Bounds for maps of an interval with one critical point of inflection type. II.

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In this paper we prove complex bounds, quasisymmetric rigidity, absence of invariant linefields and density of hyperbolicity for real analytic maps with one reflecting critical point which are orientation preserving on each branch. For real bounds, see the first part of this paper, [Le].

1 Introduction

Recently quite a few papers appeared proving complex bounds, local connectivity of Julia sets, absence of invariant linefields and quasisymmetric rigidity for real polynomial maps.

Let us first discuss the unimodal case. In 1990 Martens [Ma] proved that there are real bounds for a certain sequence of first return maps to suitable central intervals. In 1990 Sullivan showed that if the unimodal maps are real analytic, infinitely renormalizable and have bounded combinatorics, then their first return maps extend to the complex plane as polynomial-like maps, see [Su] and also [MS]. In 1994 the present authors proved that all real analytic unimodal maps allow such polynomial-like complex extensions for the 'non-central' first return maps. The proof even gave explicit numerical lower bounds for the moduli of certain annuli, [LS1]. For the quadratic case, different proofs were later provided by [GS1], [LY]. For the smooth case see [Ko]. In 1996 Sands, see [Sa], improved our numerical bounds for maps of the form $g(x^\ell)$ so that the Schwarzian derivative of g is negative. This polynomial-like structure is one of the main ingredients in many recent result on smooth unimodal maps. For example, it is heavily used in the proof that topologically conjugate quadratic maps are in fact quasisymmetrically conjugate, see [GS2], also [Ly] and more recently [Sh]. Note that this is the main step in the recent proofs that hyperbolic maps are dense in the family $z^2 + c$, $c \in \mathbb{R}$. Moreover, complex bounds lead to the proof of the local connectivity or total disconnectedness of Julia sets of some classes of real polynomials [LS1], [LS2]

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and are also needed in results on renormalization and on smoothness of conjugacies, see [Su], [MS], [McM2], and [MP], and essentially simplifies the proof of absence of invariant linefields [McM1], [LS2].

In the case of smooth critical circle homeomorphisms, the analogous statements are proved in [Ya] (complex bounds), [He2], [Yo] (quasisymmetric rigidity), [FM] (renormalization).

In this paper we deal with the next natural class, which contains smooth covering maps of the circle with a unique critical point of inflection type.. This class includes (generalized Arnol'd) maps of the form $f(x) = k \cdot x + a + b\sin(2\pi x) \bmod 1$ where $k \in \mathbb{N}$, $k \geq 2$ and b is chosen so that f has a unique cubic critical point. Other examples are certain real polynomial maps with one non-escaping critical point of inflection type, and also certain Blaschke products, see the examples below. Just as critical circle homeomorphisms are in the boundary of the set of diffeomorphisms of the circle, the maps we consider in this paper are in the boundary of the class of smooth covering maps without critical points. These covering maps without attracting or neutral periodic orbits and without critical points are hyperbolically expanding (by a theorem of Mañé), and are quasisymmetrically conjugate to each other. In this paper we will study the metric theory for maps with points of inflection. It turns out that the methods of proof differ substantially from those used in the critical circle case and also quite a bit from the unimodal case.

It turns out that to prove real and complex bounds for maps with a critical point of inflection type is more involved than for maps with a folding critical point. This is because one no longer has a dynamically relevant symmetry near the critical point. (Note that these covering maps can no longer be renormalizable.) Therefore we obtain polynomial-like extensions for a sequence of first return maps at moments which are no longer combinatorially defined. As in the case of critical circle maps, our maps are in general not quasisymmetrically conjugate to affine maps. In this case we no longer have growth of moduli (as in the quadratic case), and also no bounded geometry of the postcritical set (as in the critical circle case). In the minimal case we use instead that the dynamics does change if we 'move the map up'. In the non-minimal case, a much more general way of proving quasisymmetric rigidity is used. Our method also can be used to prove quasisymmetric rigidity and density of hyperbolicity for real analytic unimodal maps (with a folding critical point which is not necessarily quadratic) in the non-minimal case.

Let us be more formal and introduce the class **AC** which includes analytic covering maps of the circle. Let I be an open interval around the point $c = 0$ and I_i a (finite) collection of disjoint open intervals inside the interval I. **AC** is the class of maps $g \colon \cup I_i \to I$, for which the following conditions hold.

1. For every i, the map $g \colon I_i \to I$ extends to an orientation preserving real analytic homeomorphism from the closure of I_i to the closure of I.

2. For every $i \neq 0$, the map $g \colon \overline{I}_i \to \overline{I}$ is a diffeomorphism, while the map $g \colon \overline{I}_0 \to \overline{I}$ has a unique critical point (i.e., a zero of g') at the point $c = 0 \in I_0$ of odd order $\ell \geq 3$. We call $g : I_0 \to I$ *the central branch*, and I_0 the *central interval* of g.

3. All iterates of the critical point $c = 0$ under g are in $\cup I_i$ (and, hence, well defined).
4. There is an extension of g to a covering map $\bar{g}: S^1 \to S^1$ (where we identify S^1 with $\mathbb{R}/|I|\mathbb{Z}$) which is real analytic on a neighbourhood of $\cup \overline{I}_i$.

We call $g : I_0 \to I$ *the central branch*, and I_0 the *central interval* of g. In fact, in view of Proposition 2.1 below, we only use condition (4) in the proof of Theorem C' and then only in the case $I = (a_-, a_+)$, where $a_\pm$ are the left and right most fixed points of g.

Examples.

1. Consider a polynomial P with real coefficients. Assume that all except one of its critical point escape to infinity and that the non-escaping critical point c is a real point of inflection type. Moreover, assume that the orbit of c only meets components of $\{x \in \mathbb{R};\ P'(x) \neq 0\}$ on which P is orientation preserving. If we restrict P to a suitable interval of the real axis, it will satisfy the conditions 1-4 (see [LS2] for details).

2. Consider the following Blaschke product: $f(z) = \lambda z^d \frac{z - \frac{d+1}{d-1}}{1 - \frac{d+1}{d-1} z}$, where $|\lambda| = 1$, and $d \geq 3$. Then f being restricted to the circle $|z| = 1$ and written in the natural coordinate of the circle satisfies the conditions 1-4 above, with $\ell = 3$ and with $d - 1$ intervals I_i. Note that $\lambda = f(1)$ is the critical value of f on the circle.

In fact, one can easily generalize our results to more general classes of maps. For example, if g is a piecewise analytic map of the form $g: \cup_i I_i \to \cup_k I^k$ where $\cup_i I_i \subset \cup_k I^k$ and I_i (respectively I^k) are pairwise disjoint intervals satisfying the analogous properties 1-4 from the class **AC**, then the branches of first return map to I_0 containing points of $\omega(c)$ form a map from **AC** in the case that $\omega(c)$ is minimal. Otherwise one can again use methods from Sections 10-12 of this paper.

Maps from **AC** have no wandering intervals and only a finite number of attracting and parabolic orbits, see the next section. In this paper we shall mainly deal with complex bounds for such maps. By this we mean that one has geometric estimates for the (quasi) polynomial-like maps or box mappings obtained by taking complex extensions of suitable first return maps to small interval neighbourhoods of the critical point. Note that the domain of such return maps consists of infinitely many components. The interesting case is when c is recurrent (i.e., $c \in \omega(c)$), in which case c is contained in one of the components of the domain. The non-recurrent case is much easier, see Theorem A' in Section 10. As is usual in this problem one has to distinguish the persistently recurrent and the reluctantly recurrent case. Since we are in the real case, this corresponds to the distinction between $\omega(c)$ being minimal or not. (That $\omega(c)$ is minimal means that each orbit in this set is dense. In particular, $\omega(c)$ is a Cantor set in this case.) If $\omega(c)$ is minimal then there are only finitely many domains of the first return map containing points of $\omega(c)$ and in the next theorem we restrict our attention to these. The non-minimal case is discussed below the statement of theorem B.

Theorem A (complex bounds). *A map $g \in$ **AC** such that $\omega(c)$ is minimal has complex bounds: there exists a constant $C > 0$ and sequence of open topological discs V_k around the critical point such that the following properties hold.*

(i) diam$(V_k) \to 0$*, and, moreover,* $V_k \cap \mathbb{R} = U_{n_k}$ *is the range of a map* $g^{S_{n_k+1}}: U_{n_k+1} \to U_{n_k}$ *of the sequence of first return maps, see the next section.*

(ii) the (complex analytic) first return map of g *to* V_k *along the critical orbit is well defined and extends to a quasi-polynomial-like map* $P_k\colon \cup_i V_k^i \to V_k$.

(iii) the modulus of $V_k \setminus V_k^0$ *is bounded from below by* $C > 0$ *for all* $k \geq 0$.

By definition, see [LS1], $P_k\colon \cup_i V_k^i \to V_k$ is called *quasi-polynomial-like* if each V_k^i is contained in V_k (not necessarily compactly), the closures of the real traces of V_k^i, V_k^j are disjoint, if $i \neq j$, and compactly contained in the real trace of V_k, and $P_k\colon V_k^i \to V_k$ is a univalent diffeomorphism for $i \neq 0$ and a branched covering for $i = 0$. In the context of Theorem A (i.e., $\omega(c)$ minimal) we shall have that the union $\cup_i V_i^k$ is over a finite index set and that the forward orbit of c always remains in this union. Note that the domains can intersect each other. In Section 6 we construct a truly (not quasi) *polynomial-like* map ([DH], [LM]) of arbitrarily small diameter for any $g \in \mathbf{AC}$ with minimal $\omega(c)$, by 'intersecting' the quasi-polynomial-like map with an artificially constructed smooth 'polynomial-like' map (in a way which is somewhat similar to what was done in [LS1] for unimodal maps). It allows us to quasiconformally topologically conjugate this polynomial-like map to a polynomial from a family of special polynomials, see Section 7. Since the orbit of the critical point belongs to increasing branches of g, we are able to prove using holomorphic dynamics:

Theorem B (rigidity)

If two maps of the class $\mathbf{AC}$ *with* $\omega(c)$ *minimal have critical points of the same order, are topologically conjugate, and the conjugacy maps the critical point to the critical point of the other map, then the conjugacy is quasisymmetric on* $\omega(c)$.

In particular, the conjugacy is Hölder continuous. In fact, we prove a stronger statement: the two maps are quasiconformally conjugate in complex neighbourhoods of the domains of definition, and the conjugacy is conformal on the Julia sets of the polynomial-like maps mentioned above.

Any map $g \in \mathbf{AC}$ is (semi-)conjugate to a piecewise linear map. In particular, any real analytic n-covering map of a circle without attracting or neutral periodic orbits is topologically conjugate to the map $z \mapsto z^n$ on the unit circle. However, it is definitely not the case that maps from the class $\mathbf{AC}$ are always quasisymmetrically conjugate to piecewise linear maps: it is easy to construct examples of maps in the class $\mathbf{AC}$ with a sequence of longer and longer saddle-cascades. This is similar to the way one finds within families of smooth circle diffeomorphisms, maps with irrational rotation numbers for which the conjugacy with a rotation cannot be absolutely continuous, see [He1] or for example [MS].

If $\omega(c)$ is non-minimal, then similar bounds as in Theorem A hold, see Theorem A' in Section 10. In fact, if c is recurrent and non-minimal then we construct a kind of sequence of 'box mapping' sending all but one domain univalently to a fixed large scale. Note that in this case there are infinitely many domains of the return map containing points of $\omega(c)$, which shall complicate the situation considerably.

If $\omega(c)$ is non-minimal then the analogous statement to Theorem B also holds, see Theorem B' in Section 11. However, in this case we have to assume that $\omega(c)$ contains no parabolic periodic points. Note that this is always the case if $\omega(c)$ is minimal. (As mentioned before, the method in the non-minimal case is very general and also applies to real analytic unimodal maps.)

It is very easy to show that the set of Axiom A maps in the class **AC** is dense, see Section 12. It is much less obvious that Axiom A maps are dense within any analytic family. However, repeating an argument used by O. Kozlovski in his thesis [Ko], we shall deduce from Theorems A, B and A', B' (see Section 12):

Theorem C (stability)
*Within any non trivial real analytic family of regular maps $g \in$ **AC**, either each g from this family has a neutral periodic orbit or the Axiom A maps within this family are dense.*

Here we say that g is a regular map if each parabolic periodic orbit $O(p)$ of g, is either disjoint from $\omega(c)$ or one has $\omega(c) \subset O(p)$. Note that if $\omega(c)$ is minimal, then g is automatically regular. (Actually, in Section we shall prove a slight generalization to Theorem C.) It follows from Theorem C, that within families of maps as in Examples 1 and 2 above, Axiom A maps are dense (the maps as in these examples are regular). Also, g is regular if for example $Sg < 0$ or if g extends to a (generalized) polynomial-like map with one critical point of odd order.

A consequence of Theorem A is:

Corollary
1. The Julia set of polynomial-like maps as above is totally disconnected and carries no invariant line fields.
2. The Julia set of the rational function introduced in Example 2 is locally connected.
Proof: Follows as in [McM1], [LS2] from the complex bounds. □

In fact, the absence of invariant linefields proved in this corollary is a very important ingredient in the proof of Theorem B.

Another consequence of the proofs of Theorem B and B' is

Theorem B" (rigidity) *Assume that two maps of the class* **AC** *have no parabolic or attracting periodic points, and there is a conjugacy which maps the critical point of one of these maps to the critical point of the other map. Then the conjugacy is quasisymmetric. In fact, this conjugacy can be extended to a quasiconformal homeomorphism on the plane which is a conjugacy on a 'necklace neighbourhood' of the dynamical interval; this neighbourhood consists of infinitely many disjoint topological discs.*

The above results should provide the basis for a study of “attractors”, and “universality structure” in the class **AC**.

Acknowledgements. The authors wish to thank Welington de Melo, Greg Swiatek, and also Alex Eremenko and Misha Sodin for valuable discussions. We are grateful to Greg Swiatek and Feliks Przytycki for listening to detailed expositions on some results of this paper. This IHES preprint is a slightly expanded version of a paper submitted for publication (Theorem B” was added). We also would like to thank the referee for several valuable remarks, for pointing out one error in a construction used in Theorem C in the Warwick preprint version of the paper. We are also grateful to the referee for insisting that we give more detail in the proof of Theorem B' and that we make the paper more readable. The research for this paper was initiated while the first author visited the Mathematics Department of Warwick. A substantial part of the paper was written during a stay of the first author at the Mathematics Department of University of Princeton and the Mathematics Department of Penn State University (March-June 1998). The first author sincerely thanks these Institutions for their hospitality. Both authors would like to thank IHES: Theorem B” was added while they visited IHES in the latter part of September 1999.

2 Some background

Proposition 2.1 *Maps g satisfying conditions 1-3 of the class* **AC** *do not have wandering intervals and have only finitely many non-expanding periodic orbits.*

Proof: Any such map induces a covering map $\overline{g}\colon S^1 \to S^1$. Indeed, first restrict g to the interval $[a_-, a_+]$ where $a_\pm$ are the left and right most fixed points of g. Points which are mapped eventually outside this interval will also eventually map outside the domain of definition of g. Then simply extend g to a piecewise increasing map defined on $[a_-, a_+]$ with a unique critical point at c. If the (closures of) domains of g are all disjoint then we can make sure that $\overline{g}$ is C^∞. If two domains of g intersect, then $\overline{g}$ will be C^∞ except at a finite number of preimages of a fixed point (where the left and right derivative of g will differ). In particular, $\overline{g}$ will be in the class C^{1+Z} (its derivative is Zygmund) and so it follows from the comments below Theorem IV.A in [MS] that $\overline{g}$ has no wandering intervals. The map $\overline{g}$ is not in the class C^{1+z} so one cannot apply Theorem IV.B from [MS] immediately. Note however, that it fails this smoothness condition only at a finite number of preimages of a fixed point. The intervals U_n from Lemma 10.3 on page 323 of [MS] do not intersect these preimages of the fixed point, and so along the pullback $\{U_0, \ldots, U_n\}$ needed in the proof of Proposition 10.1 of [MS] one still has the required cross-ratio estimate. It follows that Theorem IV.B from [MS] still holds. Hence periodic attractors and parabolic orbits of $\overline{g}$ all have period less than some number N (which depends on g). Since g is analytic on each branch, it follows that g has only finitely many non-expanding periodic orbits. $\square$

Let us introduce some notation and background.

If $I \subset J$ are intervals then $|I|$ denotes the length of I. The left and right endpoints of I are denoted by $\partial_l I$ and $\partial_r I$. The left and right components of $J \setminus I$ are denoted by $l(J \setminus I)$ and $r(J \setminus I)$.

If $J \subset T$ are two intervals and L, R are the components of $T \setminus J$ then we define $C(T,J)$ to be the cross-ratio of this pair of intervals:

$$C(T,J) = \frac{|J||T|}{|L||R|}.$$

Often we prefer to work with $C^{-1}(T,J) = 1/C(T,J)$. Cross-ratios play a crucial role in all recent metric results in real interval dynamics. If $f|T$ is a diffeomorphism and $Sf < 0$ (where Sf is the Schwarzian derivative of f) then [MS]

$$C^{-1}(T,J) \geq C^{-1}(f(T), f(J)).$$

In our case we shall apply this to maps f of the form g^n. If $Sg < 0$ then also $Sg^n < 0$ so the previous inequality applies when we take $f = g^n$ provided $g^n|T$ is a diffeomorphism. We will use the following amusing extension of this classical fact:

Lemma 2.1 *Consider a map*

$$F(x) = [h(x)]^\ell,$$

where $\ell > 1$ is an odd number, and h is a local diffeomorphism near $x = 0$, so that $h(0) = 0$, and so that h increases cross-ratios. Then for each $\epsilon > 0$ there exists $\Delta > 0$ such that the following holds. Let L be the left component of $T \setminus J$, so that if we denote $L = (x_1, x_2)$, then

$$x_1 < 0 < x_2 \text{ and } |x_2| \geq (1+\epsilon)|x_1|$$

and assume moreover that

$$|T| < \Delta.$$

Then

$$C^{-1}(T,J) \geq C^{-1}(F(T), F(J)).$$

Proof: See [Le] and use a continuity argument. □

If h is only real analytic then a similar statement holds, see Lemma 4.3.

If I is an interval near the critical point c, then we shall denote by I^g the interval $g(I)$ near the critical value. We shall say that the smaller interval J is *well inside* I, if the length of every component of $I \setminus J$ is at least $C_* \cdot |J|$, where $C_* > 0$ is a universal constant (i.e. does not depend on the integer n introduced below). Often we shall take C_* to be equal to something like $C/10^\ell$ where C is the constant from Theorem 2.1. If we can take $C_* \geq 2$ then we say that J is *deep inside* I.

Consider now a map g as in the introduction and so that the critical point is recurrent. As usual, define a sequence of *first return maps*

$$g^{S_n} : U_n \to U_{n-1}$$

inductively as follows. We set $S_1 = 1$, $U_1 = I_0$, and $U_0 = I$. If U_{n-1} is defined, then $g^{S_n} : U_n \to U_{n-1}$ is the central branch of the first return map to U_{n-1} (i.e. U_n is the component of the domain the first return map to U_{n-1} which contains c).

Let us say that the map $g^{S_n} : U_n \to U_{n-1}$ has a *central return* if $g^{S_n}(c) \in U_n$, and a *non-central* return otherwise. Sometimes we shall simply say that $g^{S_n}: U_n \to U_{n-1}$ is central. Note that if some $g^{S_n} : U_n \to U_{n-1}$ has a central return, then by pulling back the central domain U_n by the map g^{S_n} several times, we always obtain a non-central return map $g^{S_m} : U_m \to U_{m-1}$, where $m > n$ and where the maps are the same, i.e., $S_m = S_n$. (Otherwise c is in the basin of a periodic attractor.) We always have $S_{n+1} \geq S_n$. More specifically, $S_{n+1} = S_n$ iff $g^{S_n} : U_n \to U_{n-1}$ is central, and $S_{n+1} > S_n$ otherwise.

Theorem A of this paper gives the complex analogue of the real bounds which were proved in [Le].

Theorem 2.1 *Let $g \in$ **AC** and assume that $\omega(c)$ is minimal. Also assume that g increases cross-ratios on intervals on which it acts diffeomorphically. Moreover, assume that one can write $g(x) = g(c) + F(x)$ in a neighbourhood of the critical point $c = 0$, where F is as in Lemma 2.1.*

Then there exist a positive number C, which depends on ℓ only, and an integer N, so that for every $n \geq N$, the length of every component of $U_{n-1} \setminus U_n$ is equal to at least $C \cdot |U_n|$, whenever the previous map $g^{S_{n-1}} : U_{n-1} \to U_{n-2}$ is non-central.

Let $g^{S_n} : U_n \to U_{n-1}$ be non-central. If $S_n > S_{n-1}$ (i.e., $g^{S_{n-1}} : U_{n-1} \to U_{n-2}$ is non-central again), then we set $k(n) = 0$, so that $U_{n-1} = U_{n-k(n)-1}$. Otherwise (if $g^{S_{n-1}} : U_{n-1} \to U_{n-2}$ is central), there exists a first (maximal) interval $U_{n-k(n)}$, $k(n) \geq 1$, in the chain (cascade) of central returns containing U_n, so that $S_n = S_{n-1} = \ldots = S_{n-k(n)} > S_{n-k(n)-1}$ (i.e., $g^{S_{n-k(n)-1}} : U_{n-k(n)-1} \to U_{n-k(n)-2}$ is non-central). By Theorem 2.1, $U_{n-k(n)}$ is well inside $U_{n-k(n)-1}$ and also U_{n+1} is well inside U_n. Let us call $\{U_n \subset \ldots \subset U_{n-k(n)}\}$ the maximal chain of central intervals, which contains U_n (we do not exclude the case that $k(n) = 0$).

We shall use the following statement several times.

Proposition 2.2 *Let g be as in the previous Theorem. Let $I \subset U_n$ be a component of the domain of the first return map to U_n (for example, I could be equal to the central domain U_{n+1}), and assume that the first return map restricted to I is of the form g^s for some $s > 0$. Then there exists an interval $\tilde{U}$ containing I such that $g^{s-1} : \tilde{U}^g \to U_{n-k(n)-1}$ is a diffeomorphism.*

Proof: Denote $S = S_n = \ldots = S_{n-k(n)}$. If $s = S$, then $\tilde{U} = U_{n-k}$. So assume that $s > S$. Applying g^S to U_n for $N = k(n) + 1$ times, we see that $g^S(U_n) = U_{n-1}$, $g^S(I) \subset U_{n-1} \backslash U_n$, ..., $g^{NS}(U_n) = U_{n-k(n)-1}$, $g^{NS}(I) \subset U_{n-k(n)-1} \backslash U_{n-k(n)}$. In particular, $g^{NS}(I)$ lies in a component K of the first return map to $U_{n-k(n)-1}$ other than the central one. Pulling back K to U_n, we obtain the interval $\tilde{U}$. $\square$

We shall also use the following

Theorem 2.2 *(Mañé) Let $g: N \to N$ be a C^2 map where N is a circle or an interval. Let U be an open set containing the critical points of g, the parabolic periodic points and also the immediate basins of periodic attractors. Then there exists $C > 0$ and $\lambda > 1$ such that for each x and each integer k for which $x, g(x), \ldots, g^{k-1}(x) \notin U$ one has $|Dg(x)| \geq C\lambda^k$.*

Proof: For a proof of this statement and a simplified proof of the original result of Mañé, see [MS]. □

We shall also use the following result from complex analysis. Given a bounded real interval T we shall write $D_*(T)$ for the disc which is symmetric with respect to the real line and which intersects the real line exactly in T. More generally, if T is a bounded real interval and $\alpha \in (0, \pi)$ then consider a disc D which intersects the real line exactly in T such that $D_+ = D \cap \{Im(z) > 0\}$ has an external angle with the real line of angle α. The set $D(T; \alpha)$ denotes the union of D_+ and its mirror image with respect to $\mathbb{R}$. (Note that $D(T; \alpha)$ decreases with $\alpha \in (0, \pi)$ increasing.) The reason these sets play an important role, can be explained as follows. Let $\mathbb{C}_T = \mathbb{C} \setminus (\mathbb{R} \setminus T)$. The set $\mathbb{C}_T$ carries a Poincaré metric, and with respect to this metric the set $D(T; \alpha)$ consists of all points whose distance to T is at most equal to some constant $k(\alpha)$. From this interpretation and the Schwarz lemma it follows that if $\phi: \mathbb{C}_T \to \mathbb{C}_{T'}$ is a univalent conformal mapping sending T diffeomorphically to T', then

$$\phi(D(T; \alpha)) \subset D(T'; \alpha). \tag{2.1}$$

As in [Su], we shall apply this statement in the following way:

Lemma 2.2 (Schwarz) *Let $F: \mathbb{C} \to \mathbb{C}$ be a real polynomial whose critical points are on the real line and which maps the interval T' diffeomorphically onto the interval T, then there exists a set $D \subset D(T'; \alpha)$ with $D \cap \mathbb{R} = T'$ which is mapped diffeomorphically onto $D(T; \alpha)$ by F.*

In fact, there is generalization of this lemma to the situation in which $G = F^{-1}$ is not defined on the entire region $\mathbb{C}'_T$:

Lemma 2.3 (Schwarz) *For each $\theta_0 \in (0, \pi)$ there exists $\delta > 0$ and $K > 0$ with the following properties. Let G be a univalent map defined on the unit disc $D_1(0)$, mapping the real diameter $D_1(0) \cap \mathbb{R}$ into the real line. Let T be an interval on the real line containing the origin with $|T| < \delta$ and $G(T) = T'$. Then for each $\theta \in (0, \theta_0)$,*

$$G(D(T; \pi - \theta)) \subset D(T'; \pi - \theta(1 + K|T|)).$$

Proof: See Lemma VI.5.2 in [MS]. □

3 The proof of Theorem A in the Epstein case

Let us first prove Theorem A for Epstein maps. Here we say that a map $g: \cup I_i \to I$ from the class **AC** is in the more restrictive class of Epstein maps **EC** (cf. [Su]) if

1. for every $i \neq 0$, the map $g: I_i \to I$ is a diffeomorphism such that $g^{-1}: I \to I_i$ has a univalent extension to the slit complex plane $\mathbb{C}_I$,
2. one can decompose the map $g: I_0 \to I$ as $g(x) = g(0) + [h(x)]^\ell$, where $\ell \geq 3$ is an odd integer, and $h: I_0 \to I_0'$ is a diffeomorphism with $h(0) = 0$, such that $h^{-1}: I_0' \to I_0$ has a univalent extension to $\mathbb{C}_{I_0'}$.

We remark that g from 1) and h from 2) in the above definition increase cross-ratios. Indeed, since g^{-1} and h^{-1} extend analytically to the slit complex plane and since the Poincaré metric on a disc restricted to its diameter corresponds to cross-ratios, the remark follows from the Schwarz lemma.

