{G}_0)$ and thus (16) holds with $\delta' = \delta_1$. So let us assume that there exists a maximal integer p_* with $0 \le p_* \le p$ be so that $\hat{G}_{p_* r}$ contains a critical point of $f^r|I^1$ (and hence $\hat{G}_{p_* r} \cap X \ne \emptyset$). Then by Schwarz and by Lemma 13.6, it follows that there exists an interval $G^*_{p_* r}$ with $\hat{G}_{p_* r} \subset G^*_{p_* r} \subset (1+2\delta_2)G^*_{p_* r} \subset I^0$ such that $f^{p_* r}(z) \in D_{\lambda_1\theta}(G^*_{p_* r})$, where δ_2 and λ_1 are constants depending on δ.

We may assume that $G^*_{p_* r}$ is not well-inside $I^{\hat{m}-1}$, because otherwise (17) holds for $p' = p_* - 1$. As $d(\partial X, \partial I^0)/|I^0|$ is bounded away from zero, and $G^*_{p_* r} \cap \partial X \ne \emptyset$, it follows that $|G^*_{p_* r}|/|I^0|$ is bounded away from zero. Let $k \le \hat{m} - 1$ be maximal so that $G^*_{p_* r} \subset I^k$. If $G^*_{p_* r}$ is well inside I^k then Claim 2 holds with $i = k$. Thus we may assume that $|I^k|/|I^0|$ is bounded away from 1. If $\min_{i=0,\ldots,k-1} |I^{i+1}|/|I^i|$ could be arbitrarily close to one, then there would be a map (with a critical point) in the Epstein class having an interval of fixed points (here we use that f is in the Epstein class $\mathcal{P}_b^{\tau,\sigma}$). Clearly this is impossible. This proves the claim.

So we may assume that $p \ge 10$, $\hat{m} \ge 2$, $|I^2|/|I^0| \ge \kappa_1$ and $|I^1|/|I^0| \le \gamma$ (by possibly replacing I^0 by I^i with i as in Claim 2). By replacing p by $p-1$ we may assume that $\hat{G}_{pr} \subset I^1$ (and so $f^{pr}(z) \in D_\theta(I^1)$). If $z \in D_\theta(I^1)$ then (16) holds, so we may assume that there exists a maximal q with $0 \le q < p$ and so that $f^{qr}(z) \notin D_\theta(I^1)$. Since $f^{(q+1)r}(z) \in D_\theta(I^1)$, by Lemma 13.6 there exists $\lambda_1 \in (0, 1)$ such that $f^{qr}(z) \in D_{\lambda_1\theta}(J)$ with $(1 + 2\delta_3)J \subset I^1$.

Case 1: $G_{qr} \subset I^{\hat{m}}$. Note that this implies that $G_{ir} \subset I^{\hat{m}}$ for all $0 \le i \le q$. We may assume that $q \ge 1$ (otherwise (16) holds). So $f^r(I^{\hat{m}}) \cap I^{\hat{m}} \ne \emptyset$ and therefore $|E_i|/|I^{\hat{m}}| \ge \kappa_3$.
Subcase 1.1 $G_{ir} \cap \partial(X \cup E_2) = \emptyset$ for all $i \le q$.

If $G_{qr} \subset X \cup E_2$ then by the third part of Lemma 13.4, $f^{qr}(z) \in D_{\mu\lambda\theta}(X \cup E_2)$ and hence by the last part of Lemma 13.5, $f^{(q-1)r}(z) \in D_{\mu'\lambda\theta}(K)$ where K is an interval which contains $G_{(q-1)r}$ and is well inside $I^{\hat{m}}$ (we apply Lemma 13.5 to the chains $\{H'_j\}_{j=0}^r$, $\{H_j\}_{j=0}^r$ and $\{\hat{H}_j\}_{j=0}^r$ where $H'_r = I$, $H_r = I^{\hat{m}-1}$ and $\hat{H}_r = X \cup E_2$ and $H'_0 \supset H_0 \supset \hat{H}_0 \supset G_{(q-1)r}$).

Assume that $G_{qr} \subset E_1$, and let q' be minimal so that

$$G_{q'r}, G_{(q'+1)r}, \ldots, G_{qr} \subset E_1.$$

First let us consider the case $f^r(E_1) \supset E_1$. Then $f^{(q-q')r} : G_{q'r} \to G_{qr}$ extends to a diffeomorphism onto E_1. By Lemma 13.4, $f^{qr}(z) \in D_{\mu\lambda\theta}(E_1)$, and hence by Schwarz $f^{q'r}(z) \in D_{\mu\lambda\theta}(E_1)$. We may assume that $q' \ge 2$ for otherwise (16) holds. If $G_{(q'-1)r} \subset E_2$ then by Schwarz and Lemma 13.5 $f^{(q'-2)r}(z) \in D_{\mu\lambda\theta}(K')$ with K' well-inside $I^{\hat{m}}$. Otherwise $G_{(q'-1)r} \subset X$, because f^r does not map ∂X into E_1. Let H_r be the interior of $f^r(E_1)$ and let $\{H_j\}_{j=0}^r$ be the chain with $H_0 \supset G_{(q'-1)r}$. Note that $H_0 \subset X$ because $f^r(\partial X)$ is in the boundary of H_r. Since $f^r(H_0)$ contains $H_r \setminus E_1$ and this difference is not small compared to H_r, by Lemma 13.5, $f^{(q'-1)r}(z) \in D_{\lambda'\theta}(H_0)$. Because X is well-inside $I^{\hat{m}}$, (17) holds for $p' = q' - 1$ (and appropriate choice of constants).

Now let us assume that $f^r(E_1) \not\supset E_1$. Then we let T_q be the minimal open interval which contains a component of $I^0 - I^1$ and G_{qr} and is disjoint from X. As $|T_q|/|I^0|$ is bounded away from zero, $f^{qr}(z) \in D_{\lambda'\theta}(T_q)$. It is easy to see that there exists an interval $T_{q'} \supset G_{q'r}$ such that $f^{(q-q')r} : T_{q'} \to T_q$ is a diffeomorphism. So by Schwarz, we have $f^{q'r}(z) \in D_{\lambda'\theta}(T_{q'}) \subset D_{\lambda'\theta}(I^1)$. Note that $f^r(E_1) \cap (X \cup E_2) = \emptyset$ and thus $q' \leq 2$. Therefore (16) holds.

Subcase 1.2 There exists $q' \leq q$ such that $G_{q'r} \cap \partial E_1 \neq \emptyset$. Applying Subcase 1.1 to the chain $\{G_j\}_{j=q'+1}^q$, we may assume that $f^{q'r}(z) \in D_{\lambda\theta}(I^1)$. Note that $G_{ir} \cap \partial E_1 = \emptyset$ for $i < q'$. So we can repeat the previous argument, replacing G_{pr} by $G_{(q'-1)r}$.

Case 2: $G_{qr} \not\subset I^{\hat{m}}$. Let $q' < q$ be maximal so that $G_{ir} \not\subset I^{\hat{m}}$ for $q' \leq i \leq q$. Then $f^{q'r}(z) \in D_{\lambda'\theta}(I^1)$. If $q' = 0$ then we are in case (16). Otherwise $G_{(q'-1)r} \subset I^{\hat{m}}$, $f^{(q'-1)r}(z) \in D_{\lambda''\theta}(I^1)$ and $G_{ir} \subset I^{\hat{m}}$ for $i \leq q' - 1$, and we proceed as in Case 1. □

If I is a non-terminating nice interval containing a critical point c, we define $\mathcal{C}(I) := I^m$ where $m \geq 1$ is minimal so that $R_I(c) \notin I^m$. If I is terminating we define $\mathcal{C}(I) = \emptyset$. Let us define $\mathcal{C}^k(I)$ to be equal to $\mathcal{C}(\mathcal{C}(\ldots(I)))$.

If J is a return domain to an arbitrary nice interval I, and $\{G_i\}_{i=0}^r$ is the chain with $G_r = I$, $G_0 = J$ where r is the return time of J to I, we define

$$\mathrm{Crit}(I; J) = (\bigcup_{i=0}^{r-1} G_i) \cap \mathrm{Crit}(f).$$

Similarly, when $\mathbb{G} = \{G_j\}_{j=0}^s$ is an arbitrary chain such that G_s is a pullback of I, $G_s \subset I$, and $0 = n_0 < n_1 < \cdots < n_p = s$ are the integers with $G_{n_i} \subset I$, then we define

$$\mathrm{Crit}(I; \mathbb{G}) = \bigcup_{i=0}^{p-1} \mathrm{Crit}(I; \mathcal{L}_{G_{n_i}}(I))$$

where $\mathcal{L}_{G_{n_i}}(I)$ is the return domain to I containing G_{n_i}. For any nice interval I and any critical point c', we define

$$k_{c'}(I;\mathbb{G}) = \inf\{k_{c'};\ G_j \subset \mathcal{C}^{k_{c'}}(\hat{\mathcal{L}}_{c'}(I)) \text{ for some } j = 0,1,\ldots,s-1\}$$

and

$$k(I;\mathbb{G}) = \sum_{c' \in \mathrm{Crit(I;\mathbb{G})}} k_{c'}(I;\mathbb{G}). \tag{18}$$

$k(I;\mathbb{G})$ describes the combinatorial depth of the chain $\mathbb{G}$ with respect to I. (Note that $k(I;\mathbb{G})$ is well-defined even if I does not contain a critical point.) When $c \in J \subset I$ we define

$$k(I;J) = \min\{k;\, \mathcal{C}^k(I) \subset J\}$$

and

$$\hat{k}(I,J) = \sum_{c'} k(\hat{\mathcal{L}}_{c'}(I), \hat{\mathcal{L}}_{c'}(J)). \tag{19}$$

The next proposition gives the crucial estimate describing the loss of angle when we pull-back a slitted Poincaré disc $D_\theta(\hat{G}_s) \cap \mathcal{C}_{G_s}$ with $\hat{G}_s$ well-inside I and $\mathbb{G} = \{G_j\}_{j=0}^s$ a disjoint chain. The loss of angle turns out to be only related to $k(I,\mathbb{G})$.

