Dynamics in Games

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Introduction

Game theory is a subject with a wide range of applications in economics and biology. In this dissertation, we discuss evolutionary dynamics in games, where players' strategies evolve dynamically, and we study the long-term behaviour of some of these systems. The replicator system is a well known model of an evolving population in which successful subtypes of the population proliferate. Another system known as fictitious play (also called best response dynamics) models the evolution of a population in which at any given time some individuals review their strategy and change to the one which will maximise their payoff. We will investigate these two systems and their behaviour and finally explore the link between the two.

Chapter 1

Game Theory

We begin by briefly going over the game theoretic ideas and definitions that we hope the reader is already familiar with, along with some of the biological motivation behind the technical details. The material presented here can be found in any number of books on game theory: see for example Hofbauer and Sigmund [11, §6].

1.1 Games in normal form

Generally we assume an individual can behave in n different ways. These are called *pure strategies*, denoted by P_1 to P_n, and could represent things like "fight", "run away", "play rock in a game of Rock Paper Scissors", or even "sell shares" in a more economic context. For biology, we take it that one individual plays one pure strategy. Then the make up of the population can be described by the proportions of individuals playing each pure strategy. These proportions form what is said to be simply the strategy of the population as a whole. As the proportions should clearly sum to one, the set of possible strategies is a simplex:

$$\Sigma_n = \left\{ \mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n : p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1 \right\}.$$

Here p_i can be thought of as the chance of randomly picking an individual who uses strategy i. By definition a population in which everybody plays strategy P_i corresponds to the i-th unit vector in the simplex.

We then consider games to be played by a population that evolves continuously as time progresses, describing a path in the simplex. To model some form of evolution, we need to somehow quantify what happens when one individual encounters another: which strategy will perform better? Given that we know the composition of the population, how well will one strategy perform on average over many encounters? In this context, the phrases "perform better" or "succeed" refer to an increase in evolutionary fitness: more food, more territory, more females - anything that increases the chance of reproduction.

The way to model this is to assign a "payoff": when an individual using strategy i meets a second individual following strategy j, the i-strategist will receive a payoff of a_{ij}. This creates an $n \times n$ payoff matrix $A = (a_{ij})$. Then a population with strategy $\mathbf{p}$ playing against a population with strategy $\mathbf{q}$ will receive a payoff

$$\mathbf{p} \cdot A\mathbf{q} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} p_i q_j.$$

A game described this way is called a game in *normal form*. This is as opposed to extensive form, where a game is written as a tree. Extensive form can describe games with imperfect or incomplete information, such as where players do not know the payoffs or when the payoffs are subject to (random) noise.

The next concept needed is that of a best response.

Definition 1.1 (Best responses)**.** A strategy $\mathbf{p}$ is a *best response* to a strategy $\mathbf{q}$ if for every $\mathbf{p}'$ in Σ_n, we have

$$\mathbf{p} \cdot A\mathbf{q} \geq \mathbf{p}' \cdot A\mathbf{q}.$$

We denote the set of best responses to $\mathbf{q}$ by $BR(\mathbf{q})$. Essentially, a best response to $\mathbf{q}$ is the strategy (or strategies - best responses are not necessarily unique) that will maximise the payoff. This is thus a useful concept as many of the systems we will consider will involve a population attempting to maximise its payoff.

If a strategy is a best response to itself, this is called a Nash equilibrium, named for John Forbes Nash. More formally:

Definition 1.2 (Nash equilibria)**.** A strategy $\mathbf{p}$ is a *Nash equilibrium* if

$$\mathbf{p} \cdot A\mathbf{p} \geq \mathbf{q} \cdot A\mathbf{p} \quad \forall \mathbf{p}.$$

If a strategy is the unique best response to itself, then it is said to be a *strict* Nash equilibrium.

Notice that $\mathbf{p}$ is a Nash equilibrium if and only if its components satisfy

$$(A\mathbf{p})_i = c \quad \forall\, i \text{ such that } p_i > 0,$$

for some constant $c > 0$, where

$$p_1 + \cdots + p_n = 1.$$

It is well known that every finite game has a (not necessarily unique) Nash equilibrium. Multiple proofs, including Nash's original proof from his thesis [13], may be found in a paper by Hofbauer [10, Theorem 2.1].

It seems sensible that if a strategy is a Nash equilibrium, then the population should stop evolving at this state, as it is already following the best strategy. This brings us to questions of stability of equilibria.

Definition 1.3 (Evolutionarily stable states)**.** A strategy $\hat{\mathbf{p}}$ is said to be an *evolutionarily stable state* (ESS) if

$$\hat{\mathbf{p}} \cdot A\mathbf{p} > \mathbf{p} \cdot A\mathbf{p} \tag{1.1}$$

for every $\mathbf{p} \neq \hat{\mathbf{p}}$ in a neighbourhood of $\hat{\mathbf{p}}$.

1.2 Bimatrix games

An $n \times m$ bimatrix game covers the situation where there are two players or populations, one with n strategies and one with m strategies. In normal form this can be represented by a pair of $n \times m$ matrices $[A, B]$, so that if Player A plays strategy i and Player B plays strategy j, then the payoff to Player A will be a_{ij} and the payoff to Player B will be b_{ij}. This allows for the representation of games such as the Prisoner's Dilemma or Rock, Paper, Scissors. The concepts of Nash equilibria and best responses may then be generalised to bimatrix games. For simplicity of notation we write the strategy $\mathbf{p} \in \Sigma_A$ of Player A as a row vector and the strategy $\mathbf{q} \in \Sigma_B$ of Player B as a column vector.

Definition 1.4 (Best responses)**.** A strategy $\hat{\mathbf{p}} \in \Sigma_A$ is a best response of Player A to the strategy $\mathbf{q} \in \Sigma_B$ of Player B if for every $\mathbf{p} \in \Sigma_A$ we have

$$\hat{\mathbf{p}} A\mathbf{q} \geq \mathbf{p} A\mathbf{q}. \tag{1.2}$$

Similarly a strategy $\hat{\mathbf{q}} \in \Sigma_B$ is a best response of Player B to the strategy $\mathbf{p} \in \Sigma_A$ of Player A if for every $\mathbf{q} \in \Sigma_B$ we have

$$\mathbf{p} B\hat{\mathbf{q}} \geq \mathbf{p} B\mathbf{q} \tag{1.3}$$

We denote by $\mathrm{BR}_A(\mathbf{q}) \subset \Sigma_A$ and $\mathrm{BR}_B(\mathbf{p}) \subset \Sigma_B$ the sets of best responses of Players A and B to strategies $\mathbf{q}$ and $\mathbf{p}$ respectively.

Definition 1.5 (Nash equilibria)**.** A strategy pair $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ is a *Nash equilibrium* for the bimatrix game $[A, B]$ if $\hat{\mathbf{p}}$ is a best response to $\hat{\mathbf{q}}$ and $\hat{\mathbf{q}}$ is a best response to $\hat{\mathbf{p}}$: that is, $(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \in \mathrm{BR}_A(\hat{\mathbf{p}}) \times \mathrm{BR}_B(\hat{\mathbf{q}})$.

Chapter 2

Replicator Dynamics

The replicator equation is a long-standing simple population model for the evolutionary success of different subtypes within a population. It can be found in many books on evolutionary game dynamics, including those by Hofbauer and Sigmund [11, §7] and by Fudenberg and Levine [7, §3].

If a population is very large, then one individual is evolutionarily speaking a very small insignificant part of the population. Consequently we tend to ignore the fact that populations come in discrete sizes, and assume the population changes continuously over time. Let us denote by p_i the proportion of type i individuals within the population. Then $\mathbf{p} = (p_1, \ldots, p_n) \in \Sigma_n$ can be considered as a continuous function of time t. The relative success of each type will be dependent on the overall composition of the population. Denote this fitness by $f_i(\mathbf{p})$. Then the average fitness for the population will be $\bar{f}(\mathbf{p}) := \sum_{i=1}^n p_i f_i(\mathbf{p})$.

Definition 2.1 (Replicator Dynamics)**.** Given $\mathbf{p}(t) \in \Sigma_n$, $f_i : \Sigma_n \to \mathbb{R}$, the *replicator equation* is

$$\dot{p_i} = p_i(f_i(\mathbf{p}) - \bar{f}(\mathbf{p})) \qquad i = 1, \ldots, n. \tag{2.1}$$

Notice that we can add a function $\Phi(\mathbf{p})$ to each f_i without changing the dynamics.

In the case where fitness is given by a payoff matrix A, this simplifies to

$$\dot{p_i} = p_i((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p}) \qquad i = 1, \ldots, n. \tag{2.2}$$

Lemma 2.1. *The addition of a constant c to the j-th column of A does not change the dynamics.*

Proof. Let B be the matrix A with the addition of a constant c to the j-th column. Then,

$$\begin{aligned} (B\mathbf{p})_i &= \sum_{k=1}^n a_{ik} p_k + c p_j \\ \mathbf{p} \cdot B\mathbf{p} &= \sum_{k=1}^n \sum_{l=1}^n a_{kl} p_k p_l + \sum_{k=1}^n c p_j p_k. \end{aligned}$$

Noting that $\sum_k c p_j p_k = c p_j \sum_k p_k = c p_j$, we see that

$$\begin{aligned} p_i((B\mathbf{p})_i - \mathbf{p} \cdot B\mathbf{p}) &= p_i((A\mathbf{p})_i + c p_j - (\mathbf{p} \cdot A\mathbf{p} - c p_j)) \\ &= p_i((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p}). \end{aligned}$$

Thus the dynamics do not change. □

Note that it follows from Lemma 2.1 that we may rewrite A in a simpler form, for example having only 0 in the last row.

The following useful lemma (given as an exercise in Hofbauer and Sigmund [11, p. 68]) allows us to transform the system so that we may move a point of interest such as a Nash equilibrium to the barycentre.

Lemma 2.2. *The projective transformation $\mathbf{p} \to \mathbf{q}$ given by*

$$q_i = \frac{p_i c_i}{\sum_{l=1}^n p_l c_l}$$

(with $c_j > 0$) changes (2.2) into the replicator equation with matrix $\tilde{A} = (a_{ij} c_j^{-1})$. Notice that this enables us to move a specific point $\mathbf{p} = (p_i)$ to the barycentre $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ by taking $c_i = p_i^{-1}$.