For each ϵ, take $\Delta(\epsilon)$ as in Lemma 2.1. Since $|U_n| \to 0$ as $n \to \infty$, for each $\epsilon > 0$ we shall only consider integers n which are so large that $|U_n| < \Delta(\epsilon)$.

Lemma 3.1 *There exists $\epsilon_0 > 0$ such that if for some $\epsilon \in (0, \epsilon_0)$ there exist infinitely many n's for which*

$$\max\left(|\partial_l U_n - c|, |\partial_r U_n - c|\right) < (1+\epsilon) \cdot \operatorname{dist}(\partial U_{n-1}, c)$$

then g has complex bounds.

Proof: Assume that the above inequality is satisfied. To be more concrete, assume that

$$|\partial_l U_n - c| \leq |\partial_r U_n - c| < (1+\epsilon) \cdot |\partial_l U_{n-1} - c|. \tag{3.1}$$

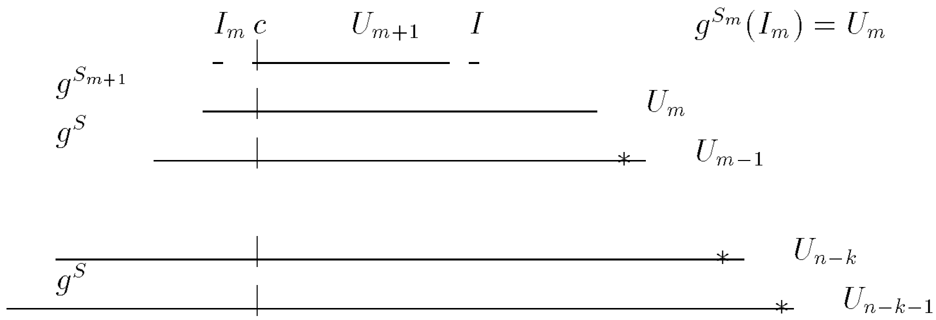
Consider two cases.

I. U_n is well inside U_{n-1}. Then we take $\Omega = D_*(U_{n-1})$ as the range of the polynomial-like map. Consider the component of the domain of the first return map to U_{n-1} which intersects U_n^g (we shall use this notation for the interval $g(U_n)$). By Schwarz and because g is in Epstein class this component is inside $D_*(U_n^g)$. If ϵ would be equal to zero, then because we are assuming in this case that U_n is well inside U_{n-1}, $g^{-1}(D_*(U_n^g))$ is compactly contained in $D_*(U_{n-1})$. Provided $\epsilon \in (0, \epsilon_0)$ and we choose ϵ_0 sufficiently small, the same is still true provided (3.1) holds. Note that the modulus of $D_*(U_{n-1}) \setminus g^{-1}(D_*(U_n^g))$ is bounded from below by a positive constant which does not depend on n. Again by Schwarz the non-central domains are mapped univalently inside $D_*(U_{n-1})$.

II. U_n is *not* well inside U_{n-1}. According to Theorem 2.1, then the previous return is central (actually, many of the previous returns are central if U_n is not well inside U_{n-1}). Hence, U_n and U_{n-1} are two consecutive intervals in a cascade of central returns. Then, on the one hand there exists a first interval U_{n-k} of this cascade, so that $S_n = S_{n-k}$ and U_{n-k} is well inside U_{n-k-1}. On the other hand, let U_m (with $m \geq n$) be the last interval of this cascade, so that $S_m = S_n$ and the map $g^{S_m} : U_m \to U_{m-1}$ is non-central. So we have

$$S_{m+1} > S := S_m = \ldots = S_n = \ldots = S_{n-k+1} = S_{n-k} > S_{n-k-1}$$

and the following situation (where $*$ are points of the form $g^{iS}(c)$)



where S_{m+1} and $S = S_m = \ldots = S_{n-k}$ are the first return times of c to U_m respectively U_{m-1} and I_m is the component of the first return to U_m, which is mapped by g^{S_m} onto U_m (which is the left of c if $f^{S_m}(c) \in r(U_{m-1} \setminus U_m)$ and to the right otherwise). Proposition 2.2 implies that there is an interval between U_m^g and U_{m+1}^g (i.e., contained in the former interval and containing the latter one) which is mapped by $g^{S_{m+1}-1}$ diffeomorphically onto U_{n-k-1}. Let U_* be the smallest symmetric interval which contains U_m. Since U_{n-1} is well inside U_{n-k-1}, provided ϵ is small, (3.1) implies that U_* is well inside U_{n-k-1}. Therefore, as we remarked before, there exists an interval U with $U_m \supset U \supset U_{m+1}$ and so that $g^{S_{m+1}-1}$ maps $g(U)$ diffeomorphically onto U_*. Since U_* is well inside U_{n-k-1}, the interval U is well inside U_m. Take as the range Ω of the desired quasi-polynomial-like map the disc $D_*(U_*)$ with two slits (cf. the proof of Lemma 3.6):

$$\Omega = D_*(U_*) \setminus (U_* \setminus U_m).$$

We put the slits in $D_*(U_*)$ because we want our quasi-polynomial-like map restricted to the real line to coincide with the first return map to Ω along the postcritical set (and so the return time of c to Ω is equal to S_{m+1}). Consider the first return map to Ω along the postcritical set. By Schwarz the g-image of the central domain Ω_0 of the first return map to Ω is contained in $D_*(g(U))$. Because $D_*(U_*)$ is a round disc centered at $c = 0$, and because U is well inside $U_m \subset U_*$, we obtain that Ω_0 is compactly contained in the range Ω (and we obtain a lower bound for the modulus of $\Omega \setminus \Omega_0$ which does not depend on n). Next consider the non-central components of the first return map. Take a non-central real domain I of the first return map R_{U_m} to U_m and let s be so that $R_{U_m}|_I = g^s$. If $s > S_m$ $(= S_n = S_{n-k})$, then there is a diffeomorphic extension $g^s : \hat{I} \to U_*$ of $g^s|_I$, and the pullback of Ω by g^s is contained in $D_*(\hat{I})$ and hence is compactly contained in Ω (here we use similar arguments as above). Consider the remaining branch $g^{S_m} : I_m \to U_m$ of the first return map to U_m. It has a diffeomorphic extension $g^{S_m-1} : \hat{I}_m^g \to U_*$ only from an interval $\hat{I}_m^g \subset U_*^g$ (so near the critical value), and the pullback of Ω by g^{S_m-1} is contained in $D_m^g := D_*(\hat{I}_m^g) \setminus (\hat{I}_m^g \setminus I_m^g)$

(with $g(c) \in (\hat{I}^g_m \setminus I^g_m)$). But since Ω is a disc centered at c (with two slits), the branch of the map g^{-1} corresponding to $g^{-1} : I^g_m \to I_m$, takes D^g_m onto some D_m, which is inside Ω, but possibly *not* compactly inside: the boundaries of Ω and D_m can coincide in a subset of the slits $U_* \setminus U_m$. Thus we obtain a quasi-polynomial-like mapping (the definition is given below the statement of Theorem A). $\square$

If we are in the situation of the previous lemma, then Theorem A follows. So we may and will assume that for each $\epsilon \in (0, \epsilon_0)$ there exists N so that for all $n \geq N$,

$$\max\left(|\partial_l U_n - c|, |\partial_r U_n - c|\right) > (1+\epsilon) \cdot \operatorname{dist}(\partial U_{n-1}, c)$$

i.e., the larger component of $U_n \setminus \{c\}$ is a definite amount longer than the shortest component of $U_{n-1} \setminus \{c\}$ (if $\epsilon = 0$ then it is already as least as long). Since $U_n \subset U_{n-1}$ this means that for each $\epsilon \in (0, \epsilon_0)$ there exists $N(\epsilon)$ so that either for all $n \geq N(\epsilon)$ $(1+\epsilon)|\partial_l U_{n-1} - c| < |\partial_r U_n - c|$ or for all $n \geq N(\epsilon)$ $(1+\epsilon)|\partial_r U_{n-1} - c| < |\partial_l U_n - c|$. In order to be specific, we will from now on make the following

STANDING ASSUMPTION: $(1+\epsilon) \cdot |\partial_l U_{n-1} - c| < |\partial_r U_n - c|$ for all $n \geq N(\epsilon)$.

In particular, we assume that for each $n \geq N(\epsilon)$ the right component of $U_n \setminus \{c\}$ is a definite amount longer than its left component. This means that for $n \geq N(\epsilon)$,

Lemma 3.2 *If $J \subset T$ are intervals such that*

$$T \subset U_{n-1}, J \subset r(U_{n-1} \setminus U_n)$$

then we can pass through the critical point without loosing on cross-ratio:

$$C^{-1}(T, J) \geq C^{-1}(g(T), g(J)).$$

Proof: Follows from Lemma 2.1 and the standing assumption. $\square$

Lemma 3.3 *Assume that the standing assumption holds. Then there are infinitely many n's such that $g^{S_n}(c) < c$.*

Proof: By the contrary, assume that $g^{S_n}(c) > c$ for all $n \geq N$. It is enough to show that $C^{-1}(U_n, U_{n+1}) \to \infty$ as $n \to \infty$ provided $S_n > S_{n-1}$ because it would contradict the standing assumption. Consider $U_n \subset U_{n-1}$ and assume that $S_n > S_{n-1}$. Since $g^{S_{n-1}}(c) > c$, g is orientation preserving on each branch and $S_n > S_{n-1}$ one has $g^{S_{n-1}}(U_n) \subset r(U_{n-2} \setminus U_{n-1})$ and by definition $g^{S_{n-1}}(U_{n-1}) = U_{n-2}$. Next apply $g^{S_{n-2}} = g^{S_{n-2}-1} \circ g$. Lemma 3.2 and the standing assumption give us

$$C^{-1}(U_{n-2}, g^{S_{n-1}}(U_n)) \geq C^{-1}(g^{S_{n-2}}(U_{n-2}), g^{S_{n-2}+S_{n-1}}(U_n)) =$$

$$= C^{-1}(U_{n-3}, g^{S_{n-2}+S_{n-1}}(U_n)).$$

Again $g^{S_{n-1}+S_{n-2}}(U_n) \subset r(U_{n-3} \setminus U_{n-2})$ and by definition $g^{S_{n-1}+S_{n-2}}(U_{n-1}) = U_{n-3}$. Hence $S_{n-1}+S_{n-2} < S_n$ because $g^{S_n}(U_n) = U_{n-1}$. Continuing in this way we apply next $g^{S_{n-3}}$, $g^{S_{n-4}}$ and so on (using each time Lemma 3.2). If we write $t(n) = S_{n-1} + \ldots + S_N$ then we get that $t(n) < S_n$, that $g^{t(n)}$ maps U_{n-1} homeomorphically onto U_{N-1} and that

$$C^{-1}(U_{n-2}, g^{S_{n-1}}(U_n)) \geq C^{-1}(U_{N-1}, g^{t(n)}(U_n)).$$

Moreover, $|g^{t(n)}(U_n)| \to 0$ because $g^{S_n}(U_n) = U_{n-1}$ tends to zero in length (here we can use the contraction principle, see for example [MS], since g has no wandering intervals). Moreover, $g^{t(n)}(U_n)$ does not tend to ∂U_{N-1} because $\omega(c)$ is assumed to be minimal. It follows that $C^{-1}(U_{N-1}, g^{t(n)}(U_n)) \to \infty$. Therefore, $C^{-1}(U^g_{n-1}, g(U_n)) \geq C^{-1}(U_{n-2}, g^{S_{n-1}}(U_n)) \to \infty$, and we are done. $\square$

Let us from now on assume that $g^{S_n}(c) < c$ for infinitely many n. So consider the (infinite) collection of integers

$$\mathcal{S} = \{n\,;\ g^{S_n}(c) < c \text{ and } S_{n+1} > S_n\}.$$

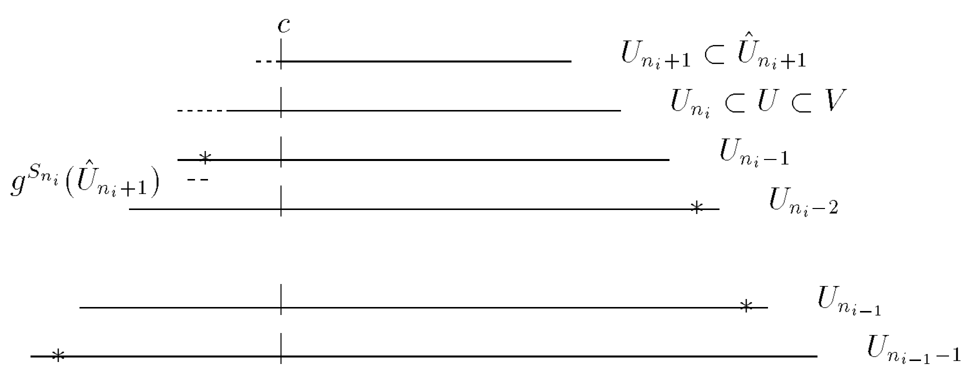
Note that for $n \in \mathcal{S}$ one cannot have that $g^{S_n}\colon U_n \to U_{n-1}$ is central, but it is possible that $g^{S_{n-1}}\colon U_{n-1} \to U_{n-2}$ is central. In other words, if there are central intervals then U_n is always the last (smallest) interval in a chain of central intervals

$$U_n \subset U_{n-1} \subset \ldots \subset U_{n-k(n)},$$

with

$$S_{n+1} > S_n = S_{n-1} = \ldots = S_{n-k(n)} > S_{n-k(n)-1}.$$

Here $k(n) \geq 0$, and $k(n) = 0$ if and only if $g^{S_{n-1}}\colon U_{n-1} \to U_{n-2}$ is again not central. Let $n_1 < n_2 < n_3 < \ldots$ be the elements of this collection $\mathcal{S}$. Below we draw the situation when $k(n_i) = 0$.



Let

$$V = [\partial_l U_{n_i-k(n_i)-1}, \partial_r U_{n_i}].$$

It will also be useful to define an interval $\hat{U}_{n_i+1}$ containing c between U_{n_i+1} and U_{n_i} with $\partial_r \hat{U}_{n_i+1} = \partial_r U_{n_i+1}$ and so that

$$g^{S_{n_i+1}}\colon \hat{U}_{n_i+1} \to [\partial_l U_{n_i-k(n_i)-1}, \partial_r U_{n_i}] = V$$

is a homeomorphism. By Proposition 2.2, $g^{S_{n_i+1}-1}\colon \hat{U}^g_{n_i+1} \to V$ is a diffeomorphism.

Lemma 3.4

$$C^{-1}(V^g, \hat{U}^g_{n_i+1}) = C^{-1}([\partial_l U^g_{n_i-k(n_i)-1}, \partial_r U^g_{n_i}], \hat{U}^g_{n_i+1}) \geq$$

$$\geq C^{-1}([\partial_l U_{n_{i-1}-1}, \partial_r U_{n_i-1}], g^{S_{n_i}}(\hat{U}_{n_i+1})).$$

Proof: Write $\hat{U} = \hat{U}_{n_i+1}$ and choose U between U_{n_i} and V (such that U, U_{n_i}, V all have the same right end point) so that U is mapped homeomorphically by $g^{S_{n_i}}$ onto $[\partial_l U_{n_{i-1}-1}, \partial_r U_{n_i-1}]$. This means that

$$g^{S_{n_i}-1}\colon \hat{U}^g \subset U^g \to g^{S_{n_i}}(\hat{U}) \subset [\partial_l U_{n_{i-1}-1}, \partial_r U_{n_i-1}]$$

is a homeomorphism between these pairs of intervals. It is enough to show that we do not have loss of cross-ratios for this map, because obviously the left-hand side of the inequality in the statement of the lemma is larger than $C^{-1}(U^g, \hat{U}^g)$. Note that $g^{S_{n_i}-1}$ is a diffeomorphism restricted to $\hat{U}^g$ and also restricted to the right component of $U^g \setminus \hat{U}^g$ (because these sets are contained in U_{n_i}). So we need to show that each time some iterate (up to time $S = S_{n_i}$) of the left component of $U^g \setminus \hat{U}^g$ meets the critical point we are in the situation described by Lemma 3.2.

The first time c enters $U_{n_i-k(n_i)-1}$ is at time

$$S = S_{n_i} = \ldots = S_{n_i-k(n_i)} \; (> S_{n_i-k(n_i)-1} \geq \ldots \geq S_{n_{i-1}}).$$

So all iterates of $\hat{U}$ up to time $S = S_{n_i}$ are outside $U_{n_i-k(n_i)-1}$. Moreover, they do not intersect ∂U_t when $t \leq n_i$ because $g^{S_{n_i}}(\hat{U}) \subset U_{n_i-1}$. Let us take $j < S_{n_i}$ and show that either g is a diffeomorphism on $g^j(U)$ or we are in the situation of Lemma 3.2 for $J = g^j(\hat{U}) \subset g^j(U) = T$. If $g^j(\hat{U})$ is outside $U_{n_{i-1}}$ then $g^j(U)$ is also outside $U_{n_{i-1}}$ (and so g is a diffeomorphism on $g^j(U)$); indeed boundary points of $U_{n_{i-1}}$ are never mapped inside $U_{n_{i-1}}$. If $g^j(\hat{U})$ is in $U_{n_{i-1}}$ but to the left of c, then $g^j(U)$ also does not contain c (because g preserves orientation and $g^{S_{n_i}-1}$ is a diffeomorphism restricted to the right component of $U^g \setminus \hat{U}^g$), and again g is a diffeomorphism on $g^j(U)$. So the only situation we have to consider is when $g^j(\hat{U}) \subset r(U_{n_{i-1}} \setminus U_{n_i-k(n_i)-1})$. In this case $g^j(\hat{U})$ is contained in one of the intervals $r(U_{n_i-s} \setminus U_{n_i-s+1})$ where $n_{i-1} \leq n_i - s < n_i - k(n_i) - 1$. We *claim* that if

$$g^j(\hat{U}) \subset r(U_{n_i-s} \setminus U_{n_i-s+1}) \tag{3.2}$$

(with s as above) then $g^j(U)$ is contained in U_{n_i-s}. So let us assume by contradiction that $j < S_{n_i}$ is the maximal integer such that there exists s as in (3.2) such that $g^j(U)$ contains a boundary point of U_{n_i-s}. It is impossible that $n_i - s = n_{i-1}$ because if $g^j(U)$ contains a boundary point of U_{n_i-s} then $g^S(U)$ would contain the left boundary of $U_{n_{i-1}-1}$ which is not the case. If $n_i - s > n_{i-1}$ then $g^{S_{n_i-s}}$ is the first iterate which maps $g^j(\hat{U})$ inside U_{n_i-s-1}. It also maps $g^j(\hat{U})$ into $r(U_{n_i-s-1} \setminus U_{n_i-s})$ (because $n_i - s < n_i - k(n_i) - 1$ and by the definition of $\mathcal{S}$). From this it follows that j is not maximal, and so by contradiction the claim is proved. But from the claim it follows that for each j with (3.2), we are in the situation of Lemma 3.2, and thus the lemma follows. $\square$

Lemma 3.5 *For each ϵ from the standing assumption there are $\delta > 0$ and infinitely many $n_i \in \mathcal{S}$ for which*

$$C^{-1}([\partial_l U^g_{n_i-k(n_i)-1}, \partial_r U^g_{n_i}], \hat{U}^g_{n_i+1}) \geq 1+\delta$$

Proof: Fix such ϵ. By the previous lemma it suffices to prove that there are $\delta > 0$ and infinitely many $n_i \in \mathcal{S}$ such that

$$C^{-1}([\partial_l U_{n_{i-1}-1}, \partial_r U_{n_i-1}], g^{S_{n_i}}(\hat{U}_{n_i+1})) \geq 1+\delta.$$

Moreover, since $g^{S_{n_i}}(\hat{U}_{n_i+1}) \subset l(U_{n_i-1} \backslash U_{n_i})$, the left hand side in the previous inequality is bounded from below by

$$C^{-1}\left([\partial_l U_{n_{i-1}-1}, \partial_r U_{n_i-1}], l(U_{n_i-1} \setminus U_{n_i})\right).$$

Normalize so that $c = 0$ and write

$$U_n = [-x_n, y_n]$$

where by our standing assumption we have $0 < (1+\epsilon)x_n < y_{n+1} < y_n$ provided $\epsilon \in (0, \epsilon_0)$. Then

$$C^{-1}\left([\partial_l U_{n_{i-1}-1}, \partial_r U_{n_i-1}], l(U_{n_i-1} \setminus U_{n_i})\right) = \frac{(x_{n_{i-1}-1} - x_{n_i-1})(y_{n_i-1} + x_{n_i})}{(y_{n_i-1} + x_{n_{i-1}-1})(x_{n_i-1} - x_{n_i})} \geq$$

$$\frac{\left(x_{n_{i-1}-1} - x_{n_i-1}\right)((1+\epsilon)x_{n_i-1} + x_{n_i})}{\left((1+\epsilon)x_{n_i-1} + x_{n_{i-1}-1}\right)(x_{n_i-1} - x_{n_i})} = \frac{\left(\lambda'_{n_i} - 1\right)}{\left((1+\epsilon) + \lambda'_{n_i}\right)} \frac{((1+\epsilon)\lambda_{n_i} + 1)}{(\lambda_{n_i} - 1)}, \tag{3.3}$$

where we denote

$$\lambda_{n_i} = x_{n_i-1}/x_{n_i}$$

and

$$\lambda'_{n_i} = x_{n_{i-1}-1}/x_{n_i-1}.$$

Take any $\kappa > 0$. Assume first that for all i sufficiently large, $\lambda_{n_i} \geq (1+\kappa)\lambda'_{n_i}$. Observe that

$$\lambda'_{n_i} = x_{n_{i-1}-1}/x_{n_i-1} \geq x_{n_{i-1}-1}/x_{n_{i-1}} = \lambda_{n_{i-1}}. \tag{3.4}$$

Hence we would have $\lambda_{n_i} \geq (1+\kappa)\lambda_{n_{i-1}}$ for every i large. It implies $\lambda_{n_i} \to \infty$, and, together with (3.4), $\lambda'_{n_i} \to \infty$. But then the expression in (3.3) would be at least $1+\epsilon/2$, for all n_i large enough, which proves the statement in the considered case with $\delta = \epsilon/2$.

Assume now that for some fixed i, $\lambda_{n_i} \le (1+\kappa)\lambda'_{n_i}$. Denote $t = \lambda'_{n_i}$. Then the expression in (3.3) is bounded from below by the function

$$A(\kappa, t) = \frac{(t-1)}{((1+\epsilon)+t)} \frac{((1+\epsilon)(1+\kappa)t+1)}{((1+\kappa)t-1)}, \tag{3.5}$$

which one can rewrite as

$$(1+\epsilon) + (2+\epsilon)\frac{(\epsilon - ((1+\epsilon)(1+\kappa)-1)t)}{((1+\epsilon)+t)((1+\kappa)t-1)}. \tag{3.6}$$

It is easy to see that the latter expression is increasing on $t > 1$. The real bounds imply that there exists $C > 1$ so that $\lambda'_{n_i} \ge C > 1$ for all i. Hence, one can assume that $t \ge C$ and $A(\kappa, t) \ge A(\kappa, C)$. On the other hand, $A(\kappa, C) \to 1 + 2\delta$ as $\kappa \to 0$, where we denote $\delta = 0.5\epsilon(C-1)/(1+\epsilon+C) > 0$. It follows that there exists $\kappa > 0$ dependent only on C and ϵ such that the expression in (3.3) is again at least $1+\delta$ provided $\lambda_{n_i} \le (1+\kappa)\lambda'_{n_i}$ for a given i. $\square$

Let us now show that the situation of the previous lemma leads to complex bounds. Let $n = n_i$ and $k = k(n_i)$. It follows immediately from the previous lemma that

(*) the length of each component of $[\partial_l U^g_{n-k-1}, \partial_r U^g_n] \setminus \{c_1\}$ is bigger than $(1+\delta)|\hat{U}^g_{n+1}|$, where $\delta > 0$ does not depend on $n = n_i$.

Let us choose (and then fix) $\Delta > 0$ dependent only on δ as follows. Let

$$V_{n-k-1}(\Delta) \subset [\partial_l U_{n-k-1}, \partial_r U_n]$$

be the interval obtained by removing from the left side of the interval $[\partial_l U_{n-k-1}, \partial_r U_n]$ its Δ-part, i.e. an interval of length $\Delta \cdot |\partial_r U_n - \partial_l U_{n-k-1}|$. We claim that for $\Delta > 0$ is sufficiently small,

(**) the length of each component of $V^g_{n-k-1}(\Delta) \setminus \{c_1\}$ is larger than $(1+\delta/2)|\hat{U}^g_{n+1}|$. Moreover, $V_{n-k-1}(\Delta)$ contains the left end of U_{n-k} and $V_{n-k-1}(\Delta)$ is well inside U_{n-k-1}.

Let us show that this is indeed possible. Let l and r be lengths of the left and the right components of $[\partial_l U^g_{n-k-1}, \partial_r U^g_n] \setminus \hat{U}^g_{n+1}$. Since U_n is well inside U_{n-k-1}, there is an absolute constant $C > 0$ such that $l \ge C \cdot |U^g_n| \ge C(r + |\hat{U}^g_{n+1}|)$. From the previous lemma, $l, r > |\hat{U}^g_{n+1}|$. This and $l \ge C(r + |\hat{U}^g_{n+1}|)$, imply that one can choose $\Delta > 0$ (dependent on C and δ) as in the first part of (**). Now, the left component of $U_{n-k-1} \setminus V_{n-k-1}(\Delta)$ has the length at least $\Delta|V_{n-k-1}(\Delta)|$. The right component of $U_{n-k-1} \setminus V_{n-k-1}(\Delta)$ is also not small compared with $V_{n-k-1}(\Delta)$ because U_{n-k} is well inside U_{n-k-1} and by the standing assumption. This gives the second part of (**).

Lemma 3.6 *Let $\delta > 0$ and Δ be as above. Then for any $n = n_i \in \mathcal{S}$, $k = k(n_i)$ with*

$$C^{-1}([\partial_l U^g_{n-k-1}, \partial_r U^g_n], \hat{U}^g_{n+1}) \geq 1 + \delta, \tag{3.7}$$

the first return map to U_n extends to a (quasi-)polynomial-like map with range

$$\Omega = D_*(V) \setminus (V \setminus U_n)$$

where V is either $[-\partial_r U_n, \partial_r U_n]$ or $V_{n-k-1}(\Delta)$.

Proof: There are two cases:

I. There exists $\rho > 0$ (not depending on n) such that the left component of $U_{n-k-1} \setminus \{c\}$ is at least $(1+\rho)$ times as long as the right component of $U_n \setminus \{c\}$. In this case $U_* = (-\partial_r U_n, \partial_r U_n)$ is well-inside U_{n-k-1}. Now proceed as in case II of Lemma 3.1 by setting $\Omega = D_*(U_*) \setminus (U_* \setminus U_n)$ (and taking $m = n$).