Proposition 11.2. *For each δ there exists $\mu \in (0,1)$ and $\delta' > 0$ with the following properties. Let I be a nice interval, and let $\mathbb{G} := \{G_i\}_{i=0}^s$ be a disjoint chain with $G_0, G_s \subset I$ with G_s a nice interval and $G_0 \cap PC(f) \neq \emptyset$. Let $\hat{G}_s$ be an interval with $G_s \subset \hat{G}_s \subset (1+\delta)\hat{G}_s \subset I$. Let $V = D_\theta(\hat{G}_s) \cap \mathbb{C}_{G_s}$ and write $U_i = \mathrm{Comp}_{G_i} f^{-(s-i)}(V)$, $i = 0,\ldots,s$. Then there exists an interval $\hat{I} \supset G_0$ with $(1+\delta')\hat{I} \subset I$ and such that*

$$U_0 \subset D_{\mu^{k(I,\mathbb{G})}\theta}(\hat{I}).$$

Here $k(I;\mathbb{G})$ is defined as in equation (18).

Proof. The proof is by induction on $N := \#\mathrm{Crit}(I;\mathbb{G})$. More precisely, we formulate the following two induction statements.

Induction Statement (N,k): There exist increasing functions $\delta \mapsto \mu_N(\delta) \in (0,1)$ and $\delta \mapsto \alpha_N(\delta) \in (0,1)$ such that for any interval I and chains

$\mathbb{G} = \{G_i\}_{i=0}^s$, $\hat{G} = \{\hat{G}_i\}_{i=0}^s$ as in the theorem for which $\mathrm{Crit}(I;\mathbb{G}) \le N$ and $k(I;\mathbb{G}) \le k$,

$$U \subset D_{\mu_N^{k(I;\mathbb{G})}\theta}(\hat{I})$$

where $\hat{I}$ is an interval with $G_0 \subset \hat{I} \subset (1+\alpha_N)\hat{I} \subset I$. Here α and μ_N do *not* depend on k.

Induction Statement N**:** Statement (N,k) holds for each $k = 0, 1, 2, \dots$.

Note that N is bounded by the number of critical points of f so it is enough to prove Statement N for each integer N. If $N = 0$ then $\{G_j\}_{j=0}^s$ only visits diffeomorphic branches of R_I, and so by Schwarz $U_0 \subset D_\theta(\hat{G}_0)$. This proves Statement 0. Therefore it is enough to prove the induction step. This is done in Lemma 11.3. $\square$

To prove the induction step, we shall use the following lemma.

Lemma 11.2. *Let I be a nice interval containing a critical point c and let $I^1 = \mathcal{L}_c(I)$. Let J be a nice interval with $J \subset I - I^1$ and $J \cap PC(f) \neq \emptyset$ and let $K = \mathcal{L}_c(J)$. Then there exists a universal constant $\hat{\rho} > 0$ such that the following hold:*

1. *$I \supset (1+2\hat{\rho})K$;*

2. *K is a $\hat{\rho}$-nice interval;*

3. *for any $x \in I$, $k \ge 0$ with $f^k(x) \in K$, $(1+2\hat{\rho})\,\mathrm{Comp}_x(f^{-k}K) \cap \mathbb{R} \subset I$.*

Proof. The second and the third statements follow from the first one by Lemma 9.7 and Lemma 9.2 respectively (redefining the constant $\hat{\rho}$). So it suffices to prove the first one.

Let m be a maximal positive integer so that the return time of c to I^{m-1} is the same as the return time r of c to I^0. Then $R_I(K) \subset I^{m-1} \setminus I^m$. We may assume that $|I^m|/|I|$ is close to 1. In particular the second assertion in Lemma 9.4 applies, and so $f^r\colon I^1 \to I^0$ extends to holomorphic branched covering $F\colon \Omega \to \Omega' = \mathbb{C}_{(1+2\rho_0)I}$ with bounded degree. Since $\mathrm{mod}(\Omega' \setminus R_I(K))$ is large, also $\mathrm{mod}(\Omega \setminus K)$ is large. Note that $\Omega \cap \mathbb{R}$ is contained in $\Omega' \cap \mathbb{R} = (1+2\rho_0)I$, it follows that $|K|/|I|$ is small, which concludes the proof of the lemma. $\square$

Lemma 11.3. *For each $N \ge 1$, Statement N-1 implies Statement N.*

Proof. Assume that Statement $(N-1)$ holds. Let I be a nice interval and $\mathbb{G}$ be a chain with $\mathrm{Crit}(I;\mathbb{G}) = N$ and $k(I;\mathbb{G}) = k$ as in the statement of Proposition 11.2.

We can and will assume that I contains a critical point c and that $\mathrm{Crit}(I;\mathbb{G}) \ni c$: otherwise simply pull-back I along the chain $\mathbb{G}$ to a nice interval I' containing a critical point (inside $\mathrm{Crit}(I;\mathbb{G})$). Applying Statement N to this new nice interval I' and then pulling-back to G_0 gives Statement N for I but with possibly a smaller μ_N.

We will prove Statement (N,k) by induction on k. The assumption of Statement $(N,0)$ is never satisfied, so the statement is correct. Let us assume that Statements $(N-1)$ and $(N,k-1)$ hold. Without loss of generality we may restrict ourselves to the case $\delta \le \hat{\rho}$, where $\hat{\rho}$ is the constant coming from Lemma 11.2.

For $\delta \in (0,\hat{\rho})$ let $\lambda(\delta)$ be the smaller of the λ coming from Lemmas 13.6 and 11.1, and let us take $\alpha(\delta)$ to be the smallest of the δ' coming from these two lemmas and the $\alpha_{N-1}(\delta)$ coming from the Induction Statement $(N-1)$. We may assume that the functions $\delta \mapsto \alpha(\delta), \lambda(\delta)$ are increasing in δ and that $\alpha(\delta) < \delta$. Let $\alpha^{\circ i}(\delta)$ denote the i-th iterate of the function $\delta \mapsto \alpha(\delta)$ (so this corresponds to applying those statements in succession i times). Now define

$$\alpha_N(\delta) = [\alpha^{\circ 10b}(\delta)]^2, \tag{20}$$

and

$$\mu_N(\delta) = [\mu_{N-1}(\alpha^{\circ 10b}(\delta))^{2N+2}\lambda(\alpha^{\circ 10b}(\delta))^5]^b\gamma, \tag{21}$$

where

$$\gamma = \lambda(\alpha_N(\hat{\rho})). \tag{22}$$

We shall prove Statement N for this choice of constants. Let $\{\hat{G}_i\}_{i=0}^s$ be the chain with $\hat{G}_0 \supset G_0$ as in Proposition 11.2 and let $s_1 < s$ be maximal so that $G_{s_1} \subset I$.

Case I: $G_{s_1} \subset I \setminus I^1$. Then define $J = \mathcal{L}_{G_{s_1}}(I)$. Let $s_1' \ge 0$ be the minimal integer such that $G_{s_1'} \subset J$ and let $k_0 = k(J, \{G_j\}_{j=s_1'}^{s_1})$. We are going to show that there exist integers $k_1, k_2 \le k_0$ with $k_1 + k_2 \le k_0 + N$ and such that

$$z \in D_{\mu_{N-1}(\hat{\alpha})^{k_1 b}\hat{\mu}_N^{k_2}\hat{\lambda}^{b+3}\gamma\theta}(\hat{I}), \tag{23}$$

where $\hat{I}$ is an interval with $G_0 \subset \hat{I} \subset (1+\hat{\alpha})\hat{I} \subset I$, and

$$\hat{\alpha} = \hat{\alpha}(\delta) := \alpha^{\circ 5b}(\delta),\ \hat{\lambda} = \lambda(\hat{\alpha}),\ \text{and } \hat{\mu}_N = \mu_N(\hat{\rho}) \ge \mu_N(\delta). \tag{24}$$

Let us first show that (23) implies the statement (N, k). It suffices to show that

$$\mu_{N-1}(\hat{\alpha})^{k_1 b} \hat{\mu}_N^{k_2} \hat{\lambda}^{b+3} \gamma \leq \mu(\delta)^k.$$

To this end, we first observe that $\mathcal{L}_c(J) \subset \mathcal{C}(I)$ and thus $k_0 \leq k-1$. From $k_1 + k_2 \leq k_0 + N$ it follows that $k_1/(k-k_2) \leq N+1$. So the inequality follows from the choice of the function μ_N by direct computation.

Let us prove (23). First, by Lemma 13.6, $U_{s_1} \subset D_{\lambda(\delta)\theta}(\hat{G}'_{s_1}) \cap \mathbb{C}_{G_{s_1}}$, where $G_{s_1} \subset \hat{G}_{s_1} \subset \hat{G}'_{s_1} \subset (1+\alpha^{\circ 1})\hat{G}'_{s_1} \subset J$. (Here and after $\alpha^{\circ i} = \alpha^{\circ i}(\delta)$.) Let $K = \mathcal{L}_c(J)$ and s_2 be the minimal integer with $0 \leq s_2 \leq s_1$ so that $G_{s_2} \subset J$ and $G_j \cap K = \emptyset$ for $s_2 \leq j < s_1$. By the choice of s_2, $\mathrm{Crit}(J; \{G_j\}_{j=s_2}^{s_1}) \subset \mathrm{Crit}(I; \{G_j\}_{j=0}^{s}) \setminus \{c\}$ and so $\#\mathrm{Crit}(J; \{G_j\}_{j=s_2}^{s_1}) < N$. Let

$$k_1 := \sum_{c' \in \mathrm{Crit}(\mathrm{I}, \mathbb{G}), c' \neq \mathrm{c}} k(\mathcal{L}_{c'}(J), \mathcal{L}_{c'}(K)).$$

Claim 1. There exists an interval $\hat{J}$ with $G_{s_2} \subset \hat{J} \subset (1+\alpha^{\circ 2b}(\delta))\hat{J} \subset J$ such that

$$U_{s_2} \subset D_{\beta\theta}(\hat{J}), \tag{25}$$

where

$$\beta = [\mu_{N-1}(\alpha^{\circ 2b})^{k_1} \lambda(\alpha^{\circ 2b})]^b \leq [\mu_{N-1}(\hat{\alpha})\hat{\lambda}]^b. \tag{26}$$

To prove this claim we shall apply the induction assumption. By choice of s_2, $G_j \cap K = \emptyset$ for all $s_2 \leq j \leq s_1$, which implies that for any critical point $c' \neq c$, there can be at most one $j \in \{s_2, s_2+1, \ldots, s_1\}$ such that $G_j \subset \mathcal{L}_{c'}(K)$. If for any $s_2 \leq j \leq s_1$ and any $c' \in \mathrm{Crit}(f) - \{c\}$, $G_j \not\subset \mathcal{L}_{c'}(K)$, then $k(J, \{G_j\}_{j=s_2}^{s_1}) \leq k_1$ and thus Statement $(N-1)$ gives us $U_{s_2} \subset D_{\mu_{N-1}(\alpha^{\circ 1})^{k_1}\lambda(\delta)\theta}(\hat{J})$ where $\hat{J}$ is an interval with $G_{s_2} \subset \hat{J} \subset (1+\alpha^{\circ 2})\hat{J} \subset J$, which implies (25). Otherwise, let p be maximal with $p < s_1$ such that $G_p \subset J$ and such that G_p enters $\mathcal{L}_{c'}(K)$ for some $c' \in \mathrm{Crit}(f) - \{c\}$ before it returns to J. Let $p' > p$ be the minimal integer such that $G_{p'} \subset J$. Then $k(J, \{G_j\}_{j=s_2}^{p'}) \leq k_1$ and thus as before, $U_{p'} \subset D_{\mu_{N-1}(\alpha^{\circ 1})^{k_1}\lambda(\delta)\theta}(\hat{J})$ with $(1+\alpha^{\circ 2})\hat{J} \subset J$. Applying Lemma 13.6 gives us $U_p \subset D_{\mu_{N-1}(\alpha^{\circ 1})^{k_1}\lambda(\delta)\lambda(\alpha^{\circ 2})\theta}(\hat{J}')$ with $(1+\alpha^{\circ 3})\hat{J}' \subset J$. Repeating this argument for at most $b-1$ times we obtain (25).

