Milnor's Conjecture on Monotonicity of Topological Entropy: results and questions*

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This note discusses Milnor's conjecture on monotonicity of entropy and gives a short exposition of the ideas used in its proof which was obtained in joint work with Henk Bruin, see [BvS09]. At the end of this note we explore some related conjectures and questions.

Motivation

In their seminal and widely circulated 1977 preprint 'On iterated maps of the interval: I,II.' Milnor and Thurston proved the following:

Theorem (Milnor and Thurston [MT77], see also [MT88])**.** *The function* $C^{2,b} \to \mathbb{R}$ *which associates to a mapping* $g \in C^{2,b}$ *its topological entropy* $h_{top}(g)$ *is continuous.*

Here $C^{2,b}$ stands for C^2 maps of the interval with b non-degenerate critical points (non-degenerate means second derivative non-zero). This theorem relies on a result of Misiurewicz and Szlenk, see [MS77, MS80] who had previously shown that $\gamma(f) := \exp(h_{top}(f))$ is equal to the growth rate of the number of laps (i.e., intervals of monotonicity) of f^n. The crucial new ingredient in the proof of Milnor and Thurson's theorem is a formula which shows that $\gamma(f)$ is also the zero of a certain meromorphic function.

*This note is based on a talk which was presented at the meeting 'Frontiers in Complex Dynamics (Celebrating John Milnor's 80th birthday)' in Banff in February 2011. The author would like to thank Charles Tresser for helpful comments on an earlier version.

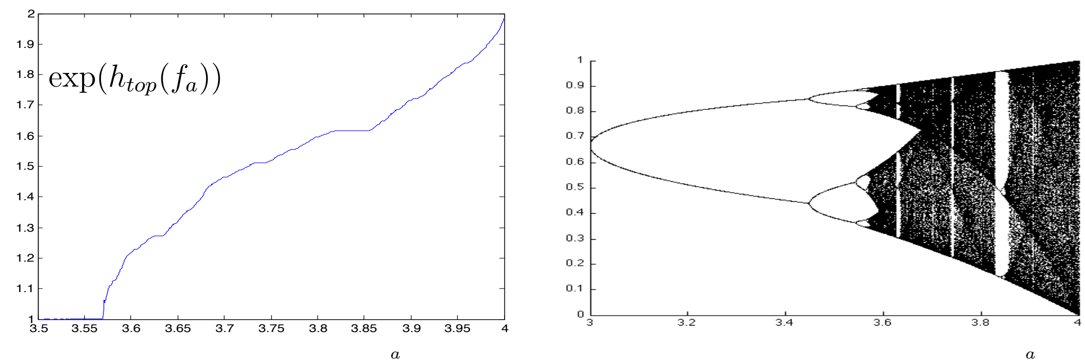

Figure 1: On the left $\exp(h_{top}(f_a))$ as a function $a \in [3.5, 4]$ for $f_a(x) = ax(1-x)$. On the right the well-known bifurcation diagram.

For the quadratic (logistic) family $q_a(x) = ax(1-x)$, the entropy function $a \mapsto h(q_a)$ looks monotone (in the weak sense), see Figure 1. Monotonicity indeed holds:

Theorem. *The topological entropy of $x \mapsto ax(1-x)$ increases with $a \in \mathbb{R}$.*

This theorem was proven in a later version of Milnor and Thurston's preprint which was published in 1988, see [MT88]. Their proof, together with other proofs of this theorem which appeared in the mid 1980's, see Douady and Hubbard [DH85, Dou95], and Sullivan [dMvS93], all rely on ideas from holomorphic dynamics. In the late 1990's, Tsujii gave a different proof which does not rely on holomorphic dynamics, see [Tsu00], but still requires the family of maps to be quadratic (or of the form $z \mapsto z^d + c$).

These proofs all show a stronger statement: periodic orbits never disappear when a increases, see the bifurcation diagram on the right of Figure 1. In this way, we have an instance of the following

Heuristic Principle. *Families of real polynomial maps undergo bifurcations in the simplest possible way.*

Later on it was shown (independently by Graczyk and Swiatek [GŚ97] and Lyubich [Lyu97]) that, within the quadratic family, hyperbolic maps are dense and so the periodic windows are dense.