Proof. This calculation is somewhat tedious but necessary as the result will be useful later. First, let us calculate

$$\begin{aligned} (\tilde{A}\mathbf{q})_i &= \sum_{j=1}^n a_{ij} c_j^{-1} q_j \\ &= \sum_{j=1}^n a_{ij} c_j^{-1} p_j c_j \frac{1}{\sum_{l=1}^n p_l c_l} \\ &= \frac{1}{\sum_{l=1}^n p_l c_l} (A\mathbf{p})_i. \end{aligned}$$

Also we have

$$\begin{aligned} \mathbf{q} \cdot \tilde{A}\mathbf{q} &= \sum_{j=1}^n a_{ij} c_j^{-1} q_i q_j \\ &= \frac{1}{(\sum_{l=1}^n p_l c_l)^2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} c_i p_i p_j. \end{aligned}$$

Then by definition,

$$
\begin{aligned}
\frac{dq_i}{dt} &= \frac{d}{dt}\left(\frac{p_i c_i}{\sum_{l=1}^n p_l c_l}\right) \\
&= \frac{\dot{p}_i c_i}{\sum_{l=1}^n p_l c_l} - \frac{p_i c_i}{\left(\sum_{l=1}^n p_l c_l\right)^2}\sum_{l=1}^n \dot{p}_l c_l \\
&= \frac{p_i c_i}{\sum_{l=1}^n p_l c_l}\left(\sum_{j=1}^n a_{ij} p_j - \sum_{i=1}^n \sum_{j=1}^n a_{ij} p_i p_j\right) \\
&\quad - \frac{p_i c_i}{\left(\sum_{l=1}^n p_l c_l\right)^2}\sum_{i=1}^n \left[p_i c_i \left(\sum_{j=1}^n a_{ij} p_j - \sum_{k=1}^n \sum_{j=1}^n a_{kj} p_k p_j\right)\right] \\
&= \frac{p_i c_i}{\sum_{l=1}^n p_l c_l}\left(\sum_{j=1}^n a_{ij} p_j - \frac{1}{\sum_{l=1}^n p_l c_l}\sum_{i=1}^n \sum_{j=1}^n a_{ij} c_i p_i p_j\right) \\
&= p_i c_i \left((\tilde{A}\mathbf{q})_i - \mathbf{q}\cdot\tilde{A}\mathbf{q}\right) \\
&= \frac{q_i}{\sum_{l=1}^n c_l^{-1} q_l}\left((\tilde{A}\mathbf{q})_i - \mathbf{q}\cdot\tilde{A}\mathbf{q}\right).
\end{aligned}
$$

This is almost the replicator equation: the only difference is the factor of $\sum_{l=1}^n p_l c_l$. This can be fixed by a rescaling of time, and we obtain the replicator equation with matrix $\tilde{A} = (a_{ij} c_j^{-1})$ as required. □

2.1 Long term behaviour and convergence

We now endeavour to say something about the long term behaviour of replicator dynamics. As one would expect, this is closely tied to the concepts of Nash equilibria, evolutionarily stable strategies and Lyapunov stability[1].

Lemma 2.3. *If $\mathbf{p} \in \Sigma_n$ is a Nash equilibrium for the game with payoff matrix A, then $\mathbf{p}$ is a fixed point of the corresponding replicator equation.*

Proof. Suppose $\mathbf{p}$ is a Nash equilibrium. Then, as discussed in the previous chapter, its components satisfy

$$(A\mathbf{p})_i = c \quad \forall\, i \text{ such that } p_i > 0, \text{ for some constant } c > 0, \text{ where } \sum_{i=1}^n p_i = 1.$$

Thus,

$$\mathbf{p}\cdot A\mathbf{p} = \sum_{i=1}^n p_i c = c.$$

Considering the replicator equation, we see that

$$
\begin{aligned}
\dot{p}_i = 0 \quad \forall\, i &\iff p_i((A\mathbf{p})_i - \mathbf{p}\cdot A\mathbf{p}) = 0 \\
&\iff p_i((A\mathbf{p})_i - c) = 0.
\end{aligned}
$$

This clearly holds when $\mathbf{p}$ is a Nash equilibrium. □

[1] It is assumed the reader is familiar with Lyapunov stability and omega limit sets: for definitions and discussion of these concepts see for example Hofbauer and Sigmund [11, §2.6]

Lemma 2.4. *Rest points $\mathbf{p} \in \mathrm{int}(\Sigma_n)$ of the replicator equation are precisely points $\mathbf{p}$ such that*

$$(A\mathbf{p})_1 = (A\mathbf{p})_2 = \cdots = (A\mathbf{p})_n \qquad \text{and} \qquad \sum_{i=1}^n p_i = 1.$$

Equivalently, a point in the interior of the simplex Σ_n is a rest point if and only if it is a Nash equilibrium.

Proof. Points in the interior have $p_i > 0$ for all i, so this follows directly from the previous lemma. □

Lemma 2.5. *If the omega limit set $\omega(\mathbf{p})$ of a point $\mathbf{p} \in \Sigma_n$ under replicator dynamics consists of one point $\mathbf{q}$, then $\mathbf{q}$ is a Nash equilibrium of the underlying matrix.*

Proof. Suppose $\{\mathbf{q}\} = \omega(\mathbf{p})$ for some $\mathbf{p} \in \Sigma_n$. Then by definition and slight abuse of notation[2], there exists a sequence t_n with $\lim_{n\to\infty} t_n = \infty$ such that $\mathbf{p}(t_n) \to \mathbf{q}$. This implies that $\dot{\mathbf{p}}(t_n) \to 0$. Suppose $\mathbf{q}$ is not a Nash equilibrium. Then there exists a standard basis vector $\mathbf{e}_i$ such that $\mathbf{e}_i \cdot A\mathbf{q} > \mathbf{q}\cdot A\mathbf{q}$. Equivalently, there exists $\epsilon > 0$ such that $\mathbf{e}_i \cdot A\mathbf{q} - \mathbf{q}\cdot A\mathbf{q} > \epsilon$. By definition, $(A\mathbf{q})_i = \mathbf{e}_i \cdot A\mathbf{q}$, thus we have

$$
\begin{aligned}
\dot{q}_i &= q_i(\mathbf{e}_i \cdot A\mathbf{q} - q \cdot A\mathbf{q}) \\
\implies \frac{\dot{q}_i}{q_i} &> \epsilon \quad \forall t \geq 0.
\end{aligned}
$$

But then $\dot{\mathbf{q}} \not\to 0$ for any $t_n \to \infty$: a contradiction. □

Lemma 2.6. *If $\mathbf{q}$ is a Lyapunov stable point of the replicator equation with matrix A, then $\mathbf{q}$ is a Nash equilibrium for the game with payoff matrix A.*

Proof. Suppose $\mathbf{q}$ is not a Nash equilibrium. Then by continuity there exists an i and some $\epsilon > 0$ such that $(A\mathbf{p})_i - \mathbf{p}\cdot A\mathbf{p} > \epsilon$ for all $\mathbf{p}$ in some neighbourhood of $\mathbf{q}$. Then $\dot{p}_i > \epsilon p_i$, so p_i increases exponentially for all $\mathbf{p}$ in some neighbourhood of $\mathbf{q}$. This contradicts the Lyapunov stability of $\mathbf{q}$. □

Notice that not every rest point of the replicator equation is a Nash equilibrium: any pure strategy will always be a rest point (representing the idea that extinct subtypes cannot come back to life), but there are certainly games where a given pure strategy is not a Nash equilibrium. Similarly, not every Nash equilibrium is Lyapunov stable under fictitious play: for example in a game where every strategy is a Nash equilibrium (take $a_{ij} = 1$ for all i, j).

Theorem 2.1. *If $\mathbf{q} \in \Sigma_n$ is an evolutionarily stable state for the game with payoff matrix A, then $\mathbf{q}$ is an asymptotically stable rest point of the corresponding replicator equation.*

Proof. Consider the function

$$F : \Sigma_n \to [0, \infty], \ \mathbf{p} \mapsto \prod_{i=1}^n p_i^{q_i}.$$

We first show that this has a unique maximum at $\mathbf{q}$. To see this, we use Jensen's Inequality:

[2] Here we use $\mathbf{p}$ to denote both the point $\mathbf{p} \in \Sigma_n$ and the solution $\mathbf{p}(t)$ to the replicator equation satisfying $\mathbf{p} = \mathbf{p}(0)$.

Theorem 2.2 (Jensen's Inequality)**.** *If a function $f : I \to I$ is strictly convex on an interval I, then*

$$f\left(\sum_{i=1}^{n} q_i p_i\right) \leq \sum_{i=1}^{n} q_i f(p_i),$$

where $\mathbf{q} = (q_i) \in \operatorname{int}(\Sigma_n)$ and $p_i \in I$, with equality if and only if $p_1 = p_2 = \cdots = p_n$.

We apply Jensen's Inequality to the function $-\log(F)$, setting $I = [0, \infty]$ and $0 \log 0 = 0 \log \infty = 0$. Notice that log is strictly increasing, so F has a maximum at $\mathbf{q}$ if and only if $\log F$ has a maximum at $\mathbf{q}$. Now,

$$\begin{aligned}\log(F(\mathbf{p})) &= \log\left(\prod_{i=1}^{n} p_i^{q_i}\right)\\ &= \sum_{i=1}^{n} \log p_i^{q_i}\\ &= \sum_{i=1}^{n} q_i \log p_i.\end{aligned}$$

By Jensen's Inequality,

$$\begin{aligned}-\log\left(\sum_{i=1}^{n} q_i \frac{p_i}{q_i}\right) &\leq -\sum_{i=1}^{n} q_i \log\left(\frac{p_i}{q_i}\right)\\ \iff \sum_{i=1}^{n} q_i \log\left(\frac{p_i}{q_i}\right) &\leq \log\left(\sum_{i=1}^{n} p_i\right).\end{aligned}$$

Since $\sum_{i=1}^{n} p_i = 1$ this gives us

$$\sum_{i=1}^{n} q_i \log p_i - \sum_{i=1}^{n} q_i \log q_i \leq 0,$$

with equality if and only if $p_i = cq_i$ for some constant c and for all i. As we must have $\mathbf{p} \in \Sigma_n$, the only viable possibility here is that $c = 1$. Thus,

$$\sum_{i=1}^{n} q_i \log p_i \leq \sum_{i=1}^{n} q_i \log q_i,$$

for all $\mathbf{p} \in \Sigma_n$, with equality if and only if $\mathbf{p} = \mathbf{q}$. So the function F has a unique maximum at $\mathbf{q}$.

