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REAL BOUNDS, ERGODICITY AND NEGATIVE SCHWARZIAN FOR MULTIMODAL MAPS

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1. Introduction and Statement of Results

Over the last 20 years, many of the most spectacular results in the field of dynamical systems dealt specifically with interval and circle maps (or perturbations and complex extensions of such maps). Primarily, this is because in the one-dimensional case, much better distortion control can be obtained than for general dynamical systems. However, many of these spectucular results were obtained so far only for unimodal maps. The aim of this paper is provide all the tools for studying general multimodal maps of an interval or a circle, by obtaining

- *real bounds* controlling the geometry of domains of certain first return maps, and providing a new (and we believe much simpler) proof of *absense of wandering intervals*;
- provided certain combinatorial conditions are satisfied, *large real bounds* implying that certain first return maps are almost linear;
- *Koebe distortion* controlling the distortion of high iterates of the map, and *negative Schwarzian derivative* for certain return maps (showing that the usual assumption of negative Schwarzian derivative is unnecessary);
- *control of distortion* of certain first return maps;
- *ergodic properties* such as sharp bounds for the number of ergodic components.

We will give historical comments below the statements of the theorems.

There are many applications and potential applications of our bounds. For example, it is clear that any future renormalization results for multimodal maps (generalizing the unimodal results

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of Sullivan, McMullen, Lyubich, de Melo, Avila ...) would require real bounds. Our real bounds are one of the key ingredients in the proof that Axiom A maps are dense within the space of real polynomials with real critical points, see [7].

Let us now be more precise. Let $M = [-1, 1]$ or $M = S^1$ and let $f : M \to M$ be a smooth map. This map is called *multimodal* if M has a partition into finitely many subintervals on which f is strictly monotone. Without loss of generality we may and will assume that $f(\partial M) \subset \partial M$. Let $c_1, \dots, c_d$ be the critical points of f, i.e., the points where f' is zero. Throughout this paper we will assume that f is C^k outside these critical points, and that f is *non-flat* at these critical point, i.e., that for $i = 1, \dots, d$ and x near c_i we can write

$$f(x) = \pm|\phi_i(x)|^{\beta_i} + f(c_i),$$

where ϕ_i is C^k, $\phi_i(c_i) = 0$ and $\beta_i > 1$. Here $k = 2$ is enough for Theorem A, B1 and C1 and $k = 3$ for the remaining theorems. In fact, for Theorem A, B1 and C1 it suffices to take $k = 1 + Zygmund$, see [18]. We denote the class of such maps by $\mathcal{A}^k$ and note that $\mathcal{A}^3 \subset \mathcal{A}^2 \subset \mathcal{A}^{1+Zygmund}$.

If $\mathcal{B}$ is a Borelean set we will denote its Lebesgue measure by $|\mathcal{B}|$. We also will use the following two definitions:

Definition 1. *Let U, V be bounded intervals such that the closure of U is contained in the interior of V. We say that V is an α-*scaled *neighbourhood of U if $|U^+| \geq \alpha|U|$ and $|U^-| \geq \alpha|U|$, where U^+ and U^- are the connected components of $V \setminus U$. We also sometimes say that U is α-*well-inside *V.*

Definition 2. *An open interval $I \subset [-1, 1]$ is called a* nice interval *if the forward orbit of its boundary does not intersect I, that is; $I \bigcap f^i(\partial I) = \emptyset$ for each $i \geq 0$.*

One reason to consider the concept of nice intervals is the fact that the domain of the first return map of a nice interval I consists of a countable union of open intervals, called *return domains*, whose boundaries are mapped into the boundary of I. Moreover, two intervals in the backward orbit of a nice interval are nested or disjoint. Intervals of the backward orbit of a nice interval are nice intervals too.

Let us now define a sequence of nice intervals around any point $x \in M$. Let $I_0 = I$ be a nice interval containing x. In the Yoccoz partition, it is customary to choose I to be an interval of the backward orbit of a component of $M \setminus f^{-1}(P)$ where P is the set of fixed points of f, and then you can take for I_0 the component of $M \setminus f^{-1}(P)$ containing x. But any other choice for I_0

is also possible in this paper. Next define inductively a sequence of nice intervals I_n containing x as follows. Let ϕ_n be the first return map to I_{n-1} and let I_n be the domain of ϕ_n containing x. If x is not in the domain of ϕ_n then we define $I_n = I_{n+1} = \cdots = \emptyset$. Of course, if x is recurrent then $I_n \neq \emptyset$ and if I_0 is periodic $I_n = I_0$. We say that $\phi_{n+1}\colon I_{n+1} \to I_n$ is *non-central* w.r.t. $x \in I_{n+1}$ if $\phi_{n+1}(x) \notin I_{n+1}$. We say that f has *real bounds* at x, if there exists $\xi > 0$ such that I_{n+1} is ξ-well-inside I_n whenever $\phi_n : I_n \to I_{n-1}$ is non-central. By the second part of Theorem A, then ALL domains of the first return map to I_{n+1} are ξ'-well-inside I_{n+1}.

1.1. **Real Bounds.** Our first theorem controls the geometry of the domains of first return maps to nice intervals, and can be used to show that first return maps to arbitrarily small neighbourhoods of x are well-controlled.

Theorem A [Real Bounds] *For each $f \in \mathcal{A}^{1+Zygmund}$, there exist $\xi_0 > 0$ and a function $\rho\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that if $x \in M$ and $\phi_1\colon I_1 \to I_0$ non-central w.r.t. x, then the following holds.*

(1) *If $\phi_n : I_n \to I_{n-1}$ is non-central w.r.t. x and $n \geq 2$, then the interval I_{n+1} is ξ_0-well-inside I_n.*
(2) *if I_{n+1} is ξ-well-inside I_n, then* all *domains of the first return map to I_{n+1} are $\xi' = \rho(\xi)$-well-inside I_{n+1}.*

We do not assume that x is recurrent, but note that if the interval I_{n+1} is empty, then we consider the statement in Theorem A trivially true. Let us also remark that for any map f and any integer n, if x is a *turning point* of f, either I_{n+1} is well-inside I_n or there exists a definite neighbourhood of I_n which one can pull-back to I_{n+1} with bounded intersection multiplicity, see Lemma 2.

The constant ξ_0 and the function ρ in the previous theorem are universal: they only depend on the constant $\gamma(f)$ associated to f defined in equations (1) and (2) in Section 2. We note that $\gamma(f)$ does *not* depend on the dynamics of f, but only on the C^2 norm of f, the number of critical points and their order.

We need to assume in part 1 of the previous theorem that $n \geq 2$ (it is easy to give a counter example if n is allowed to be equal to 1).

We should emphasize that we do *not* use the non-existence of wandering intervals in the proof of Theorem A. In fact, we obtain an independent proof for the non-existence of wandering intervals:

Corollary of the proof of Theorem A [Non-existence of wandering intervals] *Let $f \in \mathcal{A}^{1+Zygmund}$ and assume that $J \subset M$ is an interval for which all iterates $J, f(J), \dots$ are mutually disjoint. Then $\{f^n(J)\}_{n\geq 0}$ converges to an (possibly one-sided) attracting periodic orbit.*

It is well-known (and easy to show) that this implies the contraction principle: for each $\delta > 0$ there exists $\epsilon > 0$ so that if J is an interval with $|J| < \epsilon$ and not intersecting the immediate basin of a periodic attractor, then each component of $f^{-n}(J)$ has length $\leq \delta$.

If for all n larger than some n_0, $\phi_{n+1} \colon I_{n+1} \to I_n$ is central w.r.t. x, then x is contained in the periodic interval $\cap I_n$ and (if $\cap I_n$ is not a singleton) f is said to be *renormalizable* (at x). Of course, in this case one can again apply Theorem A to any nice subinterval of the periodic interval $\cap I_n$. So even if f is (infinitely often) renormalizable Theorem A is relevant. To be more precise,

Theorem A' [Real bounds on arbitrarily small scales] *For each $f \in \mathcal{A}^{1+Zygmund}$ there exists $\xi_0 > 0$ so that for any non-periodic point $x \in M$ which is not in the basin of a periodic attractor, either 1 or 2 holds.*

(1) *f is infinitely renormalizable at x of Feigenbaum-type: there exist an integer p and nice periodic intervals $K_n \ni x$ of period $p2^n$ for every $n \geq 0$, such that K_{n+1} is ξ_0-well-inside K_n. (K_n is nice, so $f^{p2^n}(\partial K_n) \subset \partial K_n$.)*

(2) *There are arbitrarily small nice intervals I around x such that the return domain J to I containing x is ξ_0-well inside I.*

Moreover, define in case (1) $I = K_n$ and $J = K_{n+1}$, and in case (2) $I \supset J := \mathcal{L}_x(I)$. Then I, J are both nice intervals. If $I_i \supset J_i$ and $I_j \supset J_j$ are pullbacks of $I \supset J$, then they are either nested or disjoint, and if $J_i \subset I_j$ then $I_i \subset I_j$.

Here, and throughout this paper, we denote by $\mathcal{L}_x(I)$ the component containing x of the domain of the first entry map to I.

Real bounds as in Theorem A, but around recurrent turning points, were proved previously by Martens in the negative Schwarzian unimodal case and by Vargas in the case of C^2 multimodal maps without inflection points, see [14], [24] and also Shen's paper [19]. If all branches of f are monotone and there is at most one critical point (of inflection type), then such bounds were proved by Levin, see [8] and also [10]. Levin's proof (in particular Proposition 4 of his paper) does not seem to work if there are several critical points. The reason that inflection points complicate matters considerably is that when a return domain J to an interval I is very small compared to I, it still need not be well-inside I. If J contains a turning point one has symmetry, and this problem does not arise. Even if there are no inflection points, our proof is quite different from previous proofs.

Non-existence of wandering intervals for one-dimensional maps was proved in a series of papers by Denjoy, Guckenheimer, Yoccoz, de Melo & van Strien, Blokh & Lyubich, Martens & de Melo & van Strien. For historical references, see [18].

1.2. **Large bounds.** In applications one often needs large real bounds, in particular for proving density of Axiom A, see [21] and [7]. If I is a nice interval containing a non-periodic recurrent critical point c then we say that J is a *child* of I if J is a component of $f^{-1}(J')$ where J' is a diffeomorphic pullback of I and J contains a critical point c' with $\omega(c) = \omega(c') \ni c, c'$. If J_1, J_2 are two children of I, then they are either disjoint or they lie nested. In the next theorem we give two ways in which one can get large bounds. Parts 1 and 2 of Theorem B were proved previously by Weixiao Shen in respectively [19] and in [21, Proposition 4.1].

Theorem B [Large Bounds] *For any* $f\colon M \to M$ *in the class* $\mathcal{A}^3$ *the following holds.*

(1) For each $C > 0$, $\delta > 0$ there exists N' so that if a nice interval I has at least N' children $K_1 \supset K_2 \supset \dots$ containing some critical point, and each of its return domains to I is δ-well-inside I, then for $I' = K_N$ with $N \geq N'$ each of return domains to I' is C-well inside I'.
(2) There exists a function $\rho\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(\xi) \to \infty$ as $\xi \to \infty$ so that if I_{n+1} is ξ-well-inside I_n, then each return domain to I_{n+1} is $\rho(\xi)$-well-inside I_{n+1}.

The proof of Theorem B1 also shows that if x is a recurrent point and $\omega(x)$ is a non-minimal set, then one can find for each $\xi > 0$, a nice interval I around x such that the component of the first return map containing x is ξ-well-inside I. (One can also argue as in the proof of Theorem A' in [10].) (That a set X is *minimal* means that each orbit in X is dense in X.)

As before, N' and ρ only depend on $\gamma(f)$ from (1) and (2) and not on f.

1.3. **Koebe and negative Schwarzian.** One of the reasons real bounds are so useful, is because of the following distortion theorem. Part 1 and 2 show that, surprisingly, one often can use Koebe even if one does NOT have disjointness of intervals.

Theorem C [Koebe and negative Schwarzian] *Let* $f\colon M \to M$ *be in the class* $\mathcal{A}^3$. *Then one has the following properties:*

(1) [Improved Macroscopic Koebe Principle] *For each $\xi > 0$, there exists $\xi' > 0$ such that if I is a nice interval, V is ξ-well-inside I and $x \in I$ and $f^k(x) \in V$ (with $k \geq 1$ not necessarily minimal), then the pullback of V along $\{x, \dots, f^k(x)\}$ is ξ'-well-inside the return domain to I containing x.*

(2) [Improved Koebe Principle] *For each $S > 0$, $\delta > 0$ and $\xi > 0$ there exists $K > 0$ such that if $J \subset T$ are intervals, with $f^n_{|T}$ a diffeomorphism, $f^n(J)$ ξ-well-inside $f^n(T)$ and either*

(i) $\sum_{i=0}^{n-1} |f^i(J)| \leq S$ or

(ii) $f^n(T) \cap B_0(f) = \emptyset$ and $dist(f^i(T), Par) \geq \delta$, $i = 0, \dots, n-1$,

then $f^n_{|J}$ has bounded distortion, i.e. for any $x, y \in J$,

$$|Df^n(x)|/|Df^n(y)| \leq K.$$

Here $B_0(f)$ is the union of immediate basins of periodic (possibly parabolic) attractors and Par is the set of parabolic periodic points of f.

(3) [Negative Schwarzian Derivative] *For each critical point c_i which is not in the basin of a periodic attractor, there exists a neighbourhood U_i so that whenever $f^n(x) \in U_i$ for some $x \in M$ and some $n \geq 0$, then the Schwarzian derivative of f^{n+1} at x is negative:*

$$Sf^{n+1}(x) < 0.$$

In part (1) no disjointness is required, and the pullback of I along $\{x, \dots, f^k(x)\}$ can even meet a critical point an arbitrary large number of times! In fact, in part (1) it is enough to assume that $f \in \mathcal{A}^{k=1+Zygmund}$. In part (2), note that the period of periodic attractors and parabolic orbits of f is bounded, see [18]. In part (2) one can choose K close to 1 if ξ is large.

The last part of Theorem C generalizes the results of Kozlovski [6] (which he proved for unimodal maps) to general multimodal maps. Since this paper was written, we learned that Duncan Sands has used our bounds to obtain the following version of Theorem C.3: assume that all periodic points are hyperbolic and repelling, then f is conjugate to a map with negative Schwarzian derivative.

1.4. **Distortion control of first entry maps.** From Theorem C we get that first entry maps have bounded distortion:

Theorem D [Control of Distortion] *Assume that $f\colon M \to M$ is in the class $\mathcal{A}^3$. Then for each $\epsilon > 0$ and for each critical point c_i there exists a nice interval V_i of diameter $< \epsilon$ such that $c_i \in V_i$*

and so that the first entry map Ψ to $\cup V_i$ is quasi-polynomial (more precisely, a composition of at most d maps of the type $L_1 \circ f \circ L_2$, where L_1, L_2 are diffeomorphisms of bounded distortion). If f is infinitely renormalizable at c_i, then one can take for V_i a periodic interval containing c_i.

In Section 8 a more precise version of this theorem will be stated. It is important to remark that the first return map to a single critical neighbourhood V_{i_0} need not be 'quasi-polynomial' (Ψ is a composition of finitely many maps of type $L_1 \circ f \circ L_2$, where L_1, L_2 are diffeomorphisms of bounded distortion): the pullback of any definite neighbourhood of V_i along a branch of the first return map to V_{i_0} can meet a critical point an arbitrary large number of times. So the situation is definitely much more complicated in the multimodal case than in the unimodal case.

1.5. **Ergodic Properties.** For the next theorem we define a partial ordering on the set of critical points **Cr**: $c_i \prec c_j$ iff either $c_i \in \omega(c_j)$ or $c_i = c_j$. We then define $c_i \sim c_k$ iff both $c_i \prec c_k$ and $c_k \prec c_i$ (so in this case $\omega(c_i) = \omega(c_k)$).

Theorem E [Ergodic Properties] *For any $f\colon M \to M$ in the class $\mathcal{A}^3$ the following properties hold.*

(1) *Any minimal set X has zero Lebesgue measure: for any point $x \in X$ there are intervals $\mathcal{N}_n \subset \mathcal{U}_n$ such that $\cap \mathcal{U}_n = \{x\}$, $(\mathcal{U}_n \setminus \mathcal{N}_n) \cap X = \emptyset$ and $\mathcal{N}_n$ is ξ-well-inside $\mathcal{U}_n$, where $\xi > 0$ does not depend on n.*

(2) *There are finitely many compact forward invariant sets $X_1, \dots, X_k$ such that $\cup B(X_i)$ has full measure in M. Here $B(X_i)$ is the basin of X_i (i.e. in $B(X_i) := \{y; \omega(x) = X_i\}$). Moreover, either*
   - *X_i is an attracting periodic orbit,*
   - *X_i is a cycle of intervals containing a turning point and such that $\omega(x) = X_i$ for a.e. $x \in X_i$, or*
   - *X_i is a minimal set containing at least one recurrent critical point.*

(3) *For any set Y of positive Lebesgue measure with $f(Y) \subset Y$, not intersecting basins of periodic attractors and not containing intervals (up to measure zero), there exists a minimal set X containing a critical point c such that $|Y \cap B(X)| > 0$ and so that for the intervals $\mathcal{U}_n \supset \mathcal{N}_n \ni c$ from part (1), $|Y \cap \mathcal{N}_n|/|\mathcal{N}_n| \to 1$.*

(4) *If X_i is not an attracting periodic orbit then $f_{|B(X_i)}$ is ergodic with respect the Lebesgue measure. The number of these X_i's is bounded by the number of equivalence classes of* $\mathbf{Cr}/\sim$ *which are* minimal *w.r.t. $\prec$ (see the definition above).*

From [18] it is known that the number of non-equivalent attracting periodic points is bounded. Here we say that two periodic points p, q are equivalent if all iterates of f are homeomorphisms on (p, q).

