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Local connectivity of the Julia set of real polynomials

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Abstract: In this paper we shall show that the Julia set of real polynomials of the form $f(z) = z^\ell + c_1$ with ℓ an even integer and c_1 real is either totally disconnected or locally connected. This answers a question of J. Milnor

1. Introduction and statements of theorems

One of the main questions in the field of complex dynamics is the question whether the Mandelbrot set is locally connected, and related to this, for which maps the Julia set is locally connected. In this paper we shall prove the following

Main Theorem. *Let f be a polynomial of the form $f(z) = z^\ell + c_1$ with ℓ an even integer and c_1 real. Then the Julia set of f is either totally disconnected or locally connected.*

In particular, the Julia set of $f(z) = z^\ell + c_1$ is locally connected for $c_1 \in \mathbb{R}$ if and only if the critical point $c = 0$ does not escape to infinity. Hence the main theorem answers a question posed by Milnor, see [Mil1]. Extending ideas and results from [McM], our methods imply that maps as in the main theorem do not have a measurable invariant linefield on their Julia set, see [LS1].

Let us make some further comments. Yoccoz [Y] already had shown that each quadratic polynomial which is only finitely often renormalizable (with

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non-escaping critical point and no neutral periodic point) has a locally connected Julia set. Moreover, Douady and Hubbard [DH1] already had shown before that each polynomial of the form $z \mapsto z^\ell + c_1$ with an attracting or neutral parabolic cycle has a locally connected Julia set. As will become clear, the difficult case is the infinitely renormalizable case. In fact, using the reduction method developed in Section 3 of this paper, it turns out that in the non-renormalizable case the Main Theorem follows from some constructions in [L3] and [L5], and similar to ones in [JS], [GS1]. A complete proof of this case is given in Section 14.

We should emphasize that if the ω-limit set $\omega(c)$ of the critical point $c = 0$ is not minimal and f is real then it not difficult to see that the Julia set is locally connected, see Section 3. If f is non-real this is not true in general: there are infinitely renormalizable non-real quadratic maps with a non-locally connected Julia set given by A. Douady, see [Mil]. (In these examples $\omega(c)$ is not minimal.) Also maps with Cremer points have a non-locally connected Julia set, see e.g. [Mil2].

Hence, the results above really depend on the use of real methods. It is worth mentioning that Petersen's result, which states that a quadratic polynomial with a Siegel disc such that the eigenvalue at the neutral fixed point is of 'constant type' has a locally connected Julia set [Pe], also uses real results (he uses the circle model).

The methods of Yoccoz break down in the infinitely renormalizable case and in the case of polynomials with a degenerate critical point. The purpose of Yoccoz's methods is to solve the well-known conjecture about the local connectedness of the Mandelbrot set and therefore, some version of our ideas might be helpful in proving this conjecture. For a survey of the results of Yoccoz, see for example [Mil] and also [L4], and in the parameter space [H].

We should note also that Hu and Jiang, see [HJ] and [Ji1] have shown that for infinitely renormalizable quadratic maps which are real and of so-called bounded type, the Julia set is locally connected. Their result is heavily based on the complex bounds for unimodal infinitely renormalizable polynomials of bounded combinatorics, which Sullivan used in his renormalization results, see [Sul] and also the last chapter and in particular Section VI.5 of [MS] (cf. also [Ji2] and see [Ly5] for special quadratic polynomials of unbounded combinatorics).

Our methods enable us to extend Sullivan's result to the class of all infinitely renormalizable unimodal polynomials *independently of the combinatorial type*. We should emphasize that these complex bounds form the most essential ingredient for the renormalization results of Sullivan [Sul], see also [MS]. In fact, in McMullen's approach to renormalization, see [McM], [McM1], these complex bounds play an even more central role. In the previous proofs of the complex bounds see [Sul], and also Section VI.5 of [MS], it is crucial that the renormalization is of bounded type and, moreover, the proof is quite intricate. Therefore we are very happy that our methods give a fairly easy way to get complex bounds independently of the combinatorial type of the map (i.e., only depend on the degree of the map):

THEOREM A (COMPLEX BOUNDS IN THE INFINITELY RENORMALIZABLE CASE). *Let f be a real unimodal infinitely renormalizable polynomial and $f^s\colon V \to V$ be an arbitrary renormalization of this map to some periodic interval $V \subset \mathbb{R}$ containing c. Then this map has a polynomial-like extension $F := f^s\colon \Omega^0 \to \Omega$ with*

- *the modulus of $\Omega \setminus \Omega^0$ is bounded from below by δ;*
- *the diameter of Ω is at most C times the diameter of V;*

Here $\delta > 0$ only depends on ℓ (and not on V) and C is completely universal. In fact, δ is asymptotically like const$/\ell$ as ℓ tends to ∞. Moreover, if f is not $s/2$-renormalizable then $\Omega^0 \cap \omega(c) \subset V$.

The notion 'polynomial-like' is defined in Section 2.

The shape of Ω is universal, up to rescaling, and in many cases we can take Ω to be a Euclidean disc; our bounds are completely independent of the combinatorial type of the map. We should emphasize that Theorem A and its proof hold for any renormalization f^s of (a maybe only finitely renormalizable map) f provided f^{2s} does not have an attracting or neutral fixed point. Also the result even holds when f is real analytic, see Section 11 (answering a question of W. de Melo).

In fact, we have the following generalization of Theorem A for any (renormalizable or not) real unimodal polynomial with $\ell \geq 4$. Namely, for each level for which one has a high return one has a small polynomial-like mapping.

THEOREM B (COMPLEX BOUNDS AT ANY HIGH RETURN). *Let $f(z) = z^\ell + c_1$ with $\ell \geq 4$ an even integer and c_1 real so that $\omega(c)$ is minimal. Consider an arbitrary symmetric interval $V = (-v, v)$ with nice boundary points: $f^n(\partial V) \cap V = \emptyset$ for all $n \geq 0$. Let $F\colon \cup V^i \to V$ be the corresponding first return map (on the real line) and assume that this map has a high return, i.e., assume that $F(V^0) \ni c$ where V^0 is the central interval. Then there exist topological discs Ω^i and Ω with $\Omega^i \cap \mathbb{R} \supseteq V^i$ and $\Omega \cap \mathbb{R} \supseteq V$ and a complex polynomial-like extension $G\colon \cup_i \Omega^i \to \Omega$ of F such that the diameter of the disc Ω is at most a universally bounded constant times the diameter of V.*

Note that this statement includes the case when V is a renormalization interval. In the non-renormalizable case, we actually have $\Omega^i \cap \mathbb{R} = V^i$ and $\Omega \cap \mathbb{R} = V$.

To prove the Main Theorem in the non-renormalizable case, we use the following result which extends ideas of [L3] and [L5] and [JS], [GS1], see also [GS3].

Theorem C (Complex Bounds at suitable return times). *Let $f(z) = z^\ell + c_1$ with ℓ an even integer and c_1 real be a non-renormalizable polynomial so that $\omega(c)$ is minimal. Then there exists a sequence of topological discs $\Omega_n \ni c$ such that $\Omega_n \cap \mathbb{R}$ is equal to the real trace of a Yoccoz puzzle piece, and such that $diam(\Omega_n) \to 0$ so that the first return map to Ω_n has a finite number of domains Ω_n^i which intersect $\omega(c)$ (all of which are topological discs) and so that the modulus of each of the annuli $\Omega_n \setminus \Omega_n^i$ is bounded away from zero by a constant which only depends on ℓ.*

For the definition of the Yoccoz pieces see e.g. [H], [Mil], and the introduction below.

We also have results for the non-minimal case, see Proposition 3.3 and its corollary.

Let us say a few words about our proofs.

Let us first sketch why the local connectivity of the Julia set follows from the complex bounds of Theorems A-C. As has become standard since the work of Yoccoz, to study the local connectivity of the Julia set one works with the Yoccoz partition: if f is non-renormalizable, consider an equipotential and the external ray through the orientation reversing fixed point of f. Next consider the partition generated by taking preimages of this first partition. If f is renormalizable, start with a region bounded by some equipotential of f and by external rays (of f) through the endpoints of a central renormalization interval; the preimages of such regions are again called a Yoccoz partition. Since the intersection of these pieces with the Julia set is connected, it suffices to show that the diameter of the pieces tends to zero. However, as we will show, the small Julia sets of the polynomial-like mappings from Theorem A-C coincide with the small Julia sets associated to certain first return maps from the Yoccoz puzzle. Since the diameter of the domains from Theorem A-C get arbitrary small, we are able to conclude local connectivity of the Julia set at the critical point. To conclude the local connectivity of the Julia set at any point, one needs that the polynomial-like mappings from Theorem A-C satisfy a certain first return condition. All this is explained in Sections 2 and 3.

Let us next explain how we prove Theorem A-B. To obtain a polynomial-like mapping, we need to show that all the preimages of the target set Ω are mapped inside itself. For this, we use a well known idea of Sullivan [Sul], [MS]. The sets Ω in Theorems A-B are inside sets of the form

$$\mathbb{C}_T = \mathbb{C} \setminus (\mathbb{R} \setminus T)$$

where $T \subset \mathbb{R}$ is some interval. This set (consisting of $\mathbb{C}$ with two infinite slits) carries a Poincaré metric. The set of points with a constant Poincaré distance to T consists of the union of two discs which are symmetric w.r.t. the real axis, intersecting the real line exactly in T and which have external angle with the real line equal to some number $\alpha \in (0, \pi)$. By the lemma of Schwarz the preimage of such a region is mapped inside a similar region, see lemma 4.1. To explain this, let us give an example. Assume that f^s maps a symmetric interval V as a unimodal map inside itself with $f(\partial V) \subset \partial V$. Let Ω be a disc with diameter V. By Schwarz, the pullback by the appropriate branch of $f^{-(s-1)}(\Omega)$ is inside a disc with diameter $f^{-(s-1)}(V) \ni c_1$. So it depends on the relative position of c_1 in this interval $f^{-(s-1)}(V)$ whether or not $f^{-s}(\Omega)$ is inside the disc with diameter V, see Section 5. Thus we have reduced the proof of Theorems A-B to checking that certain real bounds are satisfied.

To prove the real bounds, we apply a 'smallest interval' argument. They are a sharper version of those used before by Blokh, Lyubich, Martens, de Melo, Sullivan, van Strien, Świątek and others. This is done in Section 6, 7 and 8. We should emphasize that the real bounds in these sections hold for all unimodal maps with negative Schwarzian derivative.

In many cases, these bounds are immediately sufficient and we are able to work with Euclidean discs, see Section 9. However, in some other cases the real bounds we prove here, force us to study more general Poincaré domains with small external angles. A general study about preimages of such domains is given in Appendix (Section 15). Using this, we are able to show that rather weak real bounds are even sufficient to obtain the required inclusion. The reader will observe that certain cases are proved by several methods. For example, in Section 9 bounds are given in the infinitely renormalizable case when $\ell \geq 4$ and when $\ell = 2$ in the period doubling case. This case also follows from the estimates in later sections. However, the domains in Section 9 are discs and those in Section 10 and 12 are more complicated. We believe that for future purposes it might be important to have good domains, and therefore even if it was sometimes not necessary for the proofs of our theorems, we have tried to treat each case in a fairly optimal way.

The proof of Theorem C is based on an inducing method developed by Jacobson, Graczyk, Świątek and Lyubich. It shows that, in the non-renormalizable case, one can find a sequence of polynomial-like mappings associated to certain suitable intervals V_n.

Finally, a short history of this paper since several authors have partial proofs of Theorem A and the Main Theorem in the quadratic case. Firstly, we were inspired by ideas of Hu and Jiang, see [HJ] and [Ji1] where it is shown that infinitely renormalizable maps of bounded type (where Sullivan's bounds hold) have a locally connected Julia set. The first widely distributed version of our paper (dated December 31, 1994) included the proof of the Main Theorem in the quadratic case, in the infinitely renormalizable case, Theorem A (without doubling) and also some non-renormalizable cases. Subsequently, Theorem B was included in the version of this paper of January 27, 1995.

Since then two other proofs of Theorem A in the quadratic case appeared, see [GS2], [LY]. The 'quadratic' proofs of Graczyk, Świątek, Lyubich and Yampolsky of Theorem A sometimes allow one to obtain annuli with large moduli, but those proofs seem to heavily rely on the map being quadratic. (In view of the estimates in [SN] such large moduli cannot be expected to exist in the higher order case.) One advantage of our proof is that it shows that the small Julia set associated to some renormalization of a map, is contained in a definite sector which is actually based on the renormalization interval. (In many cases, the small Julia set is contained in the disc based on the renormalization interval.) After we told Lyubich about our methods to obtain local connectivity, he suggested us to look at [L3], [L5], which became the basis for Theorem C.

2. When do Julia sets of two polynomial-like mappings coincide?

We shall use the fundamental notion of polynomial-like mapping [DH] or more precisely, we need its extension due to Lyubich and Milnor from [LM]. Let $D^0, D^1, \dots, D^i$, and D be topological discs bounded by piecewise smooth curves and such that the closures $\overline{D^0}, \dots, \overline{D^i}$ are contained in the interior of D, and such that each two discs $D^0, \dots, D^i$ are pairwise disjoint. Then we call

$$R\colon D^0 \bigcup D^1 \bigcup \dots \bigcup D^i \to D$$

ℓ-polynomial-like if $R|_{D^j}$ is a univalent map onto D for each $j = 1, \dots, i$ and $R|_{D^0}$ is a ℓ-fold covering of D^0 onto D. If $i = 0$ in this definition, we obtain a polynomial-like map in the original sense of Douady-Hubbard.

The *filled Julia set* of R is said to be the set $F_R \subset \cup_{j=0}^{i} D^j$ of the points z such that $R^k(z)$ is defined for all $k = 1, 2, \dots$. The *Julia set* J_R is defined to be ∂F_R. An equivalent definition of the filled Julia set F_R is:

$$F_R = \bigcap_{k=1}^{\infty} R^{-k}(D).$$

We shall use an extension of the Straightening Theorem due to Douady and Hubbard, [DH]. This extension was also used in Lemma 7.1 of [LM], for the case that $i = 1$.

LEMMA 2.1. *Let $R: D^0 \cup \ldots \cup D^i \to D$ be a ℓ-polynomial-like map. Then R is quasi-conformally conjugate to a polynomial; the conjugacy is defined between neighborhoods of the filled Julia set F_R and the filled Julia set of the polynomial.*

Proof: Let us first pick a point $x_0 \in D \setminus (D^0 \cup \ldots D^i)$ and choose closed simple curves $\gamma_0, \ldots, \gamma_i: [0, 2\pi] \to \mathbb{C}$ such that $\gamma_j(0) = \gamma_j(2\pi) = x_0$, the curves γ_j only meet at x_0 and γ_j surrounds D^j. Moreover, we choose the function γ_j to be smooth and so that $\frac{d}{dt}\gamma_j(0)$ and $\frac{d}{dt}\gamma_j(2\pi)$ are two vectors based at x_0 having an angle $\pi/(i+1)$.

If, for example, $i = 1$ then $\gamma_0 \cup \gamma_1$ is a figure eight. Next pick a smooth curve γ in $\mathbb{C} \setminus D$ and a point $x_1 \in \gamma$. Moreover, choose a smooth function ϕ defined on a neighbourhood N of x_0 such that $\phi(x_0) = x_1$ and so that ϕ maps $\gamma_j \cap N$ diffeomorphically to $\gamma \cap \phi(N)$ for each $j = 0, 1, \ldots, i$. In local coordinates this map will have an expression of the form $z \mapsto z^{i+1}$ plus higher order terms, i.e., this map ϕ will have a critical point of order $i + 1$. Now let A_j be the open annulus between γ_j and D^j and let A be the open annulus between γ and D. Moreover, we choose a smooth map $\tilde{R}: A_j \to A$ which extends to the closure of these sets so that it agrees with R on ∂D^j and with ϕ on the neighbourhood N of x_0. Choose this extension so that $\phi: A_0 \to A$ is a ℓ-covering and $\phi: A_j \to A$ is a diffeomorphism for $j = 1, \ldots, i$. This map $\tilde{R}$ becomes an extension of R if we define it equal to R on $D^0 \cup \ldots \cup D^i$. Next choose $r > 1$ so that the circle centered at the origin with radius $r > 1$ surrounds $A_0 \cup \ldots \cup A_i$. We can extend $\tilde{R}$ smoothly to a map $\hat{R}: \mathbb{C} \to \mathbb{C}$ so that $\hat{R}(z) = z^{\ell+i}$ for $|z| \geq r$ and so that $\hat{R}$ coincides with $\tilde{R}$ on $A_0 \cup \ldots \cup A_i$. The map $\tilde{R}$ on the annulus $\{z\,;\ |z| < r\} \setminus (A_0 \cup \ldots \cup A_i)$ is a $\ell + i$-covering map to the annulus $\{z\,;\ |z| < r^{\ell+i}\} \setminus A$.

Now we use the standard trick from the Straightening Theorem. Take a standard conformal structure (i.e., the Beltrami coefficient $\mu = 0$) in a neighborhood of ∞ and extend this structure to a function $\mu: \mathbb{C} \to \{z\,;\ |z| < 1\}$ which is invariant under $\hat{R}$. Since $\hat{R}$ is conformal near infinity and on $D^0 \cup \ldots \cup D^i$, there are only a bounded number of points in each orbit of $\hat{R}$ where this map is not conformal. It follows that the L^∞-norm of $\mu(z)$ is bounded away from one, and by the Measurable Riemann Mapping Theorem, there exists a quasiconformal homeomorphism $h: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with $h(\infty) = \infty$ which has μ as its Beltrami coefficient. Since μ is invariant under $\hat{R}$, it follows that

$$h \circ \hat{R} \circ h^{-1}$$

is a holomorphic $(\ell+i)$-covering. Hence $\hat{R}$ is quasiconformally conjugate to a polynomial map P (of degree $(\ell+i)$). $\square$

A corollary is:

COROLLARY 1. *The Julia set J_R is the limit set for the preimages of any point $z \in D$ (except in one case: $i=0$ and c is the ℓ-multiple fixed point of R; in that case the assertion does not hold for c).*

We can use all this to show that the Julia set of two polynomial-like mappings coincide. In the applications of this we shall later on use for one of these the polynomial-like mapping of the Yoccoz puzzles.

PROPOSITION 2.2. *(cf. [Ji1], [McM].) Let*

$$R_1 \colon D_1^0 \bigcup D_1^1 \bigcup \ldots \bigcup D_1^i \to D_1,$$

$$R_2 \colon D_2^0 \bigcup D_2^1 \bigcup \ldots \bigcup D_2^i \to D_2$$

be two ℓ-polynomial-like mappings, such that the critical point c of these maps coincide. That is, c is contained in $D_1^0 \cap D_2^0$ and is the unique and ℓ-multiple critical point for both R_1 and for R_2. Moreover, assume that the following conditions hold:

(1) $R_1(z) = R_2(z)$ whenever $z \in \cup_{j=0}^i D_1^j \cap D_2^j$, so that R_1 and R_2 are extensions of the same map R on this set.

(2) Let C be the component of $D_1 \cap D_2$ which contains $R(c)$. Then also $c \in C$, and there exist precisely i other points $c^1, \ldots, c^i$ so that $c^j \in D_1^j \cap D_2^j$ and $R(c^j) = R(c)$, and, furthermore, $c^1, \ldots, c^i \in C$.

Under these conditions, the Julia sets of R_1 and R_2 coincide:

$$J_{R_1} = J_{R_2}.$$

If, in addition, $c \in J_{R_1}$, (and, hence, $c \in J_{R_2}$), then there exists a component of a preimage $R_2^{-n}(D_2)$, which contains c and is contained in D_1.

Proof: For $k=1,2$, let $C_k^0, \ldots, C_k^i$ be the components of $R_k^{-1}(C)$, such that $c^j \in C_k^j$ when $j \neq 0$, and $c \in C_k^0$. Firstly, $R_k \colon C_k^j \to C$ is a covering, which is just one-to-one if $j \neq 0$, and $R_k \colon C_k^0 \to C$ is a ℓ-branched cover. In particular, boundaries are mapped to boundaries. Since $R_1 = R_2$ on the common domain of definition, we get that, in fact, $C_1^j = C_2^j := C^j$, for every j. Secondly, because of 2), each component C^j has a point c^j in common with the component C. Since C^j is connected and is contained in both D_1 and D_2, it belongs to a component of $D_1 \cap D_2$ containing c^j, i.e., $C^j \subset C$. Now consider the map $R \colon C^0 \cup C^1 \cup \ldots \cup C^i \to C$, which is one-to-one on every $C^j, j \neq 0$, and ℓ-to-one on C^0. Take a point $x \in C$. Then $R^{-1}(x)$ is a subset of C and it consists of $l+i$ points (counting with multiplicities). That is,

(2.1) for any $x \in C$, the sets $R_1^{-1}(x)$ and $R_2^{-1}(x)$ coincide and belong to C.

Starting with $x_0 \in C$, we apply the corollary to Lemma 2.1 and (2.1) to get $J_{R_1} = J_{R_2} := J$. If $c \in J$, then consider a component K of J containing c. Since $K \subset D_1$, there exists a component of a preimage $R_2^{-n}(D_2)$, which contains K and is contained in D_1. □

In the sequel we will use a particular case of Proposition 2.2. Let us state it separately:

PROPOSITION 2.3. *Let*

$$R_1: D_1^0 \bigcup D_1^1 \bigcup \ldots \bigcup D_1^i \to D_1,$$

$$R_2: D_2^0 \bigcup D_2^1 \bigcup \ldots \bigcup D_2^i \to D_2$$

be two ℓ-polynomial-like mappings, such that the critical points of R_k coincide, this point $c \in D_1^0 \cap D_2^0$ and is a ℓ-multiple critical point of both R_1 and R_2. Moreover, we assume that the following conditions hold:

(1) $R_1(z) = R_2(z)$ whenever $z \in \cup_{j=0}^{i} D_1^j \cap D_2^j$, so that R_1 and R_2 are extensions of a map R.

(2) For $k = 1, 2$, all topological discs $D_k, D_k^0, \ldots, D_k^i$ are symmetric w.r.t. the real line $\mathbb{R}$ and satisfy $R_k(\overline{z}) = \overline{R_k(z)}$.

(3) Denoting $I_k = D_k \cap \mathbb{R}$ and $I_k^j = D_k^j \cap \mathbb{R}$, one has $I_2 \subseteq I_1$, $I_2^j \subseteq I_1^j$, and, for $j = 1, \ldots, i$, the (real) map $R_k: I_k^j \to I_k$ is one-to-one.

Under these conditions, the Julia sets of R_1 and R_2 coincide. If, in addition, $c \in \mathbb{R}$ lies in the Julia set of R_1 (and, hence of R_2), then there exists a component of a preimage $R_2^{-n}(D_2)$, which contains c and is contained in D_1.

3. Local connectivity of the Julia set

Let us now prove that the complex bounds from Theorems A-C imply that the Julia set is locally connected. To do this we use the Yoccoz partition. If f is non-renormalizable then consider the Yoccoz partition generated by taking preimages of the external ray through the orientation reversing fixed point of f and some equipotential. If f is renormalizable start with a region bounded by external rays (of f) through the through the endpoints of a central renormalization interval and some equipotential of f; the preimages of this region we again call a Yoccoz partition. Since intersection of these pieces with the Julia set is connected, it suffices to show that the diameter of the pieces tends to zero.

PROPOSITION 3.1. *Let $G(j): \cup_i \Omega^i(j) \to \Omega(j)$ be a sequence ℓ-polynomial-like mappings associated to a real polynomial $f(z) = z^\ell + c_1$ with $\omega(c)$ minimal such that the critical point $c = 0 \in \Omega^0(j)$ does not escape the domain of $G(j)$*

under the iterates of $G(j)$ and $\Omega(j)$ is based on the intersection of Yoccoz puzzle piece with the real line. (As before, we assume $G(j)\colon \Omega^0(j) \to \Omega(j)$ is ℓ-to-one, and that each other map $G(j)\colon \Omega^i(j) \to \Omega(j)$ is a diffeomorphism. If f is infinitely renormalizable, then $G(j)$ are polynomial-like in the sense of [DH], and we assume that the intersection $\Omega^0(j) \cap \mathbb{R}$ contains a renormalization interval V and that $G(j)|_V$ coincides with $f^s\colon V \to V$.) Assume moreover that

- *when $f^i(c) \in \Omega^0(j)$ then $f^i(c)$ is an iterate of c under $G(j)$ (we call this* **the first return condition***);*
- *the modulus of the annuli $\Omega(j) \setminus \Omega^0(j)$ is uniformly bounded away from zero;*
- *the diameter of $\Omega(j)$ tends to zero as $j \to \infty$.*

Then the Julia set of f is locally connected.

REMARK 3.2. *The proposition also holds for a complex map f, if for each j, one can find another ℓ-polynomial-like mapping $R(j)$ having as its range a critical Yoccoz piece $P(j)$, so that the two mappings $G(j)$ and $R(j)$ satisfy the conditions of Proposition 2.2 from the previous section.*

Proof: Let us first observe that we may assume that $\Omega(j)$ also satisfies the first return condition. Indeed, consider the first return map of $G(j)$ to $\Omega^0(j)$. This is again a polynomial-like mapping $\tilde{G}(j)\colon \cup_i \tilde{\Omega}^i(j) \to \Omega^0(j)$. Obviously, if $f^i(c) \in \Omega^0(j)$ then $f^i(c)$ is an iterate of c under $\tilde{G}(j)$. In addition, the modulus of $\Omega^0(j) \setminus \tilde{\Omega}^0(j)$ is equal to at least $\frac{1}{\ell} \operatorname{mod}(\Omega(j) \setminus \Omega^0(j))$. So we may replace $G(j)$ by the first return map and therefore in the remainder of the proof we can and will assume that the first return condition even holds on $\Omega(j)$. (In fact, the new map is the first return map of f to $\Omega^0(j)$ along the postcritical set of f.)

Now we use arguments similar to those in [Ji1]. If z is in the Julia set but $\omega(z)$ does not hit a critical piece then the Julia set is locally connected at z because of the contraction principle. So choose a point z from the Julia set of f so that $\omega(z)$ hits every critical piece. Fix for the moment j and let $P(j)$ be an open piece of the Yoccoz puzzle based on $\Omega(j) \cap \mathbb{R}$. Since the map f is real, there exists another ℓ-polynomial-like mapping $R(j)$ having as its range $P(j)$, which coincides with $G(j)$ on the real axis. (In the case of renormalization $f^s\colon V \to V$, a thickening of the above piece is needed, see e.g. [Mil].) There exists a large integer N such that the full preimage $R(j)^{-N}(P(j))$ is inside the domain of definition $\cup_i \Omega^i(j)$ of $G(j)$, see Proposition 2.3. Note that $R(j)^{-N}(P(j))$ consists of finitely many (open) Yoccoz pieces. Let us consider the pieces of $R(j)^{-N}(P(j))$ inside the central domain $\Omega^0(j)$, i.e.,

$$P'(j) = R(j)^{-N}(P(j)) \cap \Omega^0(j).$$

Since $\omega(z)$ hits every critical piece, there exists a *minimal* $k = k(j)$ such that $f^k(z) \in P'(j)$. In particular, the point $f^k(z)$ belongs to one of the Yoccoz pieces inside $\Omega^0(j)$. Let B_j be the branch of f^{-k} which maps a neighborhood of $f^k(z)$ to a neighbourhood of z. Let us drop j in the notation for now.
Claim 1. The map B extends to a holomorphic map on Ω.
Proof of the claim. Assume the contrary. We then get that for some minimal $r < k$ that $f^{-r}(\Omega)$ (along the same orbit) meets the critical value c_1. This means that the branch f^{-r} follows the points $c_{r+1} = f^r(c_1) \in \Omega$, $c_r = f^{r-1}(c_1)$,..., $c_2 = f(c_1)$, c_1. Among these iterations of c_1, let us mark all those $c_{j_1}, c_{j_2}, \ldots, c_{j_m}$, where $j_1 < j_2 < \ldots < j_m$, which hit the domain Ω. Because of the first return assumption there exists integers $k(1) < k(2) < \ldots$ such that $c_{j_1} = G^{k(1)}(c)$, $c_{j_2} = G^{k(2)-k(1)}(c_{j_1}) = G^{k(2)}(c)$,..., $c_{j_m} = G^{k(m)}(c)$. It follows, that $f^{-r} = f^{-(s-1)} \circ G^{-(k(m)-1)}$, where $f^{-(s-1)}$ is the branch from V to $\hat{U}$ corresponding to the restriction of G on Ω^0 (so $G|\Omega^0 = f^{s-1} \circ f$). Hence, $f^{-(r+1)}(\Omega) \subset G^{-r(m)}(\Omega) \subset \Omega^0$ and $f^{k-r-1}(z) \in f^{-(r+1)}(P'(j)) = (G|_{\Omega^0})^{-1} \circ G^{-k(m)+1}(P'(j)) \subset P'(j)$. This contradicts the minimality of k and proves the claim.

