Annals of Mathematics, **0** (1993), 1–33

Wild Cantor attractors exist

By H. Bruin, G. Keller, T. Nowicki, S. van Strien*

Abstract: In this paper we shall show that there exists a polynomial unimodal map $f\colon [0,1] \to [0,1]$ with so-called Fibonacci dynamics

- which is non-renormalizable and in particular, for each x from a residual set, $\omega(x)$ is equal to an interval; (here $\omega(x)$ is defined to be the set of accumulation points of the sequence $x, f(x), f^2(x), \ldots$);
- for which the closure of the forward orbit of the critical point c, i.e., $\omega(c)$, is a Cantor set and
- for which $\omega(x) = \omega(c)$ for Lebesgue almost all x.

So the topological and the metric attractor of such a map do not coincide. This gives the answer to a question posed by Milnor [Mil] in dimension one.

Introduction

One of the central themes in the theory of dynamical systems is the concept of attractors. However, there is no complete consensus about the 'correct' definition of this notion. In particular it is not clear whether an attractor should attract a topologically big set or a set which is large in a metric sense. So, if $f\colon M \to M$ is a dynamical system defined on a manifold M, then we could define a closed forward invariant set X to be a *topological* respectively a *metric attractor* if

(i) its basin

$$B(X) = \{x \ ; \ \omega(x) \subset X\}$$

contains a residual subset of an open subset of M, respectively $B(X)$ has positive Lebesgue measure;

(ii) there exists no closed forward invariant set X' which is strictly included in X for which $B(X)$ and $B(X')$ coincide up to a meager set respectively up to a set of measure zero.

*T.N. was supported by the DFG, NWO, KBN 2 1090 01 91. Part of this work was done during a stay of SvS at Stony Brook. SvS would like to thank M. Lyubich and F. Tangerman for some useful discussions.

Here $\omega(x)$ is the set of limit points of $f^n(x)$ as $n \to \infty$. Moreover, we say that A is a residual (resp. meager) set if it is the countable intersection (union) of open dense (closed nowhere dense) sets. We call a set X a *wild attractor* if X is a metric attractor but not a topological attractor (for a discussion on these definitions, see [Mil]).

If X is a periodic attractor, a hyperbolic attractor, a 'Feigenbaum attractor' (see for example [MS] and for the invertible case see [GST]), or one of the known strange attractors, see [BC], then X is both a metric and a topological attractor. Of course, there are some pathological cases: for example the horseshoe of a C^1 diffeomorphism can have positive Lebesgue measure and certainly is no topological attractor, see [Bow]; so this is an example of a wild attractor. There are some other – less exotic – examples. Indeed, as was shown in [Mi2], the map $\mathbb{C} \ni z \mapsto \exp(z)$ is topologically transitive. Moreover, it was shown in [L2] and [Re] that under under this map Lebesgue almost all points are attracted to X where X is the orbit of 0. Hence X is a wild attractor. Another example is given in [Kan], see also [AKYZ]: there exists a smooth topologically transitive (non-invertible) map on the annulus $S^1 \times [0,1]$ such that the basin of the boundary circles $X = S^1 \times \{0,1\}$ has full Lebesgue measure. (In fact, the basin of each circle intersects every open set in a set of positive area; this is the reason the basins are called *'intermingled'*.)

In this paper we present a polynomial interval map which has a wild attractor (a Cantor set) and such that the map is transitive on some interval. (In the terminology of [GJ] the map is said to have an *absorbing Cantor attractor.*) More precisely, in our case $M = [0,1]$ and f is a smooth unimodal interval map – this means f has one extremal point – and for simplicity we shall also assume that $f(0) = f(1) = 0$. A prototype of such a map is

$$f(x) = \lambda \left[1 - |2x-1|^{\ell}\right]$$

where $\lambda > 0$ is chosen so that f maps the interval $[0,1]$ inside itself and f has the so-called Fibonacci-type dynamics. We shall define this in the next section.

There are many publications in which it was conjectured that a smooth map $f\colon [0,1] \to [0,1]$ cannot have a wild Cantor attractor. (We should note, however, that in 1992 Misha Lyubich and Folkert Tangerman made computer estimates suggesting that wild Cantor attractor do exist for Fibonacci maps of the form $x \mapsto x^6 + c_1$.) Moreover, there are several results which prove that these sets cannot exist in particular cases, see [GJ], [JS1], [Mar], [LM] and in the general quadratic (i.e., $\ell = 2$) case [L6]. We shall show that wild Cantor attractors do exist when ℓ is a large real number.

MAIN THEOREM. *There exists $\ell_0 < \infty$ with the following property. Let $f: [0,1] \to [0,1]$ be a C^2 unimodal interval map with a critical point c of order $\ell \geq \ell_0$ and with the Fibonacci combinatorics. Then $X = \omega(c)$ is a wild Cantor attractor for f.*

In fact, X is a closed forward invariant minimal Cantor set with zero Lebesgue measure, such that its basin $B(X)$ is meagre but has positive Lebesgue measure.

Here we say that c is a critical point of a C^2 map f if $Df(c) = 0$ and the *order of the critical point* is said to be ℓ if there exists a C^2 diffeomorphism ϕ between two neighbourhoods of c such that

$$f \circ \phi(x) = f(c) - |x - c|^{\ell}$$

for x close to c.

In [Str2] the previous theorem is used to show that there exists a topologically transitive polynomial map $f: [0,1] \to [0,1]$ which has two intermingled Cantor attractors, thus showing that the phenomena from [Kan] also appear in dimension one.

The Main Theorem gives examples of polynomial maps having a wild attractor. These examples can, if required, also have additional periodic attractors.

COROLLARY 1. *There exists a unimodal polynomial map with a wild Cantor attractor X: the basin $B(X)$ of X is a meagre set of positive Lebesgue measure. One can choose this polynomial map so that it has in addition a periodic attractor and so that $B(X)$ is nowhere dense.*

Proof of Corollary. First we remind the reader that any family of unimodal maps $f_\lambda: [0,1] \to [0,1]$ for which $(x, \lambda) \mapsto f_\lambda(x)$ is smooth and for which $\lambda \mapsto f_\lambda(c)$ is onto $(0,1]$ is full. This means that within this family any combinatorial type can be found.

Clearly, the family $\lambda(x) \mapsto \lambda \left[1 - |2x-1|^{\ell}\right]$ satisfies this fullness assumption. Hence there exists a polynomial map of this form (with $\ell \geq \ell_0$ and even) which has the Fibonacci dynamics. So $X = \omega(c)$ with $c = 1/2$ is a wild Cantor attractor. In this particular case, the map will have no periodic attractors. Indeed, since this map has negative Schwarzian derivative, each periodic attractor has in its basin a critical point or a boundary point of $[0,1]$. Because the critical point is contained in a minimal Cantor set, it follows that this map has no periodic attractors. Since f has no wandering intervals, it follows that $B(X)$ is dense in this case, see Theorem AB in Chapter IV of [MS]. Moreover, see the proof of Theorem 5.2 below, the set $B(X)$ has full Lebesgue measure in this case.

To construct a similar polynomial map for which $B(X)$ is nowhere dense, notice that one can take a full family of unimodal polynomials $f_\lambda\colon[0,1]\to[0,1]$ as above such that the orienatation reversing fixed point of f_λ is always attracting and which has a critical point of even order $\geq \ell_0$. By the Main Theorem, the Fibonacci map within this family will have a wild Cantor attractor $X=\omega(c)$. By Theorem AB of Chapter IV of [MS], the union of the basin of this attracting fixed point and the basins of other periodic attractors of this map forms an open and dense set. $B(X)$ is contained in the complement of these basins and therefore nowhere dense in this case. $\square$

It is easy to show that our methods also give examples of multimodal smooth interval maps for which each critical point is quadratic and which have wild Cantor attractors: simply choose the map so that the return map near some critical point is a unimodal map of Fibonacci-type while the orbit of this critical point contains at least ℓ other critical points. However, it is not clear whether wild Cantor attractors also appear generically in one-parameter families:

Question. Does the space of smooth maps $f\colon[0,1]\to[0,1]$ with a wild Cantor attractor form a codimension-one subset of the space of all smooth interval maps?