In this survey, we will discuss a generalisation of the above theorem which solves a conjecture due to Milnor on monotonicity of entropy for more general families of one-dimensional maps.

Milnor's monotonicity of entropy conjecture

Milnor proposed a conjecture which makes the above Heuristic Principle precise in the case of polynomials of higher degree. Indeed, given $\epsilon \in \{-, +\}$,

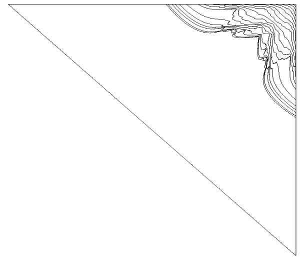

Figure 2: Level sets of the topological entropy within the space P^3_ϵ, where the axis correspond to the position of the two critical values ζ_1, ζ_2 in $[-1,1]$ where $\zeta_1 < \zeta_2$.

consider the space P^b_ϵ of real polynomials f with

1. precisely b distinct critical points, all of which real, non-degenerate and contained in $(-1,1)$;
2. $f\{\pm 1\} \subset \{\pm 1\}$;
3. with shape $\epsilon = \epsilon(f)$, where

$$\epsilon(f) = \begin{cases} +1 & \text{if } f \text{ is increasing at the left endpoint of } [-1,1], \\ -1 & \text{otherwise.} \end{cases}$$

Note that P^b_ϵ consists of polynomials of degree $d = b+1$. In particular, P^1_ϵ corresponds to quadratic maps q_c. Taking $\epsilon = +$, the required normalisation in P^1_ϵ gives $q_c(\pm 1) = 1$ from which it follows that the quadratic family takes the form $x \mapsto q_c(x) = -(c+1)x^2 + c$, where $c \in [-1,1]$ is the critical value.

P^b_ϵ forms a b-dimensional space which can be parametrized by the critical values of the maps f. So for each choice of $(\zeta_1, \ldots, \zeta_b) \in [-1,1]^b$ with $(\zeta_{i-1} - \zeta_i)(\zeta_i - \zeta_{i+1}) < 0$ for each $i = 2, \ldots, b-1$, there exists a unique map $f_{\zeta_1,\ldots,\zeta_b} \in P^d$ with critical values $\zeta_1, \ldots, \zeta_b$ and the map $(\zeta_1, \ldots, \zeta_b) \mapsto f_{\zeta_1,\ldots,\zeta_b}$ is continuous, see [dMvS93, page 120] and [MT00, page 132 and Appendix].

To formalise the above Heuristic Principle in higher dimensions, one may hope that the topological entropy of f depends monotonically on the position of its critical values, i.e., that the map $(\zeta_1, \ldots, \zeta_b) \mapsto h_{top}(f_{\zeta_1,\ldots,\zeta_b})$ is monotone in each of its components. This turns out **not** to be true, as is clear by looking at the level sets of the entropy function in Figure 2, but as can also be proved rigorously see [BvS11].

Instead, Milnor suggested that one should investigate the level sets of topological entropy, noting that monotonicity of entropy within the quadratic family q_c is equivalent to the statement that for each h_0, the level set $I(h_0) := \{c \in \mathbb{R}; h_{top}(q_c) = h_0\}$ is connected. Milnor coined these level sets **isentropes** (following the terminology used in thermodynamics for sets with a given entropy) and proposed the following conjecture.

Milnor's monotonicity of entropy conjecture. *Given $\epsilon \in \{-1, 1\}$, the set of $f \in P_\epsilon^b$ with topological entropy equal to h_0 is connected.*

Another approach to monotonicity using smooth one-parameter families of maps has been introduced by Yorke and co-workers; we review briefly this line of work in Subsection 2.1.

Milnor first posed this conjecture in the cubic case in [DGMT95, Mil92] and subsequently in joint work with Tresser in the above setting in [MT00, page 125]. At the end of this note, we state some further conjectures in this direction, due to Milnor and others.

Milnor and Tresser realized that the treatment of this conjecture in the cubic family only required a weak generalisation of the rigidity results for unimodal maps with quadratic critical points (which had been obtained independently by Lyubich [Lyu97] and Graczyk and Świątek [GŚ97]) rather than a full rigidity statement for cubic maps. Subsequently, they posed the precise analytical question which needed to be answered to experts. This question was solved successfully by Heckman, a student of Świątek, as his PhD [Hec96]. Using Heckman's work, Milnor and Tresser then solved this conjecture in the cubic case (when $b = 2$).

