The two-fixed point lemma*

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Abstract

Complicated (chaotic), global, expectations-driven business cycles in two-dimensional models have been shown to involve non-trivial intersections of stable and unstable manifolds of a (periodic) saddle steady state. Whether similar phenomena may occur in other two-dimensional dynamic economic models in discrete time is the object of this paper. In fact, it will be shown that if the dynamics is described by an invertible map of the first orthant of the plane, the stable and unstable manifolds of a *unique* steady state cannot intersect non-trivially.
Keywords: dynamic economic models, saddle steady states, Lefschetz index theorem for vector fields, (non-)intersecting stable and unstable manifolds.

1 Introduction

Complicated expectations-driven business cycles may occur in various economic models. For example, the general equilibrium models, suggested by Benhabib and Day [2] and Grandmont [8], focused on simple one-dimensional economies and required large income effects in order to get complicated cyclical behaviour. By introducing productive capital into similar types of models, which increases the dimension of the dynamics of the economy to two, one can show that both local regular (e.g. Reichlin [21], Woodford [24], Grandmont, Pintus and de Vilder [9]) and global irregular (de Vilder [22], Pintus, Sands and de Vilder [20]) cycles are compatible with dominant substitution effects. The main mechanism that accounts for the occurrence of complicated deterministic global fluctuations in the two-dimensional framework involves *intersections* of stable and unstable manifolds of a (periodic) saddle equilibrium (see de Vilder [22], Pintus, Sands and de Vilder [20] and Brock and

Hommes [4], for example).

By contrast, a broad (widely known) class of two-dimensional dynamic economic models, such as the ones studied by King, Plosser and Rebello [15] and Kydland and Prescott [16], have a unique fixed point with a saddle structure. It is not clear from the mathematical literature on the subject (see, for example, Guckenheimer and Holmes [11], Katok and Hasselblatt [14] or Palis and Takens [19]), whether complicated deterministic structures associated with intersections of stable and unstable manifolds can also be present in this widely used framework. More specifically, is it possible for the stable and unstable manifolds of a unique fixed point of a two-dimensional C^1 invertible map of the positive orthant of the plane to intersect non-trivially?

In this paper it is shown that chaos cannot arise from such intersections in these models. That is, we show that a necessary condition for stable and unstable manifolds of a saddle stationary state to intersect non-trivially is that the map has, at least, one additional steady state with positive index; we refer to this finding as *the two-fixed point lemma*. We obtain this result by exploiting the Lefschetz index theorem for vector fields [17]. So only in models with multiple steady states (see for example Hornstein [13], Farmer and Guo [6] and Boldrin and Rustichini [3]) this kind of chaos is possible.

The paper is organized as follows. In the next section we define the types of dynamic economic models we have in mind. In that section we also introduce the notion of stable and unstable manifolds as well as some related results. In section 3 we present the two-fixed point lemma and provide a sketch of the proof. In the section thereafter we give some concluding remarks. Finally, the formal proof of the two-fixed point lemma can be found in the appendix.

2 The Framework

The results of this paper apply to any two-dimensional model of the plane satisfying the axioms specified in the next subsection.

2.1 The dynamic economic model

In this paper we shall assume that the economic model satisfies the following **Standing Assumptions**.

- The phase space is a simply connected open subset U of $\mathbf{R^2}$. For example U can be the positive quadrant $\mathbf{R^2_+}$ without boundary points. Moreover, we assume that time is discrete. We denote the state variables by $(x_n, y_n) \in U, \quad n \in \mathbb{Z}$.
- The dynamics of the economy is described by $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$.
- We assume that fixed points of f are isolated.
- The map $f\colon U \to f(U)$ is C^1 and invertible.

The first three assumptions are extremely general, and are used in a broad class of models. Models that we have in mind are, for example, King et al. [15], Kydland et al. [16] and Weibull [23]. The fourth assumption is more restrictive, because in some models f is not invertible. Whether one believes that the equations of motion also allow for backward motion, is perhaps a matter of taste.