Next we show that F is a strict Lyapunov function at $\mathbf{q}$, i.e. that $\dot{F} > 0$ for all $\mathbf{p}$ in

some neighbourhood of $\mathbf{q}$, $\mathbf{p} \neq \mathbf{q}$. Notice that $F(\mathbf{p}) > 0$ for all $\mathbf{p} \in \Sigma_n$. Now

$$\begin{aligned}(\log F)^{\cdot} &= \frac{d}{dt}\left(\sum_{i=1}^{n} q_i \log p_i\right)\\ &= \sum_{i=1}^{n} \frac{d}{dt} q_i \log p_i\\ &= \sum_{i=1}^{n} q_i \frac{\dot{p}_i}{q_i}\\ &= \sum_{i=1}^{n} q_i \left((A p)_i - \mathbf{p} \cdot A\mathbf{p}\right)\\ &= \mathbf{q} \cdot A\mathbf{p} - \mathbf{p} \cdot A\mathbf{p}.\end{aligned}$$

By assumption, $\mathbf{q}$ is an ESS, and so by definition we have that $\mathbf{q} \cdot A\mathbf{p} - \mathbf{p} \cdot A\mathbf{p} > 0$ for $\mathbf{p}$ in a neighbourhood of $\mathbf{q}$. Hence, $\frac{\dot{F}}{F} > 0$ in a neighbourhood of $\mathbf{q}$. We know that F is strictly positive on Σ_n, thus we have that $\dot{F} > 0$ and F is a strict Lyapunov function. So, $\mathbf{q}$ is asymptotically stable and by previous lemmas (2.3 and 2.6), $\mathbf{q}$ is an asymptotically stable rest point of the replicator equation. □

2.2 Correspondence with Lotka-Volterra dynamics

Lotka-Volterra dynamics have been well studied and are generally one of the first examples of population modelling that one encounters. Consequently a correspondence between replicator dynamics and Lotka-Volterra systems gives us a rapid gain in understanding for comparatively little effort, and is thus very helpful.

Throughout this section, $\mathbf{p} = (p_i)_{i=1}^{n} \in \Sigma_n$ will be used to denote variables for replicator dynamics and $\mathbf{q} = (q_i)_{i=1}^{n-1} \in \mathbb{R}_+^{n-1}$ will be used to denote variables for Lotka-Volterra dynamics.

Definition 2.2 (Lotka-Volterra dynamics)**.** The generalised Lotka-Volterra system for $\mathbf{q} \in \mathbb{R}_+^{n-1}$ is given by

$$\dot{q}_i = q_i \left(r_i + \sum_{j=1}^{n-1} b_{ij} q_j\right),$$

for $i = 1, \ldots, n-1$, where the r_i and b_{ij} are constants.

Written thus, the generalised Lotka-Volterra equations and replicator dynamics are both first order non-linear ordinary differential equations. Replicator dynamics involves n equations with cubic terms on an $n-1$ dimensional space, and the Lotka-Volterra system involves $n-1$ equations with quadratic terms on an $n-1$ dimensional space. A correspondence does not seem that unreasonable.

Theorem 2.3. *There exists a smooth invertible map from $\{\mathbf{p} \in \Sigma_n : p_n > 0\}$ to $\mathbb{R}_+^{n-1}$ that maps the orbits of replicator dynamics to the orbits of the Lotka-Volterra system 2.2 with $r_i = a_{in} - a_{nn}$ and $b_{ij} = a_{ij} - a_{nj}$.*

Proof. First notice that by Lemma 2.1 we may assume without loss of generality that the last row of A consists of zeros, i.e. $(A\mathbf{p})_n = 0$. Define $q_n = 1$. Consider the transformation

$$T : \{\mathbf{p} \in \Sigma_n : p_n > 0\} \to \mathbb{R}_+^{n-1}$$
$$\mathbf{p} \mapsto \mathbf{q},$$

given by

$$q_i = \frac{p_i}{p_n}$$

for $i = 1, \dots n-1$. This has inverse given by

$$p_i = \frac{q_i}{\sum_{j=1}^{n-1} q_j + 1}$$

for $i = 1, \dots n-1$, and

$$p_n = \frac{1}{\sum_{j=1}^{n-1} q_j + 1}.$$

Clearly T is smooth and invertible. We now show that T maps the orbits of the replicator equation to the orbits of the Lotka-Volterra system (and that T^{-1} performs the reverse). Assume the replicator equation holds. Then

$$\begin{aligned}
\dot{q}_i &= \frac{d}{dt}\left(\frac{p_i}{p_n}\right) \\
&= \frac{p_i}{p_n}\left((A\mathbf{p})_i - (A\mathbf{p})_n\right) \\
&= q_i \left(\sum_{j=1}^{n} a_{ij} p_j\right) \\
&= q_i \left(a_{in} + \sum_{j=1}^{n-1} a_{ij} q_j\right) \cdot p_n.
\end{aligned}$$

By a change in velocity we can remove the extra p_n. Noting that we assumed $a_{nj} \equiv 0$ for all j, set $r_i := a_{in}$ and $b_{ij} := a_{ij}$. Then we have the Lotka-Volterra system as required.

Assume the Lotka-Volterra equations hold. Then define $a_{ij} := b_{ij}$ for $i, j = 1, \dots, n-1$. Define $a_{in} := r_i$ for $i = 1, \dots, n-1$ and $a_{nj} := 0$ for $j = 1, \dots, n$. For convenience of notation define $q_n = 1$.

Then

$$\begin{aligned}
\dot{p}_i &= \frac{d}{dt}\left(\frac{q_i}{\sum_{j=1}^{n} q_j}\right) \\
&= \frac{\dot{q}_i}{\sum_{j=1}^{n} q_j} - \frac{q_i}{\sum_{j=1}^{n} q_j} \times \frac{\sum_{j=1}^{n} \dot{q}_j}{\sum_{j=1}^{n} q_j} \\
&= \frac{q_i}{\sum_{j=1}^{n} q_j}\left(r_i + \sum_{j=1}^{n-1} b_{ij} q_j\right) - \frac{\sum_{i=1}^{n} q_i}{\sum_{j=1}^{n} q_j}\left(r_i + \sum_{j=1}^{n-1} b_{ij} q_j\right) \\
&= p_i \sum_{l=1}^{n} q_l \left(\sum_{j=1}^{n} a_{ij} p_j - \sum_{i=1}^{n} p_i \sum_{j=1}^{n} a_{ij} p_j\right) \\
&= p_i \sum_{l=1}^{n} q_l \left((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p}\right).
\end{aligned}$$

By a change in velocity we thus have the replicator equation. □

There are many other interesting results involving the replicator equation which would exceed the scope of this work. A full classification of dynamics for $n = 3$ was given by Zeeman [20], and specific examples may be found in Hofbauer and Sigmund [11, §7.4].

Chapter 3

Fictitious Play

The fictitious play process was originally defined by Brown in 1951 [2] as a method of computing Nash equilibria in zero-sum games. It is now used primarily as a model of learning (see Berger's paper [4] for more of the history). At a given moment, each player computes a best response to the average of his opponent's past strategies using a payoff matrix and instantly plays this best response. This causes each player's average to move continuously towards the best response. In terms of populations this can be interpreted as follows: in each moment of time, a proportion of the current population changes from their current strategy to the best response.

3.1 Definitions

Let $[A, B]$ be a bimatrix game. We will also use A and B to refer to the two players, but the context will make clear what is meant. Following the conventions of Sparrow et al. [18], we set the averages of the pure strategies of the players to be $\mathbf{p}^A(t)$ and $\mathbf{p}^B(t)$, defining these as row and column vectors respectively. We define the best response sets BR_A and BR_B to be the sets of best responses for each player (see Section 1.2). We also define:

$$\mathbf{v}^A(t) := A\mathbf{p}^B(t)$$
$$\mathbf{v}^B(t) := \mathbf{p}^A(t)B$$

Generically the best response sets will consist of a pure strategy. If there is more than one pure best response, then the best response set will consist of all convex combinations of these pure strategies. The strategies for which there are multiple best responses form planes in the simplex, called *indifference planes* as this is where a player is indifferent between two or more strategies.

Definition 3.1 (Indifference planes)**.** The *indifference planes* for players A and B respectively are

$$Z^B_{ij} := \left\{\mathbf{p}^A \in \Sigma_A : \mathbf{v}^B_i(\mathbf{p}^A) = \mathbf{v}^B_j(\mathbf{p}^A) = \max_k\{\mathbf{v}^B_k(\mathbf{p}^A)\}\right\} \subseteq \Sigma_A$$

$$Z^A_{kl} := \left\{\mathbf{p}^B \in \Sigma_B : \mathbf{v}^A_k(\mathbf{p}^B) = \mathbf{v}^A_l(\mathbf{p}^B) = \max_k\{\mathbf{v}^A_k(\mathbf{p}^B)\}\right\} \subseteq \Sigma_B.$$

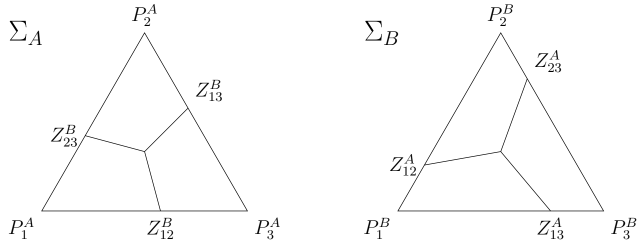


Figure 3.1: Simplices and indifference planes for the Shapley system (§3.4), $\beta \approx \frac{1}{2}$

Then the fictitious play process is defined as follows:

$$\frac{d}{dt}\mathbf{p}^A = \mathrm{BR}_A(\mathbf{p}^B) - \mathbf{p}^A \qquad (3.1)$$
$$\frac{d}{dt}\mathbf{p}^B = \mathrm{BR}_B(\mathbf{p}^A) - \mathbf{p}^B. \qquad (3.2)$$

As mentioned previously, we are here considering $\mathbf{p}^A(t)$ and $\mathbf{p}^B(t)$ to be the *averages* of the players' past strategies, or alternatively as populations many up of individuals playing pure strategies: it is not that the player is playing a mixed strategy.

