Misiurewicz maps unfold generically *

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Abstract

In this paper we shall show that in normalized families of polynomial or rational maps, Misiurewicz maps unfold generically. For example, if f_{λ_0} is critically finite with non-degenerate critical point $c_1(\lambda_0),\dots,c_n(\lambda_0)$ such that $f_{\lambda_0}^{k_i}(c_i(\lambda_0))=p_i(\lambda_0)$ are hyperbolic periodic points for $i=1,\dots,n$, then

$$\lambda\mapsto\left(f_\lambda^{k_1}(c_1(\lambda))-p_1(\lambda),\dots,f_\lambda^{k_{d-2}}(c_{d-2}(\lambda))-p_{d-2}(\lambda)\right)$$

is a local diffeomorphism for λ near λ_0. For quadratic families this result was proved previously in [DH] using entirely different methods.

1 Introduction and Statement of Results

In this paper we consider families of polynomial or rational maps f_λ depending (complex) analytically on $\lambda\in V$ where V is some neighbourhood of $\lambda_0\in\mathbb{C}^{\mathcal{N}}$ where $\mathcal{N}\in\mathbb{N}$. We say that f_λ is a *normalized* family of polynomial (resp. rational) maps if $\lambda\neq\lambda'$ implies that no affine (resp. Moebius) transformation conjugates f_λ and $f_{\lambda'}$. Let $f=f_{\lambda_0}$ and $C(f)$ be the set of critical points of f. The map f is called *critically finite* if each critical point $c_i\in C(f)$ of f eventually lands on a periodic point p_i of p. In other words,

$$f^{k_i}(c_i)=p_i \text{ and } f^{l_i}(p_i)=p_i$$

where for simplicity we shall choose k_i minimal so that $f^{k_i}(c_i)$ is a periodic point. We shall only deal with the case that p_i are hyperbolic periodic points. In that case f is in the larger class of Misiurewicz maps defined as follows. The map f satisfies the *Misiurewicz* condition if each critical point c is either eventually mapped to a periodic point which is hyperbolic (i.e., with multiplier not on the unit circle) or is in the Julia set, and moreover, $\omega(c)$ contains no parabolic or

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critical points. By a theorem of Mañé, see the remarks below Theorem [Ma2], these conditions imply that

$$P(f) = \text{the closure of } \cup_{c \in C(f)} \cup_{i>0} f^i(c)$$

is a hyperbolic set and that all components of the Fatou set of f are in the basins of super-attracting periodic orbits. Note that we only consider *forward* iterates of critical points in the definition of $P(f)$.

The aim of this paper is to prove the following theorem.

Main Theorem 1.1 *Let f_λ be a normalized family of polynomial or rational maps such that $f = f_{\lambda_0}$ is a Misiurewicz map with n critical points $c_1, \ldots, c_n$. Assume that in the definition above, the critical points c_i are all non-degenerate and that f is not a Lattès map (see definition below). Then there exists a neighbourhood $W \subset \mathbb{C}^{\mathcal{N}}$ of λ_0 such that*

- *$W \ni \lambda \mapsto c_i(\lambda)$ holomorphic;*
- *there exists a holomorphic motion $\phi: P(f) \times W \to \bar{\mathbb{C}}$ such that (writing $\phi_\lambda(z) = \phi(z, \lambda)$), for $\lambda \in U$ the set $X_\lambda = \cup_{z \in P(f)} \phi_\lambda(z)$ is mapped into itself by f_λ and so that $f_\lambda: X_\lambda \to X_\lambda$ is conjugate to $f: P(f) \to P(f)$ by a homeomorphism h_λ depending on λ and with $h_0 = id$;*
- *assuming that $f(c_i)$ are in $\mathbb{C}$, the map*

$$W \ni \lambda \mapsto (f_\lambda(c_1(\lambda)) - \phi_\lambda(f(c_1)), \ldots, f_\lambda(c_n(\lambda)) - \phi_\lambda(f(c_n)))$$

is an immersion (the derivative of this map has rank equal to n for all $\lambda \in W$).

Here we say that $\phi: X \times W \to \mathbb{C}$ is a *holomorphic motion* of $X \subset \bar{\mathbb{C}}$ parameterized by $W \ni \lambda_0$ with W a subset of some complex manifold if (writing $\phi_\lambda = \phi(z, \lambda)$) the following properties hold:

1. $\phi_{\lambda_0}(z) = z$ for each $z \in X$;
2. $P(f) \ni z \mapsto \phi_\lambda(z)$ is one-to-one for each $\lambda \in W$;
3. the map $W \ni \lambda \mapsto \phi(z, \lambda) \in \bar{\mathbb{C}}$ is holomorphic, for each $z \in X$.

In the main theorem we do not exclude the possibility that some of the points $f(c_i)$ coincide. If f is critically finite, then the proof gives that the periodic points $p_i(\lambda)$ depend holomorphically on $\lambda \in U$ and that

$$\lambda \mapsto \left(f_\lambda^{k_1}(c_1(\lambda)) - p_i(\lambda), \ldots, f_\lambda^{k_n}(c_n(\lambda)) - p_n(\lambda)\right)$$

is a local immersion.

Remarks.

1. We say that f is a *Lattès map* if the following holds. Let $n > 1$ be an integer, Λ be a lattice with $n\Lambda \subset \Lambda$ and $T^2 = \mathbb{C}^2/\Lambda$ be a torus. Let $g_n: T^2 \to T^2$

correspond to $z \mapsto n \cdot z$. Let $\pi : T^2 \to \bar{\mathbb{C}}$ be an even twofold branched covering (for example the Weierstrass function). Since g_n is odd, there exists a rational map f, called *Lattès map*, of degree n^2 on the sphere so that $\pi \circ g_n = f \circ \pi$ commutes. Since periodic points of g_α are dense, the Julia set of f is equal to $\bar{\mathbb{C}}$. The critical points of f correspond to critical values of π (critical points are branch points of π) and therefore $\#(C(f) \cup P(f)) = 4$. Since there are arcs of such maps (corresponding to Λ) consisting of quasiconformally conjugate maps, the assumption in the main theorem that f is not a Lattès map cannot be dispensed with.