Of course, it follows from the Main Theorem that on each manifold of arbitrary dimension there exists a smooth mapping with a wild Cantor attractor.

Question. Does there exist a Hénon map with a wild Cantor attractor? (Compare [GST].)

In the complex case there are related results, see [SN]:

THEOREM. *For each sufficiently large even integer ℓ there exists $c_1\in\mathbb{R}$ such that the map $f(z)=z^\ell+c_1$ has the following properties:*
(1) the set $\omega(0)$ is a Cantor set with zero Lebesgue measure;
(2) the set of points $z\in\mathbb{C}$ for which $\omega(z)$ is contained in $\omega(0)$ has positive Lebesgue measure;
(3) the set of points whose forward iterates remain bounded has no interior.

In particular, the Julia set of $z\mapsto z^\ell+c_1$ has positive Lebesgue measure. This map has the Fibonacci dynamics (to be defined in the next section).

1. Some comments on the Main Theorem and its proof

In fact, the attractor X from the Main Theorem is equal to $\omega(c)$ and this set has zero Lebesgue measure, see [Mar] and also [MS]. If the map f from the Main Theorem is a unimodal polynomial with a unique critical point in $\mathbb{C}$ (or if has negative Schwarzian derivative and f has no attracting fixed points) then $B(X)$ has full Lebesgue measure and its complement is a residual set. We should remark that a smooth map as above may have one or more periodic attractors, but that even then the attractor X has a basin which attracts a set of positive Lebesgue measure (and the critical point is density point of $B(X)$). This is not completely surprising because $\omega(c)$ is not accumulated by periodic attractors, see [MMS] and also [MS][Chapter IV].

In the theory of unimodal interval maps with negative Schwarzian derivative of f, i.e., with

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\frac{D^2 f(x)}{Df(x)} < 0$$

and for which the order of the critical point is finite, one has a well-known classification, see [Gu], [BL], [Ke] and also [MS].

- f has a stable periodic orbit O which is both a topological and metric attractor;
- f is infinitely renormalizable, i.e., there exists a nested sequence of intervals $I_n \ni c$ shrinking to c and a sequence of integers $q(n) \to \infty$ such that $I_n, \dots, f^{q(n)-1}(I_n)$ are disjoint and $f^{q(n)}(I_n) \subset I_n$. In this case $\omega(c)$ is a Cantor set of zero Lebesgue measure which is both a topological and metric attractor;
- f is not infinitely renormalizable. In this case there exists a cycle of intervals Z (a finite union of intervals) such that $B(Z)$ is dense and has full Lebesgue measure. The set Z is a topological attractor, but not necessarily a metric attractor: in principle, there could be a Cantor set $X \subset Z$ such that $B(X)$ has full Lebesgue measure (but is not dense).

From our theorem it follows that the possibility mentioned in the last case really does occur if ℓ is large. In the quadratic case, i.e. $\ell = 2$, the results of [L6] imply that Z is a metric attractor as well.

Any map with a wild Cantor attractor has no absolutely continuous invariant probability measure, because Lebesgue almost all points wander densely on the support of the measure by the Birkhoff Ergodic Theorem. If the Schwarzian derivative of f is negative and $\ell = 2$ then it is shown in [LM] that f has an absolutely continuous invariant probability measure by showing that the summability condition from [NS] is satisfied. In particular, f has no absorbing Cantor set in this case. The methods of proof in [LM] are a mixture of real tools and tools from the theory of complex analysis and hyperbolic geometry. This result was generalized in [KN]: in that paper it was shown that the same results hold for $1 < \ell \le 2 + \epsilon$ provided $\epsilon > 0$ is small. The tools in [KN] are entirely based on real estimates, and also no use is made of [NS] (because the summability condition fails if $\ell > 2$).

As mentioned, our result implies that f has no absolutely continuous invariant probability measure for ℓ large. In fact, as Henk Bruin has shown in [Br], this already follows from Proposition 3.12.

We expect that the methods of this paper can be extended to show that for Fibonacci maps of 'bounded type' (a notion which we shall discuss in the section about the combinatorial properties of Fibonacci maps) with a rather flat critical point, the same result holds.

Let us now give an outline of the proof that $\omega(x)$ is equal to the Cantor set $\omega(c)$ for Lebesgue almost all x.

- First we will show that there exists a nested sequence of intervals $(u_n, \hat{u}_n)$ containing c and that the size of the annulus $A_n = (u_n, \hat{u}_n) \setminus (u_{n+1}, \hat{u}_{n+1})$ is very small compared to the size of $(u_{n+1}, \hat{u}_{n+1})$ if the order ℓ of the critical point is large.

- Next we let $I_n, \hat{I}_n$ be the components of A_n and show that some iterate f^{S_n} of f maps I_n diffeomorphically inside $\cup_{k \ge n-2}(I_k \cup \hat{I}_k)$ and that this map is not 'too' non-linear. Because of the above this implies that 'most' points are mapped closer to c by this iterate.

- Finally, we combine the first two arguments and a kind of random walk argument to show that typical points are in the basin of $\omega(c)$. The idea of using a random walk argument based on Markovian methods had been around for several years. It was succesfully applied in [KN].

2. Combinatorial properties of the Fibonacci map

In this section we shall define and state some properties of the Fibonacci map. It is well-known that maps with these properties exist, see [HK] or [LM]. In the companion paper [SN] we shall construct such a map 'by hand'. Let $f\colon [0,1] \to [0,1]$ be a unimodal map with $f(0) = f(1) = 0$. For each $x \ne c$ there exists a 'symmetric' point $\hat{x} \ne x$ with $f(\hat{x}) = f(x)$. For $i \ge 0$ and $x \in [0,1]$, let $x_i = f^i(x)$ and choose $x_{-i} \in f^{-i}(x)$ so that the interval connecting this point to c contains no other points in the set $f^{-i}(x)$. Note that if c is not a periodic point there are always precisely two such points c_{-i} (which are symmetric with respect to each other). Let $S_0 = 1$ and define S_i inductively by

$$S_i = \min\{k \ge S_{i-1};\, c_{-k} \in (c_{-S_{i-1}}, \hat{c}_{-S_{i-1}})\}.$$

f is called a *Fibonacci* map if the sequence S_i coincides with the Fibonacci numbers: $S_0 = 1$, $S_1 = 2$ and $S_{k+1} = S_k + S_{k-1}$, i.e., the sequence $1, 2, 3, 5, 8, \dots$.

Let us denote by z_k the nearest point to c in the set $f^{-S_k}(c)$. It should be clear from the context whether z_k is to the left or right of c. Moreover, for $x \in [0,1]$ let us write

$$x^f = f(x)$$

(usually, x will be close to c and so x^f close to $c^f = f(c)$).

PROPOSITION 2.1. *A Fibonacci map $f:[0,1] \to [0,1]$ satisfies the following properties.*

(1) f is non-renormalizable;

(2) c_{S_k} and $c_{S_{k+2}}$ are on opposite sides of c.

(3) $c_{S_n} \in (c_{S_{n-1}}, \hat{c}_{S_{n-1}})$ and $c_i \notin (c_{S_{n-1}}, \hat{c}_{S_{n-1}})$ for each $0 < i < S_n$.

(4) $c_{-S_n} \in (c_{-S_{n-1}}, \hat{c}_{-S_{n-1}})$ and $c_{-i} \notin (c_{-S_{n-1}}, \hat{c}_{-S_{n-1}})$ for each $0 < i < S_n$.

(5) If T is the maximal interval adjacent to c such that $f^{S_k}_{|T}$ is monotone, then $f^{S_k}(T) = (c_{S_k}, c_{S_{k-2}})$.

(6) If $T_k \ni c^f$ is the largest interval on which $f^{S_k-1}_{|T_k}$ is monotone, then

$$T_k = (z^f_{k-1}, t^f_k)$$

where $t^f_k > c^f$ and $f^{S_k-1}(T_k) = (c_{S_{k-2}}, c_{S_{k-4}})$ (note that t^f_k is not the f-image of some point t_k, so this notation is just to suggest that t^f_k is close to c^f).