II. The situation is not as in I. Then let $V = V_{n-k-1}(\Delta)$. Since U_{n-k} is well-inside U_{n-k-1}, that the situation is not as in I means that we can assume that the left component of $U_{n-k} \setminus \{c\}$ is shorter than the right component of $U_n \setminus \{c\}$.

Consider the first return map to U_n, but extend each branch so that it maps onto $\Omega := D_*(V) \setminus (V \setminus U_n)$. Let us first consider the central domain of the first return map to U_n and its extension $g^{S_{n+1}}\colon \hat{U}(\Delta) \to V$. Let $g^{S_{n+1}}\colon D_0 \to \Omega$ be the central branch. By Schwarz, some subset of $D_*(\hat{U}(\Delta)^g)$ is mapped diffeomorphically onto $D_*(V)$ by $g^{S_{n+1}-1}$. From (**) it follows immediately that $g^{-1}(D_*(\hat{U}(\Delta)^g)) \subset D_*(V)$. To show that the modulus of $\Omega \setminus D_0$ is bounded from below by some universal positive constant, we still need to consider the real part of D_0 (because of the slits). Let $\tilde{U} \subset U_n$ be so that $g^{S_{n+1}-1} : \tilde{U}^g \to U_{n-k-1}$ is a diffeomorphism. Since V is well inside U_{n-k-1} (by (**)), the interval $\hat{U}(\Delta)$ is well inside $\tilde{U}$, i.e. also well inside U_n. All this means that the modulus of $\Omega \setminus D_0$ is bounded away from zero. Let us now show that the non-central components are also contained in Ω. Take a non-central domain I of the first return map to U_n so that the first return restricted to I coincides with g^s. If $s > S_n$, then, since forward iterates of ∂U_i never enter U_i, this branch of the first 'return' map from U_{n+1} to U_n extends diffeomorphically onto U_{n-k-1} (and the extended domain is well inside U_n). By the lemma of Schwarz, the corresponding domain is inside Ω. So let us consider the case that $s = S_n$. In this case I is the non-central interval in U_n which is mapped by g^s to U_n, and one can extend the interval $I^g = g(I)$ to the left to an interval $\tilde{I}^g = g(\tilde{I})$ so that $g^{s-1}\colon I^g \to U_n$ extends to a diffeomorphism $g^{s-1}\colon \tilde{I}^g \to V = [\partial_l U_{n-k-1}(\Delta), \partial_r U_n]$. The extended domain $\tilde{I}$ is inside $J = [\partial_l U_{n-k}, \partial_r U_n]$. Since we are in Case II, the right component of $U^g_n \setminus \{c\}$ is longer than the left component of $\tilde{I}^g \setminus \{c\}$. Therefore we can apply the lemma from the appendix and get that the component of $g^{-1}(D_*(\tilde{I}^g)) \setminus \mathbb{R}^-$ which contains I, is contained in $D_*(\tilde{I})$. So also the non-central components are contained in Ω. Therefore also in this case we obtain a quasi-polynomial-like map. $\square$

4 Real bounds for smooth maps in the minimal case

In this section, we will show that the real bounds of [Le] hold without the negative Schwarzian condition. That is, in this section we shall prove Theorem 2.1 for any C^3 map g which satisfies conditions 1-3 of the class **AC** (but is not necessary real analytic) and for which

- $g(x) = g(c) + [h(x)]^l$ in a neighbourhood of the critical point $c = 0$, where h is a C^3 local diffeomorphism and $h(0) = 0$,
- $\omega(c)$ is minimal.

Note that we shall use Theorem 2.1 for maps $g \in$ **AC** only if $\omega(c)$ is minimal. So fix such a map g.

The scheme of the proof is the following. In the first step, we prove a weaker form of Theorem 2.1, namely, the following *a priori real bounds*:

Lemma 4.1 *There exist a positive number C, and a sequence n_i, so that for every $n = n_i \to \infty$, the length of every component of $U_{n-1} \setminus U_n$ is equal to at least $C \cdot |U_n|$.*

This lemma implies that any branch $R_{U_{n_i}} : V \to U_{n_i}$ of the first return map has a diffeomorphic extension onto a definite neighbourhood of U_{n_i} (namely, to $U_{n_i - 1}$) provided V and U_{n_i} are disjoint.

In the second step we repeat the proof of Theorem 3.2 of [Ko] (replacing Lemma 3.2.3 in [Ko] by Lemma 4.1). This shall imply

Lemma 4.2 *Let $U = U_{n_i}$ be small enough, and let $g^k : W \to g(U)$ be a branch of the first return map to $g(U)$, such that $c \notin W$ and W contains points of the postcritical set. Then the Schwarzian derivative $S(g^k)$ of g^k on W is negative.*

In the third step we show how to go through the critical point arbitrary many times without loss of the cross-ratio under the conditions of Lemma 4 of [Le] or even stronger see Lemma 3.3, and Lemma 3.4. This will prove Theorem 2.1 for smooth maps as above.

In the final step, we will show that all real bounds from the previous section also hold for smooth maps g as above.

Step 1: Firstly, it turns out that it is convenient to deal with another cross-ratio $B(T, J)$ instead of $C(T, J)$ defined by

$$B(T, J) = \frac{|J||T|}{|L \cup J||R \cup J|},$$

where $J \subset T$ are intervals and L, R are connected components of $T \setminus J$. (Note that $1 + C^{-1} = B^{-1}$.) A collection of intervals $J_i \subset T_i$, $i = 1, \ldots, n$, is said to satisfy the *margins disjointness property* (which was introduced in Section 2.3 of [Ko]) if for the components L_i and R_i of $T_i \setminus J_i$,

$$L_i \cap L_j \neq \emptyset \text{ implies } R_i \cap R_j = \emptyset, \text{ for } 1 \leq i < j \leq n \tag{4.1}$$

By Lemma 2.3.2 of [Ko], a consequence of this property is that

$$\sum_{i=1}^{n} |L_i||R_i| \leq 4 \max_{1\leq i\leq n} |T_i|$$

provided all $T_i \subset [-1,1]$.

The margins disjointness property is useful in view of the following statements (which imply that we can pull-back big space provided the margins disjointness property holds):

Theorem 4.1 *[MS]*

Let $f : N \to N$ be a C^3-map of an interval or a circle N whose critical points are non-flat. Then there exists a constant $C_1 = C_1(f)$ such that

$$B(f(T), f(J)) \geq B(T, J) \exp\left(-C_1|L||R|\right),$$

where $f'(x) \neq 0$ for all $x \in T$.

Proof: See Section IV.1 from [MS]. □

Lemma 4.3 *Under the conditions of Lemma 2.1, but assuming only that h is a local C^3-diffeomorphism, we have:*

$$B(F(T), F(J)) \geq B(T, J) \exp\left(-C_1|L||R|\right),$$

where the constant C_1 depends only on the map h.

Proof: F is a composition of h and x^l. Then use Lemma 2.1 for the latter map and the previous Theorem 4.1 for h. □

In order to obtain Lemma 4.1, we observe that the proofs of Propositions 1-3 in [Le] hold for the smooth g as well, with the following changes.

The proofs in [Le] are based on combinatorial arguments (which of course do not depend on any analytic condition), the Koebe Principle (which also holds for smooth maps provided the intersection multiplicity is bounded [MS]), and the following *Interval Adding Procedure:* Given $g^{S_n} : U_n \to U_{n-1}$, let t_n^+ be a maximal interval outside U_n and to the right from U_n with a common boundary point with U_n, so that $g^{S_n}|t_n^+$ is defined and a diffeomorphism. The interval t_n^- is defined in the same way, but to the left of U_n. Let t be either t_n^+ or t_n^-. Note that one endpoint of $g^{S_n}(t)$ is in ∂U_{n-1}. If the other endpoint of $g^{S_n}(t)$ is an endpoint of the range I of the map g, then we set $k = 0$ and stop. Otherwise t is the minimal interval, such that the boundary point of t which is not in ∂U_n, is a critical point of g^{S_n}. In this case there exists i, $1 \leq i \leq S_n - 1$, so that c lies at the boundary of the interval $g^i(t)$. So in this case we will apply the following operation (called "adding the interval U_{n-1}"):

Since $g^i(U_n)$ is outside U_{n-1} and $g^i(t)$ has c in its boundary, the interval $g^i(t)$ contains also a boundary point of U_{n-1}. Hence, either $i + S_{n-1} > S_n$, or $g^{i+S_{n-1}}(t)$

contains a boundary point of U_{n-2}. Hence, again either $i + S_{n-1} + S_{n-2} > S_n$, or $g^{i+S_{n-1}+S_{n-2}}(t)$ contains a boundary point of U_{n-3}. We continue this process until we find $k \geq 1$ so that

$$S := i + S_{n-1} + ... + S_{n-k+1} < S_n,$$

while

$$S + S_{n-k} \geq S_n.$$

Then $g^S(t)$ contains a boundary point of U_{n-k}. Let us extend the interval $g^S(t)$ so that it turns into the interval $[U_{n-1}, g^S(t)]$, and call this operation "adding the interval U_{n-1}". The boundary point of t other than ∂U_n lies in a component of $U_{m-1} \setminus U_m$, for some $m \leq n$. Since ∂U_{m-1} are nice points, there exists a unique extension $\hat{t} = t_n^{\pm}$ of t inside $U_{m-1} \setminus U_m$ so that g^S maps $\hat{t}$ onto the interval $[U_{n-1}, g^S(t)]$. At the end, if $g^{S_n}(\hat{t}_n^{\pm})$ covers more than one interval of the form $g^j(U_{n-1})$, $1 \leq j \leq S_{n-1} - 1$, we shorten $\hat{t}_n^{\pm}$ so that $g^{S_n}(\hat{t}_n^{\pm})$ is the minimal interval, which either covers one interval of such form, or such that $|g^{S_n}(\hat{t}_n^{\pm})| = |U_{n-1}|$. In other words, $g^{S_n}(t_n^- \cup U_n \cup t_n^+) = \hat{U}_{n-1}$, where $\hat{U}_{n-1}$ is said to be the minimal interval containing from each side of U_{n-1} either its neighbour from the collection $g^i(U_{n-1})$, $1 \leq i < S_{n-1}$, or a 1-neighbourhood of U_{n-1} (i.e. of size $|U_{n-1}|$).

By construction, $g^{S_n}|t_n^{\pm}$ has at most $k_n^{\pm}$ critical points (see Lemma 1 of [Le] for detail). Now we shall prove the inequality of Lemma 1 of [Le]:

Lemma 4.4 *There exists $C > 0$ which only depends only on ℓ and $\max(k_n^-, k_n^+)$ so that for n large enough*

$$|\hat{t}_n^{\pm}| \geq C|U_n|. \tag{4.2}$$

Proof: Observe that the collection of intervals:

$$g^i(J) \subset g^i(L \cup J \cup R), i = 0, ..., S_n - 1$$

where $J = U_n$, $L, R = \hat{t}^{\pm}$, satisfies the margins disjointness property (4.1). Indeed, otherwise one could find $1 \leq j \leq S_n - 1$ with $g^j(t_n^-) \cap g^{S_n}(t_n^-) \neq \emptyset$ and $g^j(t_n^+) \cap g^{S_n}(t_n^+) \neq \emptyset$. Because $g^{S_n}(U_n) = U_{n-1}$ and $g^j(U_n) \cap U_{n-1} = \emptyset$, then either $g^j((t_n^-)$ or $g^j(t_n^+)$ contains a neighbourhood of U_{n-1}. If, say, $g^j(t_n^+) \supset \overline{U}_{n-1}$, then $g^{S_n}(t_n^+)$ strictly contains $g^{S_n-j}(U_{n-1})$, a contradiction with the choice of t_n^+.

The margins disjointness property implies that

$$e_n := \sum_{i=0}^{S_n} |g^i(t_n^-)||g^i(t_n^+)| \leq 4 \max_{0 \leq i \leq S_n} |g^i(T)|,$$

where $T = t_n^- \cup U_n \cup t_n^+$. Here $e_n \to 0$, because the length of the last interval $g^{S_n}(T)$ tends to zero as $n \to \infty$ (g has no wandering intervals). From the Koebe principle and the shortest interval argument, there exists a positive constant $b_0 < 1$ (depending on ℓ only) such that

$$B(\hat{U}_{n-1}, U_{n-1}) \leq b_0.$$

Now pull $\hat{U}_{n-1}$ back to T. Let $0 = j_1 < \ldots < j_r < S_n$ be the moments such that $c \in g^i(T)$. As we know, $r \leq \max(k_n^-, k_n^+)$. Write $B_i = B(g^i(T), g^i(U_n))$, $0 \leq i \leq S_n$.

Using Theorem 4.1 one can write $B_{j_r+1} \le B_{S_n} \exp(C_1 \cdot e_n) \le b_1 < 1$, where b_1 is universal (i.e., depends on ℓ only) provided n is large enough. Passing from the critical value to the critical point, $B_{j_r} \le b'_1 < 1$, where b'_1 depends only on b_1 (and on ℓ, as always). Continuing in this way, we obtain finally $B_0 = B(T, U_n) \le b'_r < 1$, where b'_r depends only on the number $r \le \max(k_n^{\pm})$. Therefore (4.2) is proved. □

Thus Propositions 1-3 of [Le] hold for smooth maps. According to Propositions 1-2 of [Le], U_m is well inside U_{m-1} whenever either U_{m-1} is the last interval in a cascade of central returns, or $g^{S_{m-3}}(c)$ and $g^{S_{m-4}}(c)$ are on opposite sides of c. (We don't need Proposition 3 since g preserves orientation. Lemma 5 of [Le] and its proof also remains unchanged.) Therefore, to complete the proof of Lemma 4.1, it is enough to have a weaker form of Lemma 4 of [Le]:

Lemma 4.5 *Assume that $g^{S_n} : U_n \to U_{n-1}$ are not central and $g^{S_n}(c) > c$ for all n. Assume also that the Standing Assumption holds. Then U_n is well inside U_{n-1} for all $n \ge 1$.*

Proof: To prove this we apply the Interval Adding Procedure. More precisely, given n, let $\hat{U}_{n-1}$ be as above. Let $T := \hat{t}^- \cup U_n \cup \hat{t}^+$ be an interval mapped onto $\hat{U}_{n-1}$ by g^{S_n} (i.e., $\hat{t}^{\pm} = \hat{t}_n^{\pm}$). We observe that because of the assumptions made in the lemma

- $\hat{t}^+$ is inside the right component of $U_{n-1} \backslash U_n$ (indeed, $g^{S_{n-2}}(U_{n-1}) \subset r(U_{n-3} \backslash U_{n-2})$ and $g^{S_n}(r(U_{n-1} \setminus U_n))$ covers $r(U_{n-3} \setminus U_{n-1})$), and, therefore $g^{S_n} : \hat{t}^+ \to \hat{U}_{n-1}$ has at most one critical point.

- $\hat{t}^-$ is inside the left component of $U_{n-1} \setminus U_n$ (because $g^{S_{n-1}}(l(U_{n-1} \setminus U_n)) \supset U_{n-1}$, i.e., g^{S_n} is not diffeomorphic on $l(U_{n-1} \setminus U_n)$). Moreover, each time an iterate of $\hat{t}^-$ passes through the critical point, more than half of this iterate of $\hat{t}^-$ lies to the right of c (as required in order to apply Lemmas 2.1 and 4.3). This follows from the construction of $t^{\pm}$: $g^S(\hat{t}^-)$ contains an end point of U_{n-k} and $\partial_l g^S(\hat{t}^-) \ge \partial_l U_{n-1}$ (see the Interval Adding Procedure).

Because of the latter observation, one can ignore that $g^i(\hat{t}^-)$ covers c from time to time, and apply the proof of (4.2) where we set $r = 2$. We therefore obtain, that $B(U_{n-1}, U_n) \le B(T, U_n) \le b'$, where $b' < 1$ depends on b_0 (and ℓ) i.e. depends finally on ℓ only. □

Thus we have proved (for the C^3-map g) Propositions 1-2 of [Le], Lemma 4.5 (instead of Lemma 4 of [Le]), and Lemma 5 of [Le]. As we have noticed it implies Lemma 4.1.

Step 2: Copying Sections 3.3-3.4 of [Ko], Lemma 4.2 also holds. Note that we can apply a theorem of Mañé (see Theorem 2.2) because $\omega(c)$ stays away from the parabolic periodic orbits of g (of which there are at most finitely many, see [MS] and Proposition 2.1).

Step 3: At last, let us prove Lemma 4 of [Le], or even stronger Lemma 3.3. We repeat its proof as follows (cf. proof of Theorem 3.1 of [Ko]). Fix an interval $T_0 = U_{n_{i_0}}$ with n_{i_0} so large that we can apply Lemma 4.2. Now, in the proof of Lemma 3.3, each time we apply g^{S_r} to the interval $T := U_r = g^{S_{n-1}+\ldots+S_{r+1}}(U_{n-1})$ (with the interval $J := g^{S_{n-1}+\ldots+S_{r+1}}(U_n)$ inside, so that $T = L \cup J \cup R$), we divide the orbit into the first M iterates and the remaining iterates. To be more precise, let $M < S_r$ be the maximal integer so that $g^{M-1}(T) \subset T_0$. Then, by Lemmas 4.2-4.3, $B(g^M(T), g^M(J)) \geq B(g(T), g(J)) \geq B(T,J)\exp(-C_1 \cdot |L| \cdot |R|)$. The iterates $g^j(T)$, $j = M, \ldots, S_r - 1$, from M to $S_r - 1$ are outside T_0 and stay away from a fixed neighbourhood of all parabolic periodic orbits of g, because $\omega(c)$ is minimal. Hence, we can use the theorem of Mañé (see 2.2) to obtain:

$$B(g^{S_r}(T), g^{S_r}(J)) \geq B(g^M(T), g^M(J)) \exp\left(-C_1 \sum_{i=M}^{S_r-1} |g^i(L)||g^i(R)|\right) \geq$$

$$B(g^M(T), g^M(J)) \exp\left(-C_2 |g^{S_r}(L)||g^{S_r}(R)|\right),$$

where the constant C_2 depends on g and the neighbourhood T_0, which is fixed. Therefore, by the Standing Assumption and Lemma 4.3, we obtain setting $k(n) = S_{n-1} + \ldots + S_N$:

$$B(g^{k(n)}(U_{n-1}), g^{k(n)}(U_n)) \geq B(U_{n-1}, U_n)$$
$$\exp\left(-C_2 \sum_{r=N}^{n} |g^{S_{n-1}+\ldots+S_{r+1}}(L)||g^{S_{n-1}+\ldots+S_{r+1}}(R)|\right).$$

The latter sum can be made arbitrarily small provided N is large enough, because the collection of intervals:

$$g^{S_{n-1}+\ldots+S_{r+1}}(U_n) \subset g^{S_{n-1}+\ldots+S_{r+1}}(U_{n-1}), r = N, \ldots., n,$$

satisfies the margins disjointness property (because $g^{S_{n-1}+\ldots+S_{r+1}}(U_n)$ lies in the right component of $U_r \setminus U_{r+1}$). Then we complete the proof as in Lemma 3.3. Thus Lemma 4 of [Le] and Lemma 3.3 are proved for smooth maps g.

Now we can proceed as in [Le] and complete the proof of Proposition 4 of [Le] and the real bounds from Theorem 2.1 for smooth maps as above.

Step 4: The only remaining real estimate we need to prove is Lemma 3.4. But under the condition and notations of the proof of Lemma 3.4, the collection of intervals

$$g^j(\hat{U}^g) \subset g^j(U^g), j = 0, \ldots, S_{n_i} - 1,$$

has the margins disjointness property, because $\hat{U}^g \subset U^g_{n_i}$, $\partial_r U^g = \partial U^g_{n_i}$ and because the intervals $g^j(U^g_{n_i}), 0 \leq j \leq S_{n_i} - 1$, are pairwise disjoint. Therefore also the inequality of Lemma 3.4 follows for smooth maps g, with a spoiling factor which tends to 1 as $n_i \to \infty$.

5 Complex bounds for real analytic maps in the minimal case: the proof of Theorem A

In this section we prove Theorem A for real analytic maps $g \in \mathbf{AC}$. This means that we have to use Lemma 2.3 instead of the Schwarz Lemma (because we only have a conformal map near the dynamical interval). Therefore we shall need that the sum of the lengths of the intervals along any branch of the first return map becomes small (because then we merely have a small loss of angles of pullbacks of the Poincaré domains). To get this, note that the real bounds from the previous section imply that $\omega(c)$ has Lebesgue measure zero (cf. [LS1] and [Ko]).

Let us go through the proofs of the results from Section 3.

First we deal with the real analytic version of Lemma 3.1.

Lemma 5.1 *Let g be real analytic as above and assume that there exist infinitely many n's for which*

$$\max\left(|\partial_l U_n - c|, |\partial_r U_n - c|\right) < (1+\epsilon)\mathrm{dist}(\partial U_{n-1}, c).$$

Then g has complex bounds.

Proof: As before assume that

$$|\partial_l U_n - c| \leq |\partial_r U_n - c| < (1+\epsilon)|\partial_l U_{n-1} - c|.$$

Let us distinguish two cases as in Lemma 3.1.

Case I. U_n is well-inside U_{n-1}. Then, as before, take $\Omega = D_*(U_{n-1})$. Let J be a domain of the first return map to U_{n-1} which intersects $\omega(c)$ and let s be its return time. We *claim* that J is well inside U_{n-1}. Indeed, if $J = U_n$ then this follows by assumption. So let us assume that J is not the central domain. $g^{s-1}\colon J^g \to U_{n-1}$ extends to a diffeomorphism $g^{s-1}\colon \hat{J}^g \to U_{n-2}$ with $\hat{J} \subset U_{n-1}$. So if U_{n-1} is well inside U_{n-2} then J is also well inside U_{n-1}. If U_{n-1} is not well inside U_{n-2}, then U_n is part of a long chain $U_{n+i} \subset \ldots \subset U_{n-k}$. But then, since $J \neq U_n$, we get that $g^{s-1}\colon J^g \to U_{n-1}$ extends to a diffeomorphism $g^{s-1}\colon \hat{J}^g \to U_{n-k-1}$ with $\hat{J} \subset U_{n-1}$. By the real bounds U_{n-1} is well inside U_{n-k-1}, and so J is well inside $\hat{J} \subset U_{n-1}$. Therefore the claim is proved in all situations. Since $|\omega(c)| = 0$, the sum of the lengths of $J, \ldots, f^{s-1}(J)$ is small when n is large. It follows from Lemma 2.3 that the pullback of $\Omega = D_*(U_{n-1})$ to J (with J either a non-central domain or equal to U_n^g) is contained in a Poincaré disc based on J with angle close to $\pi/2$ when n is large. Since U_n is well inside U_{n-1} the central domain is still contained well inside the range Ω (using the same arguments as in Case 1 of Lemma 3.1). By the claim above, a non-central domain J is well inside U_{n-1}, and so the Poincaré disc based on J with angle close to $\pi/2$ is again contained well inside Ω.

Case II. The interval U_n is not well inside U_{n-1}. Then U_n is inside a cascade of central domains. As in Lemma 3.1 define $m \geq n \geq n-k-1$ so that $S_{m+1} > S := S_m = \ldots = S_{n-k} > S_{n-k-1}$. As in Lemma 3.1 let U_* be the smallest symmetric interval containing U_m. As before, U_* is well inside U_{n-k-1}. Let $\Omega = D_*(U_*) \setminus (U_* \setminus U_m)$ and let U with $U_m \supset U \supset U_{m+1}$ be so that $g^{S_{m+1}-1}: U \to U_*$ is a homeomorphism. Since $|\omega(c)| = 0$ and the intervals $g^j(U_{m+1}), j = 0, ..., S_{m+1} - 1$ are pairwise disjoint, the sum of the lengths of these intervals is arbitrary close to zero, when n (and therefore m) is large enough. Since U_* is well inside U_{n-k-1} and has length $\leq 2|U_m| = |2 \cdot g^{S_{m+1}}(U_{m+1})|$, and since $g^{S_{m+1}-1} : U^g \to U_*$ extends to a diffeomorphism onto U_{n-k-1}, we get by the Koebe Principle that the sum of the lengths of $g^j(U), j = 0, ..., S_{m+1} - 1$, is close to zero when n and m are large. Therefore, the preimage of the disc $D_*(U_*)$ by the central branch to the critical value c_1 is contained in a set $D(U^g, \theta)$, where the angle θ is close to $\pi/2$ for n large. Since U is well inside $U_m \subset U_*$, we get that the central domain is contained well inside Ω. Let us now consider a *non-central* domain J of the return map to U_m with return time s. Let $\hat{J} \supset J$ be so that $g^s: \hat{J} \to U_*$ is a homeomorphism. The problem is that the pullback of a Euclidean disc is merely inside a Poincaré disc of angle close to $\pi/2$. Therefore we need that $\hat{J}$ is well inside U_*. This holds for all non-central domains with return time $s > S_m$, and so for these the corresponding domains in the complex plane fit inside Ω. However, unfortunately, for the non-central branch $g^{S_m} : I_m \to U_m$ this may not hold. Indeed, let $\hat{I}_m \supset I_m$ be so that $g^{S_m}: \hat{I}_m \to U_*$ is a homeomorphism. It need not be true that $\hat{I}_m$ is well inside U_*. Therefore, let us change the definition of U_* slightly. Define $U_*(\tau)$ to be the interval with boundary points $\pm(1 + \tau)\partial_r U_*$. For all $\tau \in [0, \tau_0]$ (with $\tau_0 \in (0, 1)$ some fixed small number), the interval $U_*(\tau)$ is still well inside U_{n-k-1} and there is an interval $\hat{I}_m(\tau) \ni c$ such that $g^{S-1}: \hat{I}_m(\tau)^g \to U_*(\tau)$ is a diffeomorphism. Since g has no periodic attractors or neutral periodic orbits near c (see see Proposition 2.1), we may assume that n is so large that $g^S: \hat{I}_m(\tau) \to U_*(\tau)$ has only one fixed point. This fixed point is repelling and lies in U_m.