Let $P_j(z) = B_j(P'(j))$. We want to show that the Euclidean diameters of $P_j(z)$ tends to zero as $j \to \infty$. For this, let us consider a domain $M'_j \subset \Omega(j)$ bounded by a core curve of the annulus $\Omega(j) \setminus \Omega^0(j)$. Then $\max_{y \in \partial M'_n} |f^k(z) - y| / \min_{y \in \partial M'_n} |f^k(z) - y| \leq C(m)$, $n = 1, 2, \ldots$ where m is the modulus of the annulus $\Omega(j) \setminus \Omega^0(j)$ (this follows from Koebe see e.g. [McM]). Define $E_j = B_j(M'_j)$. Since the modulus of the annulus $\Omega(j) \setminus M'_j$ is $m/2$, by Koebe's distortion theorem, $\max_{y \in \partial E_j} |z-y| / \min_{y \in \partial E_j} |z-y| \leq C_1(j)$, $j = 1, 2, \ldots$. If we assume by contradiction that $\mathrm{diam} P_j(z) \geq d > 0$ for $j = 1, 2, \ldots$, then $\min_{y \in \partial E_j} |z - y| \geq d/2C_1 = r > 0$, i.e., the disc $D_z(r) \subset E_j$. Hence, $f^k(D_z(r)) \subset M'_j$, for $j \to \infty$ and $k(j) \to \infty$. This is a contradiction with the non-normality of the family f^n at $z \in J(f)$. Thus, $\bigcap_{j>0} P_j(z) = \{z\}$ and so $J(f)$ is again locally connected at z. $\square$

If $f(z) = z^\ell + c_1$ is real and $\omega(c)$ is minimal then by Theorems A-C the conditions of the previous result are satisfied and local connectivity of the Julia set follows in this case.

So let us turn to the case that $\omega(c)$ is not minimal.

PROPOSITION 3.3. *Assume that $f(z) = z^\ell + c_1$ is real and $\omega(c)$ is not minimal. Then there exist a puzzle piece P and a neighbourhood Q of P (so that $Q \setminus P$ is an annulus) and there exist*

- *a sequence of critical puzzle pieces P_{n_k} with neighbourhoods Q_{n_k} and integers m_k such that f^{m_k} maps Q_{n_k} in a ℓ to one way to Q;*
- *if $x \in J$ and n is minimal such that $f^n(x) \in P_{n_k}$ then there exists a univalent branch of f^{-n} on Q_{n_k} to a neighbourhood of x;*
- *$f^i(\partial P) \cap (Q \setminus \overline{P}) = \emptyset$ for all $i \geq 0$.*

Proof: Since $\omega(c)$ is not minimal, it contains a point x whose forward orbit avoids some critical piece (here we use that the traces of the critical pieces on the real line tend to zero in diameter because the map is real and so preimages of the fixed point accumulate onto the critical point). In particular, this forward orbit lies in a hyperbolic set. Therefore the Yoccoz puzzle-pieces $P_n(x)$ containing x shrink down in diameter to zero. The puzzle-pieces $P_n(x)$ are mapped by some iterates of f onto a fixed critical Yoccoz piece P_N. It follows that if $\omega(c)$ is not minimal then there is a fixed critical Yoccoz piece P_N and a sequence of critical pieces P_{n_k} with $n_k \to \infty$ so that each map $f^{n_k-N}: P_{n_k} \to P_N$ is ℓ-covering.

Let a be the fixed point with more than one external argument (i.e., in the real case the orientation reversing fixed point). We will distinguish two subcases depending on whether or not the fixed point a is in $\omega(c)$. Let us first assume that $a \notin \omega(c)$. Then there exists a fixed neighbourhood Q of $P := P_N$ such that $Q \setminus \overline{P}$ is disjoint from the postcritical set $\omega(c)$ (note that the intersection of ∂P_N with J consists of finitely many preimages of a). One can even choose $Q \supset P$ so that $f^i(\partial P) \cap Q = \emptyset$ for all $i > 0$. The extension of the map $f^{n_k-N}: P_{n_k} \to P$ to a map onto Q is still ℓ to one. Hence there is a neighbourhood Q_{n_k} of P_{n_k} such that $Q_{n_k} \setminus P_{n_k}$ is an annulus whose modulus does not depend on k and does not contain any iterates of c. This proves the proposition in this case.

So consider the case that $a \in \omega(c)$. Let a_1 be the unique real preimage of a different from a. Since f is not renormalizable, there is a sequence a_n tending to a, so that $f(a_{n+1}) = a_n$, $n = 1, 2, \ldots$, and, moreover, a_{n+1} is the unique real preimage of a_n by the branch of f^{-1} fixing a (all a_n are inside of $[c_2, c_1]$, of course). Since the multiplier of a is negative, $a_3, a_5, \ldots$ are all between a_1 and a while $a_2, a_4, \ldots$ are on the other side of a.

Fix some large n_0 and let c_k be the first visit inside $[a_{n_0}, a]$ by the orbit of c. Let $n \geq n_0$ be maximal so that $c_k \in [a_n, a]$. There are two cases to distinguish.

1. $n \geq n_0 + 2$. Then c_k is the first visit inside $I_n = [a_{n-2}, a]$. This means that we can pull-back univalently I_n to c_1 by the map f^{k-1}. Moreover, $f^{n-3}(I_n) = [a_1, a]$ is the first cover of c (from the iterates of I_n) and $f^{n-3}(c_k) \in f^{n-3}([a_n, a_{n+2}]) = [a_3, a_5]$, which lies properly inside $[a_1, a]$. Hence there exists

critical pieces $P_n \subset Q_n$ which are mapped by $f^{k-1+n-3}$ onto the puzzle pieces $P \subset Q$ based on $[a_3, a_5]$ respectively $[a_1, a]$ (as a ℓ to one mapping with a unique branch point). By construction if $f^{k_n}(x)$ is the first hit of the orbit of x to P_n then there a univalent inverse branch of f^{-k_n} from Q_n to a neighbourhood of x.

2. $n = n_0$. If $I_n = [a_n, a]$, then $f^{n-1}(I_n) = [a_1, a]$, but this time $f^{n-1}(c_k) = c_{k+n-1}$ can be any point of $[a_1, a_3]$. So, if it is close to a_1, we don't have 'safe space'. The idea is to take a small portion around a_1 (containing c_{k+n-1}) and continue to iterate until this portion first hits c. Let $b_3, b_5, \dots$ be symmetric to $a_3, a_5, \dots$ with respect to c, so that $f : [a_1, b_{2i+1}] \to [a, a_{2i}]$ is one to one, $i = 1, 2, \dots$. If c_{k+n-1} is in $[b_5, b_3]$ or in $[b_3, a_3]$, then we let P be the puzzle piece based on this interval and Q the next larger puzzle piece and we obtain P_n and Q_n as before. So assume that c_{k+n-1} be in $[a_1, b_5]$. Let $j > 1$ be maximal so that $c_{k+n-1} \in [a_1, b_{2j+1}]$, i.e., $c_{k+n-1} \in [b_{2j+3}, b_{2j+1}]$. First, let us pull the interval $[a_1, b_{2j-1}]$ back to c_k: we find an interval $I_n = [a_n, a_*]$ which contains c_k and so that $f^{n-1}: I_n \to [a_1, b_{2j-1}]$ is one to one. Again, c_k is the first visit of I_n, hence, we can pull back univalently I_n to around c_1. On the other hand, $f[a_1, b_{2j-1}] = [a, a_{2j-2}]$, and further $f^{2j-3}[a, a_{2j-2}] = [a_1, a]$ covers c first time. Thus $f^{2j-2}[a_1, b_{2j-1}] = [a_1, a]$ cover c first time. But now already $f^{2j-2}(c_{k+n-1}) \in f^{2j-2}([b_{2j+3}, b_{2j+1}]) = f^{2j-3}([a_{2j+2}, a_{2j}]) = [a_3, a_5]$ proper inside $[a_1, a]$. So we can again take P, Q to be the puzzle pieces based on $[a_3, a_5] \subset [a_1, a]$. Again the proposition follows. □

COROLLARY 2. *Assume that $f(z) = z^\ell + c_1$ is real and $\omega(c)$ is not minimal. Then $J(f)$ is locally connected and zero Lebesgue measure. In particular, the Main Theorem holds in this case.*

Proof: The proof of the local connectivity of the Julia set is the same as the part of the proof of Proposition 3.1. Let us now prove that $J(f)$ has zero Lebesgue measure. Since the set of points X which do not enter some critical piece of f, is uniformly expanding (with respect to some convenient metric), X has zero Lebesgue measure. So if $J(f)$ has positive Lebesgue measure, then there exists a Lebesgue density point z of $J(f)$ with the property that $\omega(z) \ni c$. Now take some sequence of critical puzzle pieces P_{n_k} as in the previous proposition. Let r_k be the smallest integer such that $f^{r_k}(z) \in P_{n_k}$. So the puzzle piece $P_{r_k+n_k}(z)$ is mapped by f^{r_k} onto P_{n_k} and there exists a neighbourhood V_k of $P_{r_k+n_k}(z)$ which is mapped univalently onto Q_{n_k}. In particular, $f^{r_k+m_k}: V_k \to Q$ is still ℓ to one. Since $P_{r_k+n_k}(z)$ shrinks to zero in diameter (by the local connectivity of $J(f)$), and since z is a density point of $J(f)$ it follows that the relative density of $J(f)$ in P_N is arbitrarily close to one. Since $J(f)$ is closed, this implies that $J(f) \supset P_N$ what is impossible. □

4. Some tools used in interval dynamics

In the remainder of the paper we shall be concerned to associate polynomial-like mappings to real polynomials. To do this we shall need some cross-ratio estimates. In this section we shall give the necessary background on these. Let f be a real unimodal polynomial. For example, $f(z) = z^\ell + c_1$ where ℓ is even. We find it convenient to denote the critical point by c, i.e., $c = 0$. The critical value is therefore $c_1 = f(c)$ and we shall write $c_s = f^s(c)$. When $w \neq c$ then we shall define $\tau(w)$ to be the point $\neq w$ so that $f(\tau(w)) = f(w)$. For our specific map, we have $\tau(z) = -z$ but since most results in this paper do not rely on the specific form of the map f we shall write $\tau(z)$ rather than $-z$. If A, B are intervals then we shall write $[A, B]$ for the smallest interval containing A and B. Furthermore, we shall use the following notation

$$(A,B] = [A,B] \setminus A \ , \quad [A,B) = [A,B] \setminus B \text{ and } (A,B) = [A,B] \setminus (A \cup B).$$

As usual, if $J \subset T$ are two intervals and L, R are the components of $T \setminus J$ then we define $C(T, J)$ to be the cross-ratio of this pair of intervals:

$$C(T,J) = \frac{|J||T|}{|L||R|}.$$

Here $|U|$ stands for the length of an interval U. Cross-ratios play a crucial role in all recent results in real interval dynamics. Often, it suffices to use some qualitative estimates based on the so-called Koebe Principle. In our analysis, we shall need somewhat sharper estimates, which are based on direct use of the cross-ratio. For example, we shall often use the inequality that

$$|L|/|J| \geq C^{-1}(T,J).$$

If g is a map which is monotone on T and $Sg < 0$ then

$$C(gT, gJ) \geq C(T,J), \text{ i.e., } C^{-1}(T,J) \geq C^{-1}(g(T), g(J)).$$

In our case we shall apply this to maps g of the form f^n. Since $Sf < 0$ one has also that $Sf^n < 0$ so the previous inequality applies when we take $g = f^n$ and $f^n|T$ is monotone. The Koebe Principle states that if $Sg < 0$ and $J \subset T$ are intervals so that $g\colon T \to g(T)$ is a diffeomorphism and so that each component of $g(T \setminus J)$ has size $\tau|g(J)|$ (i.e., $g(T)$ is a *τ-scaled neighbourhood* of $g(J)$) then $|Dg(x)|/|Dg(y)| \leq (1+\tau)^2/\tau^2$ for each $x, y \in J$. The intervals $g(T \setminus J)$ are referred to as 'Koebe space'. We shall also use the following fact: if $Sf < 0$ and if $f^n|T$ is monotone and has a hyperbolic repelling fixed point, then $f^n(T) \supset T$.

We also need some topological facts. Let us say that W is a *symmetric* interval if it is of the form $W = [w, \tau(w)]$. The boundary point w is called *nice* if $f^i(w) \notin W$ for all $i > 0$. Note that there are plenty of nice points: each periodic orbit contains a nice point. Also, if f is not renormalizable, preimages of the orientation reversing fixed point of f can be used to find nice points (just as in the Yoccoz puzzle). Nice points were also considered in, for example, the thesis of Martens [Mar], see also Section V.1 of [MS].

If f is renormalizable, then we can take for u_n the points which are in the boundary of an interval $I_n \ni c$ which is mapped into itself in a unimodal way after $q(n)$ iterates.

If f is not renormalizable then we can construct a sequence of nice points u_n as follows. Assume that f has an orientation reversing fixed point u_0. Then we define u_n inductively as follows: let $k(n)$ be the smallest integer k such that

$$(u_{n-1}, \tau(u_{n-1})) \bigcap \left(\cup_{i=0}^{k} f^{-i}(u_0) \right) \neq \emptyset$$

and let $u_n, \tau(u_n)$ be the points in this intersection which are nearest to c. If f has no periodic attractor, then u_n is defined for each n and we get a sequence of intervals $W_n = (u_n, \tau(u_n))$. It is easy to see that each u_n is a nice point.

Let us explain why these nice points play such an important role. Let V be a symmetric interval with nice boundary points and let R_V be the first return to V. Let U be the component of the domain of R_V which contains c and take $s \in \mathbb{N}$ be so that $R_V|U = f^s$. Of course, if c is not recurrent, then U might not exist. Because V has nice boundary points, each component of the domain – except this 'central' component U – is mapped diffeomorphically by R_V onto V. Clearly, U is symmetric and also has nice boundary points. Note that $f(U), \ldots, f^s(U)$ are disjoint. If $\omega(c)$ is *minimal* then each point of $\omega(c)$ returns in a bounded time to V. Hence in this case, there are only a finite number of components of R_V which contain points of $\omega(c)$. We say that R_V has a *high return* if $R_V(U) \ni c$. (We should emphasize that this situation also includes the so-called central-high return case.) This implies that $R_V(U) = f^s(U)$ contains a component of $V \setminus \{c\}$ and therefore $f^{s+i}(U) \supset f^i(U)$ for $i \geq 1$.

It is possible that $U = V$. If this happens, we say that f is *renormalizable* and we define s to be the *period* of this interval V. In this case we can assume that $R_V : V \to V$ (which is equal to f^s in this case and consists of one fold) has a high return: otherwise this return map has a periodic attractor and therefore we do not have to consider this case. Hence, Theorem A and Theorem B can be applied in this situation.

If f is non-renormalizable and the critical point of f is recurrent, then if we define $W_{n-1} = [u_{n-1}, \tau(u_{n-1})]$ and R_n the first return map to this interval, then $W_n = [u_n, \tau(u_n)]$ is the central domain of R_n. Whenever this map R_n has a high return, Theorem B applies.

Finally, as in the complex bounds of Sullivan, we shall use the Poincaré metric on a slit region in the complex plane. Given a real interval T we shall write $D_*(T)$ for the disc which is symmetric with respect to the real line and which intersects the real line exactly in T. More generally, if T is a bounded real interval and $\alpha \in (0, \pi)$ then $D(T; \alpha)$ will denote the union of two discs which are symmetric w.r.t. the real axis, intersect the real line exactly in T and which have an external angle with the real line of angle α. The reason these sets play an important role, can be explained as follows. Let $\mathbb{C}_T = \mathbb{C} \setminus (\mathbb{R} \setminus T)$. The set $\mathbb{C}_T$ with two infinite slits, carries a Poincaré metric, and with respect to this metric the set $D(T; \alpha)$ consists of all points whose distance to T is at most equal to some constant $k(\alpha)$. From this interpretation and the Schwarz contraction principle, it follows that if $\phi: \mathbb{C}_T \to \mathbb{C}_{T'}$ is a univalent conformal mapping sending T diffeomorphically to T', then

$$\phi(D(T; \alpha)) \subset D(T'; \alpha). \tag{4.1}$$

We shall apply this statement, in the following way:

LEMMA 4.1. *Let $F: \mathbb{C} \to \mathbb{C}$ be a real polynomial whose critical points are on the real line and which maps T' diffeomorphically onto T, then there exists a set $D \subset D(T'; \alpha)$ with $D \cap \mathbb{R} = T'$ which is mapped diffeomorphically onto $D(T; \alpha)$ by F.*

Often we shall use $\alpha = \pi/2$ and so we have

$$D_*(T) = D(T; \pi/2).$$

5. How to construct a polynomial-like mapping?

In the remainder of the paper we shall be concerned with constructing polynomial like mappings. Let us first state a general principle for doing this. Let $f: \mathbb{C} \to \mathbb{C}$ be a map of the form $f(z) = z^\ell + c_1$ with c_1 real and ℓ an even positive integer. Let V be a (real) symmetric interval with nice boundary points. Let U be the component of the domain of the first return map to V containing c and let $\hat{U}$ be the component of this map containing $f(U) \ni c_1$. Take s so that $R_V|U = f^s$.

PROPOSITION 5.1. *Let $\hat{U} \supset f(U)$ be the interval which is mapped diffeomorphically onto V by f^{s-1}. Write $v^f = f(v)$, $V = [v, \tau(v)]$ and $\hat{U} = [\hat{u}^f, u^f]$. Here $u^f = f(u)$ and $\hat{u}^f$ is a point which is not the f-image of some real point. Assume*

$$K = \frac{|\hat{u}^f - c_1|}{|v^f - c_1|} < 1. \tag{5.1}$$

Moreover, assume that the critical point $c = 0$ of f is recurrent, i.e., all iterates of c under $R_V: D_V \to V$ remain in D_V and that $\omega(c)$ is minimal. Then there exists a ℓ-polynomial-like mapping

$$R: D^0 \cup \ldots \cup D^i \to D_*(V')$$

such that $c \in J_R$. *Here* $V' = V$ *if* $U \neq V$ *(i.e.* $f^s: U \to V$ *is not a renormalization), and* V' *is equal to some* ε*-neighbourhood of* V *with* $\varepsilon > 0$ *small enough if* $U = V$ *(i.e., when* $f^s: U \to U$ *is a renormalization). The map* R *is an iterate of* f *on each of the components of its domain.* $\mathbb{R} \cap D^i$ *consists of the components of* $D_V \cap V$ *containing points of* $\omega(c)$ *if* $U \neq V$*, and contains* $U = V$ *otherwise.*

Proof: Since $f: \omega(c) \to \omega(c)$ is minimal, each point $x \in \omega(c)$ is in the domain of the map R_V. By compactness, there exists therefore a finite covering of $\omega(c)$ of disjoint intervals $I^0, \dots, I^i$ consisting of components of R_V with $I^0 \ni c$. Let us first consider a component I^j with $j \neq 0$. Since then $I^j \notin c$ we get that R_V maps I^j diffeomorphically onto $R_V(I^j) = V$ and it follows that there is a region D^j contained in $D_*(I^j)$ which is mapped diffeomorphically onto $D_*(V)$ by R_V. So consider $I^0 = U = [u, \tau(u)]$. The map f^{s-1} sends $\hat{U} \supset f(U) = f(I^0) \ni c^1$ diffeomorphically onto V. Again there is a region $D' \ni c_1$ contained in $D_*(\hat{U})$ which is mapped diffeomorphically onto $D_*(V)$ by f^{s-1}. Because of (5.1), the f-inverse D^0 of $D' \subset D_*(\hat{U})$ is contained in $D_*(V)$.

In the case of renormalization, we replace V above by its ε-neighborhood V', with $\varepsilon > 0$ so small that (5.1) holds for the new points $\hat{u}^f, v^f$, and so that the new interval U is strictly inside V' (this is possible since in this case the point u is a repelling periodic point of f^s). □

Remark 5.2. *Assume that* $f^s: U \mapsto U = V$ *(i.e,* f *is renormalizable of period* s*). As we will show in the next two sections,* $K < 1$ *provided either* $\ell \geq 4$ *or* $\ell = 2$ *and* f *is also renormalizable of period* $s/2$*. If* $\ell = 2$ *then we merely obtain the estimate* $K \leq 1.1937$ *in the next two sections, but as we will show in Section 10 a modification of the previous proposition will still give us a polynomial-like map.*

6. Real bounds if the return is high

Let W be a symmetric interval with nice boundary points, let R_W be the first return map to W and let V be the domain of R_W containing c. Similarly, let R_V be the first return map to V and U the component of the domain of R_V which contains c. Let $\hat{U}$ (resp. $\hat{V}$) be the component of the domain of R_V (resp. of R_W) containing the critical value c_1. Let s, s' be so that $R_V|\hat{U} = f^{s-1}$ and $R_W|\hat{V} = f^{s'-1}$.

In this section we will assume that R_V has a high return and derive some conditions which - when satisfied - will imply that the component of the $f^{-s}(D_*(V))$ which contains c is contained in $D_*(V)$.

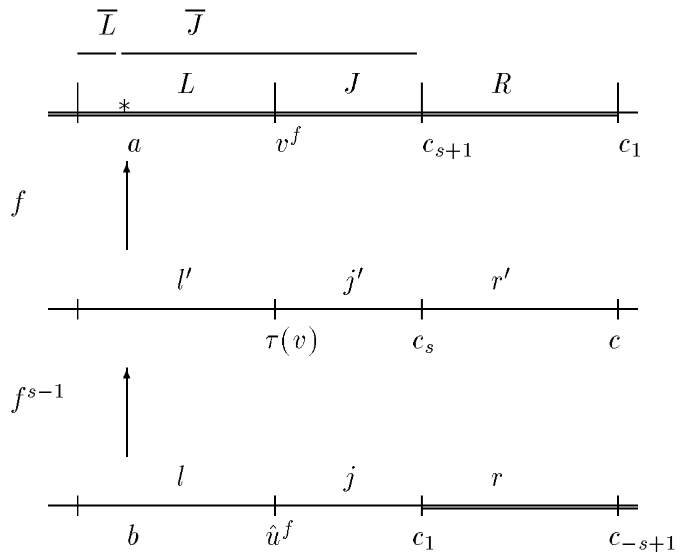
Let j be the component of $\hat{U} \setminus c_1$ which is outside $[c_1, c_2]$, i.e., $j = \hat{U} \setminus f(U)$. If R_V has a high return, $f^{s-1}|f(U)$ contains c and so we can define r to be the interval in $f(U)$ which contains c_1 and such that $f^{s-1}(r) \ni c$ (i.e., $r = [c_1, c_{-s+1}]$). Furthermore, let l be the maximal interval having a unique common point with the boundary point of $\hat{U}$ outside $[c_1, c_2]$ on which f^{s-1} is monotone. We also write,

$$U = [u, \tau(u)], \quad \hat{U} = [\hat{u}^f, f(u)] \ni c_1, \quad t = l \cup j \cup r, \quad V = [v, \tau(v)],$$
$$l' = f^{s-1}(l), \quad j' = f^{s-1}(j), \quad r' = f^{s-1}(r), \quad t' = l' \cup j' \cup r'$$

and

$$L = f^s(l), \quad J = f^s(j), \quad R = f^s(r) \text{ and } T = L \cup J \cup R.$$

Mark the typographical difference between the degree ℓ and the interval l. The situation is drawn below. (The fat lines denote the part near c_1 which is inside the interval $[c_1, c_2]$; note that the map $f^s|t$ is orientation reversing.)



The intervals of Lemma 6.1.

Given V as above, take $a \in L$ (including possibly v^f), choose $b \in l$ so that $f^s(b) = a$ and define

$$K_\ell(a) = \frac{|b - c_1|}{|a - c_1|}.$$

In the case that $a = v^f$ and $b = \hat{u}^f$ this becomes $K_\ell(v^f) = \frac{|\hat{u}^f - c_1|}{|v^f - c_1|} = \frac{|j|}{|J \cup R|}$. This number is important for our question. Indeed, if

$$K_\ell(v^f) < 1 \tag{6.1}$$

and, if $f(z) = z^\ell + c_1$ then we get that $f^{-1}(D_*(\hat{u}^f, u^f)) \subset D_*(v, \tau(v))$. As in Proposition 5.1 this allows us to get a polynomial-like extension of $R_V \colon D_V \to$

V. In this section we shall derive a condition for (6.1). Define

$$t = \frac{|c_1 - c_{s+1}|}{|T|} = \frac{|R|}{|T|},$$

$$y = \frac{|a - c_1|}{|T|}.$$

This last quantity measures the amount of 'extendibility' around $[a, c_1]$. For example, if $a = v^f$ then $y = \frac{1}{1+|L|/(|J \cup R|)}$ where $|L|/(|J \cup R|)$ is the 'space' which exists around $f(V) = J \cup R$.

LEMMA 6.1. *(See also Proposition 3.2 in [SN].) Assume that R_V has a high return and that f^{2s} has no neutral or attracting fixed point. Then*

$$K_\ell(a) \leq \frac{t(y^{1/\ell} - t^{1/\ell})}{t^{1/\ell} y(1 - y^{1/\ell})}.$$

Proof: Denote $\overline{J} = [a, c_{s+1}]$, $\overline{L} = T \setminus (R \cup \overline{J})$. Then instead of j', l' we can choose some intervals $\overline{j}', \overline{l}'$ so that $f(\overline{j}') = \overline{J}$, $f(\overline{l}') = \overline{L}$, and $\overline{l}, \overline{j}$ replace the intervals l, j, i.e., $\overline{l} = f^{-(s-1)}(\overline{l}')$, $\overline{j} = f^{-(s-1)}(\overline{j}')$). If we do this, then the intervals r', r, R do not change. Write

$$\alpha = |r'|, \overline{\beta} = |r' \cup \overline{j}'|, \gamma = |r' \cup \overline{j}' \cup \overline{l}'|,$$

and

$$Q = \frac{|\overline{J}| \cdot |T|}{|\overline{L}| \cdot |R|} \Big/ \frac{|\overline{j}'| \cdot |t'|}{|\overline{l}'| \cdot |r'|}.$$

Then, from the expansion of the cross-ratio's

$$\frac{|\overline{J}| \cdot |T|}{|\overline{L}| \cdot |R|} \geq Q \cdot \frac{|\overline{j}| \cdot |t|}{|\overline{l}| \cdot |r|} \geq Q . \frac{|\overline{j}|}{|r|},$$

and using this inequality we get

$$1/K_\ell(a) = |\overline{J} \cup R|/|\overline{j}| \geq (|\overline{J} \cup R|/|r|) \cdot Q \cdot (|\overline{L}| \cdot |R|)/(|\overline{J}| \cdot |T|)$$

$$\geq \frac{\frac{|\overline{J}| \cdot |T|}{|\overline{L}| \cdot |R|}}{\frac{|\overline{j}'| \cdot |t'|}{|\overline{l}'| \cdot |r'|}} \cdot \frac{|\overline{J} \cup R| \cdot |\overline{L}|}{|\overline{J}| \cdot |T|} = \frac{|\overline{J} \cup R| \cdot |\overline{l}'| \cdot |r'|}{|R| \cdot |\overline{j}'| \cdot |t'|}$$

$$= \frac{\overline{\beta}^\ell \cdot (\gamma - \overline{\beta}) \cdot \alpha}{\gamma \cdot (\overline{\beta} - \alpha) \cdot \alpha^\ell}.$$

Here we have used in the last inequality that $|r| \leq |R|$ which holds because $f^{2s}|r$ has no periodic attractor. Now writing $\overline{\beta}^\ell/\gamma^\ell = y$ and $\alpha^\ell/\gamma^\ell = t$ the lemma follows. □

Passing to the maximum over $t \in [0, y]$, we obtain

Corollary 3. $K_\ell(a) \le K_\ell^*(y) := \frac{(1-1/\ell)^{\ell-1}}{\ell\cdot(1-y^{1/\ell})}$.