(7) $T_k, \dots, f^{S_k-1}(T_k)$ has intersection multiplicity 3 (this means that each point of $[0,1]$ is contained in at most 3 of these intervals).

Proof. The proof of these results can be found in [KN] and [LM]. In order to be complete we give the proof here also.

(1) Suppose f is renormalizable. Then there exists an interval $J \subset [0,1]$ containing c such that $c \in f^N(J) \subset J$ for some integer N. Then for every sufficiently small neighbourhood U of c, $f^n(U) \ni c$ implies that n is a multiple of N. In particular, the numbers S_k should be multiples of N for k sufficiently large. For the Fibonacci sequence, this is clearly not the case for any $N \in \mathbb{N}$.

(2) By definition of z_k, $T = (z_k, c)$ is a maximal interval on which $f^{S_{k+1}}$ is monotone. Hence $f^{S_{k+1}}(T) = f^{S_{k-1}} \circ f^{S_k}(z_k, c) = (d_{k-1}, d_{k+1})$ contains c.

(3) These properties give an equivalent characterization of a Fibonacci map as shown in [LM]. Let us prove these properties follow from our definition: We claim that

$$d_k \in (z_{k-1}, z_{k-2}).$$

Let $T = (z_k, c)$ be as above. $T \ni z_{k+1}$. By construction $f^{S_k}(z_{k+1}) = z_{k-1}$. In fact, z_{k+1} is the pull-back of z_{k-1} along the branch $f^{S_k}_{|T}$. So $z_{k-1} \in f^{S_k}(z_k, c) = (c, d_k)$. Now suppose that also $z_{k-2} \in (c, d_k)$. Then we can pullback z_{k-2} along the branch $f^{S_k}_{|T}$, obtaining a point in $f^{-S_{k-2}-S_k}(c) \cap T$. This contradicts the definition of S_{k+1} and z_{k+1}. Hence $d_k \in (z_{k-1}, z_{k-2})$.

(4) Follows immediately from the construction of the points z_k.

(5) See the proof of (2).

(6) We have of course that $T^f_{k-1} \supset T^f_k$. Because T^f_{k-1} is a maximal interval of monotonicity, $f^{S_{k-1}-1}(T^f_{k-1}) = (c_a, c_b)$ for some $a, b < S_{k-1}$. From the proof of property (3) it follows that $c_a, c_b \notin (z_3, \hat{z}_3)$. Hence $f^{S_{k-1}-1}(T^f_{k-1}) \supset (z_3, c) \ni d_{k-1}$. Because (z_3, c) is a maximal interval of monotonicity of $f^{S_{k-2}}$, $f^{S_{k-1}-1}(T^f_k) = (z_3, c)$, and $f^{S_k-1}(T^f_k) = f^{S_{k-2}}((z_3, c)) = (d_{k-4}, d_{k-2})$.

(7) In Lemma 4.3 in [LM] the upper bound for the intersection multiplicity is given as 8. One can prove that the intersection multiplicity is in fact 3. (In fact, any upperbound would be fine for the argument.) □

From the fact that c_{-1} exists it follows that f has an orientation reversing fixed point q. Let us define inductively a sequence of points u_n as follows. Let $u_0 = q$ and let us define u_{n+1} to be the nearest point to c with

$$u_{n+1} \in f^{-S_n}(u_n)$$

so that u_{n+1} is on the same side of c as $c_{S_{n+1}}$. In particular, $u_1 = \hat{u}_0 = \hat{q}$. Moreover, let $\tilde{u}_{k+1}$ be the point in $\{u_{k+1}, \hat{u}_{k+1}\}$ which is on the same side of c as u_k. These points were also used in [KN]. In fact, Martens in his thesis [Mar] and Yoccoz in his work on local connectivity of the Mandelbrot set used the same points.

Furthermore, let

$$y_n = f^{S_n}(c_{S_{n+2}}), \quad y^f_n = f(y_n).$$

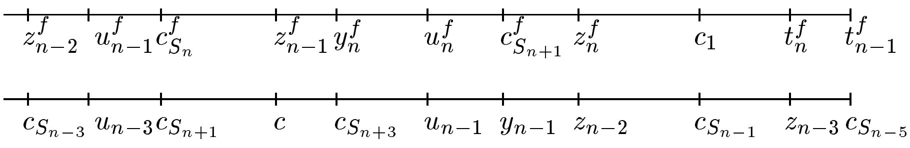

Figure 2.1: Points and their images under $f^{S_{n-1}-1}$.

PROPOSITION 2.2. *A Fibonacci map $f\colon [0,1] \to [0,1]$ satisfies the following properties.*

(1) $f^{S_n}(u_{n+1}) = u_n$ and $f^{S_n}(u_n) = u_{n-2}$;

(2) in particular, f^{S_n} maps $(\tilde{u}_{n+1}, u_n)$ diffeomorphically onto (u_n, u_{n-2}) (note that this last interval contains c);

(3) the points u^f_n, $c^f_{S_n}$, $c^f_{S_n+S_{n+2}}$, y^f_n and z^f_n are ordered as in the picture below (we state the ordering near c^f rather than near c so that we do not need to be careful about on which side of c these points lie).

Proof of Proposition 2.2. The proof of these statements can be found in [KN], but we shall include the proof here for completeness.

(1) $f^{S_n}(u_{n+1}) = u_n$ by definition, and by using the definition twice $f^{S_n}(u_n) = f^{S_{n-2}} \circ f^{S_{n-1}}(u_n) = f^{S_{n-2}}(u_{n-1}) = u_{n-2}$.

(2) Follows immediately from (1) and the relative positions of points discussed in (3).

(3) Let us restrict ourselves to the relative positions of the points in the upper part of the picture. The images under f^{S_n-1} are easily checked. The positions of z_k^f and t_k^f follow immediately from their definitions. The position of d_k^f is shown in the proof of property (3) in the previous proposition. Let us prove that $y_n^f \in (z_{n-1}^f, d_{n+1})$. $d_{n+2}^f \in (z_n^f, z_{n+1}^f)$, so $y_n^f \in f^{S_n}((z_n^f, z_{n+1}^f)) = (z_{n-1}^f, c^f)$. Futhermore $f^{S_{n-1}}(y_n^f) = d_{n+3}^f \in (z_{n+2}^f, z_{n+1}^f) \subset (c^f, z_{n-2}^f) = f^{S_{n-1}}((z_{n-1}^f, z_n^f))$. So $y_n^f \in (z_{n-1}^f, z_n^f)$. By the same argument $y_{n-1}^f \in (z_{n-2}^f, z_{n-1}^f)$. So actually $f^{S_{n-1}}(y_n^f) = d_{n+3}^f \in (z_{n+1}^f, z_{n+2}^f) \subset (y_{n-1}^f, c^f) = f^{S_{n-1}}((d_{n+1}^f, z_{n-1}^f))$, and $y_n^f \in (z_{n-1}^f, d_{n+1}^f)$.

