REAL BOUNDS IN COMPLEX DYNAMICS

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ABSTRACT. In these lectures we shall review some results in complex dynamics in which real bounds play a role. Then we shall give a survey on these real bounds and explain how to apply these.

These notes are inspired by the monograph 'One-Dimensional Dynamics' [MS3]. This monograph was written jointly with Welington de Melo. In particular the last chapter is relevant here as it deals with one of the first instances were real and complex tools in one-dimensional dynamics come together: Sullivan's renormalization result. Another source of inspiration for this paper are the papers [Su3], [ML], [Ly2], [Sw2], [SN] and [LS] were some well-known problems in complex dynamics concerning maps of the form $z \mapsto z^\ell + c_1$ with $c_1 \in \mathbb{R}$ were solved using real bounds (or real problems solved with complex methods).

These notes concentrate on maps f of the form $z \mapsto z^\ell + c_1$ where ℓ is even and c_1 is real (often $\ell = 2$). These maps f can be considered both acting on the real line as on the complex plane. The point $c = 0$ is called the critical point (we use this convention here, as it is common in real dynamics to define c to be the critical point). Hence c_1 is the critical value of f.

For convenience, we denote the length of an interval T by $|T|$. Moreover, we define $\omega(x)$ to be the set of accumulation points of the sequence $f^n(x)$, $n = 0, 1, 2, \ldots$.

1. Results on complex iterations of real maps

In this section I want to describe some results in complex dynamics in which real bounds play an important role and also results on the dynamics of interval maps in which complex methods are crucial.

1.1 Renormalization theory

The first result in interval dynamics in which complex tools play a crucial role is the renormalization result of Sullivan. We say that a unimodal interval map f is *renormalizable of period* $n > 0$ if there exists an interval I_0 containing the critical point c such that f^n maps I_0 as a unimodal map into itself. If there exists a nested sequence of intervals I_n and an increasing sequence of numbers $q(n)$ such

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that $f^{q(n)}$ maps I_n into itself then f is called *infinitely renormalizable*. In this case $q(n+1)/q(n)$ is an integer $a(n) \geq 2$.

We should emphasize that if f is of the form $z \mapsto z^\ell + c_1$ (with ℓ even and c_1 real) then this is equivalent to f being complex renormalizable of period n. In other words, for a real map f is renormalizable as above, one can prove that there exist discs $D' \supset D \ni c$ such that f^n is a $2-1$ covering map from D to D', such that $f^{in}(c) \in D$ for each $i \geq 0$ and such that $D' \setminus D$ is an annulus. In fact, $D \cap \mathbb{R}$ is an interval which contains a neighbourhood of the interval I_0 from above.

If f is infinitely often renormalizable and the sequence $a(0), a(1), \ldots$ is bounded from above by a then f is said to be of *bounded type* $\leq a$. Sullivan [Su3] proved that the geometry of the Cantor set $\omega(c)$ for such maps is rigid:

Theorem. *[Sullivan: Renormalization Result]*

Let $f, \tilde{f}$ be two C^2 unimodal maps both with a critical point of quadratic type. Assume that f and $\tilde{f}$ are conjugate and infinitely renormalizable of bounded type. Then there exists a $C^{1+\epsilon}$ homeomorphism h such that $h(\omega_f(c)) = \omega_{\tilde{f}}(\tilde{c})$, such that h conjugates $f|\omega_f(c)$ to $\tilde{f}|\omega_{\tilde{f}}(\tilde{c})$ and so that $h(c) = \tilde{c}$.

Recently it was proved by de la Llave [Lla] that in general this homeomorphism is not C^3 if $a(n) = 2$ for all $n \geq 0$ (the period doubling case studied by Feigenbaum, Coullet and Tresser).

One of the main steps in the proof of the previous theorem is the following results

Theorem. *[Sullivan's Complex Bounds]*

For each $a \in \mathbb{N}$ there exists $m(a) > 0$ with the following properties. Let $f(z) = z^2 + c_1$ be a real infinitely renormalizable map of bounded type by a. Then for each n there exists a polynomial-like extension F_n of $f^{q(n)}$. More precisely, there exist discs $D \subset D'$ such that $D' \setminus D$ has modulus $> m(a)$ and such that F maps D as a $2-1$ branched cover map onto D' with $F^i(c) \in D$ for each $i \geq 0$.

In this paper we discuss the real bounds in the proof of Sullivan's renormalization result. Using these real bounds and the complex bounds from above, Sullivan was able to develop Teichmüller theory for these maps, see Sullivan [Su2] and also Chapter VI of [MS].

1.2 Absence of Wild Attractors in the Quadratic Case

The second result on interval maps in which complex tools play a role is Lyubich's [Ly2] result about wild attractors.

Theorem. *[Lyubich: Absence of Wild Attractors]*

Let $f(z) = z^2 + c_1$ be a real map. If f is not renormalizable then there exists a set of full measure D in the interval so that $\omega(x)$ is equal to a union of intervals for each $x \in D$.

The proof of this result is based on an estimate on the relative scales of certain intervals and using certain complex methods. We shall come back to this in the final section of these notes. The first result in the direction of this last result was for the so-called Fibonacci map, see [LM]. In this Fibonacci case, Keller and Nowicki [KN] found another purely real proof of this fact. The Fibonacci map was introduced by Branner and Hubbard [BH] in the complex case, and independently by Hofbauer and Keller [HK] in the real case.

1.3 The rigidity result of Świątek

A third result in which complex results play an important role is the rigidity result of Świątek [Sw2]:

Theorem. *[Świątek: Rigidity Result]*

Let $f(z) = z^2 + c_1$ and $\tilde{f}(z) = z^2 + \tilde{c}_1$ be two conjugate real quadratic maps without periodic attractors. Then they are equal.

In fact, Świątek proves that the conjugacy between these maps has a quasiconformal extension. Using the Measurable Riemann Mapping Theorem it then can be shown using a general argument due to Sullivan that the conjugacy must then be conformal. Since the conjugacy maps has the property that $h(0) = 0$, $h(\infty) = \infty$ it follows easily that $h = id$.