Theorem E sharpens the classification of measure-theoretical attractors and ergodicity for smooth unimodal and multimodal maps obtained previously by Blokh and Lyubich, see [2] and [12].

2. Some of the tools used in this paper

Definition 3. *A sequence $\{G_i\}_{i=0}^{l}$ of open intervals in $[-1,1]$ is called a* chain *if G_i is the maximal interval such that $f(G_i) \subseteq G_{i+1}$, $i = 0, \dots, l-1$. We shall also call G_0 a* pullback *of G_l along $\{x, \dots, f^l(x)\}$ for $x \in G_0$.*

We notice that for any chain $\{G_i\}_{i=0}^{l}$, the boundary of G_i is mapped by f on the boundary of G_{i+1}. The *multiplicity of intersection of a chain* is the maximum number of intervals from it which has non-empty intersection.

Let $T \supset J$ be intervals and L, R the components of $T \setminus J$ and define $C(T, J) = (|T||J|)/(|L||R|)$ be their cross-ratio. Let $T' \supset J'$ be components of $f^{-1}(T)$ and $f^{-1}(J)$. Then if f is C^2 (in fact $C^{1+Zygmund}$ is enough) and has non-flat critical points (in the sense defined in the introduction of this paper), then there exists $\gamma(f) > 0$ (which does not depend on $T \supset J$) so that

$$\frac{C(T,J)}{C(T',J')} \geq \begin{cases} 1 - \gamma(f)|T| & \text{if } T' \text{ does not contain a critical point} \\ \gamma(f) & \text{if } T' \text{ does contain a critical point,} \end{cases} \tag{1}$$

and

$$|Df(x)| \leq \frac{1}{\gamma(f)} \frac{|f(T')|}{|T'|} \text{ for each } x \in T' \tag{2}$$

see [18]. It will be convenient to assume that if M is an interval, f is also defined on a $1/\gamma(f)$-scaled neighbourhood M' of M and that the above inequalities even hold if we take $J \subset M$ and $T \subset M'$.

From (1), (2) we get, see [18],

Lemma 1 (Koebe Distortion)**.** *Let $\{G_i\}_{i=0}^{l}$ and $\{H_i\}_{i=0}^{l}$ be chains such that $G_0 \supset H_0$ and such that G_l is a σ-scaled neighbourhood of H_l, for some $\sigma > 0$. If the multiplicity of intersection of $\{G_i\}_{i=0}^{l}$ is bounded by κ the following holds:*

(1) *G_0 is a α-scaled neighbourhood of H_0, where $\alpha > 0$ depends only on σ, κ and the constant $\gamma(f)$ from (1).*

(2) *If $G_{i_1}, \ldots, G_{i_\nu}$ are the intervals of the chain $\{G_i\}_{i=0}^{l}$ which contain critical points with $i_1 < i_2 < \cdots < i_\nu < i_{\nu+1} = l$ then the maps*

$$f^{i_{j+1}-i_j-1}|_{G_{i_j+1}} : G_{i_j+1} \to G_{i_{j+1}}, \quad \text{for any} \quad j = 1, \ldots, \nu$$

satisfy

$$\frac{|Df^{i_{j+1}-i_j-1}(x)|}{|Df^{i_{j+1}-i_j-1}(y)|} \leq K$$

for any $x, y \in H_{i_j+1}$, where $K < \infty$ depends only on σ, κ and the constant $\gamma(f)$ from (1).

(3) *$|Df^l(x)| \leq K' \frac{|H_l|}{|H_0|}$ for all $x \in H_0$*

where $K < \infty$ depends only on σ, κ and the constant $\gamma(f)$ from (1) and (2).

Proof. For the first two statement see [18]. The third assertion holds by the chain rule, the second assertion and because of (2) applied to $T = H_{i_j}$, $j = 1, \ldots, \nu$. $\square$

One of the main tools for obtaining Koebe space (which we need, together with Lemma 3, in order to apply the previous lemma) is the following analogue of a fact which is well-known if the map is unimodal, non-renormalizable and has negative Schwarzian derivative.

Lemma 2 (Existence of Koebe Space)**.** *There exists ρ_0 which only depends on the constant $\gamma(f)$ defined in (1) and (2) with the following properties. Let $I \subset M$ be a nice interval and J_1, J_2 return domains (not necessarily different) to I with return times r_1, r_2, respectively. Then there exists an interval $\mathcal{M}_i \supset I$ which is a ρ_0-scaled neighbourhood of either J_1 or J_2 and which contains at most $2^{b+1} + 2$ of the intervals $f^j(J_l)$, $j = 1, \ldots, r_l - 1$ and $l = 1, 2$. Here b is the number of turning points of f.*

Note that the assertion of the lemma does not exclude the possibility that $\mathcal{M}_i$ intersects (but not contains) one or two of those intervals. If M is an interval, and I is very close to a boundary of M, then it is possible that $\mathcal{M}_i$ contains a neighbourhood (in the real line) of a boundary point of M (this is the reason why we assumed that f is defined on a $1/\gamma(f)$ scaled neighbourhood of M). This will not cause any difficulties, because we will only consider pullbacks of $\mathcal{M}_i$. It is quite easy to prove a similar lemma for the case where we consider more than two components of the return map to I.

Proof. Define the collections $\mathcal{B}_i = \{f(J_i), \ldots, f^{r_i}(J_i)\}$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Let $I_{l,i}$ be the interval containing $f^l(J_i)$ in the pullback of I along $\{f^l(J_i), \ldots, f^{r_i}(J_i)\}$ and define the collections $\mathcal{C}_i = \{I_{l,i};\, l = 1, \ldots, r_i\}$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

Let us define integers m_1, $1 \le i \le 2$ and an interval $M_{m_1} \supset f^{m_1}(J_i)$ which we will pull backwards to get the scaled neighbourhood $\mathcal{M}_i$ of the statement. First note that two intervals from the collection $\mathcal{C}$ defined above are either disjoint or coincide. Let $f^{m_1}(J_i)$ be the smallest (shortest) interval from the collection $\mathcal{B}$. Then define $M_{m_1} := L \bigcup I_{m_1,i} \bigcup R$, where L, R are intervals adjacent to $I_{m_1,i}$ with $|L| = |R| = \rho_0' |f^{m_1}(J_i)|$. Here we can take $\rho_0' = 1$ if there are intervals from $\mathcal{B}$ on both sides of $I_{m_1,i}$, but otherwise we can take $\rho_0' = 1/\gamma(f) < 1$ where $\gamma(f)$ is so that (1) and (2) hold for intervals inside a $1/\gamma(f)$-scaled neighbourhood of M.

Note that $M_{m_1} \supset I_{m_1,i}$ is a ρ_0'-scaled neighbourhood of $f^{m_1}(J_i)$ and it contains no other interval from $\mathcal{B}_i$. Now consider the chain $\{M_t\}_{t=0}^{m_1}$ such that $M_t \supset f^t(J_i)$.

If the map $f^{m_1}_{|M_0} \colon M_0 \to M_{m_1}$ is surjective, then, for $0 \le t \le m_1$, M_t does not contain any interval from $\mathcal{C}_i$ except $I_{t,i}$ (it may contain another interval from $\mathcal{C}_j$, $j \ne i$). So in this case it follows that the chain $\{M_t\}_{t=0}^{m_1}$ has intersection multiplicity ≤ 3. Applying Lemma 1 we get a constant $\rho_0 = \rho_0(\rho_0')$ so that M_0 is a ρ_0-scaled neighbourhood of J_i and so define $\mathcal{M}_i := M_0$. This proves the lemma in this case.

However, even if the map $f^{m_1}_{|M_0} \colon M_0 \to M_{m_1}$ is not surjective, if $0 \le s, t \le m_1$ with $s \ne t$ and $M_s \supset I_{t,i}$ then $s < t$. Indeed, otherwise M_{m_1} would contain at least 2 intervals from $\mathcal{B}_i$ which is not true.

Let us first deal with the case that $J_1 = J_2 = I$ is a periodic interval of period $r = r_1 = r_2$. By the previous remark if $0 \le s, t \le m_1$ with $s \ne t$ and $M_s \supset I_{t,i}$ then $s < t$. We claim that in this case M_{m_1} *intersects* $I_{t',i}$ where $t' = t + m_1 - s (\operatorname{mod} r)$. Indeed, if $f^{\tilde{s}}(I_{t,i}) \subset I$ for some $0 \le \tilde{s} < m_1 - s$ then $f^{\tilde{s}+1}(M_s)$ intersects $f(I) \subset I_1$, and so the claim follows. Hence, each interval M_t, $t = 0, 1, \ldots, m_1$ *contains* at most 3 intervals from $\mathcal{C}$. In particular this chain has intersection multiplicity ≤ 8 in this case.

Next consider the case that I is not contained in a maximal periodic interval which is strictly contained in M. Consider those intervals $M_{n_1}, \ldots, M_{n_\zeta}$ from the collection $\{M_t\}_{t=0}^{m_1}$ which are mapped by f strictly inside M_{n_j+1}. Hence, for each $i = 1, \ldots, \zeta$, $f(\partial M_{n_i})$ consists of one point and M_{n_i} contains a turning point c_i of f such that the interval $f(M_{n_i})$ is bounded by $f(c_i)$ and $f(\partial M_{n_i})$. Assume that $M_{n_j} \cap M_{n_k} \ne \emptyset$, for some $n_1 \le n_j < n_k \le n_\zeta$. Then $I_{n_j,i}$ is not contained in M_{n_k}. Hence, if M_{n_j} contains the turning point $c_k \in M_{n_k}$ mentioned above, then because these intervals are pullbacks, $M_{n_j} \supset M_{n_k}$, which contradicts the assumption made in this paragraph. So M_{n_k} contains a turning point of f which is not contained in M_{n_j}. Moreover, the boundary of M_{n_j} is mapped by f inside $f(M_{n_k})$. Now take M_{n_j} maximal, in the sense that $M_{n_j} \cap M_{n_k} \supset M_{n_i} \cap M_{n_k}$, for each $n_i < n_k$. We claim that $M_{n_l} \cap M_{n_j} \cap M_{n_k} \neq \emptyset$ does not happen for $n_l, n_j < n_k, \quad n_l \neq n_j$. Indeed, since $f(\partial M_{n_j}) \in f(M_{n_k})$, otherwise $f(M_{n_l}) \subset f(M_{n_j})$. It follows that $M_{n_l} \subset M_{n_j}$. But again this contradicts the assumption made in this section. It follows that the multiplicity of intersection of the collection $\mathcal{D} := \{M_{n_1}, \dots, M_{n_\zeta}\}$ is at most equal to 2. In particular, $\zeta \leq 2b$.

We claim that $M_{n_{\zeta-j}}$ contains at most $j+4$ intervals from $\mathcal{C}$. Indeed, first observe that M_{n_ζ} contains at most 4 intervals from $\mathcal{C}$, otherwise it would contain at least 5 intervals and therefore M_{m_1} would contain at least 2 intervals from $\mathcal{C}$, contradicting the definition of M_{m_1}. Let us proceed by induction on j and assume that $M_{n_{\zeta-(j-1)}}$ contains at most $j+3$ intervals from $\mathcal{C}$. Then, if $M_{n_{\zeta-j}}$ contains at least $j+5$ intervals from $\mathcal{C}$ the interval $M_{n_{\zeta-(j-1)}}$ would contain at least $j+4$ intervals from $\mathcal{C}$, a contradiction again.

This also implies that for $n_{\zeta-j-1} < t < n_{\zeta-j}$, the interval M_t contains at most $j+4$ intervals from $\mathcal{C}$. Since $\zeta \leq 2b$, we conclude that any interval M_t contains at most $2b+3$ intervals from $\mathcal{C}$. Next we claim that the multiplicity of intersection of $\{M_t\}_{t=0}^{m_1}$ is bounded by $4b+7$. Indeed, notice that M_t contains $I_{t,i}$ and that the intervals $I_{1,i}, \dots, I_{m_1,i}$ are pairwise disjoint. So if $M_{t_1}, \dots, M_{t_{4b+8}}$, have a point in common, then one of these, say M_{t_s}, must contain at least $2b+4$ from $\mathcal{C}$ which again gives a contradiction.

Finally we need to deal with the case that I might be contained in a (maximal) periodic interval which is strictly contained in M. If the length of a nested chain of intervals from $\mathcal{D}$ is at most 2 we still can argue as before and get that $\zeta \leq 4b$. The multiplicity of intersection of $\{M_t\}_{t=0}^{m_1}$ in this case is at most $8b+7$. If the length of a nested chain of intervals from $\mathcal{D}$ is at least 3 then I is contained in a l-periodic interval which contains at least 3 intervals from $\mathcal{C}$. Let l be maximal so that I is contained in a periodic interval with period l, and let T be the corresponding maximal periodic interval containing I. Applying the proof of this lemma for the periodic case together with Lemma 3 to $f^l|T$, we get that the derivative of f^l on T is bounded by some universal number N. Let I_i be the shortest interval from $\mathcal{C}$ in T. If there are intervals from $\mathcal{C}$ in both components of $T \setminus I_i$ then take $m_i = i$ and M_{m_1} to be a 1-scaled neighbourhood of I_{m_1}. If I_i has only elements from $\mathcal{C}$ in one component of $T \setminus I_i$ then (since there are at least 3 intervals from $\mathcal{C}$ in T), then consider the pullback of I_i by f^{-l} or f^{-2l} and thus obtain an interval I_{m_1} from

$\mathcal{C}$ in T and a $1/M^2$-scaled neighbourhood of I_{m_1} inside T which does not contain any interval from the collection $\mathcal{C}$. So now we can apply what we did before to $f^l_{|T}$. Since this map has at most $2^b - 1$ turning points. We conclude that any interval M_t contains at most $2^{b+1}+2$ intervals from $\mathcal{C}$ and the multiplicity of intersection of $\{M_t\}_{t=0}^{m_1}$ is at most $2^{b+2}+5$. Since M_{m_1} is a scaled neighbourhood of $f^{m_1}(J_i)$, it follows from Lemma 1 that $\mathcal{M} := M_0$ is a scaled neighbourhood of J_i which satisfies our statement. $\square$

The previous lemma implies that pullbacks of $\mathcal{M}_i$ along $J_j, \ldots, f^{r_j}(J_j)$ have bounded intersection multiplicity:

Lemma 3. *Let J be a return domain of a nice interval I with return time r. Assume that $T \supset I$ contains at most e of the intervals $I_1, \ldots, I_r = I$, where I_j are the pullbacks of I with $I_j \supset f^j(J)$. Then the following holds. Let $\{T_i\}_{i=0}^r$ be the chain so that $T_r = T$ and $T_i \supset f^i(J)$. Then the multiplicity of intersection of this chain is bounded by $2(e+b(e+2))+1$. In particular, if $T_{n_1}, \ldots, T_{n_\nu}$ are the intervals from $\{T_i\}_{i=0}^r$ which contain critical points then $\nu \leq 2(e+b(e+2))+1$.*

Proof. As in the previous lemma we can assume that f is non-renormalizable (or more precisely, that the smallest periodic interval containing I is equal to M). First we prove that at most $b(e+2)$ of the intervals $T_1, \ldots, T_r$ contain turning points. Indeed, since f is non-renormalizable, if for $0 \leq i_1 < \ldots < i_j \leq r$ the intervals $T_{i_1}, \ldots, T_{i_j}$ contain the same turning point, then $T_{i_1} \subset T_{i_2} \subset \ldots \subset T_{i_j}$. Therefore T_r contains the intervals $f^{r+i_1-i_j}(J), f^{r+i_2-i_j}(J), \ldots, f^r(J)$ and so $j \leq e+2$. Since f has b turning points we conclude that at most $b(e+2)$ of the intervals $T_1, \ldots, T_r$ contain turning points. This implies that an interval T_i contains at most $e + b(e+2)$ of the intervals $I_1, \ldots, I_r = I$. (Indeed, if for example $T_i \supset I_m$ and I_m contains a turning point then T_{i+1} might not contain I_{m+1}. So each time an iterate of T_i meets a turning point, one interval could be 'lost'. So in total one can 'loose' $b(e+2)$ intervals. Hence, the intersection multiplicity of $\{T_i\}_{i=0}^r$ is bounded by $2(e+b(e+2))+1$. $\square$

By applying Lemma 2 to the case that $J = J_1 = J_2$, using the disjointness which is provided by Lemma 3 for the pullback of $\mathcal{M}_i$, we immediately get the following distortion result from Lemma 1.

Corollary 1. *Let I be a nice interval with return domain J and denote the first return map by $\phi\colon J \to I$. If $|J|/|I|$ is close to one, define $K = J$. Otherwise define $K = \phi^{-1}_{|J}(J)$. Then the restriction of ϕ to a definite neighbourhood of K is* quasi-polynomial*: it can be written as a* *composition of at most d maps of the form $L_1 \circ f \circ L_2$ where L_1, L_2 have bounded distortion, f is the original map and d is the number of critical points.*

3. First return maps

In this section we are going to prove some important metric properties of individual branches of a first return map. It will be fundamental later on, when we will be pulling back Koebe space through many different branches. So, here we fix a nice interval I with its first return map and we will be in general analyzing the action of ϕ in one return domain $J \subset I$.

Definition 4. *Let $J \subset I$ be a return domain, and let $x \in J$ and $m > 0$ be so that $x, \dots, \phi^{m-1}(x) \in J$ and $\phi^m(x) \notin J$. Then the* fundamental domain of ϕ *containing x is the maximal interval $D \ni x$ for which $D, \phi(D), \dots, \phi^{m-1}(D) \subset J$ and $\phi^m(D) \subset I \setminus J$.*

From this definition it follows that the intervals $D, \dots, \phi^m(D)$ are pairwise disjoint. If $\phi_{|J}$ is monotone, then $\phi^m(D)$ is equal to a connected component of $I \setminus J$ and the above definition of fundamental domain coincides with the usual one.