2. The main theorem implies for example that if we normalize and parameterize the space of polynomials of degree d by

$$f_\lambda(z) = z^d + a_{d-2} z^{d-2} + \ldots + a_0$$

where $\lambda = (a_{d-2}, \ldots, a_0)$ and consider $f = f_{\lambda_0}$ as above, then the map

$$\lambda \mapsto \left(f_\lambda^{k_1}(c_1(\lambda)) - p_1(\lambda), \ldots, f_\lambda^{k_{d-2}}(c_{d-2}(\lambda)) - p_{d-2}(\lambda) \right)$$

is a local diffeomorphism for λ near λ_0.

3. If f_λ and $\tilde{f}_\lambda$ are two families which are Moebius equivalent, then the derivatives at $\lambda = \lambda_0$ of

$$\lambda \mapsto (f_\lambda(c_1(\lambda)) - \phi_\lambda(f(c_1)), \ldots, f_\lambda(c_n(\lambda)) - \phi_\lambda(f(c_n))) \tag{1.1}$$

and its corresponding object for $\tilde{f}_\lambda$ coincide. This is because the right hand vanishes at $\lambda = \lambda_0$ and is the same for f_λ and $T \circ f_\lambda$ for any affine map T and because the derivative with respect to λ of (1.1) replacing f_λ by $\tilde{f}_\lambda = M_\lambda \circ f_\lambda M_\lambda^{-1}$ is the same as replacing f_λ by $T \circ f_\lambda$ where $T = DM_{\lambda_0}(0)$.

4. If f_λ is a normalized family then $\mathcal{N} \le n$. Moreover, any normalized family can be (locally) embedded in a normalized family with $\mathcal{N} = n$. This holds because one can consider the space of all polynomial (resp. rational) maps of the degree considered and define $f \sim g$ if f and g are affinely (resp. Moebius) conjugate. Each equivalence class is a two (resp. three) dimensional complex manifold. Indeed, if $f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ with $a_0 \neq 0$ and $M(z) = \alpha z + \beta$ then $M^{-1} \circ f \circ M(z) = a_0 \alpha^{n-1} z^n + \alpha^{n-2}(a_0 n \beta + \alpha^{n-2} a_1) z^{n-1} + \ldots$ which is a two-parameter family in α, β. If f is rational of degree n we can write

$$f(z) = \frac{a_0 z^n + \ldots + a_n}{b_0 z^n + \ldots + b_n}$$

such that $(a_0, b_0) \neq (0,0)$ and $(a_n, b_n) \neq (0,0)$. A Moebius transformation close to the identity map can be written as $M(z) = \frac{\alpha z + \beta}{1 + \gamma z}$ and then $M^{-1}(z) = \frac{-z+\beta}{\gamma z - \alpha}$. Then $M^{-1} \circ f \circ M$ is a locally (complex) three dimensional manifold in the space of rational maps. To see this fix three special points p_1, p_2, p_3 of f (say periodic points or critical points). Then $(\alpha, \beta, \gamma) \mapsto (M^{-1}(p_1), M^{-1}(p_2), M^{-1}(p_3)) \in \bar{\mathbb{C}}^3$ forms a complex three dimensional manifold through $(p_1, p_2, p_3) \in \bar{\mathbb{C}}^3$ consisting of corresponding special points for $M^{-1} \circ f \circ M$.

Using the results of Tan Lei [TL] one can deduce from the main theorem similarities of the Julia set and the parameter space.

2 Proof in the case that all critical points of f are eventually mapped to repelling periodic points

Even though the general case is not much more difficult, we shall first prove this case, because in this case the proof is a little more concrete. That c_i and p_i depend analytically on λ is simply a consequence of the implicit function theorem. So it remains to show that the assumption that the family is normalized implies that

$$\lambda \mapsto \Big(f^{k_1}(c_1(\lambda)) - p_1(\lambda), \ldots, f^{k_n}(c_n(\lambda)) - p_n(\lambda) \Big)$$

is a local immersion. The main ingredient for proving this is the following

Theorem 2.1 *(***Uniqueness part of Thurston's realization theorem***)* *Let $f, \tilde{f}: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be critically finite rational maps which are combinatorially equivalent. Then f and $\tilde{f}$ are conjugate by some Moebius transformation.*

Here f and $\tilde{f}$ are called *combinatorially equivalent* if there exist two orientation preserving homeomorphisms $\theta_1, \theta_2: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $f \circ \theta_1 = \theta_2 \circ \tilde{f}$, $\theta_1 = \theta_2$ on $P(f)$, $\theta_i(P(f)) = P(\tilde{f})$ and θ_1, θ_2 isotopic relative to $P(f)$. (Assume that $P(f)$ and $P(\tilde{f})$ are finite in this definition.)

We shall use this uniqueness result, by using the following hyperbolic structure near periodic points.

Lemma 2.1 *For each repelling periodic point p of period ℓ there exist an open topological disc U, two open topological discs U^0, U^1 whose closures are disjoint and contained in U and two positive integers s_1, s_2 such that $f^{s_j}: U^j \to U$ is a diffeomorphism for $j = 1, 2$. Here $p \in U^0$ and $s_0 = \ell$. We can assume that $f^{s_j} \mid U_j$ are the first return maps of f to U. Moreover, we can choose U with arbitrary small diameter.*

Proof: Take a small open Euclidean disc V around p so that one branch of $f^{-\ell}$ maps V compactly into V. Call this inverse branch g. Let $V_{-i} = g^i(V)$. So $p \in V_{-i}$ and $V_{-(i+1)} \subset V_{-i}$. Since the Julia set is perfect, and preimages of non-exceptional points are dense in the Julia set, there exists an integer i and some preimage x of p which is contained in $V_{-i} \setminus V_{-(i+1)}$. We can assume that x is not a preimage of a critical point and that the first iterate of x which enters V_{-i} is p. Let m be so that $f^m(x) = p$. Hence there exists a small disc B around x which is mapped by f^m diffeomorphically onto some $V_{i'}$ with $i' \geq i$. We can arrange it so that $f^t(B) \cap V_{i'} = \emptyset$ for $t = 0, 1, \ldots, m-1$. So define $U = V_{-i'}$, $U^0 = V_{-(i'+1)}$ and $U^1 = g^{i'-i}(B)$. Then $U^1 \subset V_{-i'} \setminus V_{-(i'+1)} = U \setminus U^0$ and $f^\ell: U^0 \to U$, $f^{m+(i'-i)\ell}: U^1 \to U$ are both diffeomorphisms. By construction, each of these maps is a first return to U and U^0, U^1 have disjoint closures. $\Box$

$F|U^j := f^{s_j}$ defines a *polynomial-like* $F: U^0 \cup U^1 \to U$. The set

$$\Lambda = \{z \in U^0 \cup U^1 \ ; \ F^j(z) \in U^0 \cup U^1\}$$

forms a hyperbolic Cantor set (by putting the Poincaré metric on U, the map two branches $(FU^i)^{-1}: U \to U^i$ become strictly contracting). The dynamics of $F: \Lambda \to \Lambda$ is conjugate to the shift $\sigma: \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ where the conjugacy is $A(z) = (a_m(z))_{m \geq 0}$ with $a_m(z)$ defined so that $f^m(z) \in U^{a_m(z)}$. The previous lemma implies that there exists a hyperbolic Cantor set Λ containing p, and that p is the limit of a sequence of periodic points $q_i \in \Lambda$. The next lemma shows that this structure persists for nearby maps.