2.2 Intersecting stable and unstable manifolds

We introduce the notion of *stable and unstable manifolds* of fixed points. That is, let $p \in U$ be a hyperbolic fixed point of f, so $f(p) = p$ and the Jacobian Df has two real eigenvalues λ_s and λ_u such that $|\lambda_s| < 1 < |\lambda_u|$. Then the stable and unstable manifolds of p are defined as follows.

$$W^s = \{x \in U \ ; \ f^n(x) \in U \text{ for all } n \geq 0 \text{ and } \lim_{n\to\infty} f^n(x) = p\}.$$

Since $f(U)$ can be not equal to U, it is possible that W^s has several connected components. Similarly, let

$$W^u = \{x \in U \ ; \ f^n(x) \in U \text{ for all } n \leq 0 \text{ and } \lim_{n\to-\infty} f^n(x) \to p\}.$$

If $f(U) \subset U$, the unstable manifold can only have one connected component, but otherwise it is possible that it has many connected components. These manifold are smooth curves passing through p, tangent to the stable and unstable eigenspaces of $Df(p)$, respectively. Contrary to a linear specified model, in a nonlinear framework, the stable and unstable manifolds of p may intersect outside the saddle; these points of intersection of stable and unstable manifolds are known as homoclinic points. These are those points x for which $f^n(x) \to p$ as $n \to +\infty$ and as $n \to -\infty$, see figure 1. If p is a saddle fixed point and the stable and unstable manifolds of p intersect at a point $q \neq p$ then q is called a *homoclinic point* of p. The orbit of a homoclinic point is called a *homoclinic orbit*; each point in it is homoclinic.

3 The Main Result

In this section we present the main result of this paper.

The two fixed-point lemma

Let $f\colon U \to f(U)$ be as in the standing assumption, and let $p \in U$ be a fixed

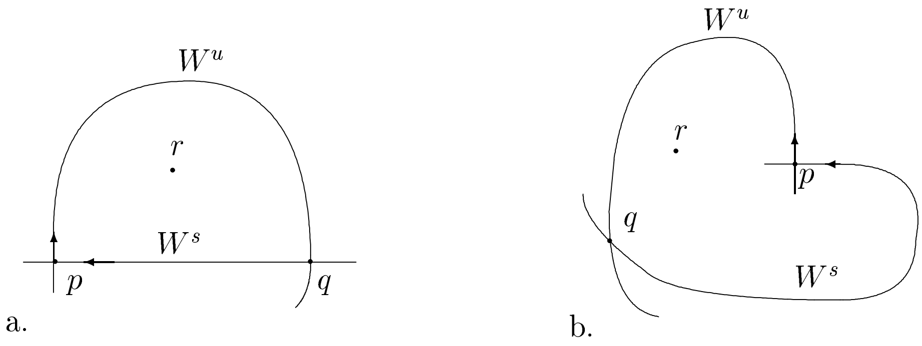

Figure 1: W^u and W^s of the saddle equilibrium p may intersect outside p. We have schematically drawn the situation where W^u intersects W^s in a point q. The point r in the figures is the additional fixed point that exists according to the two-fixed point lemma in section 3.

hyperbolic saddle point of f with positive eigenvalues. Assume that the stable and unstable manifolds of p have a point $q \neq p$ of intersection and that there are curves $\gamma_s \subset W^s$ and $\gamma_u \subset W^s$ in U connecting q and p. Then f has at least one additional fixed point r of positive index in the interior of the domain bounded by γ_s and γ_u, see figure 1 for a graphical illustration.

The index of a fixed point is defined in the appendix.

Remark: If one assumes that $f(U) \subset U$ then the statement of the lemma can be simplified: there is then no need to assume the existence of the connecting curves γ_s and γ_u. Indeed, in this case if the stable and unstable manifold intersect in some point q ($\neq p$), then there exists an integer n such that $f^n(q)$ belongs to the local stable manifolds and there is a piece of this manifold connecting p and $f^n(q)$. Moreover, the piece of the unstable manifold situated between p and $f^n(q)$ is connected because the unstable manifold is connected (here we use again $f(U) \subset U$).

We should also emphasize that we only consider fixed points in the open set U (and not on the boundary). The additional fixed point of positive index the lemma above asserts, is actually in the open set U. So if there are several saddle fixed points (with positive eigenvalues), then since these have index -1, the conclusion of the lemma still applies. If $f(U) = U$ then one can use an extension of a result of Brouwer, see the last lemma in [7]. In that case, f has no recurrent behaviour: for each point $x \neq p$ (where p is a fixed point) there is a neighborhood O so that $f^n(O) \cap f^m(O) = \emptyset$ for all $n \neq m$.