Let us continue the proof of Case I. If $s_2 = 0$ then setting $k_2 = 0$, we get the inclusion (23). So assume $s_2 > 0$. If $G_j \cap K = \emptyset$ for all $0 \leq j < s_2$ then

$f^{s_2}\colon G_0 \to G_{s_2}$ is a branch of the first entry map to J and so

$$U_0 \subset D_{\lambda(\alpha^{\circ 2b})\beta\theta}(\hat{I}) \subset D_{\hat{\lambda}\beta\theta}(\hat{J}),$$

with $G_0 \subset \hat{J} \subset (1+\alpha^{\circ 2b+1})\hat{J} \subset \mathcal{L}_{G_0}(I) \subset I$. Thus (23) holds with $k_2 = 0$. Otherwise, let s_3 with $0 \le s_3 < s_2$ be maximal so that $G_{s_3} \subset K$. Then, again $f^{s_2-s_3}\colon G_{s_3} \to G_{s_2}$ is a first entry to J and so

$$U_{s_3} \subset D_{\hat{\lambda}\beta\theta}(\hat{K})$$

where $G_{s_3} \subset \hat{K} \subset (1+\alpha^{\circ 2b+1})\hat{K} \subset K$. If there is no previous visit to K then as before

$$U_0 \subset D_{\hat{\lambda}^2\beta\theta}(\hat{I}),$$

with $G_0 \subset \hat{I} \subset (1+\alpha^{\circ 2b+2})\hat{I} \subset I$. Setting $k_2 = 0$ and we get (23) again. Otherwise, let s_4 with $0 \le s_4 < s_3$ be maximal so that $G_{s_4} \subset K$. Then

$$U_{s_4} \subset D_{\hat{\lambda}^3\beta\theta}(L)$$

where $L = \mathcal{L}_{G_{s_4}}(K)$. By Lemma 11.2, $(1+\hat{\rho})L \subset K$. By assumption, $\hat{\rho} \ge \delta$.

If $s_4 = 0$ then we get (23) (with $k_2 = 0$). So assume $s_4 > 0$ and let s_5, $0 < s_5 \le s_4$ be minimal so that $G_{s_5} \subset K$, and let $k_2 = k(K; \{G_j\}_{j=s_5}^{s_4})$. Note that $s_5 \ge s_1'$. Thus $\mathrm{Crit}(K, \{G_j\}_{j=s_5}^{s_4}) \subset \mathrm{Crit}(J, \{G_j\}_{j=s_1'}^{s_1})$, and $k_2 \le k_0$. By the induction Statement $(N, k-1)$, we obtain

$$U_{s_5} \subset D_{\hat{\lambda}^3\beta\hat{\mu}_N^{k_2}\theta}(\hat{K}'),$$

where $\hat{K}'$ is an interval with $G_{s_5} \subset \hat{K}' \subset (1+\alpha_N(\hat{\rho}))\hat{K}' \subset K$. As $f^{s_5} : G_0 \to G_{s_5}$ is a first entry map to K, by Lemma 13.6, $U_0 \subset D_{\lambda(\alpha_N(\hat{\rho}))\hat{\lambda}^3\beta\hat{\mu}_N^{k_2}\theta}(\hat{I}) \subset D_{\hat{\lambda}^{b+3}\gamma\hat{\mu}_{N-1}^{k_1 b}\hat{\mu}_N^{k_2}\theta}(\hat{I})$, where $\hat{I} = \mathcal{L}_{G_0}(K)$, and we have used (26). By Lemma 11.2, $(1+\hat{\alpha})\hat{I} \subset (1+\hat{\rho})\hat{I} \subset I$. The inclusion (23) follows.

Case II: $G_{s_1} \subset I^1$. For every point $z \in U_0$ we will find an interval K_z with $G_0 \subset K_z \subset (1+\alpha^{\circ 10b})K_z \subset I$, and such that $z \in D_{\mu_N(\delta)^k\theta}(K_z)$. By taking the union of these intervals K_z we will obtain the desired interval $\hat{I}$. So fix z and let $t \le s$ be minimal with $G_t \subset I^1$ and such that $G_j \cap (I \setminus I^1) = \emptyset$ for all $j = t, \dots, s$. If I is a terminating interval, then since G_s is a nice interval and $G_s \cap PC(f) \ne \emptyset$, G_s is also terminating and because the chain $\{G_j\}_{j=0}^{s}$

is disjoint, $s = 0$. So from now on we assume that I is non-terminating. Let $\hat{m}$ be minimal so that $R_I|I^1$ has a critical value in $I^{\hat{m}-1} \setminus I^{\hat{m}}$. Note that $G_t, \ldots, G_{s_1}$ are in the orbit of $R_I|I^1$. According to Lemma 11.1 there exists an interval K such that either

1. $f^t(z) \in D_{\lambda(\delta)\theta}(K)$ and $G_t \subset K \subset (1+\alpha^{\circ 1})K \subset I$, or
2. there exists s_2 with $t \leq s_2 < s_1$ so that $G_{s_2} \subset I^{\hat{m}}$ and $f^{s_2}(z) \in D_{\lambda(\delta)\theta}(K)$ with $G_{s_2} \subset K \subset (1+\alpha^{\circ 1})K \subset I^{\hat{m}}$.

Assume (1) holds. Then by definition of t, if $\hat{s}_1 < t$ is maximal so that $G_{\hat{s}_1} \subset I$ then $G_{\hat{s}_1} \subset I \setminus I^1$. This means that we can repeat Case I verbatim to I, $\{G_i\}_{i=0}^{\hat{s}}$, $\{\hat{G}_i\}_{i=0}^{\hat{s}}$, and get

$$z \in D_{\lambda(\delta)\mu_{N-1}(\alpha^{\circ 5b+1})^{k_1 b}\mu_N(\hat{\rho})^{k_2}\lambda(\alpha^{5b+1})^{b+3}\gamma\theta}(K_z),$$

where k_0, k_1, k_2 are non-negative integers with $k_0 \leq k-1$, $k_2 \leq k_0$, and $k_1 + k_2 \leq k_0 + N$, γ is as in (22), and K_z is an interval with $G_0 \subset K_z \subset (1+\alpha^{\circ 5b+1})K_z \subset I$. Again by the choice of μ_N and α_N, it follows that $\mu_N(\delta)^k \leq \lambda(\delta)\mu_{N-1}(\alpha^{\circ 5b+1})^{k_1 b}\mu_N(\hat{\rho})^{k_2}\lambda(\alpha^{5b+1})^{b+3}\gamma$, and $\alpha_N(\delta) \leq \alpha^{\circ 5b+1}$. Hence we are done if (1) holds.

Assume (2) holds. By definition of $\hat{m}$, there exists a critical point $c' \in \mathrm{Crit}(I, I^1) \subset \mathrm{Crit}(I, \{G_i\}_{i=0}^{s})$ such that c' enters $I - I^1$ before it enters $I^{\hat{m}}$. This implies that $\mathrm{Crit}(I^{\hat{m}}; \{G_i\}_{i=t}^{s_2}) \subset \mathrm{Crit}(I, \{G_i\}_{i=0}^{s}) - \{c\}$. Let $k_3' = k(I^{\hat{m}}; \{G_i\}_{i=t}^{\hat{s}}) < k(I; \{G_i\}_{i=0}^{s})$. It is clear that $k_3' \leq k$. By the induction assumption $N-1$ applying to $I^{\hat{m}}$, $\{G_i\}_{i=t}^{\hat{s}}$, we obtain $f^t(z) \in D_{\mu_{N-1}(\alpha^{\circ 1})^{k_3'}\lambda(\delta)\theta}(I')$, where $G_t \subset I' \subset (1+\alpha^{\circ 2})I' \subset I^{\hat{m}}$. If $t = 0$ then this completes the proof. So assume $t > 0$ and let $\hat{s}_1$ be the maximal integer with $\hat{s}_1 < t$ such that $G_{\hat{s}_1} \subset I$. Then $G_{\hat{s}_1}$ is contained in a return domain J to I with $J \subset I \setminus I^1$. Let $\hat{s}_1'$ be the minimal non-negative integer such that $G_{\hat{s}_1'} \subset J$, and let $k_0 = k(J, \{G_j\}_{j=\hat{s}_1'}^{\hat{s}_1})$. As before $k_0 \leq k-1$. Let

$$k_3 = \sum_{c' \in \mathrm{Crit}(f) - \{c\}} k(\mathcal{L}_{c'}(I^{\hat{m}}), \mathcal{L}_{c'}(J)).$$