Claim: There exists $\tau \in [0, \tau_0]$ so that $\hat{I}_m(\tau)$ is well inside $U_*(\tau)$. To prove this claim, consider $\tau_i = (i/8)\tau_0$ with $i = 1, \ldots, 8$ and let $[l_i, r_i] = \hat{I}_m(\tau_i)$. If the claim is false for τ_i then either

$$|g^S(r_i) - r_i| \leq o(n)|U_*| \text{ or } |g^S(l_i) - l_i| \leq o(n)|U_*|$$

where $o(n)$ are functions which tend to zero as $n \to \infty$. So if the claim is false for all i, then for at least four of the points $l_1, \ldots, l_8$ we have

$$|g^S(l_i) - l_i| \leq o(n)|U_*| \text{ as } n \to \infty \tag{5.1}$$

(or the same holds for four of the points r_i). Choose intervals $T \supset J$ so that $C(T, J)$ is the cross-ratio determined by these four points (say from $l_1, \ldots, l_8$). From (5.1) it follows that $C(g^S(T), g^S(J))/C(T, J)$ is close to one when n is large. But since $U_*(\tau)$ is well-inside U_{n-k-1} we get from the (real) Koebe principle that the non-linearity of $g^{S-1}: \hat{I}_m(\tau_0) \to U_*(\tau_0)$ is universally bounded. Therefore and by the definition of τ_i, $|J|$, $|l(T \setminus J)|$, $|r(T \setminus J)|$ and $|U_*|$ are all of the same order. But an explicit calculation for the map $z \mapsto z^\ell$ shows that this implies that $C(gT, gJ)/C(T, J) > 1 + \kappa$ for some universal constant $\kappa > 0$ (the intervals $J \subset T$ do not contain the critical point $c = 0$, and are not small compared to the distance to c). Since $\hat{I}_m(\tau_0) \subset U_{n-k}$ and the total length of the intervals $U_{n-k}, \ldots, f^{S-1}U_{n-k}$ tends to zero (since $|\omega(c)| = 0$), $\frac{C(g^S T, g^S J)}{C(gT, gJ)} \geq 1 - o(n)$

where $o(n) \to 0$ as $n \to \infty$. It follows that

$$\frac{C(g^S T, g^S J)}{C(T, J)} \geq 1 + \kappa - o(n).$$

But this violates (5.1). This contradiction proves the claim.

If we now define

$$\Omega(\tau) = D_*(U_*(\tau)) \setminus (U_*(\tau) \setminus U_m)$$

then the extension of all non-central domains are all contained well inside $U_*(\tau)$. Thus we obtain a quasi-polynomial-like map. □

Next we should remark that Lemmas 3.2, 3.3, 3.4 and 3.5 remain unchanged or were proved already in the previous section. So we only need to prove Lemma 3.6. To do this, simply proceed as in the proof of Lemma 5.1 (notice that in Case II we have $|V| \leq 3|U_n|$). This completes the proof of Theorem A (complex bounds).

6 Polynomial-like structure of real analytic maps in the minimal case

Given a real analytic map $g \in \mathbf{AC}$ with $\omega(c)$ minimal, we have constructed a sequence of quasi-polynomial-like mappings

$$P_k : \cup_i V_k^i \to V_k,$$

such that $\operatorname{diam}(V_k) \to 0$, the map P_k is the analytic extension from the real line of a first return map g to an interval U_{n_k} of our sequence (so that the intersection of V_k with the real axis is U_{n_k}). Quasi-polynomial-like mappings have domains which can intersect.

In this section we shall show how to construct from this polynomial-like mappings

$$P_k : \cup_i \hat{V}_k^i \to \hat{V}_k.$$

Here $\hat{V}_k^i, \hat{V}_k$ will be subsets of V_k^i, V_k, and P_k are restrictions of the quasi-polynomial-like mappings $P_k : \cup_i V_k^i \to V_k,$.

We should emphasize that the polynomial-like maps $P_k : \cup_i \hat{V}_k^i \to \hat{V}_k$ we will build here may have lost the original property of the quasi-polynomial-like maps $P_k : \cup_i V_k^i \to V_k$ that the moduli of annuli between the ranges and the central domains are uniformly bounded away from zero. In spite of this, we can conjugate the map $P_k : \cup_i \hat{V}_k^i \to \hat{V}_k$ to a polynomial, see the next Section.

The construction will go in two steps.

Step 1: Construction of a smooth polynomial-like mapping. Given the initial map $g : \cup_i I_i \to I$, one can assume that the closures of the intervals I_i are pairwise disjoint and contained in the open interval I (we can always pass to a first return map along the postcritical set, which, by the assumption that $\omega(c)$ is minimal, obeys such property). Let us choose a neighbourhood V of the interval I (with V a topological disc with analytic boundary) so close to I, that, for each $i = 0, ..., i_0$, there exists a neighbourhood U_i of the interval I_i such that for $i \neq 0$ the map $g : U_i \to V$ is a complex analytic (diffeomorphic) covering map. Choose V so that the discs U_i are disjoint. For $i = 0$ it might be impossible to extend $g|I_0$ analytically to some set U_0 in such that $g : U_0 \to V$ becomes an analytic branched covering map. (If such an extension did exist, its preimage of V would not be close to the real line.) So instead choose an analytic extension near I_0 and then extend diffeomorphically so that $g: U_0 \to V$ becomes a smooth branched covering map with critical point c. Note that U_i are not necessarily contained in V, but we can make sure that all the topological discs U_i are disjoint. Therefore, fix a topological disc A, which properly contains V and each U_i. Given i, choose a disc B_i, which contains properly U_i, is contained properly in A, and so that all B_i are pairwise disjoint. Let us artificially extend $g : U_i \to V$ to a map $\tilde{g} : B_i \to A$, which is a diffeomorphism for $i \neq 0$ and such that it is a covering map for $i = 0$. After this step, we obtain

$$\tilde{g} : \cup_i B_i \to A,$$

which is an extension of $g \cup_i U_i \to V$, such that each $\tilde{g} : B_i \to A$ is a diffeomorphism for $i \neq 0$, and $\tilde{g} : B_0 \to A$ is a covering map with a unique critical point at c. This map is the intermediate *smooth polynomial-like mapping*. Note that there is NO guarantee that $\tilde{g}$ and g coincide wherever they are both defined and that $\tilde{g}$ is not analytic. However, $g = \tilde{g}$ close to the intervals I_i.

Step 2: Intersecting the quasi-polynomial-like mapping with the smooth polynomial-like mapping. Consider the sequence of first return maps of $\tilde{g}$ along the postcritical set. Thus we obtain a sequence of smooth polynomial-like mappings

$$\tilde{g}_n : \cup_i B_i^n \to A^n,$$

such that $\tilde{g}_n$ coincides with the first return map of g to U_n on the real traces. This means that the smooth polynomial-like map $\tilde{g}_{n_k}$ coincides with the quasi-polynomial-like map P_k on the real line. We remark that one of the properties of the quasi-polynomial-like mapping P_k is that if V_k^i is one of its domains with $P_k|V_k^i = g^{s(i,k)}$ then $V_k^i, \ldots, g^{s(i,k)}(V_k^i)$ are all contained in small Poincaré neighbourhoods of the real trace of these sets. So take the component of $A^{n_k} \cap V_k$ which contains c and consider the corresponding preimages by P_k and $\tilde{g}_{n_k}$. By the above remark, $\tilde{g}_{n_k}$ coincides with P_k on these preimages. Thus we get a truly polynomial-like mapping,

$$P_k : \cup_i \hat{V}_k^i \to \hat{V}_k,$$

see also [LS1].

Note that the constructed polynomial-like maps $P_k : \cup_i \hat{V}_k^i \to \hat{V}_k$ possibly lost the property of the quasi-polynomial-like maps $P_k : \cup_i V_k^i \to V_k$, that the moduli of annuli between the ranges and the central domains are uniformly away from zero, but on the other hand we can (and will) conjugate the map $P_k : \cup_i \hat{V}_k^i \to \hat{V}_k$ with a polynomial, see next Section.

Take one of the constructed polynomial-like maps $P_k : \cup_i \hat{V}_k^i \to \hat{V}_k$. By Theorem A, it induces the quasi-polynomial-like maps $P_m : \cup_i V_m^i \to V_m$ for $m >> k$ sufficiently large (with definite bounds for the moduli of the annuli obtained by taking the difference of the range and the central domain). But as in [LS2] this is enough to obtain

Theorem 6.1 *1. Every polynomial-like map $P_k : \cup_i \hat{V}_k^i \to \hat{V}_k$ has totally disconnected Julia set.*

2. It carries no invariant line field on its Julia set.

7 Conjugation to polynomials in the minimal case

Let us fix a polynomial-like map $P : \cup_i \hat{V}^i \to \hat{V}$ from the sequence of such maps just constructed, but without any restriction on the postcritical set of the non-escaping critical point 0 except that we still require that all iterates of the critical point stay in the domain of definition of P (for example, P can have an attracting periodic orbit). All domains considered in this section will be simply connected and symmetric with respect to the real axis. We also write $U < V$, if two domains U, V like this have disjoint closures and such that the intersections of U and V with the real axis are ordered accordingly. For example, we can order the domains $\hat{V}^i$ like this. Let us relabel the domains so that in this new notation the map is:

$$P : U_{-2m^-} \cup U_{-2(m^- - 1)} \cup \ldots \cup U_0 \cup \ldots \cup U_{2(m^+ - 1)} \cup U_{2m^+} \to V,$$

where $m^{\pm}$ the number of domains to the left and right of the central domain U_0 (which contains the critical point c), and such that

$$U_{-2m^-} < U_{-2(m^- - 1)} < \ldots < U_0 < \ldots < U_{2(m^+ - 1)} < U_{2m^+}.$$

Let us add new domains $U_{-(2m^- + 1)}, U_{-(2m^- - 1)}, \ldots, U_{-1}, \ldots, U_{2m^+ - 1}, U_{2m^+ + 1}$ ordered so that

$$U_{-(2m^- + 1)} < U_{-(2m^-)} < U_{-(2m^- - 1)} < \ldots < U_{-1} < U_0 < \ldots < U_{2m^+ - 1} < U_{2m^+} < U_{2m^+ + 1}.$$

For each new domain, we choose a holomorphic isomorphism, which maps this domain onto V, such that this map is real and *reverses orientation* on the real axis.

We obtain a new polynomial-like map

$$Q : \cup_{i=-(2m^- + 1)}^{2m^+ + 1} U_i \to V$$

with a unique critical point. This critical point is $c = 0$ and is of odd order $\ell \geq 3$.

Theorem 7.1 *Q is quasiconformally conjugate to a polynomial F, in neighbourhoods of their filled-in Julia sets, where F is such that:*

- *The degree of F is equal to $d = \ell + 2(m^- + 1) + 2(m^+ + 1)$, and it is of the form $F(z) = z^\ell \cdot r(z) + C$, where $r(z) = z^{2(m^- + m^+ + 2)} + \ldots$ is a monic polynomial with real coefficients, and $C = F(0)$ is a real number.*

- *All critical points of F,*

$$x_{-(2m^-+1)} < x_{2m^-} < \ldots < x_{-1} < 0 < x_1 < \ldots < x_{2m^++1}$$

are real (and can be ordered like this), and such that

$$F(x_i) = H_- \text{ and } F(x_j) = H_+$$

for i odd and $j \neq 0$ even with $-(2m^- + 1) \leq i, j \leq 2m^+ + 1$ where $H_\pm$ are real.

- *All critical points of F except 0 escape to infinity, and, if we denote by B the Bottcher function of the basin of infinity of F (normalized so that $B(z)/z \to 1$ as $z \to \infty$), then $B(H_-) = -2$ and $B(H_+) = 2$.*

Proof: We can assume that the boundaries of the domains U_i, V are analytic curves (taking a little bit smaller disc inside V if necessary). By applying a conjugacy which maps V conformally onto a round disc $D_R = \{|z| < R\}$, for some $R > 1$, such that $R < 2 < R^d$, we can and will assume that $V = D_R$. Our map Q is defined so far on the union of U_i. Define also

$$Q(z) := z^d$$

off V. Therefore $Q(V) = D_{R^d}$. Now choose a map

$$\phi : V \setminus \cup_i \overline{U}_i \to Q(V) \setminus \overline{V},$$

such that:

1. ϕ is a smooth d-fold branched covering map (not holomorphic) of the domain $V \setminus \cup_i \overline{U}_i$ onto the standard annulus $Q(V) \setminus \overline{V}$, which extends smoothly onto the boundaries, such that it agrees with Q. That is, $\phi = Q$ on $\partial(V \setminus \cup_i \overline{U}_i)$.

2. ϕ is real on the real axis, and, moreover, $\phi(\overline{z}) = \overline{\phi(z)}$. All critical points of the cover ϕ are real (this is possible because the orientations of Q on the real traces of any two adjacent domains U_i, U_{i+1} are opposite and because $\phi = Q$ on ∂U_0).

3. Denote the critical points of ϕ by $a_{-(2m^-+1)} < a_{2m^-} < \ldots a_{-1} < a_0 < a_1 < \ldots < a_{2m^+}$, where a_i lies between U_i and U_{i+1}. Then

$$\phi(a_i) = -2, \quad \phi(a_j) = 2$$

for i odd and j even with $-(2m^- + 1) \leq i, j \leq 2m^+ + 1$.

4. ϕ is a local diffeomorphism at any point $\neq a_i$; near each critical point a_i the map ϕ is holomorphic (and non-singular).

Now we define $Q(z) = \phi(z)$, for every $z \in V \setminus \cup_i \overline{U}_i$. Thus Q is defined everywhere.

We are in a position to use the well-known trick from the Straightening Theorem [DH]. Take a standard conformal structure (i.e., the Beltrami coefficient $\mu = 0$) outside the disc $Q(V)$ and extend this structure to a function $\mu: \mathbb{C} \to \{z\,;\ |z| < 1\}$ which is invariant under Q. Since Q is conformal off V and on $\cup U^i$, there are only a bounded number of points (namely, one) in each orbit of Q where this map is not conformal. It follows that the L^∞-norm of $\mu(z)$ is bounded away from one, and by the Measurable Riemann Mapping Theorem, there exists a quasiconformal homeomorphism $H: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with $H(\infty) = \infty$ which has μ as its Beltrami coefficient. Because the structure is symmetric with respect to the real line, $H(\overline{z}) = \overline{H(z)}$. Since μ is invariant under Q, it follows that

$$F = H \circ Q \circ H^{-1}$$

is a holomorphic d-cover of the complex plane. Hence Q is quasiconformally conjugate to a polynomial map F (of degree d). By an affine conjugacy, we can assume that F is monic (it means that $H(z)/z \to 1$ at ∞), and $H(c) = 0$. Then $x_i = H(a_i)$, $i = -(2m^- + 1), ..., 2m^+$ are all simple real critical points of F. Together with $0 = h(c)$ these are the only critical points of F. The critical point $0 = H(c)$ is of order ℓ. By construction,

$$F(x_{2k+1}) = H(-2) := H_-, \quad F(x_{2k}) = H(2) := H_+.$$

H conjugates $Q(z) = z^d$ to F in the domain $\mathbb{C} \setminus V = \{|z| > R\}$, and is holomorphic there (because H preserves the standard complex structure off V). Moreover, $H(z)/z \to 1$ at ∞. Hence, the inverse map $B = H^{-1}$ coincides with the Bottcher function. That is, $B(H_-) = -2$, $B(H_+) = 2$. □

Complement to Theorem 7.1: the polynomial F can be chosen to depend continuously on the polynomial-like map P *provided the polynomial-like map P carries no invariant line fields on its filled-in Julia set $K(P)$*. (In particular, assume that P has no periodic attractor.) (Here convergence is meant in the Caratheodory topology: convergence of the domains to a kernel, and convergence of holomorphic maps uniformly on compacta, see e.g. [McM1]). Let us first observe that this implies that the polynomial F also carries no invariant line field on its filled-in Julia set $K(F)$. Indeed, the conjugacy H has to be conformal a.e. on $K(P)$ because otherwise the preimage of the standard complex structure on $K(F)$ by H gives an invariant non-standard one on $K(P)$, that is, an invariant line field. Hence H is conformal on $K(P)$, and so it takes an invariant line field of P to an invariant line field of F, and vise versa. This proves the observation. Now, let $P_n \to P$ be a sequence of polynomial-like maps tending to P. Then we can choose $Q_n: \cup_i U_i^n \to V^n$ converging to $Q: \cup_i U_i \to V$ uniformly on compact subsets of the complex plane and so that the Beltrami coefficient of Q_n on $V^n \setminus \cup_i U_i^n$ converges to the one of Q on $V \setminus \cup_i U_i$ in L^∞-norm (even in sup-norm). If H_n is the quasiconformal homeomorphism corresponding to Q_n, and $F_n = H_n \circ Q_n \circ H_n^{-1}$ is the corresponding polynomial, then the norms of the Beltrami coefficients of H_n are uniformly away from 1. Since all H_n are normalized in the same way, passing to a subsequence, $H_n \to H_*$, where H_* is quasiconformal and conformal at ∞. We get that any such subsequence F_n tends to a polynomial $F_* = H_* \circ Q \circ H_*^{-1}$ and that $H^{-1} \circ F \circ H = H_*^{-1} \circ F_* \circ H_*$. But F, F_* are monic polynomials of the same degree and H, H_* are homeomorphisms of the plane which are conformal at ∞. Then, by the last equality, the quasiconformal homeomorphism of the plane $H \circ H_*^{-1}$ (which conjugates F and F_*) is conformal off the filled-in Julia set of the polynomial F_*. If the quasiconformal homeomorphism $H \circ H_*^{-1}$ would not be not conformal a.e. on the filled-in Julia set of the polynomial F, then the complex structure, which is the image of the standard one by the map $H \circ H_*^{-1}$, is invariant under F. Hence, F carries an invariant line field on its filled-in Julia set, a contradiction. Therefore, $H \circ H_*^{-1}$ is conformal a.e.. So this map is affine and by the normalization it is equal to the identity map. Hence, from Theorem 6.1,

Corollary 7.1 *If P is a polynomial-like map constructed in the previous section with $\omega(c)$ minimal, and if $P_n \to P$ is a sequence of polynomial-like maps, then $F_n \to F$ (where F_n, F are the corresponding polynomials from the previous theorem). (The convergence in this lemma is in the Caratheodory topology).*

Proof: In the previous section we constructed polynomial-like maps P from maps g with $\omega(c)$ minimal, and showed that these do not carry invariant linefields on their filled Julia set. $\square$

8 Conjugations to expanding maps of a circle

Let $g : \cup_{i=0}^{i_0} I_i \to I$ be a real analytic map of the class we consider, but without any restriction on the postcritical set of the critical point c. Let us only assume that all iterates of the critical point stay in the domain of definition of g (for example, g can have an attracting periodic orbit). For simplicity also assume that each I_i is compactly contained in the interior of I. (This is the case for all first return maps, if we assume that $\omega(c)$ is minimal).

Let us assume for simplicity that the length of I is equal to 1. Then we can find an orientation preserving homeomorphism $G: \mathbb{R} \to \mathbb{R}$ so that $G(x+1) = G(x) + 1$ for all x and so that $G| \cup I_i = g \bmod 1$. We can do this so that each component J of $I \setminus \cup I_i$ is mapped by G affinely onto an interval of length 1. Note that G induces a piecewise smooth covering map $\hat{g}: S^1 \to S^1$. Denote by m the number of branches of the map $\hat{g}$.

Let $G_m(z) = m \cdot z \bmod 1$ be the affine m-covering of the circle $S^1 = \mathbb{R} \bmod 1$. It is well known (see e.g. [MS]), that there exists an order preserving and surjective map $h : S^1 \to S^1$, such that $h \circ \hat{g} = G_m \circ h$. This map h is unique if we insist that h maps the fixed point corresponding to $\partial I \bmod 1$ to $0 \bmod 1$. One can define h by letting $h(x)$ to be the point whose kneading invariant with respect to G_m is the same as the kneading invariant of that of x with respect to $\hat{g}$. (For example, the kneading invariant of x w.r.t. $\hat{g}$ lists for all n which component of $S^1 \setminus g^{-1}(0)$ the point $f^n(x)$ hits.) Uniqueness of h follows since the choice $h(\partial I \bmod 1) = 0 \bmod 1$ fixes the way we label these components.) Of course, h will be constant on each component of a basin of a periodic attractor of $\hat{g}$.

It is clear that h depends continuously on g. Indeed, if $\overline{g}$ is a map from our class which is close to g (in the C^0-topology), then $h_{\overline{g}}$ is close to h_g (in C^0-topology). In particular, the image $h_g(c) \in S^1$ of the critical point c of g depends continuously on g.

For any real ϵ with $|\epsilon|$ small, consider a perturbation $g_\epsilon : \cup_{i=0}^{i_0} I_i \to I^\epsilon$, where $g_\epsilon(x) = g(x) + \epsilon$, and $I^\epsilon = I + \epsilon$. Let $G_\epsilon = G + \epsilon$, and $\hat{g}_\epsilon$ be the the corresponding covering map of S^1. Then $\hat{g}_\epsilon$ has a fixed point p_ϵ depending continuously on ϵ. Without loss of generality we may assume that $I = [0,1]$ so that $p_0 = 0$. By insisting that $h_{g_\epsilon}(p_\epsilon) = 0$ (i.e. taking kneading invariants with respect to the partition $S^1 \setminus \hat{g}_\epsilon^{-1}(p_\epsilon)$ which depends continuously on ϵ when ϵ is close to zero), we have a continuous function

$$\alpha(\epsilon) = h_{g_\epsilon}(c)$$

from a closed interval $[-\epsilon_0, \epsilon_0]$ into the unit circle S^1.

Lemma 8.1 *The function $\epsilon \mapsto \alpha(\epsilon)$ is order preserving for ϵ close to zero. If the critical point of g is not in the closure of a component of the basin of a periodic attractor, then $\alpha([-\epsilon_0, \epsilon_0])$ contains an open arc of S^1.*

Proof: Take first $\epsilon > 0$ and note that

$$G_\epsilon^n(x) \geq G^n(x) + \epsilon \text{ for all } x. \tag{8.1}$$

Consider the partition generated by $g_\epsilon^{-n}(p_\epsilon)$, $n \geq 1$. Note that $p_\epsilon < p_0 = 0$ for $\epsilon > 0$ small (just look at the graphs of g_ϵ for different choices of ϵ). Fix $n > 0$ and let x_n be a solution of $g^n(x_n) = 0$. Then there exists $x_{n,\epsilon}$ depending continuously on $\epsilon > 0$ (when $\epsilon > 0$ is small) so that $g_\epsilon^n(x_{n,\epsilon}) = p_\epsilon$. We have

$$x_{n,\epsilon} < x_n \tag{8.2}$$

because otherwise (since G_ϵ is increasing and by (8.1)) $G_\epsilon^n(x_{n,\epsilon}) \geq G_\epsilon^n(x_n) > G^n(x_n) + \epsilon = p_0 + \epsilon > 0 > p_\epsilon$, a contradiction. This and (8.1) imply that

$$\epsilon \mapsto \alpha(\epsilon) = h_{g_\epsilon}(c)$$

is non-decreasing. Since g_ϵ has no wandering intervals and c is not contained in the closure of a component of the basin of a periodic attractor of $\hat{g}$, elements of the partition containing $g(c)$ shrink to zero in size. This, (8.1) and (8.2) imply that the kneading coordinates of c with respect to g and g_ϵ are not the same, i.e. $\alpha(\epsilon) > \alpha(0)$. Considering also negative ϵ, together with the continuity of α, the lemma follows. □

9 Quasiconformal rigidity in the minimal case: proof of Theorem B

Theorem 9.1 *Let g, g_1 be two real analytic maps from our class, such that the ω-limit sets of the critical points of the maps are minimal. Assume that g, g_1 are topologically* *conjugate by a homeomorphism, which maps the critical point of g to the critical point of g_1. Then some maps $G = g^{S_n}$ and $G_1 = g_1^{S_n}$ from the sequences of first return maps of g and g_1 extend analytically to polynomial-like maps. These two polynomial-like maps are quasiconformally conjugate in neighbourhoods of their Julia sets by a map which is conformal on their Julia sets.*

Proof: Provided that the quasiconformal conjugacy does exists, the latter statement (conformality of the conjugacy) follows from the fact that the polynomial-like maps induced by g and g_1 have no invariant line fields on their filled-in Julia sets. Let us prove the existence of the quasiconformal conjugacy. We can assume that the critical points of g and g_1 are at zero. As we have proved, there exists a polynomial-like extension P of some first return $g^{S_{n_1}}$ of g, and a polynomial-like extension P_1 of some first return $g_1^{S_{n_2}}$ of g_1. If, say, $S_{n_2} > S_{n_1}$, then, because the maps g, g_1 are conjugate and the conjugacy respects the critical orbits, we can take first returns to the central domains passing from P to a polynomial-like extension of $g^{S_{n_2}}$. Thus, from the beginning, we can assume that P, P_1 are polynomial-like extensions of some g^{S_n} and $g_1^{S_n}$, such that P, P_1 are topologically conjugate on the real line. Let F and F^1 be polynomials which are quasiconformally conjugate to P respectively P_1 (see Theorem 7.1). To prove the statement, it is enough to show that $F = F^1$. Define $P_\epsilon(z) = P(z) + \epsilon$ and $P^1_\epsilon(z) = P_1(z) + \epsilon$, with ϵ real and $|\epsilon|$ small. According to Lemma 8.1, there exist a sequence $\epsilon_n \to 0$ and a sequence $\epsilon'_n \to 0$, such that the orbits of the critical point 0 by the map P_{ϵ_n} and by the map $P^1_{\epsilon'_n}$ are well defined, have the same itineraries, and are pre-periodic (but not periodic). In particular, the iterates of the critical point of these maps pass through the same domains as that of P. Let q_n, q^1_n be polynomials corresponding to P_{ϵ_n} and $P^1_{\epsilon'_n}$ respectively. By Corollary 7.1, $q_n \to F$ and $q^1_n \to F^1$ as $n \to \infty$. Let us for the moment fix $q = q_n$ and $q^1 = q^1_n$. Then q, q^1 have the same degree d. The non-escaping critical point 0 of q and q^1 is preperiodic with the same itineraries with respect to the partitions on the real line given by q and q^1. Since the only non-escaping critical point of q (and q^1) is preperiodic and not periodic, the Julia set of q (and q^1) is totally disconnected [BH]. Denote by B and B^1 the Böttcher functions at infinity for q and q^1 respively. Then, by the construction of the polynomials q, q^1, the map $H = (B^1)^{-1} \circ B$ is conformal in the domain $\{z : \log |B(z)| > (\log 2)/d\}$. Since the itineraries coincide we get, step by step, that the conjugacy H extends conformally onto domains $\{z : \log |B(z)| > (\log 2)/d^n\}$, $n = 1, 2, \ldots.$ Hence it extends up to the Julia set $J(q)$ of q, and then homeomorphically to the Julia set $J(q)$ (because the Julia sets are totally disconnected). It follows that there exists a quasiconformal homeomorphism h of the plane, which is homotopic to H and coincides with H in the domain $\{z : \log |B(z)| > (\log 2)/d^m\}$, for some sufficiently large (but fixed) m, and on the postcritical set of q (which is finite). Then, by the pullback argument, there exists a quasiconformal conjugacy between q and q^1 which coincides with H outside $J(q)$ (and hence is conformal outside $J(q)$). Since the extension to $J(q)$ is unique, all this implies that H is quasiconformal. On the other hand, $J(q)$ has Lebesgue measure zero (q is geometrically finite). Therefore, H is conformal everywhere, i.e., affine. Because of the normalization at infinity, H is the identity. This means that $q = q^1$. Remembering that q and q^1 can be chosen arbitrary close to F and F^1 respectively, we get that $F = F^1$. $\square$

10 Theorem A': Complex bounds in the non-minimal case

Let $a_\pm$ be the right and left most fixed points of g. In addition, let a_i be the boundary points of immediate basins of periodic attractors. Because g is real analytic, g has only finitely many periodic attractors (see [MS]), and there are only finitely many such points $a_1, \ldots, a_s$. Every point $a_\pm, a_1, \ldots, a_s$ is either repelling or parabolic periodic. Now define $A = \{a_-, a_+, a_1, \ldots, a_s\} \cap \omega(c)$ and if this is empty let A be any repelling periodic orbit of g. (One could also use different partitions, see for example the construction described in the proof of Proposition 10.1.) Let $\mathcal{P}_n$ (or P_n^a) be the components of the complement of $g^{-n}A$. This forms a natural dynamical partition of the real line. Since $g(A) \subset A$, any component of $\mathcal{P}_{n+1}$ is contained in one and only one component of $\mathcal{P}_n$. Let $P_n(t)$ be the element of the partition which contains t (this is well-defined if t is not eventually mapped into A; if it is, take the interval whose interior is *not* contained in the basin of a periodic attractor and also does not intersect a gap between two domains of g). If two domains of g have a common end point then there is a choice, but then it will be clear from the context which interval $P_n(t)$ one should take. Since g has no wandering intervals, for each $t \in \omega(c)$, the diameter of $P_n(t)$ tends to zero as $n \to \infty$.

