Stable maps are dense in dimensional one

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Abstract

This is an exposition of our resent results contained in [KSvS03b], [KSvS03a] and [KvS06] where we prove the density of hyperbolicity for one dimensional real maps and non-renormalizable complex polynomials. The proofs of these results are very technical, so in this paper we try to show the main ideas on some simplified examples and also give some outlines of the proofs.

1 Introduction

One of the central aims in dynamical systems is to describe dynamics of a 'typical' system. In this article we will understand the word 'typical' from the topological point of view.

The nicest kind of system is one which is stable (also called *structurally stable*): this means that it is topologically conjugate to any sufficiently nearby system. This notion is closely related to that of hyperbolicity of the system (see below).

The most ambitious hope would be to show that structurally stable and hyperbolic maps are dense. Apparently, up to the late 1960's, Smale believed that hyperbolic systems are dense in all dimensions, but this was shown to be false in the early 1970's for diffeomorphisms on manifolds of dimension ≥ 2 (by Newhouse and others).

However, in dimension one hyperbolic systems are dense. This is the topic of this article.

2 Density of Hyperbolicity

The problem of density of hyperbolicity in dimension one goes back in some form to Fatou (in the 1920's). Smale gave this problem 'naively' as a thesis problem in the 1960's (to Guckenheimer and Nitecki), see [Sma00]. The problem whether hyperbolicity is dense in dimension one was studied by many people, and it was solved in the C^1 topology by Jakobson, see [Jak71] and the C^2 topology by Shen [She04].

Theorem 1 (K, Shen, vS, 2004). *Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.*

(So by changing the coefficients of the polynomial slightly, it can be made hyperbolic.) Here we say that a real one-dimensional map f is *hyperbolic* if each critical point is in the basin of a (hyperbolic) periodic point and all periodic points are hyperbolic. This implies that the real line is the union of a repelling hyperbolic set (a Cantor set of zero Lebesgue measure), the basin of hyperbolic attracting periodic points and the basin of infinity. So the dynamics of a hyperbolic map is very simple: Lebesgue almost all points are attracted to periodic cycles.

This theorem has a long history before it was proven in this full generality, see works of Yoccoz [Yoc89], Sullivan [Sul92], Lyubich [Lyu97], Światek, Graczyk [GŚ98], Kozlovski [Koz03], Blokh, Misiurewicz [BM00], Shen [She04]. Most of these works deal with the quadratic family $x \mapsto ax(1-x)$. This case is special, because in this case certain return maps become almost linear. This special behaviour does not even hold for maps of the form $x \mapsto x^4 + c$.

Note that the above theorem implies that the space of hyperbolic polynomials is an open dense subset in the space of real polynomials of fixed degree. Every hyperbolic map satisfying the mild "no-cycle" condition (critical points are not eventually mapped onto other critical points) is structurally stable.

The above theorem allows us to solve the 2nd part of Smale's eleventh problem for the 21st century.

Theorem 2. *Hyperbolic maps are dense in the space of C^k maps of the compact interval or the circle, $k = 1, 2, \ldots, \infty, \omega$.*

As mentioned, this easily implies

Corollary 3. *Structurally stable maps are dense in the space of C^k maps of the compact interval or the circle, $k = 1, 2, \ldots, \infty, \omega$.*

A similar question about density of hyperbolic maps can be asked for maps of a complex plane given by a complex polynomial. In the case of a complex polynomial, we say it is *hyperbolic* if all its critical points are in the basins of hyperbolic periodic attractors. We have only a partial result which applies to non (or finitely) renormalizable polynomials:

Theorem 4. *Any complex polynomial which is not infinitely often renormalizable, can be approximates by a hyperbolic polynomial of the same degree.*

(If we could prove this without the condition that the map is only finitely renormalizable, the complex Fatou conjecture would follows.) Here we say that a polynomial f is *infinitely renormalizable* if there exist arbitrarily large $s > 1$ (called the *period*) and simply connected open sets W containing a critical point c of f such that $f^{ks}(c) \in W$, $\forall k \geq 0$ and such that s is the first return time of c to W.