Definition 5. *Take $\alpha > 0$ and let A be an interval in I and H a connected component of $I \setminus A$. If $|H| < \alpha|A|$, then we call H an* α-small side of A *in I; otherwise we call H an* α-big side of A *in I (often we omit the reference to I if it is clear).*

Since we shall repeatedly introduce constants, we shall say that a constant α is *universal* if only depends on previous constants. Often we will omit constants which are universal, for example, if V is an α-scaled neighbourhood of U with $\alpha > 0$ universal, then we say that V is a *scaled neighbourhood* of U or also that U is *well-inside* V. The same convention will be used for α-big sides and others analogous situations.

Take a domain J on which ϕ is non-monotone. Notice that if $\phi_{|J}$ is a composition of maps with at most one turning point then $\phi_{|J}$ is special: $\phi(\partial J)$ consists of just one point in ∂I. Moreover, $\phi(J)$ is equal to $\phi(J \setminus J')$ where $J' \subset J$ is an interval and $J \setminus J'$ consists of two intervals on which ϕ is monotone (whose endpoints are turning points of $\phi_{|J}$ and boundary points of J). In any case, if we denote $J^1, \dots, J^l$ the domains of monotonicity of $\phi\colon J \to I$ labeled from left to right, then J^1, J^l are called the *external branches* of J, or the *external intervals of monotonicity* of $\phi_{|J}$. When $\phi\colon J \to I$ is surjective then it is called *orientation preserving* (resp. *reversing*) if it is so at least on both external branches (i.e., on J if ϕ is monotone). So we do not need that ϕ is monotone, for it to be orientation preserving. If $\phi\colon J \to I$ is not surjective, then this notion is simply not defined.

Lemma 4. *There exists $\delta_1 > 0$ such that for any return domain $J \subset I$ and any fundamental domain D in J which is not contained in the immediate basin of a periodic attractor, D has at least one δ_1-big side and $\phi(D)$ is contained in a δ_1-big side of D.*

Proof. Let us prove this lemma by contradiction: assume that $\phi(D)$ is contained in an α-small side of D, with $\alpha > 0$ small. Then J has an α-small side, and $\phi(D)$ is contained well-inside the interval $\mathcal{M}$ given by Lemma 2 (applied to the collection consisting of just J). In particular, the pullback of a definite neighbourhood of $\phi(D)$ is a definite neighbourhood V of D. But if α is small, this implies that $\phi(V) \subset V$ and (by the last property of Lemma 1 and Lemma 3), that $|\phi'(x)| < 1$ for each $x \in V$. Hence D is contained in the immediate basin of a periodic attractor of f, a contradiction. $\square$

Fix δ_1 as in the previous lemma.

Lemma 5. *For each $\xi > 0$ sufficiently small, there exists $\delta_2 > 0$ with the following property. Let J be a return domain of ϕ, let D be a fundamental domain in J and $k \geq 1$ be so that $D, \ldots, \phi^{k-1}(D) \subset J$ while $\phi^k(D) \cap J = \emptyset$. If D is not δ_2-well-inside I then*

- *J has at least one ξ-small side;*
- *there exists a maximal external interval of monotonicity $J^e \subset J$ of $\phi_{|J}$ which is adjacent to a ξ-small side H of J and such that $D, \ldots, \phi^{k-2}(D) \subset J^e$, and $\phi^k(D) \cap H = \emptyset$;*
- *the intervals $D, \ldots, \phi^k(D)$ lie ordered, each of the intervals $D, \ldots, \phi^{k-1}(D)$ has a ξ-small side; (by the previous lemma $\phi^{i+1}(D)$ lies in the δ_1-big side of $\phi^i(D)$ for $i = 0, \ldots, k-1$);*
- *if $k \geq 2$ then $\phi_{|J^e}$ is orientation preserving;*
- *if both sides of J are ξ-small, then $D \subset J^e$, $\phi_{|J^e}$ is orientation reversing and $k = 1$.*

Proof. Assume D is not well-inside I. Then J has at least one small side. Let $\mathcal{M}$ be the interval given by Lemma 2 associated to the collection $\{J\}$. Using the properties of this interval $\mathcal{M}$ it follows that all intervals of monotonicity of $\phi_{|J}$ restricted to $\phi_{|J}^{-1}(J)$, except possibly the external ones, are well-inside I. If both sides of J are small, then all non-external branches of $\phi_{|J}$ are well-inside I. Moreover $\phi_{|J}^{-1}(J)$ is well-inside J, whenever $\phi_{|J}$ is not monotone and $\phi_{|J}(\partial J) \in \partial I$ is contained in the closure of a big side of J.

Let us first consider the case that one side of J is big. If $k = 1$ then it follows from the previous lemma that $\phi(D)$ is contained in the big side of J, completing the proof of the lemma in this case. So let us assume that $k \geq 2$. Then the boundary point of J on the small side of J is mapped to the small side of J, because otherwise $\phi_{|J}^{-1}(J)$ and therefore D is well-inside I. So we may assume (if $k \geq 2$) D intersects the interval J^e of monotonicity of $\phi_{|J}$ which is adjacent to a small side of J and that ϕ is orientation preserving on this interval J^e. If $\phi'_{|J^e}$ is universally bounded, then it follows that D must be well-inside I, a contradiction. If this derivative is not universally bounded, then by Lemma 3 it follows that $\phi(J^e) \supset J$ (and hence $\phi_{|J^e}$ has a fixed point). Since $k \geq 2$ this implies that D is contained in J^e. Since each component of $\phi_{|J}^{-1}(J)$ except the one contained in J^e is well-inside I, $D, \ldots, \phi^{k-2}(D) \subset J^e$. By the previous lemma, for $i = 0, \ldots, k-1$, $\phi^i(D)$ has at least one δ_1-big side which contains $\phi^{i+1}(D)$. It is sufficient to prove that for $i > 0$ at least one side of $\phi^i(D)$ is small and that the other is $>> 1$ big. Let U be the interval connecting D with the fixed point p of $\phi_{|J^e}$. Since D is not well-inside I, D lies on the side $J^e \setminus \{p\}$ which is not adjacent to the small side of J and, moreover, $\phi(U)$ is large compared to U. Hence, from part 3 of Lemma 1 and Lemma 3 it follows that $\phi^{i+1}(U)$ is also large compared to $\phi^i(U)$. This shows that one side of $\phi^i(D)$ is small, and so the proof is completed in this case.

If both sides of J are small, then $\mathcal{M}$ contains a definite neighbourhood of I and $\phi'_{|J}$ is universally bounded. So adjacent fundamental domains associated to periodic points of $\phi_{|J}$ of period ≤ 2 are of comparable size. From this it follows that D is well-inside I, except possibly if $\phi(D) \cap J = \emptyset$ (i.e. $k = 1$) and D intersects an external domain J^e of monotonicity of $\phi_{|J}$ such that $\phi_{|J^e}$ is orientation reversing. □

Lemma 6. *For each $\rho > 0$ sufficiently small, there exists $\delta_3 > 0$ such that if I is a ρ-scaled neighbourhood of an interval $V \subset I \setminus J$ then J is a δ_3-scaled neighbourhood of any component A of $\phi_{|J}^{-k}(V)$ (where $k \geq 1$ is arbitrary).*

Proof. Let D be the fundamental domain containing A. If $\phi(D)$ is well-inside I, then D is well-inside J. So if the assertion of this lemma does not hold, then k is large and $\phi(D)$ is not well-inside I. From the previous lemma, it follows that the disjoint intervals $\phi(D), \ldots, \phi^{k-1}(D), \phi^k(D)$ lie ordered in I and that $\phi^{i+1}(D)$ lies in the unique δ_1-big side of $\phi^i(D)$ for $i < k$. Since $\phi^{k-1}(D)$ has a ξ-small side (which contains $\phi^{k-2}(D)$) and its δ_1-big side contains V and since V is well-inside I, the interval V is also well-inside the δ_1-big side W of $\phi^{k-2}(D)$. But then we can pullback W with intersection multiplicity ≤ 2 to a definite neighbourhood of A, and so we are done. □

4. Visited domains and jumping times

Let $I \subset M$ be a nice interval and take a point $x \in I$ which visits I infinitely many times. Let us associate to $x \in I$ the *sequence of visited domains* $\{J_i(x, I)\}_{i=0}^{\infty}$ and the *sequence of jumping times* $\{k_i(x, I)\}_{i=0}^{\infty}$ defined by taking $k_0(x, I) := 0$ and inductively for all $i \geq 0$,

$$\phi^{k_i(x,I)+j}(x) \in J_i(x, I), \quad 0 \leq j < k_{i+1}(x, I) - k_i(x, I)$$

and

$$\phi^{k_{i+1}(x,I)}(x) \notin J_i(x, I).$$

If $\phi^{k_i(x,I)+j}(x) \in J_i(x, I)$ for all $j \geq 0$, then we define $k_{i+1}(x, I) = \infty$. Note that $k_i(x, I) = \infty$ can happen, for example, when x is eventually periodic or when f is an infinitely renormalizable map and the forward orbit of x hits an interval which is mapped into itself by the first return map to I. Note also that $k_{i+1}(x, I) - k_i(x, I) > 1$ corresponds to a 'saddle-node cascade'. Unless necessary, we will denote the visited domains and the jumping times without the dependence on x and I.

Lemma 7. *Given a nice interval I and a point $x \in I$ we consider its sequences of visited domains and jumping times, respectively, $\{J_i\}_{i=0}^{\infty}$ and $\{k_i\}_{i=0}^{\infty}$. Let us assume that $k_i < \infty$ for all $i \geq 0$. Then for any $\rho > 0$ and any $n > 0$ there exists i_0 with $0 \leq i_0 \leq n$ called* stopping time *such that the following holds:*

1. *For $0 \leq i < i_0$, at least one side H_i of J_i is ρ-small and J_{i+1} lies on the other side G_i of J_i. So, only if both sides of J_i are ρ-small, J_{i+1} is allowed to lie in a ρ-small side of J_i.*
2. *For $0 \leq i < i_0$; $(J_i, J_{i+1}) \bigcap (\bigcup_{k=0}^{i} J_k) = \emptyset$.*
3. *One of the following properties holds:*
   - *Property P_1: both sides of J_{i_0} are ρ-big sides and $i_0 < n$.*
   - *Property P_2: J_{i_0} has precisely one ρ-small side, and J_{i_0+1} is contained in that side and $i_0 < n$.*
   - *Property P_3 (and not P_1 or P_2): $(J_{i_0}, J_{i_0+1}) \bigcap (\bigcup_{k=0}^{i_0} J_k) \neq \emptyset$ and $i_0 < n$. In this case we define, for later use, t to be the largest integer with $0 \leq t < i_0$ and $J_t \subset (J_{i_0}, J_{i_0+1})$.*
   - *Property P_4: $i_0 = n$ and so $(J_i, J_{i+1}) \bigcap (\bigcup_{k=0}^{i} J_k) = \emptyset$ for all $0 \leq i < n$.*

In Statement (1) it is not claimed that G_i is a ρ-big side. It is useful to have a graphic image to explain what Statements (1) and (2) of this lemma mean: the intervals are visited in a spiral fashion spiraling 'towards the center'. For example, connect J_i and J_{i+1} by a semi-circle in the upper half plane. Statement (2) says that one can draw these semi-circles without intersections. Property P_3 states that this spiral structure is broken and Property P_4 that the spiral structure is maintained until the n-th visit. Because of the spiral structure of the intervals $J_0, \dots, J_{i_0}$ (and maximality of t), if Property P_3 holds for i_0 then $H_t \cap H_{i_0} = \emptyset$ and $G_t \cap G_{i_0} \neq \emptyset$.

Proof of Lemma 7: The proof of this lemma is essentially the definition of properties P_i. $\square$

In the next lemmas we show that one can pullback space if the spiral structure is preserved as in P_4 (many visits to the same domain or a long cycle between two visited domains is allowed), and that one gets space inside I anyway if P_1, P_2 or P_3 hold.

From now on fix $\rho > 0$ smaller that $\rho_0/2$ (where ρ_0 is given by Lemma 2), so it makes sense to say that one of the properties P_i is satisfied.

4.1. **Pulling back space.**

Lemma 8. *There exists $\sigma_1 > 0$ such that if $x \in I$ satisfies Property P_4 with $i_0 = n$, and if all of the following properties are satisfied: (i) $J_0 = J_n$ has precisely one small side, (ii) $\phi(J_0)$ contains the small side of J_0 and (iii) $\phi_{|J_0}$ is not an orientation reversing surjective branch (in the sense defined above Lemma 4), then I is a σ_1-scaled neighbourhood of the pullback U of $J_{i_0} = J_n$ along $\{x, \dots, \phi^{k_n}(x)\}$.*

Proof. Let $i_0' > 0$ be minimal so that $J_{i_0'} = J_0$. We claim that the pullback of $J_{i_0'}$ along $\{x, \dots, \phi^{k_{i_0'}}(x)\}$ is well-inside I. So, in other words, we claim that it is enough to consider the case that $i_0' = n$ and so $J_0, \dots, J_{n-1}$ are pairwise disjoint.

So let us assume this and prove the lemma in this case. Since P_4 holds we can assume that $J_0, J_{n-1}, \dots, J_1$ are ordered in I from left to right. Note that U is contained in a fundamental domain D_0 which by Lemma 5 has space to its right inside I. Analogously, $\phi^{k_1}(U)$ is contained in a fundamental domain $D_1 \subset J_1$ which has space to its left inside I. It remains to show that there is space to the left of U inside I.

Let $\mathcal{M}$ be the interval from Lemma 2 associated to the collection $\{J_0, J_{n-1}\}$. There are two possibilities.

Case 1. $\mathcal{M}$ is a ρ_0-scaled neighbourhood of J_{n-1}. Then also J_1 is well-inside $\mathcal{M}$ because $J_1 = J_{n-1}$ or J_1 is to the right of J_{n-1}. As we observed before $D_1 \subset J_1$ has space on its left and if $\phi_{|J_0}$ is an orientation preserving surjective branch we can pullback this space to an space on the left of U inside J_0 (see Lemma 6). If $\phi_{|J_0}$ has at least one turning point, because $\phi(J_0)$

contains a small side of J_0, it must contain the big side of D_1 and then we get that the pullback of D_1 to x is well-inside J_0.

Case 2. $\mathcal{M}$ is a ρ_0-scaled neighbourhood of J_0. Let us consider the fundamental domain $D_{n-1} \subset J_{n-1}$ which contains $\phi^{k_{n-1}}(U)$ and let $\tilde{\mathcal{M}}$ be the pullback of $\mathcal{M}$ by the extension of $\phi_{|J_{n-1}}$. By the same reason as above, D_{n-1} has space on its left. Then $V = \phi^{-1}_{|J_{n-1}}(J_n) \subset D_{n-1}$ has space on its left and it is well-inside $\tilde{\mathcal{M}}$. Now if V is well-inside I, using Lemma 6, we can pullback it to $\phi^{k_{n-1}}(x)$ and using the disjointness of $J_0, \dots, J_{n-2}$ we get that U is well inside $J_0 \subset I$. If V is not well-inside I and $n > 2$ we have that J_1 is well-inside $\tilde{\mathcal{M}}$ and has space to its left. The same reasoning as in Case 1 can be applied. If V is not well-inside I and $n = 2$ we also can apply the same reasoning as in Case 1 to pullback to U the space on the left of V (or, of D_1 in the case that $k_2 - k_1 > 1$). $\square$

Lemma 9. *For each $\xi > 0$ there exists $\xi' > 0$ with the following property. Let $n > 0$ and let $\{J_i\}_{i=0}^n$ and $\{k_i\}_{i=0}^n$ be, respectively, the first n visited domains and jumping times of some point $x \in I$. Assume that property P_4 holds for n. Let $V \subset I$ with $\phi^{k_{n+1}}(x) \in V$ and so that V is ξ-well-inside I. Then the pullback V_0 of V along $\{x, \dots, \phi^{k_{n+1}}(x)\}$ is ξ'-well-inside J_0.*

Proof. According to Lemma 6 the pullback V_n of V along $\{\phi^{k_n}(x), \dots, \phi^{k_{n+1}}(x)\}$ is well-inside J_n. Since P_4 holds for n there exists $i_1 \le n$ so that $\phi^{i_1}(x)$ is the first visit of x to J_{i_1} and so that

$$\text{if } n - i_1 = 0 \mod 2 \text{ then } J_{i_1} = J_{i_1+2} = \cdots = J_n \text{ and } J_{i_1+1} = J_{i_1+3} = \cdots = J_{n-1}$$

and

$$\text{if } n - i_1 = 1 \mod 2 \text{ then } J_{i_1} = J_{i_1+2} = \cdots = J_{n-1} \text{ and } J_{i_1+1} = J_{i_1+3} = \cdots = J_n.$$