Lemma 2.2 *Let $f = f_{\lambda_0}$ be as before. Then there exists $K < \infty$ and a neighbourhood W of λ_0 such that for each $\lambda \in W$,*

- *there are branches of $f_\lambda^{-s_i}|U$ which are ϵ close to $(f|U_i)^{-s_i}$;*
- *if we define $U^i_\lambda := (f_\lambda^{-s_i}(U))$ then $F_\lambda|U^i_\lambda = f_\lambda^{s_i}$ still defines a polynomial-like map $F_\lambda: U^0_\lambda \cup U^1_\lambda \to U$;*
- *there exists a family of K-quasiconformal $h_\lambda: U \to U$ depending continuously on λ and with $h_0 = id$, so that h_λ sends Λ onto Λ_λ and so that $h_\lambda: \Lambda \to \Lambda_\lambda$ conjugates $F: \Lambda \to \Lambda$ to $F_\lambda: \Lambda_\lambda \to \Lambda_\lambda$;*
- *for each $z \in \Lambda$, the map $W \ni \lambda \mapsto h_\lambda(z) \in U$ is holomorphic.*

Proof: This can be proved easily by applying the λ-Lemma, see [MSS]. It is also easy to prove it by hand. Let $h^{(0)}_\lambda: U \to U$ be a family of diffeomorphisms depending holomorphically on λ which map $U \to U$, $U^0 \to U^0_\lambda$, $U^1 \to U^1_\lambda$, and so that $h^{(0)}_0(0) = id$ and so that

$$h^{(0)}_\lambda(0) \circ f^{s_i} = f_\lambda^{s_i} \circ h^{(0)}_\lambda(0) \tag{2.2}$$

on ∂U^i, $i = 1, 2$. Provided λ is close to λ_0 this is possible and moreover, one can find K such that $h^{(0)}_\lambda$ is K-quasiconformal. Next define $h^{(1)}_\lambda$ to be equal to $h^{(0)}_\lambda$ outside $U^0 \cup U^1$ and on U^i to be equal to $f_\lambda^{-s_i} \circ h^{(0)}_\lambda \circ f^{s_i}$. Because of (2.2), $h^{(1)}_\lambda$ is a continuous map on U. It is also K-quasiconformal. Continuing this pullback construction we get a sequence of K-quasiconformal homeomorphisms $h^{(n)}_\lambda: U \to U$ such that

$$h^{(n)}_\lambda \circ F = F_\lambda \circ h^{(n)}_\lambda \tag{2.3}$$

outside $\Lambda(n) = \{z \in U \ ; \ z, \ldots, F^n(z) \in U^0 \cup U^1\}$. Note that $h^{(n)}_\lambda$ agrees with $h^{(n-1)}_\lambda$ outside $\lambda^{(n-1)}$. Hence as n tends to infinity, $h^{(n)}_\lambda$ converges to a K-quasiconformal homeomorphism h_λ conjugating $F: \Lambda \to \Lambda$ to $F: \Lambda_\lambda \to \Lambda_\lambda$. For each $z \in \Lambda$, $h_\lambda(z)$ is equal to the limit of $\phi^{(a_n)}_\lambda \circ \phi^{(a_{n-1})}_\lambda \circ \ldots \circ \phi^{(a_0)}(x)$ where x is an arbitrary point $x \in U$, (a_m) is equal to the coding $A(z)$ and $\phi^{(i)}_\lambda := (f_\lambda^{s_i}|U^i_\lambda)^{-1}$. Since all these maps are contractions depending holomorphically on λ, it follows that for each fixed $z \in \Lambda$, the map $W \ni \lambda \mapsto h_\lambda(z)$ is holomorphic. $\square$

Note that the conjugacy is so that $z \in \Lambda$ and $h_\lambda(z) \in \Lambda_\lambda$ have the same coding by the map A from above (where in the definition of the coding of Λ_λ we have of course to take $a_m(\tilde{z})$ so that $f_\lambda^m(\tilde{z}) \in U_\lambda^{a_m(\tilde{z})}$ for the coding of Λ_λ). In other words, when we fix for example a periodic $q \in \Lambda$ then there exists a unique corresponding periodic point $q_\lambda = h_\lambda(q) \in \Lambda_\lambda$ for each λ sufficiently close to λ_0. The map $\lambda \mapsto q_\lambda$ is holomorphic.

Let now $f = f_{\lambda_0}$ be a rational map satisfying the assumptions from the Main Theorem. Associate polynomial-like maps $F_i \colon U^0(p_i) \cup U^1(p_i) \to U(p_i)$ to each of the periodic points $p_i = f^{k_i}(c_i)$ such that $F_i | U^j(p_i) = f^{s_{i,j}}$ are first return maps of f to $U(p_i)$. Let $\Lambda(p_i)$ be the corresponding Cantor sets and let W be a neighbourhood of λ_0 such that the previous lemma holds for each of the periodic points p_i.

Proposition 2.1 *There exists a neighbourhood W of λ_0 with the following property. Assume that $\lambda, \lambda' \in W$ and that f_λ and $f_{\lambda'}$ are so that*

$$f_\lambda^{k_i}(c_i(\lambda)) \in \Lambda_\lambda(p_i)$$

$$f_{\lambda'}^{k_i}(c_i(\lambda')) \in \Lambda_{\lambda'}(p_i)$$

are both periodic points of the same period and in fact correspond to one and the same periodic point $q_i \in \Lambda(p_i)$ of $f = f_{\lambda_0}$, i.e., are respectively equal to $h_\lambda(q_i)$ and $h_{\lambda'}(q_i)$.

Then f_λ and $f_{\lambda'}$ are conjugate via a Moebius transformation.

Note that the period of the periodic orbit from the above proposition tends necessarily to infinity when λ, λ' tends to λ_0.