Then $k_0 + k_3 \leq k + N$. Because $G_j \cap J = \emptyset$ for all $t \leq j \leq s_2$, for any $c' \in \mathrm{Crit}(f) - \{c\}$, there is at most one interval in the chain $\{G_j\}_{j=t}^{s_2}$ which enters $\mathcal{L}_{c'}(J)$. By a similar argument as in Claim 1, we get

$$f^t(z) \in D_{[\mu_{N-1}(\alpha^{\circ 2b})^{k_3}\lambda(\alpha^{\circ 2b})]^b\theta}(\hat{T}),$$

where $\hat{T}$ is an interval with $G_t \subset \hat{T} \subset (1+\alpha^{\circ 2b})\hat{T} \subset I^{\hat{m}}$. Now applying Case 1 to the chain $\{G_j\}_{j=0}^t$, we obtain two integers $0 \le k_1, k_2 \le k_0$ with $k_1 + k_2 \le k_0 + N$ such that $z \in D_{\eta\theta}(K_z)$, where

$$\begin{aligned}\eta &= [\mu_{N-1}(\alpha^{\circ 7b})]^{k_1 b}[\mu_{N-1}(\alpha^{\circ 2b})]^{k_3 b}\hat{\mu}_N^{k_2}\lambda(\alpha^{\circ 7b})^{b+3}\lambda(\alpha^{\circ 2b})^b\gamma \\ &\le [\mu_{N-1}(\alpha^{\circ 10b})^{k_1+k_3}\lambda(\alpha^{\circ 10b})^5]^b\mu_N(\delta)^{k_2}\gamma;\end{aligned}$$

and K_z is an interval with $G_0 \subset K_z \subset (1+\alpha^{\circ 7b})K_z \subset I$. Again by the choice of μ_N it follows that $z \in D_{\mu_N(\delta)^k\theta}(K_z)$. $\square$

Now we want to show that if during a pullback we visit an interval J, which is deep inside I, then we may get an improvement in angle. More precisely:

Proposition 11.3. *Let θ_0 be as in Lemma 13.4. For each N, $\delta > 0$, there exist $\mu \in (0,1)$, $C \in (0,1)$ with the following property. Let $I \ni c_0$ be a nice interval in $\mathcal{T}_\delta$, let $J \ni c_0$ be an (at most) N-modal pullback of I, and let $t \in \mathbb{N}$ be such that $J = \mathrm{Comp}_{c_0}(f^{-t}I) \cap \mathbb{R}$. Let $x \in J \cap PC(f)$, let $s \ge t$ be an integer so that $f^s(x)$ is again in J and let*

$$\nu = \#\{0 \le j \le s-t;\ f^j(x) \in J\}.$$

Let $s_0 = 0 < s_1 < \cdots < s_\nu$ be the times for which $s_j \le s-t$ and $x_{s_j} \in J$. (where we write $x_i := f^i(x)$). Take the chain $\{G_i\}_{i=0}^s$ defined by $G_s = J$, and $G_i \ni f^i(x)$. Let

$$U_s = D_\theta(I) \cap \mathbb{C}_{G_s} \text{ and } U_i = \mathrm{Comp}_{G_i} f^{-(s-i)}(U_s).$$

Then

$$U_0 \subset D_{\theta'}(J),$$

where

$$\theta' = \min\left[\mu^{\hat{k}(I;J)}\left(\prod_{j=0}^{\nu-1} C\rho_j\right)\cdot\theta\,,\,\theta_0\right],$$

where $\hat{k}(I;J)$ is defined in equation (19) and ρ_j is so that

$$(1+2\rho_j)\mathcal{L}_{x_{s_j}}J \subset J.$$

Moreover, there exist a universal (large) constant $\xi > 0$ and a positive integer ν_0 which depends on δ and N such that if

$$J \in \mathcal{T}_\xi, \ \nu \geq \nu_0$$

and if for each $c \neq c_0$,

$$\#\{0 < j \leq s; \ f^j(x) \in \mathcal{L}_c(J)\} \geq \nu_0,$$

then

$$\theta' = \min\left[\theta, \theta_0\right].$$

Proof. There exists a non-negative integer q' such that we can write

$$f^s = R_I^{q'} \circ f^t \circ f^{s_\nu} = R_I^{q'} \circ f^t \circ R_J^\nu.$$

Moreover, $R_I^{q'}$ is the first landing of $f^t(x_{s_q}) \in I$ into J. The idea is that we can use Proposition 11.2 to control the loss of angle caused by the pullback through $R_I^{q'} \circ f^t$, while the remaining pullback (through R_J^ν) gives a gain in angle if J is small compared to I and q is large.

Let us first prove that there exist constants $\lambda = \lambda(\delta, N) > 0$ and $\mu = \mu(\delta) > 0$ such that

$$U_{s_\nu+t} \subset D_{\lambda\mu^{\hat{k}(I;J)}\theta}(I). \tag{27}$$

Of course we may assume that $s_\nu + t < s$. Let $s' < s$ be the maximal integer with $f^{s'}(x) \in I$. By Lemma 13.5, using the assumption that $I \in \mathcal{T}_\delta$, we obtain

$$U_{s'} \subset D_{\lambda\theta}(\mathcal{L}_{G_{s'}}(I)).$$

Note that $(1 + 2\delta)\mathcal{L}_{G_{s'_1}}(I) \subset I$. As the chain $\{G_j\}_{j=s_\nu+t}^{s'}$ never enters the interval J, by a similar argument as in the proof of Claim 1 of Lemma 11.3, we obtain (27). Next, since J is an N-modal pullback of I and since $I \in \mathcal{T}_\delta$, by Lemma 13.5, it follows that

$$U_{s_\nu} = \mathrm{Comp}_{G_{s_\nu}} f^{-t} U_{s_\nu+t} \subset D_{\lambda^2\mu^{\hat{k}(I;J)}\theta}(J).$$

Notice that $J \in \mathcal{T}_{\delta'}$, where $\delta' > 0$ is a constant depending on δ and N. Pulling-back to $G_{s_{q-1}}$, Lemma 13.5 gives

$$U_{s_{\nu-1}} \subset D_{\lambda^2\kappa\mu^{\hat{k}(I;J)}\theta}(\mathcal{L}_{x_{s_{\nu-1}}}(J)), \tag{28}$$

where κ is a constant depending only on δ'. By assumption $(1+\rho_{q-1})\mathcal{L}_{x_{q-1}}(J) \subset J$. So replacing $\mathcal{L}_{x_{s_{\nu-1}}}(J)$ by J in (28) gives by Lemma 13.4 the following gain in angle:

$$U_{s_{\nu-1}} \subset D_{\theta^*}(J) \text{ where } \theta^* = \min\left(\lambda^2(C\rho_{q-1})\mu^{\hat{k}(I;J)}\theta, \theta_0\right),$$

where $C = C(\delta') > 0$ is a constant. Repeating this argument $\nu - 1$ times, i.e., pulling-back successively to $x_{s_{\nu-2}}, \ldots, x_{s_0}$, gives

$$U_0 \subset D_{\theta''}(J)$$

where

$$\theta'' = \min\left(\lambda^2(\prod_{j=0}^{\nu-1} C\rho_j)\mu^{\hat{k}(I;J)}\theta, \theta_0\right). \tag{29}$$

Redefining the constant proves the first part of the proposition.

Let us now prove the second part of the proposition. So assume that $J \in \mathcal{T}_\xi$ with a large ξ and that $f^j(x)$ visits each interval $\hat{\mathcal{L}}_c(J)$ many times. As we noted above, the constant C in (29) depends only on (a lower bound for) ξ. Thus provided that ξ is sufficiently large, $C\rho_j \geq 2$ for all $j = 0, 1, \ldots, \nu-1$. Next define $\hat{k} := \hat{k}(I; J)$. Then

$$k(I; J) \geq \hat{k}/b \text{ or } k(\mathcal{L}_c(I), \mathcal{L}_c(J)) \geq \hat{k}/b$$

for some critical point $c \neq c_0$. In the first case, define $K := J$ and in the second case, define $K := \mathcal{L}_c(J)$. By the (first part of the) proof of Lemma 9.6, there exists a universal number $\delta' > 0$ so that for each $x \in K$,

$$(1 + \delta')^{\hat{k}/b}\mathcal{L}_x(K) \subset K. \tag{30}$$

So for each $x \in J$ which visits K before returning to J,

$$(1 + \delta'')^{\hat{k}/b}\mathcal{L}_x(J) \subset J \tag{31}$$

and by assumption there are at least $\nu_0' = \nu_0 - N$ of such visits. This implies that $\mu^{\hat{k}(I;J)}\left(\prod_{j=0}^{\nu-1} C\rho_j\right) \geq 1$ and completes the proof of the proposition. $\square$

11.2 Proof of a n-step inclusion for puzzle pieces

The purpose of this subsection is to prove Theorem 11.1. So let I_n be the enhanced nest defined in Subsection 8.1 around the critical point c_0. Note that $I_n \in \mathcal{T}_\rho$ for each n, see Proposition 8.1.

First we show that if consecutive intervals from the collection $I_{n-M}, \dots, I_n$ are similar in size, then we get a polynomial-like extension of R_{I_n} with range $D_\theta(I_{n-M}) \cap \mathbb{C}_{I_n}$.

Proposition 11.4. *For each k there exists an integer M so that for each $\theta \in (0, \pi/2)$ one has the following. Assume that $n \le \chi$ and*

$$|I_{n-i}|/|I_{n-i+1}| \le k \text{ for all } i = 0, \dots, M. \tag{32}$$

Then the first return map R_{I_n} (restricted to $PC(f) \cap I_n$) extends to a quasi-polynomial-like map with range $D_\theta(I_{n-M}) \cap \mathbb{C}_{I_n}$ and with domains inside $D_\theta(I_{n-M+1})$.

Proof. To prove this proposition, take a domain J of R_{I_n} and let $s > 0$ be so that $R_{I_n}|J = f^s$. Let $G_s = I_n$ and $\{G_i\}_{i=0}^s$ be the chain with $G_0 = J$. Moreover, let

$$U_s = D_\theta(I_{n-M}) \cap \mathbb{C}_{G_s} \text{ and } U_i = \mathrm{Comp}_{G_i} f^{-(s-i)}(U_s).$$

We need to show that $U_0 \subset D_\theta(I_{n-M+1})$. Consider $w \in U_0$ and write $w_i = f^i(w)$.

As before, let p_{n-M} be so that $I_{n-M+1} = \mathrm{Comp}_{c_0} f^{-p_{n-M}}(I_{n-M}) \cap \mathbb{R}$. Let $s_1' < s$ be maximal so that $s - s_1' > p_{n-M}$ and so that $G_{s_1'} \subset I_{n-M+1}$ and let $s_1 < s_1'$ be maximal such that $G_{s_1} \subset I_{n-M+1}$. Because of the 4th assertion in Lemma 8.2 at most 4 of the intervals $G_{s_1}, \dots, G_s$ are contained in I_{n-M+1}. (Note that s_1 exists when $M \ge 2$, because $G_0, \dots, G_s$ visits I_{n-M+1} at least $2^{T(M-1)}$ times.) By the following Claim 1, $\hat{k}(I_{n-M}; I_{n-M+1}) \le K$. Applying Proposition 11.3 to $I = I_{n-M}$, $J = I_{n-M+1}$ and the chain $\{G_j\}_{j=s_1'}^s$, we see that there exist constants $\mu, C_1 \in (0,1)$ such that $w_{s_1'} \in D_{\mu^K C_1 \theta}(I_{n-M+1})$, and thus by Lemma 13.5

$$w_{s_1} = D_{C_2\theta}(I'_{n-M+1}),$$

where $C_2 \in (0,1)$ is a constant and $I'_{n-M+1} = \mathcal{L}_{G_{s_1}}(I_{n-M+1})$ (which is ρ-well inside I_{n-M+1}).