Finally, we come to the relative position of u_n^f. We prove by induction that $u_n^f \in (y_n^f, d_{n+1}^f)$, or equivalently $\tilde{u}_n \in (y_n, d_{n+1})$ or $\tilde{u}_n \in (y_n, \hat{d}_{n+1})$. For $n = 1$ this is true, since $u_1 = \hat{q} \in (c_7, c_3) = (y_1, d_2)$, as one can check by hand. Now for the induction step, assume that $\tilde{u}_n \in (y_n, d_{n+1}) \subset (y_n, d_{n+3})$. By definition we obtain u_{n+1} by pulling back u_n along the branch $f^{-S_n}_{|(z_{n-1},c)}$. Because $(y_{n+1}, d_{n+2}) \subset (z_{n-1}, c)$, $f^{S_n}(y_{n+1}) = d_{n+3}$ and $f^{S_n}(d_{n+2}) = y_n$, it follows that $f^{-S_n}_{|(z_{n-1},c)}(y_n, d_{n+3}) = (y_{n+1}, d_{n+1}) \ni u_{n+1}$. This concludes the proof. □ □

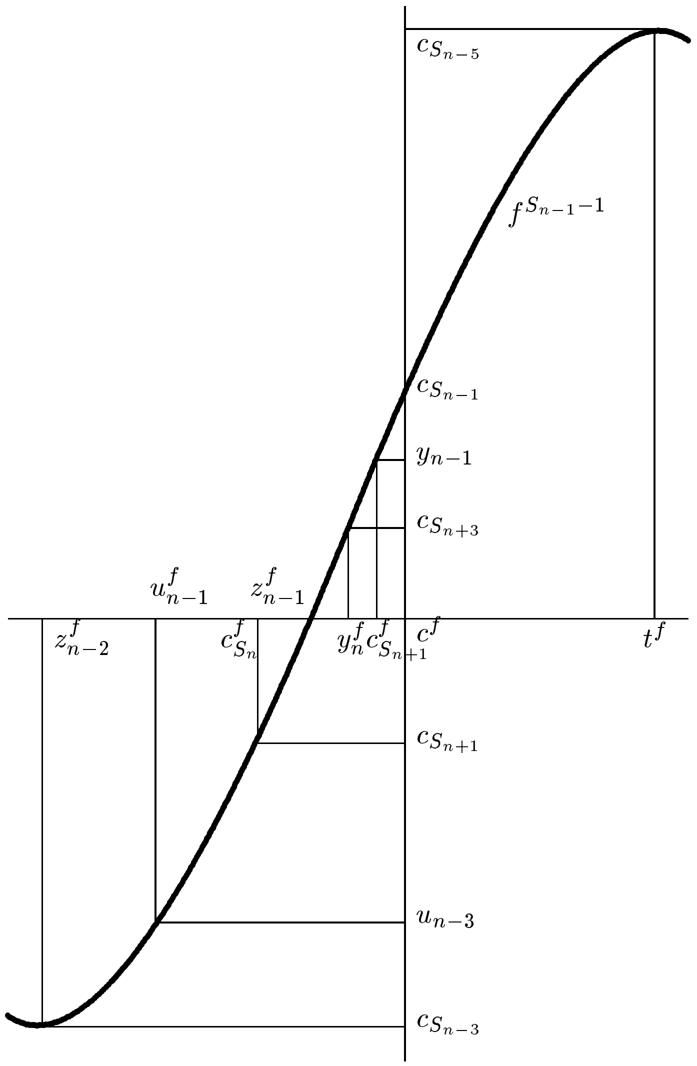

Figure 2.2.

If f has no wandering intervals (and this is the case under the present assumptions, see [MS][Chapter IV]), then $\omega(c)$ is a minimal Cantor set.

3. The estimates

In this section we shall estimate the rate of approach of the sequences $u_k^f, c_{S_k}^f, z_k^f$ to c^f. The basic tool is that of the distortion of cross-ratios.

Remark 3.1. Since f is non-flat at $x = c$ we can assume (by applying a suitable C^2 coordinate change) that f is of the form

$$f(x) = f(c) - |x - c|^{\ell}$$

near $x = c$. This will simplify some of the estimates somewhat.

Hence

$$\frac{|f(x) - f(c)|}{|x - c|} = M(x)$$

where $M(x)$ is a continuous function which is equal to $|x - c|^{\ell - 1}$ near $x = c$. Moreover,

$$\frac{Df(x)}{\ell |f(x) - f(c)|/|x - c|} = 1$$

near $x = c$. We shall use these facts repeatedly.

3a. The cross-ratio and the Koebe Principle when $Sf < 0$

Let $j \subset t$ be intervals and let l, r be the components of $t \setminus j$. Then the cross-ratio of this pair of intervals is defined as

$$C(t, j) := \frac{|t|}{|l|} \frac{|j|}{|r|}.$$

Let f be a smooth function mapping t, l, j, r onto T, L, J, R diffeomorphically. Define

$$B(f, t, j) = \frac{|T|\,|J|}{|t|\,|j|} \, \frac{|l|\,|r|}{|L|\,|R|} \; = \frac{C(T, J)}{C(t, j)}.$$

It is well known that if $Sf = f'''/f' - 3(f''/f')^2/2 \leq 0$ then $B(f, t, j) \geq 1$. Moreover, if $Sf < 0$ then $Sf^n < 0$ for each $n \in \mathbb{N}$. Most inequalities in this paper are based on versions of this cross-ratio inequality. For example, let $Sf < 0$ and t be an interval so that $f|t$ is a diffeomorphism. If $j \subset t$ is reduced to a point x then, using the notation from above,

$$\frac{|Df(x)|}{|L|/|l|} \geq |r|/|t|.$$

Similarly, if l is reduced to the point y then

$$\frac{|Df(y)|}{|J|/|j|} \leq |T|/|R|.$$

Moreover, we shall use the well-known Koebe Principle. Let us say that an interval T contains a *τ-scaled neighbourhood* of an interval $J \subset T$ if each component of $T \setminus J$ has at least size $\tau |J|$.

PROPOSITION 3.2. *[Koebe Principle if $Sf < 0$] Let f be a C^3 map with $Sf < 0$. Then for any intervals $j \subset t$ and any n for which $f^n|t$ is a diffeomorphism one has the following. If $f^n(t)$ contains a τ-scaled neighbourhood of $f^n(j)$ then*

$$\frac{|Df^n(x)|}{|Df^n(y)|} \leq \left[\frac{1+\tau}{\tau}\right]^2. \tag{3.1}$$

for each $x, y \in j$.

3b. The Koebe Principle in the C^2 case

Now we shall derive some results which we will only need if f does not satisfy the negative Schwarzian derivative condition, because otherwise the results of the previous subsection are sufficient for the rest of the paper. Therefore, the reader might want to skip this subsection at first and continue with section 3c.

Firstly we shall need that the cross-ratio cannot be decreased too much by a C^2 map f with non-flat critical points.

PROPOSITION 3.3. *Let f be a C^2 map with non-flat critical points. Then there exists a function $o(\epsilon) > 0$ with $o(\epsilon) \to 0$ as $\epsilon \to 0$ such that for any intervals $j \subset t$ and any n for which $f^n|t$ is a diffeomorphism one has the following. Let l, r be as above and let L, J, R, T be the images of l, j, r, t under f^n. Then*

$$B(f^n, t, j) = \frac{|T|}{|L|}\frac{|J|}{|R|}\frac{|l|}{|t|}\frac{|r|}{|j|} \geq \mathcal{O}_n^{-1},$$

where

$$\mathcal{O}_n = \exp\left(o(\epsilon) \cdot \sum_{i=0}^{n-1} |f^i(t)|\right)$$

and $\epsilon = \max_{i=0}^n |f^i(t)|$. (If $Sf < 0$ then $B(f^n, t, j) > 1$.)

Proof. See Theorem IV.2.1 in [MS]. □

Secondly, we shall need the Koebe Principle.

PROPOSITION 3.4. *[Koebe Principle] Let f be a C^2 map with non-flat critical points. Then there exists a function $o(\epsilon) > 0$ with $o(\epsilon) \to 0$ as $\epsilon \to 0$ such that for any intervals $j \subset t$ and any n for which $f^n|t$ is a diffeomorphism one has the following. If $f^n(t)$ contains a τ-scaled neighbourhood of $f^n(j)$ then*

$$\frac{|Df^n(x)|}{|Df^n(y)|} \leq \mathcal{O}_n \left[\frac{1+\tau}{\tau}\right]^2, \tag{3.2}$$

for each $x, y \in j$ *where*

$$\mathcal{O}_n = \exp\left(o(\epsilon)\cdot\sum_{i=0}^{n-1}|f^i(t)|\right)$$

and $\epsilon = \max_{i=0}^{n}|f^i(t)|$.