Example 6.2.
(*a*) *If the extendibility space is* 0.8, *i.e.,* $y_1 = 1/(1+0.8) = 0.55555..$ *then*

$$K_2^*(y_1) = 0.98176275...,.$$

(*b*) *If the extendibility space is* 0.6, *i.e.,* $y_2 = 1/(1+0.6) = 0.625$ *then*

$$K_2^*(y_2) = 1.19371..., \; K_4^*(y_2) = 0.951366... < 1.$$

(*c*) *If the extendibility space is* 1/3, *i.e.,* $y_3 = 1/(1+1/3) = 3/4$ *then*

$$K_2^*(y_3) = 1.8660... \text{ and } \lim_{\ell\to\infty} K_\ell^*(y_3) \to 1.2788....$$

This means that with these estimates for the extendibility space, we can apply the method suggested by Proposition 5.1 respectively for $\ell \ge 2$, $\ell \ge 4$ and not even for ℓ large in the last case. In Section 12 of this paper we shall use a slightly different method (using different Poincaré neighbourhoods) which also works when the space is equal to 1/3.

In the next two section we shall derive estimates for the number y from above.

7. Lower bounds for ‘space’ in the renormalizable case

In this and the next section we shall derive lower bounds for the number y, i.e., find lower bounds for ‘space’ by looking for a ‘smallest’ interval among a finite number of intervals. This idea is used in a large number of results in one-dimensional dynamics. In particular we were inspired by the thesis of Martens [Mar] or, specifically, by Lemma 1.2 in Section V.1 of [MS]. In this section we shall obtain quite sharp bounds, which will enable to deal with all real infinitely renormalizable maps $z \mapsto z^\ell + c_1$ of degree $\ell \ge 2$. Unfortunately, the proof splits in quite a few subcases. The main result in the section is Lemma 7.4. In the next section, we shall obtain weaker bounds which work in a more general context; these weaker bounds only apply to the case that $\ell \ge 4$.

Let $\hat{V} \supset f(V)$ be the interval which is mapped monotonically onto W by $f^{s'-1}$.

Lemma 7.1. *Let $2 \le k < s'$ and let H_1 be the maximal interval containing $f(V)$ such that $f^k|H_1$ is monotone. Then $f^k(H_1)$ contains $f^k(f(V))$ and on each side of this interval also an interval of the form $f^i(V)$ with $i \le k$.*

Proof: Let $H_{1,-}, H_{1,+}$ be the components of $H_1 \setminus f(V)$. From the maximality of H_1 it follows that there exists $i' < k$ such that $f^{i'}(H_{1,+})$ contains c.

Since $f^{i'+1}(V)$ is outside W it follows that $f^{i'}(H_{1,+})$ contains one component W_+ of $W \setminus \{c\}$. It follows that $f^k(H_{1,+})$ contains $f^{k-i'}(W) \supset f^{k-i'}(V)$. Since the same holds for $H_{1,-}$, the lemma follows. $\square$

Let $\hat{U} \supset f(U)$ be the interval which is mapped monotonically onto V by f^{s-1}.

LEMMA 7.2. *Assume that R_W has a high return and let l be one of the two maximal intervals outside $\hat{U}$ for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$. (If we take the interval which is outside $[c_1, c_2]$ then it is equal to the interval l from Lemma 6.1.) Then $L := f^s(l)$ contains an interval of the form $f^i(V)$, $1 \le i \le s'$.*

Moreover, if $U = V$ (so f is renormalizable with period s and $s' = s$) then we have the following

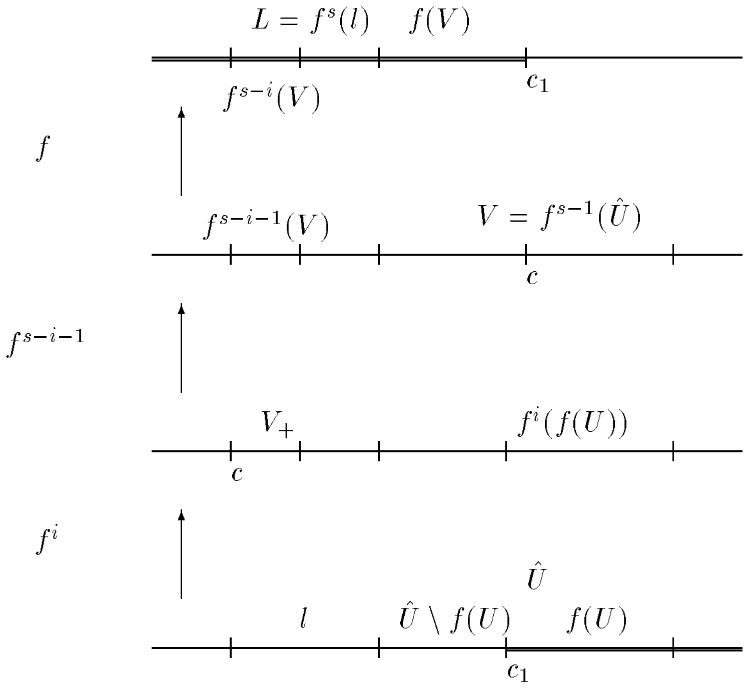
a) If f is not also renormalizable of period $s/2$ then $L := f^s(l)$ contains two distinct intervals of the form $f^{i+2}(V), f^{s-i}(V)$, with $1 \le i+2, s-i < s'$ and $i + 2 \ne s - i$ and, in addition, $l \cap \omega(c) = \emptyset$

b) If f **is** *renormalizable of period $s/2$ and l is outside $[c_1, c_2]$ then L contains two distinct intervals of the form $f^i(U)$ with $i \ne 1, s/2 + 1$.*

Proof: Let $H = l \cup \hat{U}$. By maximality of H there is i with $0 < i < s$ such that $f^i(H)$ contains c in its boundary. Choose i maximal with this property. Since $f^i(\hat{U})$ is outside V it follows that $f^i(H)$ contains one component V_+ of $V \setminus \{c\}$. Hence $f^{i+1}(l)$ contains $f(V)$ and therefore $f^{s-1}(H)$ contains $f^{s-i-1}(V)$ (and also a point in $f^{s-1}(\hat{U}) = V$). Since R_W has a high return and $f^{s'-1}(f(V))$ contains c in its interior, and since by definition $f^s|l$ is monotone, it follows that $s - i < s'$. Hence $f^{s-1}(H)$ contains one of the intervals $f(V), \ldots, f^{s'-1}(V)$.

Now we consider the case that $U = V$ (and therefore $s' = s$) and take $\tilde{H} = f^i(H) = [c, f^i(\hat{U})]$. Let us now assume $f^{s-i-1}(V)$ is the only interval from the collection $f(V), \ldots, f^{s'-1}(V)$ which is contained in $f^{s-i-1}(\tilde{H}) = f^{s-1}(H)$. Then there is no interval from the collection $f(U), \ldots, f^{s-1}(U)$ between V_+ and $f^i(f(U))$, because the image of this interval by f^{s-i-1} would then be between $f^{s-i-1}(V)$ and V (see the figure below), contradicting the previous assumption. The assumption also implies that $f^i(f(U))$ is not between $f^{s-i-1}(V)$ and V, and therefore the only remaining possibility is that $f^{i+1}(U) = f^{s-i-1}(U))$. Hence $s - i - 1 = i + 1$, i.e., $i + 1 = s - i - 1 = s/2$. It follows that $f^{s-i-1} = f^{s/2}$ maps $[\tilde{H}, \tau(\tilde{H})]$ inside itself. Since, by assumption, this interval only contains two of the intervals of the orbit $V, \ldots, f^{s-1}(V)$, it follows that f is also renormalizable of period $s/2$. Therefore $f^{s-1}(H)$ contains $f^{s-i-1}(V)$ and $f^{i+1}(V)$, if f is not renormalizable of period $s/2$. In fact, one easily sees $l \cap \omega(c) = \emptyset$: if $f^i(U)$ is the renormalizable interval next to $f(U)$ and $l = [a, b]$ contain some point b in this neighbouring interval, then since f^s fixed each of the renormalizable intervals, $f^s(a) \in f(U)$ and $f^s(b) \in f^i(U)$. Since $f^s|l$ is a homeomorphism, it follows that $f^s(l) \subset [f(U), f^i(U)]$ and therefore that f^s maps $[f(U), f^i(U)]$ into itself, which only happens if $i = s/2$ and f is renormalizable of period $s/2$.

Let us now prove the last assertion. Write $r = s/2$. Let $i < s$ be minimal so that $f^i(l)$ contains c in its closure. Then $f^i(l)$ lies on one side of $f^i(\hat{U})$ and $f^i(f^{r+1}(U))$ on the other. If $f^i(\hat{U})$ intersects $f^r(U)$ or $\tau(f^r(U))$ then $i = r-1$ (because $\hat{U}, \dots, f^{s-1}(\hat{U})$ are disjoint). But $f^i(\hat{U})$ is between $f^i(l)$ and $f^i(f^{r+1}(U))$, and therefore if $i = r-1$ we would have that $f^r(U)$ is between $f^i(l)$ and U. This is impossible since by assumption $f^i(l)$ contains c in its boundary. Hence $f^i(\hat{U})$ has no point in common with $[f^r(U), \tau(f^r(U))]$. Hence $f^i(l)$ contains one component of $[f^r(U), \tau(f^r(U))] \setminus \{c\} \supset U \setminus \{c\}$. Therefore $f^{i+1}(l)$ contains $f(U)$ and $f^{r+1}(\pm U)$. $\square$



The proof of Lemma 7.2.

Let us now assume throughout the remainder of this section that $U = V$ is a periodic interval of period $s' = s$. Many arguments also work if the interval is not periodic, but one only has a high return.

LEMMA 7.3. *Assume that $U = V$ has period $s \geq 3$ and f is not also renormalizable of period $s/2$. Consider the disjoint intervals $f(V), \dots, f^s(V)$*

and assume that $f(V)$ and $f^2(V)$ are at least as long as their neighbours. Then there exists an integer $k > 2$ such that $f^k(V)$ is shorter than its two neighbours from the collection $f(V), \dots, f^{k-1}(V)$. Take k maximal with respect to this property. Let $1 \leq i_0, i_1 < k$ be so that $f^{i_0}(V), f^{i_1}(V)$ are the neighbours of $f^k(V)$ from the collection $f(V), \dots, f^{k-1}(V)$. Let

$$Q_k = [f^{i_0}(V), f^{i_1}(V)]$$

and define $H_1 \supset f(V)$ to be the maximal interval on which f^{k-1} is monotone. Then $H_k = f^{k-1}(H_1) \supset Q_k$. Let $Z_k \subset H_k$ be the maximal interval of the form $[f^{j_0}(V), f^{j_1}(V)]$ such that each component of $Z_k \setminus Q_k$ contains at most one interval of the form $f^j(V)$ with $k < j \leq s$ (i.e., $Z_k = Q_k$ is not excluded). If we define $Z_1 \subset H_1$ so that $Z_k = f^{k-1}(Z_1)$ then

$$C^{-1}(Z_1, f(V)) \geq 0.6.$$

Proof: Consider the set of j's in $\{3, \dots, s\}$ such that $f^j(V)$ is shorter than its neighbours from the collection $f(V), \dots, f^s(V)$. This set is obviously is non-empty because j corresponding to the shortest $f^j(V)$ is in this set (note that all the intervals considered are disjoint). It follows that the largest integer k from the above set satisfies the desired properties. By assumption $k \neq 1, 2$. Let i_0, i_1 be the intervals as in the statement of the lemma. By the choice of k these neighbours are longer than $f^k(V)$. Let

$$Q_k = [f^{i_0}(V), f^{i_1}(V)] \supset f^k(V).$$

Throughout the remainder of the proof we shall consider the case that $f^{i_0}(V)$ lies to the left of $f^{i_1}(V)$. Notice that the fact that $f^{i_0}(V)$ and $f^{i_1}(V)$ are neighbours of $f^k(V)$ implies that Q_k only contains intervals of the form $f^j(V)$ with $k < j \leq s$. From the maximality of k this implies that each such interval $f^j(V) \subset Q_k$ is longer than the interval $f^k(V)$. Lemma 7.1 gives $f^{k-1}(H_1) \supset Q_k$. Write

$$H_k = f^{k-1}(H_1).$$

Let $Z_1 \subset H_1$ be as in the statement of the lemma. Let $Q_1 \supset f(V)$ be the subset of H_1 for which $f^{k-1}(Q_1) = Q_k$ and let

$$Q_{i_j} = f^{i_j-1}(Q_1)\,,\; Z_{i_j} = f^{i_j-1}(Z_1) \text{ and } H_{i_j} = f^{i_j-1}(H_1) \text{ for } j = 0, 1.$$

Since $f^{i_0}(V)$ and $f^{i_1}(V)$ are longer than $f^k(V)$ we at least have

$$C^{-1}(Z_1, f(V)) \geq C^{-1}(Q_1, f(V)) \geq C^{-1}(Q_k, f^k(V)) \geq 1/3.$$

We shall now improve this estimate, by pulling back the interval $Q = Q_k$ either to Q_{i_0} or to Q_{i_1}. In this way we shall either find another interval $f^j(V)$ inside the interval Q_k or find a lower bound for the space between the intervals $f^j(V)$ in Q_k. For this we shall distinguish between several cases depending on whether or not $Q_k = H_k$ and depending on the position of Q_{i_0} and of Q_{i_1} relative to Q_k. Often we shall even show that

$$C^{-1}(Q_k, f^k(V)) \geq 0.6.$$

Since $Z_1 \supset Q_1$ this suffices:

$$C^{-1}(Z_1, f(V)) \geq C^{-1}(Q_1, f(V)).$$

Case I. Assume that $Q_1 = H_1$. By maximality of H_1 this implies that there exist i_0 and i_1 such that $f^{k-i_0}(H_1)$ and $f^{k-i_1}(H_1)$ contain c in their boundary, see the figure below (here we also use that $k > 2$).

$$\begin{array}{ccccc}
\underline{\quad i_0 \quad} & \underline{\quad k \quad} & \underline{\quad i_1 \quad} & \qquad & f^{k-1}(Q_1) = Q_k \\[2ex]
\underline{\quad 0 \quad} & \underline{\ k - i_0\ } & \underline{\qquad\quad} & \qquad & f^{k-i_0-1}(Q_1) \\[2ex]
\underline{\qquad\quad} & \underline{\ k - i_1\ } & \underline{\quad 0 \quad} & \qquad & f^{k-i_1-1}(Q_1)
\end{array}$$

Case I.

If $f^{k-i_0}(V)$ lies closer to c than $f^{k-i_1}(V)$, then $f^{k-i_0+i_1}(V)$ lies between $f^k(V)$ and $f^{i_1}(V)$. Similarly, if $f^{k-i_1}(V)$ lies closer to c than $f^{k-i_0}(V)$, then $f^{k-i_1+i_0}(V)$ lies between $f^k(V)$ and $f^{i_0}(V)$. Therefore, since there is no $f^j(V) \subset Q_k$ with $j < k$ and $j \neq i_0, i_1$, this implies that the first possibility occurs if $i_1 > i_0$ and the second one if $i_1 < i_0$. In order to be definite, we shall assume (in this case) that $i_0 < i_1$. This implies that the situation inside Q_k is as drawn below.

$$\begin{array}{cccccc}
\underline{\quad i_0 \quad} & \underline{\quad k \quad} & \underline{k + i_1 - i_0} & \underline{\quad i_1 \quad} & \qquad & Q_k
\end{array}$$

Case I: The next interval in Q_k.

Since each of these intervals $f^{i_0}(V)$, $f^{i_1}(V)$ and $f^{k+i_1-i_0}(V)$ is at least as long as $f^k(V)$, it follows that

$$C^{-1}(Q_k, f^k(V)) = \frac{\text{left} \cdot \text{right}}{\text{middle} \cdot \text{total}} \geq \frac{1 \cdot 2}{1 \cdot 4} = \frac{1}{2}.$$

Hence

$$C^{-1}(Q_{i_0}, f^{i_0}(V)) \geq C^{-1}(Q_k, f^k(V)) \geq \frac{1}{2}$$

and therefore each of the components of $Q_{i_0} \setminus f^{i_0}(V)$ has length $\geq 1/2$ times the length of $f^{i_0}(V)$. Therefore, if Q_{i_0} does not contain any points of $f^k(V)$, then one of these components of $Q_{i_0} \setminus f^{i_0}(V)$ is contained in the gap between $f^{i_0}(V)$ and $f^k(V)$. Hence

$$C^{-1}(Q_k, f^k(V)) \geq \frac{2 \cdot 3/2}{3/2 + 1 + 2} = \frac{6}{9} > 0.6 \quad .$$

So we are finished in this case. If there exists an interval $f^j(V)$ between $f^{i_0}(V)$ and $f^k(V)$ then we also are finished, because this interval then has length $\geq |f^k(V)|$ and therefore

$$C^{-1}(Q_k, f^k(V)) \geq \frac{2 \cdot 2}{2 + 1 + 2} = \frac{4}{5} > 0.6 \quad .$$

So we shall consider the case that Q_{i_0} contains some points of $f^k(V)$ and that there exists no interval $f^j(V)$ between $f^{i_0}(V)$ and $f^k(V)$. Therefore the map $f^{k-i_0}: Q_{i_0} \to Q_k$ is orientation preserving. Indeed, otherwise the interval $[f^{i_0}(V), f^k(V)]$ would be mapped inside itself by the map f^{k-i_0}. Since there is no interval $f^j(V)$ contained in this interval $[f^{i_0}(V), f^k(V)]$, this implies that f is also renormalizable with half the period $s/2$. By assumption this is not the case. So $f^{k-i_0}: Q_{i_0} \to Q_k$ is orientation preserving. If Q_{i_0} contains (some points of) $f^k(V)$ and no points to the right of $f^k(V)$ then the gap between $f^{i_0}(V)$ and $f^k(V)$ is mapped onto $f^{k+i_1-i_0}(V)$. So if we define $W_1 \subset H_1$ so that $W_k = f^{k-1}(W_1) = [f^{i_0}(V), f^{k+i_1-i_0}(V)]$ then

$$C^{-1}(W_{i_0}, f^{i_0}(V)) \geq C^{-1}(W_k, f^k(V)) \geq \frac{1}{3}.$$

Hence the component of $W_{i_0} \setminus f^{i_0}(V)$ which is between $f^{i_0}(V)$ and $f^k(V)$ has at least length $1/3$ times the length of $|f^{i_0}(V)|$. This implies that

$$C^{-1}(Q_k, f^k(V)) \geq \frac{4/3 \cdot 2}{4/3 + 1 + 2} = \frac{8}{13} > 0.6.$$

So we finally have to consider the case that Q_{i_0} strictly contains a neighbourhood of $f^k(V)$. Then Q_k contains $f^{k+k-i_0}(V)$. Because $f^{k-i_0}|Q_{i_0}$ is a homeomorphism and $Q_{i_0} \supset f^k(V)$, we have $k + k + i_0 \leq s$. Since $k + k - i_0 > k + i_1 - i_0 > i_1$, Q_k contains three intervals of the form $f^j(V)$ to the right of $f^k(V)$ and therefore

$$C^{-1}(Q_k, f^k(V)) \geq \frac{1 \cdot 3}{1 + 1 + 3} = \frac{3}{5} = 0.6.$$

This completes case I.

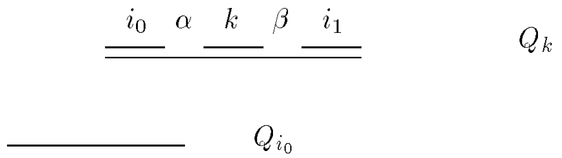
Case II and Case III. Assume that Q_1 is strictly contained in H_1. In order to be specific, let us assume that $f^{k-1}(H_1)$ contains a neighbourhood of $f^{i_1}(V)$. This information is useful since $f^j(V)$ cannot be mapped monotonically onto $f^{i_i}(V)$ by an iterate of f when $j > i_1$. In particular, since $f^{k-1}(H_1)$ contains a neighbourhood of $f^{i_1}(V)$, it follows that f^{k-i_0} cannot map $f^k(V)$ onto $f^{i_1}(V)$.

It follows that the right endpoint of Q_{i_0} cannot be inside $f^k(V)$. Therefore there are only two possibilities: II) Q_{i_0} lies to the left of $f^k(V)$ or III) Q_{i_0} contains some points to the right of $f^k(V)$. (Remember that we had assumed that $f^{i_0}(V)$ lies to the left of $f^{i_1}(V)$.)

Case II. Q_{i_0} lies to the left of $f^k(V)$.

Now we shall analyze the situation near $f^{i_1}(V)$. We shall subdivide several cases:

Case II.a. Q_{i_1} lies to the right of $f^k(V)$. Let $\alpha \cdot |f^{i_0}(V)|$ be the size of the gap between $f^{i_0}(V)$ and $f^k(V)$. Similarly, let $\beta \cdot |f^{i_1}(V)|$ be the size of the gap between $f^{i_1}(V)$ and $f^k(V)$.



Case II: Q_{i_0} lies to the left of $f^k(V)$.

Since Q_{i_0} lies to the left of $f^k(V)$,

$$\alpha \geq C^{-1}(Q_{i_0}, f^{i_0}(V)) \geq C^{-1}(Q_k, f^k(V)) \geq \frac{(1+\alpha)(1+\beta)}{3+\alpha+\beta} \geq 1/3.$$

In this case II.a, we also have a similar inequality as above for β, i.e., we have

$$\alpha, \beta \geq \frac{(1+\alpha)(1+\beta)}{3+\alpha+\beta}.$$

Since the right hand side of the above inequalities is increasing in both α and in β (and has derivative < 1 w.r.t. these variables), it follows that $\alpha, \beta \geq \kappa$ where

$$\kappa = \frac{(1+\kappa)(1+\kappa)}{3+\kappa+\kappa}, \text{ i.e., } \kappa^2 + \kappa = 1.$$

Hence

$$\alpha, \beta \geq \kappa = \sqrt{(5/4)} - 1/2 > 0.6 \quad .$$

This implies that

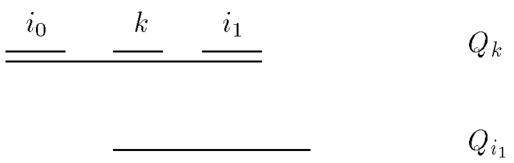
$$C^{-1}(Q_k, f^k(V)) \geq \frac{(1+\alpha)(1+\beta)}{1(3+\alpha+\beta)} \geq \kappa \geq 0.6 \quad .$$

Case II.b. Q_{i_1} contains some points of $f^k(V)$ but no point to the left of $f^k(V)$. As we remarked above, the fact that we are in Case II or Case III, implies that $f^k(V)$ cannot be mapped homeomorphically onto $f^{i_1}(V)$. It follows that f^{k-i_1} cannot map Q_{i_1} in an orientation reversing way homeomorphically onto

Q_k. Hence $f^{k-i_1}: Q_{i_1} \to Q_k$ is orientation preserving and this map sends $f^k(V)$ to $f^{i_0}(V)$. Writing $r = k - i_0$, this gives $k + (k - i_1) = i_0 + s$, i.e., $k - i_1 = s - (k - i_0) = s - r$. So

$$k = i_0 + r \text{ and } i_1 = k + (r - s) = i_0 + 2r - s. \tag{7.1}$$

Let α be so that the gap $(f^{i_0}(V), f^k(V))$ has length $\alpha|f^{i_0}(V)|$ and similarly, let β be so that the gap between $f^k(V)$ and $f^{i_1}(V)$ has size $\beta|f^{i_1}(V)|$. Let Z_k^r be the right component of $Z_k \setminus f^{i_1}(V)$ and define γ so that the size of Z_k^r is equal to $\gamma|f^{i_1}(V)|$. Similarly, let H_k^r be the right component of $H_k \setminus f^{i_1}(V)$.



Case II.b. The map $f^{k-i_1}: Q_{i_1} \to Q_k$ is orientation preserving.

Since we are in Case II, the interval H_k^r contains at least some interval of the form $f^m(V)$ (with $m < k$). If $Z_k^r \neq H_k^r$ then Z_k^r contains an interval $f^{j_1}(V)$ with $k < j_1 < s$. So in any case Z_k^r contains an interval $f^n(V)$. If the right component of $Q_{i_1} \setminus f^{i_1}(V)$ is not contained in Z_k^r, then Q_{i_1} contains a neighbourhood of $f^n(V)$ and therefore then Q_k contains an interval $f^j(V)$ between $f^k(V)$ and $f^{i_1}(V)$. Since $\alpha \geq 1/3$, this implies that

$$C^{-1}(Q_k, f^k(V)) \geq (1 + 1/3)2/(4 + 1/3) = 8/13 > 0.6.$$

Therefore, we may assume that the right component of $Q_{i_1} \setminus f^{i_1}(V)$ is contained in Z_k^r. Define $W_1 \subset Q_1$ so that $W_k = f^{k-1}(W_1) = (f^{i_0}(V), f^{i_1}(V)]$. Since we are in Case II.b, we have that f^{k-i_1} maps $f^k(V)$ to $f^{i_0}(V)$. Hence one component of $W_{i_1} \setminus f^{i_1}(V)$ is contained in the gap $(f^k(V), f^{i_1}(V))$ corresponding to β and the other in the interval Z_k^r corresponding to γ. Therefore

$$\frac{\beta\gamma}{1 + \beta + \gamma} \geq C^{-1}(W_{i_1}, f^{i_1}(V)).$$

Since

$$C^{-1}(W_{i_1}, f^{i_1}(V)) \geq C^{-1}(W_k, f^k(V)) \geq \frac{\alpha(1 + \beta)}{2 + \alpha + \beta}$$

this gives,

$$\frac{\beta\gamma}{1 + \beta + \gamma} \geq \frac{\alpha(1 + \beta)}{2 + \alpha + \beta}.$$

Hence

$$\beta(2 + \alpha + \beta) \geq \alpha(1 + \beta) + \frac{\alpha(1 + \beta)^2}{\gamma},$$

or

$$\beta^2 + 2\beta \geq \alpha + \frac{\alpha(1+\beta)^2}{\gamma},$$

i.e.,

$$\beta \geq \sqrt{1 + \alpha + \frac{\alpha(1+\beta)^2}{\gamma}} - 1.$$

Now we study the situation on the other side, around $f^{i_0}(V)$. Define $\hat{Z}_k = [f^{i_0}(V), Z_k^r]$ and let $\hat{Z}_1$ be so that $f^{k-1}(\hat{Z}_1) = \hat{Z}_k$. Then $\hat{Z}_{i_0}$ is to the left of $f^k(V)$. Indeed, $f^r = f^{k-i_0}$ maps $f^k(V)$ to $f^{k+(k-i_0)}(V) = f^{k+r}(V)$. Because of (7.1) we have $k+r > s$, that $f^r|f^k(V)$ is not monotone and $f^{k+r}(V) = f^{i_1}(V)$. Since f^r maps Z_{i_0} monotonically to Z_k (which strictly contains $f^{i_1}(V)$), we finally get that Z_{i_0} lies to the left of $f^k(V)$. Hence,

$$\alpha \geq C^{-1}(\hat{Z}_{i_0}, f^{i_0}(V)) \geq C^{-1}(\hat{Z}_k, f^k(V)) \geq \frac{(1+\alpha)(1+\beta+\gamma)}{3+\alpha+\beta+\gamma}$$

which gives that

$$\alpha^2 + 2\alpha \geq 1 + \beta + \gamma \text{ i.e., } \alpha \geq \sqrt{2+\beta+\gamma} - 1.$$

If $\gamma \geq 0.56$ then $\alpha \geq \sqrt{2.56} - 1 = 0.6$ and so we have

$$C^{-1}(Z_k, f^k(V)) \geq \frac{(1+\alpha)(1+\gamma)}{3+\alpha+\gamma} \geq \frac{1.6 \cdot 1.56}{4.16} = 0.6 \quad .$$

If $\gamma \leq 0.56$ then we have that

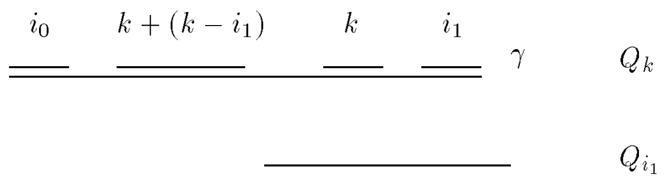
$$\frac{\alpha}{\gamma} \geq \frac{\sqrt{2+\gamma} - 1}{\gamma} \geq \frac{\sqrt{2.56} - 1}{0.56} > 1.$$

Therefore

$$\beta \geq \sqrt{1 + \alpha + 1} - 1.$$

Since we also have $\alpha \geq \sqrt{2+\beta} - 1$, and since the function $(0, \infty) \ni x \mapsto \sqrt{2+x} - 1$ has a unique attracting fixed point $\sqrt{5/4} - 1/2 > 0.61$ it follows that $\alpha, \beta > 0.61$. Again this is sufficient and this completes case II.b.