Claim 1: There exists K depending only on k so that $k(I_{n-M}, I_{n-M+1}) \leq K$ and $k(\mathcal{L}_c(I_{n-M}), \mathcal{L}_c(I_{n-M+1})) \leq K$ for each $c \neq c_0$.
Proof of Claim 1: By the real bounds, if $k(I_{n-M}, I_{n-M+1})$ is large then $|I_{n-M}|/|I_{n-M+1}|$ is large, contradicting (32). So assume that $c \neq c_0$ and that $k(\mathcal{L}_c(I_{n-M}), \mathcal{L}_c(I_{n-M+1}))$ is large. Then by Lemma 9.6 (or Lemma 9.3)

$$|\mathcal{L}_{c_0}\mathcal{L}_c(I_{n-M})|/|\mathcal{L}_{c_0}\mathcal{L}_c(I_{n-M+1})|$$

is large. Note that by construction

$$I_{n-M} \supset \mathcal{L}_{c_0}\mathcal{L}_c(I_{n-M}) \supset \Gamma(I_{n-M}) \supset I_{n-M+1}$$

and

$$I_{n-M+1} \supset \mathcal{L}_{c_0}\mathcal{L}_c(I_{n-M+1}) \supset \Gamma(I_{n-M+1}) \supset I_{n-M+2}.$$

So if $k(\mathcal{L}_c(I_{n-M}), \mathcal{L}_c(I_{n-M+1}))$ were large, then either $|I_{n-M}|/|I_{n-M+1}|$ or $|I_{n-M+1}|/|I_{n-M+2}|$ is large (or both are large), contradicting (32) and thus completing the proof of Claim 1.

Now we distinguish two cases. In the first case we shall use a method similar to one used previously in [21].

Case 1: $w_{s_1} \notin D_\theta(I_{n-M+1})$. Because of this and $w_{s_1} \in D_{C_2\theta}(I'_{n-M+1})$, Part 3 of Lemma 13.4 implies that there exists a constant C so that

$$w_{s_1} \in D_{C\theta\frac{|G_{s_1}|}{|I_{n-M+1}|}}(G_{s_1}).$$

Now $G_1, \ldots, G_s$ are disjoint and because $G_s = I_n$ belongs to $\mathcal{T}_\rho$, we get from Lemma 13.5 that

$$w_1 \in D_{C'\theta\frac{|G_{s_1}|}{|I_{n-M+1}|}}(G_1)$$

where C' depends on C and ρ.
Claim 2: There exists $\kappa > 0$ (depending only on k) so that $|G_{s_1}|/|I_n| \geq \kappa$.
Proof of Claim 2: $f^{s-s_1} \colon G_{s_1} \to G_s := I_n$ is by construction at most a 3-rd iterate of the first return map to I_{n-M+1}. Hence if $|G_{s_1}|/|G_s|$ would be small, then the derivative of $R_{I_{n-M+1}}$ would be large at some point (in one of the domains visited by $G_{s_1}, \ldots, G_s$). Because $I_{n-M+1} \in \mathcal{T}_\rho$, and because of the last part of Lemma 9.4 this would imply that one of the domains of $R_{I_{n-M+1}}$ is small compared to I_{n-M+1}. But then, by Proposition 8.1, I_{n-M+1} would have a small child. But this contradicts the assumption that (32) holds for $i = n - M + 1$, and completed the proof of Claim 2.

From this claim we get that

$$w_1 \in D_{C''\theta \frac{|I_n|}{|I_{n-M+1}|}}(G_1)$$

where C'' depends on ρ and k. By part 1 of Lemma 13.4 this gives

$$w_1 \in D_{C''\theta \frac{|I_n|}{|I_{n-M+1}|} \frac{|I^f_{n-M+1}|}{|G_1|}}(I^f_{n-M+1}),$$

where I^f_{n-M+1} is the pullback of I_{n-M} by $f^{p_{n-M}-1}$ containing $f(c_0)$. Since $I_{n-M+1} \in \mathcal{T}_\rho$, Lemma 13.2 implies

$$w \in D_{\hat{C}\theta \frac{|I_n|}{|I_{n-M+1}|} \frac{|I^f_{n-M+1}|}{|G_1|}}(I_{n-M+1}).$$

Note that

$$\hat{C}\theta \frac{|I_n|}{|I_{n-M+1}|} \frac{|I^f_{n-M+1}|}{|G_1|} \geq \frac{\hat{C}}{K}\theta \frac{|I_{n-M+1}|^{\ell-1}}{|I_n|^{\ell-1}} >> \theta,$$

where ℓ is the order of the critical point c_0. Hence $w \in D_\theta(I_{n-M+1})$. So the lemma is proved if we are in Case 1.

Case 2: $w_{s_1} \in D_\theta(I_{n-M+1})$. Then let $s_2 < s_1$ be maximal so that $s_1 - s_2 > p_{n-M+1}$ and so that $G_{s_2} \subset I_{n-M+2}$. Then again at most 4 of the intervals $G_{s_2}, \dots, G_s$ are contained in I_{n-M+2} because G_j, $j = s_1, \dots, s-1$ never enters I_{n-M+1}. As before $w_{s_2} \notin D_{C_2\theta}(I'_{n-M+2})$ and there are two cases. If $w_{s_2} \notin D_\theta(I_{n-M+2})$, the arguments in Case 1 apply (replacing s, s_1, I_{n-M} by s_1, s_2, I_{n-M+1}). By the choice of s_2 we get exactly as in Claim 2 that $|G_{s_2}|/|I_n| \geq \kappa$. So we get $w \in D_\theta(I_{n-M+1})$.

Alternatively, $w_{s_2} \in D_\theta(I_{n-M+2})$. Repeat all this, say j times, until we have to stop because either we fall in Case 1, or until $j = M-1$. In the former case $w \in D_\theta(I_{n-M+1})$ and in the latter case $w_{s_j} \in D_\theta(I_n)$ and $s_j = 0$. □

Now we are ready to prove the Main Theorem of this section:

Theorem 11.1 *There exists $\theta > 0$ and n_0 so that for all n with $\chi \geq n \geq n_0$, $\mathbf{I}_n \subset D_\theta(I_{n-n_0})$.*

Proof. Let θ_0 be as in Lemma 13.4, and fix $\theta = \min(\theta_0, \sigma)$. So $\mathbf{I}_0 \subset D_\theta(I_0)$. We shall choose n_0 in the proof below. Note that for any choice of n_0, the first induction step holds: we have $\mathbf{I}_n \subset D_\theta(I_{n-n_0})$ for $n = n_0$. So assume by induction that $\mathbf{I}_n \subset D_\theta(I_{n-n_0})$, and show that $\mathbf{I}_{n+1} \subset D_\theta(I_{n-n_0+1})$. Choose p_n so that $I_{n+1} = \mathrm{Comp}_{c_0} f^{-p_n}(I_n) \cap \mathbb{R}$ and let $\{G_j\}_{j=0}^{p_n}$ be the chain with $G_{p_n} = I_n$ and $G_0 = I_{n+1}$.

Let ξ be the constant as in Proposition 11.3. By Proposition 8.1, there exists a constant k_0 such that if $|I_i|/|I_{i+1}| \geq k_0$, then $I_{i+2} \in \mathcal{T}_\xi$. Let $M = M(k_0)$ be the constant determined by Proposition 11.4.

Claim: There exists a constant $K > 1$ such that if $n_0 \geq KM$ and if there exists $i \in \{n - n_0, \ldots, n - n_0 + M\}$ for which $|I_i|/|I_{i+1}| \geq k_0$ then $\mathbf{I}_{n+1} \subset D_\theta(I_{n-n_0+1})$.

Proof of Claim: By the induction assumption $\mathbf{I}_n \subset D_\theta(I_{n-n_0}) \cap \mathbb{C}_{I_n}$ and by construction $\mathbf{I}_{n+1} = \mathrm{Comp}_{c_0} f^{-p_n}(\mathbf{I}_n)$. To prove this claim we shall apply Proposition 11.3. Let i' be minimal such that $|I_{i'}|/|I_{i'+1}| \geq k_0$. Then $J := I_{i'+1}$ is in $\mathcal{T}_\xi$. Note that J is an N'-modal pullback of $I := I_{n-n_0}$ where $N' \leq \hat{N}^{n-n_0-i+1} \leq \hat{N}^{n_0-M}$ and where $\hat{N}$ depends on b (and is determined by the construction of the intervals $I_0, I_1, \ldots$). Also note that $\#\{0 < j < p_n; f^j(I_{n+1}) \subset J\}$ and $\#\{0 < j < p_n; f^j(I_{n+1}) \subset \mathcal{L}_c(J)\}$, $c \neq c_0$, are all at least 2^{n_0-M-1} (by Lemma 8.2), and hence is larger than the constant $\nu_0(\rho, N')$ as in Proposition 11.3 provided that n_0/M is large. Thus applying the second part of Proposition 11.3, gives $\mathbf{I}_{n+1} \subset D_{\theta'}(I_{i'+1})$ with $\theta' \geq \min(\theta, \theta_0) = \theta$. This proves the claim.

By the previous claim, we may assume that $|I_i|/|I_{i+1}| \leq k_0$ for each $i = n - n_0, \ldots, n - n_0 + M$. Therefore, by Proposition 11.4 the first return map to I_{n-n_0+M} extends to a quasi-polynomial-like map with range equal to $D_\theta(I_{n-n_0}) \cap \mathbb{C}_{I_{n-n_0+M}}$ and each of its domains is contained in $D_\theta(I_{n-n_0+1})$. But since $f^{p_n}\colon I_{n+1} \to I_n$ can be written as a composition of this first return map, $\mathbf{I}_n \subset D_\theta(I_{n-n_0})$ implies that $\mathbf{I}_{n+1} \subset D_\theta(I_{n-n_0+1})$. $\square$

11.3 A one-step inclusion for puzzle pieces

Next we prove the following

Proposition 11.5. *There exists $\theta > 0$ such that for each n with $0 \leq n \leq \chi - 1$ and for each $x \in PC(f) \cap I_n$,*

$$\mathcal{L}_x(\mathbf{I}_n) \subset D_\theta(I_n).$$

Before proving this proposition we need some lemmas. Note that if U is a successor of V other than the first one, then $R_V(U) \cap U = \emptyset$, and thus for any return domain $P \subset U$ to U, $R_U|P = R_V^q$ with $q \geq 2$.