The proof of the lemma can be found in the appendix. Here we just give a sketch. The main tool that we use to prove the two-fixed point lemma is the Lefschetz index theorem[1] for vector fields [17]. Roughly speaking, the index of a vector field V on the plane with respect to an oriented Jordan curve Γ in the plane (i.e. a continuous closed curve without self-intersections on which direction is defined) is equal to the number of full turns the vector field produces when Γ is traversed once (for a formal definition see the appendix). However, the index of V cannot be defined if it has a singularity on Γ. The index is always an integer and stays constant if one continuously deforms Γ. Provided this deformation does not create singularities for V on the curve. Assuming that these conditions are satisfied one can apply the Lefschetz index theorem for vector fields, which says the following:

Lefschetz Index Theorem
Let Γ be a Jordan curve and V a continuous vector field defined on Γ and the set bounded by Γ. Suppose that V has no singularities on Γ and that all singularities inside Γ are isolated. Then the sum of the indices of the singularities of V inside Γ is equal to the index of V on Γ.

As a vector field V we shall use $V(x) = f(x) - x$ which implies that the index of V at a singularity s is equal to the index of s as a fixed point of f. The

[1]For other applications of this theorem to Economic Theory, see for example Balasko [1], Guesnerie and Woodford [10] and Mas-Colell [18].

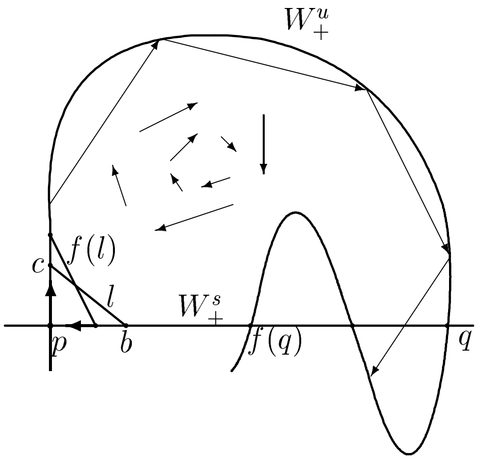

Figure 2: The closed curve Γ consists of l and the pieces $W^s(b,q)$ and $W^u(c,q)$ of stable and unstable manifolds. If the curve Γ is traversed once, V makes one full turn and so the index of V with respect to Γ is equal to $+1$.

index of a hyperbolic fixed point may be defined in terms of the eigenvalues of the Jacobian matrix evaluated at the fixed point[2], see Table 1.

To apply this theory to our map f, we shall define a curve Γ in the appendix by using segments of the stable and unstable manifolds of the fixed point p. Observe that the vector field $V(x) = f(x) - x$ is defined on Γ and its interior D, see figure 2. The final step is the observation that V restricted to Γ makes one full turn when the curve Γ is traversed once (to prove this, we shall use a deformation argument in the appendix). This implies that the index of the vector field V with respect to Γ is $+1$. Hence, from the Lefschetz Index Theorem it follows that there must be at least one singularity of V in the interior of γ of positive index. This singularity corresponds, by the definition of the vector field, with a fixed point of the map f.

[2] Although not generic we have to point out that one may construct fixed points with, for example, index $+2$

	Eigenvalues	Index of fixed point	Description
1	$\|\lambda_1\| < 1$, $\|\lambda_2\| < 1$	$Ind_{Df}(p) = 1$	contracting
2	$\|\lambda_1\| > 1$, $\|\lambda_2\| > 1$	$Ind_{Df}(p) = 1$	expanding
3	$\|\lambda_1\| = \|\lambda_2\| = 1 \neq \lambda_i$	$Ind_{Df}(p) = 1$	elliptic
4	$0 < \lambda_1 < 1 < \lambda_2$	$Ind_{Df}(p) = -1$	hyperbolic saddle
5	$\lambda_1 < -1 < \lambda_2 < 0$	$Ind_{Df}(p) = 1$	hyperbolic saddle with rotation

Table 1: The index (called $Ind_{Df}(p)$) of a fixed point p of a map f may be defined in terms of the eigenvalues of the Jacobian matrix Df evaluated at p. For orientation preserving maps, generically, five different cases can be distinguished for the index of a fixed point.