Case II.c. Q_{i_1} contains $f^k(V)$ and also some point between $f^{i_0}(V)$ and $f^k(V)$. As before, $f^{k-i_1}\colon Q_{i_1} \to Q_k$ is orientation preserving in this case, because otherwise f^{k-i_1} maps $[f^k(V), f^{i_1}(V)]$ monotonically into itself and since f has no periodic attractor, this is impossible. Hence the interval $f^{k+(k-i_1)}(V)$ lies between $f^{i_0}(V)$ and $f^k(V)$.



Case II.c.

Note that Z_k contains another interval $f^j(V)$ to the right of $f^{i_1}(V)$ because we have assumed in Cases II and III that H_k contains a neighbourhood of $f^{i_1}(V)$. If Q_{i_1} is not contained in Z_k then Q_k contains another (i.e., a fifth) interval $f^{j'}(V)$ and so $C^{-1}(Q_k, f^k(V))$ is at least 0.6. So assume that Q_{i_1} is contained in Z_k. Let γ be so that the length of the component $Z_k \setminus Q_k$ to the right of $f^{i_1}(V)$ is equal to $\gamma|f^{i_1}(V)|$. Since we have assumed that Q_{i_1} is contained in Z_k,

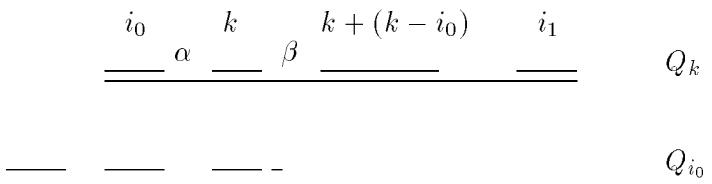
$$\gamma \geq C^{-1}(Q_{i_1}, f^{i_1}(V)) \geq C^{-1}(Q_k, f^k(V)) \geq \frac{2 \cdot 1}{4} = 0.5 \quad .$$

Hence

$$C^{-1}(Z_k, f^k(V)) \geq \frac{2(1+\gamma)}{2+1+1+\gamma} \geq \frac{2(1+0.5)}{4+0.5} = 2/3 > 0.6 \quad .$$

This completes the proof of Case II.

Case III. Q_{i_0} contains a neighbourhood of $f^k(V)$ and, moreover, H_k contains a neighbourhood of $f^{i_1}(V)$. The first assumption implies as before that $f^{k-i_0}\colon Q_{i_0} \to Q_k$ is orientation preserving. The last assumption implies that Q_{i_0} cannot have its right endpoint in some interval $f^j(V)$ with $j > i_1$ since then f^{k-i_0} cannot map $f^j(V)$ monotonically onto $f^{i_1}(V)$. Hence $f^{k+(k-i_0)}(V)$ is contained between $f^k(V)$ and $f^{i_1}(V)$. If Q_{i_0} contains points from $f^{k+(k-i_0)}(V)$, then Q_{i_0} contains this interval in its interior and therefore Q_k contains three intervals $f^j(V)$ to the right of $f^k(V)$ and so we get the required estimate. Therefore we can (and will) assume in the remainder of the proof of Case III that Q_{i_0} is to the left of $f^{k+(k-i_0)}(V)$.



Case III.

Let $\alpha > 0$ be so that the gap $(f^{i_0}(V), f^k(V))$ has size $\alpha|f^{i_0}(V)|$. Define β so that $(f^k(V), f^{k+(k-i_0)}(V))$ has size $\beta|f^k(V)|$. The gap corresponding to α is mapped to the gap corresponding to β by f^{k-i_0}. Defining $W_k = [f^{i_0}(V), f^{k+(k-i_0)}(V))$ and $W_1 \subset H_1$ so that $f^{k-1}(W_1) = W_k$, we have that

$$\alpha \geq C^{-1}(W_{i_0}, f^{i_0}(V)) \geq C^{-1}(W_k, f^k(V)) \geq \frac{(1+\alpha)\beta}{2+\alpha+\beta}.$$

This means that

$$\alpha \geq \sqrt{1+\beta} - 1.$$

Now let σ be so that $|f^{k+(k-i_0)}(V)| = \sigma|f^k(V)|$; one has $\sigma \geq 1$. Since one component of $Q_{i_0} \setminus f^k(V)$ is contained in the gap corresponding to β, and using the definition of σ, we have that

$$\beta \geq C^{-1}(Q_{i_0}, f^k(V)) \geq C^{-1}(Q_k, f^{k+(k-i_0)}(V)) \geq \frac{(2+\alpha+\beta)1}{\sigma(3+\sigma+\alpha+\beta)}.$$

This means that

$$\beta^2 + \beta(3+\sigma+\alpha-1/\sigma) \geq 2/\sigma + \alpha/\sigma. \tag{7.2}$$

Our aim is to prove that

$$C^{-1}(Q_k, f^k(V)) \geq \frac{(1+\alpha)(\beta+\sigma+1)}{3+\alpha+\beta+\sigma} \geq 0.6 \quad .$$

If $\sigma \geq 2$ or if $\alpha \geq 1/3$ then this holds. So assume that $1 \leq \sigma \leq 2$ and that $\alpha \leq 1/3$. If $\sigma \in [3/2, 2]$ then $3+\sigma+\alpha-1/\sigma \leq 5$ and so (7.2) implies that $\beta^2 + 5\beta \geq 1$, i.e., $\beta \geq 0.19$. Hence $\alpha \geq \sqrt{1.19} - 1 \geq 0.09$ and hence

$$\frac{(1+\alpha)(\beta+\sigma+1)}{3+\alpha+\beta+\sigma} \geq \frac{1.09 \times 2.69}{4.78} > 0.6 \quad .$$

If $\sigma \in [5/4, 3/2]$ then $3+\sigma+\alpha-1/\sigma \leq 4.2$ and therefore $\beta^2 + 4.2\beta \geq 4/3$ and so $\beta \geq 0.29$. Therefore, $\alpha \geq \sqrt{1.29} - 1 \geq 0.13$ and

$$\frac{(1+\alpha)(\beta+\sigma+1)}{3+\alpha+\beta+\sigma} \geq \frac{1.13 \times 2.54}{4.67} > 0.6 \quad .$$

If $\sigma \in [1, 5/4]$ then $\beta^2 + 3.8\beta \geq 8/5$ and so $\beta \geq 0.38$. Therefore, $\alpha \geq \sqrt{1.38} - 1 \geq 0.17$ and

$$\frac{(1+\alpha)(\beta+\sigma+1)}{3+\alpha+\beta+\sigma} \geq \frac{1.17 \times 2.38}{4.55} > 0.6 \quad .$$

Thus we get the required estimate in each case. This completes the proof of this lemma. □

The above lemma allows us to show that the interval $L = f^s(l)$ from Lemma 7.2 is not too short compared to $f(V)$:

Lemma 7.4. *Assume that $U = V$ has period s, that f is not renormalizable of period $s/2$ and that $f^s|U$ has no attracting fixed point. As in Lemma 7.2, let l be one of the two maximal intervals for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$. (If we take the interval which is outside $[c_1, c_2]$ then it is equal to the interval l from Lemma 6.1.) Write $L = f^s(l)$. Then*

$$|L| \geq 0.6 \cdot |f(V)|$$

and $l \cap \omega(c) = \emptyset$.

Proof: From Lemma 7.2, $L = f^s(l)$ contains at least two intervals of the form

$$f^2(V), \ldots, f^s(V).$$

That $l \cap \omega(c) = \emptyset$ is already proved in Lemma 7.2. We shall consider the disjoint intervals $f(V), \dots, f^s(V)$. Of course, $f(V)$ and $f^2(V)$ have just one neighbour in this collection, and all other have two.

First consider the case that $f(V)$ is shorter than its neighbour. Because $f^s(l)$ contains at least this neighbour, one gets $|f^s(l)| \geq |f(V)|$ which gives the required estimate. So we may assume that $f(V)$ is longer than its neighbour.

Similarly, let us consider the case that $f^2(V)$ is shorter than its nearest neighbour $f^j(V)$. Then let G be the interval containing $f(V)$ which is mapped diffeomorphically onto $[f^2(V), f^j(V)]$. Because $z \mapsto |Df(z)|$ is monotone on G and takes its maximum on $f(V)$ it follows that $G \setminus f(V)$ is longer than $f(V)$. Now since $f^s(l)$ contains a neighbour, it certainly contains $G \setminus f(V)$. In particular, as in the previous case we get $|f^s(l)| \geq |f(V)|$ and the proof is complete in this situation.

So we may assume that $f(V)$ and $f^2(V)$ are both larger than their neighbours. This implies that, as in the previous lemma, there exists a maximal integer $k > 2$ with $k \leq s$ and such that $f^k(V)$ is shorter than its two neighbours from the collection $f(V), \dots, f^{k-1}(V)$. As before, $Q_k = [f^{i_0}(V), f^{i_1}(V)] \supset f^k(V)$ contains only intervals of the form $f^j(V)$ with $k < j \leq s$ which are all longer than the interval $f^k(V)$.

Let $H_1 \supset f(V)$ be the maximal monotone interval and, as in the previous lemma, let Z_1 be the maximal interval in H_1 such that $Z_k = f^{k-1}(Z_1)$ is of the form $[f^{m_1}(V), f^{m_2}(V)]$ and so that each component of $Z_k \setminus Q_k$ contains at most one interval of the form $f^j(V)$ with $k < j \leq s$. Write

$$H_k = f^{k-1}(H_1).$$

Let $Z_{1,-}$ and $Z_{1,+}$ be the components of $Z_1 \setminus f(V)$ and for simplicity take $Z_{1,+}$ be the component which lies on the same side of $f(V)$ as L (so it lies in $[c_1, c_2]$). In order to be definite, assume that $f^{i_0}(V)$ lies to the left of $f^{i_1}(V)$ (this is just a convention) and label i_0 and i_1 so that $f^{k-1}(Z_{1,+})$ contains $f^{i_1}(V)$. If $L \supset Z_{1,+}$ then we have that

$$\frac{|L|}{|f(V)|} \geq \frac{|Z_{1,+}|}{|f(V)|} \geq C^{-1}(Z_1, f(V)) \geq C^{-1}(Z_k, f^k(V)) \geq 0.6,$$

where in the last inequality we used the previous lemma.

Therefore we may assume that L is a proper subset of $Z_{1,+} \subset H_1$. Hence $f^{k-1}|L$ is monotone. By Lemma 7.2, L and therefore $f^{k-1}(L)$ contains at least two neighbours $f^j(V)$ with $k < j \leq s$ of $f^k(V)$ (on the same side as $f^{i_1}(V)$). These intervals $f^j(V)$ are to the left of $f^{i_1}(V)$ because otherwise $f^{k-1}(L) \subset H_k$ would contain an interval $f^j(V)$ to the right of $f^{i_1}(V)$ and so $f^{k-1}(L) \supset Z_{k,+}$ and $L \supset Z_{1,+}$, a contradiction. If $f^{k-1}(L)$ contains three or more intervals $f^j(V)$ then $C^{-1}(W_k, f^k(V)) \geq 4/5 > 0.6$ and we are done. Here $W_1 = Q_{1,-} \cup f(V) \cup L$ and $W_j = f^{j-1}(W)$. So we may assume that

$f^{k-1}(L)$ contains precisely two neighbours $f^{j_1}(V)$ and $f^{j_2}(V)$ and assume for simplicity that $f^{j_1}(V)$ is to the left of $f^{j_2}(V)$. Since L contains at least two intervals of the form $f^j(V)$, this implies that we are in the situation that L contains precisely two intervals of the form $f^j(V)$. It suffices to show that

$$C^{-1}(W_k, f^k(V)) \geq 0.6.$$

We have that $f^{i_0}(V) \subset f^{i_0}(W_{1,-})$ lies to the left of $f^k(V)$. There are two remaining cases to consider.

Case 1. W_{i_0} lies to the left of $f^k(V)$. In this case choose $\alpha > 0$ so that the gap between $f^{i_0}(V)$ and $f^k(V)$ has size $\alpha|f^{i_0}(V)|$. Then

$$\alpha \geq C^{-1}(W_{i_0}, f^{i_0}(V)) \geq C^{-1}(W_k, f^k(V)) \geq \frac{(1+\alpha)2}{4+\alpha}$$

and this implies that $\alpha \geq 1/2$ and that $C^{-1}(W_k, f^k(V)) \geq 3/5 = 0.6$.

Case 2. W_{i_0} contains at least some points of $f^k(V)$. Then, as before, $f^{k-i_0}\colon W_{i_0} \to W_k$ is order preserving. If the gap between $f^{i_0}(V)$ and $f^k(V)$ contains one of the (two) intervals $f^j(V)$ inside $f^{i_0-1}(L)$ then define $\hat{W}_k = [f^{i_0}(V), f^{j_1}(V)]$, $\hat{W}_1 \subset H_1$ so that $f^{k-1}(\hat{W}_1) = \hat{W}_k$ and we get that

$$\alpha \geq C^{-1}(\hat{W}_{i_0}, f^{i_0}(V)) \geq C^{-1}(\hat{W}_k, f^k(V)) \geq \frac{(1+\alpha)1}{3+\alpha},$$

i.e., $\alpha \geq 1/3$ and $C^{-1}(W_k, f^k(V)) \geq (1+\alpha)2/(4+\alpha) > 0.6$. So assume that the gap between $f^{i_0}(V)$ and $f^k(V)$ contains no interval of the form $f^j(V)$. Since, by assumption, each of the intervals $L, \dots, f^{k-1}(L)$ contains precisely two intervals of the form $f^m(V)$, the only remaining possibility is that f^{k-i_0} maps $f^k(V)$ to $f^{j_1}(V)$ and $f^{j_1}(V)$ to $f^{j_2}(V)$. (Here we use that there is no interval of the form $f^j(V)$ between $f^{i_0}(V)$ and $f^k(V)$ and that $f^{k-1}(L)$ contains precisely two intervals of the form $f^j(V)$.) Hence

$$j_1 = k + (k - i_0) \text{ and } j_2 = k + 2(k - i_0).$$

But now we use a more precise statement from Lemma 7.2: the intervals $f^j(V)$ in L are of the form $f^{i+2}(V)$ and $f^{s-i}(V)$. Hence

$$j_1 = (i+2) + (k-1) \text{ and } j_2 = (s-i) + (k-1)$$

for some i. Writing $r = k - i_0$, combining all this gives

$$r = j_1 - k = i + 1 \text{ and } 2r = j_2 - k = s - i - 1.$$

Hence $3r = s$ and L contains the intervals $f^{r+1}(V)$ and $f^{2r+1}(V)$. But then the map $f^{k-1} = f^{r+i_0-1}$ cannot be monotone on $f^{2r+1}(V)$ (because $3r = s$ and $f^s|f(V)$ is not monotone). Therefore we get a contradiction with the assumption that $f^{k-1}|L$ is monotone. □

Finally, we shall also give in this section an estimate for the case that a map is renormalizable of period s and also of period $s/2$ (this case was not covered by the previous lemma).

LEMMA 7.5. *Assume that $U = V$ has period $s \geq 4$, that f* is *renormalizable of period $s/2$ and that $f^{s/2}|U$ has no attracting fixed point. As in Lemma 7.2, let l be the maximal interval for which $f^s|l$ is monotone, which has a unique common point with $\hat{U}$ and which is outside $[c_1, c_2]$. Write $L = f^s(l)$. Then*

$$|L| \geq 0.8 \cdot |f(V)|.$$

Proof: Let $R := [U, f^{s/2}(U)]$. From Lemma 7.2 we have that L contains two distinct intervals of the form $f^i(U)$ with $i \neq 1, s/2+1$. Take k in $\{0, \ldots, r-1\}$ corresponding to the shortest interval $f^k(R)$. If $f(R)$ is shorter than its neighbour then, because L contains this neighbour, we are done. If $f^2(R)$ is shorter than its neighbour then just like in the proof of Lemma 7.4, $f(R)$ is also shorter than its neighbour and we again are done. So we may assume that $k > 2$. There are two cases: $|f^k(U)| \leq |f^{k+r}(U)|$ or $|f^{k+r}(U)| \leq |f^k(U)|$. In the former case, let $m = k$ and in the latter case we take $m = k + r$. If $m = 1, 2$ then one has, just like in the proof of Lemma 7.4, that

$$|f^{r+1}(U)| \geq |f(U)|. \tag{7.3}$$

(Note that $f^{r+1}(U)$ is the nearest neighbour of $f(U)$ from the collection of disjoint intervals $U, f(U), \ldots, f^{s-1}(U)$ because by assumption f is also renormalizable of period $r = s/2$.) So assume that $m > 2$.

Let Q_k be the smallest interval containing $f^k(R)$ and its nearest neighbours from the collection $R, \ldots, f^{r-1}(R)$. If $m = k$ then $|f^k(U)| \leq |f^{k+r}(U)|$ and since these two intervals are both contained in $f^k(R)$, $|f^k(U)| \leq (1/2)|f^k(R)|$. Hence

$$C^{-1}(Q_k, f^k(U)) \geq \frac{2 \cdot 2}{1 + 2 + 2} = 0.8$$

(in fact, a stronger inequality is true). On the other hand, if $m = k + r$ then let $Q_{k+r} \supset f^{k+r}(U)$ be the smallest interval containing $f^k(U)$ on one side of $f^{k+r}(U)$ and the nearest neighbour $f^{j_1}(R)$ from the collection $R, \ldots, f^{r-1}(R)$ on the other side of $f^k(R) \supset f^{k+r}(U)$. As before, there is an interval $\tilde{Q}_k \supset f^k(U)$ which is mapped diffeomorphically to Q_{k+r} by f^r. In fact, one component of $\tilde{Q}_k \setminus f^k(U)$ is contained in $f^{k+r}(U)$ and the other one is contained in the gap G between $f^k(U)$ and its nearest neighbour $f^{j_2}(R)$. We have $C^{-1}(Q_{k+r}, f^{k+r}(U)) \geq \frac{2 \cdot 1}{2+1+1} = 0.5$. Hence both components of $\tilde{Q}_k \setminus f^k(U)$ are at least $1/2$ as long as $f^k(U)$. Define α by $|G| = \alpha |f^k(U)|$. Since G contains one of the components of $\tilde{Q}_k \setminus f^k(U)$, $\alpha \geq 0.5$. In particular we also have $|G|, |f^{k+r}(U)| \geq (1/2)|f^k(U)|$ and since $f^k(R)$ was the shortest interval, we also have that $|f^{j_1}(R)|, |f^{j_2}(R)| \geq |f^k(R)| \geq (3/2)|f^k(U)|$. Now consider $Q_k \supset f^k(U)$. We have

$$C^{-1}(Q_k, f^k(U)) \geq \frac{(|f^{j_2}(R)| + |G|)(|f^{j_1}(R)| + |f^{k+r}(U)|}{(|f^{j_2}(R)| + |G| + |f^k(U)| + |f^{k+r}(U)| + |f^{j_1}(R)|)|f^k(U)|}$$
$$\geq \frac{2 \cdot 2}{1+2+2} = 0.8.$$

Hence the interval $Q_1 \supset \hat{U}$ for which $f^{k-1}(Q_1) = Q_k$ satisfies $C^{-1}(Q_1, f(U)) \geq 0.8$. and in particular, components of $Q_1 \setminus f(U)$ have length $\geq (0.8)|f(U)|$. From the last part of Lemma 7.2 it follows that $L = f^s(l)$ contains a neighbour of $f(R)$ and hence L contains a component of $Q_1 \setminus f(U)$. □

8. Lower bounds for 'space' when the return is high

Let $\hat{V} \supset f(V)$ be the interval which is mapped monotonically onto W by $f^{s'-1}$. Let s'' be the smallest integer such that $f^{s''-1}(f(V)) \ni c$. In this section we assume that R_V has a high return. Hence $f^{s-1}(f(U)) \ni c$ and so $s'' \leq s$.

PROPOSITION 8.1. *Assume that R_V has a high return. Let T_0 be the smallest interval containing $f(V)$ and another disjoint interval from the collection $f^2(V), \ldots, f^{s''}(V)$. Write $L_0 = T_0 \setminus f(V)$. Then*

$$|L_0| \geq \frac{1}{3}|f(V)|.$$

Moreover, if we define l to be one of the two maximal intervals outside $\hat{U}$ for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$, then $f^s(l)$ contains L_0.

For the proof of this proposition we need two lemmas.

LEMMA 8.2. *Assume that R_V has a high return. Let $2 \leq k < s''$ and let H_1 be the maximal interval containing $f(V)$ such that $f^k|H_1$ is monotone. Then $f^k(H_1)$ contains $f^k(f(V))$ and on each side of this interval also an interval of the form $f^i(V)$ with $i \leq k$.*

Proof: Let $H_{1,-}, H_{1,+}$ be the components of $H_1 \setminus f(V)$. From the maximality of H_1 it follows that there exists $i' < k$ such that $f^{i'}(H_{1,+})$ contains c. Since $i' < k \leq s'' - 1 \leq s - 1$, the interval $f^{i'+1}(U)$ (which is contained in $f^{i'+1}(V) \subset f^{i'}(H))$ is outside V. Since $f^{i'+1}(U)$ and $f^{i'+1}(V)$ have $f^{i'+1}(c)$ as one common endpoint and the other endpoint of $f^{i'+1}(V)$ is certainly outside V (because the endpoints of V are nice), it follows that $f^{i'+1}(V)$ is outside V. Hence $f^{i'}(H_{1,+})$ contains one component of $V \setminus \{c\}$. It follows that $f^k(H_{1,+})$ contains $f^{k-i'}(V)$. □

Let $\hat{U} \supset f(U)$ the interval which is mapped monotonically onto V by f^{s-1}. In the next lemma we prove the second part of the statement of Proposition 8.1.

LEMMA 8.3. *Let l be one of the two maximal intervals outside $\hat{U}$ for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$. Then $L := f^s(l)$ contains at least one interval of the form $f^i(V)$, $1 \le i \le s''$ which is disjoint from $f^s(\hat{U}) = f(V)$.*

Proof: By maximality of l there is i with $0 < i < s$ such that $f^i(l)$ contains c in its boundary. Choose i maximal with this property. Since $f^i(\hat{U})$ is outside V it follows that $f^i(l)$ contains one component V_+ of $V \setminus \{c\}$. Hence $f^s(l)$ contains $f^{s-i}(V)$ (and also a point of $f^s(\hat{U}) = f(V)$ in its boundary). Since $f^{s''-1}(f(V))$ contains c in its interior, and since by definition $f^s|l$ is monotone, it follows that $s-i < s''$. Hence $f^{s-1}(l)$ contains one of the intervals $f(V), \ldots, f^{s''-1}(V)$ which is disjoint from $f(V)$. □

Proof of Proposition 8.1 Let L_0 be defined as in the proposition and consider the intervals $f(V), \ldots, f^{s''}(V)$. Since s'' might be larger than s', we cannot be sure that these intervals are disjoint. Even so, there exists k such that

$$|f^k(V)| \le |f^i(V)| \text{ for each } i = 1, 2, \ldots, s''.$$

If $k = 1$ then in particular the length $f(V)$ is at most equal to the length of its nearest disjoint neighbour from the collection $f(V), \ldots, f^{s''}(V)$. Since L_0 contains an interval $f^i(V)$, $1 \le i \le s''$ which is disjoint from $f(V)$ it follows that $|L_0| \ge |f(V)|$. If $k = 2$ then a similar argument applies: again the length $f^2(V)$ is at most equal to the length of its nearest disjoint neighbour from the collection $f(V), \ldots, f^{s''}(V)$. Since L_0 contains an interval $f^i(V)$, $1 \le i \le s''$ which is disjoint from $f(V)$, and since $z \mapsto |Df(z)|$ increases monotonically as z moves away from $c = 0$, it follows again that $|L_0| \ge |f(V)|$. So we may assume that $k > 2$. Because of Lemma 8.2 we can find an interval Z_1 around $\hat{V}$ such that $f^{k-1}|Z_1$ is monotone and so that $Z_k = f^{k-1}(Z_1)$ contains on each side of $f^k(V)$ an interval of the form $f^i(V)$ with $i < k$ (and which is disjoint with $f^k(V)$). Let $Z_{1,\pm}$ be two components of $Z_1 \setminus \hat{U}$ marked so that $Z_{1,+}$ intersects L_0. If $Z_{1,+} \subset L_0$, then

$$\frac{|L_0|}{|f(V)|} \ge \frac{|Z_{1,+}|}{|f(V)|} \ge C^{-1}(Z_1, f(V)) \ge C^{-1}(Z_k, f^k(V)) \ge \frac{1}{3}$$

because Z_k contains on each side of $f^k(V)$ an interval $f^i(V)$ with $i \le s''$ which is at least as long as $f^k(V)$ and because of the choice of k.