3 Quasi-conformal rigidity

The proof of these result heavily depends on complex analysis. In fact the theorems above can be derived from the following rigidity result.

Theorem 5. *Let f and $\tilde{f}$ be real polynomials of degree n which only have real critical points. If f and $\tilde{f}$ are topologically conjugate (as dynamical systems acting on the real line) and corresponding critical points have the same order, then they are quasiconformally conjugate (on the complex plane).*

A critical point c is a point so that $f'(c) = 0$. Not all critical points of a real polynomial need to be real.

If the polynomials are not real, then we need to make an additional assumption:

Theorem 6. *Let f and $\tilde{f}$ be complex polynomials of degree n which are not infinitely renormalizable and only have hyperbolic periodic points. If f and $\tilde{f}$ are topologically conjugate, then they are quasiconformally conjugate.*

This generalises the famous theorem of Yoccoz, proving that the Mandelbrot set associated to the quadratic family $z \mapsto z^2 + c$ is locally connected at non-renormalizable parameters.

4 How to prove rigidity?

First we need to associate a puzzle partition to any polynomial f which only has hyperbolic periodic points, and then use this to construct a complex box mapping $F\colon U \to V$. If f has only repelling periodic points, then the construction is a multi-critical analogue of the usual Yoccoz puzzle partition.

Definition 1 (Complex box mappings). *We say that a holomorphic map*

$$F : U \to V \tag{1}$$

between open sets in $\mathbb{C}$ is a complex box mapping if the following hold:

- *V is a union of finitely many pairwise disjoint Jordan disks;*
- *every connected component V' of V is either a connected component of U or the intersection of V' and U is a union of Jordan disks with pairwise disjoint closures which are compactly contained in V',*
- *for each component U' of U, $F(U')$ is a component of V and $F|U'$ is a proper map with at most one critical point;*
- *each connected component of V contains at most one critical point of F.*

It is possible to show that for a given polynomial which only has hyperbolic periodic point one can construct an induced complex box mapping which captures the dynamics of the polynomial, see [KvS06].

A connected component of the domain of definition of an iterate of F is called a puzzle–piece. To prove the above rigidity theorem, the main technical hurdle is to obtain a certain amount of control on the shape of these puzzle–pieces. In fact, it is not possible to obtain this control for all puzzle–pieces (and there are examples showing this), however we can prove that this control can be obtained on a combinatorially defined subsequence of puzzle–pieces:

Theorem 7 (Geometry control of puzzle–pieces). *Let F be a complex non renormalizable box mapping and c be a recurrent critical point. Then there exists $\epsilon > 0$ and a combinatorially defined sequence of puzzle–pieces $\mathbf{I}_n$ around c so that*

- *the puzzle–pieces $\mathbf{I}_n$ have ϵ–bounded geometry;*
- *for each domain A of the first return map to $\mathbf{I}_n$ one has $mod(\mathbf{I}_n \setminus A) \geq \epsilon$.*

Here we say that a simply connected domain $U \subset \mathbb{C}$ has ϵ–bounded geometry if there are two disks D_1 and D_2 such that $D_1 \subset U \subset D_2$ and the ratio of diameters of D_1 and D_2 is bounded from below by ϵ.

This control of geometry of puzzle–pieces is enough to prove the Rigidity theorems, because it allows us to apply the following new way of constructing quasiconformal conjugacies:

Theorem 8 (QC-Criterion). *For any constant $\epsilon > 0$ there exists a constant K with the following properties. Let $\phi\colon \Omega \to \tilde{\Omega}$ be a homeomorphism between two Jordan domains. Let X be a subset of Ω consisting of pairwise disjoint topological open discs X_i. Assume moreover,*

1. *for each i both X_i and $\phi(X_i)$ have ϵ-bounded geometry and moreover*
$$mod(\Omega - X_i), mod(\tilde{\Omega} - \phi(X_i)) \geq \epsilon$$
2. *ϕ is conformal on $\Omega - X_i$.*

Then there exists a K-qc map $\psi\colon \Omega \to \tilde{\Omega}$ which agrees with Ω on the boundary of Ω.