Remark: By taking the second iterate of the map f one can also account for the orientation reversing cases as well as for the case where both eigenvalues of the saddle are negative by considering the second iterate of f. In these cases the statements will be different since a priori one cannot exclude the presence of a period two orbit instead of an additional fixed point.

4 Conclusion

In this paper we have shown that stable and unstable manifolds of a saddle equilibrium of an invertible two-dimensional dynamic economic model, cannot intersect non-trivially if the model has no other steady states. This means that the system cannot have homoclinic intersections causing chaotic dynamics unless the map has a fixed point of positive index in the (open) domain of definition. So behaviour as observed by Grandmont, Brock, de Vilder and others for expectation driven business cycle models, cannot occur in that case. In other words, only if the two-dimensional dynamic economic model has multiple steady states or if the model does not satisfy the conditions stated in standing assumptions, global analysis is required. Only then one might have “unexpected” complicated deterministic structures.

5 Appendix: The proof of the two-fixed point lemma

We start by providing some useful definitions. First, we define the degree of a circle map $\phi : S^1 \to S^1$, where S^1 is equal to $\mathbf{R}$ modulo 1. We identify S^1 also with the unit circle in $\mathbf{R^2}$ which has the anti-clockwise orientation.

Definition 5.1 *Let ϕ be a continuous map from the circle S^1 into itself. Let Φ be any lift of ϕ to $\mathbf{R}$ (so $\phi(x) = \Phi(x) \pmod 1$ and Φ is continuous). The degree of ϕ is $\Phi(x+1) - \Phi(x)$, where $x \in \mathbf{R}$ is any point. The degree is independent of the choice of Φ and of x.*

A Jordan curve is an injective map $\gamma: S^1 \to \mathbf{R^2}$. We shall write $\Gamma = \gamma(S^1)$. By Jordan's theorem (see any book on topology or for example page 730 of [14]) we know that such a curve divides the plane into two components: one bounded and one unbounded (if for example U is the positive quadrant, then the 'unbounded component' is $\partial \mathbf{R}^2_+$). We shall only consider piecewise smooth curves, and say that $\gamma: S^1 \to \mathbf{R^2}$ is positively oriented if going forward along the curve, the unbounded component is on the right hand side (so positive orientation is the anti-clockwise orientation). In a similar fashion we can define the *negative* orientation of a curve. Next we define the index of a vector field with respect to γ.

Definition 5.2 *Let $\gamma: S^1 \to \mathbf{R^2}$ be a Jordan curve, $\Gamma = \gamma(S^1)$ and $V : \Gamma \to \mathbf{R^2}$ be a vector field which nowhere takes the value 0 (has no singularities). Let Γ be parameterized by some map $\gamma : S^1 \to \Gamma$ which preserves orientation. The* index *of V with respect to Γ is equal to the degree of the circle map ϕ defined by*

$$\phi : x \mapsto \frac{V(\gamma(x))}{|V(\gamma(x))|}.$$

Next we define the index of a singularity of a vector field.

Definition 5.3 *Let V be a vector field defined on an open set U and let $p \in U$ be an isolated singularity of V. Let Γ be a Jordan curve surrounding p in U, separating p from any other singularities of V. The* index of V *at p is defined to be the index of V on Γ. This index is an integer, and is independent of the choice of Γ.*

We define the vector field V by $V(x) = f(x) - x$. Then by definition, the index of a fixed point p of f is equal to the index of the vector field V at p. In the case that the Jacobian Df at p has no eigenvalues equal to 1 (the fixed point is hyperbolic), the index can be defined as $(-1)^{card(i|\lambda_i>1,\, \lambda_i \in \mathbf{R})}$ where λ_i $(i = 1, 2)$ are the eigenvalues of Df evaluated at p. In table 1 (in the core of the text) we have summarized the 5 generically occurring cases.

We are now ready to prove the two-fixed point lemma which we recall here:

The two fixed-point lemma

Let $f\colon U \to f(U)$ be as in the standing assumption, and let $p \in U$ be a fixed hyperbolic saddle point of f with positive eigenvalues. Assume that the stable and unstable manifolds of p have a point $q \neq p$ of intersection and that there are curves $\gamma_s \subset W^s$ and $\gamma_u \subset W^s$ in U connecting q and p. Then f has at least one additional fixed point r of positive index in the interior of the domain bounded by γ_s and γ_u, see figure 1 for a graphical illustration.