It remains to consider the case when $L_0 \subset Z_{1,+}$, i.e., when f^{k-1} is monotone on L_0. As we have seen above, L_0 contains some $f^j(V)$ with $j \le s''$. Hence, the interval $f^{j+k-1}(V)$ lies in Z_k and $k \le j + k - 1 \le s''$. By choice of k, the interval $f^{j+k-1}(V)$ is longer than $f^k(V)$. Hence $|f^{k-1}(L_0)| \ge |f^k(V)|$. Moreover, again by Lemma 8.2, $f^{k-1}(Z_{1,-})$ also contains an interval of the

$f^i(V)$ with $i \le s''$ and again by the choice of k this implies that $|f^{k-1}(Z_{1,-})| \ge |f^k(V)|$. Let $W_1 = Z_{1,-} \cup \hat{U} \cup L_0$ and $W_k = f^{k-1}(W_1)$. Then both components of $W_k \setminus f^k(V)$ are at least as long as $f^k(V)$ and therefore

$$\frac{|L_0|}{|f(V)|} \ge C^{-1}(W_1, f(V)) \ge C^{-1}(f^{k-1}(W_1), f^k(V)) \ge \frac{1}{3}.$$

This completes the proof of this proposition. □

9. The proof of Theorem A when $\ell \ge 4$ and in the 'doubling case' when $\ell = 2$

In this section we consider an infinitely renormalizable $f(z) = z^\ell + c_1$ map with $\ell \ge 4$. Such a map has renormalizations of period $s(n)$ where $s(n+1) = s(n) \cdot a(n)$ where $a(n) \ge 2$ is an integer. If $a(n) = 2$ then the method in this section also applies for $\ell = 2$. So for all n we have a periodic intervals $U_n = V_n$ of period $s(n)$. Because f has no wandering intervals [MS], it follows that $|U_n| \to 0$. Write $U_n = V_n = [-u_n, u_n]$ and let $s(n)$ be the period (note that for the map we consider $\tau(z) = -z$). For convenience, let us for the moment suppress the subscript n and write s for $s(n)$. Let $\hat{U} \supset f(U)$ be the interval from before and consider the diffeomorphism $f^{s-1}: \hat{U} \to V$. Let F be the inverse of this map. Since F is a diffeomorphism, it induces a univalent map

$$F: \mathbb{C}_V \to \mathbb{C}_{\hat{U}}.$$

Hence $F(D_*(V)) \subset D_*(\hat{U})$. Now we use that the space from Lemma 6.1 (outside $[c_1, c_2]$) is at least 0.6, because of the estimates of Lemma 7.4 and 7.5. If f is also renormalizable of period $s/2$ then we have even space 0.8. Hence, see the estimates below Lemma 6.1, one gets that $f^{-1}(D_*(\hat{U})) \subset D_*(V)$, i.e.,

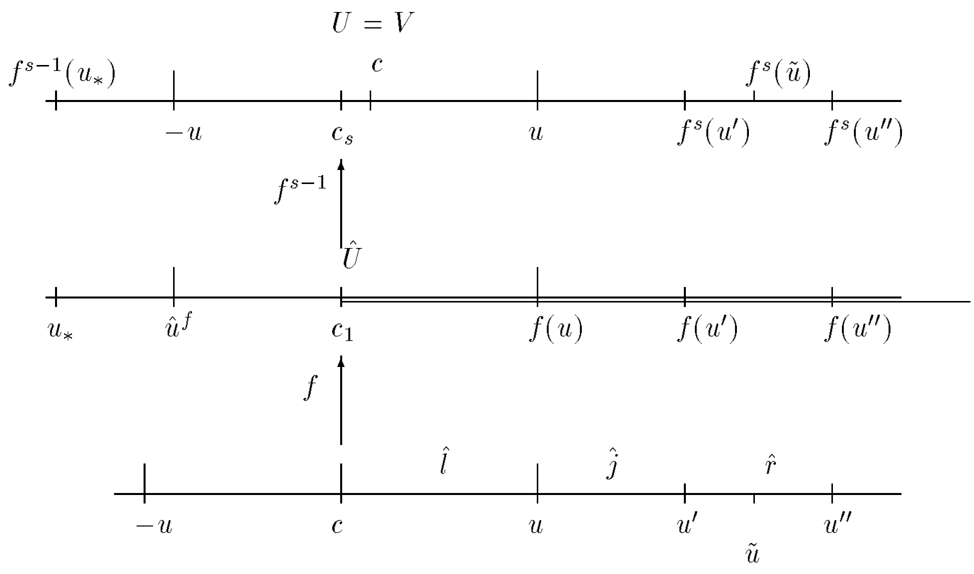
$$f^{-1} \circ F(D_*(V)) \subset D_*(V),$$

proved $\ell \ge 4$ (or $\ell \ge 2$ if f is also renormalizable of period $s/2$). From this we get that

$$f^s: D'_*(V) \to D_*(V),$$

where $D'_*(V) = f^{-1} \circ F(D_*(V))$, is a proper degree ℓ map. This is still not a polynomial-like mapping since $\partial D'_*(V) \cap \partial D_*(V)$ is not empty, so these regions intersect in the repelling periodic point u (with $f^s(u) = u$) rotated by the angles $2\pi k/\ell$, $k = 0, 1, ..., \ell - 1$. Of course, this problem can be easily solved, see Proposition 5.1.

In this way we will get a polynomial-like mapping $f^s: \Omega_n^0 \to \Omega_n$. In fact, we even want a lower bound for the modulus of the annulus $\Omega_n \setminus \Omega_n^0$. That such a lower bound exists, is not surprising since $|Df^s(u)| - 1$ is bounded from below, see Theorem B in Chapter IV of [MS] or Theorem B in [MMS]. In the Lemma below we shall use related estimates to give lower bounds for the modulus of the annulus $\Omega_n \setminus \Omega'_n$.



LEMMA 9.1. *There are universal constants $C_0, C_1, C_2 > 0$ with the following property. Let $T_1 \supset f(U)$ be the maximal interval on which f^{s-1} is monotone. Then,*

(9.1) *each component of $f^{s-1}(T_1) \setminus V$ has length $\geq \frac{C_0}{\ell}|u-c|$.*

Let T be the component of $f^{-1}(T_1) \setminus \{c\}$ which contains u. Then there exists $\tilde{u} \in T$ such that

$$\frac{C_1/2}{\ell}|u-c| \leq |f^s(\tilde{u}) - u| \leq \frac{C_1}{\ell}|u-c| \tag{9.2}$$

such that

$$|f^s(\tilde{u}) - c| \geq (1 + C_2/\ell) \cdot |\tilde{u} - c|. \tag{9.3}$$

When $\ell \geq 4$ (or $\ell = 2$, if f is $s/2$-renormalizable), there exists also $u_* \in T_1$ such that

$$f^{s-1}(u_*) = -f^s(\tilde{u}) \text{ and } |u_* - c_1| < |f(u) - c_1| \tag{9.4}$$

Proof: Let L_1 be a maximal interval on which f^s is monotone with a unique common point with $f(U)$. This interval L_1 plays the role of the interval l in Lemmas 7.4 and in 7.5. One has

$$|f^s(L_1) \setminus f(V)| \geq 0.6 \cdot |f(V)| = 0.6 \cdot |c_1 - f(u)|. \tag{9.5}$$

Since, $f(z) = (z-c)^\ell + c_1$ (where we usually take $c=0$), the first inequality (9.1) follows because there exists C_0 such that

$$1.6^{1/\ell} \geq 1 + \frac{C_0}{\ell}. \tag{9.6}$$

So we can take $u', u'' \in T$ with

$$|f^s(u') - u| = \frac{C_1}{2\ell}|c-u| \text{ and } |f^s(u'') - u| = \frac{C_1}{\ell}|c-u| \tag{9.7}$$

provided $C_1 < C_0$. We need to show that there exists $\tilde{u} \in [u', u'']$ for which (9.3) holds.

Let us first show that this would complete the proof, i.e., that (9.4) then automatically follows. Indeed, because of (9.1) one can take $u_* \in T_1$ so that $f^{s-1}(u_*) = -f^s(\tilde{u})$. By Lemma 6.1 there exists a universal constant $K_* < 1$ such that provided $\ell \geq 4$,

$$|\hat{u}^f - c_1| < K_*|f(u) - c_1|. \tag{9.8}$$

Let us show that, provided we choose C_1 sufficiently small, in (9.7), the inequality in (9.4) holds. For this define $j' = [u_*, \hat{u}^f]$ and t' the interval between $f(u)$ and the endpoint of T_1 outside $[c_1, c_2]$. Let l', r' be the components of $t' \setminus j'$ (with l' the component outside $[c_1, c_2]$. One has

(9.9)

$$\frac{|f(u) - \hat{u}^f|}{|u_* - \hat{u}^f|} = \frac{|r'|}{|j'|} \geq C^{-1}(t', j') \geq C^{-1}(f^{s-1}(t'), f^{s-1}(j')) \geq \frac{|f^{s-1}(l')|}{|f^{s-1}(j')|}\frac{|f^{s-1}(r')|}{|f^{s-1}(t')|}.$$

The first ratio in the last term will tend to infinity if we choose C_1 small (because $|f^{s-1}(l' \cup j')|$ has size $\geq 0.6^{1/\ell} \cdot |V|$ and $|f^{s-1}(j')|$ has size $(C_1/\ell) \cdot |V|$). The second factor in the last term of (9.9) is of order one. Hence, (9.9) implies that $|u_* - \hat{u}^f|/|f(u) - \hat{u}^f|$ goes to zero provided C_1 goes to zero. Combined with (9.8) this gives $|u_* - c_1| < |f(u) - c_1|$ for some universal choice of $C_1 \in (0,1)$.

Thus it remains to prove (9.3). Let $\hat{l} = [c, u]$, $\hat{j} = [u, u']$, $\hat{r} = [u', u'']$ and $\hat{t} = \hat{l} \cup \hat{j} \cup \hat{r}$. Let us compare the size of $\hat{j} \cup \hat{r}$ with that of $\hat{l}$. Define $\tau = |\hat{j}|/|\hat{l}|$. If $\tau \leq C_1/(4 \cdot \ell)$, then $|f^s(u') - c|/|u' - c| = (1 + C_1/(2\ell))/(1+\tau) \geq (1 + C_1/2\ell)/(1 + C_1/(4\ell)) \geq 1 + C_1/(8\ell)$, and (9.3) follows with $\tilde{u} = u'$ with $C_2 = C_1/8$. So we may assume that

$$\tau = \frac{|\hat{j}|}{|\hat{l}|} = \frac{|u - u'|}{|u - c|} \geq C_1/(4\ell). \tag{9.10}$$

Let us show that it suffices to show that there exists a universal constant C_3 and some $\tilde{u} \in \hat{r} = [u', u'']$ for which

$$|f^s(u) - f^s(\tilde{u})|/|u - \tilde{u}| \geq (1 + C_3). \tag{9.11}$$

Indeed, then $|\tilde{u} - f^s(\tilde{u})| \geq C_3 \cdot |u - \tilde{u}|$ and

$$\frac{|c - f^s(\tilde{u})|}{|c - \tilde{u}|} \geq 1 + (C_3)(|u - \tilde{u}|/|c - \tilde{u}|) \geq 1 + C_3\tau/(1+\tau) \geq 1 + \frac{C_1 C_3}{8\ell},$$

because of (9.10) and because $C_1 \in (0,1)$. Therefore (9.3) follows and the proof of the lemma is complete once we have shown that (9.11) holds.

We use the following observation. By the Mean Value Theorem,

$$|Df^s(\xi)| \geq \frac{|f^s(\hat{r})|}{|\hat{r}|} \text{ for some } \xi \in \hat{r} = [u', u''].$$

Using the cross-ratio inequality

$$\frac{(|f^s(u) - f^s(\xi)|/|u-\xi|)^2}{|Df^s(u)||Df^s(\xi)|} \geq 1$$

(this is just the cross-ratio inequality $C(f^s t, f^s j) \geq C(t,j)$ where we let t shrink to $j = [u, \xi]$), and using that u is a repelling periodic point,

$$\frac{|f^s(u) - f^s(\xi)|}{|u - \xi|} \geq \sqrt{|Df^s(u)||Df^s(\xi)|} \geq \sqrt{1 \cdot (\frac{|f^s(\hat{r})|}{|\hat{r}|}}). \tag{9.12}$$

In particular, we may assume

$$\frac{|f^s(\hat{r})|}{|\hat{r}|} \leq 2,$$

because otherwise (9.11) holds with $\tilde{u} = \xi$ and where we take $C_3 = \sqrt{2} - 1$. The latter inequality, together with (9.7) gives us

$$C_1/\ell \geq \frac{|\hat{r}|}{|\hat{l}|} \geq C_1/4\ell. \tag{9.13}$$

Moreover, we have

$$\frac{4\ell}{C_1} \geq \frac{|\hat{l}|}{|\hat{j}|} \geq \frac{|\hat{l}|}{|f^s(\hat{j})|} = \frac{2\ell}{C_1}, \tag{9.14}$$

because of (9.7) and (9.10) and because f^s has no attracting fixed points on $\hat{j}$.

Now, given intervals $j \subset t$ for which $t \setminus j$ has components l, r, define the cross-ratio operator

$$B(t,j) = \frac{|t||j|}{|l \cup j||r \cup j|}.$$

As with the cross-ratio operator C we defined before, one has $B(ft, fj) > B(t,j)$ if $f|t$ is monotone and has negative Schwarzian derivative and $t \setminus j$ consists of two non-degenerate intervals. Because of (9.13) and (9.14), and using the expression $f(z) = (z - \ell)^\ell + c_1$, one can easily check that

$$\frac{B(f(\hat{t}), f(\hat{j}))}{B(\hat{t}, \hat{j})} \geq 1 + C_5,$$

where $C_5 > 0$ is a universal constant.

To see this, notice that the inequalities in (9.13) and (9.14) give some positive constant $C_5 = C_5(\ell)$ for each particular ℓ, while for $\ell \to \infty$ direct calculations show that $C_5(\ell)$ stays away from zero.

It follows that

$$\frac{B(f^s(\hat{t}), f^s(\hat{j}))}{B(\hat{t}, \hat{j})} \geq 1 + C_5. \tag{9.15}$$

Let us prove (9.11). Since f^s is a repelling fixed point at u and $\hat{j}$ has u in its boundary, we have that $|f^s(\hat{j} \cup \hat{r})|/|\hat{j} \cup \hat{r}| \geq 1$. Hence

$$\frac{|f^s(\hat{j})|/|\hat{j}|}{|f^s(\hat{j} \cup \hat{r})|/|\hat{j} \cup \hat{r}|} \leq |f^s(\hat{j})|/|\hat{j}|. \tag{9.16}$$

Now we also have

$$1 \leq \frac{|f^s(\hat{t})|}{|\hat{t}|}, \frac{|f^s(\hat{j} \cup \hat{l})|}{|\hat{j} \cup \hat{l}|} \tag{9.17}$$

and

$$\frac{|f^s(\hat{t})|}{|\hat{t}|} \leq \max\left(\frac{|f^s(\hat{j} \cup \hat{l})|}{|\hat{j} \cup \hat{l}|} \quad , \quad \frac{|f^s(\hat{r})|}{|\hat{r}|}\right). \tag{9.18}$$

Here (9.17) follows from the fact that $f^s|\hat{t}$ has no attracting fixed point (and u is its fixed point), and (9.18) is a general fact about mean slopes of a monotone map. We get from (9.17) and (9.18) that

$$\frac{|f^s(\hat{t})|/|\hat{t}|}{|f^s(\hat{j} \cup \hat{l})|/|\hat{j} \cup \hat{l}|} \leq \frac{|f^s(\hat{r})|}{|\hat{r}|}. \tag{9.19}$$

Combining (9.16) and (9.19) gives,

$$\frac{B(f^s(\hat{t}), f^s(\hat{j}))}{B(\hat{t}, \hat{j})} \leq |f^s(\hat{j})|/|\hat{j}| \cdot |f^s(\hat{r})|)/|\hat{r}|.$$

On the other hand, by (9.15) we get that the left hand side of this expression is bounded from below by $1 + C_5$. From this we get that either $|f^s(\hat{j})|/|\hat{j}|$ or $|f^s(\hat{r})|)/|\hat{r}|$ is bigger than $1 + C_5/4$. In the former case we obtain (9.11) with $\tilde{u} = u'$, and in the latter case, taking into account (9.12), we can set $\tilde{u} = \xi \in [u', u'']$. $\square$

Consider the (disc) domain

$$\Omega_n = D((-f^s(\tilde{u}), f^s(\tilde{u})) = D((f^{s-1}(u_*), f^s(\tilde{u})).$$

The inverse F of f^{s-1} extends in a univalent way to $\mathbb{C}_{[-f^s(\tilde{u}), f^s(\tilde{u})]}$ and therefore

$$F(D(-f^s(\tilde{u}), f^s(\tilde{u}))) \subset D_*(u_*, f(\tilde{u}))$$

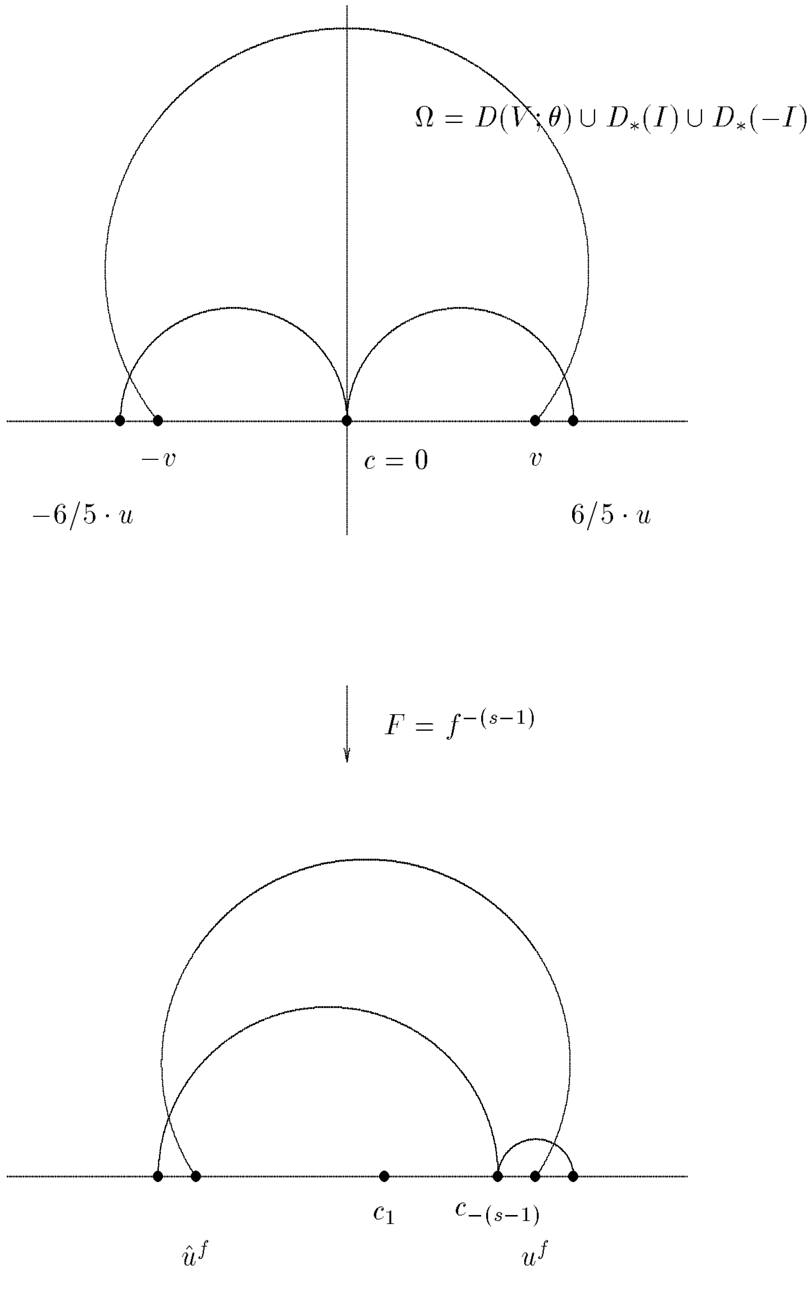
where we use that $f^{s-1}(u_*) = -f^s(\tilde{u})$. Because of (9.4) we get that

$$f^{-1}(D_*(u_*, f(\tilde{u}))) \subset D_*(-f^s(\tilde{u}), f^s(\tilde{u})) = \Omega_n.$$

From (9.3) the difference set is an annulus with a modulus $\geq C/\ell$ where C is some universal constant (not depending on the map f). In particular, writing $\Omega_n^0 = f^{-1} \circ F(\Omega_n)$, the map $f^{s(n)}\colon \Omega_n^0 \to \Omega_n$ is a polynomial-like mapping. Moreover, since f has no wandering intervals and $s(n) \to \infty$ we have that $|\Omega_n \cap \mathbb{R}| \to 0$. By construction the diameter of Ω_n is at most twice that of $V_n = [-u_n, u_n]$. By the previous lemma one also has that $\Omega_n^0 \cap \omega(c) \subset V_n$. This completes the proof of Theorem A for any $\ell \geq 4$ and in the 'doubling case' for $\ell = 2$. □

10. The proof of Theorem A when the degree is two

In this section we consider a renormalizable $f(z) = z^2 + c_1$ with a periodic interval V of the period s. As before, we may assume that R_V has a high return because otherwise f has a periodic attractor. Hence, because of Lemma 7.4, the space from Lemma 6.1 is at least 0.6. Note however, that the bound we obtain for $K_2^*(0.6)$ is equal to $1.19371... > 1$ and we cannot use the method of the previous section in the quadratic case. Therefore, we construct a domain of renormalization Ω for f^s which is different from $D_*(V)$, but with a diameter only depending on $|V|$, so that the renormalizations shrink to zero together with V.



As before, let $D(J;\theta)$ be the Poincaré neighbourhood of a real interval J, with external angle $\theta \in (0, \pi/2]$. We define the domain $\Omega = \Omega(\theta)$ as the

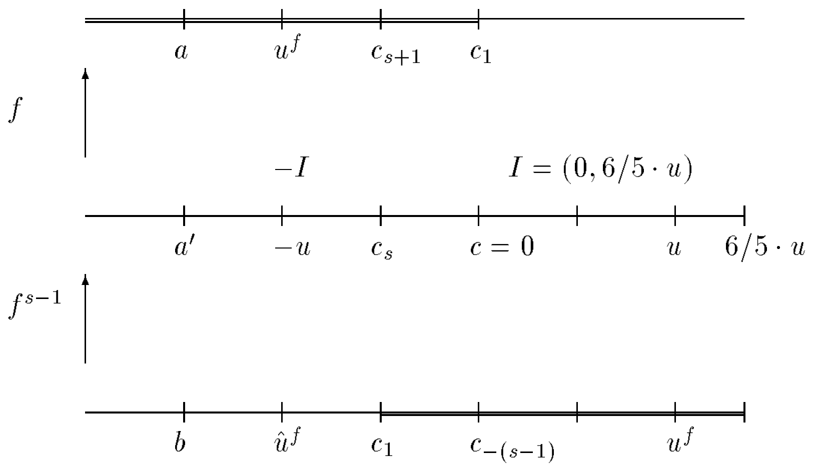
Poincare neighbourhood $D(V;\theta)$ of the periodic interval V (with the angle θ to be specified later on), united with two discs: $D_*(I)$ and $D_*(-I)$, where

$$I = (0, 6/5 \cdot u),$$

and u is the periodic endpoint of V (i.e., $f^s(u) = u$):

$$\Omega = D(V;\theta) \cup D_*(-I) \cup D_*(I),$$

see the figure above. We may and shall assume that $u > 0$. Since R_V has a high return, one has that $c_s \in [-u, 0]$.



The intervals $I, -I$.

Remark 10.1. *Consider the map $f^{s-1}\colon \hat{U} \to V$. As in Lemma 7.4, let l be the maximal interval with a unique common endpoint with $\hat{U}$ such that $f^{s-1}|l$ is monotone. From Lemma 7.4 we have that $|f^{s-1}(l)| \geq \sqrt{1+0.6} \cdot |V_+|$ where V_+ is one component of $V \setminus \{c\}$. The number $6/5$ is chosen for the following reason: $6/5 > K_2^*(0.6) = 1.19371..$, but $6/5 < \sqrt{1+0.6}$. Because of Lemma 7.4, the latter inequality shows that the pullback F of $f^{s-1}\colon \hat{U} \to V$ has a monotone extension to $\widetilde{I} = I \cup V \cup (-I)$. Let F also denote the extension of this pullback to $\mathbb{C}_{\widetilde{I}} = (\mathbb{C} \setminus \mathbb{R}) \cup \widetilde{I}$.*

Our aim is to prove that $f^{-1} \circ F(\Omega)$ is a proper domain inside of Ω. For this, it is enough to prove

Proposition 10.2. $f^{-1} \circ F(\partial\Omega) \subset \Omega$.

Proof: First of all, we observe that $f^{-1} \circ F(D_*(I))$ consists of two components D^1 and $-D^1$, where D^1 is chosen so that it contains the point u. Note that there exists an interval containing c_1 which is mapped diffeomorphically by f^{s-1} onto $\widetilde{I} = I \cup V \cup (-I)$. Therefore, $f^{-1} \circ F$ maps I homeomorphically into itself, because u is a repelling fixed point of f^s (here we use that f^s has negative Schwarzian derivative). Moreover, $c_s = f^{s-1}(c_1)$ is not in I and therefore $F(I)$ does not contain c_1. Hence, the corresponding branch of $f^{-1} \circ F$ maps $\mathbb{C}_I$ univalently into itself. Thus, D^1 is a proper domain in $D_*(I)$. By symmetry, we get also $-D^1 \subset D_*(-I)$. Thus we have shown that

$$f^{-1} \circ F(D_*(I)) \subset \Omega. \tag{10.1}$$

Next we want to show that the pullback of $D(V;\theta)$ is inside $\Omega(\theta)$ provided we choose θ conveniently. For this we shall use the following Lemma 15.1 from the Appendix:

LEMMA 10.3. *Denote $P_2(z) = z^2$. Let $K > 1$. There exists $\theta_0 = \theta_0(K) > 0$ such that, for all $\theta \in (0, \theta_0)$, the boundaries of $P_2(D((-1,1);\theta))$ and $D((-K,1);\theta)$ intersect each other at a point $Z(K,\theta)$ and at its complex conjugate. Furthermore,*

$$Z(K,\theta) \to K^2 \in \mathbb{R},$$

as $\theta \to 0$. Hence, the difference $\Delta_2(K,\theta) = D((-K,1);\theta) \setminus P_2(D((-1,1);\theta))$ tends to the interval $[1, K^2]$, as $\theta \to 0$.

Take in this lemma $K = K_2^*(0.6) = 1.19371....$ By the Schwarz contraction principle, see Lemma 4.1, $F(D(V;\theta))$ is contained in $D(\hat{I};\theta)$, where the interval $\hat{I} = (\hat{u}^f, u^f)$ is around c_1, with $|c_1 - \hat{u}^f| \le K|u^f - c_1|$. By rescaling and the above lemma, for all θ less than some positive absolute constant θ_1 the closure of the domain $F(D(V;\theta))$ is inside $f(\Omega(\theta))$. Here we have used that the lemma implies that $f^{-1}(F(D(V;\theta)))\setminus D(V;\theta)$ is inside of two domains, which converge to the intervals $\pm(u, K \cdot u)$ as $\theta \to 0$. Since $K < 6/5$, see Remark 10.1, this implies that this difference set is contained in $D_*(I) \cup D_*(-I)$ provided θ is small.

Thus,

$$f^{-1} \circ F(\partial D(V;\theta)) \subset \Omega(\theta). \tag{10.2}$$

as $\theta < \theta_1$.