Theorem (Milnor and Tresser [MT00])**.** *The entropy conjecture is true for real cubic maps.*

Note that Milnor and Tresser did not make any assumptions on the location of critical points, because any real cubic maps which is not bimodal is in fact monotone (and so has zero topological entropy). We should also remark that in the cubic case, the parameter space is two-dimensional. Accordingly, Milnor and Tresser's proof considers certain curves curves corresponding to the existence of a critical point belonging to a periodic orbit, uses that certain conjugacy classes are unique and then relies on planar topology to obtain connectedness of isentropes. As a model for the parameter space, Milnor and Tresser use stunted sawtooth maps, see Figure 3.

Some time ago, in joint work with Henk Bruin, we proved Milnor's entropy conjecture for arbitrary degree:

Main Theorem (Bruin and van Strien [BvS09])**.** *The entropy conjecture is true in general.*

In other words, fix $\epsilon \in \{-1, 1\}$, taken any integer $d \geq 1$ and any $h_0 \in \mathbb{R}$. Then the set of $f \in P_\epsilon^b$ with topological entropy equal to h_0, is connected.

The purpose of this note is to present some of the ideas of the proof of the latter theorem. At the end of this note we will state some open problems.

1 Idea of the proof

Our proof of the Main Theorem goes in four steps:

1. A generalization of the notion of hyperbolic component, namely the space $\mathcal{PH}(f)$ of maps which are **partial conjugate** to f (as defined below). A key-point is showing that these sets are **cells** (i.e., homeomorphic images of balls in some $\mathbb{R}^n$). Moreover, we give a full description of the **bifurcations** which occur at the boundary of these sets.
2. The introduction of a more suitable **parameter space**. Following Milnor and Tresser, this is done by using the space $\mathcal{S}^b$ of **stunted sawtooth maps** as a model for the space P^b. Since one can assign to each polynomial $f \in P^b$ a stunted sawtooth map $\Psi(f)$, the spaces P^b and $\mathcal{S}^b$ are naturally related. As mentioned, the space $\mathcal{S}^b$ was already used by Milnor and Tresser.
3. To have a more faithful description of the parameter space, we restrict to the class of **admissible sawtooth maps** $\mathcal{S}^b_*$. In the real case, the admissibility condition corresponds to the absence of wandering intervals; for general polynomials this would correspond to the 'absence of Levy cycles'. A non-trivial result is that isentropes within the space $\mathcal{S}^b_*$ of admissible sawtooth maps are **contractible**.
4. A proof that $\Psi\colon P^b \to \mathcal{S}^b_*$ is 'almost' a **homeomorphism**. This statement (which we will make precisely below) is the main rationale for introducing the space $\mathcal{S}^b_*$ rather than the much more pleasant space $\mathcal{S}^b$.

First ingredient: rigidity and the partial conjugacy class

The first step in the proof is to introduce the notion of **partial conjugacy class** and to show that one has generic bifurcation at the boundary of these sets.

Let $B(f)$ consists of all points x so that $f^n(x)$ tends to a (possibly one-sided) periodic attractor. Let $f, g\colon [-1, 1] \to [-1, 1]$ be two d-modal maps.

Definition. We say that two f, g are **partially conjugate** if there is a homeomorphism $h\colon [-1,1] \to [-1,1]$ such that

- h maps $B(f)$ onto $B(g)$;
- h maps the i-th critical point of f to the i-th critical point of g;
- $h \circ f(x) = g \circ h(x)$ for all $x \notin B(f)$.

Definition. We denote by $\mathcal{PH}(f)$ the class of maps which are partially conjugate to f, also called the **partial conjugacy class** associated to f. Furthermore, we denote by $\mathcal{PH}^o(f)$ the set of maps $g \in \mathcal{PH}(f)$ with

- only hyperbolic periodic points and
- no critical point of g maps to the boundary of a component of $B(g)$.