Proof of Proposition 2.1:

Step 1. Take $U(p_i)$ so that they are all disjoint and take the family of homeomorphisms $h_\lambda \colon U(p_i) \to U(p_i)$ from Lemma 2.2. Next let $\theta = h_{\lambda'} \circ h_\lambda^{-1}$. This homeomorphism conjugates the polynomial-like maps

$$F_{i,\lambda} \colon U_\lambda^0(p_i) \cup U_\lambda^1(p_i) \to U(p_i)$$

to

$$F_{i,\lambda'} \colon U_{\lambda'}^0(p_i) \cup U_{\lambda'}^1(p_i) \to U(p_i).$$

Note that $\theta | \partial U(p_i) = id$ and $F_{\lambda'} \circ \theta = \theta \circ F_\lambda$ on $\cup_{i,j} U_\lambda^j(p_i)$.

Step 2. We have $F_\lambda | U_\lambda^j(p_i) = f_\lambda^{s_{i,j}}$. Let $W_{i,\lambda}$ be the union of the first $s_{i,0} - 1$ iterates of $U_\lambda^0(p_i)$ together with the first $s_{i,1} - 1$ iterates of $U_\lambda^1(p_i)$. (Note that we may assume that all these iterates are disjoint, since $F_\lambda | U_\lambda^j(p_i)$ is a first return map.) Define $W_{i,\lambda'}$ similarly. Extend θ to a homeomorphism $\cup W_{i,\lambda} \to \cup W_{i,\lambda'}$ so that $f_{\lambda'} \circ \theta(z) = \theta \circ f_\lambda(z)$ whenever $z \in \cup W_{i,\lambda}$.

Step 3. Take a disc $V_{i,j}$ of radius r centered at $f^j(c_i)$ for $j = 0, \ldots, k_i - 1$. Provided λ, λ' are sufficiently close to λ_0, one has $f_\lambda^j(c_i(\lambda)), f_{\lambda'}^j(c_i(\lambda')) \in V_{i,j}$. Assume that $r > 0$ is so small so that all these sets $V_{i,j}$ are pairwise disjoint (or coincide) and are disjoint from $\cup W_{i,\lambda}$ and $\cup W_{i,\lambda'}$. Now choose a $\tilde{K}$-quasiconformal homeomorphism $\theta: V_{i,j} \to V_{i,j}$ which is equal to the identity on the boundary of $V_{i,j}$ and so that it maps $f_\lambda^j(c_i(\lambda))$ onto $f_{\lambda'}^j(c_i(\lambda'))$. Next define $\theta = id$ outside $\bigcup(V_{i,j} \cup W_i)$. Thus $\theta: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ becomes a $\tilde{K}$-quasiconformal homeomorphism which sends the post-critical set of f_λ to that of $f_{\lambda'}$. Note that we can choose $\tilde{K}$ close to one (i.e., θ close to a Moebius map), provided λ, λ' are close to λ_0.

Step 4. Let us now assume that the neighbourhood W is small and $\tilde{K}$ is close to one. Then define a conjugacy ϕ between f_λ and $f_{\lambda'}$ as the limit of a sequence of quasiconformal homeomorphisms θ_n. Here we define θ_n inductively as follows. Let $\theta_0 = \theta$ and given a $\tilde{K}$-quasiconformal homeomorphism θ_n which sends the post-critical set of f_λ to that of $f_{\lambda'}$, define a homeomorphism θ_{n+1} with the same properties as follows. Take z and consider the set $f_{\lambda'}^{-1} \circ \theta_n \circ f_\lambda(z)$. Provided $z \notin V_{i,0}$ this set contains precisely one point z' which is close to z. (Provided $\epsilon > 0$ is sufficiently small, there exists a neighbourhood W of λ_0 such that if $\lambda, \lambda' \in W$, then θ_n is close to the identity map and so there is a unique point $z' \in f_{\lambda'}^{-1} \circ \theta_n \circ f_\lambda(z)$ which is ϵ-close to z. This does not hold when $z \in V_{i,0}$ because $V_{i,0}' := \theta(f_\lambda(V_{i,0}))$ contains a critical value of $f_{\lambda'}$.) So define $\theta_{n+1}(z) = z'$ in this way for $z \notin \cup V_{i,0}$. Thus θ_{n+1} is a homeomorphism sending the complement of $\cup V_{i,0}$ to the complement of $\cup V_{i,0}'$. Since θ_{n+1} is now already defined from $\partial V_{i,0}$ to $\partial V_{i,0}'$, this map extends uniquely to a homeomorphism sending $V_{i,0}$ to $V_{i,0}'$ with $\theta_{n+1}(z) \in f_{\lambda'}^{-1} \circ \theta_n \circ f_\lambda(z)$ (since the map is prescribed on the boundary, continuity of θ_{n+1} forces uniqueness in the choice for the preimages). Note that θ_{n+1} is again a $\tilde{K}$-quasiconformal homeomorphism sending the post-critical set of f_λ to that of $f_{\lambda'}$ and so that $\theta_{n+1} \circ f_{\lambda'} = \theta_n \circ f_\lambda(z)$. Now take a subsequence of θ_n which converges to some $\tilde{K}$-quasiconformal homeomorphism ϕ we get $\phi \circ f_{\lambda'} = \phi \circ f_\lambda(z)$. From Thurston's uniqueness result, it follows that $f_{\lambda'}$ and f_λ are conjugate by a Moebius transformation. □

We shall use the following

Lemma 2.3 *Let $G = (G_1, \ldots, G_n): V \to \mathbb{C}^n$ be a complex analytic map where $\lambda_0 \in V \subset \mathbb{C}^n$. Assume that G has an isolated zero at λ_0. Then there is (locally) a univalent parameterization*

$$\Psi: \mathbb{C} \to \{z \; ; \; G_2(z) = \ldots = G_n(z) = 0\}$$

with $\Psi(0) = \lambda_0$.

Proof: This is a consequence of the Uniformization Theorem, see for example [AGLV]. □

Proof of the Main Theorem in the critically finite case: Assume that f_λ is a normalized family depending on $\lambda \in W \subset \mathbb{C}^n$, and consider the map $G: W \to \mathbb{C}^n$ defined by

$$\lambda \mapsto \left(f_\lambda^{k_1}(c_1(\lambda)) - p_1(\lambda), \ldots, f_\lambda^{k_n}(c_n(\lambda)) - p_n(\lambda) \right).$$

Let us first show that G has an isolated zero at λ_0. Indeed, otherwise there exist a sequence of parameters $\lambda(j)$ converging to λ_0 such that $G(\lambda(j))$ vanishes. This means that for each fixed i, each of these parameters $f^{k_i}_{\lambda(j)}(c_i(\lambda(j)))$ corresponds to the same periodic point. Hence by Proposition 2.1 the maps $f_{\lambda(j)}$ all coincide up to a Moebius transformation, contradicting the assumption that f_λ is a normalized family.