In this section we shall prove the following theorem:

Theorem A' (complex bounds in the recurrent non-minimal case). *Let $g \in \mathbf{AC}$ and assume that $\omega(c)$ is recurrent and non-minimal. Then g induces a Box Mapping: there exists a sequence $C_k \to \infty$, θ close to zero, and a sequence of open topological discs $V_k := D(V_k \cap \mathbb{R}, \pi - \theta)$ around the critical point such that the following properties hold.*

1. *diam$(V_k) \to 0$ and $V_k \cap \mathbb{R}$ is a critical piece of the above partition.*
2. *the (complex analytic) first return map to V_k restricted to domains containing points of $\omega(c)$ is a box mapping $F_k : \cup_i V_k^i \to V_k$ (see the definition below the statement of the theorem).*
3. *the modulus of the annulus $V_k \setminus V_k^i$ is bounded from below by C_k.*

   Passing to the first return map to the central domain V_k^0, we then obtain again a box mapping $F_k : \cup_i V_k^i \to V_k$ (where this time V_k is the central domain V_k^0 and were the domains V_k^i are the new domains) with all listed properties, and additionally such that:

4. *each branch $F_k|V_k^i$ extends to a univalent map $F_k|W_k^i$ for $i \neq 0$ (and for $i = 0$ to an analytic covering map which is branched at c), such that the modulus of $W_k^i \setminus V_k^i$ is at least C_k, and such that for $i \neq j$ either W_k^i is contained in $W_k^j \setminus V_k^j$ or the other way around W_k^j is contained in $W_k^i \setminus V_k^i$ (or W_k^i and W_k^j are disjoint). Moreover, W_k^i is contained in V_k for each i.*

Here we say that $F_k\colon \cup_i V_k^i \to V_k$ is a *box mapping* (following terminology of [GS2]) if the countably many domains V_k^i, V_k^j with $i \neq j$ are pairwise disjoint, V_k^i is contained in V_k for all i and if $F_k|V_k^i \to V_k$ is a diffeomorphism for $i \neq 0$ while $F_k|V_k^0 \to V_k$ is a ℓ-branched covering.

It is very easy to prove an analogous theorem in the non-recurrent case, see [Str] and [MS]. In that case all branches are diffeomorphic (i.e. V_k^0 simply does not exist).

Proof: The proof uses ideas of Proposition 3.3 from [LS1]. Assume that $\omega(c)$ is recurrent and non-minimal. For each integer i one has $|P_n(g^i(c))| \to 0$ as $n \to \infty$. First, let us show the existence of a sequence of (real) critical pieces $P_{n_k}(c)$, such that for the (real) central component U_k of the first return map to $P_{n_k}(c)$, we have: $C^{-1}(P_{n_k}(c), U_k) \to \infty$ as $k \to \infty$. Since $\omega(c)$ is non-minimal, there exists $x \in \omega(c)$ such that forward iterates of x avoid some neighbourhood of the critical point. Therefore, there exists N such that for an infinite sequence of k, then we choose the interval $P_k(x) \in \mathcal{P}_k$ so that it is mapped by g^{k-N} diffeomorphically onto some fixed interval $P := P_N \in \mathcal{P}_N$ contained in one of the domains of g and such that $g^i(P_k(x))$ does not contain c for $i = 0, 1, \ldots, k-N-1$. Considering the first entry of c into $P_k(x)$, we then obtain a sequence of intervals $P_{n_k}(c)$ with diameters shrinking to zero and integers $n_k \to \infty$ so that $g^{n_k - N}\colon g(P_{n_k}(c)) \to P$ is a diffeomorphism. Fix k for a moment. By the construction, the iterates $g(c), \ldots, g^{n_k-N-1}(c)$ are outside $P_{n_k}(c)$. Let $s > n_k - N - 1$ be the smallest integer for which $g^s(c) \in P_{n_k}(c)$. Let U_k be the pullback of $P_{n_k}(c)$ from $g^s(c)$ to c.

Claim 1: The interval $g^{n_k-N}(U_k)$ is very deep inside P, i.e., $C^{-1}(P, g^{n_k-N}(U_k)) \to \infty$ as $k \to \infty$.
Proof of claim 1: The length of $g^s(U_k) = P_{n_k}(c)$ tends to zero as $k \to \infty$ and therefore the length of $g^{n_k-N}(U_k)$ tends to zero as $k \to \infty$ (here we use that $n_k - N < s$ and that g has no wandering intervals). If ∂P is a preimage of a repelling periodic point, then we can go to big scale with bounded distortion, and so if $g^{n_k-N}(U_k)$ is close to the boundary of P then $g^{n_k-N}(U_k)$ is very deep inside P (if it is not close to one of the boundary points there is nothing to prove).

But in fact, we do not even need to use that ∂P is in a hyperbolic repelling set. Indeed, let ∂P be a preimage of the periodic point a, which is parabolic, and assume that again $g^{n_k-N}(U_k)$ is close to a boundary point x of P, but the distance between $g^{n_k-N}(U_k)$ and x is comparable to the length of $g^{n_k-N}(U_k)$. Then, when one end point of $g^{n_k-N}(U_k)$ goes to a fixed (large) distance from a by an iterate g^r, then the length of $g^r(g^{n_k-N}(U_k))$ is also large (i.e., independent of k). (We see this in a standard local coordinate at the parabolic point.) But this is impossible, because $g^s(U_k)$ tends to zero.

Claim 2: $C^{-1}(P_k(x), g^{n_k-k}(U_k))$ tends to infinity as $k \to \infty$.
Proof of Claim 2: The proof of this is essentially the same as that of Claim 1. Indeed, g^{k-N} maps $P_k(x)$ diffeomorphically onto $P \in \mathcal{P}_N$. If $\omega(x)$ does not contain parabolic periodic points, then x is contained in a hyperbolic set because of Mañé's theorem 2.2. So in this case g^{k-N} has bounded distortion on $P_k(x)$ and the proof follows immediately

from Claim 1. If $\omega(x)$ contains a parabolic periodic point, then we can assume that x is one of these (finitely many) parabolic periodic points, and g^{n_k-k} simply maps in interval $P_k(x)$ periodically along the parabolic periodic orbit $O(x)$ to the big interval P (where P also contains a point from $O(x)$ as a boundary point by the construction of the partition). By the previous claim, both components of $P \setminus g^{n_k-k}(U_k)$ contain a very large number of adjacent fundamental domains of the parabolic periodic orbit $O(x)$. But then the same holds for the components of $P_k(x) \setminus g^{n_k-k}(U_k)$. Since neighbouring fundamental domains of a parabolic periodic point have approximately the same size, it follows that $g^{n_k-k}(U_k)$ is very deep in $P_k(x)$. This concludes the proof of Claim 2.

The intervals $g^i(P_{n_k}(c)), 0 \leq i \leq n_k - k$, are disjoint, because $g^{n_k-k}(P_{n_k}(x))$ is the first visit of some iterate of $P_{n_k}(c)$ to $P_k(x)$. Hence we can apply the Koebe Principle to estimate the cross-ratio and obtain $C^{-1}(P_{n_k}(c), U_k)$ tends to ∞ as $k \to \infty$. To prove the theorem, fix now k and consider the first return map to $P_{n_k} := P_{n_k}(c)$. Then each (and not only the central) component of this map is very deep inside P_{n_k} (we apply the margins disjointness property (4.1) this time which holds because the components of the first return map are disjoint). Define $V_k = D(P_{n_k}; \pi - \theta)$, where θ is small enough, and consider the pullback of V_k along some branch of the latter first return. Because we consider first returns, the sum of the lengths of the preimages of P_{n_k} along this pullback is uniformly bounded (by the length of the range of g). Hence choosing θ small enough, we obtain by Lemma 2.3, that the pullback is well-defined and is contained deep inside V_k. Disjointness of the domains holds because by Lemma 2.3 the g-images of all the domains (which are near $g(c)$) are contained in sets of the form $D(g(V_k^i) \cap \mathbb{R}; \pi - \theta')$ with $\theta' > 0$ small. This implies the first three assertions of the theorem.

In property 4 the range of the extension can be taken to be the range of the previous map. $\square$

It will be useful later on to modify the range of the polynomial-like map constructed in Theorem A'. The reason for this is that later on we will consider complex perturbations of the map. Therefore we will want to modify the range of the polynomial-like map in a more dynamical way (to deal with complex eigenvalues), so that nearby maps still have an associated polynomial-like map.

Addendum to Theorem A': Assume that $\omega(c)$ contains no parabolic periodic points; in particular, assume that the periodic points in the set A are hyperbolic. Then, given k, we can modify the range $V = V_k = D(P_{n_k}; \pi - \theta)$ in the following way (imitating in some sense external rays to periodic points in the standard Yoccoz puzzle construction). Let us do this construction in the case that $a \in A$ is a fixed point (the periodic case goes in the same way).

1. Denote $\sigma = g'(a)$. Then $\sigma > 1$ and the branch g^{-1} (so that $g^{-1}(a) = a$) is conformally conjugate in a fixed neighbourhood Z of a to the map $w \mapsto \sigma^{-1}w$. It follows that, given a unit vector v at the point a there exists a unique (analytic) curve l_v in Z which starts at the point a and so that v is the tangent vector of l_v at a, such that $g^{-1}(l_v) \subset l_v$. It gives a foliation of $Z \setminus \{a\}$.

2. Fix two neighbourhoods Z_1, Z_2 of the end points a_1, a_2 of the real trace of V,

such that $g^M : Z_i \to Z$, $i = 1, 2$, are well-defined and injective. For $b_i \in \partial V \cap Z_i$ let l_{v_i} be the leaf of the foliation which contains $g^M(b_i)$. If we choose point $b_i \in \partial V \cap Z_i$ sufficiently close to a_i, then the vector v_i is close to one of the two (complex conjugate) tangent vectors of $g^M(\partial V \cap Z_i)$ at a and also the end point of l_{v_i} (other than a) is non-real.

3. Replace the small arc in ∂V connecting b_i and a_i by $g^{-M}(l_v)$ (and do the same for its complex conjugate). We obtain in this way a boundary of some domain $\tilde{V}$

Then the domain $\tilde{V}$ can be chosen instead of V in the construction of the box mapping. Indeed, $\partial\tilde{V}$ is contained between two domains $D(P_{n_k}; \pi - \theta_-)$ and $D(P_{n_k}; \pi - \theta_+)$, where $\theta_\pm$ are close to θ. Hence, all the pullbacks are well-defined and form a box mapping just as in Theorem A'.

We shall need also the following.

Proposition 10.1 (Construction of Cantor repeller) *Assume that g is as in the statement of Theorem A' and in addition assume that $\omega(c)$ contains no parabolic periodic points. Fix a critical piece $P := P^a_m(c)$ from the partition defined above, and let $F\colon \cup_i P^i \to P$ be the restriction of the first return map to P to domains containing points of $\omega(c) \cap P$. Let X be the closure of the end points of P^i. Then there exists a hyperbolic Cantor set $\hat{X} \supset X$ which is contained in the set of non-escaping points (a Cantor repeller) of some 'polynomial-like mapping'*

$$G : \cup B(J) \to \cup D(I; \pi - \beta),$$

(without critical points and with a finite number of different ranges)

Proof: Here P^i are (some of the) pieces of the partition $\mathcal{P}^a$ inside P, and, as usual, P^0 is the critical one. Observe that the forward iterates (by g) of the points $x \in X$ avoid a neighbourhood U of the parabolic cycles of g, because $\omega(c)$ is a closed set not containing such cycles, P^i are pieces covering $\omega(c) \cap P$, and ∂P^i are pre-images of points $a \in A$. Define $\hat{X}$ to be the set of points y from the domain of g such that all iterates $g^j(y)$, $j \geq 0$ are well-defined and lie outside the piece P^0 and outside the neighbourhood U of the parabolic cycles and outside the basins of attraction. Note that $X \subset \hat{X}$.

Let us now choose a repelling periodic point b inside the central piece P^0 and let $B = O(b)$. We would like to construct a partition using the backward orbit of B. However, preimages of B do not (necessarily) accumulate from both sides to points from A. There are several ways to remedy this. For example, consider the extension $\overline{g}\colon S^1 \to S^1$ of g from assumption 4 in the definition of the class **AC**. Modify g possibly even further on neighbourhoods of periodic attractors (and parabolic points) by gluing-in additional branches inside immediate basins. By choosing the derivative large, this can be done so that for the new map $\tilde{g}\colon S^1 \to S^1$ the points from the set A are no longer in the basins of periodic attractors. Then simply let $\mathcal{P}^b_n$ be the partition generated by $\tilde{g}^{-n}(B)$. It is absolutely crucial to observe that we shall actually only consider elements of the partition on which the C^∞ map $\tilde{g}$ coincides with the real analytic map g. We do this by choosing n_0 so large that $\tilde{g}$ coincides with g on each element of the partition $\mathcal{P}^b_{n_0}$ containing points of the set $\hat{X}$. Fix n_0 also so large that the critical piece $P^b_{n_0}(c)$ of this partition is completely inside the critical piece P^0 of the previous partition (determined by the set A).

One can assume also that the neighbourhood U of the parabolic cycles of g consists of finitely many pieces of $\mathcal{P}^b_{n_0}$. The set $\hat{X}$ is compact (because g has finitely many domains of definition) and forward invariant under g and a positive distance away from the end points of the pieces of $\mathcal{P}^b_{n_0}$ (since these end points hit $b \in P^0$ eventually). For every $n > n_0$, consider the elements of the partition $\mathcal{P}^b_n$ which intersect $\hat{X}$. Denote these intervals by J^k_n. Note that for n large, the remaining (finitely many) pieces have a definite distance to the boundary of the pieces of the finite partition $\mathcal{P}^b_{n_0}$. Let us choose a sufficiently small angle $\beta > 0$, and consider Poincaré neighbourhoods $D(I^i; \pi - \beta)$, where I^i are the elements of the partition $\mathcal{P}^b_{n_0}$ except for the critical piece $P^b_{n_0}(c)$ of this partition. Take $n > n_0$ and any interval $J = J^k_n$. Then $g^{n-n_0}(J)$ is equal to some interval $I^{i(J)}$ (we write $I^{i(J)}$ to emphasize that the interval depends on J), and moreover the iterates $g^j(J)$, $j = 0, 1, \dots, n - n_0$ are all disjoint from the piece $P^b_{n_0}(c)$ and from the neighbourhood U of the parabolic points. Hence, by Mañé's theorem 2.2, the sum of the lengths of these intervals is bounded by a universal constant, and, as we know (for any n large and β small fixed), there exists a domain $B(J)$, such that the map $g^{n-n_0} : B(J) \to D(I^{i(J)}; \pi - \beta)$ is well-defined, $B(J)$ is inside $D(J; \pi - C\beta)$, where $C > 0$ is some universal constant. The closed set $\hat{X}$ is contained in the (finite) union of the interiors of I^i. It follows that as $n \to \infty$, each of the intervals J (the partition elements of $\mathcal{P}^b_n$ intersecting $\hat{X}$) becomes deeper and deeper inside the (appropriate) interval I^i. Therefore, provided n is large enough, the domains $B(J)$ are well inside the domains $D(I^i; \pi - \beta)$ and pairwise disjoint.

Thus we end up with a map

$$G : \cup B(J) \to \cup D(I; \pi - \beta),$$

where J runs over the finitely many intervals J^k_n (from $\mathcal{P}^b_n$) and I runs over the finitely many intervals I^i (i.e, every piece of $\mathcal{P}^b_{n_0}$ except the central piece one). If two domains $B(J)$ touch then we shrink $D(I; \pi - \beta)$ slightly, and then the new domains $B(J)$ will no longer touch. The above map is a polynomial-like map (without critical points and with a finite number of different ranges).

Note that $X \subset \hat{X}$, and that points from $\hat{X}$ never leave the domain of definition of G under iterates of G. $\square$

11 Proof of Theorem B': quasisymmetric rigidity of maps in the non-recurrent and the non-minimal case

Theorem B' (rigidity)

Consider two maps of the class **AC** *with critical points of the same order, such that for both maps c is not attracted to an attracting or parabolic periodic orbit, $\omega(c)$ is non-minimal or non-recurrent, and contain no parabolic periodic points. If the conjugacy on $\omega(c)$ maps the critical point of one map to the critical point of the other map, then the conjugacy is quasisymmetric on $\omega(c)$.*

Proof: In the non-recurrent case, this is an exercise from Section III.6.2 of [MS]. So let us assume that we are in the recurrent non-minimal case. Fix maps $g, \tilde{g}$ as in Theorem B'. Denote the box mapping from Theorem A' by

$$F : \cup_{i=0}^{\infty}\Omega_i \to \Omega \text{ and } \tilde{F} : \cup_{i=0}^{\infty}\tilde{\Omega}_i \to \tilde{\Omega}.$$

By assumption, F and $\tilde{F}$ are conjugate on $\omega(c)\cap\Omega$, such that the conjugacy maps the critical point of F to that of $\tilde{F}$. Hence, one can choose the generating real partitions in the construction of the box mappings (see Section 10) in such a way that $F, \tilde{F}$ are the same iterate of $g, \tilde{g}$ on corresponding components. In what follows, $\tilde{\Omega}_i$ denotes the component of $\tilde{F}$ which corresponds to the component Ω_i of F. Let us try to extend the conjugacy on $\omega(c)$ to a quasiconformal conjugacy between F and $\tilde{F}$, as in [LS1]. The problem is that in this case we do not apriori have a natural external conjugacy, i.e., a quasiconformal map from

$$A := \text{closure}(\Omega \setminus \cup_{i=0}^{\infty}\Omega_i)$$

onto

$$\tilde{A} := \text{closure}(\tilde{\Omega} \setminus \cup_{i=0}^{\infty}\tilde{\Omega}_i),$$

which respects the dynamics of F and $\tilde{F}$ on the boundaries. (We can assume that F and $\tilde{F}$ are defined dynamically on the closures of Ω_i and $\tilde{\Omega}_i$.) In the case of polynomials or (generalized) polynomial-like mappings such external conjugacy always exists: it is a composition of the Böttcher functions at the basins of infinity or external maps [DH]. Then all steps of the proof below are greatly simplified. In the general case we will proceed as follows.

In Steps 1-2 we construct a quasiconformal external conjugacy $H : A \to \tilde{A}$. In Step 3 we construct extensions of $H\colon A \to \tilde{A}$ using a pullback construction and describe some of the geometric structure involved. Finally, in Step 4 we consider the dynamical extension of H to a map $\cup_{n=0}^{\infty}F^{-n}(A) \to \cup_{n=0}^{\infty}\tilde{F}^{-n}(\tilde{A})$ (this is trivial, just pullback dynamically), and then show that this map can be extended quasiconformally to Ω as a conjugacy.

The box mapping F is one of the box mappings F_k of Theorem A'. We will fix k and therefore drop k in the notation. In particular, we denote by $W_i = W_k^i$ the neighbourhood of $\Omega_i = \Omega_k^i$, and by $C = C_k$ the constant from property (4) of Theorem

A'. To simplify the picture, we choose the map F so that the constant C is so large that every annulus $W_i \setminus \Omega_i$ contains a standard round annulus surrounding Ω_i with the modulus at least $C/2$. Similar for $\tilde{F}$.

Given a set Y of the plane, denote $Y^+ = \{z \in Y; \Re(z) \geq 0\}$ and $Y^- = \{z \in Y; \Re(z) \leq 0\}$.

Step 1: In this step, we shall show that the boundary of each of the sets $A^\pm$ (and similarly for $\tilde{A}$) is a quasi-circle using the Ahlfors-Beurling criterion [LeVi]. Pick any two points z_1, z_2 in, say, $\partial(A^+)$, and consider one of the two subarcs $C = C(z_1, z_2)$ of $\partial(A^+)$ between z_1 and z_2 of a smaller diameter. According to the Ahlfors-Beurling criterion, we need to prove the existence of a constant L independent of z_1, z_2 such that C lies inside a circle of diameter $L \cdot |z_1 - z_2|$. Let us consider several possibilities for z_1 and z_2. (a) If z_1, z_2 are both in the boundary of either Ω or some Ω_i, such L exists (which does not depend on Ω_i). Indeed, L exists for Ω, because $\partial\Omega$ is a piecewise analytic curve without turning points. For the same reason, L exists for the central component Ω_0. Any other Ω_i is an image of Ω under a branch of F^{-1}, which is univalent in a fixed neighbourhood W of Ω. Thus L is independent on i. (b) If z_1, z_2 are in the boundaries of different Ω_i, Ω_j, then we notice that from Theorem A' one gets that the diameter of these topological discs is comparable to their real trace, that the discs are roughly round (cf. Fact 11.3 (1d) below). This and the case (a) imply that one has a uniform choice for L in this case. (c) These considerations also settle the case when one or both points z_1, z_2 are real, or belong to $\partial\Omega$.

Thus $\partial(A^\pm)$ and similarly $\partial(\tilde{A}^\pm)$ are quasi-circles. So there exist K-quasiconformal homeomorphisms $\phi^\pm : D \to A^\pm$, $\tilde{\phi}^\pm : D \to \tilde{A}^\pm$, where D stands for the unit disc. We choose $\phi^\pm$ to be so that $\overline{\phi^+(z)} = \phi^-(\overline{z})$ (and similarly for $\tilde{\phi}$).

Step 2: In this step we show that there exists a quasiconformal conjugacy $H : A \to \tilde{A}$.

Note that the boundary $\partial(A^+)$ of A^+ consists of the set $(\partial A)^+$ and some (possibly empty) open subset O of $\overline{\Omega} \cap \mathbb{R}$ (for example, O can contain an open set of points which are attracted by an attracting periodic orbit of g).

We are going to construct the map $H : \partial A \to \partial\tilde{A}$ to be symmetric (i.e. $\overline{H(z)} = H(\overline{z})$) and so that $H|(\partial A)^+$ is a restriction of some map $H^+ : \partial(A^+) \to \partial(\tilde{A})^+$ to the subset $(\partial A)^+$ of $\partial(A^+)$. Thus we need to construct H^+ (such that $H^+(x)$ is real for x real). Then $H^- : \partial(A^-) \to \partial(\tilde{A}^-)$ is defined symmetrically: $H^-(\overline{z}) = \overline{H^+(z)}$, so that $H(z) = H^+(z)$ iff $\Re(z) \geq 0$ and $H(z) = H^-(z)$ iff $\Re(z) \leq 0$.

First define $H^+ : (\partial\Omega)^+ \to (\partial\tilde{\Omega})^+$ as follows. For both endpoints y of $\Omega \cap \mathbb{R}$ fix a neighbourhood Y such that some iterate g^N maps Y univalently onto the neighbourhood Z of the fixed point a of g (see Addendum to Theorem A'). Similar for $\tilde{F}$: $\tilde{g}^N : \tilde{Y} \to \tilde{Z}$. By the construction of Ω, $g^N((\partial\Omega \cap Y)^+) = l_v^+$ and $\tilde{g}^N((\partial\tilde{\Omega} \cap Y)^+) = \tilde{l}_v^+$, where $l_v, \tilde{l}_v$ are invariant arcs from Addendum to Theorem A'. Now fix a quasiconformal symmetric conjugacy K between $g|Z$ and $\tilde{g}|\tilde{Z}$, such that $K(l_v) = \tilde{l}_v$. Then define H^+ on the arc $g^{-N}(l_v^+) = (\partial\Omega \cap Y)^+$ of $\partial\Omega$ as $H^+ = \tilde{g}^{-N} \circ K \circ g^N$. Extend H^+ to the remainder of $(\partial\Omega)^+$ so that it distorts (Euclidean) distances by at most a uniformly bounded factor.

Next, define H^+ on each $(\partial\Omega_i)^+$ dynamically as follows. For $z \in (\partial\Omega_i)^+$, let

$H^+(z) = \tilde{F}^{-1}_{|\tilde{\Omega}_i} \circ H^{\pm} \circ F_{|\Omega_i}(z)$, where $H^{\pm} = H^+$ iff $F(z) \in (\partial\Omega)^+$ and $H^{\pm} = H^-$ iff $F(z) \in (\partial\Omega)^-$. It is clear also which branch of $\tilde{F}^{-1}_{\tilde{\Omega}_0}$ to take on the central domain. If n is such that $g^n(\Omega_i) = \Omega$, then extend $H^+_{|g^k(\partial\Omega_i)^+} = \tilde{g}^{-(n-k)} \circ H^+_{(\partial\Omega)^+} \circ g^{n-k}$, $k = 1, ..., n-1$.

Finally, the remainder of $(\partial A)^+$ is the set $X \cup O$ on the real line, where X is the Cantor set introduced in Proposition 10.1. Remember that $X \subset \hat{X}$, where $\hat{X}$ is the Cantor set on the real axis invariant under g and is contained in the set of non-escaping points of the expanding map $G : \cup B(J) \to \cup D(I; \pi - \beta)$ (without critical points and with a finite number of different ranges). Let $\tilde{X}$, $\hat{\tilde{X}}$, $\tilde{G} : \cup B(\tilde{J}) \to \cup D(\tilde{I}; \pi - \beta)$ be the corresponding sets and the map of $\tilde{F}$ chosen so that G and $\tilde{G}$ are combinatorially conjugate (it is done by constructing G and $\tilde{G}$ as in the proof of Proposition 10.1 simultaniously for F and $\tilde{F}$ in a combinatorially the same way). Then $G, \tilde{G}$ are quasiconformally conjugate: there exists a quasiconformal symmetric map $\tau : \cup D(I; \pi - \beta) \to \cup D(\tilde{I}; \pi - \beta)$ such that $\tau \circ G = \tilde{G} \circ \tau$ on $\cup B(J)$. In particular $\tau(X) = \tilde{X}$. One can assume that τ is extended to a quasiconformal map of the plane. Define finally $H^+ = \tau$ on $X \cup O$. This completes the definition of $H^+ : \partial(A^+) \to \partial(\tilde{A}^+)$.