Lemma 11.4. *There exist N and $\theta_1 > 0$ such that for each $0 \le n \le \chi - 3$ there exists $m \in \{n, n+3\}$ such that for each $x \in PC(f) \cap I_m$ one has* $\mathrm{Comp}_x \mathrm{Dom}(R^N_{\boldsymbol{I}_m}) \subset D_{\theta_1}(I_m)$.

Proof. By Theorem 11.1, there exists a constant $\theta \in (0, \pi/2)$ such that $\mathbf{I}_n \subset D_\theta(I_{n-n_0})$ for every $0 \le n \le \chi$.

Let ξ be a constant as in Proposition 11.3. By Claim 1 in proposition 11.4, there exists a positive integer $\hat{k}$ (depending on n_0) such that if $\hat{k}(I_{n-n_0}; I_n) \ge \hat{k}$, then $|I_n|/|I_{n+1}|$ is large and thus by Proposition 8.1 (3), $I_{n+2} \in \mathcal{T}_\xi$. By choosing ξ larger if necessary we may assume that $R_{I_{n+2}}$ has a quasi-polynomial-like extension with range $D_\theta(I_{n+2})$.

Let us first consider the case that $\hat{k}(I_{n-n_0}; I_n) \ge \hat{k}$. Let $m = n+3$. Note that for every $y \in I_m$, the orbit $y, f(y), \ldots, R_{I_m}(y)$ visits each of the sets $\hat{\mathcal{L}}_c(\mathbf{I}_{m-1})$ for every critical point c. Let $x \in I_m \cap PC(f)$, let $N \ge 100$ be a large positive integer, and let $y = R^{10}_{I_m}(x)$. Applying the second part of Proposition 11.3 to $I = I_{n-n_0}$ and $J = I_{m-1}$ we see that provided that N is sufficiently large, we visit every critical landing domain of $R_{I_{m-1}}$ many times. Hence the pull back of $\mathbf{I}_m \subset D_\theta(I_{n-n_0}) \cap \mathbb{C}_{I_m}$ along the orbit $\{y, f(y), \ldots, R^{N-10}_{I_m}(y)\}$ is contained in $D_\theta(I_{m-1})$. As we can express $R^{10}_{I_m}$ as $R^q_{I_{m-1}} \circ f^{p_{m-1}}$ with $q \ge 0$, and $R_{I_{m-1}}$ has a quasi-polynomial-like extension with range $D_\theta(I_{m-1})$, it follows that $\mathrm{Comp}_x(\mathrm{Dom}(R^N_{\mathbf{I}_m})) \subset D_{\theta_1}(I_m)$.

Now assume that $\hat{k}(I_{n-n_0}; I_n) \le \hat{k}$. Let $m = n$. Let $t = p_{n-1} + p_{n-2} + \cdots + p_{n-n_0}$ (so that $f^t(\mathbf{I}_n) = \mathbf{I}_{n-n_0}$). For any $x \in I_n \cap PC(f)$ and any N sufficiently large ($N \ge 6$), if s is such that $R^6_{I_n}(x) = f^s(x)$, then $s \ge t$ and so by the first part of Proposition 11.3, we have $\mathrm{Comp}_x \mathrm{Dom}(R^N_{\mathbf{I}_n}) \subset D_{\theta_1}(I_n)$. $\square$

Lemma 11.5. *There exists a constant $\mu \in (0,1)$ such that for each $N \ge 2$, each $\theta > 0$, and each $0 \le n \le \chi - 2$, if $\bigcup_{x \in PC(f) \cap I_n} \mathrm{Comp}_x \mathrm{Dom}(R^N_{\boldsymbol{I}_n}) \subset D_{\theta_1}(I_n)$ then $\bigcup_{x \in PC(f) \cap I_{n+1}} \mathrm{Comp}_x \mathrm{Dom}(R^{N-1}_{\boldsymbol{I}_{n+1}}) \subset D_{\mu\theta}(I_{n+1})$.*

Proof. As before, let p_n be so that $f^{p_n}\mathbf{I}_{n+1} = \mathbf{I}_n$. Take $x \in I_{n+1} \cap PC(f)$, let $U = \mathrm{Comp}_x \mathrm{Dom}(R^{N-1}_{\mathbf{I}_{n+1}})$ and let s be so that $R_{\mathbf{I}_{n+1}}|U = f^s$. Then $U = \mathrm{Comp}_x f^{-s}(\mathbf{I}_{n+1}) = \mathrm{Comp}_x f^{-s-p_n}(\mathbf{I}_n)$. Let k be so that $f^s|f^{p_n}(U) = R^k_{\mathbf{I}_n}$. Because of Lemma 8.2, $\hat{r}(I_n) \le \frac{1}{2^{T-1}} r(I_{n+1})$. Hence $k \ge 2^{T-1}(N-1)$ and so $k \ge N$. Therefore the assumption in the lemma implies $f^{p_n}U \subset D_\theta(I_n)$. Hence $U = \mathrm{Comp}_x f^{-p_n} f^{p_n} U \subset \mathrm{Comp}_x f^{-p_n}(D_\theta(I_n)) \subset D_{\mu\theta}(I_{n+1})$ (by Lemma 13.5). $\square$

Proof of Proposition 11.5. The two previous lemmas imply Proposition 11.5. $\square$

Proof of Proposition 8.3. Let $0 \le n \le \chi - 1$. By construction, there exists a positive integer $\nu = \nu_n$ such that $f^{\nu_n} : \mathcal{B}(\mathbf{I}_n) \to \mathbf{I}_n$ is a proper map with bounded degree and $f^\nu(\mathcal{A}(\mathbf{I}_n)) \subset \mathcal{L}_{f^\nu(c)}(\mathbf{I}_n)$. From Lemma 11.5 we have

$$\mathcal{L}_{f^\nu(c)}(\mathbf{I}_n) \subset D_\theta(I_n).$$

Since $I_n \in \mathcal{T}_\rho$ (and the pullback only meets the critical point at most b^2 times, see Lemma 8.1), we get

$$\mathcal{A}(\mathbf{I}_n) \subset D_{\lambda\theta}(\mathcal{B}(I_n)) \tag{33}$$

and

$$\frac{\operatorname{diam}(\mathcal{A}\mathbf{I}_n)}{|\mathcal{A}I_n|} \le C(\theta) \max\left\{1, \left(\frac{\operatorname{diam}(\mathbf{I}_n)}{|I_n|}\right)^{1/2}\right\},$$

where we use that since $I_n \in \mathcal{T}_\rho$, $f^{r'}(\mathcal{L}_{f^\nu(c)}(I_n))$ is not small compared to I_n, Koebe on the diffeomorphic pieces and Lemma 13.1. (Here r' is the return time of $\mathcal{L}_{f^\nu(c)}(I_n)$ to I_n.) As $\mathcal{B}(I_n) \in \mathcal{T}_{\rho'}$, it also follows that there exists a topological disc $\mathcal{A}'(\mathbf{I}_n) \supset\supset \mathcal{A}(\mathbf{I}_n)$ so that $\mod(\mathcal{A}'(\mathbf{I}_n) \setminus \mathcal{A}(\mathbf{I}_n)) > \xi$ and so that $(\mathcal{A}'(\mathbf{I}_n) \setminus \mathcal{A}(\mathbf{I}_n)) \cap PC(f) = \emptyset$, where $\xi > 0$ is a constant. Let p'_n be such that $f^{p'_n}(\mathbf{I}_{n+1}) = \mathbf{I}_n$. Then $f^{p'_n} : \mathbf{I}_{n+1} \to \mathcal{A}(\mathbf{I}_n)$ is a proper map with bounded degree. Set $\mathbf{I}'_{n+1} = \operatorname{Comp}_{c_0}(f^{-p'_n}\mathcal{A}'(\mathbf{I}_n))$. Then $\mathbf{I}'_{n+1} - \mathbf{I}_{n+1}$ is an annulus disjoint from $PC(f)$ and its modulus is bounded away from zero. Moreover,

$$\frac{\operatorname{diam}(\mathbf{I}_{n+1})}{|\mathbf{I}_{n+1}|} \le C(\theta) \max\left\{1, \left(\frac{\operatorname{diam}(\mathbf{I}_n)}{|I_n|}\right)^{1/2}\right\}.$$

From this inequality, and from the assumption that $\mathbf{I}_0 \subset D_\sigma(I_0)$ it follows that $\operatorname{diam}(\mathbf{I}_n)/|I_n|$, $0 \le n \le \chi$, is uniformly bounded from above.

$\square$

12 Appendix 1: A criterion for the existence of quasiconformal extensions

In this appendix, we shall prove a result which we used to prove the quasiconformality of certain maps. This result is inspired by Smania [40] and Heinonen and Koskela [13].

Recall that a topological disk P has *ε-bounded geometry* if there is a point $x \in P$ such that $B(x, \varepsilon \operatorname{diam}(P)) \subset P$.

QC-Criterion. *For any constants $0 \le k < 1$ and $\varepsilon > 0$ there exists a constant K with the following property. Let $\phi : \Omega \to \tilde{\Omega}$ be a qc homeomorphism between two Jordan domains. Let X be a subset of Ω consisting of pairwise disjoint topological disks. Assume that the following hold:*

1. *if P is a component of X, then both of P and $\phi(P)$ have ε-bounded geometry, and moreover*
$$\operatorname{mod}(\Omega - P) \ge \varepsilon, \ \operatorname{mod}(\tilde{\Omega} - \phi(P)) \ge \varepsilon;$$
2. *$|\bar{\partial}\phi| \le k|\partial\phi|$ holds a.e. on $\Omega - X$.*

Then there exists a K-qc map $\psi : \Omega \to \tilde{\Omega}$ such that $\psi = \phi$ on $\partial\Omega$.