In fact, the result on puzzle by Yoccoz [Yo2], stated below, implies this result of Świątek in the case that f and $\tilde{f}$ are not renormalizable. Moreover, Sullivan's renormalization result implies this result in the case that f and $\tilde{f}$ are infinitely renormalizable of bounded type. The main contribution of Świątek is to show that when two real quadratic, infinitely renormalizable maps of unbounded type, are conjugate, then this conjugacy extends to a quasiconformal homeomorphism on the Riemann sphere. To prove this, Świątek proves that $|I_n|/|I_{n+1}|$ gets large when $a(n)$ is large (using the notation from above). This implies that $f^{q(n)}|I_n$ tends – up to scaling – to a quadratic map. Lyubich, using different methods proved similar estimates in his proof of the absence of wild attractors in the quadratic case.

1.4 Local connectivity of the Julia set

Let us now consider the question of local connectivity. Using his puzzle's and a tableau argument which originated in the work of Branner and Hubbard, and as part of his work on the local connectivity of the Mandelbrot set, Yoccoz [Yo2] proved

Theorem. *[Yoccoz, Puzzle Theorem]*

Each non-renormalizable real quadratic map $f(z) = z^2 + c_1$ has a Julia set which is locally connected (when the critical point does not escape).

Genadi Levin and the author, see [LS], recently showed that this result can be extended to the infinitely renormalizable case provided the map is real:

Theorem. *[Levin and van Strien: Local Connectivity]*

Each real quadratic map has a Julia set which is locally connected or – if the critical point escapes to infinity – totally disconnected.

In fact, we show that all real infinitely renormalizable maps of the form $z \mapsto z^\ell + c_1$ (with ℓ even) have a locally connected Julia set. The methods we use for this result are in some sense quite different from those of Yoccoz. We shall discuss the real bounds used for this theorem in this paper. We should note that, from our proof, it follows (see [LS]) that one can get an improved version of the previously mentioned complex bounds of Sullivan:

Theorem. *[Levin and van Strien: Sullivan's Complex Bounds Generalized]*

For each even $\ell \in \mathbb{Z}$ there exists $m > 0$ such that each real infinitely renormalizable map $f(z) = z^\ell + c_1$ of periods $q(n)$ has the following properties. There exists a sequence of polynomial-like extension F_n of $f^{q(n)}$. More precisely, there exist discs $D_n \subset D'_n$ such that $D'_n \setminus D_n$ has modulus $> m$ and such F_n maps D_n as a branched cover map onto D'_n.

Note that the bound for the modulus of the annulus is independent of the combinatorial type of the map.

1.5 The Lebesgue measure of the Julia set

Extending the arguments of Yoccoz, and relating the modulus of a multiple connected annulus with a statement about areas, Lyubich [Ly1] and Shishikura [Shi] proved independently

Theorem. *[Lyubich and Shishikura: Zero Lebesgue Measure]*

Each non-renormalizable real quadratic map $f(z) = z^2 + c_1$ has a Julia set which has Lebesgue measure zero.

When f has a periodic attractor or when c escapes to infinity, then this result was known before. The situation for quadratic maps is quite different from that of maps of the form $z \mapsto z^\ell + c_1$ when ℓ is large:

Theorem. *[Van Strien and Nowicki: Positive Lebesgue Measure]*

There exists ℓ_0 such that for each even $\ell \geq \ell_0$ there exists c_1 such that the Julia set of the real polynomial $z \mapsto z^\ell + c_1$ has a Julia set of positive Lebesgue measure.

We should note that the Julia set of each polynomial has no interior. (We should also note that it is not known whether there exists quadratic maps with a Julia set of positive Lebesgue measure. The above theorem uses in an essential way, that the degree of the polynomial is large.)

1.6 Existence of Wild Attractors

The previous result about a Julia set with positive Lebesgue measure uses estimates and ideas concerning so-called Fibonacci maps. In the real case, one has the following result on such interval maps:

Theorem. *[Bruin, Keller, Nowicki and van Strien: Existence of Wild Attractors]*

There exists ℓ_0 such that for each even $\ell \geq \ell_0$ there exists a real parameter c_1 such that $f(z) = z^\ell + c_1$ is non-renormalizable and has a wild attractor:

1. *there exists a set of positive Lebesgue measure D such that for $x \in D$ one has $x \in \omega(c)$ where $\omega(c)$ is a Cantor set;*
2. *the set of points x such that $x \in \omega(c)$ is of first category (for a set of points x of second category one has that $\omega(x)$ is equal to an interval).*

2. Real Bounds

In complex dynamics, one often obtains bounds using the Koebe Lemma. Another ingredient is the use of the modulus of an annulus. Furthermore, one often uses the Poincaré metric and the Schwarz Lemma. We shall describe some analogous tools which exist in real dynamics.

2.1 The Poincaré metric and cross-ratios

One of the main tools in complex dynamics is the use of the Poincaré metric. In the case of the unit disc $\mathbb{D}$ in $\mathbb{C}$ this metric is given (in integrated form) by the following expression. If $z, w \in \mathbb{D}$ then

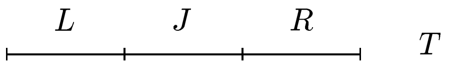
$$d(z,w) = \log \frac{|w - z_\infty|}{|z - z_\infty|} \frac{|w_\infty - z|}{|w_\infty - w|}$$

where z_∞ and w_∞ are the intersections of the unit circle with the circle (or line) that contains z, w and which is orthogonal to the unit circle. Here z_∞ is the point of intersection such that the arc on this circle (or line) in $\mathbb{D}$ bounded by z and z_∞ does not contain w. If $S \subset \bar{\mathbb{C}}$ is a set such that $\bar{\mathbb{C}} \setminus S$ contains at least three points then the universal covering of S is holomorphically equivalent to $\mathbb{D}$ and so S carries a hyperbolic metric by lifting the Poincaré metric from above from $\mathbb{D}$ to S.