So we may assume that the zero of G is isolated. Let $G_1, \ldots, G_n$ be the coordinate functions of G and let Z_i be the set of $\lambda \in W$ for which $G_j(\lambda) = 0$ for all j with $j \neq i$. Let $\Psi_i\colon \mathbb{C} \to Z_i$ be the univalent parameterization of Z_i with $\Psi_i(0) = \lambda_0$.

Claim: $(G_i \circ \Psi_i)'(0) \neq 0$.
Proof of Claim: Assume the claim is false. Choose a periodic point $q_i(\lambda) \in \Lambda_\lambda$ distinct from $p_i(\lambda)$ such that $R_i(\lambda) := q_i(\lambda) - p_i(\lambda)$ is close to zero for all $\lambda \in W$. The existence of such periodic points follows from Lemmas 2.1 and 2.2. Note that $p_i(\lambda) \neq q_i(\lambda)$ for all $\lambda \in W$. Note that $G_i(\lambda) = f^{k_i}_\lambda(c_i(\lambda)) - p_i(\lambda)$ and write

$$\tilde{G}_i(\lambda) := f^{k_i}_\lambda(c_i(\lambda)) - q_i(\lambda) = \left(f^{k_i}_\lambda(c_i(\lambda)) - p_i(\lambda)\right) + (p_i(\lambda) - q_i(\lambda))$$

and

$$U_i(\lambda) = (p_i(\lambda) - q_i(\lambda))\,.$$

Then $\tilde{G}_i = G_i + U_i$. Choose $\delta > 0$ so small that $\Psi(t) \in W$ for all $t \in B_\delta(0)$. Since we have assumed that the claim does not hold, $G_i \circ \Psi_i$ has a zero of multiplicity $d \geq 2$ at 0. Hence $G_i \circ \Psi_i(t) = t^d \cdot g(t)$ where $g(t)$ is some holomorphic function on the disc $B_\delta(0)$ with $g(0) \neq 0$. By choosing δ sufficiently small, we may assume that $g(t) \neq 0$ for all $t \in B_\delta(0)$. Let us show that $B_\delta(0) \ni t \mapsto \tilde{G}_i \circ \Psi_i(t)$ has several distinct zeros, provided we choose $q_i(\lambda)$ sufficiently close to $p_i(\lambda)$ (which means that U_i is close to 0). Note that

$$0 = \tilde{G}_i \circ \Psi_i(t) = G_i \circ \Psi_i(t) + U_i \circ \Psi_i(t) = t^d \cdot g(t) + U_i \circ \Psi_i(t)$$

is equivalent to

$$t^d \cdot \frac{g(t)}{u_i(t)} = -1$$

where $u_i(t) = U_i \circ \Psi_i(t)$ (note that $u_i(t) \neq 0$ for $t \in B_\delta(0)$). Since $\frac{g(t)}{u(t)} \neq 0$ for all $t \in B_\delta(0)$, it follows that the closed curve $\partial B_{\delta/2}(0) \ni t \mapsto \varrho(t) := \frac{g(t)}{u_i(t)}$ has winding number zero, and that $\partial B_{\delta/2}(0) \ni t \mapsto t^d \cdot \varrho(t)$ has winding number equal to d. Moreover, the image of $\partial B_{\delta/2}(0) \ni t \mapsto t^d \cdot \varrho(t)$ is outside the ball $B_3(0)$ (provided we choose $q_i(\lambda)$ close enough to $p_i(\lambda)$ because then $u(t)$ is small but not zero in $B_\delta(0)$). It follows that there are d points $t \in B_{\delta/2}(0)$ such that $t^d \cdot \varrho(t) = -1$. To show that these points do not occur with higher multiplicity we need the following
Subclaim: -1 is not a critical value of the map $B_{\delta/2}(0) \ni t \mapsto t^d \cdot \varrho(t) = -1$ provided we choose $\delta > 0$ sufficiently small and q_i sufficiently close to p_i (which means that $u(t)$ is close to zero).

Proof of subclaim: The homeomorphism h_t which conjugates $f_{\Psi_i(0)}$ to $f_{\Psi_i(t)}$ is K-quasiconformal and maps $p_i(\Psi_i(0)), q_i(\Psi_i(0))$ to $p_i(\Psi_i(t)), q_i(\Psi_i(t))$. By normalizing we can assume that $\tilde{h}_t$ maps $u(0)$ into $u(t)$ and that $\tilde{h}_t$ fixes $0, 1, \infty$. Therefore, according to Lemma 4.1 in the appendix, there exists K' so that for all $t \in B_\delta(0)$,

$$\frac{1}{K'}|u(0)|^{1/K} \leq |u(t)| \leq K'|u(0)|^{1/K}$$

and (going along a line through 0 to $\frac{u(0)}{|u(0)|}$) this implies that for $s \in (0,1)$,

$$|\arg(\tilde{h}_t(s\frac{u(0)}{|u(0)|})) - \arg(\tilde{h}_t(\frac{u(0)}{|u(0)|}))| \leq K' \cdot \frac{1}{-\log s}.$$