It is enough to show that the pullback V_{i_1} of V_n along $\{\phi^{k_{i_1}}(x), \dots, \phi^{k_n}(x)\}$ is well-inside J_{i_1} because the further pullback along $\{x, \dots, \phi^{k_{i_1}}(x)\}$ of J_{i_1} is disjoint ($\phi^{k_{i_1}}(x)$ is the first visit to J_{i_1}). If there exists an integer $i_1 \le i'_1 \le n$ for which $\phi^{k_{i'_1}}(x)$ is contained in a fundamental domain of $J_{i'_1}$ which is δ_2 well-inside $J_{i'_1}$ then is enough to pulling back this space along $\{\phi^{k_{i_1}}(x), \dots, \phi^{k_{i'_1}}(x)\}$. This means that we may assume that $\phi^{k_{i_1}}(x), \dots, \phi^{k_{i'_1}}(x)$ are contained in fundamental domains of respectively $J_{i_1}, \dots, J_{i'_1}$ which are not δ_2 well-inside $J_{i_1}, \dots, J_{i'_1}$. This means that we can apply Lemma 5 throughout the remainder of this lemma. In particular $\phi^{k_{i_1}}(x), \dots, \phi^{k_{i'_1}}(x)$ are all contained in fundamental domains of $J_{i_1}, \dots, J_{i'_1}$ which intersect the maximal external interval of monotonicity. Because of Lemma 6 it is also enough to assume that $i'_1 - i_1$ is large. So let us assume for example that $i'_1 - i_1 \ge 6$ (we take i_1 the biggest as possible with this property), so that $J_{i_1} = J_{i_1+2} = \dots$ and $J_{i_1+1} = J_{i_1+3} = \dots$ (where $\dots$ refers to intervals up to $J_{i'_1}$ or $J_{i'_1-1}$). Then $\phi^{k_{i_1}}(x), \phi^{k_{i_1+2}}(x), \dots, \phi^{k_{i'_1-3}}(x)$ are all in a orientation preserving external branch of the first return to J_{i_1}. We can assume that the pullback $V_{i'_1-1}$ of V_n along $\{\phi^{k_{i'_1-1}}(x), \dots, \phi^{k_n}(x)\}$ is well-inside J_{i_1} and using Lemma 6 the lemma follows. $\square$

Lemma 10. *There exists $\sigma_2 > 0$ so that if $x \in I$ satisfies Property P_1 or P_2 for some $i_0 < n$ then the pullback of J_{i_0+1} along $\{x, \dots, \phi^{k_{i_0+1}}(x)\}$ is σ_2-well-inside I. If $i_0 > 0$ this pullback is σ_2-well-inside J_0.*

Proof. If P_2 holds then Lemma 4 implies that I is a δ_2-scaled neighbourhood of the fundamental domain D containing $\phi^{k_{i_0}}(x)$. Obviously the same happen if P_1 holds. In either case it follows from Lemma 9 that the pullback of D along $\{x, \dots, \phi^{k_{i_0}}(x)\}$ into J_0 is well-inside I. If $i_0 > 0$ we get it well-inside J_0. $\square$

Lemma 11. *There exists $\sigma_3 > 0$ such that if $x \in I$ satisfies Property P_3 for some $i_0 < n$, then I is a σ_3-scaled neighbourhood of the pull-back $U \subset J_t$ of J_{i_0+1} along $\{\phi^{k_t}(x), \dots, \phi^{k_{i_0+1}}(x)\}$.*

Proof. In order to be definite assume that J_t is to the left of J_{i_0}. Let G_{i_0} be the left side of J_{i_0} (it contains J_t) and let H_t be the left side of J_t (which is small and contains J_{i_0+1}). Consider the first return map ψ to G_{i_0}, and let D, D' be the fundamental domains containing x of ϕ and ψ respectively. Then $D' \subset D$ and $D', \dots, \psi^{k_{t+1}-k_t-1}(D')$ are in J_t and $\psi \circ \psi^{k_{t+1}-k_t-1}(D') \subset J_{t+1}$. By Lemma 5 if D' is not well-inside G_{i_0}, then $D', \dots, \phi^{k_{t+1}-k_t}(D')$ lie ordered, and always lie on a big side of the previous interval. This implies that there is space to the left of D'. So it remains to show that there is space to the right of D' in I. So we may assume that both sides of J_t are small. But then it follows from the last part of Lemma 5 that D has space to its right inside I. Combined this shows that D' is well-inside I. $\square$

The previous lemmas can be combined:

Proposition 1. [Part 1 of Theorem C] *For each $\xi > 0$ there exists $\xi' > 0$ with the following property. Let V be an interval ξ-well-inside I and $x \in I$ such that $\phi^k(x) \in V$, for some $k \geq 1$. Then the pullback of V along $\{x, \dots, \phi^k(x)\}$ is ξ'-well-inside the return domain to I containing x.*

Proof. If P_4 holds then the lemma follows from Lemma 9. If P_1, P_2 or P_3 hold, then this follows from the previous Lemmas 10 and 11. $\square$

4.2. **Corollary: There exist no wandering intervals.** Of course this corollary is not new (it was proved in different generalities by Guckenheimer, de Melo & van Strien, Lyubich, Blokh & Lyubich and Martens & de Melo & van Strien. Let us show that our methods provide a new (and we believe simpler) proof of this fact.

Proof of the Corollary: Let us introduce some notation. We call a finite partition $\mathcal{P}$ of M *nice*, if boundary points of partition elements are mapped into boundary points. (Such a partition is generated by preimages of a finite number of periodic orbits.) Denote by $\mathcal{P}(x)$ the element of the partition $\mathcal{P}$ containing x (this is unique if x is not on a boundary of partition elements). Furthermore, denote by $\mathcal{L}_x(I)$ the component of the domain of the first entry map to I containing x.

Let W be an interval so that all its forward iterates are disjoint and assume by contradiction that W does not converge to a periodic orbit. Let z be an accumulation point of $f^n(W)$, and take a nice partition $\mathcal{P}_0$ and let $I_0 = \mathcal{P}_0(z)$. (We can assume that f has a nice partition. Indeed, if f has no fixed points, then by modifying f on the wandering interval W we can construct a new map $\tilde{f} \in \mathcal{A}^{1+Zygmund}$, with possibly more turning points, with a wandering interval and with at least one fixed point.) Note that infinitely many iterates of W are contained in I_0 and that, since all iterates of W are disjoint, every point in W simultaneously visits the same return domains of I_0. We *claim* that there exists a constant $\rho > 0$ so that the first iterate of W entering I_0 is contained in a nice interval I_0' which is ρ-well-inside I_0. Here I_0' is a component of a nice partition $\mathcal{P}_1$ which refines $\mathcal{P}_0$.

Let us first show that this proves the Corollary. From the claim it follows that there exists a sequence of refining nice partitions $\mathcal{P}_0, \mathcal{P}_1, \ldots$ such that for each $n \geq 0$, the first iterate of W entering $I_n = \mathcal{P}_n(x)$ is contained in a component I_n' of $\mathcal{P}_{n+1}$ which is ρ-well-inside I_n. By pulling back this space (using Lemma 1) we get

$$\mathcal{L}_W(I_n') \text{ is } \rho'\text{-well-inside } \mathcal{L}_W(I_n).$$

Moreover, since I_n' contains the first entry of W to I_n and since both I_n' and I_{n+1} are components of $\mathcal{P}_{n+1}$,

$$\mathcal{L}_W(I_{n+1}) \subset \mathcal{L}_W(I_n').$$

Combining this gives that for each n, $W \subset \mathcal{L}_W(I_n)$ is $n\rho'$-well-inside $\mathcal{L}_W(I_0)$. This gives a contradiction with the compactness of the dynamical space M.

So let us prove the claim. Notice that as in Lemma 7 either one of the properties P_1, P_2, P_3 holds at some moment, or P_4 holds for ever, or W eventually only visits one return domain.

Case 1. P_1, P_2, P_3 holds. Then the previous two lemmas and Proposition 1 imply that we can take for I_1 a pullback of a domain of I_0.

Case 2. P_4 holds for ever. If W keeps spiraling towards the center of I_0 visiting infinitely many different return domains in I_0 then eventually it visits a domain which is well-inside I_0 and so again Claim 1 holds (using Proposition 1). So the final two possibilities are that (during visits to I_0) either W eventually only visits one return domain J or that P_4 holds forever and W eventually only visits two domains J, J' in I_0 both having one large side. Let us consider both of these possibilities next.

Case 3. All iterates of W (entering I_0) eventually visit only one return domain J of the first return map to I_0. So W eventually enters a periodic interval $J' \subset J$ (i.e. $\phi(J') \subset J'$ and $\phi(\partial J') \subset \partial J'$). Let Q be the set of fixed points of $\phi\colon J' \to J'$. From Lemma 2, each component of $J' \setminus \phi^{-1}(Q)$ which contains a turning point of ϕ is well-inside J'. This lemma also implies that $|\phi'|$ is universally bounded on J' and so all fundamental domains associated to fixed points of ϕ are well-inside J'. It follows that some iterate of W is contained in a component of $J' \setminus \phi^{-k}(Q)$ which is well-inside $J' \setminus I_0$. By Proposition 1, the first iterate of W is contained in an element of $\mathcal{P}_1$ where $\mathcal{P}_1$ is the refinement of $\mathcal{P}_0$ obtained by adding the periodic points associated to Q and appropriate preimages by f.

Case 4. W eventually only visits two return domains J_1, J_2 of I_0, then we argue as follows. If either $|J_1|/|I_0|$ or $|J_2|/|I_0|$ are close to one, then $|\phi'|$ is bounded on J_1 or J_2, and so fundamental domains of $\phi\colon J_1 \to I$ or of $\phi\colon J_2 \to I$ are well-inside I, and we can complete the argument as before. So assume that both these domains do not occupy almost all of I_0. Then consider the first return map to $J' = J_1$. If for this first return map, we are in Case 1 or Case 2 then we argue as before. If we are again in Case 3, then consider the two domains J_1', J_2' in J'. If either $|J_1'|/|J'|$ or $|J_2|/|J'|$ are close to one, then we are done as before. But if this does not hold, then either J_1' or J_2' is well-inside I_0 (here we use that $|J_1|/|I_0|$, $|J_2|/|I_0|$, $|J_1'|/|J'|$ and $|J_2'|/|J'|$ are all away from one). This means that we can pullback this space as before.

This completes the proof of the claim and of the corollary. $\square$

5. Theorems A and A': Real bounds at arbitrary non-periodic points

Let I be a nice interval and $x \in I$ be a recurrent point. Let us show that one has real bounds around x (often one will take for x a critical point). Write $I_0 = I$, let ϕ_1 be the first return map to I_0 and let I_1 be the domain of this map containing x. Inductively one gets in this way a sequence of intervals I_n and first return maps ϕ_{n+1} to I_n so that I_{n+1} is the domain of ϕ_{n+1} containing x. We say that the first return map ϕ_{n+1} to I_n is *non-central w.r.t.* x (if clear we will omit the reference to x) if $\phi_{n+1}(x) \notin I_{n+1}$. Of course either x is contained in a periodic interval or otherwise there are infinitely many n's for which $\phi_{n+1}\colon I_{n+1} \to I_n$ are non-central. If x is not recurrent then I_{n+1} might be empty, so the results below become trivial.

Lemma 12. *Assume that I_{n-1} is well-inside I_{n-3}. Then*

(1) *if $\phi_{n-1}\colon I_{n-1} \to I_{n-2}$ is non-central then I_n is well-inside I_{n-1};*

(2) *If I_n is ξ-well-inside I_{n-1} then for each $j \geq 1$, there exists $\xi(j) > 0$ such that I_{n+j} is $\xi(j)$-well-inside I_{n+j-1}.*

Proof. Let us first prove that (1) holds. Note that $\phi_{n-1}(I_n)$ is in a component of $I_{n-2} \setminus I_{n-1}$ because $\phi_{n-1}\colon I_{n-1} \to I_{n-2}$ is non-central and that $\phi_{n-2} \circ \phi_{n-1}(I_n)$ is an iterate of I_n before it returns to I_{n-1}. Hence from Proposition 1 and the assumption that I_{n-1} is well-inside I_{n-3} it follows that $\phi_{n-1}(I_n)$ is well-inside I_{n-2}. But this implies that I_n is well-inside I_{n-1}.

Let us next proof (2). If ϕ_n is non-central then $\phi_n(I_{n+1})$ is an iterate of I_{n+1} before it returns to I_n, and so we can again apply Proposition 1 and that I_n is well-inside I_{n-1} to conclude that $\phi_n(I_{n+1})$ is well-inside I_{n-1}. This implies that I_{n+1} is well-inside I_n. If ϕ_n is central we have that $\phi_n(I_{n+1}) \subset I_n$ and $\phi_n(I_n) \subset I_{n-1}$ and that I_{n+1} is well-inside I_n follows immediately. $\square$

Lemma 13. *If $\phi_n : I_n \to I_{n-1}$ and $\phi_{n+1}\colon I_{n+1} \to I_n$ are non-central returns then I_{n+2} is well-inside I_n.*

Proof. Take a point $x \in I_{n+1} \subset I_n$ such that $\phi_{n+1}(x) \notin I_{n+1}$ and $\phi_n(x) \notin I_n$. Let $k > 1$ be the smallest integer such that $\phi_n^k(x) \in I_n$ and consider i_0 the stopping time of x such that $k_{i_0} \leq k$.

First observe that, if there exists a fundamental domain D which is well-inside I_{n-1} such that $\phi_n^{k_i}(x) \in D$ for some $k_i \leq k$, then from Proposition 1 it follows that the pullback of D to x is an interval well-inside I_{n-1}. Because this interval contains I_{n+1} we get that I_{n+1} is well inside I_{n-1} and this implies the lemma. So we can assume that P_1 and P_2 do not hold. If Property P_3 holds it follows from Lemma 11 that the pullback $U \subset J_t$ (remember that t is the largest integer with $0 \leq t < i_0$ and $J_t \subset (J_{i_0}, J_{i_0+1})$) of J_{i_0+1} to $\phi_n^{k_t}(x)$ is well-inside I_{n-1} and by the same reasoning as before, now playing with U instead of D, we are done in the same way. If Property P_4 holds we have that $k_{i_0} = k$. Then if I_n satisfies the hypothesis of Lemma 8 it follows that the pullback

$U \subset I_n$ of I_n along $\{x, \phi_n(x), \ldots, \phi_n^k(x)\}$ is well-inside I_{n-1} and the same reasoning as before applies again.

So we assume from now on that $k_{i_0} = k$, $J_{i_0} = I_n$, Property P_4 holds for x but I_n does not satisfy the hypothesis of Lemma 8. It means that I_n has two small sides, or $\phi(I_n)$ does not contain a small side of I_n or $\phi_{n|I_n}$ is an orientation reversing surjective branch.

Then we set $y = \phi_n(x)$ and define the smallest $s > 0$ such that $\phi_n^s(y) \in V$, where $V = \phi_{n|I_n}^{-1}(J_0(y))$. If such t does not exists we set $s = \infty$. From Lemma 7 we know that one of the Properties P_1, P_2, P_3 or P_4 holds for y with $k_{i_0(y)} \leq s$. If $k_{i_0(y)} \leq s$ and one of the Properties P_1, P_2 or P_3 holds it follows from one of the situations analysed above that the pullback of $J_{i_0(y)+1}$ to y is well-inside I_{n-1}. Pulling back once more to x we get an interval which contains I_{n+2} and is well-inside I_n and the statement follows in this case. Let us assume that P_4 holds and $k_{i_0(y)} = s$ which means that $J_{i_0(y)} = I_n$.

We still need to consider two cases: (i) The visited domain $J_{i_0(y)-1}$ lies between I_n and $J_0(y)$. Then, because $\phi_n^s(y) \in V$, we have that P_3 holds for $z = \phi_m^{s-1}(y)$ (with $k_{i_0(z)} = 2$). So we can use Lemma 11 and get that $\phi_{n|J_{i_0(y)-1}}^{-1}(V)$ is well-inside I_{n-1} and we are done again. (ii) The visited domain $J_{i_0(y)-1}$ is equal to $J_0(y)$ which means that there is no visited domain between I_n and $J_0(y)$. We can assume that $J_0(y)$ does not satisfy the hypothesis of Lemma 8 otherwise this lemma would imply that the pullback W of V to y is well-inside I_{n-1}. Then the pullback of W to x would be well-inside I_n what would implies that I_{n+2} is well-inside I_n and the statement would follow. So we can assume that both, I_n and $J_0(y)$ do not satisfy the hypothesis of Lemma 8. But then we consider the first return map to I_n whose domain I_{n+1} (containing x) either satisfies the hypothesis of Lemma 8, is well-inside I_{n-1}, or it has two small sides. In each of these situations we get the required statement. $\square$

Lemma 14. *Let $\phi_{m-1} \colon I_{m-1} \to I_{m-2}$ be non-central, while for some $n > m$, $\phi_k \colon I_k \to I_{k-1}$ is central for $k = m, \ldots, n-1$ and $\phi_n \colon I_n \to I_{n-1}$ is non-central. Then I_{n+1} is well-inside I_n.*

Proof. By assumption $\phi_m = \phi_{m+1} = \cdots = \phi_n$. Of course it is enough to show that $\phi_n(I_{n+1})$ is well-inside a component of $I_{n-1} \setminus I_n$, Let us prove this. Since $\phi_m = \phi_{m+1} = \cdots = \phi_n := \phi$, one gets that a component of $I_{n-1} \setminus I_n$ is mapped by ϕ^j to a component of $I_{n-1-j} \setminus I_{n-j}$ for $j = 0, \ldots, n-m$ and all these iterates are pairwise disjoint. So it is enough to show that $\phi^{n-m}(I_{n+1})$ is well-inside a component of $I_{m-1} \setminus I_m$ (note that $\phi^{n-m}(I_{n+1})$ is an iterate of I_{n+1} before it has returned to I_n). Note that $\phi_{m-1} : I_{m-1} \to I_{m-2}$ and $\phi_m : I_m \to I_{m-1}$ are non-central w.r.t. $\phi_n^{n-m}(I_{n+1})$ and the previous lemma implies that the return domain to I_m which contains $\phi_n^{n-m}(I_{n+1})$ is well-inside I_{m-1}. It follows that $I_n \subset I_{m+1}$ is well-inside I_{m-1}. Hence by Proposition 1 we get that I_{n+1} is well-inside I_n. $\square$

5.1. **Proof of Theorem A.** Let us assume now, for $m > p$, that $\phi_p \colon I_p \to I_{p-1}$ and $\phi_m \colon I_m \to I_{m-1}$ are non-central and prove the first statement. It follows from Lemma 13 or 14 that I_{m+2} is well-inside I_m. Because of this, if for $n > m$, $\phi_n \colon I_n \to I_{n-1}$ is also non-central, Lemma 12 or 14 implies that I_{n+1} is well-inside I_n and the statement follows.