Let us now show that the map $(\tilde{\phi}^+)^{-1} \circ H^+ \circ \phi^+ : \partial D \to \partial D$ is quasisymmetric. In order to do this, take three nearby points $z_1, z_2, z_3 \in \partial D$ with z_3 the midpoint between z_1 and z_2. Then the points $w_i = \phi^+(z_i) \in \partial(A^+)$, $i = 1, 2, 3$, are roughly equidistant because ϕ^+ is quasiconformal. ("Roughly equidistant" means that the distances between the points are roughly the same; we say that numbers a, b are roughly the same: $a \asymp b$ (resp. a is much bigger than b) if $L^{-1} \le a/b \le L$ (resp. $a/b \ge l$), for some universal $L > 1$).

We need to show that the points $\tilde{w}_i = H^+(w_i), i = 1, 2, 3$, are roughly equidistant as well.

Assume w_1, w_2, w_3 all lie on the real line or all belong to the boundary of either Ω or of some Ω_k. Then we are done because H^+ is quasisymmetric on the real line, distorts distances by a bounded factor on $\partial\Omega$, and the map $F : \Omega_i \to \Omega$ has definite Koebe space.

Hence, we can assume that at least one and at most two of the points w_i lie in one of the domains $\partial\Omega_k$. Denote by $B_i, i = 1, 2, 3$, the domain Ω_j of F for which $w_i \in \partial B_i$ (for example, one of the sets B_i is Ω_k; if w_i is outside all the domains Ω_j, then $w_i \in \mathbb{R}$ and we take B_i to be the point w_i). Let $U_i(x)$ be a neighbourhood of an endpoint x of $B_i \cap \mathbb{R}$, such that $F(U_i(x)) = Y$, where Y is the above mentioned neighbourhood of one of the endpoints of $\Omega \cap \mathbb{R}$ from the domain of definition of $H^+_{|(\partial\Omega)^+}$. There are two main cases to distinguish.

(a) all points w_i lie in $U_j(x)$, for some j between 1 and 3. Then pass to big scale. The map $g^N \circ F = g^{N_1} : U_j(x) \to Z$ has bounded distortion, and Z is the neighbourhood of the repelling fixed point a of g. Then the point $z_j = g^{N_1}(w_j)$ lies in the arc l_v^+. So we can iterate the map g (keeping bounded distortion) until the distances of images of w_i become larger than a universal positive number. (We use that there is a universal $\epsilon > 0$ such that the Euclidean distance between any two components Ω_i, Ω_j is larger than $\epsilon \cdot \{$minimal diameter of $\Omega_i, \Omega_j\}$, see Fact 11.3 for detail; the same holds for the images of Ω_i, Ω_j until one of the images is Ω. Hence, $g^{N_1}(B_i), i \ne j$, lie in a $(1 + \epsilon)$-

neighbourhood of Z, i.e. still close to the fixed point a). Hence we can transport all this by the map K, and then by an inverse to $\tilde{g}^{N_1}$, and get again that the property that these three points are roughly equidistant is preserved.

(b) for every $j = 1, 2, 3$, at least one of the points w_1, w_2, w_3 is outside the neighbourhood $U_j(x)$. To avoid to deal with many scales and cases (diameters of B_1, B_2, B_3 and mutual distances between B_i), we do the following. Remember that $H^+_{|\partial(A^+)\cap\mathbb{R}}$ is the restriction of the quasiconformal map τ of the plane. Hence, the points $\tilde{w}'_i = \tau(w_i), i = 1, 2, 3$, are roughly equidistant, and we finish the proof by showing that for every i, j, the distances $\tilde{l} = |\tilde{w}_i - \tilde{w}_j|$ and $\tilde{l}' = |\tilde{w}'_i - \tilde{w}'_j|$ are roughly the same: $\tilde{l} \asymp \tilde{l}'$. Let $\tilde{B}_i$ be the component of $\tilde{F}$ with $\partial\tilde{B}_i = H^+(\partial B_i)$, and $\tilde{B}'_i = \tau(B_i)$. Denote also $d_i, \tilde{d}_i, \tilde{d}'_i$ the diameters of $B_i, \tilde{B}_i, \tilde{B}'_i$, and $D, \tilde{D}, \tilde{D}'$ the distances between B_i and B_j, $\tilde{B}_i$ and $\tilde{B}_j$, $\tilde{B}'_i$ and $\tilde{B}'_j$ respectively. Consider three simple subcases. (1) if $\max(d_i, d_j) \asymp D$, then all numbers $\max(\tilde{d}_i, \tilde{d}_j)$, $\tilde{D}$, $\max(\tilde{d}'_i, \tilde{d}'_j)$, $\tilde{D}'$ are roughly the same (because the real traces of $B_i, \tilde{B}_i, \tilde{B}'_i$ are comparable with their diameters, and the map τ is quasisymmetric on $\mathbb{R}$). Therefore $\tilde{l}$, $\tilde{l}'$ are roughly equal to D; (2) if d_i and d_j are much smaller than D, the same holds for $\tilde{d}_i, \tilde{d}_j$ and $\tilde{D}$, and for $\tilde{d}'_i, \tilde{d}'_j$ and $\tilde{D}'$. But $\tilde{D} \asymp \tilde{D}'$, hence, $\tilde{l}, \tilde{l}'$ are roughly $\tilde{D}$; (3) d_i is much bigger than d_j and than D. Let x be an endpoint of $B_i \cap \mathbb{R}$ closer to B_j. The neighbourhood $U_i(x) = F^{-1}_{|B_i}(Y)$ is roughly a round disc, and the diameters of $U_i(x)$ and B_i are roughly the same. Hence, there exists a round disc $U'(x)$ inside $U_i(x)$ and centered at x, with radius roughly d_i, such that $w_i \notin U'(x)$, because otherwise all points w_1, w_2, w_3 would be in $U_i(x)$. Hence $l = |w_i - w_j| \asymp d_i$. Also, $\tilde{w}'_i \notin \tau(U'(x))$, hence $\tilde{l}' \asymp \tilde{d}'_i$. Moreover, since $\tilde{w}_i = \tilde{F}^{-1} \circ H^+_{|(\partial\Omega)^+} \circ F(w_i)$, the point $\tilde{w}_i$ is outside a definite neighbourhood of $\tilde{x} = H^+(x)$, and again $\tilde{l} \asymp \tilde{d}_i$. But $\tilde{d}_i \asymp \tilde{d}'_i$ (they are comparable with common real trace of $\tilde{B}_i, \tilde{B}'_i$), and we are done.

Thus $(\tilde{\phi}^+)^{-1} \circ H^+ \circ \phi^+ \colon \partial D \to \partial D$ is quasisymmetric, and therefore there exists a quasiconformal extension $H^+ \colon \mathrm{closure}(A^+) \to \mathrm{closure}(\tilde{A}^+)$ which is a conjugacy whenever this makes sense. The quasiconformal external conjugacy $H \colon \mathrm{closure}(A) \to \mathrm{closure}(\tilde{A})$ is thus constructed: $H(z) = H^+(z)$ iff $\Re(z) \geq 0$ and $H(z) = H^-(z) = \overline{H^+(\overline{z})}$ iff $\Re(z) \leq 0$.

Step 3: In this step we construct a sequence of quasiconformal extensions of $H \colon \mathrm{closure}(A) \to \mathrm{closure}(\tilde{A})$.

Fact 11.1

$$F^{-1}(E) \cup \mathrm{closure}(\cup_i \partial\Omega_i \cap \mathbb{R})$$

is closed for any closed subset E of $\mathrm{closure}(\Omega)$.

Proof: Let $x_n \in F^{-1}(E)$ and $x_n \to a$. If a subsequence of $\{x_n\}$ belongs to one component Ω_j, then $F(a) \in E$, because $F : \mathrm{closure}(\Omega_j) \to \mathrm{closure}(\Omega)$ is continuous. Hence, $a \in F^{-1}(E)$ in this case. If there are infinitely many components Ω_i containing points of the sequence, then a is the only limit point of such sequence of components because their diameters tend to zero. Hence, $a \in \mathrm{closure}(\cup_i \partial\Omega_i \cap \mathbb{R})$. $\square$

For $n \geq 0$, let

$$A_n = A \cup F^{-1} \cup \ldots \cup F^{-n}(A) \text{ and } A_\infty = \cup_{n\geq 0} A_n.$$

Since A is closed and closure$(\cup_i \partial\Omega_i \cap \mathbb{R}) \subset A$, we have from Fact 11.1 that A_n is closed. Moreover, let

$$U_0 = \Omega, U_1 = F^{-1}(\Omega) = \cup_i \Omega_i, \ldots, U_n = F^{-n}(\Omega).$$

Since $U_1 \subset U_0$ this is a decreasing sequence of open sets. These sets form the analogue of the Yoccoz puzzle pieces: the components of $U_n = F^{-n}(\Omega)$ are said to be the pieces of level n. It is easy to see that

$$A_n \cup U_{n+1} = \text{closure}(\Omega), n = 0, 1, \ldots.$$

One can extend $H\colon A \to \tilde{A}$ to a conjugacy

$$H\colon A_\infty \to \tilde{A}_\infty,$$

see the following fact. Restrictions of H to A_n even extend to a quasiconformal homeomorphism (but the dilatation of ϕ_n can depend on n):

Fact 11.2 *$H\colon A \to \tilde{A}$ can be extended:*

1. *For each $n = 0, 1, \ldots$, $H\colon A \to \tilde{A}$ extends to a conjugacy $H\colon A_n \to \tilde{A}_n$. (We shall also denote this homeomorphism by $H_{|A_n}$.)*

2. *For every $n = 0, 1, \ldots$, there exists K_n such that $H\colon A_n \to \tilde{A}_n$ extends to a K_n-quasiconformal homeomorphism ϕ_n of Ω onto $\tilde{\Omega}$.*

Proof: Use the pullback construction (see e.g. [MS]). More precisely, given n one can choose a K_n-quasiconformal mapping $\phi : \Omega \to \tilde{\Omega}$ which is symmetric with respect to $\mathbb{R}$ such that $\phi_{|\partial A} = H$ and such that $\phi(F^r(c)) = \tilde{F}^r(\tilde{c})$, $r = 1, \ldots, n+1$, where c, $\tilde{c}$ are the critical points of F and $\tilde{F}$. (Note that K_n could tend to infinity as $n \to \infty$.) Next set $\phi_n = H|_{A_n}$ on A_n and define $\phi_n = \tilde{F}^{-n-1} \circ \phi \circ F^{n+1}$ on U_{n+1} in such a way that ϕ_n becomes continuous and so that $\tilde{x} = \phi_n(x)$ means that $F^k(x) \in \Omega_{i_k}$ iff $\tilde{F}^k(\tilde{x}) \in \tilde{\Omega}_{i_k}, k = 1, \ldots, n$. (Continuity of ϕ_n determines which preimage of the branched covering map of $\tilde{F}\colon \tilde{\Omega}_0 \to \Omega$ one should take and also implies that ϕ_n will be symmetric with respect to $\mathbb{R}$.) Then $\phi_n\colon \Omega \to \tilde{\Omega}$ is a homeomorphism which is K-quasiconformal on interior$(F^{-k}(A))$, $k = 0, \ldots, n$, and K_n-quasiconformal on U_{n+1}. $\phi_n\colon A_n \to \tilde{A}_n$ conjugates F and $\tilde{F}$. Let us consider the remaining set $L_n := \cup_{k=0}^n F^{-k}(\partial A) = \Omega \setminus (\cup_{k=0}^n \text{interior}(F^{-k}(A)) \cup U_{n+1})$. Note that by Fact 11.1, L_n is closed in closure(Ω). Moreover, L_n is a countable union of

(a) rectifiable arcs: $\partial\Omega_i$ and components of $F^{-k}(\partial\Omega_i)$

(b) subsets of components of $F^{-k}(\mathbb{R} \cap \Omega)$.

Therefore, L_n is a closed subset of the plane with σ-finite linear measure, and, by [Stre], or see Theorem V.3.2 of [LeVi], L_n is removable for any homeomorphism which is quasiconformal outside L_n. $\square$

Since the dilatation K_n can tend to infinity, we need to study the geometric structure of the sets Ω_n in more detail. To do this, if $x \in U_n$ let $\Omega^n(x)$ be the component of U_n containing this point (so $\Omega^n(x)$ is the level n piece which contains x). Note that for each $n > 0$, there are infinitely many pieces of level n, and (what is worse) their 'diameters' are, in general, incomparable with each other. The following fact describes the geometry of these sets in more detail. Let c be the critical point. For $n \geq 0$, let $\Omega^n(c)$ be the critical piece (i.e., the piece of level n containing c). Then define the *base point* $c_{\Omega^n(c)}$ of this critical piece to be c. For any non-critical piece Ω^n define its base point c_{Ω^n} inductively so that $F(c_{\Omega^n}) = c_{F(\Omega^n)}$.

Denote by $\mathrm{mod}(C)$ the modulus of an annulus C, by $\mathrm{diam}(M)$ the Euclidean diameter of a planar set M, and by $d(M,N)$ the Euclidean distance between two planar sets M, N.

Fact 11.3 *For each $n \geq 1$, there exist positive constants M_n, R_n and ϵ_n, $\epsilon_{n,n-1}$, such that the following properties hold.*

1. *Every level n piece Ω^n has a simply connected neighbourhood $W(\Omega^n)$ for which*

   a) $\mathrm{mod}(W(\Omega^n) \setminus \Omega^n) \geq M_n$,

   b) F maps $W(\Omega^n)$ into $W(F(\Omega^n))$ and restricted to $W(\Omega^n) \setminus \Omega^n$ is an (unbranched) covering onto its image,

   c) $W(\Omega^n)$ is a subset of the unique level $(n-1)$ piece which contains Ω^n,

   d) Ω^n is roughly a disc:

$$\sup_{z \in \partial\Omega^n} |z - c_{\Omega^n}| / \inf_{z \in \partial\Omega^n} |z - c_{\Omega^n}| < R_n.$$

2. *a) for every piece Ω^n of level n and every piece Ω^{n-1} of level $(n-1)$, if $\Omega^n \subset \Omega^{n-1}$, then $d(\Omega^n, \partial\Omega^{n-1})/diam(\Omega^n) \geq \epsilon_{n,n-1}$.*

   b) for every two different pieces Ω_1^n, Ω_2^n of level n, if $diam(\Omega_1^n) \geq diam(\Omega_2^n)$, then $d(\Omega_1^n, \Omega_2^n)/diam(\Omega_2^n) \geq \epsilon_n$.

Proof: Let us first prove assertions 1a)-1d) by induction on $n \geq 1$. Property (4) of Theorem A' implies these assertions hold for $n = 1$. So assume 1a)-1d) hold for $1, 2, \ldots, n-1$. Take a piece Ω^n of level n, and consider the piece $\Omega^{n-1} = F(\Omega^n)$ of level $(n-1)$. We need to show that M_n and R_n do not depend on Ω^n (but only on n). If $W(\Omega^{n-1})$ does not contain the critical value c_1, then define $W(\Omega^n)$ to be the preimage of $W(\Omega^{n-1})$ by F (remember that all branches of F^{-1} are defined on a fixed neighbourhood of Ω). Statements 1a)-1d) then hold; (for statement 1d) use the induction assumption and Koebe). Now consider the case that $c_1 \in W(\Omega^{n-1})$. Let $a > 0$ be the distance between c_1 and $\partial\Omega^{n-1}(c_1)$. By the induction assumption 1d) for $n-1$, there are only finitely many pieces of level $(n-1)$ with diameter $\geq a$. Hence, one can choose corresponding neighbourhoods so that 1a)-1d) hold for these finitely many pieces.

If the diameter of Ω^{n-1} is smaller than a and $\Omega^{n-1} \cap \Omega^{n-1}(c_1) = \emptyset$, then one can choose $M'_{n-1} \leq M_{n-1}$ and a smaller neighbourhood W' of Ω^{n-1} inside $W(\Omega^{n-1})$ such that $\mathrm{mod}(W' \setminus \Omega^{n-1}) \geq M'_{n-1}$ and so that $\mathrm{diam}(W') \leq (3/2)a$. Then $c_1 \notin W'$ and $W(\Omega^n)$ can be taken as the diffeomorphic preimage of W'. This proves 1a)-1c) for n

with $M_n = M'_{n-1}$. From Koebe, induction and 1a)-1b) we get a constant K_n in 1d) which does not depend on Ω^n. If finally $\Omega^{n-1} = \Omega^{n-1}(c_1)$ then 1a)-1c) follows with $M_n = M_{n-1}/\ell$. Again 1d) follows.

Statement 2a) is a direct corollary of the statements 1a) and 1d) we just proved.

Let us now prove assertion 2b) by induction. Let $n = 1$. Let Ω_i, Ω_j be two different level 1 pieces, and W_i, W_j the corresponding neighbourhoods, see Theorem A' (4). If W_i, W_j are disjoint, then some uniform choice for $\epsilon_1 > 0$ exists because the distance between the pieces is comparable or larger than their diameters diameters. If W_i, W_j are not disjoint, then $W_j \subset W_i$ (or $W_i \subset W_j$ if the diameter of the pieces is comparable), and again the distance between the pieces is larger than the distance between Ω_j and ∂W_j, which is larger than $\epsilon_1 \cdot \mathrm{diam}(\Omega_j)$ where $\epsilon_1 > 0$ depends only on the lower bound for the $\mathrm{mod}\,(W_j \setminus \Omega_j)$. (We always use that $\mathrm{mod}\,(W_j \setminus \Omega_j)$ is universally bounded from below.) This proves 2b) for $n = 1$. So assume 2b) hold for $n - 1$. If Ω_1^n, Ω_2^n are in different pieces of level $n-1$, then one can put $\epsilon_n = \epsilon_{n-1}$. If $\Omega_1^n, \Omega_2^n \subset \Omega^{n-1}$, then apply F and use the induction assumption (one should consider two cases: Ω^{n-1} is critical or not). $\square$

Step 4: In this step we approximate the desired conjugacy by the K-quasiconformal mappings H_n constructed below.

To do this, we need the following fact which uses that there are pieces $\Omega_{i(m)}$ of level 1 with $\mathrm{diam}(\Omega_{i(m)}) \to 0$ as $m \to \infty$ which intersects $\omega(c)$ (because $\omega(c)$ is non-minimal). This allows us to prove a statement similar to Proposition 3.3 of [LS1]:

Fact 11.4 *There exist a sequence of integers $n_m \to \infty$, a neighbourhood Q of the range Ω, such that, for every n_m:*

(a) there exists a neighbourhood Q_{n_m} of the critical piece $\Omega^{n_m}(c)$, such that F^{n_m} maps Q_{n_m} onto Q in a ℓ to one fashion, and $F^{n_m}(c)$ lies in the central component Ω_0 of the map F;

(b) if x is so that $F^j(x) \in \Omega^{n_m}(c)$ with j minimal, then there exists a univalent branch of F^{-j} on Q_{n_m} to a neighbourhood of x.

Proof: Pick such a small disc $\Omega_{i(m)}$, and let $k(m)$ be minimal so that $F^{k(m)}(c) \in \Omega_{i(m)}$. Note that $k(m) \to \infty$. Let Ω' be a component of $F^{-k(m)}(\Omega_{n(m)})$ containing c. Then $F^{k(m)} : \Omega' \to \Omega_{i(m)}$ is a ℓ to one cover. Since c is recurrent, there is a minimal $k' \geq 1$ such that $F^{k(m)+k'}(c)$ belongs to the domain Ω_0. One can assume $k' \geq 2$ (for infinitely many $k(m)$). Then we can define $n_m = k(m) + k'$. The neighbourhood Q of Ω exists because of Property 4 of Theorem A'. This completes the proof of Fact 11.4. $\square$

Denote

$$\Omega_0(m) = \Omega^{n_m}(c).$$

On the boundary of $\Omega_0(m)$ (and on part of its interior) H is already defined by the previous pullback construction. We now modify the definition of H on $\Omega_0(m)$ and also on all its preimages. We do this as follows. Choose a quasiconformal homeomorphism $\hat{H}: \Omega \to \tilde{\Omega}$ which coincides with H on the boundary of Ω and which maps $F^{n_m}(c)$ to

$\tilde{F}^{n_m}(\tilde{c})$. Since $F^{n_m}(c)$ is roughly in the middle of Ω (and the same holds for the other map), one can choose $\hat{H}$ so that its quasiconformal dilatation is at most some universal number K. Now redefine the partially defined map H to a map $H_m: \Omega_0(m) \to \tilde{\Omega}_0(m)$ so that $H_m = \tilde{F}^{-n_m} \circ \hat{H} \circ F^{n_m}$. Since $\hat{H}(F^{n_m}(c)) = \tilde{F}^{n_m}(\tilde{c})$, H_m is well-defined and has quasiconformal dilatation $\leq K$. Next define H_m from a preimage of $\Omega_0(m)$ to the corresponding preimage of $\tilde{\Omega}_0(m)$ as a conjugacy, but only consider those preimages $F^{-j}(\Omega_0(m))$ whose forward iterates up to $j-1$ do not enter $\Omega_0(m)$. Outside all these preimages let H_m coincide with the map H constructed by the pullback construction in the previous section.

Fix m and partition Ω into the following 4 sets:

$X_m(a)$ is the set of points x such that there exists $n \geq 0$ so that $F^n(x)$ lies in the (open) central piece $\Omega_0(m)$;

$X_m(b)$ is the set of x such that there exists a minimal $n \geq 0$ so that $F^n(x)$ lies in the interior of the annulus A and $F^k(x) \notin \Omega_0(m), k = 0, \ldots, n-1$;

$X_m(c)$ is the set of x such that there exists a minimal $n \geq 0$ so that $F^n(x)$ lies in the boundary of A and $F^k(x) \notin \Omega_0(m), k = 0, \ldots, n-1$;

Finally, $X_m(d)$ is the set of x such that $F^n(x)$ is well-defined and lies outside $\Omega_0(m)$, for every $n = 0, 1, 2, \ldots$.

Until now, H_m has been defined on the sets $X_m(a), X_m(b), X_m(c)$. Extend H_m naturally to the set $X_m(d)$ as follows. Any point $x \in X_m(d)$ is contained in a nested sequence of the puzzle pieces $\Omega^n(x), n > n_m$, such that each $\Omega^n(x)$ is mapped by F^{n-n_m} univalently onto some non-critical piece of some fixed level n_m (where n_m does not depend on x but only on m), and moreover, this map extends univalently onto one level up, i.e., from $\Omega^{n-1}(x)$ onto some piece of level $n_m - 1$. Note that the distortion of this map does not depend on the point x (but only on m) because the modulus of the annuli between $n_m - 1$ and n_m levels is bounded from below by some positive constant m_- (which is fixed if m is fixed). All n_m-components are roughly discs with respect to their base points, see Fact 11.3(1d). We conclude that all $\Omega^n(x)$ are also roughly discs (with respect to their base points): the inequality of Fact 11.3(1d) holds for them but with a constant R_n which only depends m (i.e. it does not depend on n and x). In particular, the (Euclidean) diameters of $\Omega^n(x)$ tend to zero as $n \to \infty$ uniformly for all $x \in X_m(d)$. For $x \in X_m(d)$ define $H_m(x)$ dynamically as the unique point $y \in \tilde{\Omega}$ with the same itinerary, i.e., for every $n \geq 0$, $F^n(x) \in \Omega_{i_n}$ if and only if $\tilde{F}^n(y) \in \tilde{\Omega}_{i_n}$. Thus we have that

$$H_m : \Omega \to \tilde{\Omega}$$

is well-defined homeomorphism, which is a conjugacy between F and $\tilde{F}$ everywhere except in the central domain $\Omega_0(m)$. Observe that the sequence $\{H_m\}_{m \geq 1}$ converges uniformly to a conjugacy h between $F, \tilde{F}$.

So we now need to show that h is quasiconformal. To this end, fix again m and consider H_m at different points. First of all, H_m is K-quasiconformal on the open sets $X_m(a), X_m(b)$, where K does not depend on m (because H_m is K-quasiconformal on $\Omega_0(m)$; on the interior of A one has that $H_m = H$ is K-quasiconformal). Thus it is enough to prove the following three facts (where the dilatations of H_m in the last two facts can depend on m).

Fact 11.5 *$X_m(c)$, $X_m(d)$ are of zero Lebesgue measure.*

Fact 11.6 *H_m is quasiconformal at any point from $X_m(d)$.*

Fact 11.7 *H_m is quasiconformal at any point from $X_m(c)$.*

By quasiconformality in the last two of these statements we mean that (***) below holds at any point from $X_m(c)$ and $X_m(d)$.

Let us first show that these three facts imply Theorem B'. Note that these facts imply (cf. [GS2]) that the homeomorphism H_m is K-quasiconformal everywhere (see e.g. Theorem V.3.3 of [LeVi]). On the other hand, the sequence $\{H_m\}_{m\geq 1}$ converges uniformly to a conjugacy $h\colon \Omega \to \tilde{\Omega}$ between F and $\tilde{F}$. Hence h is quasiconformal on Ω. So let us prove these facts.

Proof of Fact 11.5: That the Lebesgue measure of $X_m(c)$ is zero is trivial: $X_m(c)$ is equal to $\cup_{n\geq 0} F^{-n}(\partial A)$. Here ∂A has Lebesgue measure zero since it is a countable union of rectifiable curves $\partial\Omega_i$ and a subset of the real axis.

So let us show by contradiction that $|X_m(d)| = 0$. So let y be a Lebesgue density point of $X_m(d)$. Consider the same nested sequence of pieces $\Omega^n(y)$, $n > n_m$, as before. We know that the distortion of $F^{n-n_m} : \Omega^n(y) \to \Omega^{n_m}(y_n)$, where $y_n = F^{n-n_m}(y)$, is uniformly bounded (it only depends on m), and all $\Omega^n(y)$ are roughly discs. Therefore (since $X_m(d)$ is forward invariant under F) the density of $X_m(d)$ in $\Omega^n(y)$, i.e., $|\Omega^n(y)\cap X_m(d)|/|\Omega^n(y)|$ and also $|\Omega^{n_m}(y_n) \cap X_m(d)|/|\Omega^{n_m}(y_n)|$ both tend to 1 as $n \to \infty$. On the other hand, each piece $\Omega^{n_m}(y_n)$ of level n_m is mapped by an F-iterate for the first time either to a central piece (of a level $< n_m$) or to Ω, and the distortion of this map is uniformly bounded because n_m is fixed. Therefore, there exists a fixed piece Q which is either a critical piece of level $< n_m$ or which is equal to Ω so that the density of the set $X_m(d)$ in Q is equal to one. In particular, $X_m(d) \cap \Omega_0(m) \neq 0$, a contradiction. This completes the proof of Fact 11.5. □

Proof of Fact 11.6: To prove that H_m is quasiconformal we are going to use the following recent surprising result by Heinonen-Koskela, see [HK].