Write $u(t) = \exp(v(t))$ (since $u(t)$ is close to zero, $\Re(v(t))$ is close to $-\infty$). Choose $v(t)$ so that $\Im(v(0)) \in [0, 2\pi)$. The previous inequalities show that

$$K \cdot \Re(v(0)) \leq \Re(v(t)) \leq \frac{1}{K} \cdot \Re(v(0)) \text{ and } |\Im(v(t))| \leq K' \cdot |\Re(v(0))| + K'' \leq K''' \cdot |\Re(v(0))|$$

provided $u(0)$ is close enough to 0 (which means that $\Re(v(0))$ is close to $-\infty$). In particular,

$$|v(t)| \leq K_1 \cdot |\Re(v(0))| \tag{2.4}$$

for $t \in B_\delta(0)$. Because of the Cauchy integral formula this implies

$$|v'(t)| \leq K_2 \cdot |\Re(v(0))| \tag{2.5}$$

for $|t| \leq \delta/2$. So let us by contradiction assume that $t^d \cdot \varrho(t) = -1$ and that t is a critical point of $t \mapsto t^d \cdot \varrho(t)$. Since $\varrho(t) = \frac{g(t)}{u(t)}$ this gives

$$0 = dt^{d-1}\varrho(t) + t^d \frac{g'(t)u(t) - g(t)u'(t)}{[u(t)]^2},$$

i.e.,

$$0 = dg(t) + tg'(t) - tg(t)\frac{u'(t)}{u(t)} = dg(t) + tg'(t) - tg(t)v'(t).$$

Since g is non-zero on $B_\delta(0)$, this means that $|tv'(t)| \geq K_3 > 0$ for $t \in B_\delta(0)$ provided $\delta > 0$ is sufficiently small (where K_3 only depends on g and not on which point q_i we choose). From (2.5) this gives that

$$|t| \cdot |\Re(v(0))| \geq K_4 > 0$$

and, since $-1 = t^3\varrho(t)$,

$$1 = |t|^d \cdot |\varrho(t)| \geq |t|^d \cdot \exp(-K_5\Re(v(0)))$$

(where we used (2.4)). Clearly these two inequalities cannot both hold when $|u(0)|$ is small (because then $\Re(v(0))$ is close to $-\infty$). This completes the proof of the subclaim.

From all this it follows that $t^d \cdot \frac{g(t)}{r(t)} = -1$ has d distinct solutions inside the disc $B_{\delta/2}(0)$. In particular, if $d > 1$ then $\tilde{G}_i \circ \Psi_i$ has at least two distinct zeros $t, t' \in B_{\delta/2}(0)$. It follows that $\lambda = \Psi_i(t)$ and $\lambda' = \Psi_i(t')$ are distinct zeros of $\tilde{G}_i$. Using Proposition 2.1 we again obtain a contradiction if $d > 1$. This concludes the proof of the claim.

From the claim we get that $\Psi_i'(0) \neq 0$ and that Z_i is a smooth curve in $\mathbb{C}^n$ through $\lambda_0 \in \mathbb{C}^n$. Hence we also get from the claim that $DG_i(v_i) \neq 0$ where v_i is any unit tangent vector in $T_{\lambda_0} Z_i \subset T_{\lambda_0}\mathbb{C}^n$. Since G maps Z_i to the i-th coordinate plane in $\mathbb{C}^n$ it follows that the tangent lines at λ_0 of the analytic curves Z_i are linearly independent, i.e. $v_1, \ldots, v_n$ forms a basis of $T_{\lambda_0}\mathbb{C}^n$. It follows that $DG'(\lambda_0) \neq 0$ and that G is a local diffeomorphism near λ_0. $\square$

3 Proof in the Misiurewicz case

The main tools in the Misiurewicz case are the following two theorems of Mañé.

Theorem 3.1 *Let $f\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map. If a point $x \in J(f)$ is not a parabolic periodic point and is not contained in the ω-limit set of a recurrent critical point, then for all $\epsilon > 0$ there exists a neighbourhood U of x such that*

1. *for every $k \geq 0$, each connected component of $f^{-k}(U)$ has spherical diameter $\leq \epsilon$;*
2. *there exists $N > 0$ depending only on f such that for all $k \geq 0$ and every connected component V' of $f^{-k}(U)$,*
$$\operatorname{diam}(f^k\colon V' \to U) \leq N;$$
3. *for all $\epsilon_1 > 0$ there exists k_0 such that every connected component of $f^{-k}(U)$ for $n \geq k_0$ has spherical diameter $\leq \epsilon_1$.*

Proof: See [Ma2] and also [ST]. $\square$

Theorem 3.2 *Let $f\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map and $\Lambda \subset J(f)$ a compact invariant set not containing critical points and parabolic periodic points. Then Λ is either an expanding hyperbolic set or $\Lambda \cap \omega(c) \neq \emptyset$ for some recurrent critical point c of f.*

Proof: See [Ma2] and also [ST]. $\square$

In order to prove the main theorem we shall generalize all the steps from the previous section. First note that a Misiurewicz map cannot have a Siegel disc or a Hermann ring F. Indeed, since no critical point of a Misiurewicz map is recurrent, we can apply Theorem 3.1 to any point x in ∂F. But property

3 from that theorem clearly cannot hold for a neighbourhood U of x since U would intersect the Siegel disc or Herman ring. So a Misiurewicz map f cannot have non-hyperbolic periodic points (i.e. all eigenvalues have norm $\neq 1$). Let us show that f can also have no periodic points with eigenvalues $|\mu| < 1$. Indeed, a periodic attractor has a critical point in its immediate basin. Since f is a Misiuremicz map, this critical point is eventually mapped into a periodic orbit. This is only possible if the critical point actually lies on the attracting periodic orbit. It follows that for a Misiurewicz map the only Fatou components of a Misiurewicz map f correspond to basins of super attracting periodic points.

Theorem 3.3 *(***Uniqueness part of Thurston's realization theorem in the Misiurewicz case***)*

Let $f, \tilde{f}\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be Misiurewicz maps which are K-quasiconformally conjugate. Moreover, assume that $f, \tilde{f}$ are not Lattès maps. Then f and $\tilde{f}$ are conjugate by some Moebius transformation.

Proof: Let h be the quasiconformal conjugacy between f and $\tilde{f}$. We may assume that the restriction of h to the Fatou set is conformal. Indeed, let $c_1, \dots, c_s$ the critical point of f which are not in the Julia set and let $\tilde{c}_i = h(c_i)$. Since the maps are Misiurewicz, these critical points are either periodic or eventually mapped to a periodic critical point. Hence there exists an conformal conjugacy h_1 of f from a neighbourhood O of the f-forward orbits of $\{c_1, \dots, c_s\}$ to a neighbourhood $\tilde{O}$ of the $\tilde{f}$-forward orbits of $\{\tilde{c}_1, \dots, \tilde{c}_s\}$ with $\tilde{h}(c_i) = \tilde{c}_i$. (Here we use that a super-attracting periodic orbits is locally conformally conjugate to a map of the form $z \mapsto z^\ell$ and moreover that critical points of f have the same order as corresponding critical points of $\tilde{f}$ since these maps are conjugate.) Let H be a K'-quasiconformal homeomorphism which is equal to h on a neighbourhood of $J(f)$ and coincides with h_1 near the f-forward orbits of $\{c_1, \dots, c_s\}$. Next we can define a sequence of quasiconformal homeomorphisms H_n such that $H_{n+1} \circ \tilde{f} = f \circ H_n$. Since $H_n = H_{n+1}$ outside the basins of the critical points c_i and (and also on $f^{-n}(O)$), H_n converges to a conjugacy H_∞ which is also K'-quasiconformal. By construction H_∞ is conformal on the basin of super critical periodic points, and therefore outside the Julia set.