The points z_∞, z, w and w_∞.

In this subsection we shall introduce the analogous notion in real dynamics. As we shall explain in some cases one can improved estimates which there might be no complex analogue.

If a map $f\colon \mathbb{R} \to \mathbb{R}$ has critical points we cannot hope to get a bound for its non-linearity: the ratio of the length of a pair of adjacent intervals can be distorted by an unbounded amount. So, instead of considering three consecutive points, we consider four points and we measure their position by their cross-ratio. Then we determine how a map distorts the cross-ratio of such a configuration of four points. Cross-ratios are connected with hyperbolic geometry. The hyperbolic distance between two points in the hyperbolic space is given by the logarithm of the cross-ratio of four points in the following sense:



Definition. Let $J \subset T$ be open and bounded intervals in $\mathbb{R}$ such that $T \setminus J$ consists of intervals L and R. Define the *cross-ratio* of these intervals as

$$D(T,J) = \frac{|J||T|}{|L||R|},$$

(where $|I|$ denotes the length of the interval I). This cross-ratio is related to the hyperbolic metric. Indeed, let T be an open and bounded interval on $\mathbb{R}$. For $x, y \in T$ let

$$\rho_T(x,y) = \log \frac{|L \cup J||J \cup R|}{|L||R|} = \log(1 + D(T,J))$$

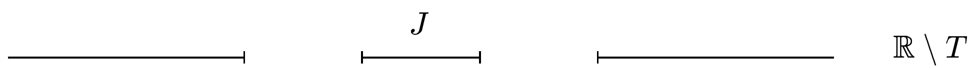
where J is the interval bounded by the points x, y.

If we define $\mathbb{D}$ to be the disc in the complex plane $\mathbb{C}$ which is symmetric w.r.t. the real line and intersects $\mathbb{R}$ exactly in T then it follows from the above expressions that $\rho_T(x,y)$ is precisely the hyperbolic (Poincaré) distance $d(x,y)$ between $x, y \in \mathbb{D}$.

Of course, the distance $\rho_T(x,y)$ can also be expressed in terms of the distance of x and y in terms of the Poincaré metric on the slit region $\mathbb{C}_T = (\mathbb{C} \setminus \mathbb{R}) \cup T$. Moreover, the cross-ratio $D(T,J)$ (and so $d(x,y)$) also completely determines the modulus of the (degenerate) annulus

$$(\mathbb{C} \setminus \mathbb{R}) \cup (T \setminus J).$$

(If $D(T,J)$ is large then the modulus of the annulus $(\mathbb{C} \setminus \mathbb{R}) \cup (T \setminus J)$ is small.) We shall not give these formulae here, but refer to [Ah1] and [Ah2].



The degenerate annulus $(\mathbb{C} \setminus \mathbb{R}) \cup (T \setminus J)$.

Cross-ratios were introduced by de Melo and myself [MS1], [MS2] to prove an analogue of the Denjoy result on circle diffeomorphism in the unimodal interval case. A version of these cross-ratios had already been used by Yoccoz [Yo1] for circle homeomorphisms. Slightly later, but independently, Świątek also introduced cross-ratios for his study of circle maps with flat spots or with critical points [Sw1]. For a more thorough discussion see [MS3].

2.2 The analogue of the Schwarz Lemma

In the complex case, a conformal map from a hyperbolic space S into itself contracts the Poincaré metric. This follows from the Schwarz Lemma: if $f\colon S \to S'$ is a conformal map between two hyperbolic surfaces then f either strictly contracts the Poincaré metric or is a covering map.

In the real case, one has something similar provided one works with maps which expand cross-ratios or (equivalently) satisfy the so-called negative Schwarzian condition:

Definition. If $g: T \to \mathbb{R}$ is continuous and monotone and $J \subset T$ as above then we define the cross-ratio distortion of g as

$$B(g, T, J) = \frac{D(g(T), g(J))}{D(T, J)}$$

(if J has a common boundary point with T then we take $\limsup B(g, T, J_n)$ where J_n is a sequence of intervals with $J_n \uparrow J$).

If $f: I \to I'$ and $g: I' \to I''$ are continuous and monotone then

$$B(g \circ f, T, J) = B(g, f(T), f(J)) \cdot B(f, T, J)$$

when $J \subset T \subset I$. So the space of maps which expands cross-ratios is closed under composition. The inverse of such a map contracts cross-ratios and therefore the Poincaré metric. In this sense one can view such maps as the real analogue of conformal maps in the complex case.

Often, rather than checking that a map expands cross-ratios one checks whether its Schwarzian derivative is negative. Here the Schwarzian derivative of a C^3 map $g: T \subset \mathbb{R} \to \mathbb{R}$ is defined to be

$$Sg = \frac{D^3 g}{Dg} - \frac{3}{2}\left(\frac{D^2 g}{Dg}\right)^2.$$

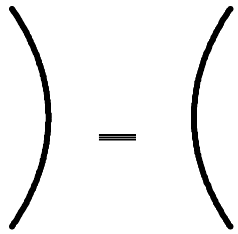
It is easy to see that if $Sg < 0$ then g expands cross-ratios and the reverse implication also essentially holds, see Section IV.1 in [MS3]. Again the negative Schwarzian condition is closed under composition: $S(g \circ f) = (Sg \circ f) \cdot (Df)^2 + Sf$. The boundary of the space of maps with negative Schwarzian (those with zero Schwarzian) is the group $\mathcal{M}$ of Möbius transformations of the real line, namely, $\phi \in \mathcal{M}$ if $\phi(x) = \frac{ax+b}{cx+d}$ where a, b, c, d are real numbers with $ad - bc \neq 0$. These maps preserve cross-ratios.