4.1 The strategy of the proof of QC-rigidity

So the proof of the rigidity theorem relies on the following steps:

First we associate to the polynomial f a suitable sequence of partitions $\mathcal{P}_n$. Let Ω_n be a union of puzzle piece containing the critical points, defined using the puzzle-pieces $\mathbf{I}_n$ from Theorem 7. Because of the geometric properties of $\mathbf{I}_n$, one has control on the domains of the first return map to Ω_n, in the manner required by the previous criterion. This is only true provided one constructs the sequence of partitions $\mathcal{P}_n$ very carefully. Let X_n be the domain of the first return map to Ω_n.

We can do the same for the topologically conjugate polynomial $\tilde{f}$. Now f and $\tilde{f}$ are conformally conjugate near ∞ (by the Böttcher coordinates). Since $\partial\Omega_n$ consists of pieces of external rays and equipotentials, one can show that there exists a qc homeomorphism $\phi_n\colon \Omega_n \to \tilde{\Omega}_n$ (which on the boundary of Ω_n preserves the natural parametrisation induced by the Böttcher coordinates). Moreover, $\phi_n(X_n) = \tilde{X}_n$ and ϕ_n is conformal outside X_n. Hence, because of the control on the geometry of puzzle pieces, the QC-criterion gives a K-qc homeomorphism $h_n\colon \Omega_n \to \tilde{\Omega}_n$ which preserves the natural parametrisation on the boundary defined by the Böttcher coordinates. Here K does not depend on n.

Because $h_n\colon \Omega_n \to \tilde{\Omega}_n$ is natural on the boundary, the above qc map h_n can be extended to a global homeomorphism h_n which is K-qc and so that

$$h_n \circ f(x) = \tilde{f} \circ h_n(x)$$

for each $x \notin \Omega_n$. (So h_n is a conjugacy everywhere except on the small set Ω_n.)

Since K-qc homeomorphisms form a compact space, we can extract a K-qc limit h from the sequence h_n. As Ω_n shrinks to the set of critical points, the limit h is a K-qausi-conformal conjugacy between f and $\tilde{f}$.

5 Enhanced nest construction

As we have mentioned before the geometry estimates do not hold for all puzzle–pieces and we have to find a way to combinatorially construct a subsequence of puzzle–pieces where this property holds. This is achieved through a powerful construction which we call "enhanced nest".

For simplicity of the exposition let us consider a unicritical box mapping $F : U \to V$ (*ie* F has a unique critical point) and write $U = \cup U_i$ where U_i are the connected components of the domain U. In this case we can assume that U is a subset of V and V is connected. Let U_0 be a component of U containing the critical point. Consider the critical value $F(c)$ and iterates of F near the critical value. Let us only discuss the case when c is recurrent (the non-recurrent case is much easier). It can happen that there are infinitely many domains W_i containing $F(c)$ and n_i such that F^{n_i} maps W_i univalently onto V for a suitable choice of n_i. This case is called *reluctantly recurrent*. This case is easy: c is recurrent, so there are infinitely many n_{i_j} such that $F^{n_{i_j}+1}(c)$ is inside U_0. The pullback of U_0 by $F^{n_{i_j}+1}$ is a puzzle–piece and then one can easily show (using the Koebe lemma) that its geometry depends only on U_0.

The opposite case, when this infinite sequence of iterates $F^{n_i} : W_i \to V$ does not exist, is called *persistently recurrent*. The enhanced nest construction applies to this case.

If the infinite sequence as above does not exist, then we can consider a minimal domain W around $F(c)$ such that W is univalently mapped onto V by some F^n. This domain W has several nice properties.