We shall prove a slightly stronger result. For a homeomorphism $\phi : \Omega \to \tilde{\Omega}$ and for $x \in \Omega$, let

$$\underline{H}(\phi, x) = \liminf_{r \to 0} \frac{\sup_{|y-x|=r} |\phi(y) - \phi(x)|}{\inf_{|y-x|=r} |\phi(y) - \phi(x)|} \in [1, \infty].$$

Lemma 12.1. *Let $H > 1$ and let $\varepsilon \in (0, 1)$ be constants. Let $\phi : \Omega \to \tilde{\Omega}$ be an orientation-preserving homeomorphism between two Jordan domains. Let X_0, X_1 be disjoint subsets of Ω such that $m(X_0) = m(\phi(X_0)) = 0$, where m denotes the 2-dimensional Lebesgue measure. Assume that the following hold:*

1. *for each $x \in \Omega - (X_0 \cup X_1)$, we have*
$$\underline{H}(\phi, x) < H;$$
2. *for each $x \in X_0$, we have*
$$\underline{H}(\phi, x) < \infty;$$
3. *there exists a family $\mathcal{P}$ of pairwise disjoint topological disks which form a covering of X_1, such that for each $P \in \mathcal{P}$, we have*
    - *both P and $\phi(P)$ have ε-bounded geometry;*

- $$\mathrm{mod}(\Omega - \overline{P}) \geq \varepsilon, \ \mathrm{mod}(\tilde{\Omega} - \phi(\overline{P})) \geq \varepsilon.$$

Then there exists a K-qc homeomorphism $\psi : \Omega \to \tilde{\Omega}$ such that $\psi|\partial\Omega = \phi|\partial\Omega$, where $K \geq 1$ is a constant depending only on H and ε.

Proof. For any four points a, b, c, d which lie in an anticlockwise order on the $\partial\Omega$, denote by $\Gamma(a, b; c, d)$ the family of rectifiable curves which join the arc ab and cd and lie inside Ω, and let $\lambda(a, b; c, d)$ the extremal length of this family. Similarly we define $\tilde{\Gamma}(\tilde{a}, \tilde{b}; \tilde{c}, \tilde{d})$ and $\tilde{\lambda}(\tilde{a}, \tilde{b}; \tilde{c}, \tilde{d})$. It is a basic fact that we only need to show that $\Gamma(a, b; c, d)$ is bounded away from zero when $\tilde{\Gamma}(\tilde{a}, \tilde{b}; \tilde{c}, \tilde{d}) = 1$. See [1].

By means of the Riemann mapping theorem and Möbius transformations, we may assume that $\Omega = \tilde{\Omega} = \mathbb{D}$, and also that $a = \tilde{a} = 1$, $b = \tilde{b} = i$, $c = \tilde{c} = -1$. Then $\tilde{d} = -i$. Note that by the Koebe distortion theorem, the image of P under a conformal map defined on Ω has ε'-bounded geometry, where $\varepsilon' > 0$ is a constant depending only on ε. To bound $\lambda(a, b; c, d)$ from below, we shall define an open covering $\mathcal{E}$ of the unit disk with the following properties:

- $\sum_{E \in \mathcal{E}} \chi_E \leq C$,
- $\sum_{E \in \mathcal{E}} \mathrm{diam}(\phi(E))^2 \leq C$,
- for each $E \in \mathcal{E}$, either $E \in \mathcal{P}$, or E is a round disk such that $2E \subset \mathbb{D}$,

where C is a constant depending only on ε.

Let us prove that this covering completes the proof. For any $P \in \mathcal{P}$, choose a point $x = x_P \in P$ such that the round disk $B(x, \varepsilon \,\mathrm{diam}\, P)$ is contained in P. We denote by P' this round disk and also set $\hat{P} = B(x, 2 \,\mathrm{diam}\, P)$. For any $E \in \mathcal{E} - \mathcal{P}$, let $E' = E$ and $\hat{E} = 2E$. Note that for each $E \in \mathcal{E}$ and for each rectifiable curve γ which joins two points in ∂D, if $\gamma \cap E \neq \emptyset$, then the length of $\gamma \cap \hat{E}$ is at least $\varepsilon' \,\mathrm{diam}(E)$, where $\varepsilon' > 0$ depends only on ε. Now define

$$\rho = \sum_{E \in \mathcal{E}} a_E \chi_{\hat{E}},$$

where

$$a_E = \frac{\mathrm{diam}(\phi(E))}{\mathrm{diam}(E)}.$$

Then, for each $\gamma \in \Gamma(1, i; -1, d)$, we have

$$\begin{aligned}
\int_\gamma \rho(z)|dz| = \sum_{E\in\mathcal{E}} a_E \int_\gamma \chi_{\hat{E}}|dz| \qquad &\geq \sum_{E;E\cap\gamma\neq\emptyset} \operatorname{diam}(\phi(E)) \frac{\operatorname{length}(\gamma\cap\hat{E})}{\operatorname{diam}(E)} \\
\geq \varepsilon' \sum_{E;E\cap\gamma\neq\emptyset} \operatorname{diam}\phi(E) \quad &\geq \sqrt{2}\varepsilon'.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int \rho^2 dxdy = \int (\sum_{E\in\mathcal{E}} a_E \chi_{\hat{E}})^2 \qquad &\leq C' \int (\sum_{E\in\mathcal{E}} a_E \chi_{E'})^2 \\
\leq C' \int \sum_{E\in\mathcal{E}} a_E^2 \chi_{E'} \sum_{E\in\mathcal{E}} \chi_{E'} \quad &\leq C' \int \sum_{E\in\mathcal{E}} a_E^2 \chi_{E'} \sum_{E\in\mathcal{E}} \chi_E \\
\leq C'C \int \sum_{E\in\mathcal{E}} a_E^2 \chi_{E'} \qquad &= C'C \sum_{E\in\mathcal{E}} a_E^2 m(E') \\
\leq C'C \sum_{E\in\mathcal{E}} a_E^2 m(E) \qquad &= C'C \sum_{E\in\mathcal{E}} \operatorname{diam}(\phi(E))^2 \frac{m(E)}{\operatorname{diam}(E)^2} \\
\leq C'' \sum_{E\in\mathcal{E}} \operatorname{diam}(\phi(E))^2 \qquad &\leq C''C,
\end{aligned}$$

where in the second inequality we use the following lemma, which is a consequence of the Hardy-Littlewood maximal function theory. See [3].

Lemma 12.2. *For each constant $\tau > 1$, there is a constant C with the following property. Let $\{B_i\}_{i\in\Lambda}$ be a family of Euclidean disks, and let $a_i \geq 0$ be constants. Then*

$$\int (\sum_i a_i \chi_{\tau B_i})^2 \leq C \int (\sum_i a_i \chi_{B_i})^2.$$

In particular, we have an upper bound on $\int \rho^2$. Together with the lower bound on $L_\rho(\Gamma(1, i; -1, d))$, this implies that $\lambda(1, i; -1, d)$ is bounded away from zero.

It remains to construct the covering $\mathcal{E}$. It will be the union of three families of topological disks:

$$\mathcal{E} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{P},$$

where $\mathcal{A}$, $\mathcal{B}$ will be coverings of X_0 and $\Omega - X_0 \cup X_1$ respectively, and $\mathcal{P}$ is as in the lemma.

Let us first define $\mathcal{B}$. For each $x \in \Omega - X_0 \cup X_1$, there is a $r_x > 0$ such that $\phi(B(x, r_x))$ has H^{-1}-bounded geometry. Moreover, we can assume that $B(x, 2r_x) \subset \mathbb{D}$. By Besicovic's theorem, we can find countably many $x_i \in \Omega - X_0 \cup X_1$ such that

- $\bigcup_i B(x_i, r_{x_i}) \supset \Omega - X_0 \cup X_1$;
- $\sum_i \chi_{B(x_i, r_{x_i})} \leq C$, where C is a universal constant.

We define $\mathcal{B}$ to be the family of the balls $B(x_i, r_{x_i})$.

To define the covering $\mathcal{A}$, we first decompose the set X_0 into countably many disjoint subsets $X_0^n := \{x \in X_0 : n \leq \underline{H}(\phi, x) < n+1\}$. For $n \in \mathbb{N}$, let us fix a small neighborhood of U_n of X_0^n such that

$$m(U_n) \leq (n+1)^{-4}, \ m(\phi(U_n)) \leq (n+1)^{-4}.$$

For each $x \in X_0^n$, we choose a small $r_x > 0$ such that $B(x, r_x) \subset U$, $B(x, 2r_x) \subset \mathbb{D}$; and such that $\phi(B(x, r_x))$ has $(n+1)^{-1}$-bounded geometry. Let $\mathcal{D}_n$ denote the family of such Euclidean disks $B(x, r_x)$. Then $\mathcal{D} = \bigcup_n \mathcal{D}_n$ is a covering of X_0. Apply Besicovic's theorem once again, we choose a countable subfamily $\mathcal{A}$ such that

- $\bigcup_{A \in \mathcal{A}} A \supset X_0$;
- $\sum_{A \in \mathcal{A}} \chi_A \leq C$, where C is a universal constant as before.

Note that

$$\begin{aligned}
\sum_{A \in \mathcal{A}} \mathrm{diam}(\phi(A))^2 = \sum_{n=1}^{\infty} \sum_{A \in \mathcal{A} \cap \mathcal{D}_n} \mathrm{diam}(\phi(A))^2 \quad &\leq \sum_n \sum_{A \in \mathcal{A} \cap \mathcal{D}_n} (n+1)^2 m(\phi(A)) \\
\leq \sum_{n=1}^{\infty} C(n+1)^2 m(\phi(\bigcup_{D \in \mathcal{D}_n} D)) \quad &\leq C \sum_{n=1}^{\infty} (n+1)^2 m(\phi(U_n)) \\
\leq C \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \quad &\leq C.
\end{aligned}$$

Note also that

$$\sum_{B \in \mathcal{B}} \mathrm{diam}(\phi(B))^2 \leq H^2 \sum_B m(B) \leq CH^2,$$

and

$$\sum_{P \in \mathcal{P}} \mathrm{diam}(\phi(P))^2 \leq \sum_P m(\phi(P))/\varepsilon^2 \leq 1/\varepsilon^2.$$

$\square$

Proof of QC-Criterion. Let X_0 be the subset of $\Omega - X_0$ consisting of points z at which ϕ is not differentiable, and let $X_1 = X$. It follows from the assumption (2) that $H(\phi, z) \leq (1+k)/(1-k)$. Since ϕ is qc, $H(\phi, z) < \infty$ for any $z \in \Omega$ and moreover X_0 and $\phi(X_0)$ both have measure zero. Applying Lemma A.1 completes the proof. $\square$

13 Appendix 2: Some basic facts about Poincaré discs

We say that a real-symmetric holomorphic map g is in the *Epstein class* if for any interval $J \subset \mathbb{R}$ for which $g\colon J \to g(J)$ is a diffeomorphism, $g^{-1}\colon g(J) \to J$ extends to a univalent map defined on $\mathbb{C}_{g(J)}$.