To complete the proof, it remains to show that for all angles θ less than some other positive constant θ_2 the domain $F(D_*(-I))$ is contained properly inside $P_2(\Omega(\theta))$, where $P_2(z) = z^2$. Note that $P_2(-I) = P_2(I) = (c_1, a)$, where $|c_1 - a| = (6/5)^2|c_1 - u^f| < (1+0.6)\cdot|c_1 - u^f|$, i.e. $a \in L$. Hence, by Lemma 6.1, the interval $F(-I)$ is inside an interval (b, u^f), where

$$|b - c_1|/|u^f - c_1| = (|b - c_1|/|a - c_1|) \cdot (6/5)^2 < K_2^*(y_0) \cdot (6/5)^2 := K_1,$$

where, using the notation of Lemma 6.1, $y_0 = |a - c_1|/|T| = (6/5)^2/(1+0.6) \in (0.6, 1)$. It follows, $F(D_*(-I))$ is inside of the ball $D_*(b, u^f)$. By rescaling, we need to show that $P_2(D((-1,1);\theta))$ contains a fixed ball $D_*(-K_1, 1)$, if θ is small. It is a not difficult exercise. Hence

$$f^{-1} \circ F(D_*(-I)) \subset f^{-1}(D_*(b, u^f)) \subset D_*(U;\theta).$$

Together with (10.1) and (10.2) this implies Proposition 10.2. □

Let us now prove Theorem A for $\ell = 2$. For this we make use of Lemma 9.1 (or rather its proof): for every small enough $\varepsilon > 0$ there exists a constant $C, 0 < C < 1$ (depending only on ε), and a point $\widetilde{u}$ in the interval $I \setminus V$ such that the image $f^s(\widetilde{u})$ lies in the ε-neighbourhood of the point $\frac{6}{5}u$ and

$$|\widetilde{u} - c| \le C|f^s(\widetilde{u}) - c|. \tag{10.3}$$

With this ε small enough (but fixed) and the corresponding point $f^s(\widetilde{u})$, which replaces the previous point $\frac{6}{5}u$, we can construct the domain Ω (see Remark 10.1), with the same angle θ_0. Then the modulus of the annulus $\Omega \setminus (f^{-1} \circ F)\,(D(V;\theta_0) \cup D_*(-I))$ is bounded from below by a positive absolute constant. On the other hand, by (10.3), two preimages $f^{-1} \circ F(D_*(I))$ are also on a proportionally definite distance from the boundary of Ω. This completes the proof of Theorem A. □

11. Theorem A holds for real analytic maps

Let us show that Theorem A also holds for real analytic maps. This means that the complex bounds which Sullivan proved for infinitely renormalizable Epstein maps of bounded type, even hold for arbitrary infinitely renormalizable maps which are analytic on the dynamical interval. This answers a question of W. de Melo and gives the possibility to extend certain renormalization results of Sullivan and McMullen to the class of real analytic maps.

THEOREM 11.1. *Theorem A holds for a real analytic unimodal map f which is infinitely renormalizable: there exists $N(f)$ such that when V_n is a periodic central interval of f of period $s(n) \ge N(f)$, then there exists a polynomial-like extension $F_n\colon \Omega_n' \to \Omega_n$ of $f^{s(n)}\colon V_n \to V_n$ such that the modulus of $\Omega_n \setminus \Omega_n'$ is universally bounded from below by some positive number which only depends on ℓ and so that the diameter of Ω_n is of the same order as that of V_n.*

Proof: First we prove that the real bounds from Sections 5 and 6 still hold if f is a of class $C^{1+\text{zygmund}}$ with a non-flatness condition at c (see Section IV.2.a in [MS]). Because of Theorem IV.2.1 from [MS], this means that if $J \subset T$ are intervals such that $f^s|T$ is diffeomorphic then

$$C(f^sT, f^sJ) \geq \prod_{i=0}^{s-1} \Big(1 - o(|f^i(T)|)\Big) \, C(T, J)$$

where $o(t)$ is some function such that $o(t) \to 0$ as $t \to 0$. Now, as in Section 6, let l be maximal so that as before $f^s|l$ is monotone and - this is new - $L = f^s(l)$ contains at most 5 iterates of V. Therefore, in the renormalizable case, the orbit of $T, \ldots, f^k(T)$ in the proof of Lemma 6.1 has intersection multiplicity bounded by 15. Moreover, because the map has no wandering intervals, one has $\max_{i=0,\ldots,k} |f^i(T)|$ tends to zero if the period tends to infinity. (Note also that if I_n, I_{n+1} are consecutive central intervals of f then

$$|I_{n+1}| \leq \lambda |I_n| \tag{11.1}$$

where $\lambda < 1$ is independent of n. This follows from the extension given by Proposition 8.1 and the Koebe Principle.) In particular, the inequality proved in Lemma 6.1 still holds with a spoiling factor O_s such that $O_s \to 1$ as the period s tends to infinity. Now take in Lemmas 7.2-7.5 also the same definition for l. Then these lemmas still holds with a spoiling factor O_s. In Lemma 7.3 simply note that if Q_k contains more than 5 iterates of V then one simply takes $\tilde{Q}_1 \supset f(V)$ so that $\tilde{Q}_k = f^{k-1}(\tilde{Q}_1)$ contains $f^k(V)$ and precisely 5 iterates of V. Then, because the intersection multiplicity of the orbit $\tilde{Q}_1, \ldots, f^{k-1}(\tilde{Q}_1)$ is bounded by 15 and in the same way as before we get $C^{-1}(Q_1, f(V)) \geq C^{-1}(\tilde{Q}_1, f(V)) \geq O_s C^{-1}(\tilde{Q}_k, f^k(V)) \geq O_s \cdot 0.6$. If Q_k contains less than 5 intervals then we can obtain $C^{-1}(Q_1, f(V)) \geq O_s \cdot 0.6$ in the same cases as before. Only in cases II.b and II.c we used the interval Z_1. But now notice that the arguments used there also apply if replace Z_1 by the maximal interval $\tilde{Z}_1 \subset Z_1$ in H_1 so that each component of $f^{k-1}(\tilde{Z}_1) \setminus Q_k$ contains at most one iterate of V. Since the intersection multiplicity of $\tilde{Z}_1, \ldots, f^{k-1}(\tilde{Z}_1)$ is bounded by 18 we get that Lemma 7.3 still holds with a spoiling factor. In Lemmas 7.4 and 7.5 exactly the same remarks apply. Now in Lemma 9.1 we redefine T_1 as the maximal interval such that $f^{s-1}|T_1$ is monotone and such that each component of $f^{s-1}(T_1) \setminus V$ contains at most 5 iterates of V. So we can still apply Lemma 7.4 in the proof of Lemma 9.1 to this T_1 and so this lemma still holds.

All this implies that the same real bounds can be still used in Sections 9 and 10. Now of course, the Schwarz Lemma (that the pullback of some Poincaré domain with angle θ maps inside a similar region with the same angle θ) which we used in these sections does not hold anymore, because the map f is only analytic on a small neighbourhood of the dynamical interval. However, in Lemmas VI.5.2 and VI.5.3 of [MS] it is proved that we can still essentially obtain the same inclusion but with a slightly worse angle. According to Lemmas VI.5.2 and VI.5.3 the loss in the estimate tends to zero if the size of the interval tends to zero. Therefore we still get the same estimate in the proof of Theorem A. □

Let us note that the number $N(f)$ from the previous theorem is uniformly bounded when f runs over a compact space of maps. Let us clarify this: consider a real analytic unimodal map $f: [-1,1] \to [-1,1]$ with a unique critical point at 0 and with $f(\pm 1) = 1$. Such a map f can be written in the form $f = Q \circ \phi$, where where Q is a real unimodal polynomial (of the form $z^\ell + c_1$ up to a conjugation by $z \mapsto a \cdot z$ so that 1 becomes the orientation preserving fixed point), and $\phi: [-1,1] \to [-1,1]$ is a real analytic diffeomorphism with $\phi(0) = 0$. Now fix $K > 0$ and $\delta > 0$ and consider the space of all infinitely renormalizable maps f such that the C^2 norm of ϕ and its inverse on some δ neighbourhood of $[-1,1]$ is bounded by K, and so that g extends to a univalent map on a δ neighbourhood of $[-1,1]$. Then there exists a number N which only depends on K and $\delta > 0$ such that the assertion of the theorem holds. This is true since in this case, the estimate from (11.1) is uniform for all maps in the compact space defined above.

12. The proof of Theorem B

In this section we prove Theorem B.

As before, let W be a symmetric interval with nice boundary points, let R_W be the first return map to W and let $V = [v, \tau(v)]$ be the domain of R_W containing c. Similarly, let R_V be the first return map to V and $U = [u, \tau(u)]$ the component of the domain of R_V which contains c. Let $\hat{U} = [\hat{u}^f, u^f]$ be the component of R_V containing the critical value c_1. Let s be so that $R_V|\hat{U} = f^{s-1}$.

We assume that R_V has a high return, i.e., $R_V(U) = f^s(U) \ni c$.

We cannot use Lemmas 7.3-7.4 since V above is not necessarily a periodic interval. However we can use Proposition 8.1 (which also holds for renormalizable f). Let us state it quickly again. Let T_0 be a minimal interval containing $f(V)$ and its immediate neighbour among the disjoint intervals $f^2(V), \dots, f^{s'}(V)$. Write $L_0 = T_0 \setminus f(V)$. By Lemma 7.2, there exists an interval l on either side of $\hat{U}$ which has a unique common point with $\hat{U}$ such that $f^s: l \to L_0$ is one-to-one and by Proposition 8.1:

Lemma 12.1.
$$|L_0| \geq \frac{1}{3}|f(V)|.$$

Let us fix $\ell \geq 4$. Given the interval $V = (-v, v)$ such that R_V has a high return we construct an ℓ-polynomial-like map inside the domain $\Omega = \Omega(\ell, V)$, where Ω is either the disc $D_*(V)$ based on the diameter V, or

$$\Omega = D(V;\theta) \bigcup D(I;k_0\theta) \bigcup D(-I;k_0\theta),$$

where $\theta = \theta_0$ is some absolute constant (angle) to be determined later on,

$$I = (0, v), \quad \textit{and} \quad k_0 = 0.89.$$

(In fact, the constant k_0 is not rigid: 0.9 instead of 0.89 is also fine, though calculations are slightly more complicated then.)

As before, F is the branch of $f^{-(s-1)}$ from V onto $\hat{U}$. Also $F(v) = u^f \in (c_1, v^f)$ and we may assume that $v > 0$.

Again it is enough to prove

PROPOSITION 12.2. *There exists $\theta_0 > 0$, such that, for any $0 < \theta < \theta_0$ and for any $\ell \geq 4$,*

$$\overline{f^{-1} \circ F(\partial\Omega)} \subset \Omega \cup \{\pm v\}.$$

This implies that one has a polynomial-like mapping:

PROPOSITION 12.3. *Proposition 12.2 implies Theorem B.*

Proof: There are two cases to consider: either V is a periodic interval, or not. In the first case v is a repelling fixed point of f^s, and the domain $(1+\epsilon)\Omega$, where $\epsilon > 0$ is small enough, is a range of the desired polynomial-like map. In the second case we construct the ℓ-polynomial-like mapping inside the domain Ω. Since $f\colon \omega(c) \to \omega(c)$ is minimal, each point in $\omega(c)$ eventually is mapped into V and therefore is in the domain of definition of the map R_V. There exists a finite collection of disjoint intervals $I^0 = U$ and $I^1, \ldots, I^i$ in V, which form the domain of definition of the map R_V (see Proposition 5.1 and its proof). More precisely, every $x \in \omega(c) \cap V$ belongs to some I^j or to U, the map $R_V\colon I^j \to V$ is a diffeomorphism for $j = 1, 2, \ldots, i$, and $U \in c$ with $R_V|_U(U) = f^s(U) \ni c$ (the high return).

By Lemma 4.1, there exists a domain Ω_j inside $D(I^j;\theta) \subset D(I;\theta) \subset \Omega$ which is mapped diffeomorphically by a map R^j (an iteration of f) onto Ω. We have constructed Ω_j for each $j = 1, 2, .., i$, and each $\theta \in (0/\pi/2)$. Assuming Proposition 12.2, we fix the angle θ_0 and find a domain $\Omega_0 \subset \Omega$ such that $R^0 = f^s\colon \Omega_0 \to \Omega$ is a proper ℓ-cover. Since the domains $\Omega_j, j = 0, 1, \ldots, i$ may intersect each other, we modify so that they become ℓ-polynomial-like. To do this, let us consider a Yoccoz piece P_V containing the ends of the interval V on its boundary. Let D be a component of $P_V \cap \Omega$ containing V. For every $j = 0, 1, \ldots, i$, there exists a domain $D_j \subset \Omega_j$ such that $R_j\colon D_j \to D$ is a diffeomorphism if $j > 0$, and an ℓ-cover if $j = 0$. Since all D_j lie in different Yoccoz pieces, we obtain the ℓ-polynomial-like map. $\square$

Remark 12.4. In the proof of the above proposition one trims the domains in order to obtain a polynomial-like map (with disjoint domains). For this reason one does not obtain a lower bounds for $D \setminus D_0$. However, for all our applications the notion 'quasi-polynomial-like mapping' seems to be equally useful. (By this we mean in the definition of ℓ-polynomial-like mapping the discs Ω^i are allowed to intersect each other, but instead the all iterates of the critical point have to be well defined.) Because of Proposition 14.5 one has $|W| \geq \lambda |V|$ for some universal $\lambda > 1$ (only depending on ℓ) provided the previous return was non-central (this certainly happens infinitely often). From this and the construction below one easily gets that the modulus of $\Omega \setminus \Omega^0$ is universally bounded from below (by a number which only depends on ℓ).

The proof of Proposition 12.2 is somewhat similar to the proof of the Main Theorem in the infinitely renormalizable case for degree 2. The main new ingredient is contained in the following lemma:

LEMMA 12.5. *Either the disc $D_*(V)$ is a domain of the ℓ-polynomial-like mapping, i.e., $f^{-1} \circ F(D_*(V)) \subset D_*(V)$, or otherwise, for any $\theta > 0$, $F(D(-I;\theta))$ lies inside $D(I^f;\theta)$, where I^f is an interval around c_1:*

$$I^f = (c_1 - 1.52|v^f - c_1|, c_1 + 0.68|v^f - c_1|).$$

Proof: Remember that the constant $K_\ell(v^f) = |\hat{u}^f - c_1|/|v^f - c_1|$ depends not only on the extendibility space (which is $1/3$), but on the parameter $t = |c - c_{s+1}|/|T|$ as well (see Lemma 6.1). If $t \geq 0.51$, then

$$K_\ell(v^f) \leq \frac{0.51((3/4)^{1/\ell} - 0.51^{1/\ell})}{0.51^{1/\ell}(3/4)(1-(3/4)^{1/\ell})} \leq$$

$$\leq \frac{0.51((3/4)^{1/4} - 0.51^{1/4})}{0.51^{1/4}(3/4)(1-(3/4)^{1/4})} \leq 0.991818... < 1,$$

so that we apply Proposition 5.1. Thus, we can assume $t < 0.51$. The right end of the interval $F(-I)$ is just the point c_{-s+1}, which belongs to the interval (c_1, c_{s+1}) (since we have a high return). Hence,

$$\frac{|c_1 - c_{-s+1}|}{|c_1 - v^f|} \leq \frac{|c_1 - c_{s+1}|}{|c_1 - v^f|} \leq 0.51 \times (4/3) = 0.68.$$

The left end of the interval $F(-I)$ is the point $\hat{u}^f$, and we obtain from Corollary 5.1, where we put $y = 1/(4/3) = 3/4$, that $K^*_\ell(y) \leq K^*_4(y) = 1.51983... < 1.52$. $\square$

If the first alternative in the lemma holds then Proposition 12.2 holds. So we will assume in the remainder of the proof that the second alternative holds.

In order to prove Proposition 12.2, we need to find some $\theta = \theta_0 > 0$, such that, for any $\ell \geq 4$ one has $f^{-1} \circ F(\partial\Omega) \subset \Omega$. To do this, consider the pullback $F(\Omega)$. Let us rescale Ω such that the interval $V=(-v,v)$ turns into the interval $(-1,1)$. We call the obtained domain Ω^*. Let us rescale also $F(\Omega)$ by shifting first by $-c_1$, and then rescaling it so that the interval (c_1,v^f) turns into the interval $(0,1)$. We call the obtained domain $\widetilde{\Omega}^*$. Introduce also the scaled map F^* corresponding to the map F (the pullback of f^{s-1}):

$$F^*(z)=\frac{F(vz)-c_1}{v^f-c_1}.$$

So $\widetilde{\Omega}^*=F^*(\Omega^*)$. It is enough to find $\theta=\theta_0$ such that the closure of $P_\ell^{-1}(\widetilde{\Omega}^*)$ is inside Ω^* united with the endpoints ±1, if we are in the case of renormalization, i.e. $f^s(v)=v$. (As above, $P_\ell(z)=z^\ell$.) By our choice,

$$\Omega^*=D((-1,1);\theta)\cup D((0,1);k_0\theta)\cup D((-1,0);k_0\theta).$$

Let us look at each piece of $P_\ell^{-1}\circ F$. Given $z\neq0$, define $E_i(z)$ to be the unique point w such that $P_\ell(w)=w^\ell=z$ and $\arg(w)\in[(2i-1)\pi/\ell,(2i+1)\pi/\ell)$, $i=0,1,\dots,l-1$. The restriction of F^* to the real axis is certainly defined on the interval $(-1,1)$. Moreover, the real map $E_0\circ F^*\colon(0,1)\to\mathbb{R}$ is a homeomorphism and it sends the interval $(0,1)$ into itself (since R_V has a high return and f has no attracting periodic orbit). It follows (as before, from the Schwarz contraction principle) that

$$E_0\circ F^*(D((0,1);k_0\theta)) \text{ is a proper subset of } D((0,1);k_0\theta), \tag{12.1}$$

for any angle $\theta\in(0,\pi/2]$.

Let us now show that

$$E_0(F^*(D((-1,1);\theta)))\subset\Omega^*, \tag{12.2}$$

in other words, that $F^*((D((-1,1);\theta)))$ is covered by the set $P_\ell(\Omega\cap\{z;\ -\pi/\ell<\arg z<\pi/\ell\})$. To see this, first note that by Corollary 6.1, $F^*((D((-1,1);\theta)))\subset D((-K_0,1);\theta)$ with the constant

$$K_0=K_\ell^*(3/4)\le K_4^*(3/4)=1.51983...<1.52.$$

We are going to use several purely geometric statements (Lemmas 15.2- 15.5) which are proved in the Appendix. First, we use the following Lemma 15.2 from the Appendix:

LEMMA 12.6. *Let us denote for simplicity $D(\theta)=D((-1,1);\theta)$. Set*

$$\Pi_\ell(\theta)=\{z\in D(\theta):0\le\arg z\le\pi/\ell\}.$$

As before, $P_\ell(z)=z^\ell$. Fix $0<\theta<\pi/2$. Then

$$P_\ell(\Pi_\ell(\theta))\subset P_{\ell+2}(\Pi_{\ell+2}(\theta)),\quad \ell=2,4,\dots.$$

By this lemma, the difference $\Delta_\ell(K_0,\theta) = D((-K_0,1);\theta)\setminus P_\ell(\Pi_\ell(\theta))$ is contained in $\Delta_2(K_0,\theta)$. Hence (12.2) follows from the following Lemma 15.3 from the Appendix:

LEMMA 12.7. *For all θ less than an absolute positive θ_0, and for $k_0 = 0.89$, $K_0 = 1.52$,*

$$\Delta_2(K_0,\theta) \subset P_\ell\left(D((0,1);k_0\theta)\cap\{z\,;\ 0<\arg z<\pi/\ell\}\right), \ell = 4,6,\ldots \quad .$$

Because of (12.1) and (12.2), in order to conclude that $E_0\circ F^*(\Omega)\subset\Omega$, we only have to show that $E_0\circ F^*(D((-1,0);k_0\theta))\subset\Omega^*$, for θ small. Lemma 12.5 says that $F^*(D((-1,0);k_0\theta))\subset D((-1.52,0.68);k_0\theta)$. Therefore, by the Lemma 12.6, the statement follows from the following Lemma 15.5 from the Appendix:

LEMMA 12.8. *If θ is small, and $k_0 = 0.89$, then*

$$D((-1.52,0.68);k_0\theta)\subset P_2(D((-1,1);\theta)). \tag{12.3}$$

Thus, we have proved that $E_0\circ F^*(\Omega^*)$ is inside Ω^*. By symmetry of Ω^*, $E_{\ell/2}\circ F^*(\Omega^*)$ is inside Ω^* too, for the same θ and ℓ.

Now we consider $E_1\circ F^*(\Omega^*)$. First note that the domain $F^*(\Omega^*)$ is contained in $D((-1.52,1);k_0\theta)$. Therefore, for $\ell=4$, and for θ sufficiently small, the domain $P_4(D((-1,1);\theta)\cap\{z\,;\ \pi/4\le\arg z\le 3\pi/4\})$ and its complex conjugate contain $D((-1.52,1);k_0\theta)$ since the diameter of the latter domain grows as $const/\sin(k_0\theta)$ while the distance between the origin and the boundary of the former domain grows as $const/\sin^4\theta$ as $\theta\to 0$ (with a positive $const$). If $\ell = 6,8,...$, we use that

$$P_{\ell-2}(D((-1,1);\theta)\cap\{z:\pi/(\ell-2)\le\arg z\le 3\pi/(\ell-2)\})$$

is contained in

$$P_\ell(D((-1,1);\theta)\cap\{z:\pi/\ell\le\arg z\le 3\pi/\ell\}),$$

for $\ell = 4,6,...$ (the proof of this inclusion is identical to the proof of Lemma 15.2).

This is enough to end the proof of Proposition 12.2. Indeed, each other $E_i\circ F^*(\Omega^*)\subset E_i\circ F^*(D((-1.52,1);\theta))$, $(i\ne 0,\ell/2)$, is contained in $D((-1,1);\theta)$, since $D((-1,1);\theta)$ is invariant under the rotation $z\mapsto\exp(i\cdot 2\pi/\ell)z$, for z in the first quadrant. $\square$

Thus the proof of Theorem B is complete.

13. Remark on the 'small' Julia sets

Let $f^s : V\to V$ be a renormalization of f on the real axis (so that an end point v of the interval V is fixed by f^s). By Theorem A, there exists a polynomial-like extension $F = f^s : \Omega_0 \to \Omega$. Let J_F be the Julia set of this mapping (the "small" Julia set of f). Note that the end points $\pm v$ of V belong to J_F. As we have shown, the set J_F is contained in the open disc $D_\star(V)$ (based on the interval V) together with the end points $\pm v$, in the following cases: either $\ell \geq 4$, or $\ell = 2$ and f is $s/2$-renormalizable. Let us show now that in the remaining cases the set J_F again is contained in a definite domain based on the interval V only. So, let f be s-renormalizable, but not $s/2$-renormalizable, and $\ell = 2$. By Lemmas 7.4-7.5, the extendibility space is 0.6. Let us show that even the space 0.5 is enough (showing that weaker estimates would suffice if we would try to generalize the Main Theorem to for example multimodal maps).

Firstly, define Ω as in the beginning of Section 12 (Proof of Theorem B) with the difference that we take the constant $k_0 = 0.8$ now. In Lemma 12.5, we replace 1.52 by 1.37 (which corresponds to the extendibility space 0.5 and the degree $\ell = 2$) and 0.68 by 0.6 taking $\ell = 2$ and $t = 0.4$ in its proof. Lemma 15.2 is not needed. In Lemma 15.3 we have to take $\ell = 2$ in the statement. In the proof we take $K_0 = 1.37$ and $k_0 = 0.8$ (see the Remark after the proof of Lemma 15.3). Finally, in Lemma 15.5 we have to put 1.37 instead of 1.52, and 0.6 instead of 0.68 (see Remark after the proof of Lemma 15.5).

Thus, in all cases the "small" Julia set of the renormalizations are contained in a universal domain (in particular, in a universal sector), based on the periodic interval.

14. Proof of Theorem C

This section is in some sense an elaboration of Section 5 of [L3] and Lemmas 14 and 15 in [L5] and is also related to the inducing construction developed by Jakobson, Świątek and Graczyk [JS], [GS1] (see also [GS3]). We wish to thank Edson Vargas for several discussions on [L3] and [L5].

Let $\mathcal{E}(T^0)$ be the collection of mappings $g\colon \cup T_i^1 \to T^0$ where T^0 is some symmetric interval around c with nice boundary points and where T_i^1 is a collection of disjoint closed subintervals of T^0 for which

- for $i \neq 0$ the map $g\colon T_i^1 \to T^0$ is a diffeomorphism of the form $f^{j(i)}$ and the inverse map $(g|T_i^1)^{-1}$ has a univalent extension to $\mathbb{C}_{T^0}$;
- writing $T^1 = T_0^1$ we have that $g|T^1$ is a unimodal map of the form f^j and with $g(\partial T^1) \subset \partial T^0$; one can write $g|T^1 = h \circ f$ where h^{-1} has also a univalent analytic extension to $\mathbb{C}_{T^0}$;
- all iterates of the critical point $c = 0$ under g are in $\cup T_i^1$.

Next we define the subset $\mathcal{E}(T^0, T^{-1})$ of this class. Here T^{-1} is a symmetric interval T^{-1} containing T^0. Here $g \in \mathcal{E}(T^0)$ is in this subset if when writing as before $g|T_i^1 = h_i \circ f$, the map h_i^{-1} has a univalent analytic extension from $\mathbb{C}_{T^{-1}}$ into $\mathbb{C}_{H_i}$ where H_i is some interval near c_1 such that $f^{-1}(H_i) \cap \mathbb{R} \subset T^0$. Note that we do not require that the map can be extended diffeomorphically from a non-central domain to T^{-1}; the extension is only required from the f-image of the non-central domains. To simplify the exposition we shall say that a branch of $g \in \mathcal{E}(T^0)$ of the form $g|D = f^i|D$ *extends to some interval* V *via* c_1 if $f^{i-1}|f(D)$ extends to a diffeomorphism mapping onto V.

An example of a map g which is of type $\mathcal{E}(T^0, T^{-1})$ is the first return map to an interval $W_n = [u_n, \tau(u_n)]$ as in Section 2. Indeed, the boundary points of W_n are nice and there are no forward iterates of ∂W_n in $W_{n-1} \setminus W_n$. So take a domain $I \cap W_n = \emptyset$ of the first return map R to W_n and a maximal interval T containing I so that $R|I = f^i$ is monotone. By maximality of T for each component T_+ of $T \setminus I$ there exists some $j < i$ so that $f^j(T_+)$ contains c in its boundary and since R is the first return map $f^j(I) \cap W_n = \emptyset$. Hence $f^j(T_+)$ contains a boundary point of W_n and therefore $f^i(T_+)$ contains a point of ∂W_{n-1}. Since this holds for both components of $T \setminus I$ this gives $f^i(T) \supset W_{n-1}$. So on each component of the domain $I \subset W_n$ of R, one can write $R = h \circ f$ and h extends as a diffeomorphism from some neighbourhood of $f(I) \ni c_1$ onto W_{n-1}. If we assume that $\omega(c)$ is minimal, then there are only finitely many components of the domain which contain points from $\omega(c)$ and we can also consider the corresponding restriction of the return map.

We say that $g \in \mathcal{E}(T^0)$ has a *low return* if $g(T^1)$ does not contain the critical point c. A low return is *central* if $g(T^1) \cap T^1 \neq \emptyset$ and *non-central* otherwise. Let us define in the low return case the renormalization $\mathcal{R}g$ of g and show that $\mathcal{R}g \in \mathcal{E}(T^0)$.