Proof of the two fixed-point lemma: Let us introduce some notation that we will use later on. If x, y are points on W^u then $W^u(x, y)$ denotes a piece of W^u bounded by the points x and y. The same notation is applied to W^s.

Let W^s_+ and W^u_+ be components of the stable and unstable manifolds of p which intersect. Remember that W^u_+ is contained in U. We have the 2 curves γ_s and γ_u connecting p and q. Moreover, we can assume that these curves intersect only in the points p and q. Indeed, if it is not the case, then we can take the first point q' of intersection of γ_u with γ_s so that

$W^u(p,q') \cap \gamma_s = \{p, q'\}$. Now, if we denote $\gamma_u' = W^u(p,q')$ and $\gamma_s' = W^s(p,q')$, we obtain two curves intersecting only in p and q'.

Let us now define a closed Jordan curve Γ and a domain D bounded by Γ. Let O be a neighborhood of p on which Hartman-Grobman linearization is possible (see Katok and Hasselblatt [[14], ch.6, p.260]) and so that $W^u(p, f(q)) \cap O$ and $W^s(p,q) \cap O$ have only one component. Take a straight line segment $l = [b, c]$ close to p in O connecting $c \in W^u_+$ with $b \in W^s_+$, where the points b and c are very close to the origin and such that $f(l)$ and $W^s(p, f^{-1}(b))$ are also in O. The curve $\Gamma = W^u(c,q) \cup W^s(q,b) \cup l$ forms a closed Jordan curve in the simply connected domain U and by Jordan's theorem Γ bounds a simply connected region (i.e. a disc) D which is contained in U. We choose on Γ a positive orientation. The origin may or may not be in D, see figure 1 for the two possible cases.

Next we define a vector field V on the closure of D by considering $V(x) = f(x) - x$. Any zero of V is a fixed point of f. By construction V has no zeroes on the boundary Γ of D. This implies that the index $ind_\Gamma(V)$ is well-defined, where the index of a vector field is defined as in definition 5.2. Our aim is to show that $ind_\Gamma(V)$ is equal to $+1$. Using the Lefschetz index theorem for vector fields (see the core of the text) this implies the following proposition

Proposition 5.1 *There is a singularity of V in the interior of D of positive index corresponding to a fixed point $r \neq p$ of f.*

Proof: By the Lefschetz index theorem for vector fields the sum of the indices of singularities of V in the interior of D is equal to $ind_\Gamma(V)$ which is equal to $+1$ as we shall show below. By the definition of V we have that a singularity of V corresponds to a fixed point of f. Hence, if $p \notin D$ then this gives the result immediately. If $p \in D$ then since the index of the fixed point p is -1 (it is a saddle with positive eigenvalues, see table 1), we must have other fixed points in D in order to have that the sum of the indices equals $+1$. $\square$

To prove that the index of V w.r.t. the boundary of D is equal to $+1$ we will continuously deform the vector field on the boundary without creating new singularities. We first define the notion of a *rotational vector field.*

Definition 5.4 *A vector field V defined on a Jordan curve Γ is called* rotational *if it has no singularities on Γ and if for any $x \in \Gamma$ the point $x + V(x)$ also belongs to Γ.*

Proposition 5.2 *A rotational vector field has index $+1$.*

Proof: By an isotopy of the curve (and a corresponding one of the vector field) we may assume that Γ is a circle. Define a new vector field N which at a base point x points to the center of the circle. Next consider a deformation of the vector field defined by

$$V_\lambda(x) = (1 - \lambda)V(x) + \lambda N(x),$$

where $0 \leq \lambda \leq 1$. Notice that $V_0(x) = V(x)$ and that $V_1(x) = N(x)$ and that $V_\lambda(x)$ is not equal to 0 for $0 < \lambda < 1$ for all $x \in \Gamma$. Indeed, if this would not be true then $(1-\lambda)V(x)+\lambda N(x) = 0$ and so $V(x) = -\lambda/(1-\lambda)N(x)$ which would mean that the vector field points outside the unit disc, a contradiction. Since the index of a vector field V_λ is an integer and depends continuously on the parameter, the index of V_λ is a constant. In particular, the indices of $V_0 = V$ and $V_1 = N$ are equal. Obviously the index of N is equal to $+1$, it follows that the index of V is also $+1$. $\square$