It is very easy to see that for a Misiurewicz map $J(f)$ either is equal to $\bar{\mathbb{C}}$ or has zero Lebesgue measure. Indeed, assume that the Lebesgue measure of $J(f)$ is positive. Take a Lebesgue density point y of $J(f)$ and a limit point x of $f^{n_i}(y)$. Let $|A|$ denote the Lebesgue measure of a measurable set A. Fix two discs $U' \subset U$ centered at x where U is taken so small so that Theorem 3.1 holds. Using Mañé's Theorem 3.1 and applying Koebe at most N times, we get that the component B_i of $f^{-n_i}(U')$ which contains y is 'approximately round', and so this gives $|B_i \cap J(f)|/|B_i| \to 1$ and that

$$|U' \cap J(f)|/|U'| \to 1 = |f^{n_i}(B_i) \cap J(f)|/|f^{n_i}(B_i)| \to 1.$$

Hence if $J(f)$ has positive Lebesgue measure then $J(f)$ contains an open set, and so $J(f) = \bar{\mathbb{C}}$. So if $J(f)$ is not equal to $\bar{\mathbb{C}}$ then H_∞ is almost everywhere conformal and therefore a Moebius transformation. So assume $J(f) = \bar{\mathbb{C}}$. If the quasiconformal distortion of H_∞ does not vanish almost everywhere then it induces a measurable line field μ on the Julia set of f which is invariant under f. Now we argue as in [McM] see also [LS2]: take a density point y of the support of μ at which μ is almost continuous (i.e., for any $\varrho > 0$ and any sequence of balls B_i centered in y we have the following: the set where the angle of the linefield $\mu|B_i$ differs more than ϱ from some constant linefield has Lebesgue measure $\theta_i \cdot |B_i|$ where θ_i tends to zero). Using again Mañé's Theorem 3.1 and Koebe in the same way as above, we obtain a small neighbourhood U' of x on which μ defines a smooth foliation with at most N singularities (because by Mañe's Theorem one can pass through the critical point at most N times). From Lemma 3.16 in [McM] it follows that f is a Lattès map, a contradiction. $\square$

We want to generalize Lemma 2.1 by showing that each critical value is a non-isolated point in some hyperbolic Cantor set (assuming that f is a Misiurewicz map).

Lemma 3.1 *(***Analogue of Lemma 2.1***) Assume that f is a Misiurewicz map. Then there exists a neighbourhood O of the set of critical points such that*

$$X_O = \{z \; ; \; f^n(z) \notin O \text{ for all } n \geq 0\}$$

is compact hyperbolic set and for each critical point c there exists a sequence of preperiodic points in X_O converging to $f(c)$.

Proof: Since f is Misiurewicz, any $x \in \omega(c)$ satisfies the assumptions of Theorem 3.1 and 3.2 (f has no neutral periodic points). Theorem 3.2 gives that X_O is hyperbolic for any neighbourhood O of $C(f)$. So let us prove the last assertion from this lemma. Since f is Misiurewicz map, there exists $\epsilon > 0$ such that the distance of all forward iterates of critical points to $C(f)$ is at least 2ϵ. If c is eventually mapped to a repelling periodic orbit then Lemma 2.1 implies that one can choose O so that $f(c)$ is a non-isolated point in X_O. So assume this is not the case and take $x \in \omega(c) \subset J(f)$. Let U be the neighbourhood of x from Theorem 3.1 corresponding to the above choice of ϵ. Let y be a repelling periodic point in $U \cap J(f)$ and let $\epsilon_1 \in (0, \epsilon)$ so that the orbit of y has distance $\geq \epsilon_1$ to $C(f)$. Let O be an $\epsilon_1 > 0$ neighbourhood of $C(f)$. Since $x \in \omega(c)$, there exist arbitrarily large $\hat{k}$ with $f^{\hat{k}}(c) \in U$ and the preimages of U have diameter $\leq \epsilon$. In particular, there exists a point $y' = y'(\hat{k})$ in the component of $f^{-\hat{k}}(U)$ containing c such that $f^{\hat{k}}(y') = y$ (i.e., y' is preperiodic) and so that $\mathrm{dist}\,(f^i(y'), f^i(f(c))) \leq \epsilon$ for $i = 0, \ldots, \hat{k}$. In particular, this implies that $f^i(y')$ has distance $\geq \epsilon'$ to $C(f)$ for all $i \geq 1$ (by the definition of ϵ and ϵ'). From part (3) of Theorem 3.1, $y'(\hat{k})$ tends to c as $\hat{k} \to \infty$. The assertion of the lemma follows. $\square$

The persistence of X_O under (small) changes of the parameter follows immediately from the hyperbolicity of X_O, see for example Theorem 1.6 in [MS]. This proves the analogue of Lemma 2.2.

Lemma 3.2 *Let $f = f_{\lambda_0}$ be as before and let X be a hyperbolic set. Then there exists a neighbourhood W of λ_0 such that for $\lambda \in W$,*

- *there exists a family of K-quasiconformal $h_\lambda \colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ depending continuously on λ and with $h_0 = id$, so that $X_\lambda = h_\lambda(X)$ and so that the restriction of $h_\lambda \colon X \to X_\lambda$ conjugates $F \colon X \to X$ to $F_\lambda \colon X_\lambda \to X_\lambda$;*
- *for each $z \in X$, the map $W \ni \lambda \mapsto h_\lambda(z) \in U$ is holomorphic.*

Proof: Follows using Theorem III.1.6 in [MS] and the lambda-Lemma (compare also with [McM]). □

Proof of Main Theorem: Consider the map

$$G(\lambda) = (f_\lambda(c_1(\lambda)) - \phi_\lambda(f(c_1)), \ldots, f_\lambda(c_n(\lambda)) - \phi_\lambda(f(c_n)))$$

and let G_i be its coordinate functions. G has an isolated zero at λ_0 because otherwise we get a sequence of Misiurewicz maps $f_{\lambda(n)}$ with $\lambda(n) \to \lambda_0$ and $G(\lambda(n)) = 0$. Because of Lemma 3.2 all these maps $f_{\lambda(n)}$ are quasiconformally conjugate to G_{λ_0} and using Theorem 3.3 they are conformally conjugate. This contradicts the assumption that f_λ is a normalized family.