3.2 Existence and uniqueness of solutions

It is important to note that as the best response is not necessarily unique, fictitious play is a *differential inclusion*. Consequently we must be very careful regarding existence and uniqueness of solutions. Firstly we notice that there is only a problem on the indifference planes defined previously. These form a codimension-one subset of $\Sigma_A \times \Sigma_B$. Outside of this subset, the best responses of each player are unique and so we have existence of uniqueness of solutions.

Lemma 3.1 (Solutions of fictitious play)**.** *If* $\mathrm{BR}_A(\mathbf{p}^B) = P^A_i$ *and* $\mathrm{BR}_B(\mathbf{p}^A) = P^B_j$, *then fictitious play is a differential equation with solutions*

$$\mathbf{p}^A(t) = (\mathbf{p}^A(0) - P^A_i)e^{-t} + P^A_i$$
$$\mathbf{p}^B(t) = (\mathbf{p}^B(0) - P^B_j)e^{-t} + P^B_j,$$

for $t \in [0, \epsilon)$ *for some* $\epsilon > 0$.

It is worth noting that the system can be reparametrised in such a way that - assuming existence and uniqueness of solutions - the players would reach their best response at time 1. This is particularly useful when modelling the system numerically.

Lemma 3.2 (Reparametrisation)**.** *The fictitious play system (3.1) can be reparametrised as described above.*

Proof. Set $s = 1 - e^{-t}$. Then

$$\begin{aligned}\frac{d}{ds}\mathbf{p}^A &= \frac{d\mathbf{p}^A}{dt}\frac{dt}{ds}\\ &= e^t(P_i^A - \mathbf{p}^A(t))\\ &= \frac{1}{1-s}(P_i^A - \mathbf{p}^A(-\ln(1-s))).\end{aligned}$$

From the solution above, we can calculate

$$\mathbf{p}^A(-\ln(1-s)) = (\mathbf{p}^A(0) - P_i^A)(1-s) + P_i^A.$$

Then we have

$$\begin{aligned}\frac{d}{ds}\mathbf{p}^A &= \frac{1}{1-s}(P_i^A - P_i^A - (1-s)(\mathbf{p}^A(0) - P_i^A))\\ &= P_i^A - \mathbf{p}^A(0).\end{aligned}$$

Following the same procedure for $\mathbf{p}^B$, this gives us solutions

$$\mathbf{p}^A(s) = (1-s)\mathbf{p}^A(0) + sP_i^A \tag{3.3}$$

$$\mathbf{p}^B(s) = (1-s)\mathbf{p}^B(0) + sP_j^B, \tag{3.4}$$

for $s \in [0, \epsilon)$ for some $\epsilon > 0$. □

This is all very well when the best response is unique. However, most solutions will cross an indifference plane eventually. Hence we endeavour to extend existence and uniqueness to the set where at most one player is indifferent. This is done rigorously in the 2008 paper by Sparrow et al. [18]. Existence and uniqueness problems with differential inclusions are explored far more fully in the book by Aubin and Cellina [1].

Denote by Z^* the set where both players are indifferent between two or more strategies. Then we have the following theorem.

Theorem 3.1. *Solutions to the fictitious play process (3.1) given in Lemma 3.1 extend continuously to* $(Z^*)^c$ *provided the following* transversality condition *is satisfied:*

For any $(\mathbf{p}^A, \mathbf{p}^B) \notin Z^*$ *such that* $\mathbf{p}^A \in Z_{ij}^B$ *for some* i, j *and* $\mathrm{BR}_A(\mathbf{p}^B) = P_k^A$, *we require that the vector towards* P_k^A *at the point* $\mathbf{p}^A$ *is not parallel to the indifference plane* $Z_{ij}^B \subset \Sigma_A$. *We also have the corresponding condition for* $\mathbf{p}^B \in Z_{ij}^A$ *and* $\mathrm{BR}_B(\mathbf{p}^A) = P_k^B$.

Proof. Suppose that $\mathbf{p}^A(0) \in Z_{ij}^B$ for some i, j and $\mathrm{BR}_A(\mathbf{p}^B(0)) = P_k^A$. We expect that due to transversality, for $t > 0$ we have $\mathrm{BR}_B(\mathbf{p}^A(t)) = P_i^B$ and for $t < 0$ we have $\mathrm{BR}_B(\mathbf{p}^A(t)) = P_j^B$ (of course possibly with i, j interchanged depending on the system). The system we hope only has uniqueness issues "momentarily" at $t = 0$, and as such we hope to define a continuous extension to the unique solutions that exist before and after t=0. To do this properly, we would need to define concepts like continuity for set-valued maps. This can be done quite naturally, but a full discussion of differential inclusions is somewhat beyond the scope of this project. As such, this is something of a sketch proof: the details of differential inclusions may be found in Aubin and Cellina [1] and the application thereof in Sparrow et al. [18, Proposition 3.1]. Continuing, we see

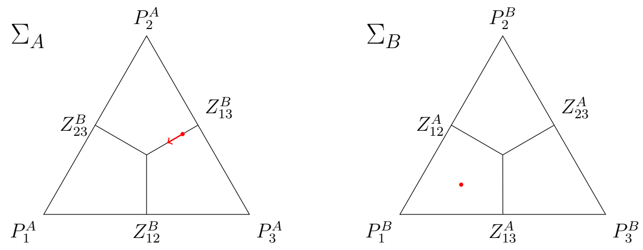


Figure 3.2: Demonstration of the failure of the transversality condition in the Shapley system with $\beta = 0$ (see §3.4)

that as $\mathrm{BR}_A(\mathbf{p}^B(0)) = P_k^A$, there exists $\epsilon > 0$ such that for all t with $|t| < \epsilon$, we have $\mathrm{BR}_A(\mathbf{p}^B(t)) = P_k^A$.

Transversality means that $((1-\lambda)\mathbf{p}^A(0) + \lambda P_k^A) \notin Z_{ij}^B$ for any non-zero value of λ. Thus, we may choose δ with $0 < \delta < \epsilon$ such that $\mathrm{BR}_B((1-\delta)\mathbf{p}^A(0) + \delta P_k^A)$ is unique and $\mathrm{BR}_B((1+\delta)\mathbf{p}^A(0) - \delta P_k^A)$ is unique. Then we may define the solution as follows:

$$\mathbf{p}^A(t) = tP_k^A + (1-t)\mathbf{p}^A(0) \qquad \text{if } |t| < \epsilon$$

$$\mathbf{p}^B(t) = \begin{cases} \mathbf{p}^B(0) & \text{if } t = 0\\ t\mathrm{BR}_B(\mathbf{p}^A(\delta)) + (1-t)\mathbf{p}^B(0) & \text{if } 0 < t < \epsilon\\ t\mathrm{BR}_B(\mathbf{p}^A(-\delta)) + (1-t)\mathbf{p}^B(0) & \text{if } -\epsilon < t < 0\end{cases}$$

This is well defined and satisfies the fictitious play equations (3.1). It is equal to the (already known) unique solution for $t \neq 0$ and is clearly a continuous extension of that unique solution. Thus this is the unique solution for $|t| < \epsilon$. □

3.3 Games with convergent fictitious play

The main class of games in which fictitious play is known to converge is zero-sum games, as shown by Robinson [15]. Two more classes of games for which fictitious play has been extensively studied are quasi-supermodular games (also known as games with strategic complementarities) and games with diminishing returns. It was originally thought that fictitious play would converge for all games - it was after all designed as a method of finding Nash equilibria. This was later shown by Shapley to be false, but considerable effort has gone into showing convergence for various classes of games. It is conjectured that non-degenerate quasi-supermodular games are one such class, but attempts at a proof have so far been unsuccessful. However, steps have been made in this direction, including the following types of games:

- Generic $2 \times n$ non-zero-sum games ([5] Berger, 2005).
- 3×3 non-degenerate quasi-supermodular games ([9] Hahn, 1999).

- $3 \times m$ and 4×4 non-degenerate quasi-supermodular games ([6] Berger, 2007).

This section will go through (some of) the ideas, results and proofs from these papers. Unfortunately the proofs have essentially employed a brute force method which does not extend to higher dimensions. The idea is to show that when a player changes strategy, they must change to a "better" strategy. Then we use the properties of quasi-supermodularity to show that there does not exist a cycle of strategies to follow with each better than the last, and thus fictitious play must converge.

We begin as always with definitions, which may be found in any of the papers above.

Definition 3.2 (Quasi-supermodular games)**.** An $n \times m$ bimatrix game $[A, B]$ is said to be *quasi-supermodular* if given $i, i' \in N = \{1, \dots, n\}$ and $j, j' \in M = \{1, \dots, m\}$ with $i < i'$ and $j < j'$, we have that

$$\begin{aligned} a_{i'j} > a_{ij} &\implies a_{i'j'} > a_{ij'} \\ b_{ij'} > b_{ij} &\implies b_{i'j'} > b_{i'j}. \end{aligned}$$

The idea is that as one player increases his strategy, it becomes increasingly advantageous for the other to also increase their strategy to a higher numbered one.

Definition 3.3 (Non-degeneracy)**.** An $n \times m$ bimatrix game $[A, B]$ is said to be *degenerate* if either for some $i, i' \in N$ with $i \neq i'$, there exists j such that $a_{i'j} = a_{ij}$ or for some $j, j' \in M$ with $j \neq j'$, there exists i such that $b_{ij'} = b_{ij}$. A game is said to be *non-degenerate* if it is not degenerate.

The crux of the proofs of convergence of fictitious play hinged on the concept of an improvement step.

Definition 3.4 (Improvement steps)**.** We say that $(i, j) \to (i', j')$ (called an *improvement step*) if either $i = i'$ and $b_{ij'} > b_{ij}$ or $j = j'$ and $a_{i'j} > a_{ij}$. A sequence of improvement steps forms an *improvement path* and a cyclical improvement path is called an *improvement cycle*.

This simple idea is intrinsically linked to the solutions of fictitious play in that the sequence of best responses must follow an improvement path.

Definition 3.5 (Fictitious play paths)**.** A strategy path $(i_t, j_t) \in N \times M$ for $t \in [0, \infty)$ is a (continuous) fictitious play path if for almost every $t \geq 1$ we have

$$(i_t, j_t) \in \mathrm{BR}_A(\mathbf{q}(t)) \times \mathrm{BR}_B(\mathbf{p}(t)), \tag{3.5}$$

where the path $(\mathbf{p}(t), \mathbf{q}(t))$ solves the fictitious play equations (3.1).

With this definition, we may consider the indifference planes as being the places where the fictitious play path *switches*.