First we define the intermediate renormalization $\mathcal{C}g$ as follows. Let $s(g) \geq 0$ be the minimal integer so that $g^{s(g)+1}(c) \in T^0 \setminus T^1$. Of course, $s(g)$ is finite since otherwise g would have a periodic attractor. If $s(g) = 0$ then the return is non-central and then we define $T^{1,0} = T^1$ and $\mathcal{C}g = g$. If $s(g) \geq 1$ then first define $\mathcal{C}g$ so that it coincides with g on $\cup_{i \neq 0} T_i^1$. Define a nested sequence of intervals $T^1 = T^{1,0} \supset T^{1,1} \supset \ldots \supset T^{1,s(g)}$ as follows. For $i < s(g)$, the two intervals $T^{1,i} \setminus T^{1,i+1}$ consist of those points in T^1 which are mapped by g^{i+1} for the first time outside T^1. Define

$$\mathcal{C}g|(T^{1,i} \setminus T^{1,i+1}) = g \circ g^{i+1}$$

for $i < s(g)$ and define

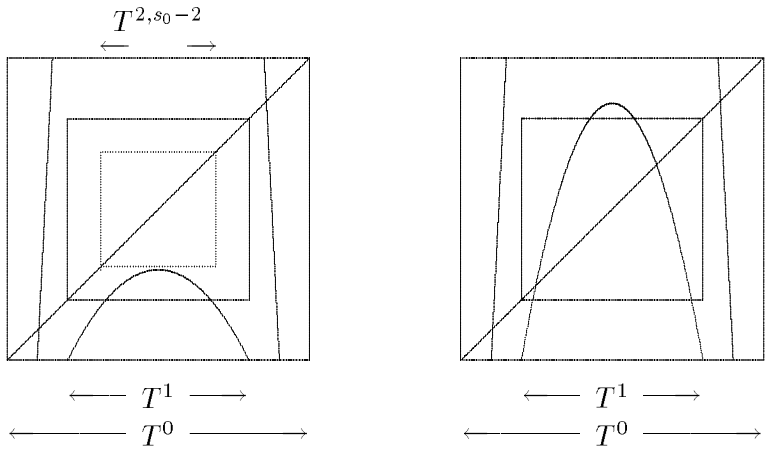
$$\mathcal{C}g|T^{1,s(g)} = g^{s(g)+1}.$$

Then, $\mathcal{C}g$ has a non-central low return with central domain $\tilde{T}^1 = T^{1,s(g)}$.

Now we define the actual renormalization $\mathcal{R}g$. First define $\mathcal{R}g|(T^0 \setminus \tilde{T}^1) = \mathcal{C}g$. Next let $\tilde{T}^1(i)$ be the domain of $(\mathcal{C}g)^i$ intersected with $\tilde{T}^1$ and let $\tilde{s} \geq 0$ be minimal so that $(\mathcal{C}g)^{\tilde{s}}(\tilde{T}^1(\tilde{s})) \cap \tilde{T}^1 \neq \emptyset$. Because c is recurrent, the integer $\tilde{s}$ exists. We can define the new central domain of $\mathcal{R}g$ to be $\tilde{T}^1(\tilde{s})$ and also denote this interval as $T^2 = \tilde{T}^1(\tilde{s})$. We define $\mathcal{R}g|T^2 = (\mathcal{C}g)^s$. For $x \in \tilde{T}^1 \setminus T^2$, let $s(x) \leq \tilde{s}$ be the smallest integer for which $(\mathcal{C}g)^{s(x)}(x)$ and $(\mathcal{C}g)^{s(x)}(c)$ are in different components of $\cup_i \tilde{T}^1_i$ (the non-central domains of $\mathcal{C}g$) and define $\mathcal{R}g(x) = (\mathcal{C}g)^{s(x)+1}(x)$ for $x \in \tilde{T}^1 \setminus T^2$. Let us call the intervals $T^{1,1}, T^{1,1}, ..., T^{1,s(g)}, T^2$ the *symmetric intervals* of the renormalization $\mathcal{R}g$.

As we shall show below $\mathcal{R}g \in \mathcal{E}(T^0)$ so if $\mathcal{R}g$ has also a low return then we can again define $\mathcal{R}^2 g \in \mathcal{E}(T^0)$ and so on.

In the figure below we have drawn on the left a central low return with $s_0 = 3$ and $T^{2,0} = T^1$ and on the right a map with a non-central high return.



If g has a *high return* (i.e., not a low return) then let x be the orientation preserving fixed point of $g|T^1$ and z the boundary of T^0 on the same side of c as x. Next take preimages of $z_0 = z, z_1, z_2, \ldots$ of z along this branch of $g|T^1$ (so $[z_1, \tau(z_1)] = T^1$ and $z_k \to x$ as $k \to \infty$). Define $U_k = [z_k, \tau(z_k)]$ and choose k minimal so that $gU_k \supset U_k$. Such an integer k exists because we have assumed that f is not renormalizable. The interval $V^1 := U_k$ is the *escape interval* associated to a map g with a high return. For $i = 0, 1, \ldots, k-1$ define the new map $\widetilde{\mathcal{W}}g$ on $U_i \setminus U_{i+1}$ as $g^i \circ g$ and define $\widetilde{\mathcal{W}}g$ on $U_k = V^1$ as the first return map of g to U_k. Note that $\widetilde{\mathcal{W}}g \notin \mathcal{E}(T^0)$ because the image of the central branch T^2 is contained in the interior of T^0 (and does not stretch over to ∂T^0). However,

$$\mathcal{W}g = \widetilde{\mathcal{W}}g \,|\, V^1 \text{ is contained in } \mathcal{E}(V^1, T^0),$$

as we will show in the lemma below. Note that the domain W^1 of $\mathcal{W}g$ is smaller the domain T^0 of g, and that the extension associated to $\mathcal{W}g$ includes the domain of the original map g.

LEMMA 14.1.

If $g \in \mathcal{E}(T^0)$ has a low return then $\mathcal{R}g \in \mathcal{E}(T^0)$. Moreover, if x and some $g^n(x)$, $n > 0$, lie in the interval $T^{1,s(g)}$ (the central domain of the map $\mathcal{C}g$), then $g^n(x)$ is some iterate of x by the renormalization $\mathcal{R}g$: there exists $i > 0$, such that $g^n(x) = (\mathcal{R}g)^i(x)$. If $g \in \mathcal{E}(T^0)$ has a high return then $\mathcal{W}g = \widetilde{\mathcal{W}}g \mid V^1$ is contained in $\mathcal{E}(V^1, T^0)$.

Proof: First assume that g has a low return. It is clear that the non-central branches of $\mathcal{C}g$ map onto T^0. Write $g|T^1 = f^j$ and note that $\mathcal{C}g$ on its the central domain $\tilde{T}^1$ is equal to $\mathcal{C}g|\tilde{T}^1 = g^{s(g)+1}$. Let us show that $g^{s(g)+1}|\tilde{T}^1$ extends via c_1 onto a component of $T^0 \setminus \tilde{T}^1$. Indeed, since $g \in \mathcal{E}(T^0)$, the Koebe space of $f^{j-1}|f(T^1)$ spreads over T^0 (by this we mean that there is a neighbourhood of $f(T^1)$ is mapped diffeomorphically onto T^0 by f^{j-1}). In particular, this Koebe space contains one of the components of $T^0 \setminus \{c\}$. By definition, $\tilde{T}^1, g(\tilde{T}^1), \ldots, g^{s(g)}(\tilde{T}^1) \subset T^1$ and therefore applying $g^{s(g)}|T^1$ to $g(\tilde{T}^1) = f^{j-1}f(\tilde{T}^1)$ it follows that the Koebe space of $g^{s(g)} \circ f^{j-1}|f(\tilde{T}^1)$ spreads over one of the intervals connecting ∂T^0 to $g^{s(g)}(c)$ (which by definition is in T^1). In particular this space contains a component of $T^0 \setminus \tilde{T}^1$. The same argument shows that $k < s(g)$, the Koebe space of $g^{k+1}|(T^{1,k} \setminus T^{1,k+1})$ extends to the component of $T^0 \setminus T^1$ containing $g^{k+1}(T^{1,k} \setminus T^{1,k+1})$.

Let us now show that $\mathcal{R}g \in \mathcal{E}(T^0)$. Write $\mathcal{R}g|T^2 = f^m$ and $\mathcal{C}g|\tilde{T}^1 = f^k$. As we just proved, the Koebe space of $f^{k-1}|f(\tilde{T}^1)$ spreads over one of the components of $T^0 \setminus \tilde{T}^1$. But since $(\mathcal{C}g)^i(\tilde{T}^1(i)) \cap \tilde{T}^1 = \emptyset$ for $i < \tilde{s}$ (and by definition $(\mathcal{C}g)^i(\tilde{T}^1(i))$ is contained in some interval $\tilde{T}^1_j$, $j \neq 0$) and $(\mathcal{C}g)$ maps each such interval $\tilde{T}^1_j$ diffeomorphically onto T^0, the Koebe space of $f^{m-1}|f(T^2)$ spreads over T^0.

The second assertion also follows from the construction.

If g has a high return then restricted to the central branch V^1, $\widetilde{\mathcal{W}}g$ can be written as $h \circ f$, where h has a diffeomorphic extension to T^0 (because this central branch is just g and because $g \in \mathcal{E}(T^0)$). Therefore, the two non-central branches of $\mathcal{W}g$ of the form $\{x \in V^1\,;\ \widetilde{\mathcal{W}}g(x) \in V^1\}$ are mapped diffeomorphically by $\mathcal{W}g$ onto V^1, and are of the form $h \circ f$, where h has a diffeomorphic extension to T^0. On $\{x \in V^1\,;\ \widetilde{\mathcal{W}}g(x) \notin V^1\}$ the map $\mathcal{W}g$ is a composition of the map $h \circ f$, where h has a diffeomorphic extension to T^0 and iterates of the non-central branches (in $T^0 \subset V^1$) of $\widetilde{\mathcal{W}}g$ (which map diffeomorphically onto T^0). This proves that $\mathcal{W}g \in \mathcal{E}(V^1, T^0)$. □

To prove Theorem C, we need the two lemmas below.

LEMMA 14.2. *Assume that $g \in \mathcal{E}(T^0)$ is a first return map to a symmetric interval T^0 around c with nice boundary bounds. Let $g, \mathcal{R}g, \ldots, \mathcal{R}^{k-1}g$ have low returns and let $T_0, T_1, \ldots, T_P$ be the symmetric intervals corresponding to the renormalizations $g, \mathcal{R}g, \ldots, \mathcal{R}^k g$ (i.e., the symmetric intervals from the* *construction of $(\mathcal{C}g),(\mathcal{C}g)^1,\dots,(\mathcal{C}g)^k$, see the definition above), so that $T_0 = T^0$, $T_1 = T^1$, and $T_P = T^{k+1}$ (the central domain of the last map $\mathcal{R}^k g$), and ordered so that $T_{p'} \subset T_{p''}$ whenever $p' > p''$. Write $\mathcal{R}^k g|T^{k+1} = f^m$ with $m \in \mathbb{N}$ and let $A \supset f(T^{k+1}) \ni c_1$ be the interval which is mapped by f^{m-1} diffeomorphically onto T^0. Then*

(1) for each $i < m$ one has that $f^i(c)$ lies between c and $f^i(T^{k+1})$ whenever this last interval intersects T^0;

(2) there exists a sequence of integers $P \geq p_1 > p_2 > \dots > p_r := 1$ and integers $n(p_1) < n(p_2) < \dots < n(p_r)$ such that $A^{p_i} := f^{n(p_i)}(A) \subset T_{p_i-1} \setminus T_{p_i}$. Moreover, writing $s(p_i) := n(p_{i+1}) - n(p_i) > 0$ for each $i = 1,\dots,r-1$, we have that $f^{s(p_i)}$ maps diffeomorphically A^{p_i} onto $A^{p_{i+1}}$, and the map $f^{s(p_i)-1}\colon fA^{p_i} \to A^{p_{i+1}}$ has Koebe space spread over $T_{p_{i+1}-1}$;

(3) the Koebe space of $f^{n(p_1)}\colon A \to A^{p_1}$ spreads over T_{p_1-1};

(4) $p_r = 1$ and the Koebe space of $f^{m-n(p_r)-1}\colon fA^1 \to T^0$ spreads over T^{-1}.

Proof: To prove (1), notice that $\mathcal{R}^i g$ is an iterate of the first return map g, and that the central branch of each lower iterate does not cover c (by definition of the number s).

To prove (2)-(4), we use the following observations $a)-c)$. Here $a)$ and $b)$ describe the dynamics on the sets ("annuli") $C(p) := T_{p-1} \setminus T_p$, $p = 1,\dots,P$, $C(P+1) := T_P$, and $c)$ describes the relation between g and the renormalizations.

Denote by $C(l_j)$ the annulus containing the critical value $\mathcal{R}^{j-1}g(c)$ of the renormalization $\mathcal{R}^{j-1}g$, $j = 1,...,k$. In other words, $C(l_j) = T_{l_j-1} \setminus T_{l_j}$, where T_{l_j} is the central domain of $\mathcal{R}^{j-1}g$ in the non-central low case and the central domain of the intermediate renormalization $\mathcal{C}\mathcal{R}^{j-1}g$ otherwise.

a). g diffeomorphically maps: (i) each component of the domain of definition of g in $C(1)$ diffeomorphically onto T^0, (ii) the annulus $C(j)$ onto $C(j-1)$ for $j = 2,\dots,l_1$, and (iii) the interval T_{l_1} into the annulus $C(l_1)$. Note that $T^0 = C(1) \cup \dots \cup C(l_1) \cup T_{l_1}$.

b). Similarly, for $j = 1,\dots,k$, $\mathcal{R}^j g$ diffeomorphically maps: (i) each component of the domain of definition of $\mathcal{R}^j g$ in $C(l_j+1)$ onto T^0, (ii) each annulus $C(i)$ onto $C(i-1)$ for $i = l_j+2,\dots,l_{j+1}$, and (iii) the interval $T_{l_{j+1}}$ into the annulus $C(l_{j+1})$. Note that $\mathcal{R}^j g$ restricted to $T^0 \setminus T_{l_j}$ coincides with the previous renormalizations. In particular, $\mathcal{R}^j g$ maps each component of this set diffeomorphically onto T^0. Note also that we have a partition

$$T^0 = (T^0 \setminus T_{l_j}) \cup C(l_j+1) \cup C(l_j+2) \cup \dots \cup C(l_{j+1}) \cup T_{l_{j+1}},$$

that is, the above describes the behaviour of $\mathcal{R}^j g$ on the whole interval T^0.

c). The previous lemma implies that if $x, g^n(x) \in T_{l_j}$, then $g^n(x)$ is some iterate of x by the map $\mathcal{R}^j g$.

By definition, $\mathcal{R}^k g|T^{k+1} = f^m$ is some iterate g^{N+1} of g. Hence if we define e so that $g|T^1 = f^e$ and $\tilde{A} = f^{e-1}(A) \ni g(c)$ then we have that $g^N\colon \tilde{A} \to T^0$ is a diffeomorphism. Even though $g(A)$ is not defined, it will make life a little simpler if we write $g(A)$ instead of $\tilde{A}$. It follows, $g^N(c) \in g^N(A) \subset C(1) = T^0 \setminus T^1$.

Now we can define the sequence $1 \le q_1 < q_2 < \ldots < q_r \le P$ inductively as follows. Set $q_1 = 1$ and $t_1 = 1$. If $q_1 < \ldots < q_i$ and $t_1 < \ldots < t_i$ are already defined, we let t_{i+1} be the *minimal* $t > t_i$, such that $g^{N-t}(g(c))$ lies in T_{p-1} for some $p > q_i$ and let q_{i+1} be the largest such p. If there is not such t_{i+1} then we stop the inductive definition and define $r = i$.

The set $g^{N-1}(g(A))$ coincides with a monotone component of the map g. Since $g \in \mathcal{E}(T^0, T^{-1})$, this branch of g extends to T^{-1} via $f(c)$. This is the first step. Now we have the following by induction on $i = 1, 2, \ldots, r$:

When $i < r$ then $g^{N-t_i}(g(A)) \subset C(q_i)$ and if we denote

$$f^{s(q_i)} = g^{(N-t_i)-(N-t_{i+1})}|g^{(N-t_{i+1})}(g(A)),$$

then the Koebe space of the diffeomorphism

$$f^{s(q_i)-1}\colon f(g^{(N-t_{i+1})}(g(A)) \to g^{(N-t_i)}(g(A))$$

extends over T_{q_i-1}. For $i = r$, denote

$$f^{s(q_r)} = g^{N-t_r+1},$$

so that the Koebe space of the diffeomorphism

$$f^{s(q_r)-1}\colon f(A) \to g^{(N-t_r)}(g(A))$$

spreads over T_{q_r-1}.

Note that if we replace g here by iterates of f, then we obtain (2)-(4) of the Lemma by defining $p_i := q_{r+1-i}, i = 1, \ldots, r$.

So the lemma holds once we have proved the above induction step. We already proved that the assertion holds $i = 1$. So assume that the assertion holds for i and show that the assertion then also holds for $i+1$. In fact, we will only use the induction assumption to prove that $g^{N-t_{i+1}}(g(A)) \subset C(q_{i+1})$. Consider the point $g^{N-t_i}(g(c))$. It lies, by definition, in the annulus $C(q_i) = T_{q_i-1} \setminus T_{q_i}$. Let us find the maximal l_j such that the interval $T_{q_i-1} \subset T_{l_j}$. It means that $C(q_i) \subset T_{l_j} \setminus T_{l_{j+1}}$. Consider $g^{N-t_{i+1}}(g(c))$ if $i < r$ (and simply $c = 0$ if $i = r$). By definition, $g^{N-t_{i+1}}(g(c))$ lies in $C(q_{i+1}) \subset T_{q_i}$. By the property c), the point $g^{N-t_i}(g(c))$ is some n_1-iterate $(\mathcal{R}^j g)^{n_1}(c)$ of c by the renormalization $\mathcal{R}^j g$, and $g^{N-t_{i+1}}(g(c))$ is some n_2-iterate of c by $\mathcal{R}^j g$, where $n_2 < n_1$. Consider the backward orbit

$$(\mathcal{R}^j g)^{n_1}(c), (\mathcal{R}^j g)^{n_1-1}(c), \cdots, (\mathcal{R}^j g)^{n_2+1}(c), (\mathcal{R}^j g)^{n_2}(c),$$

where (by the definition) all points except for the last one lie in the annuli $C(p)$ with $p \le q_i$. Denote $Q_n = (\mathcal{R}^j g)^n(c)$. Let us consider two cases. If $n_2 = n_1 - 1$, then the statement follows from the fact that $\mathcal{R}^j g \in \mathcal{E}(T^0)$. If $n_2 < n_1 - 1$, then Q_{n_1-1} lies in $C(p)$ for some $p \leq q_i$. By property b), there is a component of the domain of $\mathcal{R}^j g$ in $C(p)$, which is mapped by $\mathcal{R}^j g$ diffeomorphically onto T^0, which of course contains T_{q_i-1}. Then this component is pulled back by $\mathcal{R}^j g$ univalently to Q_{n_1-2}. Repeating this argument proves the induction step. $\square$

REMARK 14.3. *The above proof shows that the actual Koebe space of the map $f^{s(p_i)-1}\colon f(A^{p_i}) \to A^{p_i+1}$ is, in fact, T^0. We shall not use this.*

LEMMA 14.4. *There exists $K > 0$ and given $\lambda > 1$ there exists $\sigma \in (0,1)$ with the following property. Assume that $g \in \mathcal{E}(T^0, T^{-1})$ as above and that $|T^{-1}| \geq \lambda |T^0|$.*

(1) If all renormalizations $g, \mathcal{R}g, \mathcal{R}^2 g, \ldots$ have low returns then there exists k such that $\mathcal{R}^k g$ has a polynomial-like extension $G\colon \cup D_i \to D_*(T^0)$;

(2) If k is minimal such that $\mathcal{R}^k g$ does **not** *have a low return then either $\mathcal{R}^k g$ has a polynomial-like extension $G\colon \cup D_i \to D_*(T^0)$ such that the modulus of $D_*(T^0) \setminus D_0$ is at least $\sigma > 0$ or $(|T^0|/|W^{k+1}|) \geq (1-\sigma)^{-1} \cdot (|T^{-1}|/|T^0|)$ where W^{k+1} is the escape interval associated to $\mathcal{R}^k g$. Note that $\widetilde{\mathcal{W}}\mathcal{R}^k g \in (W^{k+1}, T^0)$.*

(3) If $|T^{-1}|/|T^0| > K$ and k is minimal such that $\mathcal{R}^k g$ does not have a low return iterate then $\mathcal{R}^k g$ has a polynomial-like extension $G\colon \cup D_i \to D_*(T^0)$ such that the modulus of $D_*(T^0) \setminus D_0$ is at least $\sigma > 0$.

Proof: First assume that $\mathcal{R}^i g$ has low returns for all i. The first return map R to T^{k+1} (which is equal to the first return map of $\mathcal{R}^k g$ to T^{k+1}) is in class $\mathcal{E}(T^{k+1}, T^0)$. Since f has no wandering intervals we have that $|T^{k+1}|/|T^0|$ tends to zero as k tends to infinity and therefore G restricted to the central branch U is equal to a composition of z^ℓ and an almost linear map. In particular, since this composition sends a neighbourhood of U inside T^{k+1} onto an interval connecting $G(c)$ with T^0, U is much smaller than T^{k+1}. From all this one gets that R has a polynomial-like $R\colon \cup D_i \to D_*(T^{k+1})$. The central domain D_0 of R is almost a round disc and intersects the real line in U. The non-central domains D_i are because of Schwarz inside discs with real trace equal to the non-central real domains of G. This completes the proof of the first case.

So from now assume that $g, \ldots, \mathcal{R}^{k-1} g$ have low returns and that $\mathcal{R}^k g\colon \cup I_i \to T^0$ has a *high return.* As before, the Schwarz Lemma implies that the pullbacks of $D_*(T^0)$ under the extensions of the monotone branches $I_i \to T^0$ of $\mathcal{R}^k g$ fit inside $D_*(T^0)$. So let us consider the pullback associated to the inverse of the map $\mathcal{R}^k g | T^{k+1} = f^m$ on the central interval T^{k+1}.

First we notice that since $g \in \mathcal{E}(T^0, T^{-1})$ there exists because of the Koebe Principle a constant $\rho > 0$ which depends on $\lambda = |T^{-1}|/|T^0| > 1$ such that the ρ-scaled neighbourhood of each domain T_i^1 is still contained in T^0. Here we use that the interval H from the definition of $\mathcal{E}(T^0, T^{-1})$ is mapped inside T^0 by f^{-1}. In particular,

$$|T^1| < (1+\rho)^{-1}|T^0|, \tag{14.1}$$

where T^1 is the central domain of g.

Let $A \supset f(T^{k+1})$ be so that f^{m-1} maps A diffeomorphically onto T^0. Let $R^0 = \mathcal{R}^k g(T^{k+1}) = f^m(T^{k+1})$ and $I^0 = T^0 \setminus R^0$. We want to compare the sizes of the pullbacks in $I', R' \subset A$ by $f^{m-1}: A \to T^0$ of the 'real' and 'imaginary' pieces $R^0, I^0 \subset T^0$. That is, $R' = f(T^{k+1})$. Let p_i be as in the previous lemma and let $R^i = f^{n(p_i)}(R')$ and $I^i = f^{n(p_i)}(I')$ be the partition of A^i corresponding to I' and R'. Write $\mu_{i+1} = \frac{|T_{i+1}|}{|T_i|} \in (0,1)$ where T_i, $i = 0, 1, ..., P$ is the sequence of symmetric intervals defined in the previous lemma. Remember that $T_0 = T^0$ and $T_1 = T^1$.

If R, I are two intervals in $T_i \setminus T_{i+1}$ with a unique common point then using the fact that $f(z) = z^\ell + c_1$ it easy to see that

$$\frac{|R|}{|I|} \geq [\mu_{i+1}]^{\ell-1} \cdot \frac{|f(R)|}{|f(I)|}. \tag{14.2}$$

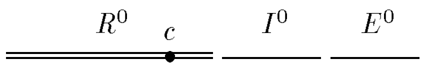
Indeed, if $I = (a, x)$ and $R = (x, b)$, where $0 < a < x < b$, then (14.2) follows from the inequality:

$$\frac{b^{\ell-1} + b^{\ell-2}x + \cdots + x^{\ell-1}}{x^{\ell-1} + x^{\ell-2}a + \cdots + a^{\ell-1}} \leq \frac{b^{\ell-1}}{a^{\ell-1}}.$$

Because of (14.1), one has for $i = 1$ even that there exists $\tau > 0$ (depending on ρ only) so that

$$\frac{|R|}{|I|} \geq [\mu_1]^{\ell-1-\tau} \cdot \frac{|f(R)|}{|f(I)|} \quad \text{or} \quad \frac{|R|}{|I|} \geq 1. \tag{14.3}$$

Now we will start pulling back the intervals R^0, I^0. Let E^0 be the component of $T^{-1} \setminus T^0$ which is adjacent to I^0. We will pull-back the interval $R^0 \cup I^0 \cup E^0$.



$T^0 = I^0 \cup R^0$

Since $\mathcal{R}^k g$ has a *high* return, $R^0 = f^m(T^{k+1}) \supset W^{k+1}$ where W^{k+1} is the escape interval associated to $\mathcal{R}^k g$. Let us consider the case that the inequality in assertion (2) in the statement of the lemma does not hold. Then, writing $t = |T^0|/|W^{k+1}| > 1$, we have

$$(|T^0|/|W^{k+1}|) \leq (1-\sigma)^{-1} \cdot (|T^{-1}|/|T^0|). \tag{14.4}$$

Hence the relative size of the intervals R^0, I^0 and E^0 can be estimated as in the figure above. Therefore, if (14.4) holds,

$$\begin{aligned} C^{-1}([E^0, R^0], I^0) &\geq \frac{(1+1/t)(t(1-\sigma)-1)}{(1-1/t)(1+t(1-\sigma))} \\ &= 1 + \frac{-2\sigma t}{(t-1)[t(1-\sigma)+1]} \geq 1-\epsilon(\sigma) \end{aligned}$$

where $\epsilon(\sigma)$ is some function so that $\epsilon(\sigma) \to 0$ as $\sigma \to 0$. Here we have used that $t \geq |T^0|/|T^1| > 1$ is bounded away from 1 because of (14.1). Hence, using the map $f^{m-n(p_r)-1}$ which sends the interval $H \supset R^1 \cup I^1$ diffeomorphically onto $R^0 \cup I^0 \cup E^0$ we get $|f(R^1)|/|f(I^1)| \geq C^{-1}(H, I^1) \geq 1-\epsilon(\sigma)$. Because of (14.1) we can apply (14.3) and get

$$\frac{|R^1|}{|I^1|} \geq (1-\epsilon(\sigma))\mu_1^{\ell-1-\tau} = (1-\epsilon(\sigma))\mu_{p_r}^{\ell-1-\tau}. \tag{14.5}$$

Let us now compare $|R^{p_{i+1}}|/|I^{p_{i+1}}|$ with $|R^{p_i}|/|I^{p_i}|$. Here $k+1 \geq p_1 > \ldots > p_r := 1$ are defined as in the previous lemma. There exists an interval $K^{p_i} \supset f(A^{p_i})$ which is mapped monotonically onto $T_{p_{i+1}-1}$ by $f^{s(p_i)}$. Let $E^{p_{i+1}}$ be the component of $T_{p_{i+1}-1} \setminus A^{p_{i+1}}$ containing c (again the extension). Then because $I^{p_{i+1}} \cup R^{p_{i+1}} = A^{p_{i+1}} \subset T_{p_{i+1}-1} \setminus T_{p_{i+1}}$ and $E^{p_{i+1}} \supset T_{p_{i+1}}$ and because $I^{p_{i+1}}$ is between c and $R^{p_{i+1}}$ (see the first assertion of the previous lemma), we have

$$\frac{|f(R^{p_i})|}{|f(I^{p_i})|} \geq C^{-1}(K^{p_i}, f(I^{p_i})) \geq$$

$$\geq C^{-1}([E^{p_{i+1}}, R^{p_{i+1}}], I^{p_{i+1}}) \geq \frac{|R^{p_{i+1}}|}{|I^{p_{i+1}}|}\frac{|T_{p_{i+1}}|}{|T_{p_{i+1}-1}|} \geq \mu_{p_{i+1}}\frac{|R^{p_{i+1}}|}{|I^{p_{i+1}}|}.$$

Hence, using (14.2),

$$\frac{|R^{p_i}|}{|I^{p_i}|} \geq \mu_{p_{i+1}}\mu_{p_i}^{\ell-1}\frac{|R^{p_{i+1}}|}{|I^{p_{i+1}}|} \tag{14.6}$$

All this implies that if (14.4) holds then combining (14.5) and (14.6) we get

$$\frac{|R^{k+1}|}{|I^{k+1}|} = \frac{|R^{p_1}|}{|I^{p_1}|} \geq (1-\epsilon(\sigma))(\mu_{p_r}\cdot\ldots\cdot\mu_{p_2})^{\ell}\mu_{p_1}^{\ell-1}\mu_1^{-\tau}$$

and applying an estimate as the one above (14.6) to the map $f^{n(p_1)}\colon A \to A^{p_1-1}$ we obtain,

$$\frac{|R'|}{|I'|} \geq (1-\epsilon(\sigma))(\mu_{p_r}\cdot\ldots\cdot\mu_{p_1})^{\ell} \geq (1-\epsilon(\sigma))\left(\frac{|T^{k+1}|}{|T^0|}\right)^{\ell}\left(\frac{|T^0|}{|T^1|}\right)^{\tau}. \tag{14.7}$$

It follows from (14.7) that there exists $\kappa > 1$ such that provided $\sigma > 0$ is sufficiently small

$$\frac{|R'|}{|I'|} \geq \kappa \cdot \left(\frac{|T^{k+1}|}{|T^0|}\right)^{\ell} \tag{14.8}$$

Since $|R'| = |T^{k+1}|^{\ell}$ and the pullback under f^{m-1} of $D_*(T^0)$ is inside $D_*([R', I'])$ where c_1 is the unique common point of R' and I', the last inequality implies that the pullback of the disc $D_*(T^0)$ (the central domain D_0) along the central branch fits again inside $D_*(T^0)$. By Schwarz, the non-central domains are inside discs based on their real trace. All these domains are disjoint, because the f-images of these (near c_1) are inside discs based on disjoint intervals in the real line. By construction the modulus of the annulus $D_*(T^0)\setminus D_0$ is universally bounded away from zero.