Firstly, F^n maps the critical value $F(c)$ into the critical domain U_0. Indeed, otherwise $F^{n+1}(c)$ would be in some domain U_i which is mapped univalently onto V by F; then $F^{n+1} : \mathrm{comp}_{F(c)}(F^{-n}(U_i)) \to V$ is a univalent map which contradicts the minimality of W. Here the notation $\mathrm{comp}_x(U)$ denotes a connected component of U containing x. Secondly, the annulus $W \setminus \tilde{W}$, where $\tilde{W} = \mathrm{comp}_{F(c)}(F^{-n}(U_0))$ is a pullback of the central domain, does not contain points of the postcritical set. Suppose the contrary, so there is $k > 0$ such that $F^k(c) \in W \setminus \tilde{W}$ and let k be minimal with this property. Since $F^k(c)$ is not in $\tilde{W}$, the point $F^{k+n}(c)$ is in some non central domain U_j. Let X be a pullback of U_j by F^{k+n-1} along the orbit of the critical value $F(c)$, so $F(c) \in X$. Notice that $F^n : F^{k-1}(X) \to U_j$ is univalent and that $F^{k-1}(X) \subset W \setminus \tilde{W}$. Moreover, $F^{k-1} : X \to F^{k-1}(X)$ is also univalent because of the minimality of k. Hence the map $F^{k+n} : X \to V$ is univalent and this again contradicts the minimality of W.

The pullback of the domain W by $F\colon U \to V$ to the critical point we call the *smallest successor* of V and denoted by $\mathcal{B}(V)$. The corresponding pullback of $\tilde{W}$ will be denoted by $\mathcal{A}(V)$. From the construction the smaller successor we know that $\mathcal{A}(V)$ has some space outside which contains no postcritical points and the $\mathcal{B}(V)$ has some space inside near the boundary free of the postcritical set. Thus, if we combine both operations, we see that $\mathcal{B}(\mathcal{A}(V))$ has some space inside and outside free of the postcritical set. The size (in terms of moduli) of this 'empty' space can be easily estimated if one has estimates from below on $\inf_i \mathrm{mod}(V \setminus U_i)$.

Obviously, this property of having some space around the boundary of a domain free of postcritical set is very important: if $F^m : X \to Y$ is a univalent map between two simply connected domains and $Y' \supset Y$ is another simply connected domain such that the annulus $Y' \setminus Y$ does not contain points of the postcritical set, then there is a domain $X' \supset X$ so that F^m extends to X', $F^m(X') = Y'$ and the map $F^m : X' \to Y'$ is univalent. If one can control the modulus of $Y' \setminus Y$, then the distortion of $F^m|_X$ can be controlled by the classical Koebe lemma.

In the unimodal case we define $\Gamma(W) = \mathcal{B}(W)$. Now, the enhanced nest construction goes as following: given V, let

$$\mathbf{I}_0 := V \text{ and } \mathbf{I}_{i+1} := \Gamma^T(\mathcal{B}(\mathcal{A}(\mathbf{I}_i))),$$

where T only depends on the order of the critical point. We have already explained the rationale behind taking $\mathcal{B}(\mathcal{A}(\mathbf{I}_i))$. The Γ operation is used to control the return times of the critical point to the domains $\mathbf{I}_i$ and is - in some sense - a rather minor technical point.

This is a full description of the enhanced nest in the unicritical case. The construction in the general case is slightly more complicated and then the definitions of $\mathcal{A}(V)$ and $\mathcal{B}(V)$ are based on the following lemma:

Lemma 1. *Let $F : U \to V$ be a persistently recurrent box mapping, c be a critical point of F and $Y \ni c$ be some pullback of a connected component of V by an iterate of F. Then there is a positive integer ν with $F^\nu(c) \in Y$ such that* *the following holds. Let $X_0 = comp_c(F^{-\nu}(Y))$ and $X_j = F^j(X_0)$ for $0 \le j \le \nu$. Then*

1. $\#\{0 \le j \le \nu - 1 : X_j \cap Crit(F) \neq \emptyset\} \le b^2$;
2. $X_0 \cap PC(F) \subset comp_c\left(F^{-\nu}(\tilde{Y})\right)$;

where $Crit(F)$ denotes the set of critical points of F, $PC(F)$ is the postcritical set, b is the number of critical points counted with their multiplicity and $\tilde{Y}$ is a connected component of the domain the first return map to Y containing c.

6 Small Distortion of Thin Annuli

To control the shape of the puzzle–pieces we must control the amount of space around a puzzle–piece which is free of points of the postcritical set. As the previous construction of the enhanced nest shows we should estimate the modulus of pullbacks of various annuli.