Example. Consider the quadratic family $q_c(x) = -(c+1)x^2 + c$, $c \in [-1,1]$ and let a_n be the first period doubling bifurcation creating a periodic orbit of period 2^{n+1}. Then all the maps corresponding to $c \in (-1, a_0]$ are partially conjugate. Similarly, all the maps corresponding to $c \in (a_n, a_{n+1}]$ are partially conjugate. If a polynomial $f \in P^b$ has no periodic attractors, then $g \in P^b$ is only partially conjugate to f if it is topologically conjugate. Hence, in this case, $\mathcal{PH}^o(f) = \mathcal{PH}(f)$ and, by Theorem 1 below, $g = f$ and $\mathcal{PH}(f) = \{f\}$.

If all critical points of f are in the basin of hyperbolic periodic attractors, then $\mathcal{PH}^o(f)$ agrees with Douady and Hubbard's hyperbolic component, but the above definition also makes sense if not all critical points are in basins of periodic attractors. The following three theorems from [BvS09] generalise Douady and Hubbard's theorem that hyperbolic components for the family $z \mapsto z^2 + c$ are cells.

Theorem 1. *Let $f \in P^b_\varepsilon$. Then*

- *$\mathcal{PH}^o(f)$ is a submanifold with dimension equal to the number of critical points in $B(f)$.*
- *$\mathcal{PH}(f) \subset \overline{\mathcal{PH}^o(f)}$.*

In particular, if no critical point of f is in the basin of a periodic attractor then $\mathcal{PH}(f)$ is a single point. In fact, the description of $\mathcal{PH}^o(f)$ is more detailed:

Theorem 2. *$\mathcal{PH}^o(f)$ is parametrized by (finite)* **Blaschke** *products and, for example, critical relations unfold transversally.*

Here Blaschke products are maps of the $\mathbb{D}$ of the type $z \mapsto z \prod_{i=1}^{n-1} \frac{z-a_i}{1-\bar{a}_i z}$. If each periodic attractor has precisely one critical point in its basin, then this description simplifies: $\mathcal{PH}^o(f)$ is parametrized by **multipliers** at the periodic attractors (so this corresponds to Douady and Hubbard's result).

That critical relations unfold transversally in special cases (when a critical point is eventually periodic) was proved previously in [vS00] and [BE09]. In addition to the above theorem we need the following additional transversality properties at the boundary of partial conjugacy classes (and consequently at the 'boundary' of the space of real Blaschke products):

Theorem 3. *Bifurcations at $f \in \mathcal{PH}(f) \setminus \mathcal{PH}^o(f)$ are always generic in the following sense:*

- **saddle-node***: creation of one-sided attractor, which then becomes becomes an attracting + repelling pair;*
- **pitchfork***: a two-sided attractor, which becomes repelling and spins off a pair of attracting orbits;*
- **period-doubling***: multiplier -1 with a creation/destruction of a attractor of double the period;*
- **homoclinic bifurcation***: a critical point in the basin is (eventually) mapped to the boundary of the basin.*

In other words, near $f \in \mathcal{PH}(f) \setminus \mathcal{PH}^o(f)$ several of these bifurcations can occur at the same time, but each of these is generic. So, for example, in the saddle-node bifurcation case, $(f^n)''$ is non-zero at a periodic point of period n. In [BvS09] a slightly weaker version of Theorem 3 is proved, as this suffices for the proof of Milnor's conjecture. Theorem 3 will be proved elsewhere.

The proofs of Theorems 1, 2 and 3 rely on complex methods, in particular quasiconformal rigidity for maps within the space P^b: two topologically conjugate maps in P^b are quasiconformally-conjugate. This result was proved by Kozlovski, Shen and van Strien in [KSvS07a]. As was shown in [KSvS07b] it implies density of hyperbolicity within one-dimensional systems.

Second Ingredient: the space of stunted sawtooth maps as a model for the parameter space

One can assign to each map $f \in P^b$ a so-called kneading map in the following way:

- Given a piecewise monotone d-modal map f with turning points $c_1, \ldots, c_b$, associate to $x \in [-1, 1]$ its **itinerary** $i_f(x)$ consisting of a sequence of symbols $i_{f,n}(x)$, $n \geq 0$ from the alphabet $\mathcal{A} = \{I_0, c_1, I_1, c_2, \ldots, c_b, I_b\}$ (where $i_{f,n}(x)$ is the symbol s in $\mathcal{A}$ iff $f^n(x)$ belongs to the corresponding interval or singleton).
- As is well-known, $x \mapsto i_f(x)$ is **monotone** with respect to a variant of the lexicographic ordering (the signed lexicographical ordering).
- So the following is well-defined:

$$\nu_i := \lim_{x \downarrow c_i} i_f(x)$$

- The **kneading invariant** $\nu(f)$ of f is defined as

$$\nu(f) := (\nu_1, \ldots, \nu_b).$$

  Any kneading sequence which is realized by some piecewise monotone d-modal map is called *admissible*.