Let us connect the real to the complex situation more tightly. Firstly, it is easy prove that if F is a real polynomial such that all zero's of F' (i.e., all critical points) are on the real line then $SF(x) < 0$ for $x \in \mathbb{R}$, see Exercise 1.7 on page 285 of [MS3]. So such a map expands cross-ratios of intervals on which F is a diffeomorphism. The inverse of F contracts cross-ratios on such intervals. An alternative way of seeing this is through the Schwarz Lemma: let $\mathbb{C}_T = \mathbb{C} \setminus (\mathbb{R} \setminus T)$ and define $D_*(T)$ to be the disc which is symmetric w.r.t. to real line and intersects the real line in T. This set is equal to the set of points whose distance (w.r.t. the hyperbolic metric on $\mathbb{C}_T$) is at most $\log \tan(\pi/2 - \pi/8)$, see page 486 of [MS3]. If F is a real polynomial whose critical points are on the real line and which is a diffeomorphism from T to T' then, by this interpretation of sets of the form $D_*(W)$ and by the Lemma of Schwarz, the appropriate branch of F^{-1} is a conformal map from $D_*(T')$ into $D_*(T)$. Hence $F^{-1}: D_*(T') \to D_*(T)$ is a contraction (in terms of the Poincaré metrics on these discs). Hence, $F^{-1}: T' \to T$ contracts cross-ratios.

In other words, for a real polynomial F whose critical points are all real, estimates on the cross-ratio distortion $B(F,T,J)$ are simply estimates on the distortion of the Poincaré distance of these points. A more geometric way of expressing this, is by considering the conformal map

$$F^{-1}: (\mathbb{C} \setminus \mathbb{R}) \cup (T' \setminus J') \to (\mathbb{C} \setminus \mathbb{R}) \cup (T \setminus J).$$

The image of the domain is in general an annulus which is a proper subset of $\mathbb{C}\setminus\mathbb{R})\cup(T\setminus J)$. If $B(F,T,J) > 1+\epsilon$ then the modulus of the annulus $(\mathbb{C}\setminus\mathbb{R})\cup(T\setminus J)$ is at least $(1+\epsilon)$ times the modulus of $(\mathbb{C} \setminus \mathbb{R}) \cup (T' \setminus J')$.



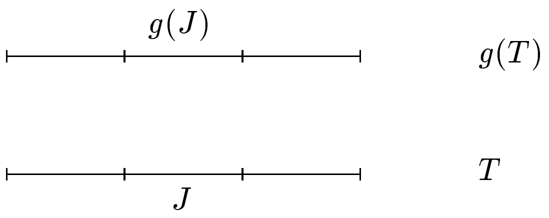
The image of the annulus $(\mathbb{C} \setminus \mathbb{R}) \cup (T \setminus J)$ *(schematically)*

Frequently one only considers mappings which expand cross-ratios. However, certainly not all polynomial maps $F: \mathbb{R} \to \mathbb{R}$ have this property. Even so, one can often show that high iterates of F restricted to certain small intervals do not contract cross-ratios too much (this is based on disjointness arguments (similar to those used in the proof of Denjoy's theorem) or using 'telescope' arguments (see for example Strien (1987), (1990)). Here we shall confine ourselves with the case that a map expands cross-ratios.

2.3 The Koebe Principle

Now we shall discuss one of the most powerful tools in one-dimensional dynamics: the Koebe Principle. For maps with negative Schwarzian derivative one version of this principle was already used and proved in Van Strien (1981) and later rediscovered by Johnson and Guckenheimer, see Guckenheimer (1987). It was extended in Van Strien (1987), (1990) for maps which do not satisfy the negative Schwarzian derivative condition. The simplest form of this principle states that a negative Schwarzian map has bounded non-linearity 'away from its critical values'. If a map satisfies the negative Schwarzian derivative condition then also iterates satisfy this condition. So in that case we can apply the next result to each iterate.

Let $U \subset V$ be two intervals. We say that V *is an ϵ-scaled* neighbourhood of U if both components of $V \setminus U$ have length $\epsilon \cdot |U|$.



Theorem 2.3 ("Koebe Principle"). *Let $J \subset T$ be intervals and $g\colon T \to g(T)$ a diffeomorphism with negative Schwarzian derivative. If $g(T)$ contains a τ-scaled neighbourhood of $g(J)$, then*

$$\frac{1}{K(\tau)} \leq \frac{Dg(x)}{Dg(y)} \leq K(\tau), \quad \forall x, y \in J,$$

where $K(\tau) = \dfrac{(1+\tau)^2}{\tau^2}$.

Proof. See [MS]. $\square$

Of course, this is the precise analogue of the corresponding result for univalent maps.

In fact, in the interval one is often interested in creating space. In that case, instead of the Koebe Principle one often can pullback space using the estimate

$$|L|/|J| \geq C^{-1}(T, J).$$

Indeed, if g is a map which is monotone on T and satisfies $Sg < 0$ then

$$|L|/|J| \geq C^{-1}(T, J) \geq C^{-1}(gT, gJ).$$

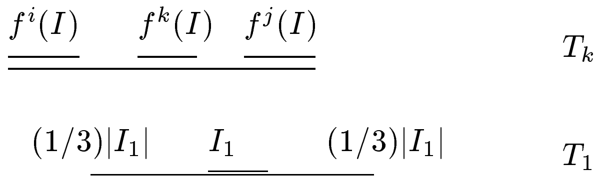
So if $g(T)$ contains a 1-scaled neighbourhood of $g(J)$ then $C^{-1}(gT, gJ) \geq 1/3$ and so $|L| \geq (1/3)|T|$.

3. Application of Real Bounds in Complex Dynamics

Several bounds on the complex dynamics of real mappings depend on the fact that the real line has an order structure. We shall only give two instances of ideas which are based on this. The first one exploits the obvious fact, that among a finite number of disjoint intervals one is smaller than its neighbours. The second circle of ideas use that in real dynamics one often is able to say that certain dynamically defined points are between certain others; from this one can sometimes get inductive estimates.