Proof. See Theorem IV.3.1. in [MS]. □

We should remark that if we take t_n to be the maximal interval containing c_1 on which f^{S_n-1} is a diffeomorphism, then from Proposition 2.1,

$$\sum_{i=0}^{S_n-1}|f^i(t_n)| \le 3. \tag{3.3}$$

So this implies that we get a universal upper bound for $\mathcal{O}_n$ in the above estimates. In fact, the next lemma will show that $\max_{i=0}^{S_n-1}|f^i(t_n)| \to 0$ as $n \to \infty$. This and the previous proposition implies that the numbers $\mathcal{O}_n$ from the above results satisfy

$$\mathcal{O}_n \to 1. \tag{3.4}$$

LEMMA 3.5. *For each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $f^n(I)$ *is not contained in an immediate basin of a periodic attractor and* $|f^n(I)| \le \delta$, *then* $\max_{i=0}^{n-1}|f^i(I)| \le \epsilon$.

Proof. We recall that, by the Contraction Principle, see [MS][IV.5.1] if I is an interval with $\inf_{i\ge 0}|f^i(I)| = 0$ then either I is completely contained in the basin of a periodic attractor or a wandering interval. Suppose by contradiction that there exists a sequence of intervals I_i with $|I_i| \ge \epsilon$ and a sequence $n(i)$ with $|f^{n(i)}(I_i)| \to 0$ where $f^{n(i)}(I_i)$ is not completely contained in the immediate basin of some periodic attractor. By taking subsequences, there exists an interval I such that $\inf_{i\ge 0}|f^i(I)| = 0$ and such that I is not completely contained in the basin of a periodic attractor. This is impossible because f has no wandering intervals, see [MS][Chapter IV, Theorem A]. □

In fact, we shall also have to apply the Koebe Principle to iterates of f which are large compositions of maps of the form f^{S_i} and we have to find a bound for the constant $\mathcal{O}$ which then appears. In this case we do not have a bound as in (3.3). However, we shall be able to bound the term $\mathcal{O}_n$ from the Koebe Principle above in this situation by using the next proposition.

PROPOSITION 3.6. *Let* f *be a* C^2 *map with non-flat critical points defined on some compact interval. Then for each* $\tau, S > 0$ *there exist constants* $\delta, \tau, S' > 0$ *such that the following holds. Let* x *be a recurrent point of* f, *let* U *be an interval neighbourhood around* x *of size at most* δ *and* $D \subset U$ *be some* *union of disjoint open intervals I_i. Let F be a map defined on $D = \cup_i I_i$ such that for each interval I_i there exists an integer $j(i)$ and an interval $T_i \supset I_i$ such that*

(1) $F|I_i = f^{j(i)}$, $F(I_i) \cap I_j \neq \emptyset$ implies that $F(I_i) \supset I_j$, $F(T_i) \supset T_j$ and $F(I_i)$ contains at least two of those intervals I_k;

(2) $f^{j(i)}(T_i)$ contains a τ-scaled neighbourhood of each interval $I_k \subset f^{j(i)}(I_i)$;

(3) for each interval $I_k \subset F(I_i)$ one has $|I_k| \le (1 - \frac{1}{S}) \cdot |F(I_i)|$;

(4) $\sum_{m=0}^{j(i)-1} |f^m(T_i)| \le S$.

Then for each $n \in \mathbb{N}$ and each component J of the domain of F^n one has $F^n|J = f^s$ for some $s \in \mathbb{N}$, there exists an interval $T' \supset J$ for which $F^n(T')$ contains a τ'-scaled neighbourhood of each element $I_k \subset F^n(J)$ and for which

$$\sum_{m=0}^{s-1} |f^m(T')| \le S'.$$

Proof of Proposition3.6. The idea of the proof of this proposition is essentially the same as in [Str]. The proof of this proposition can be substantially simplified if f has negative Schwarzian derivative: in this case it is not necessary to choose δ small. (In fact we do not even need this proposition in that case.) However, in the general case, f could for example have a periodic interval (corresponding to basins of periodic attractors). This complicates matters to some extend.

It is no restriction to assume that f is defined on an interval of size at most one. Fix τ and K. Since f is C^2 each periodic point p of f of sufficiently large period k is repelling, see [MS][Theorem IV.B]. In particular, provided $\delta > 0$ is sufficiently small, each periodic point in the δ-neighbourhood U of the recurrent point x is repelling. For this reason we shall be able to apply Lemma 3.5.

Let $\mathcal{I}$ be the partition of the domain of F of the intervals I_i and define inductively $\mathcal{I}_0 = \mathcal{I}$ and

$$\mathcal{I}_n = \mathcal{I}_0 \vee F^{-1}\mathcal{I}_0 \vee \ldots \vee F^{-(n-1)}\mathcal{I}_0.$$

Each element J of $\mathcal{I}_n$ is mapped by F^{n-1} diffeomorphically onto some interval I_k and each of the intervals I_i is contained in U. Because of property 1) there exist integers $0 \le \hat{s} < s$ with $F^n|J = f^s$ and $F^{n-1}|J = f^{\hat{s}}$. Let us first show that it is sufficient to prove that there exists a universal upperbound $S' < \infty$ such that

$$\sum_{m=0}^{\hat{s}} |f^m(J)| \le S' \text{ for each } J \in \mathcal{I}_n. \tag{3.5}$$

To prove that this is sufficient, let $\hat{T} \supset J$ be an interval such that $F^{n-1}|\hat{T}$ is a diffeomorphism and such that $F^{n-1}(\hat{T})$ is a τ-scaled neighbourhoods of

$F^{n-1}(J) = f^{\hat{s}}(J) = I_k \in \mathcal{I}_0$. If (3.5) holds, then one has from Lemma 8.3 in [Str] that there exists an interval T with $J \subset T \subset \hat{T}$ for which

$$f^{\hat{s}}(T) \text{ is a } \tau/2\text{-scaled neighbourhood of } F^{n-1}(J) = f^{\hat{s}}(J) \tag{3.6}$$

and

$$|f^m(T)| \le K|f^m(J)| \tag{3.7}$$

for all $m = 0, 1, \ldots, \hat{s}$. Here K depends on f, τ and $\sum_{m=0}^{\hat{s}} |f^m(J)|$. Next, by property 4),

$$\sum_{m=\hat{s}}^{s-1} |f^m(T_k)| \text{ is universally bounded} \tag{3.8}$$

where $T_k \supset I_k$ is as above. Hence by the Koebe estimate (3.2) there exist a universal number $\tau' > 0$ and an interval T_k' with $T_k \supset T_k' \supset I_k$ which is contained in a $\tau/2$-scaled neighbourhood of I_k and such that $F(T_k')$ contains a τ'-scaled neighbourhood of each interval I_i which is contained in $F(I_k) = F^n(J)$. Therefore the interval $T \supset T' \supset J$ for which $F^{n-1}(T') = T_k'$ is so that $F^n(T')$ contains a τ'-scaled neighbourhood of each interval I_k in $F^n(J)$ and because of (3.7) and (3.8), we also have that

$$\sum_{m=0}^{s-1} |f^m(T')|$$

is universally bounded. Thus we have shown that it is sufficient to prove (3.5).

In order to prove that (3.5) holds, we first claim that there exists $\kappa < 1$ such that

$$\text{if } J \in \mathcal{I}_1 \text{ is contained in } I_i \in \mathcal{I}_0 \text{ then } |J| \le \kappa |I_i|. \tag{3.9}$$

This holds since $F(J)$ is equal to an interval $I_k \in \mathcal{I}_0$ while properties 2) and 3) imply that there exists an interval J' with $J \subset J' \subset I_i$ for which $F(J')$ is contained inside a $\tau/2$-scaled neighbourhood of $F(J) = I_k$ and for which a definite proportion of $F(J')$ is outside $F(J) = I_k$. Moreover, because of 4) and the Koebe Principle there exists a universal constant $K_0 < \infty$ such that

$$\sup_{x,y \in J'} \frac{|DF(x)|}{|DF(y)|} \le K_0.$$

Combining this proves (3.9).