Let $G : U \to V$ be a holomorphic surjective map and the domains $A \subset U$, $B \subset V$ be simply connected so that $G(A) = B$. We would like to have some estimates from below of the modulus of the annulus $U \setminus A$ in terms of the modulus of $V \setminus B$. If G is univalent map, this is the best case scenario: $\mathrm{mod}(U \setminus A) = \mathrm{mod}(V \setminus B)$. Now suppose that G has some critical points and all of them are in A. Then $G : U \setminus A \to V \setminus B$ is an unbranched covering, hence $\mathrm{mod}(U \setminus A) = \mathrm{mod}(V \setminus B)/d$, where d is the degree of G. If d is large, the modulus can deteriorate quite a lot and one can do nothing about it.

An important case is when G has relatively small number of critical points in A and possibly a large number of critical points in $U \setminus A$. Simple examples show that if the annulus $V \setminus B$ was fat (has large modulus), the modulus of its pullback $U \setminus A$ can drop a lot. However there is a special case when this does not happen: if the annulus $V \setminus B$ is thin, the map G is real and all the domains are symmetric with respect to the real line. More precisely the following lemma holds:

Lemma 2 (Small Distortion of Thin Annuli). *For every $K \in (0,1)$ there exists $\kappa > 0$ such that if $A \subset U$, $B \subset V$ are simply connected domains symmetric with respect to the real line, $G : U \to V$ is a real holomorphic branched covering map of degree D with all critical points real which can be decomposed as a composition of maps $G = g_1 \circ \cdots \circ g_n$ with all maps g_i real and either real univalent or real branched covering maps with just one critical point, the domain A is a connected component of $G^{-1}(B)$ symmetric with respect to the real line and the degree of $G|_A$ is d, then*

$$mod(U \setminus A) \ge \frac{K^D}{2d} \min\{\kappa, mod(V \setminus B)\}.$$

It is not possible to drop the condition of G being real and the domains being symmetric. If V is a disk and B spirals around its centre (and therefore not symmetric with respect to the real line), it is possible to construct G so that the lemma does not hold.

This lemma allowed us to considerably simplify the original proof of the real Geometry control of puzzle–pieces theorem (which initially used many sophisticated real pullback arguments). The basic idea how to use the lemma in order to prove Theorem 7 is this: let $\mu_n = \inf \operatorname{mod}(\mathbf{I}_n \setminus A)$ where the infimum runs over all domains A of the first return map to $\mathbf{I}_n$. Now consider the iterate G of F which maps $\mathbf{I}_n$ to $\mathbf{I}_{n-M}$. When M is large, the degree D of this map is large. However, it turns out that

- the degree d of $G|_A$ remains bounded, independently of M;
- the set $\mathbf{I}_{n-M} \setminus G(A)$ contains many 'previous annuli', and using this we get: $\operatorname{mod}(\mathbf{I}_{n-M} \setminus G(A)) \geq K'(\mu_{n-M-1} + \cdots + \mu_{n-5})$, where K' is independent of M.

Now fix M so large that $K'(\mu_{n-M-1} + \cdots + \mu_{n-5}) \geq 8d\, \mu_{n-M-1,n-5}$ where $\mu_{n-M-1,n-5} = \min\{\mu_i;\ i = n-M-1, \ldots, n-5\}$. Next choose $K \in (0,1)$ so close that $K^D \geq 1/2$. Using the previous lemma we then get some $\kappa > 0$ so that $\operatorname{mod}(\mathbf{I}_n \setminus A) \geq \frac{1/2}{2d} \min(\kappa, 8d\ \mu_{n-M-1,n-5})$. From this one easily proves recursively a lower bound for μ_n. The proof of Theorem 7 follow then easily.

The proof of the previous lemma is relatively simple and is based on the following idea. We can cut B into two symmetrical pieces by the real line and pullback just a half of B by maps g_i. All the pullbacks are going to lay in a half complex plain, and it is possible to provide good moduli estimates for this case. When the half of B is pullbacked all the way to U we can reconstruct A from it by the symmetry. In this last operation we loose only factor of one half.