So assume that G has an isolated zero, let Z_i be the set of $\lambda \in W$ with $G_j(\lambda) = 0$ for all $j \neq i$. As before, let $\Psi_i \colon \mathbb{C} \to Z_i$ be univalent parametrizations of Z_i with $\Psi_i(0) = \lambda_0$. We again need to show that $(G_i \circ \Psi_i)'(0) \neq 0$. Let us consider a few cases.

Case 1. The critical point c_i is in the Julia set. In this case, by Lemma 3.1, there is a hyperbolic set containing hyperbolic periodic points q_i arbitrarily close to $f(c_i)$. In this case, the proof is identical to the proof in the previous section.

Case 2. The critical point c_i is periodic (i.e., a super attracting periodic point) of, say, period $l = l_i$. In this case, let $p_i(\lambda)$ be the periodic point of f_λ with $p_i(\lambda_0) = c_i$. Note that $p_i(\lambda)$ and $c_i(\lambda)$ depend analytically on λ because of the implicit function theorem. Note that by the chain rule

$$(f_\lambda^l)'(p_i(\lambda)) = g(\lambda) \cdot (p_i(\lambda) - c_i(\lambda))$$

because the critical point of f is non-degenerate. Here $g(\lambda)$ is non-zero. Now if

$$t \mapsto \Psi_i \circ G_i(t) = (p_i(G_i(t)) - c_i(G_i(t)))$$

has derivative zero at $t = 0$ then there are at least two parameters $\lambda = G_i(t)$ and $\lambda' = G_i(t')$ arbitrarily close to λ_0 such that

$$(f_\lambda^l)'(p_i(\lambda)) = (f_{\lambda'}^l)'(p_i(\lambda'))$$

(and are close to 0, i.e. the points are attracting). But then we conformally conjugate the restriction of f_λ to a neighbourhood of $p_i(\lambda)$ to the restriction of $f_{\lambda'}$ to a neighbourhood of $p_i(\lambda')$ in such a way that these neighbourhoods contain $f_\lambda(c_i(\lambda))$ respectively $f_{\lambda'}(c_i(\lambda'))$ and so that the conjugacy maps the first critical point to the second. Then one can extend the conjugacy so that it becomes conformal on the basin of these attracting fixed points. But since the forward orbits of other critical points still remain outside neighbourhoods of the critical points (or are still periodic), we can then extend this to a quasiconformal conjugacy between f_λ and $f_{\lambda'}$, but using the same proof as Theorem 3.3 this gives that in fact f_λ and $f_{\lambda'}$ are conjugate via a Moebius transformation. $\square$

4 Appendix

In the next lemma we prove that a quasiconformal homeomorphism fixing 0 sends a line 0 to a curves through 0 which spirals no faster than logarithmically.

Lemma 4.1 *For each $K \geq 0$ there exist $K' < \infty$ with the following property. Let $h\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a K-quasiconformal homeomorphism fixing 0, 1 and ∞. Then for $z \in \mathbb{C}$, $s > 0$, $\phi \in \mathbb{R}$,*

$$\frac{1}{K'}|z|^K \leq |h(z)| \leq K'|z|^{1/K} \tag{4.6}$$

$$|\arg(h(se^{i\phi})) - \arg(h(e^{i\phi}))| \leq K' \cdot \frac{1}{-\log|s|}. \tag{4.7}$$

(The difference of the arguments is well-defined if we take a continuous lift of $(0,1) \ni s \mapsto \arg(h(se^{i\phi}))$.)

If we write for $r > 0$, $\phi \in \mathbb{R}$, $h(r \cdot e^{\phi i}) = h_1(r,\phi) \cdot e^{h_2(r,\phi)i}$ with $h_1(r,\phi) \geq 0$ and $h_2(r,\phi) \in \mathbb{R}$, then (4.7) gives

$$|h_2(r,\phi) - h_2(1,\phi)| \leq \frac{K'}{-\log r}. \tag{4.8}$$

Proof: Let $\mathcal{F}$ be the family of K-quasiconformal homeomorphisms fixing $0, 1, \infty$. By Theorem 2.1 of [Le], $\mathcal{F}$ is an equicontinuous (and normal) family. From this it follows (see Theorem 2.4 of [Le]), that there exists $c(K) > 0$ such that for any $r > 0$,

$$\frac{\max_\phi |h(re^{i\phi})|}{\min_\phi |h(re^{i\phi})|} \leq c(K). \tag{4.9}$$

According to Theorem II.3.1 of [LV], there exists a Hölder function $\phi_K\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for any K-quasiconformal homeomorphism $\tilde{h}$ fixing 0 and sending the unit disc into itself, one has

$$|\tilde{h}(z)| \leq \phi_K(|z|). \tag{4.10}$$

Combining (4.9) and (4.10), gives (4.6). This also shows that the h-image of an annulus $\{1/2 \leq |z| \leq 1\}$ is contained in an annulus of the form $\{1/M \leq |z| \leq M\}$ where M only depends on K. Now consider the h-image of a line of the form $se^{i\phi}$ with $s \in [1/2, 1]$. By the normality of $\mathcal{F}$ the maximal variation of

$[1/2, 1] \ni s \mapsto \arg(h(se^{i\phi}))$ is bounded by some number M' which only depends on K (otherwise there would be a sequence of maps $h_n \in \mathcal{F}$ for which the corresponding M'_n is at least n; such a sequence could not have a subsequence converging to a homeomorphism mapping $\{1/2 \le |z| \le 1\}$ into $\{1/M \le |z| \le M\}$, contradicting the normality of $\mathcal{F}$). Now consider the h-image of an annulus $\{1/2^{k+1} \le |z| \le 1/2^k\}$. By rescaling h as follows $\tilde{h}(w) = \frac{h(w/2^k)}{h(1/2^k)}$ we get that $\tilde{h} \in \mathcal{F}$. Hence we can apply the previous argument to $\tilde{h}$ and this gives that the variation of $[1/2^{k+1}, 1/2^k] \ni s \mapsto \arg(h(se^{i\phi}))$ is again bounded by M'. This means that the variation of $[1/2^k, 1] \ni s \mapsto \arg(h(se^{i\phi}))$ is bounded by $M'k$ which proves the second inequality in the lemma. $\square$

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