3.1 The smallest interval trick

One of the most fruitful techniques to get real bounds is the smallest interval trick. The purpose of this trick is to get space around an interval. Abstractly the idea of this is the following: let $I, \ldots, f^n(I)$ be disjoint intervals. Usually one considers the situation that I contains the critical point. Then for each $2 < i \le n$ the interval $f^i(I)$ is between $f(I)$ and $f^2(I)$. Now one of these intervals, say $f^k(I)$ must be the smallest. Let us assume $2 < k \le n$. With some luck there exists an interval $T_1 \supset f(I)$ such that $f^{k-1}|T$ is monotone and so that $T_k = f^{k-1}(T)$ contains on either side of $f^k(I)$ an interval from the collection of $I, \ldots, f^n(I)$. Hence $T_k = f^{k-1}(T)$ is a 1-scaled neighbourhood of $f^k(I)$. Because of the last inequalities in the previous section it follows that T_1 contains a 1/3-scaled neighbourhood of $f(I)$.



The smallest interval trick.

Application 1. One of the places where the smallest interval trick plays a role is in the proof of Sullivan's renormalization result. More precisely, assume that f is an infinitely renormalizable map of bounded type $a(n) \le a$. Then $\omega(c)$ is a Cantor set which can be written in the form $\cap_{n\ge 0} F_n$ where

$$F_n = \triangle_n \cup \triangle_n^1 \cup \cdots \cup \triangle_n^{q(n)-1},$$

$\triangle_n^j = f^j(\triangle_n)$ and $\triangle_n^0$ is an interval containing the critical point of period $q(n)$.

F_0 has 2 and F_1 has $3 \cdot 2$ components.

Hence each $\triangle_n^j$ is a connected component of F_n and it is called an *interval of generation* n. Each connected component of $F_{n-1} \setminus F_n$ is called a *gap of generation* n. An interval or a gap of generation n is called an element of generation n. Using the smallest interval trick, Sullivan proves that provided f is C^2 (or something even slightly weaker) $(F_n)_{n\ge 0}$ has bounded geometry: there exist $0 < \lambda < \mu < 1$ such that if I is an interval of generation n and $J \subset I$ is either an interval of generation $n+1$ or a gap of generation $n+1$ then

$$0 < \lambda < \frac{|J|}{|I|} < \mu < 1.$$

Now let f and $\tilde{f}$ be conjugate maps as above. This implies that both F_n and $\tilde{F}_n$ have bounded geometry and consist of the same number of components. Using this it is not hard to show that there exists a quasiconformal homeomorphism h which preserves the real line and sends F_n to $\tilde{F}_n$ for each n. Thus we get a quasiconformal homeomorphism of the Riemann sphere which is a conjugacy between $f\colon \omega_f(c) \to \omega_f(c)$ and $\tilde{f}\colon \omega_{\tilde{f}}(\tilde{c}) \to \omega_{\tilde{f}}(\tilde{c})$. This is the starting point for the entire complex machinery that Sullivan uses to prove his renormalization result.

Example 2. Another application of this trick is a result due to M. Herman (based on work of Świątek [Sw1]) which states that a smooth circle homeomorphism with a unique critical point is quasi-symmetrically conjugate to a rotation provided its rotation number is of bounded type. (Its continued fraction expansion is bounded.) We shall not elaborate on this example, but just mention that it was used by C. Peterson [Pe] in his proof that a quadratic map with a neutral fixed point with eigenvalue $e^{2\pi t}$ has a locally connected Julia provided t is constant type.

Example 3. A third example of an application, is the proof in [LS] of the local connectivity of the Julia set of a map f of the form $z \mapsto z^2 + c_1$ with c_1 real. Firstly, in that paper the $1/3$ estimate from above had to be improved somewhat. This was done by a careful analysis of the dynamics which shows that certain dynamical interval are contained in others. Let us not dwell on the details of these estimates, but explain how these estimates are used to prove local connectivity. To do this, it enough to prove that there exist a sequence of discs $D_n \subset D'_n$ containing c with diameter tending to zero with the following additional property. Let n be minimal so that $f^n(c) \in D'$. Then $D' \subset f^n(D)$ (for a more precise statement see [LS]). Let us explain the ingredients which are necessary to prove the last inclusion.

Consider a real interval T, take $\alpha \in (0, \pi)$ and define $D(T;\alpha)$ to be the union of two half-discs which are symmetric w.r.t. the real axis, intersect the real line exactly in T and which have an external angle α with the real line. The reason these sets play an important role, can be explained as follows. Let $\mathbb{C}_T = \mathbb{C} \setminus (\mathbb{R} \setminus T)$. The set $\mathbb{C}_T$ with two infinite slits, carries a Poincaré metric, and with respect to this metric the set $D(T;\alpha)$ consists of all points whose distance to T is at most equal to some constant $k(\alpha)$. From this interpretation and the Schwarz contraction principle, it follows that if $\phi\colon \mathbb{C}_T \to \mathbb{C}'_T$ is a univalent conformal mapping sending T diffeomorphically to T', then

$$\phi(D(T;\alpha)) \subset D(T';\alpha).$$

The set $D(T;\alpha)$.

In other words,

Lemma. *Let $P: \mathbb{C} \to \mathbb{C}$ be a real polynomial whose critical points are on the real line and which maps T' diffeomorphically onto T, then there exists a set $D \subset D(T';\alpha)$ with $D \cap \mathbb{R} = T'$ which is mapped diffeomorphically onto $D(T;\alpha)$ by P.*