If G is not real, the situation is more complicated because as we mentioned there is not (and cannot be) an analogue of the previous lemma. However, it is still possible to control moduli if one pullbacks two annuli instead of one. The following powerful lemma is due to Kahn and Lyubich, see [KL05]:

Lemma 3. *For any $\eta > 0$ and $D > 0$ there is $\epsilon = \epsilon(\eta, D) > 0$ such that the following holds: Let $A \subset A' \subset U$ and $B \subset B' \subset V$ be topological disks in $\mathbb{C}$ and let $G : (A, A', U) \to (B, B', V)$ be a holomorphic branched covering map. Let the degree of G be bounded by D and the degree of $G|_{A'}$ be bounded by d. Then*

$$\mathit{mod}(U \setminus A) > \min(\epsilon,\ \eta^{-1}\mathit{mod}(B' \setminus B),\ C\eta d^{-2}\mathit{mod}(V \setminus B)),$$

where $C > 0$ is some universal constant.

7 Approximating non renormalizable complex polynomials

If the complex Rigidity theorem were proven in full generality, then using the standard Sullivan technique one could show that the hyperbolic polynomials are dense in the space of complex polynomials of fixed degree. We have proven the complex Rigidity theorem in the case of finitely renormalizable polynomials, so some extra work is needed to show that such polynomials can be approximated by hyperbolic ones.

To simplify the exposition we will show how to do this in the case of cubic non renormalizable polynomial whose both critical points are recurrent. We can normalise a cubic polynomial so it is $f(z) = z^3 + az + b$.

The Rigidity theorem implies that there are no other normalised polynomials qc conjugate to f. Fix some neighbourhood W of f in the space of cubic normalised polynomials. For $g \in W$ let $c_k(g)$, $k = 1, 2$, denote the critical points of g.

First we claim that there are polynomials in W which have a critical relation, *ie* there are k_1, k_2, n such that $g^n(c_{k_1}(g)) = c_{k_2}(g)$. Indeed, if this was not the case, all preimages of the critical points would move holomorphically as functions of $g \in W$. Then using Lambda lemma we can extend this holomorphic motion to the whole $\mathbb{C}$ and get that all polynomials in W are qc conjugate.

The neighbourhood W can be chosen arbitrarily small, and therefore there are polynomials arbitrarily close to f having a critical relation. Any critical relation gives an algebraic curve in the space of normalised cubic polynomials (which is $\mathbb{C}^2$), this curve contains all polynomials having the same critical relation.

Consider one of these curves. Since it is an algebraic curve it has just finitely many singular points, we can remove them from this curve and get a holomorphic one dimensional manifold. Take some connected component of the intersection of W and this manifold which will be denoted by M_1 and take a polynomial $f_1 \in M_1$. Arguing as before we can see that either all polynomials in M_1 are qc conjugate or there is polynomial in M_1 having another critical relation. If a cubic polynomial has two critical relations, then it is hyperbolic. So if the second alternative holds, we are done because we have found a hyperbolic polynomial in W. If all polynomials in M_1 are qc conjugate, we cannot apply the Rigidity theorem because we do not know whether f_1 is finitely renormalizable or not. Instead we should do the following.

Take a sequence of polynomials f_i having a critical relations and converging to f. Let $M_i \ni f_i$ denote connected components of intersection of W and the corresponding manifolds as in the previous paragraph. We can assume that all polynomials in M_i are qc conjugate (otherwise we are done). The closure of each M_i has non empty intersection with the boundary of W because M_i is a part of an algebraic curve and such curves cannot have compact components in $\mathbb{C}^2$. Therefore we can find $\tilde{f} \in \partial W$, a subsequence i_j and $\tilde{f}_{i_j} \in M_{i_j}$ so that $\tilde{f}_{i_j}$ converges to $\tilde{f}$. The maps f_{i_j} and $\tilde{f}_{i_j}$ are qc conjugate and $f_{i_j} \to f$, so it is possible to show (though it is not completely straightforward) that the maps f and $\tilde{f}$ are qc conjugate as well. Now we can apply the Rigidity theorem because f is non renormalizable and we can see that such the polynomial $\tilde{f}$ cannot exist.

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