Let us now prove assertion (3) of the lemma. Since the last return is high the expression in (**??**) can be replaced by $1 \times K/(2+K)$ which tends to 1 as $K \to \infty$. Hence then $|R'|/|I'| \geq K/(2+K)\left(|T^{k+1}|/|T^0|\right)^{\ell}\left(\frac{|T^0|}{|T^1|}\right)^{\tau}$ becomes a definite factor larger than $\left(|T^{k+1}|/|T^0|\right)^{\ell}$ when K is large. Again we get a polynomial-like mapping with a definite lower bound for the modulus of the annulus formed by the target domain $D_*(T^0)$ minus the central domain D^0. □
The proof of Theorem C Let W_n be the sequence of intervals as in Section 2 and let R_n be the corresponding first return maps. As before, we say that the return is *low* if the image of R_n of the central component W_{n+1} does not contain c and it is called *central* if $R_n(c) \subset W_{n+1}$. One has the following a priori bounds (these bounds were first obtained by Martens, see [Mar]):

LEMMA 14.5. *There exists a universal number $\lambda > 1$ (only depending on ℓ) the following property. Assume that we are in one of the following situations: 1) either R_{n-1} or R_n has a non-central low return; 2) the return of R_{n-1} is non-central high. Then $|W_n| \geq \lambda |W_{n+1}|$.*

Proof: If R_n has a non-central low return then $|W_n| \geq \lambda |W_{n+1}|$ according to the corollary on page 345 in [MS]. In the same way, if the return to W_{n-1} is non-central low then again $|W_{n-1}| \geq \lambda |W_n|$. Now let the first return map to W_n restricted to W_{n+1} be equal to f^s. There exists an interval neighbourhood T of $f(W_{n+1})$ such that f^{s-1} maps T diffeomorphically onto W_{n-1} and so that $f^{-1}(T) \cap \mathbb{R} \subset W_n$. Hence, using the Koebe Principle it follows that $f^{-1}(T) \cap \mathbb{R} \subset W_n$ is a definite neighbourhood of W_{n+1}. So in this case we are done also. □

Of course, there are infinitely many integers n for which the map R_{n-2} has a non-central return. Choose such an n and write $T^0 = W_n$, $T^{-1} = W_{n-1}$.

From the lemma above, there exists a universal $\lambda > 1$ (only depending on ℓ) such that

$$|T^{-1}| \geq \lambda |T^0|. \tag{14.9}$$

Now consider the return map $g \in \mathcal{E}(T^0, T^{-1})$ to T^0 with central domain $T^1 = W_{n+1}$. From Lemma 14.4 it follows that for some k either (i) $\mathcal{R}^k g$ has a polynomial-like extension such that the central domain D_0 is inside a disc based on $T^{k+1} = D_0 \cap \mathbb{R}$ and such that T^0 is a definite amount larger than T^{k+1} or (ii) it does not have a low return iterate. In the first case, one obviously has that the modulus of $D_*(T_0) \setminus D_0$ is bounded away from zero. Let $G\colon \cup D_i \to D$ be the polynomial-like mapping which we just constructed. This map does not necessarily have the required first return property. However, if we let $\hat{G}$ be the first return map of G to the central domain, then we obtain again a polynomial-like mapping $\hat{G}\colon \hat{D}_i \to \hat{D} = D_0$. The modulus of $\hat{D} \setminus \hat{D}_0$ is equal or larger to $1/\ell$ times the modulus of $D \setminus D_0$. Because of (2) of Lemma 14.4, if $\hat{D}_i$ is a non-central domain of $\hat{G}$ then the modulus of $\hat{D} \setminus \hat{D}_i$ is also universally bounded away from zero.

In case (ii), the new map $\mathcal{W}\mathcal{R}^k g$ (as defined above Lemma 14.2) is then defined on a smaller domain V^{k+1}, i.e., $\mathcal{W}\mathcal{R}^k g \in \mathcal{E}(V^{k+1}, T^0)$. In fact, this map is the first return map of g to V^{k+1}. If $\mathcal{R}^k g$ has no polynomial-like extension then because of the second assertion in the previous lemma, the corresponding 'Koebe space' $|T^0|/|V^{k+1}|$ increases by a definite factor $(1-\sigma)^{-1} > 1$ (compared to the previous Koebe space $|T^{-1}|/|T^0|$). Applying this process several times, either one obtains a polynomial extension at some stage or the Koebe space becomes arbitrarily large (compared to the size of the domains). But from the last assertion of Lemma 14.4 one then also obtains a polynomial-like extension with the required properties. □

Remark. Theorem C holds for a wider class of maps, namely, for any polynomial f with the real coefficients, such that: one critical point $c = 0$ of an even multiplicity ℓ has a bounded orbit, while the other critical points escape to infinity, and such that f is not renormalizable on the real axis. To prove this, we argue as follows.

First, we construct an ℓ-polynomial-like mapping as follows. Fix a level curve Γ of the Green function of the polynomial f, such that Γ is connected and not critical, i.e., all preimages $f^{-n}(\Gamma)$ are smooth curves. Since all but one critical points of f escape, one can pick up a full preimage $\gamma = f^{-k}(\Gamma)$ such that γ bounds finitely many topological discs $V_0, V_1, ...,$ such that $f : V_0 \to f(V_0)$ is ℓ-covering, and all other $f : V_i \to f(V_i)$ are one-to-one. Denote $V_0 = \Omega$ (the only domain containing the critical point $c = 0$ of f). The desired ℓ-polynomial-like mapping $G\colon \cup_{i=0}^{i_0} \Omega^i \to \Omega$ will be the *first return map* of the points of the set $\omega(c) \cap \Omega$ to Ω. Since $\omega(c)$ is minimal, the definition is correct.

G obeys (by construction) the first return property: if $P^n(c) \in \Omega$, then $P^n(c)$ is an iterate of G. Moreover, the map G is real since it is the first return of a real map. Now we want to quasi-conformally conjugate G to a real polynomial with all critical points real (to insure that the inverse branches of the new map have univalent extensions to corresponding slit complex planes). To this end, let us consider the restriction G to $\mathbb{R}$ of the map $G\colon \cup_{i=0}^{i_0}\Omega^i \to \Omega$ to the real axis, and add (finitely many) components to the domain of definition of G in such a way, that the new map $\hat{G}$ is again a real ℓ-polynomial-like map, with the property that the graph of the new map on the real axis "looks like" a graph of a polynomial with all critical points being real, i.e., the monotone increasing and monotone decreasing branches alternate each other. Now we can use the construction of Straightening Theorem from Section 2 to conjugate the polynomial-like map $\hat{G}$ with a polynomial $\hat{f}$, so that all critical points of $\hat{f}$ are real. Moreover, the quasi-conformal conjugacy between $\hat{f}$ and $\hat{G}$ will transfer any polynomial-like structure induced by $\hat{f}$ to a one induced by $\hat{G}$. On the other hand, since we consider the induced maps the first returns along the postcritical set, and since the $\hat{G}$-orbit of the critical point visits only the branches of the original map G, the maps induced by $\hat{G}$ are induced by G (and so by the original f as well). This means that we can assume that the map G belongs to a class of the maps $\mathcal{E}(T)$, with the difference that the map f in the definition of $\mathcal{E}(T)$ is not exactly of the form $z^\ell + c_1$. To handle this, we make the following change in the proof of Lemma 14.4. In (14.2) a factor of the form $1 + \phi(L)$ will appear, where L is the length of the interval $R \cup I$, and $|\phi(L)| \le K \cdot L$, where K is a constant, which depends on f only. Consequently, in (14.7) a product of the factors like this will appear, which will tend to 1 as the interval T will shrink to the critical point c. Hence Theorem C applies in this case also.

15. Appendix: Geometry of Poincaré discs

As before, denote $D(J;\theta)$ the Poincaré neighbourhood of a real interval J, with external angle $\theta \in (0, \pi/2]$.

LEMMA 15.1. *Denote $P_2(z) = z^2$. Let $K > 1$. There exists $\theta_0 = \theta_0(K) > 0$ such that, for all $\theta \in (0, \theta_0)$, the boundaries of $P_2(D((-1,1);\theta))$ and $D((-K,1);\theta)$ intersect each other at a point $Z(K,\theta)$ and at its complex conjugate. Furthermore,*

$$Z(K,\theta) \to K^2 \in \mathbb{R},$$

as $\theta \to 0$. Hence, the difference $\Delta_2(K,\theta) = D((-K,1);\theta) \setminus P_2(D((-1,1);\theta))$ tends to the interval $[1, K^2]$, as $\theta \to 0$.

Proof: Consider a possible intersection point

$$z_1 \in P_2(\partial D((-1,1);\theta)) \cap \partial D((-K,1);\theta).$$

That is, $z_1 = P_2(z_2) = z_2^2$ with

$$z_2 \in \partial D((-1,1);\theta)).$$

Since these sets are symmetric with respect to the real axis and since $D((-1,1);\theta)$ is also symmetric with respect to the imaginary axis, we may consider the case that z_1 is in the upper half plane, and that z_2 is in the first quadrant.

Since $z_2 \in \partial D((-1,1);\theta)$,

$$z_2 = 1 + \frac{i\exp(i\theta)}{\sin(\theta)}(1-\exp(i\alpha)) \tag{15.1}$$

where $\alpha \in (0, 2\pi - 2\theta)$ is the angle between the vectors $z_2 - C, 1 - C$, with C the centre of the circle $D((-1,1);\theta)$. Similarly, since $z_1 \in \partial D((-K,1);\theta)$,

$$z_1 = 1 + \frac{K+1}{2}\cdot\frac{i\cdot\exp(i\theta)}{\sin(\theta)}\cdot(1-\exp(i\phi)). \tag{15.2}$$

Taking the square of (15.1),

$$z_2^2 = 1 + \frac{2\exp(i\theta)}{\sin^2(\theta)}(1-\exp(i\alpha))\left\{i\sin(\theta) - \frac{1}{2}\exp(i\theta)(1-\exp(i\alpha))\right\}.$$

We have in the brackets $\{...\}$ term:

$$i\sin(\theta) - \frac{1}{2}\exp(i\theta)(1-\exp(i\alpha)) = \frac{1}{2}(\exp(i\theta) - \exp(-i\theta) - \exp(i\theta) + \exp(i(\theta+\alpha))) =$$

$$\frac{\exp(i\alpha)}{2}(\exp(i\theta) - \exp(-i(\theta+\alpha))).$$

So,

$$z_2^2 = 1 + \frac{2\exp(i\theta)}{\sin^2(\theta)}(1-\exp(i\alpha))\frac{\exp(i\alpha)}{2}(\exp(i\theta) - \exp(-i(\theta+\alpha))) =$$

$$1 + \frac{2\exp(i\theta)}{2\sin^2(\theta)}\exp(i\alpha)\{\exp(i\theta) - \exp(-i(\theta+\alpha)) - \exp(i(\theta+\alpha)) + \exp(-i\theta)\} =$$

$$1 + \frac{\exp(i(\theta+\alpha))}{\sin^2(\theta)}\{2\cos(\theta) - 2\cos(\theta+\alpha)\}.$$

If $z_2^2 = z_1$, then we compare the last expression with (15.2), cancel 1 on both sides of the equation, divide them by $\exp(i\theta)$ and multiply by $\sin(\theta)$, and after that separate the real and imaginary parts:

$$\frac{2(\cos(\theta) - \cos(\theta+\alpha))}{\sin(\theta)}\cos(\alpha) = \frac{K+1}{2}\sin(\phi),$$

$$\frac{2(\cos(\theta)-\cos(\theta+\alpha))}{\sin(\theta)}\sin(\alpha)=\frac{K+1}{2}(1-\cos(\phi)).$$

Now divide the second equality by the first one:

$$\tan(\alpha)=\tan(\phi/2).$$

Here $0<\phi/2<\pi$, so either $\alpha=\phi/2$, or $\alpha=\phi/2+\pi$. The latter case is impossible, since z_2 is in the first quarter, i.e. $0<\alpha+\theta<\pi/2$.

Thus,

$$\alpha=\phi/2.$$

Now we substitute $\phi=2\alpha$ in the equality, say, for the real parts:

$$\frac{2(\cos(\theta)-\cos(\theta+\alpha))}{\sin(\theta)}\cos(\alpha)=\frac{K+1}{2}\sin(2\alpha),$$

or

$$4\sin(\alpha/2)\sin(\theta+\alpha/2)\cos(\alpha)=\frac{K+1}{2}2\sin(\alpha)\cos(\alpha)\sin(\theta),$$

or

$$\sin(\theta+\alpha/2)=\frac{K+1}{2}\cos(\alpha/2)\sin(\theta),$$

or

$$\cos(\theta)\sin(\alpha/2)=\frac{K-1}{2}\cos(\alpha/2)\sin(\theta),$$

or, at last,

$$\tan(\alpha/2)=\frac{K-1}{2}\tan(\theta).$$

If θ here is small, we find a unique α in the admissible interval $(0,\pi-\theta)$ for the angle α (this is because $\alpha=\phi/2\in(0,\pi-\theta)$). So there is the unique point of intersection of the curves $\partial P_2(D((-1,1);\theta))$ and $\partial D((-K,1);\theta)$ in the upper half-plane. If $\theta\to 0$, then $\phi\sim 2(K-1)\theta$ and $z_1\sim K^2$. The rest of the lemma follows easily. □

The next lemma will allow us to apply Lemma 15.1 for any degree $\ell\geq 4$ (and not just for $\ell=2$). Let us denote for simplicity $D(\theta)=D((-1,1);\theta)$. Set

$$\Pi_\ell(\theta)=\{z\in D(\theta):0\leq\arg z\leq\pi/\ell\}.$$

As before, $P_\ell(z)=z^\ell$.

Lemma 15.2. *Fix $0<\theta<\pi/2$. Then*

$$P_\ell(\Pi_\ell(\theta))\subset P_{\ell+2}(\Pi_{\ell+2}(\theta)),\quad \ell=2,4,\ldots.$$

Proof: It is easy to see that the reverse inclusion cannot hold (this can be seen consider the points in these sets on the negative real axis). Now assume the contrary. Then the boundaries of $P_\ell(\Pi_\ell(\theta))$ and $P_{\ell+2}(\Pi_{\ell+2}(\theta))$ have a common non-real point, i.e., $z^\ell = u^{\ell+2}$, for some $z \in \partial D(\theta)$, and $u \in \partial D(\theta)$. Hence, $u = z^t$, with $t = \ell/(\ell+2)$ between zero and 1. The point z^t belongs to an arc

$$\Gamma(1, \theta_1; 0, \Lambda_0) = \{z = \exp\{\Lambda \cos(\theta)/\sin(\theta)\} \exp(i\Lambda) : 0 < \Lambda < \Lambda_0\}$$

of a logarithmic spiral starting at the points 1 and ending at $z \in \partial D(\theta)$ and crossing the circle $\partial D(\theta)$ at the other point u. Note that $\theta_1 \le \theta$ is not possible since then the spiral $\Gamma(1, \theta_1; 0, \infty)$ would not cross the disc $D(\theta)$ at all. So consider the case $\theta_1 > \theta$. Then $\Gamma(1, \theta_1; 0, \Lambda)$ is inside $D(\theta)$, if Λ is small. Hence, there are two points z_1, z_2 of the intersection of the arc $\Gamma(1, \theta_1; 0, \Lambda_0)$ with $\partial D(\theta)$, such that $\arg z_1 < \arg z_2$, and this arc leaves the disc $D(\theta)$ at z_1 and again enters it at z_2. The angle under which the logarithmic spiral intersects the rays via zero is constantly θ_1. Hence, the angle between the vector z_1 and the circle $D(\theta)$ is at least θ_1, and the angle between the vector z_2 and the circle $D(\theta)$ is at most θ_1. This is a contradiction with the fact that the angle between a vector $w \in \partial D(\theta)$ and the tangent to $\partial D(\theta)$ at w is increasing as $w \in \partial D(\theta)$ moves from 1 to -1 (in fact, it increases from θ to $\pi - \theta$). □

Denote $\Delta_\ell(K_0, \theta) = D((-K_0, 1); \theta) \setminus P_\ell(\Pi_\ell(\theta))$

LEMMA 15.3. *For all θ less than an absolute positive θ_0, and for $k_0 = 0.89$, $K_0 = 1.52$,*

$$\Delta_2(K_0, \theta) \subset P_\ell\left(D((0,1); k_0\theta) \cap \{z;\ 0 < \arg z < \pi/\ell\}\right), \ell = 4, 6, \ldots \quad .$$

Proof: By virtue of the previous lemma, it is enough to find the constant θ_0 for $\ell = 4$ only. So, assume that this is not the case. That is, for some sequence $\theta \to 0$, $z_1 = z_2^4$, for some $z_1 \in \partial D((-K_0, 1); \theta)$ and $z_2 \in \partial D((0,1); k_0\theta)$. Moreover, $\arg z_2 < \pi/4$, and, what is crucial, since we were able to apply Lemma 15.1, z_1 tends to a point of the real interval $[1, K_0^2]$ (for some subsequence) as $\theta \to 0$. We have

$$z_1 = 1 + \frac{K_0 + 1}{2} \cdot \frac{i \cdot \exp(i\theta)}{\sin(\theta)} \cdot (1 - \exp(i\phi)). \tag{15.3}$$

where $\phi \in (0, 2\pi - 2\theta)$ is the angle between the vectors $z_1 - C, 1 - C$, with C the centre of the circle $D((-K_0, 1); \theta)$. For $z_2 \in \partial D((0,1); k_0\theta)$ we have a similar expression:

$$z_2 = \frac{\sin(k_0\theta + \gamma)}{\sin(k_0\theta)} \cdot \exp(i\gamma), \tag{15.4}$$

where $\gamma \in (0, \pi/4)$ is an argument of z_2. Since z_1 tends to a real point in $[1, K_0^2]$, it follows from (15.3), that ϕ/θ tends to a non-negative finite constant B, as $\theta \to 0$, and $1 + \frac{K_0+1}{2} B \le K_0^2$, i.e.,

$$0 \le B \le 2(K_0 - 1). \tag{15.5}$$

Hence, from the condition $z_1 = z_2^4$ and from (15.4), γ/θ tends to a finite $D \geq 0$. Separating now real and imaginary parts of the equality $z_1 = z_2^4$, we obtain the following system for B and D:

$$1 + \frac{K_0+1}{2} \cdot B = (1+D_1)^4 \tag{15.6}$$

$$\frac{K_0+1}{2} \cdot B \cdot (1+\frac{B}{2}) = k_0(1+D_1)^4 \cdot 4 \cdot D_1, \tag{15.7}$$

where

$$D_1 = D/k_0,$$

with

$$k_0 = 0.89 \quad \textit{and} \quad K_0 = 1.52.$$

Note that we have that $B, D > 0$, because otherwise consider B and D as infinitely close to zero when $\theta \to 0$. Then (15.6)-(15.7) imply $\frac{K_0+1}{2}B \sim 4 \cdot D_1$ resp. $\frac{K_0+1}{2}B \sim 4k_0 \cdot D_1$, what is impossible. Therefore, $B > 0$ and $D > 0$.

Dividing (15.7) by (15.6) and substituting the obtained expression for D_1 into (15.6), we come to the equation:

$$1 + \frac{K_0+1}{2} \cdot B = \{1 + \frac{1}{4 \cdot k_0} \cdot \frac{K_0+1}{2} \cdot \frac{B \cdot (1+\frac{B}{2})}{1+\frac{K_0+1}{2} \cdot B}\}^4, \tag{15.8}$$

where

$$k_0 = 0.89,$$

and

$$K_0 = 1.52.$$

With these K_0 and k_0, the equation (15.8) has no positive solutions.

In order to see this, let us make a linear change of the variable: define $x = 1 + \frac{K_0+1}{2}B$, so that $1 \leq x < \infty$. Then

$$\frac{K_0+1}{2}\frac{B(1+\frac{B}{2})}{1+\frac{K_0+1}{2}B} = 1 + \frac{x-2}{K_0+1} - \frac{K_0}{K_0+1}\frac{1}{x} := T(x)$$

(the latter equality is just notation).

The equation takes the form:

$$x = (1 + \frac{1}{4k_0}T(x))^4,$$

or

$$(1 + \frac{1}{4k_0}T(y^4)) - y = 0, \tag{15.9}$$

where $x = y^4$. Let us show that the latter equation has no solutions $y > 1$. Since $y = 1$ is always a solution, it is enough to check that the derivative of the left-hand side of (15.9) is positive for $y > 1$, i.e. the equation

$$\frac{1}{k_0}\frac{1}{K_0+1}(1+K_0t^8) - t^3 = 0 \tag{15.10}$$

has no solutions $0 < t < 1$ (we have used here that $T'(x) = \frac{1}{K_0+1}(1+K_0\frac{1}{x^2})$ and set $t = 1/y$). Really, the function in the left-hand side of (15.10) has the unique minimum at the point

$$t_0 = [\frac{3k_0(K_0+1)}{8K_0}]^{1/5} = 0.888374...,$$

and the minimum itself is equal to 0.0076767... which is indeed positive. □

REMARK 15.4. *Firstly, let us note that the constant* 0.9 *gives a negative minimum of the derivative* (*15.10*)*; this is precisely the reason we chose* $k_0 =$ 0.89*. Secondly, the lemma remains true for* $\ell = 2$*, if we take* $K_0 = 1.37$ (*which corresponds to the extendibility space* 0.5)*, and* $k_0 = 0.8$*. In the proof we should take the equation*

$$\frac{1}{k_0}\frac{1}{K_0+1}(1+K_0t^4) - t = 0 \tag{15.11}$$

instead of (*15.10*)*. The minimum of the left-hand side becomes equal to*

$$\frac{1}{k_0}\frac{1}{K_0+1} - \frac{3}{4}[\frac{k_0(K_0+1)}{4K_0}]^{1/3}, \tag{15.12}$$

and this value is again positive, because

$$k_0 = 0.8 < \frac{4K_0^{1/4}}{3^{3/4}(K_0+1)} = 0.801034.... \tag{15.13}$$

LEMMA 15.5. *If* θ *is small, and* $k_0 = 0.89$*, then*

$$D((-1.52, 0.68); k_0\theta) \subset P_2(D((-1,1);\theta)). \tag{15.14}$$

Proof: Since $1/0.68 = 1.47059 > 1.47$ and $1.52/0.68 = 2.2352941... <$ 2.5, it is enough to prove (after rescaling) that

$$D((-2.5,1); k_0\theta) \subset P_2(D((-(1.47)^{1/2}, (1.47)^{1/2}); \theta)). \tag{15.15}$$

For a possible point Z of intersection of the boundaries, we obtain an equation

$$1+\frac{2.5+1}{2}\cdot\frac{i\exp(ik_0\theta)}{\sin(k_0\theta)}\cdot(1-\exp(i\phi)) = 1.47\cdot(1+\frac{\exp(i(\theta+\alpha))}{\sin^2\theta}\cdot(2\cos\theta-2\cos(\theta+\alpha))).$$

If $\theta \to 0$ and ϕ, α is a solution of this equation, then ϕ/θ and α/θ tend to finite constants $M \geq 0$ and $N \geq 0$ respectively (proof: $P_2(D((-(1.47)^{1/2}, (1.47)^{1/2}); \theta))$ certainly contains $P_2(D((-1,1);\theta))$ while the boundary of the latter domain intersects $D((-K,1);k_0\theta)$ at a point $Z(K,\theta)$ of an angle ϕ_1 such that ϕ_1/θ is bounded as $\theta \to 0$, see proof of Lemma 15.1.) We have the following equations for M and N:

$$1 + \frac{2.5+1}{2} \cdot \frac{M}{k_0} = 1.47 \cdot (1+N)^2 \tag{15.16}$$

$$\frac{2.5+1}{2} \cdot \frac{M}{k_0} \cdot (k_0 + \frac{M}{2}) = 1.47 \cdot 4 \cdot \frac{N}{2} \cdot (1 + \frac{N}{2}) \cdot (1+N). \tag{15.17}$$

This system has no non-negative solutions (M,N). A way to see this is to reduce the system to a polynomial equation. For this, we first set $M_1 = M/k_0$ and then denote $M+1 = x$, $N+1 = y$. Then

$$\frac{x-1}{2} = \frac{1.47y^2 - 1}{2.5+1} \tag{15.18}$$

$$\frac{x+1}{2} = \frac{1}{k_0} \frac{1.47(y^2-1)y}{1.47y^2 - 1}. \tag{15.19}$$

This implies that y is a zero of the polynomial

$$h(y) := y^4 - \frac{2.5+1}{1.47k_0} y^3 + \frac{2.5-1}{1.47} y^2 + \frac{2.5+1}{1.47k_0} y - \frac{2.5}{1.47^2}, \tag{15.20}$$

where, as before $k_0 = 0.89$. However, the polynomial $h(y)$ does not have solutions $y \geq 1$. Indeed, the second derivative of this polynomial $h''(y)$ is a parabola with zero's at $y = .142276$ and at $y = 1.195337$. If follows that $h'(y)$ is a cubic function with a local maximum at $y = .142276$ and a local minimum at $y = 1.195337$. An explicit calculation shows that $h'(1.195337) = .4790869... > 0$ and therefore, one has that $h'(y) > 0$ for each $y \geq 1$. Hence $h(y) \geq h(1) > 0$ for each $y > 1$. $\square$

REMARK 15.6. *The statement remains true with the constants* $K_0 = 1.37$ (*instead of* 1.52), $k_0 = 0.8$, *and* 0.6 (*instead of* 0.68). *Since* $1/0.6 = 5/3$ *and* $1.37/0.6 = 2.2833... < 2.3$, *the polynomial* h *will take the form:*

$$h(y) := y^4 - \frac{2.3+1}{(5/3)k_0} y^3 + \frac{2.3-1}{(5/3)} y^2 + \frac{2.3+1}{(5/3)k_0} y - \frac{2.3}{(5/3)^2}, \tag{15.21}$$

where $k_0 = 0.8$. *The minimum of the derivative for* $y > 1$ *will be equal to* $h'(1.12159...) = .527977... > 0$.

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