Since the space of kneadings with the natural topology is not connected, following Milnor and Tresser, we find it easier to work in another space, namely the space of stunted sawtooth maps. These are stunted versions of some fixed sawtooth map S with slope $\pm\lambda$, where $\lambda > 1$, as drawn in Figure 3. Stunted sawtooth maps T are modifications of S with each peak 'stunted' (i.e., replaced by a plateau). In other words, T has slopes $\pm\lambda, 0$. The space of **stunted sawtooth maps** is denoted by $\mathcal{S}^b$ and can be parametrized by the parameters ζ_i as in Figure 3.

To each map $f \in P^b$ we will assign a *unique* stunted sawtooth map $\Psi(f) \in \mathcal{S}^b$. Let $\nu(f) = (\nu_1, \ldots, \nu_b)$ be the kneading invariant of f, and let s_i be the *unique point* in the $(i+1)$-th lap I_i of S such that

$$\lim_{y \downarrow s_i} i_S(y) = \nu_i := \lim_{x \downarrow c_i} i_f(x).$$

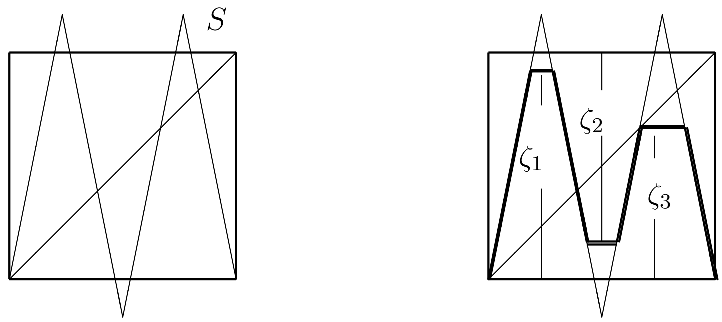

Figure 3: The sawtooth map S on the left and a stunted sawtooth map $T \in \mathcal{S}^3$ on the right (drawn in bold) with 3 plateaus. Adjacent plateaus of maps in $\mathcal{S}^b$ are allowed to touch.

Such a point s_i exists because all kneading sequences are realized by S. It is **unique** since S is expanding and so distinct points have different different kneading sequences. Next we associate to $f \in P^b$ the stunted sawtooth map $\Psi(f) \in \mathcal{S}^b$ which

- is constant on a plateau Z_i with right endpoint s_i and
- agrees with S outside $\cup Z_i$.

Although all critical points of any map $f \in P^b$ are distinct, several plateaus of $\Psi(f) \in \mathcal{S}^b$ can touch (and so the number of genuine plateaus in $\Psi(f) \in \mathcal{S}^b$ can be less than b). We should emphasise that the map

$$P^b \ni f \mapsto \Psi(f) \in \mathcal{S}^b$$

is non-continuous, non-surjective and also non-injective. Nevertheless, as we will see, weaker versions of the properties hold. Moreover, the space $\mathcal{S}^b$ has the following useful property: let ζ_i describe the **height** of the i-th plateau of T as in Figure 3 then $T \mapsto h_{top}(T)$ is monotone increasing in each parameter ζ_i. Using this, one can show **isentropes within $\mathcal{S}^b$ are connected** (and even contractible). We should emphasise that the approach we mentioned in this subsection goes back to [DGMT95, MT00]. As our proof exploits the map $\Psi\colon P^b \to \mathcal{S}^b$, our next ingredient is to consider a subspace of $\mathcal{S}^b$.

Third ingredient: addressing non-surjectivity of Ψ by introducing the space $\mathcal{S}^b_*$ of non-degenerate sawtooth maps

Polynomial maps have **no wandering intervals**. Hence if the endpoints of an interval containing two distinct critical points have the same itineraries, then the interval is contained in the basin of a periodic attractor. Analogously, $\mathcal{S}^b_* \subset \mathcal{S}^b$ consists of maps T so that if

- an interval J contains two plateaus **and**
- $n > 0$ is so that $T^n(J)$ is a point,
- **then** J is contained in the basin of a periodic attractor of T.