To apply this proposition to the vector field V we have to continuously deform V into a rotational vector field without creating singularities. To see why

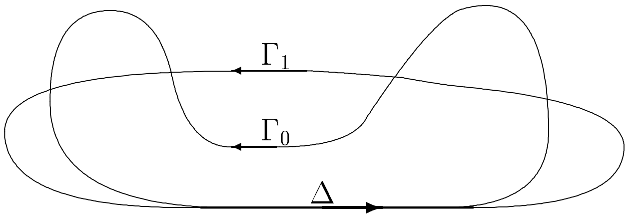

Figure 3: The curves Γ_i from proposition A.3 and the deformation of the vector field V.

it fails to be rotational, notice that a point in $\Gamma \cap W^s_+$ close to q, is not mapped into Γ. Therefore we deform V, and subdivide Γ in four segments $W^u(c, f^{-1}(q))$, $W^u(f^{-1}(q), q)$, $W^s(q, f^{-1}(b))$ and $W^s(f^{-1}(b), b) \cup l$. Obviously V is well defined on these segments and it has the rotational property on the segments $W^u(c, f^{-1}(q))$ and $W^s(q, f^{-1}(b))$. It remains to be shown that V can be deformed such that it also has the rotational property on the remaining two segments. We use the following proposition that is a standard result from topology

Proposition 5.3 *Take two Jordan curves (Γ_0 and Γ_1) which have the same orientation, suppose that these two curves have a common segment Δ and moreover suppose that along this segment Γ_0 and Γ_1 are oriented in the same way (see figure 3 Then there exists a homotopy from the curve $\Gamma_0 \setminus \Delta$ to the curve $\Gamma_1 \setminus \Delta$ without creating intersections with the segment Δ.*

Proof: Since we assume that Γ_i are piecewise smooth, there are curves $\tilde{\Gamma}_i$ near Γ_i homotopic to Γ_i which intersect transversally. Then proceed as in chapter 8 of [12]. $\square$

Notice that Γ_0 and Γ_1 are allowed to intersect in some points which are not in Δ.

We next apply this result to the curves $\Gamma_0 = W^u(p,q) \cup W^s(p,q)$, $\Gamma_1 = W^u(p,f(q)) \cup W^s(f(q),p)$ and $\Delta = W^s(f(q),p) \cup W^u(p,q)$. Γ_0 and Γ_1 are closed curves without self-intersection because of the choice of the point q. Notice also that $\Gamma_1 = f(\Gamma_0)$. If we orient these curves in the direction of the unstable manifold, then they will have the same orientation because f is an orientation preserving. Moreover, these curves have the same orientation along Δ. Thus we can apply the previous proposition and obtain a homotopy from $W^u(q,f(q))$ to $W^s(q,f(q))$ without crossing $W^u(p,q)$. Let us denote this homotopy by $\psi_\lambda : W^u(q,f(q)) \to \mathbf{R^2}$, $\lambda \in [0,1]$, so $\psi_0 = Id$, $\psi_1(W^u(q,f(q))) = W^s(q,f(q))$.

Now we can define a deformation of the vector field V on the segment $W^u(f^{-1}(q),q)$ by

$$V_\lambda(x) = \psi_\lambda(f(x)) - x,$$

where $x \in W^u(f^{-1}(q),q)$ and $0 \leq \lambda \leq 1$. Obviously, $V_0 = V$ because $\psi_0 = Id$, V_λ is never singular because $\psi_\lambda(x)$ is never in $W^u(p,q)$ and V_1 satisfies the rotational property on the segment $W^u(f^{-1}(q),q)$ for the curve $\Gamma = \partial D$.

Since in the small neighborhood O of p (the segments $l = [b,c]$, $f(l)$ and $W^s(p,f^{-1}(b))$ are contained in O), f is almost a linear map, it is easy to see that one can continuously deform the vector field V on $l \cup W^s(f^{-1}(b),b)$ without creating singularities in such a way that the vector field becomes rotational: simply again use proposition A.3.

So, we have shown that V can be continuously deformed on the curve $\Gamma = \partial D$ to a rotational vector field without creating singularities. Once again, since the index depends continuously on the parameter this implies that the index of V on Γ is equal to the index of the rotational vector field which is $+1$.

Together with proposition 5.1 this proofs the lemma. □

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