Definition 3.6 (Switching)**.** A fictitious play path (i_t, j_t) is said to *switch* from (i, j) to (i', j') at time t_0 if $(i, j) \neq (i', j')$ and there exists $\epsilon > 0$ such that

$$(i_t, j_t) = (i, j) \quad \text{for } t \in [t_0 - \epsilon, t_0) \tag{3.6}$$

$$(i_t, j_t) = (i', j') \quad \text{for } t \in (t_0, t_0 + \epsilon]. \tag{3.7}$$

Now that we have arrived at this set up, we simply show that all switches must be improvement steps. This result is the combination of the Improvement Principle due to Sela [16] and its analog called the Second Improvement Principle due to Berger [6].

Lemma 3.3 (Improvement Principle)**.** *Suppose a fictitious play path switches from (i, j) to (i', j') at time t_1.*[1] *Then either there are improvement steps $(i, j) \to (i, j') \to (i', j')$ or there are improvement steps $(i, j) \to (i', j) \to (i', j')$.*

Proof. At time t_1, we have that $(i, j), (i', j') \in \mathrm{BR}_A(\mathbf{q}(t_1)) \times \mathrm{BR}_B(\mathbf{p}(t_1))$. The players are indifferent between both strategies. By definition of a switch, there exists $\epsilon > 0$ such that $(i_t, j_t) = (i, j)$ for all $t \in [t_0, t_1)$ where $t_0 = t_1 - \epsilon$. Then we may write

$$(\mathbf{p}(t_1), \mathbf{q}(t_1)) = \lambda(P_i^A, P_j^B) + (1 - \lambda)(\mathbf{p}(t_1), \mathbf{q}(t_1)),$$

where $\lambda \leq 1$. Rearranging,

$$\begin{aligned} (P_i^A, P_j^B) &= \frac{1}{\lambda}(\mathbf{p}(t_1), \mathbf{q}(t_1)) + \frac{\lambda - 1}{\lambda}(\mathbf{p}(t_1), \mathbf{q}(t_1)) \\ &= c(\mathbf{p}(t_1), \mathbf{q}(t_1)) + (1 - c)(\mathbf{p}(t_0), \mathbf{q}(t_0)), \end{aligned}$$

where $c = \frac{1}{\lambda} \geq 1$.

Considering only the part in Σ_B and multiplying by A,

$$AP_j^B = cA\mathbf{q}(t_1) + (1 - c)A\mathbf{q}(t_0)$$

Now subtract the i-th component from the i'-th component to get

$$\begin{aligned} a_{i'j} - a_{ij} &= c\underbrace{[A\mathbf{q}(t_1)_{i'} - A\mathbf{q}(t_1)_i]}_{0} + (1 - c)[A\mathbf{q}(t_0)_{i'} - A\mathbf{q}(t_0)_i] \\ &= \underbrace{(1 - c)}_{\leq 0 \text{ as } c \geq 1} [A\mathbf{q}(t_0)_{i'} - A\mathbf{q}(t_0)_i] \\ &\geq 0. \end{aligned}$$

□

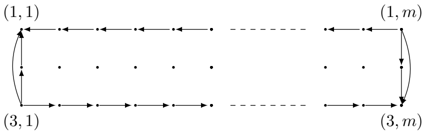
Now that we have the Improvement Principle, we are able to prove the following theorem:

Theorem 3.2. *The fictitious play process converges for every $3 \times m$ non-degenerate quasi-supermodular game (NDQSMG)*

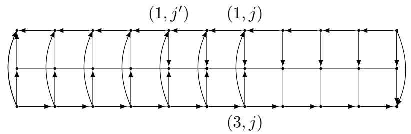
Proof. Without loss of generality we may assume that there are no dominated strategies. Then in a NDQSMG, we begin with the following set up of improvement steps.

Then we look for a step that goes "up": to be precise, a step of the form $(a, j) \to (b, j)$ where $a > b$. Clearly a cycle must contain such a step. There are only three possibilities for going "up": a cycle must contain either a step $(3, j) \to (1, j)$, a step $(2, j) \to (1, j)$ or a step $(3, j) \to (2, j)$. We consider each case.

[1] Notice that here we are assuming solutions exist

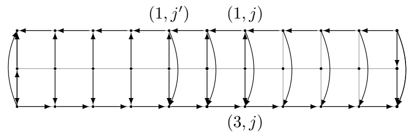


Case 1 Suppose the cycle contains a step $(3,j) \to (1,j)$. Then by quasi-supermodularity, we have steps $(3,k) \to (1,k)$ for all $k = 1,\dots,j$. Then from $(1,j)$ the cycle must continue by going left or down. We do not want to reach the equilibrium $(1,1)$, therefore at some point in the cycle there must be a step downwards from $(1,j')$ where $j' \le j$. This step cannot be to $(3,j')$ because we know there is a step $(3,j') \to (1,j')$. So, there must be a step $(1,j') \to (2,j')$. By quasi-supermodularity there are steps $(1,k) \to (2,k)$ for all $k = j',\dots,m$ and specifically for $k = j$. This implies there is a step $(3,j) \to (2,j)$, and similar steps for $k = 1,\dots,j$. Having thus extracted as much information as we can from our assumption and quasi-supermodularity, we have the following picture.



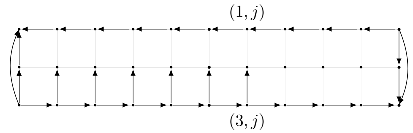
Now consider how we arrived at the point $(3,j)$. It is clear from the picture that the improvement path must have come from the bottom left corner $(3,1)$. This point cannot be part of an improvement cycle. Thus there can be no such step $(3,j) \to (1,j)$ in an improvement cycle.

Case 2 Suppose the cycle contains a step $(2,j) \to (1,j)$. Quasi-supermodularity gives us steps $(2,k) \to (1,k)$ for $k = 1,\dots,j$. From $(1,j)$ we must go left or down, and to avoid the equilibrium and follow a cycle we must hence have a step $(1,j') \to (3,j')$, where $1 < j' \le j$. This implies there are steps $(2,k) \to (3,k)$ for $k = j',\dots,m$ as shown.



Then it is clear that the next steps from $(3,j')$ must be straight to $(3,m)$, as by the picture we cannot move to row 2 and by Case 1 we cannot move to row 1. $(3,m)$ cannot be part of an improvement cycle, and so there can be no such step $(2,j) \to (1,j)$ in an improvement cycle.

Case 3 The case of a step $(3,j) \to (2,j)$ follows instantly from the preceding cases: it can be seen from the picture that any path must either involve a step as in Case 2, or will end in an equilibrium.



Thus, it is not possible to go "up". Thus any improvement path must be finite, and so fictitious play must converge. □

This method of proof is somewhat brute force. It can be applied to many situations, but is inadequate to prove convergence for general $n \times m$ NDQSMGs: indeed, in [6] Berger found an example of a 4×5 NDQSMG which *does* have an improvement cycle (but conjecturally its fictitious play converges nonetheless).

3.4 The Shapley system

Originally it was thought that fictitious play of all games would converge. Then in 1964, Lloyd Shapley presented an example where fictitious play does not converge, but rather cycles between strategies delineating a triangle in the simplex [17]. This example was extended by Sparrow et al. [18] to a one-parameter family of games, and numerical observations suggest that periodic orbits can be found in other games as well. The Shapley system has payoff matrices

$$A = \begin{pmatrix} 1 & 0 & \beta \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -\beta & 1 & 0 \\ 0 & -\beta & 1 \\ 1 & 0 & -\beta \end{pmatrix}, \quad \text{for some } \beta \in (-1,1).$$

Shapley's original example was this system with $\beta = 0$. Generally we here take β between -1 and 1. This has Nash equilibrium at the barycentre of $\Sigma_A \times \Sigma_B$:

$$[E^A, E^B] = \left[(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T \right] \in \Sigma_A \times \Sigma_B.$$

It has been shown by Sparrow, van Strien and Harris [18] that for $\beta \in (-1,\sigma)$ there is an attracting periodic orbit. This orbit is actually a hexagon in the four-dimensional space $\Sigma_A \times \Sigma_B$, but the projection to the simplices shows the orbit as triangles around E^A and E^B to be followed clockwise (see Figure 3.3). Similarly, there is a periodic orbit for $\beta \in (\tau, 1)$ which projects as triangles to be followed anti-clockwise. Here $\tau \approx 0.9$ is a root of a polynomial. Sparrow et al. further showed that the Shapley system undergoes a Hopf-like bifurcation at $\beta = \sigma$, whereupon the system becomes chaotic for $\beta \in (\sigma, \tau)$ (shown rigorously in Strien et al. [19]). These results are stable under small perturbations, and similar behaviour has been empirically observed in other games.

The existence of the periodic orbit is proved by looking for fixed points of the first return map to an indifference plane. The calculations involved are only practicable due to the simplifying effect of the symmetry of the orbit. Even so, the calculations are somewhat tedious and not instructive, hence the interested reader is referred to the paper for the full details.

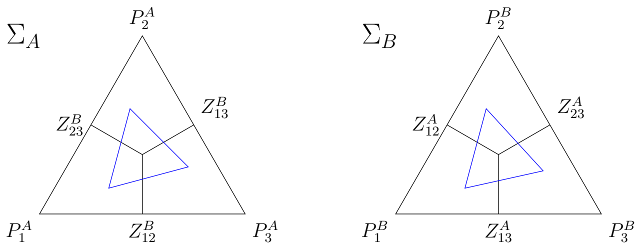


Figure 3.3: Periodic orbit for the Shapley system, $\beta = 0$.

3.5 Open questions

There are many open questions pertaining to fictitious play. As discussed in Section 3.3, there are many classes of games for which the dynamics are unknown. We now consider some of the questions related to the Shapley system as in the previous section.

3.5.1 Transition diagrams

The state space of the Shapley system is $\Sigma_A \times \Sigma_B$. This creates an issue with visualisation: it is very difficult to mentally picture things in a four dimensional space. To help with this problem, Sparrow et al. [18] thought of a way of representing $\Sigma_A \times \Sigma_B$ that enables one to see quickly and easily what indifference planes are crossed in what order. It is a similar idea to the diagrams used by Hahn [9] and Berger [6] but also incorporates the concept of indifference planes.