Let us assume that I_{n+1} is well-inside I_n and prove the second statement of the theorem. Let J be a domain of the first return map ϕ_{n+2} to I_{n+1}. If $\phi_{n+1}(J) \subset I_{n+1}$ then the first part of the second statement holds because $\phi_{n+2|J} = \phi_{n+1|J}$. If $I_{n+1} \cap \phi_{n+1}(J) = \emptyset$, then by Proposition 1 the first part of the second statement holds because $\phi_{n+1}(J)$ is well-inside a domain of ϕ_n because it is a pullback of I_{n+1} by some iterate of ϕ_n and I_{n+1} is well-inside I_n. To prove that ξ' tends to infinity as ξ tends to infinity, one needs also to consider the situation that the pullback of I along the orbit $\{x, \ldots, \phi_n(x)\}$ has high intersection multiplicity. $\square$

5.2. **Proof of Theorem A'.** Let us first show that $\omega(x)$ contains a dense orbit. To prove this we use a slight adaptation of a well-known argument showing that if a map is transitive on a compact set X then there is a dense subset of X consisting of points with dense orbit in X. Since $\omega(x)$ is compact, there exists a collection of open subsets U_n of M such that $\cap_{n\geq 1} \cup_{k \geq n} U_n = \omega(x)$ and such that $\operatorname{diam}(U_n) \to 0$. By the definition of $\omega(x)$, for each non-empty open subset U, V intersecting $\omega(x)$ there exists $m \geq 0$ such that $f^{-m}(U) \cap V \neq \emptyset$. So $\cup_{m<0} f^m(U_n) \cap \omega(x)$ is dense in $\omega(x)$ (and of course open). So by Baire $X = \cap_{n \geq 0} \cup_{m<0} f^m(U_n) \cap \omega(x)$ is also dense, and in particular non-empty. Points in X have dense orbits in $\omega(x)$. It follows that either x is eventually mapped to a periodic point y, or that $\omega(x)$ contains a non-periodic recurrent point y.

Let us distinguish three cases.

(I) The point y is a non-periodic recurrent point and there exists arbitrarily small periodic intervals J around y. If x is not contained in the orbit of the periodic intervals around y, then the domain of the first return map to a small neighbourhood of x is empty, and the assertion is trivial. So we may as well assume that $y = x$. Let ϕ be the first return map to J. Because of Lemma 2, $\phi \colon J \to J$ has good distortion properties (because the pullback of a definite neighborhood of J has bounded intersection multiplicity). In particular, the derivative of ϕ is bounded by some universal number. Let P be the set of fixed points of $\phi \colon J \to J$, and let $\hat{I}_0$ be the component of $J \setminus \phi^{-1}(P)$ containing x. It is not hard to see that if $\phi^2(x) \notin \hat{I}_0$ then the domain $\hat{I}_1$ of the first return map to $\hat{I}_0$ containing x, is well-inside $\hat{I}_0$. If $\phi^2(x) \in \hat{I}_0$, then define $\hat{I}_i \ni x$ inductively as before. Either

(Ia) for each i, $\phi^{2i}(x) \in \hat{I}_0$; in this case x is contained in a periodic interval $J_1 \subset \hat{I}_0 \subset J$ of period two, i.e., with $\phi^2(J_1) \subset J_1$ and $\phi^2(\partial J_1) \subset \partial J_1$, or

(Ib) there exists (a minimal) $k \geq 0$ so that $\phi^2(x) \notin \hat{I}_k$. As before, it is not hard to see that this implies that $\hat{I}_{k+1}$ is ξ_0-well-inside $\hat{I}_k$, with ξ_0 universal.

In case (Ia), $I := K_0 := I_0$ and $J := K_1 := J_1$ are both periodic nice intervals of ϕ-period 1 respectively 2, while in case (Ib) $I := I_0$ and $J := J_1$ are nice intervals. The required properties hold for these intervals. To show that either (1) or (2) from the statement of Theorem A' hold, replace $\phi\colon J \to J$ by $\phi^2\colon J_1 \to J_1$ in the above if case (Ia) holds, and repeat the same argument. It follows that either f is Feigenbaum-like at x (from some moment only case (Ia) occurs), or we can take infinitely small nice intervals $J \subset I$ with $J = \mathcal{L}_x(I)$.

(II) The point y is a non-periodic recurrent point and the period of periodic intervals $J_i \ni y$ is bounded (so f is finitely often renormalizable at y). Then let J be the smallest periodic interval containing y (possibly $J = M$), let ϕ be the first return map to J and P be the set of fixed points of $\phi\colon J \to J$. Let I_0 be the component of $J \setminus \phi^{-1}(P)$ containing y. Then the sequence of return maps $\phi_{i+1}\colon I_{i+1} \to I_i$ around y (defined above) has infinitely many non-central returns (because y is non-periodic), and so by Theorem A there exists $\xi_0 > 0$ and arbitrarily small nice intervals I' containing y such that all its first return domains are ξ_0-well-inside I'. Notice that $I = \mathcal{L}_x(I')$ is nice, and that I is small if I' is small. Let $z \in I$, let $t(z) > 0$ be minimal so that $z' = f^{t(z)}(z) \in I'$. Then $\mathcal{L}_z(I)$ is contained in the pullback by $f^{t(z)}$ of $\mathcal{L}_{z'}(I')$. It follows that for each z its first return domain $\mathcal{L}_z(I)$ to I is ξ_0'-well-inside I. (Of course it is conceivable the domain of the first return map to I is empty, in which case the assertion of the Theorem is trivial.)

(III) y is a (repelling) periodic point with period p. Let $\lambda = Df^p(y)$ be its multiplier.

Claim 1: there exists $\xi_0 > 0$ and arbitrarily small nice intervals I' around y such that each fundamental domain of $R_{I'}$ restricted to $\mathcal{L}_y(I')$ is ξ_0-well-inside I'. To see this, we first remark that there exists a sequences $z_i \uparrow y$, $z_i' \downarrow y$ of points which are eventually mapped into periodic orbits $O \not\ni y$ respectively $O' \not\in\ni y$. Here we take z_i, z_i' so that $f^p(z_{i+1}) = z_i$ and $f^p(z_{i+1}') = z_i'$. If $|\lambda|$ is not large, then neighbouring fundamental domains are comparable in length, so the claim holds for $I' = (z_i, z_j')$ regardless of the choice for i and j (provided they are large). If $\lambda < 0$, then $I' := (z_i, f^p(z_i))$ is the required nice interval: each fundamental domain of $R_{I'}$ is well-inside

I' (because we assumed that $|\lambda|$ large this is trivial). If $\lambda > 0$ is large, then choose j minimal so that $|z'_j - y| < |z_i - y|$. If $|z'_j - y| \geq 10|z_{i+1} - y|$ then take $I' = (z_i, z'_j)$ and otherwise take $I' = (z_{i+1}, z'_j)$. In both cases $\mathcal{L}_y(I')$ is well-inside I', so again the claim holds.

Claim 2: There exists ξ_1 so that provided we take I' sufficiently small, for each $z \in I' \setminus \mathcal{L}_y(I')$, $\mathcal{L}_z(I')$ is ξ_1-well-inside I'. To see this, take two intervals $I' \subset I''$ as above with I' well-inside I'' and so that for some $t \geq 1$, $f^t \colon I' \to I''$ is a diffeomorphism. Some iterate z' of z will enter $I'' \setminus \mathcal{L}_y(I'')$ before returning to I'. Since I' is well-inside I'', by Proposition 1, $\mathcal{L}_{z'}(I')$ is well-inside I'' (and contained in the fundamental domain $I'' \setminus \mathcal{L}_y(I'')$). If $|\lambda|$ is not large, then $\mathcal{L}_{z'}(I')$ is well-inside two neighbouring fundamental domains, and so it follows that $\mathcal{L}_z(I')$ is well-inside I'. If $|\lambda|$ is large, then $f^t|I'$ has bounded distortion, and therefore $\mathcal{L}_z(I') = (f^t|I')^{-1}(\mathcal{L}_{z'}(I')$ is well-inside I'.

Now taking $I = \mathcal{L}_x(I')$ the required statements hold (here we use that x is pre-periodic, but not periodic, which implies that the return domains of I are pullbacks of return domains to I' which are contained in $I' \setminus \mathcal{L}_y(I')$). So this completes the case that y is periodic.

The final part of assertion of Theorem A' holds because both I and J are nice. $\square$

6. Theorem C: Improved Koebe Principles and Negative Schwarzian derivative

Before proving Theorem B, we shall prove Theorem C. First we state and prove an improved Koebe Principle. The usual Koebe Lemma applies to a situation when one has an interval $T_0 \supset J$ such that $f^n_{|T_0}$ is a diffeomorphism, and so that either in addition $T_0, \dots, f^n(T_0)$ are pairwise disjoint or that the Schwarzian derivative of f is negative. In the following version we do not require that $T_0, \dots, f^n(T_0)$ are pairwise disjoint, have finite total length or even that $f^n_{|T_0}$ is a diffeomorphism, but only an upperbound for $\sum_{i=0}^{n-1} |f^i(J)|$.

Proposition 2. [Koebe Principle requiring less disjointness] *Assume that f is C^2. Then there exist constants $K > 0$ and a function $O(\epsilon)$ with $O(\epsilon) \to 0$ as $\epsilon \downarrow 0$ with the following properties. Let n be an integer and J an interval so that $f^n_{|J}$ is a diffeomorphism. Let T be a δ-scaled neighbourhood of $f^n(J)$ for some $\delta > 0$ and $T_0, \dots, T_n := T$ the pullback of T along $J, \dots, f^n(J)$.*

- *Let $\mathcal{N} \subset \{0, \dots, n-1\}$ be the set of integers i for which T_i contains a critical point and let $\epsilon = \max |T_i|$. Then for each $x, y \in J$,*

$$\frac{|Df^n(x)|}{|Df^n(y)|} \leq \exp\left(O(\epsilon)\sum_{i=0}^{n-1}|f^i(J)|\right) \cdot \left[\frac{1+\delta}{\delta}\right]^2 \cdot \exp\left(K \cdot \sum_{m \in \mathcal{N}} \frac{|f^m(J)|}{dist(f^m(J), \mathbf{Cr})}\right). \tag{3}$$

(If there are no such integers i then we take the sum $\sum_{m\in\mathcal{N}}$ above to be equal to zero.) If f is C^3 then there exists $\hat{K} > 0$ such that $O(t) \leq \hat{K}t$ for each $t > 0$.

- *If $f^n_{|T}$ is a diffeomorphism, then T is a $\delta'(\epsilon, \sum_{i=0}^{n-1}|f^i(J)|)$-scaled neighbourhood of J. Here δ' is a function which is specified in the proof of the proposition.*

In the proof of this proposition we need to use the following cross-ratios, see [18]. Let $j \subset t$ be intervals, and let l, r be the components of $t \setminus j$. Define

$$A(t,j) = \frac{|t| \cdot |j|}{|l \cup j| \cdot |j \cup r|} \text{ and } B(t,j) = \frac{|t| \cdot |j|}{|l| \cdot |r|}$$

and if $f_{|T}$ is monotone, define

$$A(f,t,j) = \frac{A(f(t), f(j))}{A(t,j)} \text{ and } B(f,t,j) = \frac{B(f(t), f(j))}{B(t,j)}.$$

It is well-known that if f is C^2, under the assumption stated above, see [18] and also [23], there exists function $O(\epsilon)$ with $O(\epsilon) \to 0$ as $\epsilon \downarrow 0$, so that for any intervals $j \subset t$ as above with $f_{|t}$ a diffeomorphism, $A(f,t,j) \geq \exp(-|r| \cdot O(|l|))$ and $B(f,t,j) \geq \exp(-|t| \cdot O(|t|))$. If f is C^3 there is even a universal constant $C > 0$ so that $A(f,t,j) \geq \exp(-C \cdot |l| \cdot |r|)$ and $B(f,t,j) \geq \exp(-C|t|^2)$.

Proof. The main idea of this lemma appeared before in [23] (for example equation (2.5) in the proof of Theorem 2.3 in that paper), see also [5]. Let $\epsilon > 0$ be so that each interval $T_0, \dots, T_n$ has at most length ϵ. Take any point x in the interior of J. The point x divides the intervals J into two intervals J^-, J^+. Similarly split $J_i := f^i(J)$ into $J_i^+ = f^i(J^+)$ and $J_i^- = f^i(J^-)$ and also split T_i into $T_i^- \supset J_i^-, T_i^+ \supset J_i^+$. We either have

$$\frac{|f^n(J^-)|}{|J^-|} \geq \frac{|f^n(J)|}{|J|} \text{ or } \frac{|f^n(J^+)|}{|J^+|} \geq \frac{|f^n(J)|}{|J|}.$$

Without loss of generality, suppose the latter holds. Let $n_s < \dots < n_1 < n_0 := n$ be the integers i so that T_i^- contains a critical point. Fix $i \in \{1, \dots, s\}$ and take $r = J^+_{n_i+1}$, $l = T^-_{n_i+1} \cup J^-_{n_i+1}$, $j = \{f^{n_i+1}(x)\}$ and $t = l \cup j \cup r$. Applying A to $t = l \cup j \cup r$ and j gives,

$$A\left(f^{n_{i-1}-n_i-1}, l \cup j \cup r, j\right) \geq \exp\left(-\sum_{k=0}^{n_{i-1}-n_i-1} |f^k(r)| \cdot O(|f^k(t)|)\right) \geq C_0(n_i, n_{i-1}),$$

where $C_0(a,b) := \exp\left(-O(\epsilon)\cdot\sum_{k=a}^{b-1}|f^k(J)|\right)$. Using the definition of the cross-ratio A, and that $|t|\geq|l|$,

$$|Df^{n_{i-1}-n_i-1}(f^{n_i+1}(x))| \geq C_0(n_{i-1},n_i)\cdot\frac{|f^{n_{i-1}}(J^+)|}{|f^{n_i+1}(J^+)|}\cdot\frac{|T^-_{n_{i-1}}\cup J^-_{n_{i-1}}|}{|T^-_{n_{i-1}}\cup J^-_{n_{i-1}}\cup J^+_{n_{i-1}}|}.$$

By the non-flatness of the critical points, there exists a universal constant $K_2>0$ so that

$$|Df(f^{n_i}(x))| \geq \frac{|f^{n_i+1}(J^+)|}{|f^{n_i}(J^+)|}\cdot\exp\left(-K_2\cdot\frac{|J_{n_i}|}{\mathrm{dist}(J_{n_i},\mathbf{Cr})}\right),$$

where $\mathbf{Cr}$ is the set of critical points. Also,

$$\frac{|T^-_{n_{i-1}}\cup J^-_{n_{i-1}}|}{|T^-_{n_{i-1}}\cup J^-_{n_{i-1}}\cup J^+_{n_{i-1}}|} \geq \frac{\mathrm{dist}(J_{n_{i-1}},\mathbf{Cr})}{\mathrm{dist}(J_{n_{i-1}},\mathbf{Cr})+|J_{n_{i-1}}|} \geq \exp\left(-K_3\frac{|J_{n_{i-1}}|}{\mathrm{dist}(J_{n_{i-1}},\mathbf{Cr})}\right).$$

Combining all these inequalities,

$$|Df^n(x)| \geq C_0(0,n)\cdot\frac{|f^n(J^+)|}{|J^+|}\cdot\frac{|T^-_{n_0}\cup J^-_{n_0}|}{|T^-_{n_0}\cup J^-_{n_0}\cup J^+_{n_0}|}\cdot\prod_{i=1}^{s}\exp\left(-(K_2+K_3)\cdot\frac{|J_{n_i}|}{\mathrm{dist}(J_{n_i},\mathbf{Cr})}\right).$$

By assumption $\frac{|f^n(J^+)|}{|J^+|}\geq\frac{|f^n(J)|}{|J|}$. Hence, by the definition of δ in the statement of the theorem, and by the disjointness of the orbit of J, we obtain

$$|Df^n(x)| \geq C_0(0,n)\cdot\frac{\delta}{1+\delta}\cdot\frac{|f^n(J)|}{|J|}\cdot\exp\left(-K\cdot\sum_{m\in\mathcal{N}}\frac{|f^m(J)|}{\mathrm{dist}\left(f^m(J),\mathbf{Cr}\right)}\right). \tag{4}$$

Let us now prove the reverse inequality. To do this, we consider the cross-ratio B. Again, either

$$\frac{|f^n(J^-)|}{|J^-|}\leq\frac{|f^n(J)|}{|J|}\quad\text{or}\quad\frac{|f^n(J^+)|}{|J^+|}\leq\frac{|f^n(J)|}{|J|}.$$

Suppose the first holds and take an interval $V\subset J^-$. Since

$$B(f^n,J^-,V)\geq\exp\left(-O(\max_{i\leq n}|f^i(J)|)\cdot\sum|f^i(J)|\right)\geq C_0(0,n)$$

by taking V converging to J^- we get that

$$|Df^n(x)|\leq C_0(0,n)\frac{1}{|Df^n(a)|}\cdot\left(\frac{|f^n(J^-)|}{|J^-|}\right)^2,$$

where a is the boundary point of J^- which is not inside J. Using (4) (with $x=a$) to bound $|Df^n(a)|$ and using $\frac{|f^n(J^-)|}{|J^-|}\leq\frac{|f^n(J)|}{|J|}$, we get (3).