Lemma 13.1 (Schwarz Inclusion). *Let f be in the Epstein class. Let $J \subset \mathbb{R}$ and $f^s\colon J \to f^s(J)$ be diffeomorphic and $\theta \in (0, \pi)$. Then the Schwarz inclusion holds:*

$$U := \mathrm{Comp}_J\, f^{-s}(D_\theta(f^s J)) \subset D_\theta(J).$$

Proof. This simply follows from the fact that (i) if d_P is the Poincaré metric on $\mathbb{C}_J$ then the set of points z with $d_P(z, J) = c$ lies on the boundary of two (symmetric) circles with real trace J, (ii) f^{-s} maps $\mathbb{C}_{f^s(J)}$ univalently into $\mathbb{C}_J$ and (iii) from the Schwarz inclusion theorem. $\square$

Lemma 13.2. *Let $\ell \geq 2$ be even and $P(z) = z^\ell$.*

- *For each $A > 1$ there exists $\lambda \in (0,1)$ such that for each $\theta \in (0, \pi/2)$,*

$$P^{-1}(D_\theta([-A, 1])) \subset D_{\lambda\theta}([-1, 1]).$$

- *For each $\delta > 0$ there exists $\delta' > 0$ and $\lambda \in (0,1)$ so that for each $\theta \in (0, \pi/2)$ one has the following. Let G_1, I be a real intervals with $(1+\delta)G_1 \subset I$ and let G_0 be a component of $P^{-1}(G_1) \cap \mathbb{R}$. Then*

$$\mathrm{Comp}_{G_0}\, P^{-1}(D_\theta(G_1)) \subset D_{\lambda\theta}(G')$$

where G' is an interval with $G_0 \subset G' \subset (1+\delta')G' \subset \mathrm{Comp}_{G_0}\, P^{-1}(I) \cap \mathbb{R}$.

Proof. The first part was proved in Lemma 7.4 for [37] for $\ell = 2$, and the extension for $\ell > 2$ follows then from the appendix of [17]. The second part immediately follows from the first part. $\square$

Lemma 13.3. *Let $\ell \geq 2$ be an even integer and let $P(z) = z^\ell$. For any $A > 0$ and any $\theta \in (0, \pi)$ there exists $\theta' \in (0, \pi)$ such that*

$$P^{-1}(D_\theta((-A, 1))) \supset D_{\theta'}((-1, 1)).$$

Part one of the next lemma shows that we improve the angle if we are allowed to replace the base by a larger base. Part two and three of this lemma will allow use to capture 'an escaping part' by a Poincaré domain based on a suitable smaller base.

Lemma 13.4. *One can compare Poincaré discs in the following ways:*

1. *There exists θ_0 such that for each $A > 1$ and each $\theta > 0$,*

$$D_\theta([-1, 1]) \subset D_{\min(\theta A/2, \theta_0)}([-A, A]).$$

2. *For each $\lambda \in (0, 1)$ and $\delta > 0$ there exists $\lambda' \in (0, 1)$ such that for each $\theta > 0$,*

$$D_{\lambda\theta}([-1, 1]) \setminus D_\theta([-1 - \delta, 1 + \delta]) \subset D_{\lambda'\theta}([-1 - \delta, -1]).$$

3. *For each $\lambda \in (0, 1)$ and $\delta > 0$ there exists $\lambda' \in (0, 1)$ such that for each interval $J \subset [-1, 1]$ and each $\theta > 0$,*

$$D_{\lambda\theta}([-1, 1]) \setminus D_\theta([-1 - \delta, 1 + \delta]) \subset D_{\lambda'\theta|J|}(J).$$

Proof. The first part holds because for $\theta \in (0, \pi/2)$, the upper part of $\partial D_\theta([-1, 1])$ is a circle with centre $i/\tan(\theta)$ and of radius $1/\sin(\theta)$, so its boundary intersects the imaginary axis in

$$\frac{i}{\tan(\theta)} + \frac{i}{\sin(\theta)} = \frac{i}{\sin(\theta)}\left[\cos(\theta) + 1\right],$$

while the upper part of $\partial D_{A\theta/2}([-A, A])$ is a circle with centre $iA/\tan(\theta A/2)$ of radius $A/\sin(\theta A/2)$, so it intersects the positive imaginary axis

$$\frac{Ai}{\tan(\theta A/2)} + \frac{Ai}{\sin(\theta A/2)} = \frac{Ai}{\sin(\theta A/2)}\left[\cos(\theta A/2) + 1\right].$$

When θA is small, $A/\sin(\theta A/2) \geq 1.5/\sin(\theta)$, so the inclusion holds.

So let us prove the second and third part. The upper-part of $D_{\lambda\theta}([-1,1])$ is (part of) a ball centered at $A = i/\tan(\lambda\theta)$ and radius $r_1 = 1/(\sin(\lambda\theta))$ and the upper-part of $D_\theta([-1-\delta, 1+\delta])$ is (part of) a ball centered at $B = i(1+\delta)/\tan(\theta)$ and radius $r_2 = (1+\delta)/(\sin(\theta))$. Note that we can assume that $\lambda(1+\delta) < 1$ (otherwise $D_{\lambda\theta}([-1,1])$ contains $D_\theta([-1-\delta, 1+\delta])$). To compute the intersection points of these balls, note that the upperparts of these balls are given by

$$\begin{array}{rcll} x^2 + (y - \frac{1}{\tan(\lambda\theta)})^2 & = & \frac{1}{\sin^2(\lambda\theta)} & \text{and } y > 0 \\ x^2 + (y - \frac{1+\delta}{\tan(\theta)})^2 & = & \frac{(1+\delta)^2}{\sin^2(\theta)} & \text{and } y > 0. \end{array}$$

Subtracting these equations and rearranging, gives

$$y = \frac{\delta(2+\delta)}{2\left(\frac{1}{\tan(\lambda\theta)} - \frac{1+\delta}{\tan(\theta)}\right)} \geq L\theta$$

provided $\theta > 0$ is small where $L > 0$ is a constant. From this the second and third part of the lemma easily follow. $\square$

In this paper we often have to pullback a Poincaré domain along a chain is contained in a Poincaré domain whose angle is not too small. For the remainder of this appendix assume that f is in the class $\mathbf{P}_b^{\tau,\sigma}$ defined in Section 4. The constants appearing in the following two lemmas depend on b, τ and σ.

Lemma 13.5. *For each $\delta > 0$ and $N \in \mathbb{N}$ there exists $\lambda \in (0,1)$ with the following property. Let $\{H_j\}_{j=0}^s$ and $\{H_j'\}_{j=0}^s$ be two chains with $H_j \subset H_j'$ for all j. Assume that the chain $\{H_j'\}_{j=0}^s$ has order $\leq N$ and that $|f^s(H_0)| \geq \delta|H_s|$. Let $\theta \in (0,\pi)$, $V = D_\theta(H_s)$ and $U = \mathrm{Comp}_{H_0}(f^{-s}V)$. Then*

$$U \subset D_{\lambda\theta}(H_0).$$

Moreover, for every $\delta' > 0$ there exists $\lambda' > 0$ the following holds. Let $\{\hat{H}_j\}_{j=0}^s$ be a chain with $\hat{H}_j \subset H_j$. Let T be the component of $f^s(H_0) \setminus \hat{H}_s$ containing a boundary point of H_s in its closure and assume that $|T| \geq \delta'|H_s|$. Then for each $\theta > 0$,

$$\hat{U} = \mathrm{Comp}_{\hat{H}_0} f^{-s} D_\theta(\hat{H}_s) \subset D_{\lambda'\theta}(H')$$

where H' is an interval with $\hat{H}_0 \subset H' \subset (1+\lambda')H' \subset H_0$.

Proof. Let us prove the first assertion. Let $0 \leq s_1 < s_2 < \cdots < s_k < s$ be all the integers such that H'_{s_i} contains a critical point. Then by Schwarz, $f^{s_k+1}(U) \subset D_\theta(H_{s_k+1})$. By Koebe, $|f^{s_k+1}(H_0)|/|H_{s_k+1}|$ is bounded away from zero, and in particular, so is $|f(H_{s_k})|/|H_{s_k+1}|$. Let Ω_0 be the component of the domain of f which contains H_{s_k} and $\Omega = f(\Omega_0)$. Then $f|\Omega_0$ can be written as $P \circ \varphi$, where $P(z) = z^2$ and φ is a real symmetric conformal map defined on Ω_0. By Lemma 13.2, $\varphi(f^{s_k}U) \subset D_{\lambda\theta}(\varphi H_{s_k})$. By the τ-extendibility condition, φ extends to a conformal map onto a topological disk Ω' so that $\mathrm{mod}(\Omega' - \Omega)$ is bounded away from zero. Applying the Koebe distortion theorem, we get $U_{s_k} \subset D_{\lambda'\theta}(H_{s_k})$. Repeating the argument and redefining the constant, we complete the proof.

To prove the second assertion, we only need to consider the case that $\hat{H}_s$ contains $H_s \setminus f^s(H_0)$ and $\hat{H}_s \cap f^s(H_0)$ is not small compared to $f^s(H_0)$. By the first part of this lemma, $\hat{U} \subset D_{\lambda'\theta}(\hat{H}_0)$. Since $\hat{H}_0$ is well-inside H_0 the second assertion of the lemma follows. □

Lemma 13.6. *For each $\delta > 0$ there exists $\delta' > 0$ and $\lambda \in (0,1)$ such that the following holds. Let I be a nice interval, J a domain of the first entry map R_I and s an integer so that $R_I|J = f^s$. Let $\{H_j\}_{j=0}^s$ be a chain with $(1+\delta)H_s \subset I$ and $H_0 \subset J$. Then there exists an interval H' with $H_0 \subset H' \subset (1+\delta')H' \subset J$ and so that*

$$\mathrm{Comp}_{H_0} f^{-s}(D_\theta(H_s)) \subset D_{\lambda\theta}(H').$$

Proof. Let $\{G_j\}_{j=0}^s$ and $\{G'_j\}_{j=0}^s$ be the chains with $H_j \subset G_j \subset G'_j$ and $G'_s = I$ and $G_s = (1+\delta/2)H_s$. Let $H' = G_0$. Then the order of the chain $\{G'_j\}_{j=0}^s$ is bounded from above. Moreover, $f^s(G_0)$ contains a component of $H_s - G_s$ and thus its length is comparable to that of G_s. By Koebe H' is well inside $G'_0 = J$ and by the previous lemma, $U \subset D_{\lambda\theta}(H')$. □

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