By using a 'telescope argument' we can improve this statement and show by induction that there exists $\kappa < 1$ such that for each $n \in \mathbb{N}$ there exists $\delta > 0$ such that if $|U| < \delta$, $J \in \mathcal{I}_n$ and J is contained in $I_i \in \mathcal{I}_0$ then

$$|J| \le \kappa^n |I_i|. \tag{3.10}$$

For $n = 0$ there is nothing to prove. So assume the statement holds for $n-1$ and consider $J \in \mathcal{I}_n$. If $F^n|J = f^j$ and $T \supset J$ so that $f^j|T$ is a diffeomorphism and $f^j(T)$ is a τ-scaled neighbourhood of $f^j(I) = I_k \in \mathcal{I}_0$ then

$$\sum_{i=0}^{j-1} |f^i(J)| \le n \cdot S \text{ and } \max_{i=0,\dots,j-1} |f^i(T)| = o(|f^j(T)|), \tag{3.11}$$

where $o(t)$ is a function so that $o(t) \to 0$ if $t \downarrow 0$. Here we have used respectively property 4) and the previous Lemma 3.5. (We should note that $f^n(J) \subset U$ and so $|f^j(J)| = |F^n(J)| \le (1+2\tau)|U| \le (1+2\tau)\delta$.) Hence, by the Koebe Principle, there exists K_1 (which only depends on τ) such that for each n

$$\frac{|DF^n(x)|}{|DF^n(y)|} \le K_1 \tag{3.12}$$

for all $x, y \in J$ provided δ is sufficiently small. (To get K_1 uniform we shrink δ for increasing n; by (3.11) and (3.2) this avoids the constants in the Koebe Principle to grow.) Now F^{n-1} maps each element of $\mathcal{I}_{n-1}$ diffeomorphically onto some element of $\mathcal{I}_0$ and each element of $\mathcal{I}_n$ onto an element of $\mathcal{I}_1$. From this, (3.12) and (3.9) it follows that each element J of $\mathcal{I}_n$ is a definite factor smaller than the element $I \in \mathcal{I}_{n-1}$ containing J. This proves (3.10).

Now of course (3.10) does not suffice because δ (and therefore the size of U) depends on n. Therefore, let us fix n_0 so large that

$$\kappa^{-n_0} \ge 4K_2 \text{ where } K_2 = \left[\frac{1+\tau}{\tau}\right]^2$$

and write $G = F^{n_0}$. If J is an element of $\mathcal{I}_{kn_0}$ and $G^i(J) \supset J$ for some $0 \le i \le k$ then

$$|DG^i(x)| \ge 2 \text{ for all } x \in J. \tag{3.13}$$

Indeed, we may assume that i is minimal and then $J, \dots, G^{(i-1)}(J)$ are disjoint. If $G^i = f^j$ then this gives that $J, \dots, f^j(J)$ have intersection multiplicity bounded by n_0. (This means that each point is contained in at most n_0 of these intervals.) Therefore, and since f^j maps some interval $T \supset J$ onto a τ-scaled neighbourhood of $f^j(J)$, it follows from the Koebe Principle that

$$|DG^i(x)| \ge \exp\left(-o(\epsilon) \cdot \sum_{i=0}^{j-1} |f^i(J)|\right) \frac{1}{K_2} \frac{|G^i(J)|}{|J|} \tag{3.14}$$

$$\ge \exp\left(-o(\epsilon) \cdot n_0\right) \frac{1}{K_2} \frac{|G^i(J)|}{|J|} \ge \frac{1}{2}\frac{1}{K_2}\kappa^{-n_0} \ge 2 \tag{3.15}$$

for each $x \in J$ provided $|f^j(J)| = |G^i(J) \le |U| \le \delta$ is sufficiently small. (This last inequality implies that $\epsilon = \max |f^i(J)|$ is small when δ is small.) Hence, if some interval returns then its size has increased by a uniform factor; as we shall now show this implies the total length of the intervals remains bounded. Indeed, consider again $J \in \mathcal{I}_{kn_0}$. Then

$$\sum_{i=0}^{k-1} |G^i(J)| \le 2. \tag{3.16}$$

This is because $G^{i_1}(J) \cap G^{i_2}(J) \neq \emptyset$ with $i_1 < i_2 \le k$ implies that $G^{i_1}(J) \subset G^{i_2}(J)$. Moreover, if $G^{i_1}(J), G^{i_2}(J) \subset G^{i_3}(J)$ and $i_1 < i_2 \le i_3 \le k$, then there exists $J' \supset G^{i_1}$ (which is an interval from a partition of the form $\mathcal{I}_{hn_0}$ with $h \in \{0, 1, \dots, k\}$) such that $G^{i_2 - i_1}(J') = G^{i_3}(J)$. Hence, by (3.13)

$$|G^{i_1}(J)| \le \frac{1}{2} |G^{i_2}(J)|.$$

Using this it follows that the total length of the interval $J, \dots, G^{k-1}(J)$ contained in one interval $G^{i_3}(J)$ is at most $\sum_{i \ge 0} 2^{-i} = 2$ times the length of $G^{i_3}(J)$. This implies (3.16) since we have assumed that f lives on an interval of length at most one. Now (3.16) gives that

$$\sum_{m=0}^{j-1} |f^m(J)| \le 2n_0$$

where $f^j = G^k = F^{kn_0}$. From this (3.5) follows easily. $\square$

3c. Two step bounds

For simplicity define

$$d_n = c_{S_n}.$$

We shall use boldface letters to indicate the distance to the critical point (or value), so

$$\mathbf{d}_n = |d_n - c|, \text{ and } \mathbf{d}_n^f = |d_n^f - c^f|.$$

This notation will also be used for the points we defined before, namely t_n^f is the critical point of the monotone branch of $f^{S_n - 1}$ near c^f lying on the other side of c^f than c (and therefore than z_n^f as well). The critical value corresponding to t_n^f is $c_{S_{n-4}} = f^{S_n - 1}(t_n^f)$. Define as before

$$z_n = c_{-S_n} \text{ and } z_n^f = f(z_n)$$

where z_n could be either to the left or the right of c depending on the context. Moreover, remember that we defined

$$y_n = f^{S_n}(c_{S_{n+2}}) \text{ and } y_n^f = f(y_n)$$

in Proposition 2.2. In the next lemmas the constant $\mathcal{O}$ from Proposition 3.3 will be written as $\mathcal{O}_n$, in order to indicate its dependence on S_n. Notice that

$\mathcal{O}_n \to 1$ as $n \to \infty$ because of inequality (3.4). If $Sf < 0$ then we simply have $\mathcal{O}_n \geq 1$.

Lemma 3.7. *(See [KN]) Let* $\lambda_n^f = \mathbf{d}_{n-2}^f / \mathbf{d}_n^f$ *then* $\lambda_n^f > 3.85$ *and*

$$\ln(\mathbf{d}_{n-4}^f / \mathbf{d}_n^f) > 2.7$$

for sufficiently large n.

Proof. Applying the cross-ratio inequalities we have

$$\begin{aligned} \frac{\mathbf{d}_n - \mathbf{y}_n}{\mathbf{d}_{n+2}^f} \frac{\mathbf{d}_{n-4}}{|t|} &= \frac{|J|}{|j|} \frac{|T|}{|t|} \geq \mathcal{O}_n \frac{|L|}{|l|} \frac{|R|}{|r|} \\ &\geq \mathcal{O}_n \frac{\mathbf{y}_n}{|\mathbf{z}_n^f - \mathbf{d}_{n+2}^f|} \frac{\mathbf{d}_{n-4} - \mathbf{d}_n}{|r|} \end{aligned}$$

where t, j, l, r are chosen as in the figure below.

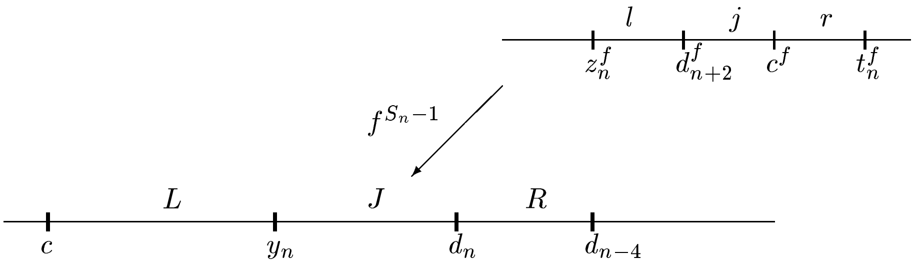

Figure 3.1.