To prove the second part of the proposition, choose $j,j,r\subset T$ so that $l\cup j=J$, $|f^n(l)|=|f^n(j)|$ and so that $f^n(r)$ is one of the components of $f^n(T\setminus J)$. As before, $A(f^n,t,j)\geq C_0(0,n)$. Using the definition of A and δ then gives

$$\frac{|r\cup j|}{|j|}\geq\frac{|l\cup j|}{|j|}\frac{|r\cup j|}{|t|}\geq C_0\frac{2(1+\delta)}{1(2+\delta)}.$$

Since $C_0 = \exp(-x)$ with $x = O(\epsilon) \sum_{i=0}^{n-1} |f^i(J)|$ and $\exp(-x) \geq 1 - x$ for $x \geq 0$, this gives $\frac{|r|}{|j|} \geq \frac{-2x+\delta(1-2x)}{2+\delta}$. Since by the first part of this proposition, the length of $l \cup j$ is at most $2\frac{1}{C_0}[(1+\delta)/\delta]^2$ times the length of j, the result follows. $\square$

Let us now prove that the Schwarzian derivative of high iterates of f is negative. In order to do this, we assume that f is C^3 and prove the following proposition.

Proposition 3. *Assume that f is C^3. For each integer N, and each $\xi > 0$, $\delta > 0$, $S > 0$, there exists $\tau > 0$ with the following property. Let n be an integer and J an interval so that $\sum_{0 \leq i < n} |f^i(J)| \leq S$. Let T_n be a δ-scaled neighbourhood of $f^n(J)$ and let $T_0, \ldots, T_n$ be the pullback with $T_i \supset f^i(J)$. Let $\mathcal{N} \subset \{0, \ldots, n-1\}$ be the set of integers for which T_i contains a critical point. Assume that*

- *$\#\mathcal{N} \leq N$;*
- *$|f^n(J)| \geq \xi \cdot dist(f^n(J), \mathbf{Cr})$, where $\mathbf{Cr}$ is the set of critical points.*
- *$|f^n(J)| \leq \tau$.*

Then

$$Sf^{n+1}(x) < 0 \textit{ for all } x \in J.$$

Proof. Since f is C^3 and all its critical points are non-flat, there exists $C > 0$ such that $Sf(x) < C$ for all x, and that there exists a neighbourhood U of $\mathbf{Cr}$ and a constant $C' > 0$ with

$$Sf(y) < -\frac{C'}{[\mathrm{dist}(y, \mathbf{Cr})]^2} \text{ for all } y \in U. \tag{5}$$

Let $n_s < \cdots < n_0 = n$ be the integers m so that $|f^m(J)| \geq \xi \cdot \mathrm{dist}(f^m(J), \mathbf{Cr})$ (for example, when $f^m(J)$ contains a critical point). Because of the composition formula for the Schwarzian derivative it suffices to show the following claim for each $i \leq s$.

Claim: $Sf^{k_i}(x) < 0$ for any $x \in f^{n_{i+1}+1}(J)$ where $k_i = n_i - n_{i+1}$, where $k_0 = n_s + 1$ and $n_{s+1} = -1$.

Proof of Claim: Fix i, write $n' = k_i$ and let us show that $Sf^{n'}(y) < 0$ for any $y \in f^{n_{i+1}+1}(J)$. By the composition formula for Schwarzian derivative,

$$Sf^{n'}(y) = |Df^{n'-1}(y)|^2 \cdot \left(Sf(f^{n'-1}(y)) + \sum_{i=0}^{n'-2} Sf(f^i(y)) \cdot |Df^{n'-1-i}(f^i(y))|^{-2} \right).$$

From (5) this is at most

$$|Df^{n'-1}(y)|^2 \cdot \left(\frac{-C'}{[\mathrm{dist}(f^{n'-1}(y), \mathbf{Cr})]^2} + \sum_{i=0}^{n'-2} \left(C \cdot |Df^{n'-1-i}(f^i(y))|^{-2} \right) \right).$$

Because of the assumptions and from the previous proposition this is at most

$$\leq |Df^{n'-1}(y)|^2 \cdot \left(\frac{-C'\xi^2}{|f^{n_{i-1}}(J)|^2} + C'' \sum_{j=n_i}^{n_{i-1}-1} \frac{|f^j(J)|^2}{|f^{n_{i-1}}(J)|^2} \right) =$$

$$= \frac{|Df^{n'-1}(y)|^2}{|f^{n_{i-1}}(J)|^2} \left(-C'\xi^2 + C'' \sum_{j=n_i}^{n_{i-1}-1} |f^j(J)|^2 \right).$$

Here C'' depends on the cardinality of $\mathcal{N}$, C and on ξ and S (through the exponential term in (3)). Provided $|f^n(J)|$ is sufficiently small, all $|f^i(J)|$, $0 \leq i < n$, will be small (because there are no wandering intervals). Because of this, and by the bound for $\sum_{0 \leq i < n} |f^i(J)|$, we get $Sf^{n'}(y) < 0$ for $y \in f^{n_{i+1}+1}(J)$. Combining this, and using that compositions of maps with negative Schwarzian again have negative Schwarzian, completes the proof. □

6.1. **Proof of Theorem C.** Part 1 of Theorem C is already proved in Proposition 1. Before proving Part 2, let us prove Part 3. According to Theorem A' there are two cases:

Case 1. The critical point c is not in the basin of a periodic attractor (so in particular non-periodic) and f is not of Feigenbaum type at c. According to Theorem A' there exists (an arbitrarily small) nice intervals T containing c with the property that each first return to T is well-inside T. (Here we use that c is not in the basin of a periodic attractor.) Let $\mathcal{L}_y(T)$ be the component of the domain of this first entry map to T containing y (this set can be empty). Let $V \subset T$ be $\mathcal{L}_c(T)$ or if that is empty, let V be any neighbourhood of c which is well-inside T. Now take any x so that, for some n, $f^n(x) \in V$. We want to show that $Sf^{n+1}(x) < 0$. Let $\tilde{n} \leq n$ be minimal so that $f^{\tilde{n}}(x) \in T$ and either (i) one of the intervals in the pullback of $\mathcal{L}_{f^{\tilde{n}}(x)}(T)$ along $\{x, \ldots, f^{\tilde{n}}(x)\}$ contains a critical point or (ii) so that $f^{\tilde{n}}(x) \in V$. Since $f^n(x) \in V$ such an integer exists. If (i) holds then define $J = \mathcal{L}_{f^{\tilde{n}}(x)}(T)$ and if (ii) holds then define $J = V$. In both cases, J is well-inside T, and one of the intervals in its pullback along $\{x, \ldots, f^{\tilde{n}}(x)\}$ contains a critical point. Take $\hat{n}$ maximal with $0 \leq \hat{n} < \tilde{n}$ so that $f^{\hat{n}}(x) \in T$ (if $\hat{n}$ does not exist let $\hat{n} = 0$). Let $J_0, \ldots, J_{\tilde{n}}$ and $T_0, \ldots, T_{\tilde{n}}$ be the pullback of respectively J and T along $\{x, \ldots, f^{\tilde{n}}(x)\}$ and let n' be minimal so that $J_{n'}$ contains a critical point. Note that $\hat{n} < n' \leq \tilde{n} \leq n$ and that $T_{\hat{n}} = \mathcal{L}_{f^{\hat{n}}(x)}(T)$. By the minimality of $\tilde{n}$, none of the intervals $T_0, \ldots, T_{\hat{n}-1}$ contains a critical point; moreover, $T_{\hat{n}}, \ldots, T_{n'-1}$ is pairwise disjoint and the same holds for $J_{\hat{n}}, \ldots, J_{n'-1}$. In particular, since J is well-inside T, also $J_{n'}$ is well-inside $T_{n'}$. Finally by definition $J_{n'}$ contains a critical point. From all this, and Proposition 3 it follows that $Sf^{n'+1}(x) < 0$. If $n' = n$ then we are done. If $n' < n$, then define $x' = f^{n'+1}(x)$ and we repeat the argument. Since the composition of maps with negative Schwarzian derivative again has negative Schwarzian derivative, we are done. This completes the proof of Part 3 of Theorem C in this case.

Case 2. The critical point c is of Feigenbaum type (as defined in Theorem A'). Let $\{K_i\}_{i=0}^{\infty}$ be a nested sequence of nice $p2^i$-periodic intervals containing c. We want to show that if $f^n(x) \in K_{i+1}$ where i is large, then $Sf^{n+1}(x) < 0$. For the moment fix i and let $n' \leq n$ be minimal so that $f^{n'}(x) \in K_{i+1}$ and $\hat{n} \leq n'$ be minimal so that $f^{\hat{n}} \in K_i$. Then $n' = \hat{n} + kp2^i$ for some integer k. If $k \leq 2$, then the intersection multiplicity of the pullback of K_i along $\{x, \dots, f^{\hat{n}'}(x)\}$ is at most 3, and so we can apply the previous proposition, replacing n by n' and taking $f^{n'}(J) = K_{i+1}$ and $T_{n'} = K_i$ (provided we take i sufficiently large). If $k > 2$, then $f^{\hat{n}+(k-2)p2^i}(x)$ is a contained in a fundamental domain F_i associated to the repelling fixed point of $f^{p2^i} \colon K_i \to K_i$ in ∂K_i. Because of the Corollary at the end of Section 2, this fundamental domain is comparable in size to K_i (here we use that $f^{\hat{n}+kp2^i}(x) \in K_{i+1}$ and that K_{i+1} is not small compared to K_i in this Feigenbaum situation). Moreover, it is comparable in size to its two neighbouring fundamental domains $F_{i,l}, F_{i,r}$. So replacing in the proposition n by $\tilde{n} := \hat{n} + (k-2)p2^i$ and taking $f^{\tilde{n}}(J) = F_i$ and $T_{\tilde{n}} = F_{i,l} \cup F_i \cup F_{i,r}$, we again obtain that $Sf^{\tilde{n}+1}(x) < 0$ provided i is sufficiently large.

Part 2 of Theorem C now follows easily: let U be a nice neighbourhood of a critical points as in Part 3 such that $f^n(T) \subset U$. Since U is nice and $f^n|T$ is a diffeomorphism, $T, \dots, f^n(T)$ are either contained in U or outside U. Let s be maximal with $0 \leq s < n$ so that $f^s(T) \subset U$. If there exists no such s, then $T, \dots, f^n(T)$ are disjoint and the result follows from the previous proposition. If there exists such an integer s, then $f^{s+1} \colon T \to f^{s+1}(T)$ has negative Schwarzian. By a theorem of Mañé [13] any compact forward-invariant set not containing parabolic or critical points, is hyperbolic. Combined, this implies that $f^{s+1} \colon T \to f^{s+1}(T)$ and $f^{n-s-1} \colon f^{s+1}(T) \to f^n(T)$ have bounded distortion, and we obtain Part 2. (In fact, one can derive this result of Mañé also from our real bounds.) $\square$

7. Theorem B: Large Bounds

Let us now prove large bounds. This will be done in the two propositions in this section. If I is a nice interval containing a non-periodic recurrent critical point c then we say that J is a *child* of I if J is a component of $f^{-1}(J)$ where J' is a diffeomorphic pullback of I and J contains a critical point c' with $\omega(c) = \omega(c') \ni c, c'$.

Let us first prove part 1 of Theorem B. It will be convenient to define $\mathcal{L}_x(J)$ to be the domain of the first entry map to J containing x, and to define inductively $\mathcal{L}_x^{i+1}(J) = \mathcal{L}_x(\mathcal{L}_x^i(J))$.

Proposition 4. *For each $C > 0$, $\delta > 0$ there exists N' so that if a nice interval I has at least N' children $K_1 \supset K_2 \supset \dots$ around some critical point, and has each of its domains in I δ-well-inside I, then for $I' = K_N$ with $N \geq N'$ each of the first return domains to I' is C-well inside I'.*

Note that because of Theorem A1 there exists many nice intervals I so that each of the domains of the first return map to I is δ-well-inside I for some $\delta > 0$.

Proof. Assume that I is a nice interval as above and let R_I be the first entry map to I. Let $s(i)$ be so that K_i is a unimodal (or homeomorphic if d is an inflection point) pullback of I under $f^{s(i)}$. Then $f^{s(i)}(K_{i+1})$ is contained in a first return domain to I. It follows that K_{i+1} is δ''-well-inside K_i for each $i = 1, \dots, N$.

Next consider a point $u \in I' = K_N$ in a return domain to I' and let f^s be the unimodal (or homeomorphic) map so that K_N is a component of $f^{-s}(I)$ (this is possible because K_N is a child of I). Note that $f(K_N), \dots, f^{s-1}(K_N)$ do not intersect $I' = K_N$ (otherwise K_N could not be a unimodal pullback of I). Let $v = f^s(u) \in I$. If $v \in I'$, then $I \supset K_1 \supset K_2 \supset \dots \supset K_N := I'$ and since I' is very deep inside I we can pullback this space through the unimodal map $f^s \colon I' \to I$. If $v \notin I'$ then pick an integer $i \in \{1, \dots, N\}$, let R_{K_i} be the first entry map to K_i and let $x_i = R_{K_i}(v)$. Then applying Theorem B1 (taking I to be K_i and V to be K_{i+1}) shows that $\mathcal{L}_{x_i}(K_{i+1})$ is well-inside K_i, and so by Koebe we get that $\mathcal{L}_v(K_{i+1})$ is $\tilde{\delta}$-well-inside $\mathcal{L}_v(K_i)$. Hence $\mathcal{L}_u\mathcal{L}_v(K_{i+1})$ is $\tilde{\delta}'$-well-inside $\mathcal{L}_u\mathcal{L}_v(K_i)$. Since this holds for $i = 1, \dots, N$, and since $\mathcal{L}_u\mathcal{L}_v(K_1)) \subset I'$ and $\mathcal{L}_u\mathcal{L}_v(K_N) = \mathcal{L}_u\mathcal{L}_v(I') = \mathcal{L}_u(I')$ (because $v \notin I'$), it follows that $\mathcal{L}_u(I')$ is very deep inside I' when N is large. □

Let J be a nice interval and let J_1 be its first return domain containing a critical point c. Define $J_i = \mathcal{L}_c^i(J)$. Assume that k is maximal so that the first return map $\phi_{J_{i-1}}(c) \in J_i$ for $1 \leq i < k$ and $\phi_{J_{k-1}}(c) \notin J_k$ and define $\mathcal{C}_c(J) = J_k$ (so $\mathcal{C}_c(J)$ is next non-central return in the cascade). In addition, we say that a pair of nice intervals $W \supset V$ is *nice* if no forward iterates of ∂V enters the interior of W. Note that any pullback $W' \supset V'$ of a nice pair $W \supset V$ is again a nice pair.

Lemma 15. *Let $J \subset W$ be a return domain to W which contains a point c and $J \cap U = \emptyset$, then any pullback of U which contains c is contained in $\mathcal{C}_c(W)$.*

Proof. No pullback of U can contain a boundary point of $\mathcal{C}_c(W)$, because otherwise U would contain a boundary point of W, a contradiction. □

Lemma 16. *There exists a function $\rho_1\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho_1(\xi) \to \infty$ as $\xi \to \infty$ with the following properties. Let W be a nice interval containing a critical point c and let $J = \mathcal{L}_c(W)$. Assume that either c is a turning point or that the first return map $\phi\colon J \to W$ is monotone. Let V be an interval which is ξ-well-inside W and so that $W \supset V$ is a nice pair. Let $s > 0$ and assume that $f^s(z) \in V$ (not necessarily for the first time). Denote by V_i, W_i the pullback of V, W along $f^i(z), \dots, f^s(z)$. Then there exists $0 \le t < s$ and W_t^1 with $V_t \subset W_t^1 \subset \mathcal{L}_{f^t(z)}W$ and so that*

(1) *V_t is $\rho_1(\xi)$-well-inside W_t^1, and if $t > 0$ $V_t \subset W_t$ is a nice pair, and*
(2) *each pullback of W_t^1 intersecting c is contained in $\mathcal{C}_c(W)$.*

Proof. Observe that by Part 2 of Theorem C, there exists a function $\hat\rho$ with $\hat\rho(\xi) \to \infty$ as $\xi \to \infty$ so that the pullback of V by any first entry map to W is $\rho(\xi)$-well-inside its domain. So we can assume that $z \in W$. For simplicity assume $\xi' := \rho(\xi) < \xi$ and that ξ is large.

Let $0 := t_0 < t_1 < t_2 < \ldots < t_k = s$ be the integers such that $f^{t_j}(z) \in W$. By the above observation $V_{t_{k-1}}$ is ξ'-well-inside $W_{t_{k-1}}$. Hence, if $f^{t_{k-1}}(z) \notin J$, then by the previous lemma all the required properties hold for $t = t_{k-1}$, taking $W_t^1 = \mathcal{L}_{f^t(z)}(W)$ and $\rho_1(\xi) = \hat\rho(\xi)$. So from now on we assume that $f^{t_{k-1}}(z) \in J$, and let $k' \ge 0$ be minimal so that $f^{t_{k'}}(z), \dots, f^{t_{k-1}}(z) \in J$. Let $k'' \ge k'$ be minimal so that $W_{t_{k''}} \subset \cdots \subset W_{t_{k-1}} = J \subset W_{t_k} = W$. If $W_{t_{k-2}} \not\ni c$ then define $k^* = k-2$. If $W_{t_{k-2}} \ni c$, $k'' > k'$ and if there exists some $i = k', \dots, k''-1$ with $f^{t_i}(z) \in W_{t_{k-1}} \setminus W_{t_{k-2}}$, then let k^* be the maximal such integer (and otherwise leave k^* undefined). We can assume that J is not $\sqrt{\xi}$-well-inside W, because otherwise the lemma also holds: if $k' > 0$ then pull-back once more (the required properties hold for $t = t_{k'-1}$ setting $W_t^1 = \mathcal{L}_{f^t(z)}(W)$), whereas if $k' = 0$ simply set $t = 0$ and $W_t^1 = W$. This means that we can also assume that $V \subset W \setminus J$ (since $V \subset W$ is a nice pair, otherwise $V \supset J$, and J would be a ξ-well-inside W).