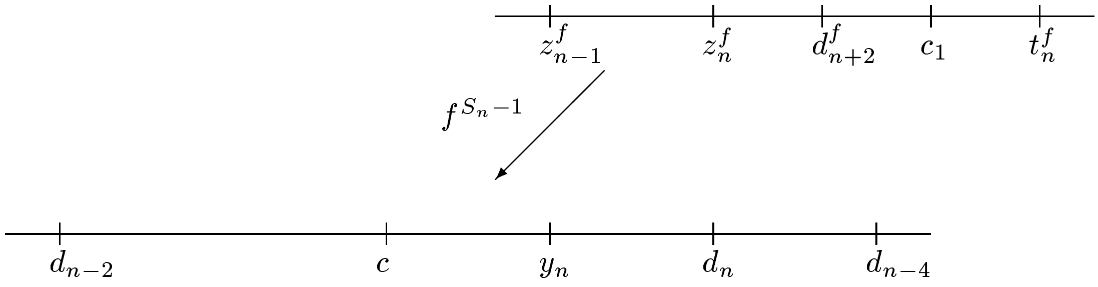
In particular, take D a set of the form $D(T;\alpha)$ where T is a real interval neighbourhood of the critical point c and let n be minimal so that $f^{n-1}(c_1) \in D$. Take $T_1 \ni c_1$ so that f^{n-1} maps T_1 diffeomorphically onto T (since n is minimal such T_1 exists). From the lemma we have that f^{n-1} maps some set which is contained in $D(T_1;\alpha)$ diffeomorphically onto $D(T;\alpha)$. So the problem remains to prove that f^{-1} maps the set $D(T_1;\alpha)$ (which contains the critical value of f) into the set $D(T;\alpha)$. But this is essentially a question of real estimates. For example, assume that one of the boundary points of T is a periodic point and that T is of the form $T = [-q, q]$ with, say, $q > 0$. Now T_1 is of the form $[c_1 - d, c_1 + q^\ell]$ with $d, q > 0$. Then $f^{-1}(D(T_1;\pi/2))$ is contained in $D(T;pi/2)$ precisely if $d < q^\ell$. (In the figure below the set $D(T;\pi/2)$ is drawn as a dotted circle and $f^{-1}(D(T_1;\pi/2))$ is drawn for $d > q^\ell$ and for $d < q^\ell$.)

The preimage of $D(T_1;\pi/2)$ by f depending on d.

3.2 The inductive trick

Next we want to describe another method proving space which requires some explicit knowledge about the ordering of some iterates of the critical point. This precise way in which this method is applied depends highly on the situation. Let us illustrate in a few examples.

Example 4. In this example we shall prove some a priori bounds for space using induction and the relative position of points. We illustrate this in the case of a Fibonacci map. We shall present this argument abstractly (and refer to for example [ML], [KN], [BKNS] and [SN] for the definition of a Fibonacci map and the proofs concerning the position of these points). Let us simply say that S_n is the sequence of Fibonacci numbers: $S_0 = 1$, $S_1 = 2$ and $S_{n+1} = S_n + S_{n-1}$. Moreover, define $d_n = f^{S_n}(c)$ and $d_n^f = f(d_n)$. The points are ordered as in the figure below.



The aim is to show that some restriction of the diffeomorphism

$$f^{S_n-1}: [z_{n-1}^f, t_n^f] \to [d_{n-2}, d_{n-4}] \ni c$$

have bounded non-linearity. For example, one would like to estimate the non-linearity of the restriction of this map to the interval $[z_n^f, c_1]$ (where z_n^f is by definition a point so that $f^{S_n-1}(z_n^f) = c$). To this, one would like to have Koebe space. In other words, one would like a lower bound for $|d_{n-2} - d_n|/|d_n - c|$. In the next lemma instead

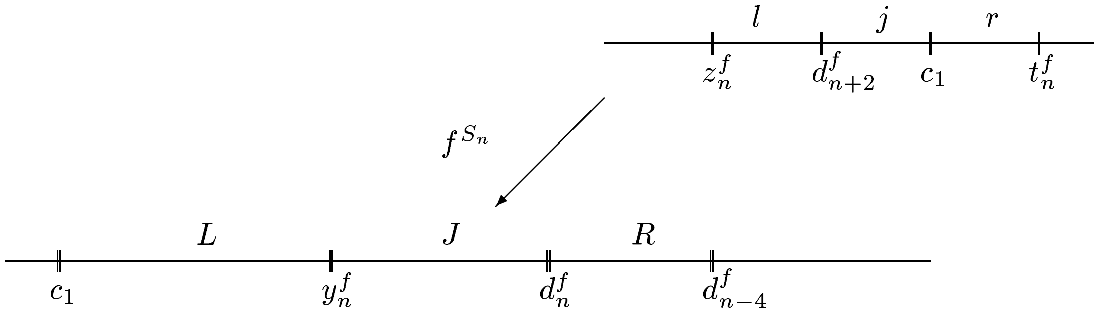
$$\frac{|d_{n-2}^f - c_1|}{|d_n^f - c_1|}$$

is estimated (which leads immediately to an estimate of the previous ratio).

Proposition. *(See Keller and Nowicki [KN])*

Let $\lambda_n^f = |d_{n-2}^f - c_1|/|d_n^f - c_1|$ *then* $\lambda_n^f > 3.85$.

Proof. Let l, r, t and L, J, T as in the picture.



Then, using the cross-ratio estimate

$$\frac{|d^f_{n-4}-c_1|-|d^f_n-c_1|}{|d^f_{n-4}-c_1|}=\frac{|R|}{|T|}\le\frac{|J|}{|L|}\frac{|l||r|}{|j||t|}\le$$

$$\le\frac{|J|}{|L|}\frac{|l|}{|j|}=\frac{|d^f_n-c_1|-|y^f_n-c_1|}{|y^f_n-c_1|}\frac{||z^f_n-c_1|-|d^f_{n+2}-c_1||}{|d^f_{n+2}-c_1|}.$$

Now we use the relative position: d^f_{n+1} is between c_1 and y^f_n and similarly z^f_n is between d^f_{n+1} and d^f_{n+2}. Hence one can estimate this from above by

$$\le\left(\frac{|d^f_n-c_1|}{|d^f_{n+1}-c_1|}-1\right)\cdot\left(\frac{|d^f_{n+1}-c_1|}{|d^f_{n+2}-c_1|}-1\right).$$