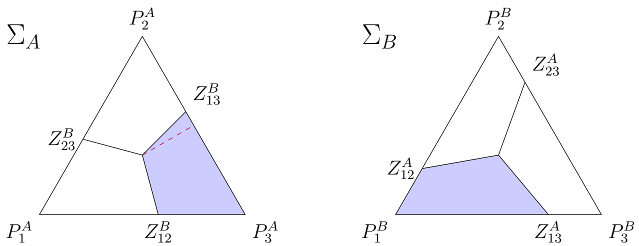


Figure 3.4: Simplices and indifference planes for the Shapley system, $\beta \approx \frac{1}{2}$

Given the simplices as above, we consider the (four-dimensional) regions where $A \to i$ and $B \to j$. There are three possibilities for the best responses of each of A and B, thus giving us nine such *preference regions*. We represent these by the grid as follows, identifying the top edge with the bottom and the left edge with the right.

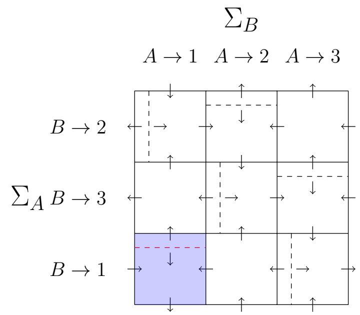
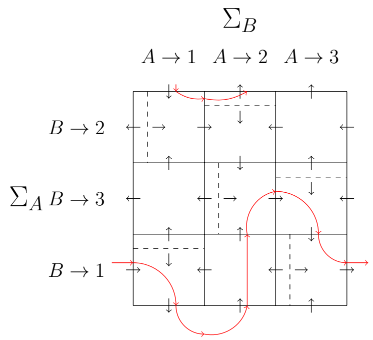


Figure 3.5: Transition diagram for $\beta > 0$

In Figures 3.4 and 3.5 above, the region where $A \to 1$ and $B \to 1$ is indicated in blue. Looking at the simplices, one can see that from a position in Σ_A that is in the blue region but *above* the red dashed line, the next indifference plane crossed will be Z_{13}^B and *not* Z_{12}^B. This is represented by the arrows and the red dashed line shown in the transition diagram. The transition diagrams are only a representation: certainly orbits in the simplex are *not* in bijective with paths drawn in the transition diagrams. These representations are, however, useful, as they provide an easy way of depicting various questions: for example, do there exist orbits of the following type?



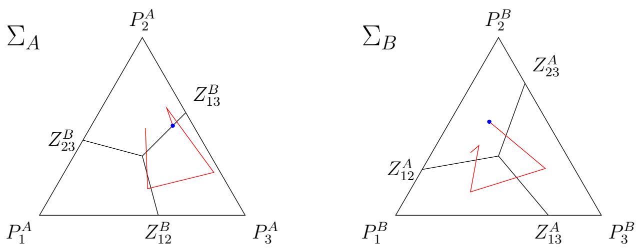
The proper phrasing of this question is as follows: for the Shapley system with $\beta > 0$, do there exist orbits beginning on Z_{13}^B that cross the following indifference planes in this exact order:

$$Z_{13}^B, Z_{23}^A, Z_{13}^B, Z_{13}^A, Z_{12}^B, Z_{12}^A, Z_{12}^B, Z_{13}^B.$$

Equivalently, do there exist orbits such that the sequences of players' best responses is as follows:

$$\begin{matrix} A \\ B \end{matrix}\begin{pmatrix} 2 & 3 & 3 & 1 & 1 & 2 & 2 \\ 3 & 3 & 1 & 1 & 2 & 2 & 1 \end{pmatrix}.$$

Drawn in the simplices, an orbit of this type would look approximately like this, beginning at the blue dots:



The transition diagrams make it possible to instantly see which regions an orbit may move into. However, these arrows only describe possible movements from one region to another: sequences of arrows are another matter entirely. The diagram does *not* mean that for any path drawn following the arrows there actually exists an orbit (or orbits) that follows this sequence. This is the question I have been considering, focusing specifically on the paths shown above before attempting to generalise. Numerical experimentation suggests that such an orbit is impossible, and indeed this appears to be the case.

The first thing to notice is that it is sufficient to consider an orbit starting on the boundary $\partial(\Sigma_A \times \Sigma_B)$.[2] This is because the system may be projected onto the boundary (see [19, §3]).

Thus, for the initial point we take $\mathbf{p}^A(0) = (0, \frac{1+\beta}{2+\beta}, \frac{1}{2+\beta}) = (a_1, a_2, a_3)$. Now we endeavour to find conditions on the initial point $\mathbf{p}^B(0) = (\mathbf{p}_1^B(0), \mathbf{p}_2^B(0), \mathbf{p}_1^B(0)) = (b_1, b_2, b_3)$ such that the orbit is of the desired type. These conditions can be essentially seen by eye: for example in Figure 3.4, a starting point in the blue region that in Σ_A is above the red dashed line will next cross Z_{12}^B whereas one that is below the red dashed line in Σ_A will cross next Z_{13}^B. Thus whether the orbit has passed the dashed lines becomes important when looking for a particular sequence of indifference planes. We hence draw in the relevant lines as in Figure 3.6.

Notice the numbered points are the points at which a player crosses an indifference plane. This is because it causes the other player to change direction: thus to proceed to the next indifference plane in the sequence, the players must be in the correct place to allow this to happen. The conditions are listed thus:

- At 0, require $\mathbf{p}_1^B < \mathbf{p}_2^B$ to next cross Z_{23}^A rather than Z_{12}^A.

[2]Notice that $\partial(\Sigma_A \times \Sigma_B) = (\Sigma_A \times \partial\Sigma_B) \cup (\partial\Sigma_A \times \Sigma_B)$. This is not the same as $\partial\Sigma_A \times \partial\Sigma_B$.

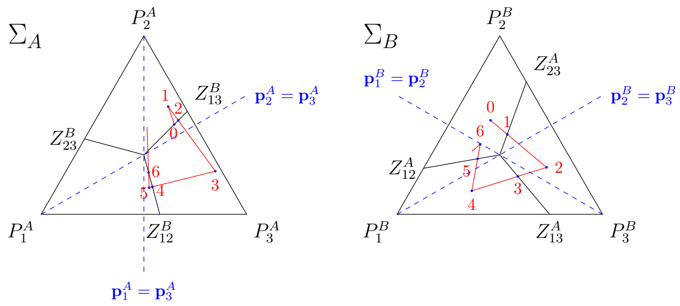


Figure 3.6: Orbit with numbers depicting the points at which an indifference plane is traversed.

- At 2, require $\mathbf{p}_2^B < \mathbf{p}_3^B$ to next cross Z_{13}^A rather than Z_{23}^A.
- At 3, require $\mathbf{p}_2^A < \mathbf{p}_3^A$ to next cross Z_{12}^B rather than Z_{13}^B.
- At 4, require $\mathbf{p}_1^B > \mathbf{p}_3^B$ to next cross Z_{12}^A rather than Z_{13}^A.
- At 5, require $\mathbf{p}_1^A < \mathbf{p}_3^A$ to next cross Z_{12}^B rather than Z_{23}^B.

With point 2, for example, this could be more explicitly written as follows: let $t_2 > 0$ be the time such that $\mathbf{p}^A(t_2) \in Z_{13}^B$. Then we require that $\mathbf{p}_2^B(t_2) < \mathbf{p}_3^B(t_2)$. For ease of calculation we use the reparametrised system (3.2). Let us now commence a numerical investigation of the orbit.

As an example, we calculate the first few inequalities explicitly. The first is clear by inspection: we require $\mathbf{p}_1^B(0) < \mathbf{p}_2^B(0)$. Then, let the time at which the orbit reaches Z_{23}^A be $t_1 > 0$. On this first leg of the orbit $(0 < t < t_1)$, we have $\mathrm{BR}_A(\mathbf{p}^B(t)) = P_2^A$, $\mathrm{BR}_B(\mathbf{p}^A(t)) = P_3^B$. Thus the equation of the orbit is

$$\mathbf{p}^A(t) = (1-t)\mathbf{p}^A(0) + tP_2^A$$
$$\mathbf{p}^B(t) = (1-t)\mathbf{p}^B(0) + tP_3^B$$

Hence t_1 solves $(A\mathbf{p}^B(t_1))_2 = (A\mathbf{p}^B(t_1))_3$. Writing this explicitly and solving using the equations above, we see

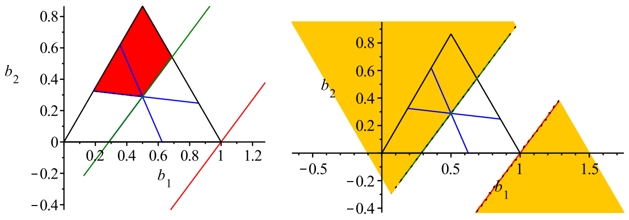
$$\beta\mathbf{p}_1^B(t_1) + \mathbf{p}_2^B(t_1) = \beta\mathbf{p}_2^B(t_1) + \mathbf{p}_3^B(t_1) \tag{3.8}$$

$$\beta(1-t_1)b_1 + (1-t)b_2 = \beta(1-t_1)b_2 + (1-t_1)b_3 + t_1 \tag{3.9}$$

$$\implies t_1 = \frac{\beta(b_1 - b_2) + (b_2 - b_3)}{1 + \beta(b_1 - b_2) + (b_2 - b_3)} \tag{3.10}$$

$$\iff t_1 = \frac{\beta(b_1 - b_2) + b_1 + 2b_2 - 1}{\beta(b_1 - b_2) + b_1 + 2b_2}. \tag{3.11}$$

The last line comes from substituting $b_3 = 1 - b_1 - b_2$ and is performed because it is then easier to get Maple to plot the correct diagrams.[3] Hence t_1 may be considered as a function of b_1 and b_2. Clearly we require $t_1 > 0$. Plotting this in Maple[4], we have



(a) $t_1 = 0$ and $t_1 = \infty$ (b) $t_1 > 0$

Figure 3.7: Plots for t_1

With some thought, these plots of Σ_B are sensible. The blue lines are (extended) indifference planes and the red area is the region where $\mathrm{BR}^A(\mathbf{p}^B(0)) = P_2^A$. The green line denotes $t_1 = 0$ and coincides with the indifference plane Z_{23}^A: if the initial point $\mathbf{p}^B(0)$ is on Z_{23}^A, then it will take no time to get there. The red line denotes $t_1 = \infty$: if the initial point $\mathbf{p}^B(0)$ is in fact P_3^B, then $\mathbf{p}^B(t)$ will be constant for $t \in [0, t_1]$ and will thus never reach Z_{23}^A. The orange region on the right hand figure is where $t_1 > 0$. Notice that in plotting this we are currently ignoring the previous conditions on b_1 and b_2 (for example the condition for $\mathrm{BR}_A(\mathbf{p}^B(0)) = P_2^A$) and are actually extending the function $t_1(b_1, b_2)$ from its original domain determined by these conditions to the entirety of $\mathbb{R}^2$.