Although we have no control on the distortion of $\phi : J \to W$, Lemma 2 gives that $\phi\colon W_{t_{k-2}} \to J$ is quasi-polynomial: it is composition of at most $\#\mathbf{Cr}$ maps of the type $L \circ f$, where L is a diffeomorphism with universal bounded distortion.

Claim: There exists a function ρ_1 as above, so that $V_{\hat t}$ is $\rho_1(\xi)$-well-inside $\hat W$; where either (i) $\hat t = t_{k'}$ and $\hat W = J$ or (ii) $\hat t = t_{k''}$ and $\hat W = W_{\hat t}$, or (iii) k^* is defined, $\hat t = t_{k^*}$ and $\hat W = \mathcal{L}_{f^{\hat t}(z)}(J)$.

This claim implies the lemma. If (i) holds, take $t = t_{k'-1}$, $W_t^1 = \mathcal{L}_{f^t(z)}(W)$ if $k' > 0$ and $t = 0$, $W_t^1 = \mathcal{L}_z(W)$ if $k' = 0$. If (ii) holds, set $t = t_{k''-1}$, $W_t^1 = \mathcal{L}_{f^t(z)}(W_{t_{k''}})$ if $k'' > 0$ and $t = 0$, $W_t^1 = \mathcal{L}_z(W)$ if $k'' = 0$. If (iii) holds, set $t = t_{k^*}$ and $W_t^1 = \mathcal{L}_{f^t(z)}(J)$. By the previous lemma, the required properties hold.

In order to prove the claim we distinguish some cases.

Case I: $\phi\colon J \to W$ is monotone (so $k' = k''$). For simplicity assume that ϕ is orientation preserving (otherwise consider the 2nd iterate of ϕ) and p a (the) fixed point of ϕ. Let us introduce some notation: set $W_{-i} := W_{t_{k-i}}$, $V_{-i} = V_{t_{k-i}}$. Let $W_{-i,\pm}$ be the components of $W_{-i} \setminus \{p\}$, labeled so that $W_{-i,+}$ are all on the same side of p as V.

Case I.1: The union (of two adjacent fundamental neighbourhoods) $W_{0,+} \setminus W_{-2,+}$ contains a $\sqrt{\xi'}$-scaled neighbourhood of V. Then we are done (because their pullback has intersection multiplicity bounded by 3).

Case I.2: Case I.1 does not hold, but $|W_{-1,+} \setminus W_{-2,+}| \geq \sqrt{\xi'}|W_{-2,+}|$. Then $|W_{0,-}| \geq \frac{\xi' - 2\sqrt{\xi'}}{2\sqrt{\xi'}}|W_{-1,+}| \geq C_1\sqrt{\xi'}|W_{-1,+}|$ (because V is ξ'-well-inside W and Case I.1 does not hold). It follows that V_{t_i} is $C\sqrt{\xi'}$-well-inside W for each $i = k', \dots, k-1$. Using this statement for $i = k'+1$, and pulling back once more, we get that $V_{t_{k'}}$ is $\hat{\rho}(C_1\sqrt{\xi'})$-well-inside J.

Case I.3: $|W_{-1,+} \setminus W_{-2,+}| \leq \sqrt{\xi'}|W_{-2,+}|$. Since $\phi\colon W_{-2,+} \to W_{-1,+}$ is quasi-polynomial, this gives $\phi' \leq C_2\sqrt{\xi'}$ on $W_{-2,+}$. Hence, pulling back the external ξ'-space of V_1 in W_{-1}, we get that V_{-1} is $\rho_1(\xi')$-well inside $W_{-1,+} \setminus W_{-3,+}$. As in Case I.1 the claim follows.

Thus the first alternative in the claim holds in the monotone case.

Case II: c is a turning point. If $c \notin W_{-2}$, the 3rd alternative holds, whereas if $c \in W_{-2}$ both components of $W_{-1} \setminus W_{-2}$ are comparable in length. So if $|W_{-1} \setminus W_{-2}| \geq \sqrt{\xi}|W_{-2}|$ then V_{t_i} is $\sqrt{\xi}$-well-inside W_{-1} for $i = k'', \dots, k'$, and either the first or 3rd alternative of the claim holds. Otherwise $|W_{-1} \setminus W_{-2}| \leq \sqrt{\xi}|W_{-2}|$ and exactly as in Case I.3, V_{-1} is very deep inside a component of $W_{-1} \setminus W_{-3}$. Since the intersection multiplicity of the pullback of this component along $f^{t_{k''}}(z), \dots, f^{t_k}(z)$ is bounded by 3 (here we use that $W_{t_{k''}} \subset \dots \subset W$), the 2nd alternative of the claim holds. ☐

Proposition 5. *For each $\tilde{\xi} > 0$ there exists $\xi > 0$ such that if I_{n+1} is ξ-well-inside I_n, then each return domain to I_{n+1} is $\tilde{\xi}$-well-inside I_{n+1}.*

Proof. Take $x \in I_{n+1}$ and let t be its return time to I_n. Then choose the the minimal $s > 0$ such that $f^s(x)$ is in a return domain to I_n which, for simplicity we still denote by I_{n+1}, is ξ-well-inside I_n. Now let $W_s^0 := I_n$, and $V_s = I_{n+1}$ and let W_i^0, V_i, $i = 0, \dots, s$ be the pullbacks of W_s^0 and V_s containing $f^i(x)$. Note that V_0 is the first return domain of I_{n+1} containing x. Let s_1' be maximal with $0 \leq s_1' < s$ and so that $W^0_{s_1'}$ contains a critical point (if none of the intervals $W_0^0, \dots, W_{s-1}^0$ contains a critical point, then set $s_1' = 0$). By Part 2 of Theorem C, there exists a function $\hat{\rho}$ with $\lim_{\xi\to\infty}\hat{\rho}(\xi) = \infty$ so that $W^0_{s_1'}$ is a $\hat{\rho}(\xi)$-scaled neighbourhood of $V_{s_1'}$. If $s_1' = 0$ then $x \in V_0 \subset W_0^0 \subset I_{n+1}$ and the proof is finished. If $s_1' > 0$, let c_1' be the critical point in $W^0_{s_1'}$.

Note that $W^0_{s'_1} \supset V_{s'_1}$ is also a nice pair. If the branch of the first return map to $W^0_{s'_1}$ around c_1 is monotone we set $c_1 = c'_1$ and $s_1 := s'_1$. If it has a turning point, let $s_1 \le s'_1$ be maximal for which $W^0_{s_1}$ contains a turning point c_1. Now let t_1 and $W^1_{t_1}$ be as in the previous lemma. So V_{t_1} is $\rho_1(\hat\rho(\xi))$-well-inside $W^1_{t_1}$ and $W^1_{t_1} \supset V_{t_1}$ is a nice pair if $t_1 > 0$. If $t_1 = 0$ this completes the proof. If $t_1 > 0$ any pullback of $W^1_{t_1}$ which intersects c_1 is contained in $\mathcal{C}_{c_1}(W^1_{s_1})$. Repeating this construction inductively, we get a sequence $t = t_0 \ge s_1 > t_1 > \cdots > s_{\kappa-1} > t_{\kappa-1} \ge s_\kappa \ge t_\kappa = 0$ and nice pairs $W^i_{t_i} \supset V_{t_i}$, $i = 0, 1, \dots, \kappa-1$ (so that $W^i_{t_i}$ is a pullback of I_n), so that

(1) for $i = 1, \dots, \kappa$, $W^i_{s_i}$ is a $\hat\rho((\rho_1 \circ \hat\rho)^{i-1}(\xi))$-scaled neighbourhood of V_{s_i};
(2) for $i = 0, \dots, \kappa$, $W^i_{t_i}$ is a $(\rho_1 \circ \hat\rho)^i(\xi)$-scaled neighbourhood of V_{t_i};
(3) if $W^i_{t_i}, W^j_{t_j}$ $(i < j)$ both contain the same critical point c then $W^j_{t_j} \subset \mathcal{C}_c(W^i_{t_i})$.

Note that any pullback of I_n intersecting x is contained in I_{n+1}. Hence I_{n+1} is a $(\rho\circ\hat\rho)^\kappa(\xi)$-scaled neighbourhood of $V_0 = \mathcal{L}_x(I_{n+1})$. So if κ is not large, the proposition immediately follows. If κ is large, we will use a different argument: by the last property there is a critical point c, and a sequence $i(1) < i(2) < \cdots < i(r)$ (with $r \ge \kappa/(d-1)$ where d is the number of critical points of f), such that

$$W^{i(1)}_{n_{i(1)}} \supset \mathcal{C}_c(W^{i(1)}_{n_{i(1)}}) \supset W^{i(2)}_{n_{i(2)}} \supset \mathcal{C}_c(W^{i(2)}_{n_{i(2)}}) \supset W^{i(3)}_{n_{i(3)}} \supset \cdots \supset W^{i(r)}_{n_{i(r)}} \supset V_{n_{i(r)}} \cup \{c\}.$$

From the first part of Theorem A it follows that for $j = 1, \dots, r-1$, $\mathcal{C}_c(W^{i(j+1)}_{n_{i(j+1)}})$ is δ-well-inside $\mathcal{C}_c(W^{i(j)}_{n_{i(j)}})$. By the first part of Theorem B it follows that $\mathcal{L}_x(\mathcal{C}_c(W^{i(j+1)}_{n_{i(j+1)}}))$ is δ'-well-inside $\mathcal{L}_x(\mathcal{C}_c(W^{i(j)}_{n_{i(j)}}))$. Since this holds for $j = 1, \dots, r-1$, $\mathcal{L}_x(W_1)$ contains a $(1+\delta')^{r-2}$-scaled neighbourhood of $\mathcal{L}_x(W_{n_{i_r}})$. Note that $V_0 \subset \mathcal{L}_x(W_{n_{i_r}})$ (it is the pullback of $V_{n_{i_r}} \subset W_{n_{i_r}}$ containing x) and that $\mathcal{L}_x(W_{n_1}) \subset I_{n+1}$ (because W_{n_1} is a pullback of I_n). Hence I_n is a $(1+\delta')^{r-2}$-scaled neighbourhood of $I_{n+1} = V_0$. This completes the argument if κ (and therefore r) is large. □

8. Theorem D: Bounded distortion

Let us state here a more complete version of Theorem D.

Theorem D' [Control of Distortion] *There is $\xi > 0$ such that for any $\epsilon > 0$ the following holds. Assume that c_1 is a recurrent critical point which is minimal w.r.t. the partial ordering $\prec$ on* **Cr** *and let $c_2, \dots, c_k$ be the collection of critical points with $\omega(c_i) \ni c_1$, $i = 2, \dots, k$. Then there exists a nice interval $W_1 \ni c_1$ with $|W_1| < \epsilon$ such that $W_i := \mathcal{L}_{c_i}(W_1)$ are pairwise disjoint nice intervals and $orb^+(c) \cap W_1 = \emptyset$ for any critical point $c \in \mathbf{Cr} \setminus \{c_1, \dots, c_k\}$. If c_i is non-recurrent* *then $W_i \cap orb^+(c_i) = \emptyset$. If f is infinitely renormalizable at c_1, then we can take for W_1 a periodic interval (with $f^q(\partial W_1) \subset \partial W_1$ where q is the period of W_1). Denoting the first entry map to W_1 by ψ, the following properties hold.*

(1) *Let $V_i \ni c_i$ be the maximal interval such that $\psi(V_i) \subset W_1$ is contained in the return domain to W_1 containing $\psi(c_i)$, i.e., V_i is maximal so that $\psi(V_i) \subset \mathcal{L}_{\psi(c_i)}(W_1)$. Then the first entry map Ψ to $\cup V_i$ is a composition of at most d maps of the type $L_1 \circ f \circ L_2$, where L_1, L_2 are diffeomorphisms of bounded distortion. (Here d is the number of critical points of f.)*

(2) *If f is infinitely renormalizable of Feigenbaum type at c_1, then the period q is of the form $q = p2^n$. Let $V_1 \subset W_1$ be the maximal interval containing c_1 with period $p2^{n+1}$. For $i = 2, \ldots, k$, let $V_i \ni c_i$ be the maximal interval such that $\psi(V_i) \subset \mathcal{L}_{\psi(c_i)}(V_1)$. Then again the first entry map to $\cup V_i$ is a composition of at most d maps of the type $L_1 \circ f \circ L_2$, where L_1, L_2 are diffeomorphisms of bounded distortion.*

Proof. Let us first consider the case when f is not Feigenbaum-like at c_1. Let $W_1 \supset V_1 \ni c_1$ be neighbourhoods of c_1 as in Theorem A', case 2. Take W_1 so small, so that each critical point which does not accumulate on c_1 remains outside W_1. Then take V_i, W_i as above, and let $s_i \geq 0$ be so that $\psi|W_i = f^{s_i}$. It follows that the intervals $W_i, \ldots, f^{s_i}(W_i)$ are pairwise disjoint and V_i is well-inside W_i.

Now take $x \in M$ for which exists the minimal $t \geq 0$ so that $f^t(x) \in \cup V_j$. So let us assume that $f^t(x) \in V_{j_0}$ and define the chains $\{\mathcal{V}_i\}_{i=0}^t$ and $\{\mathcal{W}_i\}_{i=0}^t$ such that $f^i(x) \in \mathcal{V}_i \subset \mathcal{W}_i$, $\mathcal{V}_t = V_{j_0}$ and $\mathcal{W}_t = W_{j_0}$. We claim that each critical point is contained in at most one of the intervals from $\{\mathcal{W}_i\}_{i=0}^t$. Indeed, if for $0 \leq i_1 < i_2 \leq t$ the intervals $\mathcal{W}_{i_1}$ and $\mathcal{W}_{i_2}$ contain the same critical point c_l then $\mathcal{W}_{i_1} \subset \mathcal{W}_{i_2}$ and this would imply that $f^{i_1}(x) \in \mathcal{W}_{i_1} \subset V_l$ which contradicts the definition of t. Consider the intervals $\mathcal{W}_{n_1}, \ldots, \mathcal{W}_{n_\nu}$ from $\{\mathcal{W}_i\}_{i=0}^t$ which contain critical points. The previous theorem implies that (shrinking W_1 if necessary) the map $f^{n_{j+1}-n_j}_{|\mathcal{W}_{n_j+1}}$ (for $0 \leq j \leq \nu - 1$, take $n_0 = 0$) has negative Schwarzian and this implies that the diffeomorphism $f^{n_{j+1}-n_j-1}_{|\mathcal{V}_{n_j+1}}$ has bounded distortion. Hence, the first entry map to $\cup V_i$ is a composition of at most d maps of the type $L_1 \circ f \circ L_2$, where $L_, L_2$ are diffeomorphisms of bounded distortion.

If f is Feigenbaum-like at c_1, then Let $W_1 \ni c_1$ be a neighbourhood of c_1 as in Theorem A', case 1, i.e. of period $p2^i$. Because W_1 is periodic, $\mathcal{L}_{\psi(c_i)}(W_1) = W_1$ so we cannot proceed precisely as above. But because of Corollary 1 (at the end of Section 2), for each $z \in W_1$, $\mathcal{L}_z(V_1)$ is well-inside W_1. With this modification, the proof goes as before. □

Theorem D simply follows from Theorem D'.

9. Theorem E: Measure of invariant sets and ergodicity

First let us prove the following result (which was proved previously in the negative Schwarzian unimodal case by Martens, see [14] and [18]).

Proposition 6. *If $f(Y) \subset Y$, and Y has positive Lebesgue measure, then either*

- *there exists an interval H such that $|H| = |H \cap Y|$, or*
- *for almost every $y \in Y$ there exists a subset $C = C(y)$ of the critical set of f so that $\Omega := \omega(c) = \omega(\tilde{c})$ for each $c, \tilde{c} \in C$, each $c \in C$ is recurrent, Ω is a minimal set (every forward orbit is dense), $f^n(y) \to \Omega$ as $n \to \infty$, and $\lim_{n\to\infty} r_n(y) \to 0$.*

Here $r_n(y)$ is defined as follows. Let n be a positive integer and let $T_n(y)$ be the maximal interval such that $f^n_{|T_n(y)}$ is a diffeomorphism. Then $r_n(y)$ is the length of the smallest component of $f^n(T_n(y) \setminus \{y\})$.

Proof. Let us assume that the first alternative does not hold. Then Y does not have full measure in any interval.