Using the non-flatness of c and the previous inequality (and $|t| > |r|$) we get

$$\begin{aligned} \frac{1}{\ell}\left(1 - \frac{\mathbf{d}_n^f}{\mathbf{d}_{n-4}^f}\right) &\leq \frac{\mathbf{d}_{n-4} - \mathbf{d}_n}{\mathbf{d}_{n-4}} < \mathcal{O}_n \frac{\mathbf{d}_n - \mathbf{y}_n}{\mathbf{y}_n} \frac{|\mathbf{z}_n^f - \mathbf{d}_{n+2}^f|}{\mathbf{d}_{n+2}^f} \cdot 1 \\ &< \mathcal{O}_n \left(\frac{\mathbf{d}_n}{\mathbf{d}_{n+1}} - 1\right) \cdot \left(\frac{\mathbf{d}_{n+1}^f}{\mathbf{d}_{n+2}^f} - 1\right) \qquad (3.17) \\ &< \mathcal{O}_n \frac{1}{\ell} \left(\frac{\mathbf{d}_n^f}{\mathbf{d}_{n+1}^f} - 1\right) \cdot \left(\frac{\mathbf{d}_{n+1}^f}{\mathbf{d}_{n+2}^f} - 1\right). \end{aligned}$$

Here we have used $0 < \ell a^{\ell-1} < (b^\ell - a^\ell)/(b-a) < \ell b^{\ell-1}$. Using $(a-1)(b-1) \leq (\sqrt{ab} - 1)^2$ we can bound the last term in (3.17) from above and thus we obtain

$$\left(1 - \frac{\mathbf{d}_n^f}{\mathbf{d}_{n-4}^f}\right) \leq \mathcal{O}_n \left(\sqrt{\frac{\mathbf{d}_n^f}{\mathbf{d}_{n+2}^f}} - 1\right)^2 .$$

Hence
$$1-(\lambda_n^f\lambda_{n-2}^f)^{-1}\le \mathcal{O}_n(\sqrt{\lambda_{n+2}^f}-1)^2$$
which yields the analogous inequality for $\lambda_\infty^f=\liminf\lambda_n^f$, and thus $\lambda_\infty^f>3.85$ and $\liminf\ln(\mathbf{d}_{n-4}^f/\mathbf{d}_n^f)=2\ln\lambda_\infty^f>2.7$. One can obtain better estimates for large ℓ using $\ell(\lambda-1)/\lambda\le\ln\lambda^f\le\ell(\lambda-1)$. $\square$

Lemma 3.8. *Let $a\in(z_n,\hat z_n)$, $a^f=f(a)$, $b=f^{S_n}(a)$ and $b^f=f(b)=f^{S_n}(a^f)$. Then for n large enough*
$$|Df^{S_n}(a^f)|\le\frac{\mathbf{b}^f}{\mathbf{a}^f}\ln(\frac{\mathbf{d}_{n-4}^f}{\mathbf{b}^f})\ln(\frac{\mathbf{d}_n^f}{\mathbf{b}^f})\left(\frac{\mathbf{d}_{n-4}^f}{\mathbf{b}^f}\right)^{\frac{1}{\ell}}.$$

Proof. We use the cross-ratio for f^{S_n-1} with l shrunk to a point $l=\{a^f\}$ and $j=(a^f,c^f)$ and $r=(c^f,t_n^f)$.

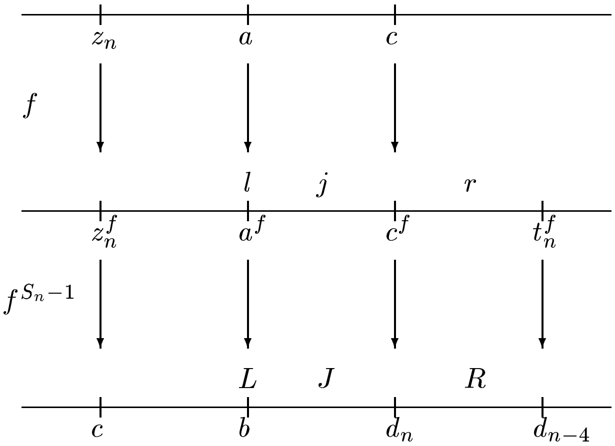

Figure 3.2.

In the cross-ratio inequality we can use $|r|<|l|+|j|+|r|$ and have
$$\begin{aligned}|Df^{S_n}(a^f)| &= |Df(b)|\,|Df^{S_n-1}(a^f)|\\ &\le \mathcal{O}_n\ell\frac{\mathbf{b}^f}{\mathbf{b}}\frac{\mathbf{d}_n-\mathbf{b}}{\mathbf{a}^f}\frac{\mathbf{d}_{n-4}-\mathbf{b}}{\mathbf{d}_{n-4}-\mathbf{d}_n}\\ &= \mathcal{O}_n\frac{\mathbf{b}^f}{\mathbf{a}^f}\ell\cdot\frac{\mathbf{d}_n-\mathbf{b}}{\mathbf{d}_n}\ell\cdot\frac{\mathbf{d}_{n-4}-\mathbf{b}}{\mathbf{d}_{n-4}}\frac{1}{\ell\cdot\frac{\mathbf{d}_{n-4}-\mathbf{d}_n}{\mathbf{d}_n}}\frac{\mathbf{d}_{n-4}}{\mathbf{b}}\\ &\le \mathcal{O}_n\frac{\mathbf{b}^f}{\mathbf{a}^f}\frac{\ln(\mathbf{d}_n^f/\mathbf{b}^f)\ln(\mathbf{d}_{n-4}^f/\mathbf{b}^f)}{\ln(\mathbf{d}_{n-4}^f/\mathbf{d}_n^f)}\left(\frac{\mathbf{d}_{n-4}^f}{\mathbf{b}^f}\right)^{\frac{1}{\ell}},\end{aligned}$$

which implies the statement as $\mathcal{O}_n\to1$ and $\ln\frac{\mathbf{d}_{n-4}^f}{\mathbf{d}_n^f}>2$. Here we have used Lemma 3.7 and the obvious inequalities $\frac{x-1}{x}\le\ln(x)\le x-1$. $\square$

Remark 3.9. We shall use this lemma several times. In order to simplify the notation let us introduce $\varrho_n^f := \max\{\mathbf{d}_k^f/\mathbf{d}_{k+1}^f \ ; \ n - N_0 \le k < n\}$, where $N_0 \le 10$ may change from one lemma to another. Suppose that b in the previous lemma satisfies $\mathbf{d}_{n+i} \le \mathbf{b}$ for some $i \le 6$, then $\mathbf{d}_n^f/\mathbf{b}^f \le (\varrho_{n+i}^f)^i$, $\mathbf{d}_{n-4}^f/\mathbf{b}^f \le (\varrho_{n+i}^f)^{i+4}$ and

$$|Df^{S_n}(a^f)| \le \frac{\mathbf{b}^f}{\mathbf{a}^f} \cdot (4+i) \cdot \ln(\varrho_{n+i}^f) \cdot i \cdot \ln(\varrho_{n+i}^f)(\varrho_{n+i}^f)^{\frac{4+i}{\ell}} ,$$

where in fact ϱ^f could have been taken as $\max\{\mathbf{d}_k^f/\mathbf{d}_{k+1}^f \ ; \ n-i-4 \le k < n\}$.