Performing similar calculations to find $t_2 \in (0,1)$ such that $\mathbf{p}^A(t_1 + t_2) \in Z_{13}^B$, we find

$$t_2 = \frac{-1 - \beta b_2 + b_1 + 2b_2 + \beta b_1}{-1 - \beta^2 b_2 + 2b_1 + 3\beta b_1 + 4b_2 + \beta^2 b_1}. \tag{3.12}$$

As seen in Figure 3.6, the orbit must reach Z_{13}^B after $\mathbf{p}_2^B(t) = \mathbf{p}_3^B(t)$ in order to next cross Z_{13}^A rather than Z_{23}^A. Thus we calculate $s_2 \in (0,1)$ such that $\mathbf{p}_2^B(t_1 + s_2) = \mathbf{p}_3^B(t_1 + s_2)$.

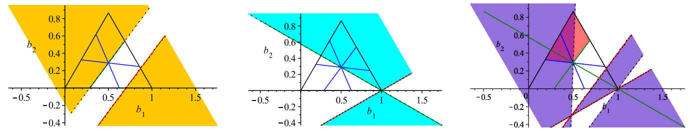
$$s_2 = -\frac{\beta(b_1 - b_2)}{b_1 + 2b_2}. \tag{3.13}$$

Now we require $t_2 > s_2$. This forms the purple region in Figure 3.8:

Once again, the blue lines denote extended indifference planes, the green lines denote where the relevant quantity is 0, and the red lines where the relevant quantity is ∞. The additional edges of the purple region visible in Figure 3.8 (c) are where $t_2 = s_2$. This plot tells us that there do exist regions such that both $t_2 > s_2$ and the initial conditions are satisfied. Omitting calculations, we continue plotting the further conditions: if there

[3] Plotting in the simplex can be achieved in Maple by creating a standard plot with b_1 against b_2 and then using the *transform* function in the *plottools* library to send (x, y) to $(1 - x - \frac{1}{2}y, \frac{\sqrt{3}}{2}y)$. This method proved easier than correctly orienting a 3d plot.

[4] Plots shown are for $\beta = \frac{2}{5}$; other values look similar.

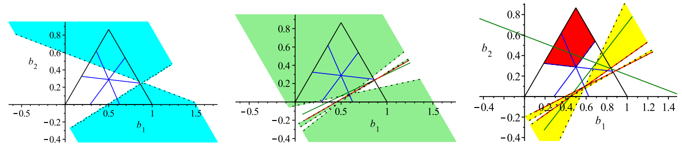


(a) $t_2 > 0$ (orange) (b) $s_2 > 0$ (cyan) (c) $t_2 > s_2$ (purple)

Figure 3.8: Plots for t_2 and s_2

exists a region such that all of the conditions are satisfied at once, then the orbit is possible. If upon overlapping the plots there is no such region, then the orbit is impossible.

We thus get the following plots:

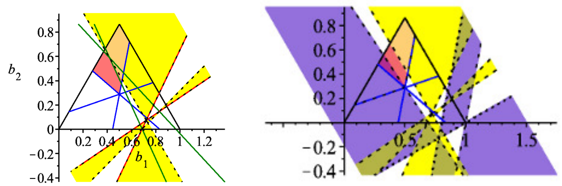


(a) $s_3 < t_3$ (cyan) (b) $s_4 < t_4$ (green) (c) $s_5 > t_5$ (yellow)

Figure 3.9: Plots for t_3, t_4 and t_5.

It is clear that for this particular value $\beta = \frac{2}{5}$, the yellow region where $s_5 > t_5$ does not intersect the red initial region. Thus for $\beta = \frac{2}{5}$, no orbit may follow the desired sequence of indifference planes.

For other values of beta, it is slightly less clear. For example for $\beta = \frac{4}{5}$ we instead get the plot Figure 3.10 (a). However, considering also the plot for $s_2 < t_2$ as in Figure 3.10 (b), we see that there is still no feasible region.



(a) $s_5 > t_5$ (yellow) (b) Plots for t_5 and t_2 overlaid

Figure 3.10: Plots for $\beta = \frac{4}{5}$.

Here, there is clearly nowhere where the initial conditions are satisfied (red), $t_2 > s_2$ (purple), *and* $s_5 > t_5$ (yellow). Thus the orbit is still impossible. This appears to happen for all values $\beta \in (0,1)$.

Of course, the above reasoning is only numerical and does not constitute a fully rigorous proof. However, it provides a good intuition as to why this particular sequence cannot be realised by a concrete orbit of fictitious play.

Chapter 4

The link between replicator dynamics and fictitious play

As we have seen, there are quite some similarities between fictitious play and replicator dynamics, and a bit of numerical experimentation lends credence to the idea of there being some sort of link between the two. As noted by Gaunersdorfer and Hofbauer [8] and in the book by Hofbauer and Sigmund [11], the general rule of thumb is that in the long term, the time averages of solutions to the replicator equation behave as solutions for fictitious play. This was made more precise in Benaïm et al. [3], and this chapter will carefully go through the ideas and proofs presented in that paper. The main statement is that the time averages of solutions to replicator dynamics approximate the fictitious play process, and that this approximation gets better as $t \to \infty$.

We begin by generalising. Instead of considering replicator dynamics and fictitious play as involving matrices at all, we instead just focus on one player (or population). This player receives a stream of outcomes: if he plays strategy i at time t, he will receive the payoff $U_i(t)$. The key point here is that the player has no idea of what is behind these outcomes; no idea even of how many other players he is facing, let alone what they are doing, what matrices are possibly involved, whether there is some sensible mechanism behind the calculations of the payoffs or whether it is just random. For simplicity, we assume the payoffs to be bounded and measurable. Formally, with n the number of available strategies and $c > 0$ a bound on the payoffs, we have an outcome process:

$$\mathcal{U} = \{U(t) \in [-c, c]^n, t \geq 0\}.$$

As noted above, $U_i(t)$ is the payoff the player receives from playing strategy i at time t. Then the time average of the payoffs is $\bar{U}(t) = \frac{1}{t}\int_0^t U(s)\,\mathrm{d}s$.

We next define best responses: for $U \in [-c, c]^n$,

$$\mathrm{BR}(U) = \{\mathbf{p} \in \Sigma_n : \langle \mathbf{p}, U\rangle = \max_{\mathbf{q}\in\Sigma_n} \langle \mathbf{q}, U\rangle\}.$$

Now we define generalised fictitious play for $\mathcal{U}$ by

$$\dot{\mathbf{p}}(t) \in \frac{1}{t}[\mathrm{BR}(\bar{U}(t)) - \mathbf{p}(t)]. \tag{4.1}$$

If we set $U(t) = A\mathbf{q}(t)$ where $\mathbf{q}(t)$ is the opponent's strategy at time t, then we recover the original fictitious play equation (3.1) for player one.

We define generalised replicator dynamics by

$$\dot{\mathbf{p}}_i(t) = \mathbf{p}_i(t)[U_i(t) - \langle \mathbf{p}(t), U(t)\rangle] \qquad i \in 1,\dots,n. \tag{4.2}$$

Here we may set $U(t) = A\mathbf{p}(t)$ to recover the original version.

To show the time averages of solutions to replicator dynamics approximate the fictitious play process, we first need a sensible description of an approximation to the fictitious play process. We do this by approximating the best responses:

Definition 4.1 (Approximation to best responses)**.** Given $U \in [-c,c]^n$, $\eta > 0$, define the *η-neighbourhood of* $\mathrm{BR}(U)$ as follows:

$$[\mathrm{BR}]^{\eta}(U) = \{\mathbf{p} \in \Sigma_n \,|\, \exists\, \mathbf{q} \in \mathrm{BR}(U) \,:\, \|\mathbf{p}-\mathbf{q}\|_\infty < \eta\}, \tag{4.3}$$

where $\|\cdot\|_\infty$ is the supremum norm.

We will now show that the logit map can be used to find approximate best responses. Then, we will use the logit map to find solutions of the replicator equation for a given $\mathcal{U}$.

Definition 4.2 (Logit map)**.** Define $L : \mathbb{R}^n \to \Sigma_n$ by

$$(L(U))_i = \frac{\exp(U_i)}{\sum_{j=1}^n \exp(U_j)}. \tag{4.4}$$

Lemma 4.1. *[12, Proposition 3.1] For every $U \in [-c,c]^n$ and every $\epsilon > 0$, there exists a function $\eta(\epsilon)$ with $\eta(\epsilon) \to 0$ as $\epsilon \to 0$ such that*

$$L\left(\frac{U}{\epsilon}\right) \in [\mathrm{BR}]^{\eta(\epsilon)}(U).$$

Proof. Given $\eta > 0$, define

$$D^{\eta}(U) = \{\mathbf{p} \in \Sigma_n \,|\, U_i + \eta < \max_{j=1,\dots,n} U_j \implies \mathbf{p}_i \le \eta \text{ for } i = 1,\dots,n\}. \tag{4.5}$$

This rather awkward-looking definition is actually quite simple once fully understood. To illustrate the idea, consider the following example.

Example 1. Suppose that

$$U = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in [-1,1]^3$$

and suppose that $\eta < 1$. Then

$$D^{\eta}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \left\{ \begin{pmatrix} \lambda\delta \\ (1-\lambda)\delta \\ 1-\delta \end{pmatrix} \in \Sigma_3 \,|\, \delta < \eta, \lambda \in [0,1] \right\}.$$

Here, $\max_j U_j = U_3 = 1$, and we have $U_1 + \eta = \eta < 1$ and $U_2 + \eta = \eta < 1$. Thus we require that $p_1, p_2 < \eta$, and furthermore we want the *set* of all such $\mathbf{p} \in \Sigma_3$ with $p_1, p_2 < \eta$. This gives the set above.

Now it happens that $D^{\eta}(U)$ is a subset of $[\mathrm{BR}]^{\eta(\epsilon)}(U)$ and it is conveniently easy to show that $L\left(\frac{U}{\epsilon}\right) \in D^{\eta}(U)$. This is the idea of the proof.