Using $(a-1)(b-1)\le(\sqrt{ab}-1)^2$ we can bound the last term from above by

$$\left(1-\frac{|d^f_n-c_1|}{|d^f_{n-4}-c_1|}\right)\le\left(\sqrt{\frac{|d^f_n-c_1|}{|d^f_{n+2}-c_1|}}-1\right)^2.$$

Hence

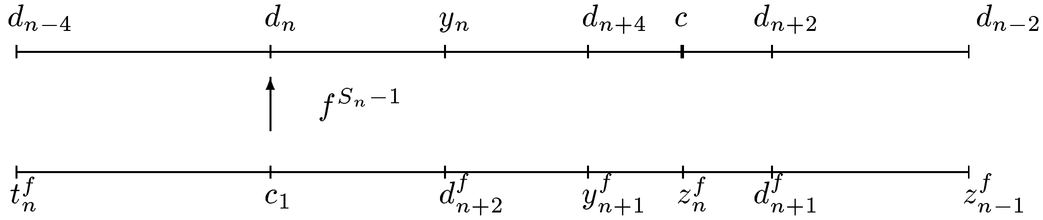
$$1-(\lambda^f_n\lambda^f_{n-2})^{-1}\le(\sqrt{\lambda^f_{n+2}}-1)^2$$

which yields the analogous inequality for $\lambda^f_\infty=\liminf\lambda^f_n$, and thus $\lambda^f_\infty>3.85$. $\square$

We shall use the a priori estimates in the next example.

Example 5. In this example we shall show that certain scaling laws in the Fibonacci map in the quadratic case ($\ell=2$) and in the case that $\ell>2$ are fundamentally different. Since these scaling laws are so fundamental in many results mentioned in the first section, we shall give an idea how to obtain them. (For example, they play a crucial role in the results of Lyubich on the non-existence of wild attractor in the quadratic case, in Świątek's rigidity result, and in the results on heavy Julia sets [SN] and its real analogue [BKNS].)

Let us take the Fibonacci map and mark some dynamical points and their images under the map f^{S_n-1}. For a proof of the fact that these points are ordered in this way, see for example [LM], [KN], [BKNS] or [SN]. Here $d_n=f^{S_n}(c)$ and the other points are defined by the picture. (In fact, one also has $|d_i-c|<|d_j-c|$ when $i>j$).



Points and their images under f^{S_n-1}.

By the result in the previous example,

$$|d_{n-4}-d_n|/|d_n-c|, |d_{n-2}-d_{n+2}|/|d_{n+2}-c| \geq (3.85)^{2/\ell}-1$$

for large n. Note that for large ℓ this is of the order $\geq C/\ell$ so little Koebe space is available for large ℓ. In fact, it is proved in [SN] that asymptotically for ℓ large this bound is really sharp. In the next proposition and its corollary, we shall indicate why the asymptotic behaviour of $|d_n-c|/|d_{n-2}-c|$ is different for $\ell=2$ and for $\ell>2$.

The estimates we give below (and a version of the a priori bounds given in the previous example) were found by Tomasz Nowicki and myself in the spring of 1991 and is used in [KN]. (Better estimates are given in [BKNS] but they are harder to explain.) A similar estimate was used by Tangerman and Veerman in a preprint of 1990, in the context of circle diffeomorphisms. These latter estimates inspired Lyubich and Milnor to get similar estimates for the Fibonacci map. In Lyubich's paper [Ly5] and also Świątek [Sw2] similar estimates also play an important role.

Proposition. *Define*

$$D_n := |Df^{S_n}(f(c))| \text{ and } \mathbf{d}_n = |d_n - c|.$$

Then

$$\frac{D_{n+1}}{[D_n \cdot D_{n-1}]^{1/\ell}} \approx \ell^{2(1-1/\ell)}$$

and

$$D_{n+1}/(\frac{\mathbf{d}_{n-1}}{\mathbf{d}_{n+1}}) \approx \ell^2.$$

Here we use the symbol $\approx$ in the following way. The above expressions on the left hand side of $\approx$ are bounded and bounded away from zero by constants which are uniform in n but may depend on ℓ and, moreover, the lower and upperbounds are close to the right hand side of these expressions if $\mathbf{d}_{n-2}/\mathbf{d}_n$ and $\mathbf{d}_{n+1}/\mathbf{d}_{n-1}$ are sufficiently large.

Write

$$\lambda_n = \frac{\mathbf{d}_{n-2}}{\mathbf{d}_n}.$$

Then the second expression in the proposition becomes

$$D_{n+1}/\lambda_{n+1} \approx \ell^2.$$

Substituting this in the first expression in the proposition,

$$\frac{D_{n+1}}{[D_n \cdot D_{n-1}]^{1/\ell}} \approx \ell^{2(1-1/\ell)},$$

gives

$$\frac{\lambda_{n+1}}{[\lambda_n \cdot \lambda_{n-1}]^{1/\ell}} \approx \ell^{4/\ell},$$

i.e., the left hand side is bounded and bounded away from zero by constants which are uniform in n but may depend on ℓ; these bounds are close to ℓ^2 if $\mathbf{d}_{n-2}/\mathbf{d}_n$ is sufficiently large.

Corollary. *If $\ell = 2$ and λ_n is sufficiently large for some n then for k large*

$$\lambda_k \sim [2^{2/3}]^k.$$

If $\ell > 2$ then λ_k remains bounded.

Proof. Take $\ell = 2$. Then, by the previous observation, there exists for each $\epsilon \in (0,1)$ some λ such that

$$\lambda_{k+1} \geq (2-\epsilon)\sqrt[2]{\cdot \lambda_k \cdot \lambda_{k-1}}$$

provided $\lambda_k \geq \lambda$. Assume that $\lambda_n \geq \lambda^2$. Since $\lambda_{n-1} \geq 1$ then $\lambda_{n+1} \geq \lambda$. Therefore, $\lambda_{n+2} \geq \lambda$. In this we get by induction that $\lambda_k \geq \lambda$ for each $k \geq n$. Hence

$$\lambda_{k+1} \geq (2-\epsilon)\sqrt[2]{\cdot \lambda_k \cdot \lambda_{k-1}}$$

for each $k \geq n$ and therefore one easily sees that $\lambda \to \infty$ and, in particular, that for large k one can take ϵ going to zero. This gives that λ_k grows like $[2^{2/3}]^k$.