> (***) A homeomorphism φ of the plane (more general, of $\mathbb{R}^d$) is quasiconformal, if there exists L such that for any point x there exists a sequence of radii $r_n \to 0$, such that, for any n,
>
> $$\sup_{|y-x|=r_n} |\varphi(y)-\varphi(x)| / \inf_{|y-x|=r_n} |\varphi(y)-\varphi(x)| < L. \tag{11.1}$$
>
> So it makes sense to all φ quasiconformal at a point x if (***) holds.

In our case we shall take $\varphi = H_m$ and $x \in X_m(c) \cup X_m(d)$. Note that we do not mind if L from (***) does depend on m. (In fact, for points in $X_m(c)$ we can take any sequence $r \to 0$.) In order to prove (***), we will use that H_m is strongly related to quasiconformal extensions ϕ_n of $H\colon A_n \to A_n$ when n is large.

So fix m and consider a point $x \in X_m(d)$. Our aim is to show that (***) is satisfied when we take r_n equal to $2d_n(x)$, where $d_n(x)$ is the diameter of $\Omega^n(x)$. (The factor 2 is because we need space between $\Omega^n(x)$ and the circle $\{y : |y - x| = 2d_n\}$, since x can be close to the boundary of $\Omega^n(x)$.)

Note that all 'universal' constants below as well as statements on the shapes of discs can depend on m but these constants will not depend on x.

As before, there exists $m_- = m_-(m) > 0$ so that for each $n \geq 0$ and each $x \in X_m(d)$, the modulus of the annulus $\Omega^{n-1}(x) \setminus \Omega^n(x)$ is bounded from below by m_-. (These moduli may not be bounded from above.) Hence there exists j (which only depends on m), such that for any n, the modulus of the annulus $\Omega^{n-j}(x) \setminus \Omega^n(x)$ is so big that the Euclidean disc $B_n := B(x, 2d_n(x))$ centered at x with radius equal to $2d_n(x)$ is contained in $\Omega^{n-j}(x)$ (and certainly contains $\Omega^n(x)$). This is because $\operatorname{mod}(B_n \setminus \Omega^n(x))$ lies between two universal positive numbers.

F^{n-j-n_m} maps B_n to a set between pieces of levels $n_m + j$ and n_m. Note that these iterates of B_n do not intersect the critical piece $\Omega_0(m)$, because the latter piece has level n_m. Since the map F^{n-j-n_m} has universally bounded distortion on $\Omega^{n-j}(x)$, the image $B' = F^{n-j-n_m}(B_n)$ is roughly a Euclidean disc centered at

$$x' = F^{n-j-n_m}(x),$$

is contained in a piece of level n_m, and its diameter is comparable with the diameter of

$$P := \Omega^{n_m+j}(x').$$

One can take everything back by the corresponding branch of $\tilde{F}^{-(n-j-n_m)}$ with bounded distortion to $\tilde{x} = H_m(x)$, and hence (***) holds for $x \in X_m(d)$ (and therefore Fact 11.6 is proved) provided we can prove the following

Claim 1: *For each set B' as above, the image $\tilde{B}' = H_m(B')$ is roughly a disc centered at $\tilde{x} = H_m(x')$.*

Proof of Claim 1: Note that

$$P = \Omega^{n_m+j}(x') \subset B' \subset \Omega^{n_m}(x').$$

The levels of the two puzzle pieces only depends on m, but since their diameters can be incomparable, this does not help much. The idea of the proof of the claim is to compare the set B' with a larger set which has dynamical meaning by adding to B' certain puzzle pieces. Thus $H_m(B')$ is contained in a set whose boundary is the image by some map ϕ_k (the quasiconformal extension of the pullback map $H\colon A_\infty \to \tilde{A}_\infty$ from Step 3). The quasiconformality of ϕ_k will then allow us to conclude the claim.

Let us start by noting that B' is roughly a disc around x' and $\operatorname{mod}(B' \setminus P)$ is universally bounded from above and from zero (as before universal at least for a given choice of m). Hence, there exists universal $r > r' > 0$ such that

$$r' \cdot \operatorname{diam}(P) \leq d(y, P) \leq r \cdot \operatorname{diam}(P),$$

for every $y \in \partial B'$.

Fix a point $y \in \partial B'$, and consider its iterates up to $F^j(y)$. The following three cases can occur:

1. There exists a minimal i_0, $0 \leq i_0 \leq j$, such that $F^{i_0}(y) \in A_{n_m-1}$ and $F^i(y) \notin \Omega_0(m) := \Omega^{n_m}(c)$ for $0 \leq i \leq i_0$ (as remarked before $A_{n_m-1} = \cup_{k=0}^{n_m-1} F^{-k}(A)$ is the complement to the pieces of level n_m, and $\Omega_0(m) = \Omega^{n_m}(c)$ is the critical piece of the level n_m). One can write:

$$H_m(y) \ = \tilde{F}^{-i_0} \circ H|_{G_{n_m-1}} \circ F^{i_0}(y) = H|_{A_{n_m-1+i_0}}(y) = \phi_{n_m-1+i_0}(y) = \phi_{n_m+j+1}(y),$$

where ϕ_n is the quasiconformal extension of $H|_{A_n}$ to Ω constructed in Fact 11.2. In this case define $Q(y) = \emptyset$.

2. For each i with $0 \leq i \leq j$ the iterate $F^i(y)$ is well-defined, $F^j(y) \in U_{n_m}$ (i.e., lies in $\Omega^{n_m}(F^j(y))$), and $F^i(y) \notin \Omega_0(m)$ for $0 \leq i \leq j$. Then $y \in Q'(y) := \Omega^{n_m+j}(y)$. Note that $Q'(y) \cap P = \emptyset$ because otherwise two pieces of the same level $n_m + j$ coincide which is impossible because P does not intersect $\partial B'$. Next let us look at one level down and consider two subcases.

(a) y is ouside the pieces of level $n_m + j + 1$. Then $F^{j+1}(y)$ is outside the pieces of level n_m, and

$$H_m(y) = \phi_{n_m+j+1}(y).$$

Then define $Q(y) = \emptyset$.

(b) y is inside a piece $\Omega^{n_m+j+1}(y)$ of level $(n_m + j + 1)$. Then define $Q(y) := \Omega^{n_m+j+1}(y)$. Note that $H_m(y) \in \tilde{\Omega}^{n_m+j+1}$, where the latter is the piece of $\tilde{F}$ of level $(n_m + j + 1)$ for which $\partial\tilde{\Omega}^{n_m+j+1} = H|_{A_{n_m+j+1}}(\partial Q(y))$. In other words,

$$H_m(y) \in \phi_{n_m+j+1}(Q(y)).$$

It is useful to note at this point that the sets $F^i(P), 0 \leq i \leq j$, do not intersect $\Omega_0(m)$ (because the iterates of x do not meet $\Omega_0(m)$). Therefore, as above,

$$H_m(x') \in H_m(P) = \phi_{n_m+j}(P) = \phi_{n_m+j+1}(P),$$

where $x' = F^{n-j-n_m}(x)$ as before.

3. There exists a minimal $i_0, 0 \leq i_0 \leq j$, such that $F^{i_0}(y) \in \Omega_0(m)$. Then y lies in $\Omega^{n_m+i_0}(y)$, which is a preimage of the critical piece $\Omega_0(m)$ by a branch of F^{-i_0}. Consider again two subcases.

(a) $\mathrm{diam}(\Omega^{n_m+i_0}(y)) \leq (1/10) \cdot r' \cdot \mathrm{diam}(P)$. Then define $Q(y) = \Omega^{n_m+i_0}(y)$. Note that $H_m(y) \in \tilde{\Omega}^{n_m+i_0}$, where $\partial\tilde{\Omega}^{n_m+i_0} = H|_{A_{n_m+i_0}}(\partial Q(y)) = H|_{A_{n_m+j+1}}(\partial Q(y))$. In other words, again

$$H_m(y) \in \phi_{n_m+j+1}(Q(y)).$$

(b) If $\mathrm{diam}(\Omega^{n_m+i_0}(y)) > (1/10) \cdot r' \cdot \mathrm{diam}(P)$, then define $Q(y) = \emptyset$. By construction H_m is K-quasiconformal on $\Omega^{n_m+i_0}(y)$, and $\phi_{n_m+j+1} = H_m$ on the boundary of $\Omega^{n_m+i_0}(y)$; Moreover, there are at most finitely many $\Omega^{n_m+i_0}(y)$ as in this subcase, see Fact 11.3(1d) and Fact 11.4. In addition, $\Omega^{n_m+i_0}(y) \cap P = \emptyset$ because otherwise the level $n_m + j$ piece P is contained in the level $n_m + i_0$ piece $\Omega^{n_m+i_0}(y)$ and then $F^{i_0}(x') \in F^{i_0}(P) \subset F^{i_0}(\Omega^{n_m+i_0}(y)) = \Omega_0(m)$, a contradiction.

Now *define a new map* ϕ'. Let ϕ' be equal to ϕ_{n_m+j+1} everywhere in Ω except for pieces $\Omega^{n_m+i_0}(y)$ as in 3(b), on which we define $\phi = H_m$. Define also

$$C' = \partial B' \cup \{Q(y);\ y \in \partial B'\}.$$

We have proved that

$$H_m(\partial B') \subset \phi'(C')$$

and

$$H_m(x') \in \phi'(P).$$

Observe that ϕ' is a homeomorphism of Ω onto $\tilde{\Omega}$, which is K_{n_m+j+1}-quasiconformal on any open set U for which $\phi'|U = \phi_{n_m+j+1}$ and K-quasiconformal when $\phi'|U = H_m$. Hence, ϕ' is K_{n_m+j+1}-quasiconformal everywhere except along finitely many piecewise analytic curves (the boundaries of the pieces $\Omega^{n_m+i_0}(y)$ as in Case 3b). Therefore, ϕ' is K'-quasiconformal on Ω, where $K' = K_{n_m+j+1}$ depends only on m. In order to use this, we need to show that C' has a good geometric structure:

Claim 2: *There exist two universal constants* $0 < b < a$*, such that, for every* $z \in C'$*,*

$$b \cdot \mathrm{diam}(P) \leq d(z, P) \leq a \cdot \mathrm{diam}(P).$$

Before we prove Claim 2, let us show that it implies Claim 1. Indeed, take $x'' = (\phi')^{-1} \circ H_m(x') \in P$. Assuming claim 2 is true, for every $z \in C'$,

$$b \cdot \mathrm{diam}(P) \leq d(z, x'') \leq (1+a) \cdot \mathrm{diam}(P).$$

Hence, $H_m(\partial B')$ surrounds the set $\Gamma_1 := \phi'(B(x'', b \cdot \mathrm{diam}(P)))$ and is contained in the set $\Gamma_2 := \phi'(B(x'', (1+a) \cdot \mathrm{diam}(P)))$, where $B(z,r) = \{y; |y-z| \leq r\}$. Let $\tilde{x} = H_m(x') = \phi'(x'')$. Since ϕ' is K'-quasiconformal, there exists $D = D(K')$, such that

$$B(\tilde{x}, \tilde{r}_1) \subset \Gamma_1 \subset B(\tilde{x}, D \cdot \tilde{r}_1),$$

where $\tilde{r}_1 = \min\{|\tilde{x} - y|; y \in \Gamma_1\}$, and

$$B(\tilde{x}, \tilde{r}_2/D) \subset \Gamma_2 \subset B(\tilde{x}, \tilde{r}_2),$$

where $\tilde{r}_2 = \max\{|\tilde{x} - y|; y \in \Gamma_2\}$. Hence

$$\mathrm{mod}\,(\Gamma_2 \setminus \Gamma_1) \geq 2\pi \cdot \log(\frac{\tilde{r}_2/D}{D\tilde{r}_1}).$$

We need to prove that $\tilde{r}_2/\tilde{r}_1$ is universally bounded from above. But since ϕ' is K' quasiconformal and $(\Gamma_2 \setminus \Gamma_1)$ is the ϕ' image of the annulus $\{z\,;\ b \cdot \mathrm{diam}(P) \leq |z - x''| \leq (1+a) \cdot \mathrm{diam}(P)\}$, we get

$$(2\pi/K') \cdot \log((1+a)/b) \leq \mathrm{mod}\,(\Gamma_2 \setminus \Gamma_1) \leq (2\pi \cdot K') \cdot \log((1+a)/b).$$

This and the above inequality for $\log(\Gamma_2 \setminus \Gamma_1)$ imply that the ratio $\tilde{r}_2/\tilde{r}_1$ cannot be larger than $D^2 \cdot [(1+a)/b]^{K'}$. So let us prove Claim 2. First of all, it holds for any

$z \in \partial B'$: take $b = r'$ and $a = r$. It also holds for any $z \in Q(y)$ for any $Q(y)$ (as above) for which $\operatorname{diam}(\Omega(y)) \leq (1/10) \cdot r' \cdot \operatorname{diam}(P)$. This is because in that case it is enough to put $b = 9/10 \cdot r'$ and $a = r + 1/10 \cdot r'$. In particular it holds for all $z \in Q(y)$ with $Q(y)$ as in case 3(a).

So it remains to consider $z \in Q(y)$ where $Q(y)$ is as in 2(b). To prove the right-hand side inequality of claim 2, we need to show that any such level $n_m + j + 1$ piece $Q(y), y \in \partial B'$, is not too large compared to $\operatorname{diam}(P)$. Indeed, for the level $n_m + j$ piece $Q'(y)$ defined in case 2, we have $Q'(y) \cap P = \emptyset$ and, by Fact 11.3(2a),

$$\epsilon_* \cdot \operatorname{diam}(Q(y)) \leq d(Q(y), \partial Q'(y)) \leq d(Q(y), P) \leq d(y, P) \leq r \cdot \operatorname{diam}(P),$$

where $\epsilon_* = \epsilon_{n_m+j+1, n_m+j}$. Hence

$$\operatorname{diam}(Q(y)) \leq (r/\epsilon_*) \cdot \operatorname{diam}(P)$$

which prove the right-hand inequality of Claim 2. Let us now prove the left-hand side inequality. Since we consider $Q(y)$ such that $\operatorname{diam}(Q(y)) \geq (1/10) \cdot r' \cdot \operatorname{diam}(P)$, we get by Fact 11.3(2b):

$$\begin{aligned} d(Q(y), P) &\geq d(Q', P) \geq \epsilon_{n_m+j} \cdot \min\{\operatorname{diam}(Q'), \operatorname{diam}(P)\} \geq \\ &\geq \epsilon_{n_m+j} \cdot \min\{\operatorname{diam}(Q), \operatorname{diam}(P)\} \geq \epsilon_{n_m+j} \cdot \min(1, r'/10) \cdot \operatorname{diam}(P). \end{aligned}$$

This proves that one can also choose $b > 0$ uniformly in the left-hand side inequality of Claim 2. This completes the proof of claim 2 and therefore also the proof of claim 1. □

Proof of Fact 11.7: Now consider H_m at points $x \in X_m(c)$. By definition, for some $n \geq 0$, $y := F^n(x)$ lies in the inner boundary of A, i.e. in the closure of the union of $\partial\Omega_i$, and $F^k(x) \notin \Omega_0(m)$, $k = 0, \ldots, n$. There are two substantially different cases:

(a) y is not real, belongs to the boundary of some Ω_j, and $F(y)$ is not real as well,

(b) either $y \in \partial A \cap \mathbb{R}$ or $y \in \Omega^1(c)$ and $F(y)$ is real (i.e. one of the end points of $\Omega \cap \mathbb{R}$).

In turn, for both (a) and (b), there are two subcases:

(1) there exists k, $0 \leq k \leq n$, such that $F^k(x) \in \partial\Omega_0(m)$

(2) $F^k(x) \notin \Omega_0(m)$, for all $k = 0, \ldots, n$.

In the cases (a1)-(a2) there exists a neighbourhood U of $z = F^k(x)$ in case (a1) and of $z = y$ in case (a2), such that $U \subset A_{n-k}$ in (a1) and $U \subset A_1$ in (a2) (remember that $A_k = \cup_{i=0}^k F^{-i}(A)$). Hence, the homeomorphism H_m is quasiconformal at each point of U except for a piecewise analytic arc (which is either $U \cap \Omega_0(m)$ or $U \cap \Omega_j$). Since such arcs are removable, H_m is quasiconformal at $z \in U$

In case (b), the subcases (1)-(2) are very similar. Let us first consider the most difficult case (b2) assuming also that $y \in \partial A \cap \mathbb{R}$ (the second possibility that $y \in \Omega^1(c)$ with $F(y)$ real is easier). Since $F^k(x) \notin \Omega_0(m)$ for $k \leq n$, we have $H_m = \tilde{F}^{-n} \circ H_m \circ F^n$ in a small neighbourhood of the point x (the size of the neighbourhood depends on n). Hence, it is enough to show that H_m is quasiconformal at $y \in \partial A \cap \mathbb{R}$. In particular, y

is not inside any Ω_i but can be a point of $\partial\Omega_i \cap \mathbb{R}$. Take a round disc $B = B(y, r)$ with the center at y and a small radius r. Similar to the proof of Claim 1 above, consider

$$C = \partial B \cup \{\Omega^2; \Omega^2 \cap \partial B \neq \emptyset\},$$

where Ω^2 are the level 2 pieces. We have: $H_m(y) = \phi_2(y)$ and $H_m(\partial B) \subset \phi_2(C)$. Since $\phi_2 : \Omega \to \tilde{\Omega}$ is quasiconformal, it is enough to prove that C is roughly a 'circle' around y, i.e. that there exist $r_2 > r_1 > 0$ such that $C \cap \{z; |z - y| = r_i\} = \emptyset$, $i = 1, 2$ and such that r_2/r_1 is universally bounded from above. To prove this, let $\Omega^2 \cap \partial B \neq \emptyset$, and let Ω^1 be the level 1 piece containing Ω^2. Note that $y \notin \Omega^1$. As it follows from Fact 11.3(1c), for example, there exists a universal $\epsilon_m > 0$, such that

$$d(\Omega^2, \partial\Omega^1) \geq \epsilon_m \cdot \mathrm{diam}(\Omega^2).$$

This implies that $\mathrm{diam}(\Omega^2) \leq r/\epsilon_m$ because $d(\Omega^2, \partial\Omega^1) > \epsilon_m \cdot (r/\epsilon_m) = r$ and since $y \notin \Omega^1$ this would imply that $(\Omega^2, y) > r$, a contradiction. Similarly, $d := d(\Omega^2, y) \geq r \cdot \epsilon_m/(1 + \epsilon_m)$ because otherwise we would have: $d = d(\Omega^2, y) \geq d(\Omega^2, \partial\Omega^1) \geq \epsilon \cdot \mathrm{diam}(\Omega^2) \geq \epsilon \cdot (r - d) > r \cdot \epsilon/(1 + \epsilon)$, i.e. a contradiction. Thus one can take $r_1 = r \cdot \epsilon/(1 + \epsilon)$ and $r_2 = r + r/\epsilon = r \cdot (1 + \epsilon)/\epsilon$.

If y is as in case b(1) then $H_m = \tilde{F}^{-k} \circ H_m \circ F^k$ and so it is enough to show that (***) holds for H_m near $z = F^k(y) \in \partial\Omega_0(m)$. Using again a set C as above, (***) again follows at z.

Thus we have proved that (***) holds at any point of $X_m(c)$ and befor we proved (***) for any point of $X_m(d)$ with a special choice of $r_n \to 0$. The proof of Theorem B' is completed. $\square$

Remark. Combining the considerations used in the cases $x \in X_m(d)$ and $x \in X_m(c)$, one can show that (***) holds at $x \in X_m(d)$ for an arbitrary small enough r_n, i.e. show that H_m is quasiconformal without the use of the deep result of [HK].

12 Proof of Theorem C: Density of Axiom A maps in families

Let $g_\lambda : \cup_{i=0}^{i_0} I^i(\lambda) \to I$ be a family of maps of the class **AC**, where $\lambda \in \Lambda := [\Lambda_1, \Lambda_2]$. Denote $c_\lambda \in I^0(\lambda)$ the critical point of g_λ, and $\omega_\lambda(c)$ the omega-limit set of $c(\lambda)$ by g_λ. Assume that:

a) the itineraries of the critical point of g_{Λ_1} and g_{Λ_1} are different;

b) $g_\lambda(x)$ is real analytic on (λ, x) in a neighbourhood of every point (λ_0, x_0), where $\lambda_0 \in (\Lambda_1, \Lambda_2)$ and $x_0 \in \mathrm{closure}(\cup_{i=0}^{i_0} I^i(\lambda_0))$;

c) there exists $\lambda \in (\Lambda_1, \Lambda_2)$ such that g_λ has no parabolic periodic orbits.

Let $A = \{\lambda \in \Lambda : g_\lambda \textit{ is Axiom A map }\}$ and S be its complement. Given $\lambda_0 \in S$, let $S(\lambda_0)$ be connected component of S containing λ_0. In this section, we address the important question whether the set A is dense (or equivalently, $S(\lambda) = \{\lambda\}$ for every $\lambda \in S$).

Observe that if $g_\lambda(x) = g_0(x) + \lambda$, then A is dense in this family, because all the branches "move up" with λ (see Section 8 and also Lemmas 12.2-12.3 below). In particular, Axiom A maps are dense in the family $x \mapsto kx+\lambda-k/2\pi \sin(2\pi x), \lambda \in \mathbb{R}$, of Arnold's maps, and in the family of Blaschke products of Example 2 of the Introduction. For the same reason, *Axiom A maps are dense in the space of all maps* $g \in \mathbf{AC}$.

Let us consider this problem for a family g_λ satisfying the conditions a)-c) from the beginning of this section. It is very easy to see that $S(\lambda_0) = \{\lambda_0\}$ if c is attracted to an attracting or parabolic periodic orbit of g_{λ_0} (see Lemma 12.2) or eventually periodic (use condition a) from above and Lemma 12.3).

The cases when $\omega(c)$ is infinite are much more difficult to consider. We use Theorems A, B, A', B', and a method of [Ko, Sect. 7.2, p. 69-70] to prove the following slight generalization of Theorem C:

Theorem C'. *Assume $\omega_{\lambda_0}(c)$ is infinite.*

C. If $\omega_{\lambda_0}(c)$ is minimal, then $S(\lambda_0) = \{\lambda_0\}$, i.e., g_{λ_0} is approximated by Axiom A maps g_λ.

C'. If $\omega_{\lambda_0}(c)$ is not minimal, and the following condition (*) holds:

(*) $\omega_\lambda(c)$ does not contain parabolic periodic orbits, where λ is at least one end point of $S(\lambda_0)$,

then again $S(\lambda_0) = \{\lambda\}$.

For more information see Lemmas 12.4-12.5.

In particular, we have

Corollary 12.1. *Let P_λ be a holomorphic family of polynomials of Example 1 from the introduction. Then either all polynomials P_λ are conformally conjugate or Axiom A maps are dense in this family.*

The proof of Theorem C' is contained in the following Lemmas. Let us start by fixing a closed subinterval $[e_1, e_2]$ of (Λ_1, Λ_2).

Lemma 12.1. *For every $n \geq 1$, the set $N_n = \{\lambda \in [e_1, e_2] : g_\lambda^n$ has a parabolic fixed point$\}$ is finite. In particular, the set $N = \cup_{n\geq 1} N_n$ is at most countable.*

Proof: Use b) and c). (Actually we have much more, see Theorem IV.B' in [MS].) $\square$

Lemma 12.2. *Fix arbitrary $\lambda_0 = 0 \in (e_1, e_2)$. If g_0 has an attracting or parabolic periodic orbit P_0, which attracts c_0, then there is λ arbitrary close to λ_0 from either side of λ_0, such that g_λ is Axiom A map.*

Proof: Consider several cases.

1). P_0 is attracting. Then for every λ close enough to λ_0, g_λ has an attracting periodic orbit P_λ, which changes continuously with λ, tends to P_0 as $\lambda \to \lambda_0$, and attracts c_λ. By Lemma 12.1, one can choose λ arbitrary close to λ_0, such that g_λ has no parabolic periodic orbits. Then, by Mane's theorem [MS, p. 222-223], this g_λ is Axiom A.

2). P_0 is a two-sided attracted parabolic periodic orbit (like 0 for $x - x^3$). Then, for any λ close enough to λ_0, there exists an attracting periodic orbit (close to P_0) which attracts c_λ. For this λ, we apply 1).

3). P_0 is a one-sided attracted parabolic periodic orbit (like 0 for $x + x^2$). There are two subcases:

3a). For λ arbitrary close to λ_0, g_λ has a nearby to P_0 periodic orbit(s). Then one of these orbits is attracting, and it attracts the critical point. Then we apply 1).

3b). If 3a) does not hold, then we proceed as in [Le1]. Let p be the period of P_0. Then $g_0^{pi}(c_0) \to a_0$ as $i \to +\infty$, where $a_0 \in P_0$. Moreover, since P_0 is the one-sided attracted orbit, one can assume that $g_0^{pi}(c_0) < a_0$, $i = 1, 2, \ldots$. On the other hand, since there are no wandering intervals, there exists a sequence $g_0^{-n_i}(c_0)$ of preimages of c_0, which tends to a_0 from the right side. By continuity, for every small $\epsilon > 0$ and for every i_0, there exists $\delta > 0$, such that for every $\lambda' \in (\lambda_0 - \delta, \lambda_0 + \delta)$,

$$g_{\lambda'}^{-n_{i_0}}(c_{\lambda'}) \in (a_0 - \epsilon, a_0 + \epsilon).$$

But since there are no fixed points of g_λ^p in a neighbourhood of a_0, if $\lambda \neq \lambda_0$ is close to λ_0, there exists (for the above λ') i_1, such that

$$g_{\lambda'}^{pi_1}(c_{\lambda'}) > a_0 + \epsilon.$$

Changing now λ from λ_0 to λ', we must meet λ such that

$$g_\lambda^{-n_{i_0}}(c_\lambda) = g_\lambda^{pi_1}(c_\lambda).$$

Therefore c_λ is periodic, and we can again apply 1). $\square$

Lemma 12.3. *If the itinerary of c_λ is not constant for any λ in a neighbourhood of λ_0, then there exists λ arbitrary close to λ_0, such that c_λ is attracted by an attracting or neutral periodic orbit of g_λ.*

Proof: By the continuity of the itinerary on λ, one can choose λ arbitrary close to λ_0, so that the itinerary is periodic. It implies g_λ has a periodic interval containing the critical point. $\square$

Lemma 12.4. *Assume that $[\lambda_1, \lambda_2]$ is a closed subinterval of (e_1, e_2), such that:*

1) the itinerary of c_λ is the same for each $\lambda \in [\lambda_1, \lambda_2]$ and not periodic,

2) $\omega_\lambda(c)$ does not contain parabolic periodic orbits, for λ equal to either λ_1 or to λ_2.