Proposition 1. $D^{\eta}(U) \subseteq [\mathrm{BR}]^{\eta}(U)$.

Proof. Suppose $\mathbf{p} \in D^{\eta}(U)$. Then, given ℓ such that $U_\ell = \max_{j=1,\dots,n} U_j$, clearly we have that the ℓ-th unit vector is a best response, i.e. $P^{\ell} = (0,\dots,0,1,0,\dots,0) \in \mathrm{BR}(U)$.

Then for $\mathbf{p} \in D^{\eta}(U)$, we know that $\mathbf{p}_\ell \ge 1-\eta$. Thus:

$$\begin{aligned} |(P^{\ell})_\ell - \mathbf{p}_\ell| &= |1 - \mathbf{p}_\ell| \\ &\le |1 - (1-\eta)| = \eta. \end{aligned}$$

For $i \ne \ell$,

$$|(P_\ell)_i - \mathbf{p}_i| = |0 - \mathbf{p}_i| = |\mathbf{p}_i| \le \eta.$$

Hence, for every $\mathbf{p} \in D^{\eta}(U)$, there exists $\mathbf{q} \in [\mathrm{BR}]^{\eta}(U)$ such that

$$\|\mathbf{q} - \mathbf{p}\| \le \eta.$$

Thus $D^{\eta}(U) \subseteq [\mathrm{BR}]^{\eta}(U)$. □

Continuing the proof of Lemma 4.1, let $\epsilon(\eta)$ satisfy

$$\exp(-\frac{\eta}{\epsilon}) = \eta.$$

Then for all $i, k \in \{1,\dots,n\}$,

$$\begin{aligned} L^i\left(\frac{U}{\epsilon}\right) &= \frac{\exp(\frac{U_i}{\epsilon})}{\sum_j \exp(\frac{U_j}{\epsilon})} \\ &= \frac{\exp(\frac{U_i}{\epsilon})}{\sum_j \exp(\frac{U_j}{\epsilon})} \cdot \frac{\exp(-\frac{U_k}{\epsilon})}{\exp(-\frac{U_k}{\epsilon})} \\ &= \frac{\exp(\frac{(U_i - U_k)}{\epsilon})}{\sum_{j\ne k} \exp(\frac{U_j - U_k}{\epsilon})}. \end{aligned}$$

This holds for every $i, k \in \{1,\dots,n\}$. Specifically, this holds for $k = \ell$ such that $U_\ell = \max_j U_j$. Then if $U_i + \eta < U_\ell$, we have

$$\begin{aligned} & U_i - U_\ell < -\eta \\ \implies & \exp\left(\frac{(U_i - U_\ell)}{\epsilon}\right) < \exp\left(-\frac{\eta}{\epsilon}\right) = \eta. \end{aligned}$$

Hence, for a given $\eta > 0$, we may take $\eta(\epsilon)$ to satisfy the inverse of the equation for $\epsilon(\eta)$, that is let $\eta(\epsilon)$ satisfy

$$\epsilon = -\frac{\eta}{\log \eta}.$$

Then for $\epsilon < \epsilon(\eta)$, we have

$$L\left(\frac{U}{\epsilon}\right) \in D^{\eta}(U) \subset [\mathrm{BR}]^{\eta(\epsilon)}(U),$$

where $\eta(\epsilon) \to 0$ as $\epsilon \to 0$. □

We now use the logit map to describe solutions of replicator dynamics, called "*continuous exponential weight*" (CEW) by Hofbauer, Sorin, and Viossat [12].

Definition 4.3 (Continuous exponential weight)**.** Given $\mathcal{U}$, define

$$\mathbf{p}(t) = L\left(\int_o^t U(s)\,\mathrm{d}s\right). \tag{4.6}$$

L maps $\mathbb{R}^n$ to Σ_n, so $\mathbf{p}(t) \in \Sigma$. Notice also that $\mathbf{p}(0) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. Later we will

Theorem 4.1. *[12, Proposition 4.1] Continuous exponential weight satisfies the replicator equation.*

Proof. By definition of L, we have

$$\mathbf{p}_i(t) = \frac{\exp\left(\int_0^t U_i(s)\,\mathrm{d}s\right)}{\sum_j \exp\left(\int_0^t U_j(s)\,\mathrm{d}s\right)}.$$

Differentiating $\log(\mathbf{p}_i(t))$,

$$\begin{aligned}
\frac{\dot{\mathbf{p}}_k(t)}{\mathbf{p}_k(t)} &= \frac{d}{dt}\left[\log\left(\exp\left(\int_0^t U_i(s)\,\mathrm{d}s\right)\right) - \log\left(\sum_j \exp\left(\int_0^t U_j(s)\,\mathrm{d}s\right)\right)\right] \\
&= \frac{d}{dt}\int_0^t U_i(s)\,\mathrm{d}s - \frac{d}{dt}\log\left(\sum_j \exp\left(\int_0^t U_j(s)\,\mathrm{d}s\right)\right) \\
&= U_i(t) - \frac{\frac{d}{dt}\sum_j \exp\left(\int_0^t U_j(s)\,\mathrm{d}s\right)}{\sum_\ell \exp \int_0^t U_\ell(s)\,\mathrm{d}s} \\
&= U_i(t) - \sum_j U_j(t)\left(\frac{\exp\left(\int_0^t U_j(s)\,\mathrm{d}s\right)}{\sum_\ell \exp\left(\int_0^t U_\ell(s)\,\mathrm{d}s\right)}\right) \\
&= U_i(t) - \langle \mathbf{p}(t), U(t)\rangle.
\end{aligned}$$

□

Theorem 4.2. *[12, Proposition 4.2] CEW satisfies*

$$\mathbf{p}(t) \in [\mathrm{BR}]^{\delta(t)}(\bar{U}(t)), \tag{4.7}$$

for some $\delta(t) \to 0$ as $t \to \infty$.

Proof. By Lemma 4.1,

$$\begin{aligned}
\mathbf{p}(t) &= L\left(\int_0^t U(s)\,\mathrm{d}s\right) \\
&= L(t\bar{U}(t)) \in [\mathrm{BR}]^{\eta(\frac{1}{t})}(\bar{U}(t)).
\end{aligned}$$

Then we simply set $\epsilon = \frac{1}{t}$ and $\delta(t) = \eta(\frac{1}{t})$. □

4.1 Time averages

Definition 4.4. Define the *time average* of $\mathbf{p}(t)$ by

$$\mathbf{P}(t) = \frac{1}{t}\int_0^t \mathbf{p}(s)\,\mathrm{d}s. \tag{4.8}$$

For $\mathbf{p}(t)$ given by CEW, we have

$$\begin{aligned}
\dot{\mathbf{P}}(t) &= \frac{d}{dt}\left(\frac{1}{t}\int_0^t \mathbf{p}(s)\,\mathrm{d}s\right) \\
&= \frac{1}{t}\mathbf{p}(t) - \frac{1}{t^2}\int_0^t \mathbf{p}(s)\,\mathrm{d}s \\
&= \frac{1}{t}(\mathbf{p}(t) - \mathbf{P}(t)).
\end{aligned}$$

Thus,

$$\dot{\mathbf{P}}(t) \in \frac{1}{t}\left([\mathrm{BR}]^{\delta(t)}(\bar{U}(t)) - \mathbf{P}(t)\right), \tag{4.9}$$

with $\delta(t) \to 0$ as $t \to \infty$. This clearly looks like an approximation to fictitious play. However, under continuous exponential weight as defined previously, $\mathbf{p}(0) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, whereas we would like to discuss solutions with any given initial condition.

Theorem 4.3. *[12, §4.4] The solution of the replicator process (4.2) with initial condition $\mathbf{p}(0) \in \mathrm{int}(\Sigma_n)$ is given by*

$$\mathbf{p}(t) = L\left(U(0) + \int_0^t U(s)\,\mathrm{d}s\right), \tag{4.10}$$

where $U_i(0) = \log(\mathbf{p}_i(0))$. Then the time average process satisfies

$$\dot{\mathbf{P}}(t) \in \frac{1}{t}\left([\mathrm{BR}]^{\alpha(t)}(\bar{U}(t)) - \mathbf{P}(t)\right), \tag{4.11}$$

with $\alpha(t) \to 0$ as $t \to \infty$.

Proof. We may check that (4.10) does indeed satisfy the replicator process (4.2) with the correct initial condition. Then it can be seen that

$$\mathbf{p}(t) = L\left(U(0) + \int_0^t U(s)\,\mathrm{d}s\right) // \quad = L(t\bar{U}(t) + t\cdot\frac{1}{t}U(0)) // \in [\mathrm{BR}]^{\eta(\frac{1}{t})}(\bar{U}(t) + \frac{1}{t}U(0)),$$

where $\eta\left(\frac{1}{t}\right) \to 0$ as $t \to \infty$. This can be rewritten as

$$\dot{\mathbf{P}}(t) \in \frac{1}{t}\left([\mathrm{BR}]^{\alpha(t)}(\bar{U}(t)) - \mathbf{P}(t)\right),$$

with $\alpha(t) \to 0$ as $t \to \infty$ as required. □

This intuitively looks like an approximation to the fictitious play process, and indeed this idea is broadly correct. The full details involve stochastic approximations of differential inclusions (see [3]) and even stating these results becomes rather complicated and as such is beyond the scope of this discussion. In essence the limit set of the time averages of the replicator process is a subset of the limit set of the fictitious play process.

Chapter 5

Conclusion

Replicator dynamics and fictitious play are two interesting examples of evolutionary games. We have seen in Chapter 2 that the behaviour of the replicator system in the long term ties in with the concept of Nash equilibria and that orbits of the replicator system are equivalent to those of the Lotka-Volterra system. In Chapter 3, we have seen that fictitious play is known to converge for zero-sum games and various sizes of non-degenerate quasi-supermodular games (§3.3), but for the Shapley system (§3.4) the behaviour of fictitious play orbits is highly complicated and not yet completely understood. Finally in Chapter 4, using the logit map and continuous exponential weight, we have seen that as proved in Benaïm et al. [3], the time averages of solutions to the replicator system approximate the fictitious play process, with this approximation improving as $t \to \infty$. These are only some examples of dynamics in games. There are many other systems modelling different evolutionary behaviour, such as the more general setting of adaptive dynamics [11, §9]. It is a wide area with many questions for the future.

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