When $\ell > 2$ it is easy to see from the recursive inequalities that the numbers λ_k remain bounded as $k \to \infty$. $\square$

In other words, once one is able to show that λ_n is large for some n then the results proved here, give the rates for $\ell = 2$. Of course, one could check this by computer by finding first an estimate for the parameter c_1 for which $z^2 + c_1$ has Fibonacci dynamics and then simply computing $\mathbf{d}_n$ for a few values of n. A more satisfactory point of view was taken in [LM]. There the starting condition was proved in [LM] by complex methods using a very nice trick. They quasisymmetrically conjugate the map is some sense to a polynomial-like with a good starting condition. In [KN] the starting condition was proved by real methods by showing that one has a certain kind of 'cocycle' condition.

To obtain the proof of the proposition, let us first prove the last inequality.

Lemma.

$$\frac{D_{n+1}\mathbf{d}_{n+1}}{\mathbf{d}_{n-1}} \approx \ell^2.$$

Proof. Write D_{n+1} as

$$f^{S_n}(f(c)) \cdot Df^{S_{n-1}}(d_n) = Df^{S_n-1}(f(c))Df(d_n) \cdot Df^{S_{n-1}-1}(d_n^f)Df(d_{n+1}) =$$

$$= \left(\frac{|Df^{S_n-1}(f(c))|}{\frac{\mathbf{d}_{n+2}+\mathbf{d}_n}{\mathbf{d}_{n+1}^f}} \cdot \frac{\mathbf{d}_{n+2}+\mathbf{d}_n}{\mathbf{d}_n} \cdot \frac{\mathbf{d}_{n+1}\cdot Df(d_{n+1})}{\mathbf{d}_{n+1}^f} \cdot \frac{\mathbf{d}_n}{\mathbf{d}_{n+1}} \right) \cdot$$

$$\left(\frac{|Df^{S_{n-1}-1}(f(d_n))|}{\frac{\mathbf{d}_{n+1}+\mathbf{d}_{n-1}}{\mathbf{d}_n^f}} \cdot \frac{\mathbf{d}_{n+1}+\mathbf{d}_{n-1}}{\mathbf{d}_{n-1}} \cdot \frac{\mathbf{d}_n\cdot Df(d_n)}{\mathbf{d}_n^f} \cdot \frac{\mathbf{d}_{n-1}}{\mathbf{d}_n} \right).$$

The third terms in both these factors ( ) are asymptotically equal to ℓ since $Df(x) \approx \ell(x-c)^{\ell-1}$ and since $d_n \to c$ as $n \to \infty$. The second terms in these factors ( ) are between 1 and 2 (and in fact close to one if $\mathbf{d}_{n-2}/\mathbf{d}_n$ and $\mathbf{d}_{n+1}/\mathbf{d}_{n-1}$ are sufficiently large). The first term in these factors can easily be estimated: the average slope of the diffeomorphism $f^{S_n-1}\colon [c_1, d^f_{n+1}] \to [d_n, d_{n+2}]$ is comparable to the derivative of f^{S_n-1} at c_1 because one has Koebe space: the map extends diffeomorphically to one onto $[d_{n-4}, d_{n-2}]$. Here we use the a priori bound from the proposition in the previous example. Note that the amount of Koebe space depends on ℓ. If $\mathbf{d}_{n-4}/\mathbf{d}_n$ and $\mathbf{d}_{n-2}/\mathbf{d}_{n+2}$ are large then the constants in the Koebe Principle are close to one (because one has much space then). The lemma follows. $\square$

Proof of Proposition By the chain rule,

$$D_{n+1} = [D_{n+1}]^{1-1/\ell} \cdot \left[Df^{S_n}(f(c)) \cdot Df^{S_{n-1}}(f(c_{S_n}))\right]^{1/\ell} =$$

$$= [D_{n+1}]^{1-1/\ell} \cdot \left[Df^{S_n}(f(c)) \cdot Df^{S_{n-1}}(f(c)\right]^{1/\ell} \cdot \left[\frac{Df^{S_{n-1}}(f(c_{S_n})}{Df^{S_{n-1}}(f(c))}\right]^{1/\ell}.$$

Hence

$$D_{n+1} = [D_n \cdot D_{n-1}]^{1/\ell} \cdot [D_{n+1}]^{1-1/\ell} \cdot \left[\frac{Df^{S_{n-1}}(f(c_{S_n}))}{Df^{S_{n-1}}(f(c))}\right]^{1/\ell}.$$

Let us rewrite the last factor in the [] parenthesis:

$$\frac{Df^{S_{n-1}}(f(c_{S_n}))}{Df^{S_{n-1}}(f(c))} \frac{Df(d_{n+1})}{\mathbf{d}_{n+1}^{\ell-1}} \cdot \frac{\mathbf{d}_{n-1}^{\ell-1}}{Df(d_{n-1})} \cdot \left(\frac{\mathbf{d}_{n+1}}{\mathbf{d}_{n-1}}\right)^{\ell-1}$$

Again by Koebe space, the first ratio of derivatives is bounded from below and above. The second and third factor tend to one as $n \to \infty$ because then $d_n \to c$ and because $Df(x) = \ell(x-c)^{\ell-1}$. Hence, up to some multiplicative factor which bounded and bounded away from zero uniform in n (but not uniformly in ℓ), we have

$$D_{n+1} = [D_n \cdot D_{n-1}]^{1/\ell} \cdot [D_{n+1}]^{1-1/\ell} \cdot \left[\left(\frac{\mathbf{d}_{n+1}}{\mathbf{d}_{n-1}}\right)^{\ell-1}\right]^{1/\ell}.$$

Using the previous lemma, the proposition follows. $\square$

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