Then the interval $[\lambda_1, \lambda_2]$ is not maximal obeying these properties.

In other words, if $[\lambda_1, \lambda_2]$ is a maximal non-trivial interval satisfying 1), then: $\omega_\lambda(c)$ is non-minimal for each $\lambda \in [\lambda_1, \lambda_2]$, contains a periodic orbit, and in addition, $\omega_\lambda(c)$ contains a parabolic periodic orbit for both λ equal to λ_1 and to λ_2.

Moreover, we have

Lemma 12.5. *Assume that $[\lambda_1, \lambda_2]$ is as in the previous lemma and $\Lambda' \subset (\lambda_1, \lambda_2)$ is an open interval such that $\omega_\lambda(c)$ contains no parabolic cycles for all $\lambda \in \Lambda'$. Then for every $\lambda \in \Lambda'$, there is a first return map G_λ to a nice interval around c (depending on λ), such that all G_λ are included in a family of box-mappings depending complex-analytically on λ in a complex neighbourhood Ω of the open interval Λ', $\Omega \cap \mathbb{R} = \Lambda'$, and all maps $G_\lambda, \lambda \in \Omega$, are pairwise quasi-conformally conjugate.*

Proof of Lemmas 12.4-12.5. If $\omega(c)$ is minimal, we use Theorems A and B, and the further proof is a repetition of the proof of the infinitely renormalizable case of real analytic family of real analytic unimodal maps, see Sect. 7.2 of [Ko]. Note that condition 2) holds automatically in this case.

If $\omega(c)$ is not minimal, there is a substantial difference caused by the fact that the box mapping we constructed in Theorem A' has infinitely many branches, and in order to apply an idea of [Ko], we need to include this map in a complex analytic (on the parameter) family of such maps.

Let us assume that $\lambda_2 = 0$ and that $\omega_0(c)$ contains no parabolic cycles. Then the map g_0 satisfies the conditions of Theorems A' and its addendum, and Theorem B'. Therefore, the first return map of g_0 to a nice interval $P(0)$ around c, $R_0 : \cup P^i(0) \to P(0)$, can be extended to a box mapping $F_0 : \cup V^i(0) \to V(0)$, such that $V^i(0) \cap \mathbb{R} = P^i(0)$, $V(0) \cap \mathbb{R} = P(0)$. 'Zero' in the notations indicates that all this is constructed for the map g_0. Let us note that Theorem A' shows that all intervals $P^i(0)$ are very deep inside $P(0)$, so that we can take $V(0)$ to be $D(P(0); \pi - \theta)$, where $\theta > 0$ is small. We need the modification of $V(0)$ as made in the addendum to Theorem A', see Section 10. Namely, we replace (and fix, from now on) $V(0)$ by $\tilde{V}$, where $\tilde{V}$ is $D(P(0); \pi - \theta)$, except for two small neighbourhoods Z_1, Z_2 of the end points a_1, a_2 of the real trace $P(0)$ of $\tilde{V}$, so that $\partial \tilde{V}$ in Z_1, Z_2 is g_0^M-preimage of two leaves l_{v_1}, l_{v_2} (invariant under an iterate of g_0) of the foliation defined in Section 10. As explained in the addendum to Theorem A', with the new range $V(0)$, the first return map to $V(0)$ is still well-defined and forms the initial box map. Let us box mapping by $F_0 : \cup V^i(0) \to V(0)$ (forgetting about the box mapping before this modification).

Let us consider the Yoccoz puzzle pieces of the map F_0 as defined in the Step 3 of the proof of Theorem B' (see the previous Section): the pieces of level $j \geq 0$ are by definition the components of $F_0^{-j}(V(0))$. In particular, we consider the critical pieces $V_j(0)$, $j = 1, 2, \ldots$, of this partition (i.e., $V_j(0)$ is the component of $F^{-j}(V(0))$ containing c). Denote by $F_{j,0} : \cup_i V_j^i(0) \to V_j(0)$ the first return map to $V_j(0)$ containing the points of $\omega(c) \cap V_j(0)$. Our aim is to show that, provided j is large enough, one of the maps $F_{j,0}$ can be included in a holomorphic family $F_{j,\lambda}$ (induced by g_λ), for λ in a small (complex) neighbourhood of $\lambda = 0$. The idea is the following. We include each individual component of $F_{j,0}$ in a holomorphic motion which agrees with the g_λ-dynamics. On the other hand, provided j is large, the dynamics of $F_{j,0}$ is concentrated on a small neighbourhood of the real axis, and this persists for any individual motion. If now two motions intersect each other for some λ, then this intersection can be transferred to a fixed neighbourhood of the periodic point a_λ of g_λ. Since the local dynamics of g_λ repels a domain to a fixed distance away from the real axis, we obtain a contradiction.

1. Let us include $\partial V(0)$ in a holomorphic motion $\phi_\lambda : \partial V(0) \to \partial V(\lambda)$, $|\lambda| < \epsilon$, as follows.

Denote by a_λ the periodic point of g_λ, which is a holomorphic extension of the periodic point $a = a_0$ of g_0, which was used to generate the real partition. Since $\omega_0(c)$ contains no parabolic points, a_0 is repelling. One can even assume that a_0 is a fixed point.

First of all, using that the linearization Böttcher coordinates around a_λ depend holomorphically on λ, the leaves l_{v_1}, l_{v_2} are naturally included in motions $l_{v_1}(\lambda)$, $l_{v_2}(\lambda)$, so that $l_{v_i}(\lambda)$ are invariant under g_λ^{-1} in a neighbourhood Z of a_λ which is fixed once and for all. This holomorphic motion is transferred by some branches of g_λ^{-M} to neighbourhoods of $a_i(\lambda)$, where the branches are chosen so that $g_0^{-M}(a_0) = a_i(0)$ with $a_i(0)$ the end points of $P(0)$. In particular, the end points $Y_i(\lambda)$ of $g_\lambda^{-M}(l_{v_i}(\lambda))$ (other than $a_i(\lambda)$) are holomorphic functions of λ. The rest of $\partial V(0)$ between $Y_i(0)$, $i = 1, 2$, is an arc of the circle (and its complex conjugate). It can be included in a holomorphic motion as well within a family of arcs of circles joining $Y_i(\lambda)$. We choose a complex neighbourhood W_0 of $\lambda = 0$ so small that the end points of l_{v_i}, $i = 1, 2$, are some definite distance away from the real axis for all $\lambda \in W$. Then any point $x \in l_{v_i}$, $x \neq a_\lambda$ is repelled by iterates of g_λ inside the neighbourhood Z eventually to a definite distance $h > 0$ away from the real axis, where h is independent of $\lambda \in W_0$. Note that here we use that we work with the modified range V with pieces of 'external rays' in the boundary, rather than the original range.

We shall constantly use the following obvious fact: Consider a univalent map $g_0^n : A_0 \to B_0$, where A_0, B_0 are domains with piecewise analytic boundaries and let $\phi_\lambda : \partial B_0 \to \partial B_\lambda$ be a holomorphic motion. Then, for any λ in a complex neighbourhood of $\lambda = 0$, the map $g_\lambda^n : A_\lambda \to B_\lambda$ is also a well-defined univalent map such that the domain A_λ tends to A_0 as $\lambda \to 0$. In particular, Step 1 gives a holomorphic motion of each individual domain $V_j^i(0)$ defined by $g_\lambda^{-n_{i,j}} \circ \phi_\lambda \circ g_0^{n_{i,j}} : \partial V_j^i(0) \to \partial V_j^i(\lambda)$, where $g_0^{n_{i,j}}$ maps $V_j^i(0)$ onto $V(0)$. However, *a priori*, each such motion is defined in its own neighbourhood of $\lambda = 0$.

2. Choose j so large that the following holds: if $V_j^i(0)$ is any component of the map $F_j(0)$, then all iterates of $V_j^i(0)$ by g_0 until the range $V^j(0)$ of $F_j(0)$ lie within a $h/10$-neighbourhood of the real axis (where $h > 0$ has been defined in the step 1). To see this, use Fact 11.1, where we set $j = r(m)$ large enough. Indeed, then the moduli of the annuli $Q_{r(m)} \setminus V_{r(m)}$ are uniformly away from zero, and in particular $V_{r(m)}(0)$ shrinks to the critical point. By Fact 11.1 (b), with $x \in \mathbb{R}$, the statement follows.

3. With chosen j as in the previous step, we forget about all other maps. So let us write F_0, $V^i(0)$ instead of $F_{j,0}$, $V_j^i(0)$ (and so on). If (as for the minimal case) the number of components $V^i(0)$ is finite, then finitely many holomorphic motions already give us the desired extension. In our case, we also will obtain a holomorphic motion, because all, except finitely many, components $V^i(0)$ of F_0 are concentrated around a real set $X(0)$, on which g_0 is expanding, and this situation persists for all λ complex near 0.

To make this precise, we consider the piece $P = P(0)$, and for this piece we consider the sets $X(0) = X$ and $\hat{X}(0) = \hat{X}$ defined in the end of Section 10, and the corresponding complex map

$$G(0) : \cup B(J) \to \cup D(I; \pi - \beta),$$

where $B(J)$ are finitely many pairwise disjoint topological discs all contained properly inside $\cup D(I; \pi - \beta)$. Note that all $G(0)$-iterates of points of the set $\hat{X}(0)$ remain in the domain of definition of $G(0)$.

Now, let us construct the family F_λ. For any component $V^i(0)$ of the map F_0, choose $m_i \geq 0$ minimal, such that $g_0^{m_i}(V^i(0))$ is *not properly* contained in any domain $B(J)$. Then, since we have fixed n_0 in the construction of the map G in Section 10 and since all iterates of all $V^i(0)$ until the range $V(0)$ of F_0 are either disjoint or coincide, and are roughly discs, the number of different domains $g_0^j(V^i(0))$, $j = m_i, \ldots, N(i)$ (where $N(i)$ is such that $g_0^{N(i)}(V^i(0)) = V(0)$) is bounded (over all $V^i(0)$). Hence, all inverse branches of g_λ leading from $V(\lambda)$ to these finitely many domains can be well-defined in some small complex neighbourhood of $\lambda = 0$. On the other hand, all other branches (mapping from $g_0^{m_i}(V^i(0))$ to $V^i(0)$) persist for any other map g_λ, provided λ is in a small complex neighbourhood of $\lambda = 0$. The reason for this is that the map $G(0)$ is included in a complex analytic family $G(\lambda)$, where λ is close to $\lambda = 0$, where we fix the range of $G(\lambda)$ for all such λ, because the number of components of the domain of the map $G(0)$ is finite and their closures are pairwise disjoint. Therefore $G(\lambda)$ is uniformly expanding (where the expansion factor is bounded by some constant which does not depend on λ and on the points). It follows, that there exists a small enough complex neighbourhood of $\lambda = 0$, such that any $V^i(0)$, $g_0^{m_i}(V^i(0)) = V(0)$, is included in a holomorphic motion $V^i(\lambda)$, and $g_\lambda^{m_i}(V^i(\lambda)) = V(\lambda)$, and all $g_\lambda^k(V^i(\lambda)), k = 0, 1, \ldots, m_i$, remain in a $h/5$-neighbourhood of the real axis.

4. It remains to show that these holomorphic motion of the individual components are pairwise disjoint with each other and with $\partial V(\lambda)$ for every λ in a fixed neighbourhood of $\lambda = 0$. Assume that the boundaries of $V^i(\lambda)$ and $V^j(\lambda)$ are not disjoint, for some λ (close to 0). In this case it is impossible that the sizes of both $V^i(\lambda)$, $V^j(\lambda)$ are smaller than some $r > 0$ (independent on λ). This holds, because iterates of $G(\lambda)$ map such domains to disjoint domains. It is also impossible that both $V^i(\lambda)$, $V^j(\lambda)$ are larger than some constant $r' > 0$ when λ is close to zero, because there are only finitely many large components. So, the only possible case for which we could have an intersection would be that $V^i(\lambda)$ is one of the $\tilde{N}$ big components (where $\tilde{N}$ does not depend on λ), while $V^j(\lambda)$ is small (and the diameter of $V^j(\lambda)$ tends to zero as $\lambda \to 0$). In particular, $g_\lambda^{K_i}(V^i(\lambda)) = V(\lambda)$, where K_i are bounded from above by some universal K. To obtain a contradiction, observe that in this situation, the iterates $g_\lambda^k(V^j(\lambda))$, $k = 0, 1, \ldots, K$ lie in a small neighbourhood of the real interval, and that this neighbourhood even shrinks to the real axis as $\lambda \to 0$ (and the size of $V^j(\lambda)$ tends to zero as well). Therefore, $g_\lambda^{K_i+M}(V^j(\lambda))$ lies in a small neighbourhood of the fixed point a_λ of g_λ, and, moreover intersects, say, the leaf $l_{v_1}(\lambda)$ of the foliation (because we assumed that $V^i(\lambda), V^j(\lambda)$ did intersect and because $l_{v_1}(\lambda)$ is a piece of the boundary of $g_\lambda^{K_i+M}(V^i(\lambda)))$. It follows that iterates of g_λ repel a point of the domain $g_\lambda^{K_i+M}(V^j(\lambda))$ (staying inside the neighbourhood Z of a_λ) to h-distance from the real axis before returning to $V(\lambda)$. However, as shown in Step 3 this does not happen, and so we proved by contradiction that domains remain disjoint. The same argument also shows that domains are disjoint form the boundary of the range, provided λ is sufficiently close to 0.

The rest of the proof repeats the consideration of Sect.7.2 of [Ko] (the infinitely renormalizable case), using Theorem B'. If c is not recurrent, then we replace F by the map G constructed in Proposition 10.1 (so that it is not related to the piece P anymore, but the Cantor repeller of G contains $\omega(c)$ is this case), and again repeat the construction of [Ko]. □

13 Proof of Theorem B"

In this section we want to prove Theorem B", which states that one has a quasisymmetric conjugacy on the entire dynamical interval when there are no parabolic or attracting periodic points.

Proof of Theorem B": Let us for simplicity assume that the maps $g, \tilde{g}$ are induced by covering maps of the circle $\mathbb{R}/\mathbb{Z}$. If there are gaps in the domain of these maps, then the proof is even simpler. The proof goes in four steps.

Step 1. Let us first deal with the minimal case. Then in Theorem A we constructed a complex extension of the first return map to some nice interval U_k, but only considered the domains containing points in $\omega(c)$. Now consider the domains associated to extensions of ALL real domains of the first return map to the nice interval U_k. Thus we get as in the proof of Theorem A a quasi-box mapping $P_k\colon \cup V_k^i \to V_k$. That the domains $\hat{V}_k^i$ intersecting $\omega(c)$ are contained in V_k follows from $|\omega(c)| = 0$, see the proof of Theorem A. In order to see that the other domains are also contained in V_k, we need an improvement to the Schwarz Lemma 2.3, which is due to E. de Faria and W. de Melo, see Lemma 2.4 in part II of [FM]. This improvement states that there exists $\epsilon > 0$ so that the spoiling factor in the Schwarz Lemma in the angles of the Poincaré domains when going from $g^{j+1}(V_k^i)$ to $g^j(V_k^i)$ is at most $(1 + K|g^j(V_k^i)|^{1+\epsilon})$. Since the orbit of V_k^i up to its first entry in V_k is disjoint and since the maximum size of the intervals in this orbit is small when k is large, we get that $\prod_{j=1,\dots,p}(1 + K|g^j(V_k^i)|^{1+\epsilon})$ can be assumed to be as close to 1 as we want (here p is the return time of V_k^i). Hence V_k^i is inside a disc with angle close to $\pi/2$ (with some slits). Because the real trace of V_k^i is well inside the real trace of V_k, we then obtain that V_k^i is inside V_k. Thus we obtain a quasi-box mapping with infinitely many domains, which can intersect and are contained (but not necessarily properly) inside V_k. The range V_k is a disc with some slits: $V_k \cap \mathbb{R} = U_k$. Let us from now on fix k and remove the k from the notation. Consider a real one-parameter family of ranges $V(t)$ by slightly changing the radius of the disc in such a way that $V(0) = V$ but so that the real trace is independent of t. Then we get a family of quasi-box mappings $P(t)\colon \cup_i V^i(t) \to V(t)$ for $t \in \mathbb{R}$ sufficiently close to 0. Now we want to obtain a box mapping from this. For this we proceed as in Section 6. As in Step 1 of Section 6, construct a smooth box mapping $\hat{g}\colon \cup_i \hat{B}_i \to \hat{A}$, where $\hat{B}_i \cap \mathbb{R}$ are the (finitely many) domains of the definition of g. As above, consider the sequence of (all branches of the) first return maps to the central domains $\hat{g}_j\colon \cup_i \hat{B}_i^j \to \hat{A}^j$. In fact, we can choose $\hat{g}$ so that $\hat{A}^j \cap \mathbb{R} = U_j$ for all j. To do this, take an invariant curve through one of the fixed points p of g (transversal to $\mathbb{R}$), then choose $\hat{A}$ so that close to p, $\partial \hat{A}$ coincides with the invariant curve, and so that $\hat{g}$ coincides with g for all points in $\hat{A}$ close to the real line. Then extend g smoothly as in Section 6. Next, as in Step 2 of Section 6, intersect the smooth box mapping $\hat{g}_j$ with the quasi-box mapping $P(t)$ (where we assume that j is chosen so that two ranges $\hat{A}^j$ and V correspond to the same level). By the Theorem of Sard (which states that the set of critical values of a smooth mapping $h: \mathbb{R} \to \mathbb{R}$ has Lebesgue measure zero), it follows that we can find t close to zero, so that the ranges of $P(t)$ and $\hat{g}_j$ (which are both piecewise smooth curves) are transversal. In fact, the range $P(t)$ had two slits, but since we made sure that $\hat{A}^j \cap \mathbb{R} = U_j$, we get that the component of $\hat{A}^j \cap V(t)$ containing c is a topological circle consisting of a finite number of smooth arcs and without cusps (this follows from the transversality). So we obtain a box mapping $P: \cup_i \Omega^i \to \Omega$ so that $\partial\Omega$ is a quasi-circle. P is an iterate of g on each of the domains, the domains are all disjoint, and properly contained in Ω. We do not claim that the modulus of the annuli $\Omega \setminus \Omega^i$ can be bounded away from zero uniformly in i. However, by passing to the first return map to the central domain this modulus condition will be satisfied: the 'safe-space' condition 4 from Theorem A' in Section 10 holds. If we are in the non-minimal case, then we can also extend the box mapping from Theorem A' to a box mapping $P: \cup_i \Omega^i \to \Omega$ such that the closure of $\cup_i \Omega^i \cap \mathbb{R}$ again contains $\Omega \cap \mathbb{R}$. To prove this, we do not need the smooth polynomial-like mappings, but only that the real domains $\Omega^i \cap \mathbb{R}$ are deep inside $\Omega \cap \mathbb{R}$ (see the proof of Theorem A'). Both in the minimal and non-minimal case let A^+, A^- be the set of points in the upperhalf (lowerhalf) plane which are inside Ω but outside all the domains Ω^i and let $A = A^+ \cup A^-$.

Step 2: Now observe that the boundaries of A^+ and of A^- are quasicircle. The proof of this is identical to that in Step 1 of Section 11 because the boundary of Ω is a quasicircle, and because each of the domains Ω^i is mapped with uniformly bounded distortion to Ω.

Step 3: In Theorems B and B' we did obtain a quasiconformal map $\phi: \Omega \to \tilde{\Omega}$ conjugating $P: \cup_{i\in J} \Omega^i \to \Omega$ and $\tilde{P}: \cup_{i \in J} \tilde{\Omega}^i \to \tilde{\Omega}$ where the index set J corresponds to the domains intersecting $\omega(c)$. Take any quasiconformal map $\psi: \Omega \to \tilde{\Omega}$ extending the restriction of ϕ to $\partial\Omega$. Let K be its quasiconformal dilatation. By pulling back once, we obtain also K-quasiconformal maps $\psi_i: \Omega^i \to \tilde{\Omega}^i$ for each $i \notin J$. As in Step 2 of the proof of Theorem B' we can find a quasiconformal conjugacy $H: A \to \tilde{A}$ which agrees with ϕ on $\partial\Omega^i$ for $i \in J$ and with ψ_i on $\partial\Omega^i$ for $i \notin J$. Now extend $H: A \to \tilde{A}$ to $H: \Omega \to \tilde{\Omega}$ as follows. On $\cup_{i\in J}\Omega^i$ let H be equal to the conjugacy ϕ and on Ω^i with $i \notin J$ let H be equal to ψ_i. As in Fact 11.2, we obtain that $H: \Omega \to \tilde{\Omega}$ is quasiconformal. H is a conjugacy on the domains Ω^i intersecting $\omega(c)$. If g is minimal, then the domains where H is a conjugacy cover $\omega(c) \cap \Omega$. So by taking subsequent pullbacks (as in the proof of Fact 11.2 but since we already have a quasiconformal conjugacy on $\omega(c)$ the dilatation now will not depend on n), we obtain a sequence of quasiconformal homeomorphisms $H_n: \Omega \to \tilde{\Omega}$ with the same quasiconformal dilatation and converging to a conjugacy Φ between $P: \cup\Omega^i \to \Omega$ and $\tilde{P}: \tilde{\Omega}^i \to \tilde{\Omega}$. If g is non-minimal, then the construction of a conjugacy is already done in Steps 3 and 4 of the proof of Theorem B' (simply also include the domains which do not intersect $\omega(c)$; the proof is identical).

Step 4: There exists a minimal N so that the g^N maps $\Omega \cap \mathbb{R}$ injectively onto the whole dynamical interval. By taking, if necessary, puzzle pieces of deeper level in Ω which intersect the real segment $\Omega \cap \mathbb{R}$, we can make sure that the image under g^N of all these puzzle pieces are disjoint. Their union is the necklace neighbourhood mentioned in the statement of the theorem. Of course the conjugacy can be naturally defined (extending the previous conjugacy) quasiconformally to this neighbourhood by using $\tilde{g}^N \circ \Phi \circ g^{-N}$. Using the same composition, it also follows that any point of the real line has a neighbourhood on which the conjugacy extends quasiconformally. By compactness of the dynamical interval, it follows that this quasiconformal map is defined on a neighbourhood of the dynamical interval. Therefore, the conjugacy on the real line is a quasisymmetry. $\square$

14 Appendix

As before, let J be a real interval and $D_*(J)$ the disc with diameter J. Consider the inverse by $P(z) = z^\ell$ of $D_*([-1, K_0^\ell]) \setminus \mathbb{R}^-$ with $K_0 > 1$. The next lemma states that the component of this inverse which contains $[0, K_0]$ is contained in $D_*([-1, K_0])$.

Lemma 14.1 *Let $P_\ell(z) = z^\ell$ where $\ell \geq 2$, $-1 < 0 < 1 < K_0$ and*

$$W = \{z \in \mathbb{C};\ \arg(z) \notin [\pi(1 - 1/\ell), \pi(1 + 1/\ell)]\}.$$

Then

$$P_\ell^{-1}(D_*([-1, K_0^\ell])) \cap W \subset D_*([-1, K_0]).$$

In particular, some subset of

$$D_*([-1, K_0]) \cap \{z \in \mathbb{C};\ \arg(z) \in [-\pi/\ell, \pi/\ell]\}$$

is mapped diffeomorphically onto

$$D_*([-1, K_0^\ell]) \setminus \mathbb{R}^-.$$

Proof: Rescale $[-1, K_0]$ to $[-\Delta, 1]$ and so consider the disc D with diameter $[-\Delta, 1]$, where $\Delta < 1$, and define $P(z) = z^2$. Let us show that D is contained in $P(D)$ (and that the only intersection point is $z = 1$). Let us parameterize the boundary by $\alpha \mapsto r(\alpha) \cdot e^{i\alpha}$. We want to show that $r^2(\alpha) > r(2\alpha)$ except that equality holds if $r = 1$ (i.e. $\alpha = 0$).

Using the cosine rule we get

$$r^2(\alpha) + \frac{(1-\Delta)^2}{4} - r(\alpha)(1-\Delta)\cos(\alpha) = \frac{(1+\Delta)^2}{4}, \text{ i.e.,}$$

$$r^2(\alpha) - r(\alpha)(1-\Delta)\cos(\alpha) - \Delta = 0.$$

Hence,

$$\cos(\alpha) = \frac{r^2(\alpha) - \Delta}{r(\alpha) \cdot (1-\Delta)} \text{ and}$$

$$\cos(2\alpha) = 2\left(\frac{r^2(\alpha)-\Delta}{r(\alpha)\cdot(1-\Delta)}\right)^2 - 1.$$

Similarly, the point $r(\alpha)\cdot e^{i2\alpha}$ satisfies

$$r^2(2\alpha) - r(2\alpha)(1-\Delta)\cos(2\alpha) - \Delta = 0.$$

Let us write $A = r^2(\alpha)$ and $B = r(2\alpha)$ and substitute the expression for $\cos(2\alpha)$ in the last expression. Then we get

$$B^2 - B(1-\Delta)\left[2\frac{(A-\Delta)^2}{A(1-\Delta)^2} - 1\right] - \Delta = 0. \tag{14.2}$$

This means

$$\frac{(1+B)(B-\Delta)}{B(1-\Delta)} = \frac{B^2-\Delta}{B(1-\Delta)} + 1 = \frac{2(A-\Delta)^2}{A(1-\Delta)^2}.$$

If $A = B := C$ then this gives the following linear equation in C

$$(1+C) = 2\frac{(C-\Delta)}{(1-\Delta)}$$

which is only satisfied if $C = 1$. This means that the only intersection of D with $P(D)$ is at $z = 1$. To show that $\partial P(D)$ is outside ∂D (except at $z = 1$), let us analyze the situation close to $z = 1$. So regard in (14.2) B as a function of A and differentiate B w.r.t. A at $A = 1$ (and so also take $B = 1$). This gives

$$2B'B - B'(1-\Delta)\left[2\frac{(A-\Delta)^2}{A(1-\Delta)^2} - 1\right] - B\frac{2}{(1-\Delta)}\left[1-\Delta^2/A^2\right] = 0,$$

i.e., since $A = B = 1$,

$$2B' - B'(1-\Delta) - 2\frac{(1-\Delta^2)}{(1-\Delta)} = 0$$

which gives $B' = 2$ and so $B(A) < A$ for $A < 1$.

This, and Lemma 15.2 from [LS1] imply the proof of this lemma. □

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