(This corresponds to absence of a Levy-cycle obstruction.) This space $\mathcal{S}^b_*$ will be crucial in our discussion. Note that $\mathcal{S}^b_* = \mathcal{S}^b$ when $b = 1, 2$. The space $\mathcal{S}^b_*$ is messier than the original space $\mathcal{S}^b$, but still has the (rather non-trivial property) property that:

Theorem 4. *Isentropes within $\mathcal{S}^b_*$ are connected and even contractible.*

The proof of the analogous statement for the space $\mathcal{S}^b$ is much simpler and was already given in Milnor and Tresser [MT00]. Even though the map $P^b \ni f \mapsto \Psi(f) \in \mathcal{S}^b_*$ still is not surjective, it turns out that there is an interpretation (making this map set-valued) in which it does become surjective. Indeed, define the *plateau-basin* $\mathcal{B}(T)$:

$$\mathcal{B}(T) = \{y; T^k(y) \in \text{interior}(\cup_{i=1}^b Z_{i,T}) \text{ for some } k \geq 0\}.$$

In order to ignore what happens within the basins of periodic attractors, define the equivalence class

$$\langle T \rangle = \{\tilde{T} \in \mathcal{S}^b; \mathcal{B}(\tilde{T}) = \mathcal{B}(T)\}$$

and also define

$$[T] = \text{closure}(\langle T \rangle).$$

Using this, we get surjectivity of Ψ:

Theorem 5 (‘Surjectivity’ of Ψ)**.** *For each $T \in \mathcal{S}^b_*$ there exists $f \in P^b$ so that $T \in [\Psi(f)]$.*

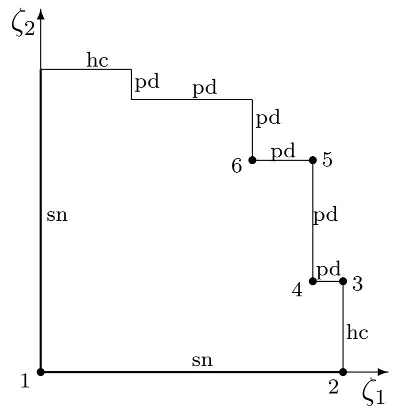

Figure 4: The case of a periodic component W of $W(T)$ of period $s_1 + s_2$ so that W and the component W' of $\mathcal{B}(T)$ containing $T^{s_1}(W)$ both contain a plateau.

Fourth ingredient: Ψ is almost injective and almost continuous

To prove Milnor's conjecture we need to show that isentropes are connected within the space P^b. Since, by Theorem 4, the corresponding statement is true within the space $\mathcal{S}_*^b$ of non-generate sawtooth maps we want to show that the spaces P^b and $\mathcal{S}_*^b$ are essentially homeomorphic. As we have shown in Theorem 5, $f \mapsto [\Psi(f)]$ is surjective. The next two propositions show that this map is essentially homeomorphic.

Theorem 6 (Injectivity of Ψ). *If $f_1, f_2 \in P^b$ and $[\Psi(f_1)] \cap [\Psi(f_2)] \neq \emptyset$ then $\overline{\mathcal{PH}(f_1)} \cap \overline{\mathcal{PH}(f_2)} \neq \emptyset$.*

Theorem 7 (Continuity of Ψ). *Suppose $f_n \in P^b$ converges to $f \in P^b$. Then any limit of $\Psi(f_n)$ is contained in $[\Psi(f)]$.*

The proof of Theorems 6 and 7 relies strongly on the fact that on the boundary of a set $\mathcal{PH}^o(f)$ one has generic bifurcations, see Theorems 2 and 3. Basically, this involves a description of the boundary of the set $[\Psi(f)]$ and show what bifurcations occur at this boundary, see Figure 4.

From the previous four theorems it easily follows that isentropes in P^b are connected, thus proving Milnor's conjecture.

2 Open problems

In this section we will pose some further conjectures and questions on monotonicity of entropy. For other questions and a broader survey on one-dimensional dynamics, see [vS10].