Claim 1: y cannot accumulate onto parabolic or fixed points of f. To prove this we use an argument similar to the one given on page 482-483 of [9]. More precisely, assume by contradiction that y accumulates onto a parabolic or fixed point a. Let a have period s and for simplicity assume that f^s is orientation preserving near a. Note that the period of parabolic points is bounded, see [18]. Let $b \neq a$ be a periodic point for which there exists a sequence b_n converging monotonically to a (from the same side as $y, f(y), f^2(y), \ldots$ accumulates) so that $f^{sn}(b_n) = b$. Then $[b_n, b_{n+1}]$ are fundamental domains of the periodic point a. Fix n_0 large, and let $f^k(y)$ be the first visit to $[b_{n_0}, a]$ and let n be maximal so that $f^k(y) \in [b_n, b_{n+1}] \subset [b_n, a]$. If $n \geq n_0 + 1$, then the pullback of $[b_{n-1}, b_{n+2}] \supset [b_n, b_{n+1}] \ni f^k(y)$ along $\{y, \ldots, f^k(y)\}$ is disjoint. So the pullback of $[b_0, b_3]$ along $\{y, \ldots, f^{k+s(n-1)}(y)\}$ has intersection multiplicity ≤ 3. Since $[b_1, b_2] \ni f^{k+s(n-1)}(y)$ is well-inside $[b_0, b_3]$ (the sizes of adjacent intervals fundamental domains associated to a are of the same order), by letting n tend to infinity, and using that y is a Lebesgue density point of Y, it follows that Y contains either $[b_0, b_1]$ or $[b_2, b_3]$. (Note that the pullback of $[b_0, b_3]$ to y might not be monotone, but that its image under $f^{k+s(n-1)}$ contains one of the components of $[b_0, b_3] \setminus f^{k+s(n-1)}(y)$.) So in this case, Y contains an interval and the claim is proved by contradiction. If on the other hand, $n = n_0$ then $f^{k+s(n-1)}(y) \in [b, b_1] = [b_0, b_1]$ and then choose a sequence U_i of adjacent fundamental domains associated to b with $U_i \to b$ as $i \to \infty$ and with $b_1 \in U_0 \subset [b, b_1]$. If $f^{k+s(n-1)}(y) \in U_0$ then the previous argument can still be used to prove that Y contains an interval (because U_0 is well-inside $[b_0, b_3]$). Otherwise let m be maximal so that $y' = f^{k+s(n-1)}(y) \in U_m$ and so by iterating y' to successive fundamental domains of b, there exists $k' > k + s(n-1)$ with $f^{k'}(y) \in U_1$. Now we can pullback $[U_2, U_0] \ni f^{k'}(y)$ along $\{y, \dots, f^{k'}(y)\}$ with intersection multiplicity ≤ 3 and as before obtain that Y contains an interval, by letting n tend to infinity. This contradiction, concluded the proof of claim 1.

Claim 2: $r_n(y) \to 0$ for almost all $y \in Y$. (This fact, is well known, but for completeness we prove it again.) Indeed, if this is not the case then there are $\delta > 0$, a subsequence n_i, $z \in Y$ and a Lebesgue density point y of Y with $r_{n_i}(y) \geq \delta$ and $f^{n_i}(y) \to z$. If the orbit of y stays away from parabolic periodic points, then by by part 2(ii) of Theorem B, and because y is a Lebesgue density point and Y is forward invariant, Y contains (up to measure zero) a $\delta/2$ neighbourhood of z, contradicting that Y does not contain intervals.

Claim 3: Let y be a point of Lebesgue density of Y. Then $\omega(y) = \cup_{c \in C} \omega(c)$ where $C = C(y)$ is the set of critical points such that $c \in \omega(y)$. By definition for each n there is a critical point c so that f^n is a diffeomorphism on some interval $[a, y]$ such that $f^k(a)$ is a critical point c, and so that $f^{n-k}(c, f^k(y))$ is a diffeomorphism. Since $r_n(y) \to 0$ and since f has no wandering intervals, it follows that both $|c - f^k(y)|$ and $f^{n-k}(c), f^n(y)|$ tend to zero when n tends to infinity. Together this implies claim 3.

Claim 4: If y and $C = C(y)$ are as in Claim 3 and so that y is not eventually mapped onto a critical point, then $c \in \omega(c')$ for any $c, c' \in C$. This follows from the argument given in Proposition 3.3 of [9] (we use here that Y does not have full measure in any interval). To prove this, let us first assume by contradiction that for some $c \in C$ there exists one or more critical points $c' \in C$ such that $c \notin \omega(c')$. Let C' be the set of such critical points in C. Note that it is not impossible that $c \in C'$. Define $\hat{C} = C \setminus C'$. Let U be a neighbourhood of c so that each $c' \in C'$ avoids U. Let I be a small nice interval, containing c and well inside U, taken from the partition generated by the fixed points of f and so that each first return domain to I is well-inside I. This is possible by Theorem A. Now fix I throughout the remainder of the proof of this claim and write $I(c) = I$. For each each critical point c^* such that $c \in \omega(c^*)$, let $k \geq 0$ be the smallest integer such that $f^k(c^*) \in I(c)$, and let $I(c^*)$ be the pullback of $I(c)$ along $\{c^*, \dots, f^k(c^*)\}$. Then each return domain to $I(c^*)$ is also well-inside $I(c^*)$. Now fix some $c' \in C'$. Take arbitrarily small nice neighbourhoods U_j of c' from the partition generated by the fixed points of f. We can take these neighbourhoods so that the forward orbit of their boundary points do not intersect $I(c)$ and if $k_j \geq 0$ is minimal so that $f^{k_j}(y) \in U_j$ then $k_j \to \infty$. Let $n_j > k_j$ be minimal so that

$$f^{n_j}(y) \text{ is in a component } I(\tilde{c}), \text{ of } \hat{I} = I(c) \bigcup \cup_{\hat{c}\in\hat{C}} I(\hat{c}).$$

Let $V_j^0, \ldots, V_j^{n_j} = I(\tilde{c})$ be the chain along $\{y, \ldots, f^{n_j}(y)\}$. Since k_j is minimal and $I(\tilde{c})$ is nice, $V_j^0, \ldots, V_j^{k_j}$ are pairwise disjoint. Moreover, since the iterates of $c' \in C'$ avoid U, by the minimality of n_j, because $r_n(y) \to 0$ and the orbit of y does not come close to critical points in the complement of $C = C(y)$, $\quad f^{n_j - k_j} \colon V_j^{k_j} \to \hat{I}$ is a diffeomorphism onto $I(\tilde{c})$. From Claim 1 we can assume that $\omega(y)$ does not contain fixed points of f and so $f^{n_j}(y)$, $j = 1, 2, \ldots$ are all contained in closed intervals $W' \subset W \subset I(\tilde{c})$ which are all properly contained in each other. By assumption Y does not have full Lebesgue measure in any subinterval and therefore the density of Y in W is less than one. Since $f^{n_j}(y) \in W'$, since $f^{n_j - k_j} \colon V_j^{k_j} \to I(\hat{c})$ is a diffeomorphism, and since $V_j, \ldots, V_j^{k_j}$ are disjoint, we get that the density of Y in the pullback of W to y is also uniformly bounded away from one. This contradicts the assumption that y is a Lebesgue density point of Y and therefore proves the claim.

Claim 5: For y as above, each $x \in \omega(y)$ accumulates on C. Indeed, assume not. Take an arbitrary $c \in C$ and take $I(c) = I_n$ as in the proof of claim 4, so that the orbit of x never enters the neighbourhood $I(c)$. Let $I(c^*)$, $c^* \in C$, be the corresponding pullback of $I(c)$. Then take partition elements U_j around x. Defining k_j, n_j as in Claim 4 one gets again a contradiction.

Claim 6: Each point $x \in \omega(y)$ has an orbit which is dense in $\omega(y)$. Indeed, by the previous claim iterates of x accumulate on some critical point $c \in C$. From Claim 4, they then also accumulate on any other critical point, and so $\omega(x) = \cup_{c\in C}\omega(c)$. From Claim 2 the claim then follows.

From the previous claims the proposition follows. □

Assuming that X is a closed set which is forward invariant ($f(X) \subset X$), define

$$B(X) = \{y; \omega(y) \subset X\}.$$

Proposition 7. *Let X be a forward invariant set which is minimal. Then X has zero Lebesgue measure and for each $x \in X$ there exists a sequence of intervals $\mathcal{N}_n \subset \mathcal{U}_n$ such that $\cap \mathcal{U}_n = \{x\}$, $(\mathcal{U}_n \setminus \mathcal{N}_n) \cap X = \emptyset$ and $\mathcal{N}_n$ is well-inside $\mathcal{U}_n$. Moreover, if $Y \subset B(X)$ has positive Lebesgue measure then X is equal to $\omega(c)$ for at least one recurrent critical point c, and $|\mathcal{U}_n \cap Y|/|\mathcal{U}_n| \to 1$ as $n \to \infty$.*

Proof. Take any point $x \in X$ and let $W_1 \ni x$ be a nice interval such that all its return domains are well-inside. Let $V_i \subset W_i$ be the intervals given by Theorem D' and let ψ be the first entry map to $\cup V_i$. For a connected component J of the domain of ψ, let $\psi(J) \subset V_{j_0}$ and $T(J) \supset J$ be the pullback of W_{j_0}. Because of the choice of intervals $V_i \subset W_i$, see Theorem A', if J_1, J_2 are connected components of the domain of ψ such that if $T(J_1) \cap T(J_2) \neq \emptyset$ then either $T(J_1) \subset T(J_2)$ or $T(J_2) \subset T(J_1)$. Moreover, if $J_2 \cap T(J_1) \neq \emptyset$ then $T(J_2) \subset T(J_1)$. So let us take a minimal component J in the sense that $X \cap J \neq \emptyset$ and such that $T(J) \subset T(\tilde{J})$ for any other component $\tilde{J}$ of $D(\psi)$ for which $X \cap \tilde{J} \neq \emptyset$. Since X is compact this can be done. Because of this minimality, $T(J)$ contains at most 2^b components $\tilde{J}$ of $D(\psi)$ with $\tilde{J} \cap X \neq \emptyset$. Here b denotes the number of turning points of f. Each one of these components $\tilde{J}$ is well-inside $T(J)$ and $T(\tilde{J}) = T(J)$. It is not hard to see that this implies that there are intervals $N \subset U$ such that $U \subset T(J)$, N is well-inside U and $X \cap (U \setminus N) = \emptyset$ and $N \cap X \neq \emptyset$. The choice for N depends on the size and position of the components $\tilde{J}$ inside $T(J)$; it is not obvious that N and U are nice.

Since X is minimal, for each $z \in X$ there exists the minimal $s \geq 0$, such that $f^s(z) \in U$. In fact, we have that $f^s(z) \in N$ because $X \cap (U \setminus N) = \emptyset$. Because s is minimal, the pullback $z \in N_s \supset U_s$ of $N \supset U$ along $\{z, \ldots, f^s(z)\}$ is disjoint. So N_s is well-inside U_s and $X \cap (U_s \setminus N_s) = \emptyset$ (here we use Lemma 1). So, taking a sequence of W_1 shrinking to a point of X we get a sequence $\mathcal{N}_n \subset \mathcal{U}_n$ as in the first statement.

From the previous proposition, it follows that if $Y \subset B(X)$ has positive Lebesgue measure and X is minimal, then X is equal to $\omega(\tilde{c})$ for at least one recurrent critical point $\tilde{c}$ (which is in X). Take $z = \tilde{c}$, let $\mathcal{U}_n \supset \mathcal{N}_n \ni \tilde{c}$ be as above and let C be the set of critical points c with $\omega(c) \ni \tilde{c}$. Let y be a Lebesgue density point of Y. From the previous proposition, there exists t so that $f^t(y) \in \mathcal{N}_n$. Let $\mathcal{U}_n^i \supset \mathcal{N}_n^i \ni f^i(y)$, $i = 0, \ldots, t$ be the pullbacks of $\mathcal{U}_n \supset \mathcal{N}_n$ along $\{f^i(y), \ldots, f^t(y)\}$. From the properties of $\mathcal{U}_n \supset \mathcal{N}_n$ stated above, for $i < j \leq t$ with $\mathcal{U}_n^i \cap \mathcal{U}_n^j \neq \emptyset$ one either has $\mathcal{U}_n^i \subset \mathcal{U}_n^j$ or $\mathcal{U}_n^i$ is contained in a component of $\mathcal{U}_n^j \setminus \mathcal{N}_n^j$. Because of the minimality of t, the first case is impossible. Since $\mathcal{U}_n \cap X \subset \mathcal{N}_n$, for each critical point $c \in X$ there is at most one $i \leq t$ with $c \in \mathcal{N}_n^i \subset \mathcal{U}_n^i$. Moreover $\mathcal{U}_n^i \cap X \subset \mathcal{N}_n^i$. Because of the contraction principle, provided n is large, any critical point c which is not in X is not contained in any interval $\mathcal{U}_n^i$. From Part (2) of Theorem C and since y is a Lebesgue density point of Y, there exists $i(n) \leq t$ and a critical point $c \in X$ for which $c \in \mathcal{N}_n^{i(n)}$ such that $|Y \cap \mathcal{N}_n^{i(n)}|/|\mathcal{N}_n^{i(n)}|$ goes to one. $\square$

From the previous theorems and Propositions 6 and 7, Theorem E follows.

Remark 1. *It is NOT true that any Cantor set X with $f(X) = X$ has Lebesgue measure zero. Indeed, assume now that $\omega(c)$ is a Cantor set, and that its basin $B(\omega(c))$ has positive Lebesgue measure. We claim that in this situation, there is a Cantor set X of positive Lebesgue measure with $X \subset B(\omega(c))$ and $f(X) = X$. (Examples of this situation are infinitely renormalizable maps, and non-renormalizable maps with an absorbing Cantor attractor, see* [4]*.) To prove the claim, let $X_0 \subset B(\omega(c))$ be a Cantor set of positive Lebesgue measure and let X_1 be the closure of $\cup f^i(X_0)$. Then $f(X_1) \subset X_1$. Since X_0 is a Cantor set, and $X_0 \subset B(\omega(c))$, the set X_1 is closed, equal to $\cup f^i(X_0) \cup \omega(c)$, and X_1 has no isolated points. X_1 contains no intervals, because $B(\omega(c))$ contains no intervals. Hence X_1 is a Cantor set of positive Lebesgue measure with $f(X_1) \subset X_1$. In order to find a Cantor set with $f(X) = X$, take X_0 so that it lies in the range of f and add appropriate preimages of X_1 to the set. For examples, if f is unimodal, then we can define X to be the union of X_1 and all x so that for integer $n \geq 1$, $x, \ldots, f^{n-1}(x)$ lies to the left of the critical point, while $f^n(x) \in X_1$. Again X is a Cantor set of positive Lebesgue measure, but now $f(X) = X$.*

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References

[1] A. M. Blokh and M. Yu Lyubich, Measure and dimension of solenoidal attractors of of one-dimensional dynamical systems, *Comm. Math. Phys.* **127**, (1990), 573-583.

[2] A. M. Blokh and M.Yu. Lyubich, *Measurable dynamics of S-unimodal maps of the interval*, Ann. Sci. École Norm. Sup. 4e srie, **24**, (1991), 737-749.

[3] A.M. Blokh and M.Yu. Lyubich, *Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. II. The smooth case*, Ergod. Th. Dyn. Sys., **9**, (1989), 751–758.

[4] H. Bruin, G. Keller, T. Nowicki and S. van Strien, *Wild Cantor attractors exist*, Annals of Math. **143**, (1996), 97-130.

[5] O. Kozlovskii, *Axiom A maps are dense in the space of unimodal maps in the C^k topology*, Ann. of Math. **157** (2003), no. 1, 1-43.

[6] O. Kozlovski, *Getting rid of the nagative Schwarzian derivative condition*, Annals of Math. **152**, (2000), 743-762.

[7] O. Kozlovski, W. Shen and S. van Strien, *Rigidity for real polynomials*, Preprint 2003, available from http://maths.warwick.ac.uk/∼strien/Publications

[8] G. Levin, *Bounds for maps of an interval with one reflecting critical point. I*, Fundamenta Math. **157**, (1998), 287-298.

[9] G. Levin and S. van Strien, *Local connectivity of the Julia set of real polynomials*, Annals of Math. **147**, (1998), 471-541.

[10] G. Levin and S. van Strien, *Bounds for maps of an interval with one reflecting critical point. II.*, Inventiones Math. **141**, (2000), 399-465.

[11] M.Yu. Lyubich, *Non-existence of wandering intervals and structure of topological attractors of one dimensional dynamical systems: 1. The case of negative Schwarzian derivative*, Ergod. Th. & Dyn. Sys. **9**, (1989), 737-749.

[12] M.Yu. Lyubich, Ergodic theory for smooth one-dimensional dynamical systems, Stony Brook pre-print 1991/11.

[13] R. Mañé, *Hyperbolicity, Sinks and Measure in One Dimensional Dynamics*, Commun. Math. Phys. **100**, (1985), 495-524.

[14] M. Martens, *Interval dynamics,* Thesis, Delft Technical University, (1990) and *Distortion results and invariant Cantor sets of unimodal maps*, Ergod. Th. Dynam. Sys, **14**, 1994, 331-349.

[15] M. Martens, W. de Melo, and S. van Strien, *Julia-Fatou-Sullivan theory for real one-dimensional dynamics*, Acta Math. **168**, 1992, 273-318.

[16] W. de Melo and S. van Strien, *A structure theorem in one-dimensional dynamics*, Ann. of Math. (2), **129**, 1989, 519-546.

[17] W. de Melo and S. van Strien, *A structure theorem in one-dimensional dynamics*, Ann. of Math. (2), **129**, 1989, 519-546.

[18] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **25**, Springer Verlag, Berlin (1993).

[19] W. Shen, *Bounds for one-dimensional maps without inflection critical points*, Preprint June 2000 and J. Math. Sci. Univ. Tokyo. **10**, (2003), 41-88

[20] W. Shen, *On the measurable dynamics of real rational functions*, Ergod. Th. Dyn.Sys. **23**, 2003, 957-983.

[21] W. Shen, *On the metric properties of multimodal interval maps and* C^2 *density of Axiom A*, Invent. Math, **156**, 2004, 301–403.

[22] G. Świątek and E. Vargas, *Decay of geometry in the cubic family*, Ergod. Th. & Dyn. Sys. **18**, (1998), 1311-1329.

[23] S. van Strien, *Hyperbolicity and invariant measures for general* C^2 *intervals maps satisfying the Misiurewicz condition* – Commun. Math. Phys. **128**, (1990), 437-495.

[24] E. Vargas, *Measure of minimal sets of polymodal maps*, Ergod. Th. and Dynam. Sys. **16**, (1996), 159-178.

Abstract. We consider smooth multimodal maps which have finitely many non-flat critical points. We prove the existence of real bounds. From this we obtain a new proof for the non-existence of wandering intervals, derive extremely useful improved Koebe principles, show that high iterates have 'negative Schwarzian derivative' and results on ergodic properties of the map. One of the main complications in the proofs is that we allow f to have inflection points.

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