Two questions: Are isentropes contractible? Are hyperbolic maps dense within 'most' isentropes?

The following questions are due to Milnor [MT00, page 125]:

Question (Milnor)**.** *Fix $\epsilon \in \{-1, 1\}$. Are isentropes in P^b_ϵ contractible and cellular?*

Note that the isentropes in the space of admissible stunted sawtooth maps $\mathcal{S}^b_*$ are contractible (as mentioned, this is by no means a trivial result). Obviously, even though the map $\Psi\colon P^b_\epsilon \to \mathcal{S}^b_*$ is 'almost' a homeomorphism, it is not easy to use the subtle deformation in $\mathcal{S}^b_*$ to construct one in the space P^b_ϵ. We believe that this conjecture is true, but this is work in progress.

A somewhat different conjecture was posed (in an email) by Thurston:

Question (Thurston)**.** *Does there exist a dense set of level sets $H \subset [0, \log(d)]$ (where $d = b + 1$) so that for any $h_0 \in H$, the isentrope $I(h_0)$ in P^b_ϵ contains a dense set of hyperbolic maps?*

As usual, we say that a map is **hyperbolic** if each critical point is in the basin of a periodic attractor. As mentioned, it is known that hyperbolic maps are dense within P^b_ϵ, see [KSvS07a]. Solving Thurston's question most probably requires an ability to perturb a map to a hyperbolic map while staying inside an isentrope.

Question: Are isentropes always non-locally connected

Even though all isentropes are connected, we have:

Theorem 8 ([BvS11])**.** *When $b \geq 4$, not all isentropes within P^b are non-locally connected.*

A related theorem is due to [FT86] for maps of the circle. The above theorem only works when $b \geq 4$, and it is possible that all isentropes are locally connected within P^b when $b = 3$. The previous theorem is in some sense the analogue of Milnor's theorem stating that the connectedness locus for cubic maps (in the complex plane) is not locally connected.

Conjecture. *When $b \geq 4$, no isentrope within P^b is locally connected. The boundary of the isentropy corresponding to zero-entropy is non-locally connected.*

Conjecture: Isentropes within the space of real polynomials are connected

In another direction, Tresser posed the following conjecture. Consider the space Pol_ϵ^d of real polynomials f of degree d, not necessarily with all critical points on the real line, but still with $f(\{\pm 1\}) \subset \{\pm 1\}$ and $\epsilon(f) = \epsilon$ as in the definition of P^b.

Conjecture (Tresser)**.** *Fix $\epsilon \in \{-1, 1\}$. Isentropes in Pol_ϵ^d are connected.*

To explain this conjecture, let us take $d = 4$ and consider the subspace $Pol_\epsilon^{4,1}$ of real degree 4 polynomials $f \in Pol_\epsilon^4$, with one critical point c_1 on the real and the other two critical points c_2, c_3 (with $c_2 = \bar{c}_3$) off the real line. Maps in $Pol_\epsilon^{4,1}$ are unimodal. So consider the question whether isentropes within $Pol_\epsilon^{4,1}$ are connected. To be specific, consider the set $\Sigma_{\log(2)}$ of maps $f \in Pol_\epsilon^{4,1}$ with $f(c_1) = 1$. Such maps are, restricted to the real line, conjugate to $x \mapsto 4x(1-x)$ and have topology entropy $\log(2)$. The situation seems good, because we can prove that any two maps $f, \tilde{f} \in \Sigma_{\log(2)}$ are quasi-symmetrically conjugate on the real line. However this does not imply that one can connect $f, \tilde{f}$ by an arc within $\Sigma_{\log(2)}$. Indeed, the dynamics of $f, \tilde{f}$ on the complex plane are entirely unrelated, and so f and $\tilde{f}$ are in general certainly not quasiconformally conjugate. It seems therefore hopeless to use quasiconformal surgery to prove that $f, \tilde{f}$ can be connected by a path in $\Sigma_{\log(2)}$. On the other hand, the set $\Sigma_{\log(2)}$ forms a codimension-one algebraic subset of the (real) three dimensional parameter space. By a somewhat tedious explicit calculation this algebraic set can be shown to be connected.

In spite of these difficulties, we recently proved in joint work with Cheraghi the following:

Theorem 9 ([CvS])**.** *Isentropes within the space $Pol_\epsilon^{4,1}$ are connected.*

The main ingredient in our proof is the property that critically finite polynomials with a given combinatorial type are unique.

Conjecture: Isentropes within more general unimodal families are connected

Of course, one can ask what happens if one considers wider classes of functions. In recent work with Lasse Rempe, we have recently been able to prove results of the following type:

Theorem 10. *[RvS10] The topological entropy of the map* $f_a \colon [0,1] \to [0,1]$ *defined by* $f_a(x) = a \cdot \sin(\pi x)$ *depends monotonically on* a*.*

In the proof of this theorem it is heavily used that $x \mapsto \sin(x)$ is a transcendental map with some additional geometric properties, for a proof and a much more general theorem see [RvS10] where the following theorem is proven

Theorem 11. *[RvS10] Each isentrope within the space of trigonometric polynomials is connected.*

However, it is far from clear how to obtain results without relying on complex tools. For example, the following well-known conjecture has been open for the last 30 years:

Conjecture. *Take* $\ell > 1$ *not an integer. Then the topological entropy of the map* $x \mapsto -(c+1)|x|^\ell + c$ *depends monotonically on* c*.*

When ℓ is an integer, the corresponding statement can be proved as in the quadratic case. More generally, the following conjecture was posed by Nusse and Yorke:

Conjecture. *Let* f *be* S*-unimodal of the unit interval and symmetric, i.e.,* $f(1-x) = f(x)$*. Does the topological entropy of the map* $f_a = a \cdot f$ *depend monotonically on* a*?*

Note that if one drops the assumption that f is symmetric then this the conjecture definitely does not hold, as was shown by Zdunik, Nusse and Yorke, Kolyada and others (for references see [dMvS93]). In fact, to prove this conjecture it is enough to show that, under the above assumptions, periodic orbits of $[0,1] \ni x \mapsto a \cdot f(x) \in [0,1]$ can never be destroyed as a increases. It should be noted that some partial results towards this can be obtained by applying the notion of rotation number, see [GT92], [Blo94] and [BM97]; under mild conditions periodic orbits with particular types of combinatorics (and uncountably many types of aperiodic behaviour) do not disappear as a increases. The previous conjecture is subtle: there are C^3 close maps $f, g\colon [0,1] \to [0,1]$ of this type for which $f \leq g$ and $h_{top}(f) > h_{top}(g)$, see [Bru95].

2.1 Question: is antimonotonicity common?

We should note that even though isentropes within the space of cubic polynomials are connected, isentropes are complicated non-locally connected topological sets, see [BvS11]. Related to this, one has the following:

Theorem 12 ([BvS09])**.** *It is well-known that cubic polynomials can be parametrized by their critical values. However, the entropy of a cubic polynomial does not depend monotonically on these parameters separately.*

Dawson, Grebogi, Kan, Koçak and Yorke proposed that this phenomenon holds more generally, by stating the following general conjecture:

Antimonotonicity Conjecture ([DG91, DGY$^+$92, DGK93])**.** *A smooth one-dimensional map depending on one parameter has antimonotone parameter values whenever two critical points have disjoint orbits and are contained in the interior of a chaotic attractor.*

A further discussion about the relation between connectedness of isentropes and the above antimonotony conjecture can be found in [MT00].

Question: Are isentropes within families of higher dimensional maps connected?

As we have seen, even though isentropes for cubic maps are connected, one has antimonotonicity. Motivated by this, we pose the following question.

Question (Hénon maps)**.** *Let $H(x,y) = (1 - ax^2 + by, y)$ be the family of Hénon maps. It is known, see [KKY92] and [DGY$^+$92] that for fixed b, the set of parameters $\{a; h_{top}(H_{a,b}) = h_0\}$ is not connected. However, is it possible that isentropes $I(c) = \{(a,b); h_{top}(H_{a,b}) = h_0\}$ are connected? For all h_0? For some h_0? For $h_0 = 0$.*

For preliminary results in this direction and further references, see [GT91]. A positive answer for the case when $h_0 = 0$ would mean that the boundary of chaos (as defined by positivity of topological entropy) is connected. If so, a decent picture emerges of how one can move from simple (i.e., zero entropy dynamics) to chaotic dynamics.

Of course, the difficulty in resolving the last question is that one can no longer rely on holomorphic dynamics.

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