LEMMA 3.10. *We have the following estimate*

$$|Df^{S_m}(d_{m+1}^f)| \le 160 \cdot \frac{\mathbf{d}_{m+2}^f}{\mathbf{d}_{m+1}^f} \ln^4(\varrho_{m+2}^f) \cdot (\varrho_{m+2}^f)^{\frac{13}{\ell}} .$$

Proof. We decompose $Df^{S_m}(d_{m+1}^f) = Df^{S_{m-2}}(y_{m-1}^f) Df^{S_{m-1}}(d_{m+1}^f)$ and use the previous lemma and remark to both factors. First we put in the lemma $n = m-2$, $a = y_{m-1}$, $b = c_{S_{m+2}}$ and $i = 4$ in the remark.

$$|Df^{S_{m-2}}(y_{m-1}^f)| \le \frac{\mathbf{d}_{m+2}^f}{\mathbf{y}_{m-1}^f} \cdot 32 \cdot \ln^2(\varrho_{m+2}^f) \cdot (\varrho_{m+2}^f)^{\frac{8}{\ell}}$$

Then we put $n = m-1$, $a = d_{m+1}$, $b = y_{m-1}$ and as $d_m \in (y_{m-1}, \hat{y}_{m-1})$ we have $i = 1$.

$$|Df^{S_{m-1}}(d_{m+1})| \le \frac{\mathbf{y}_{m-1}^f}{\mathbf{d}_{m+1}^f} \cdot 5 \cdot \ln^2(\varrho_m^f) \cdot (\varrho_m^f)^{\frac{5}{\ell}} .$$

The result follows taking ϱ^f depending on 9 consecutive k: $m - 7 \le k < m + 2$. ☐

The next lemma prepares the last tool in this subsection. It describes the estimation (both ways) of $Df^{S_m}(c^f)$.

LEMMA 3.11. *We have for large m*

$$\frac{\mathbf{d}_m^f}{\mathbf{d}_{m+1}^f}(\varrho_m^f)^{-\frac{4}{\ell}} \le |Df^{S_m}(c^f)| \le 2 \cdot \frac{\mathbf{d}_m^f}{\mathbf{d}_{m+2}^f} \cdot \ln(\varrho_{m+1}^f) \cdot (\varrho_{m+1}^f)^{\frac{1}{\ell}}.$$

Proof. We use the same trick as in Lemma 3.8.

$$|Df^{S_m}(c^f)| = \ell \frac{\mathbf{d}_m^f}{\mathbf{d}_m} |Df^{S_m - 1}(c^f)| .$$

For one side we use the cross-ratio for $f^{S_m - 1}$ on $l = (z_m^f, c^f)$, $r = (c^f, t_m^f)$.

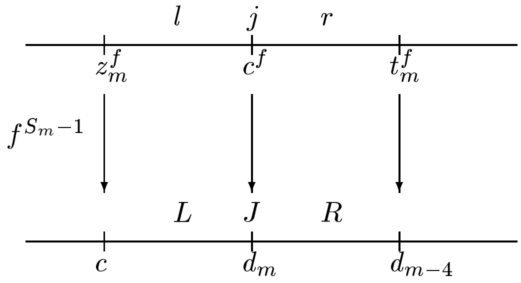

Figure 3.3.

Then we obtain,

$$Df^{S_m}(c^f) \geq \mathcal{O}_m \frac{\mathbf{d}_m^f}{\mathbf{d}_m} \ell \frac{\mathbf{d}_{m-4} - \mathbf{d}_m}{\mathbf{d}_{m-4}} \frac{\mathbf{d}_m}{\mathbf{z}_m^f} = \mathcal{O}_m \ell \left(\frac{\mathbf{d}_{m-4}}{\mathbf{d}_m} - 1 \right) \frac{\mathbf{d}_m}{\mathbf{d}_{m-4}} \frac{\mathbf{d}_m^f}{\mathbf{z}_m^f}.$$

Using the inequality $\ln(x) \leq x - 1$, $x \geq 1$, the last can be bounded from below by

$$\geq \mathcal{O}_m \ln \frac{\mathbf{d}_{m-4}^f}{\mathbf{d}_m^f} \frac{\mathbf{d}_m}{\mathbf{d}_{m-4}} \frac{\mathbf{d}_m^f}{\mathbf{z}_m^f} > \mathcal{O}_m \left(\frac{\mathbf{d}_m^f}{\mathbf{d}_{m-4}^f} \right)^{\frac{1}{\ell}} \frac{\mathbf{d}_m^f}{\mathbf{d}_{m+1}^f},$$

where we have used Lemma 3.3 to bound the second factor and have used that $\mathbf{d}_{m+1}^f < \mathbf{z}_m^f$. For the other side we take $l = (z_m^f, d_{m+2}^f)$, $j = (d_{m+2}^f, c^f)$ and $r = (c^f)$.

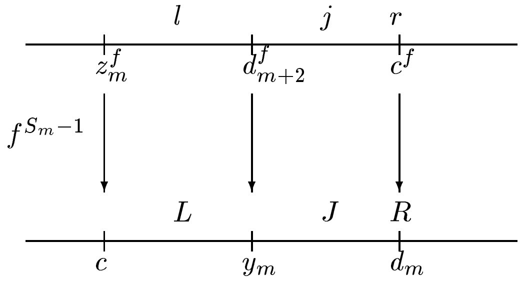

Figure 3.4.

We obtain, using $\mathbf{d}_{m+1} \leq \mathbf{y}_m$,

$$\begin{aligned} |Df^{S_m}(c^f)| &\leq \mathcal{O}_m \frac{\mathbf{d}_m^f}{\mathbf{d}_m} \ell \frac{\mathbf{d}_m - \mathbf{y}_m}{\mathbf{d}_{m+2}^f} \frac{\mathbf{d}_m}{\mathbf{z}_m^f} \frac{\mathbf{z}_m^f - \mathbf{d}_{m+2}^f}{\mathbf{y}_m} \\ &\leq \mathcal{O}_m \frac{\mathbf{d}_m^f}{\mathbf{d}_{m+2}^f} \ell \frac{\mathbf{d}_m - \mathbf{d}_{m+1}}{\mathbf{d}_m} \frac{\mathbf{d}_m}{\mathbf{d}_{m+1}} \\ &\leq \mathcal{O}_m \frac{\mathbf{d}_m^f}{\mathbf{d}_{m+2}^f} \ln(\varrho_{m+1}^f)(\varrho_{m+1}^f)^{\frac{1}{\ell}} . \end{aligned}$$

And again ϱ^f could have been taken as $\max\{\mathbf{d}_k^f / \mathbf{d}_{k+1}^f \; ; \; m - 4 \leq k \leq m\}$. $\square$

3d. The 1-step bounds

We can get a better upper estimate if we combine the previous calculations.

PROPOSITION 3.12. *For n and ℓ large enough the derivatives $Df^{S_n}(c^f)$ and the proportions $\mathbf{d}_n^f/\mathbf{d}_{n+1}^f$ are bounded from above and separated from below from 1 by constants independent of n and ℓ.*

Proof. Consider the following decomposition:

$$Df^{S_n}(c^f) = Df^{S_{n-2}}(d_{n-1}^f)\cdot Df^{S_{n-1}}(c^f) = Df^{S_{n-2}}(d_{n-1}^f)Df^{S_{n-3}}(d_{n-2}^f)\cdot Df^{S_{n-2}}(c^f).$$

By Lemma 3.10 used twice with $m = n-2$ and $m = n-3$ in the two first factors and by Lemma 3.11 used for $m = n-2$ in the third one we have

$$\begin{aligned} |Df^{S_n}(c^f)| &\leq 160^2 \cdot 2 \cdot \frac{\mathbf{d}_n^f}{\mathbf{d}_{n-1}^f}\frac{\mathbf{d}_{n-1}^f}{\mathbf{d}_{n-2}^f}\frac{\mathbf{d}_{n-2}^f}{\mathbf{d}_n^f} \cdot \ln^{(4+4+1)}(\varrho_n^f)(\varrho_n^f)^{\frac{13+13+1}{\ell}} \\ &= 51200 \cdot \ln^9(\varrho_n^f)\cdot(\varrho_n^f)^{\frac{27}{\ell}}. \end{aligned}$$

This and the other part of Lemma 3.11 gives

$$\frac{\mathbf{d}_n^f}{\mathbf{d}_{n+1}^f} \leq 51200 \cdot \ln^9(\varrho_n^f)\cdot(\varrho_n^f)^{\frac{31}{\ell}} ,$$

where again ϱ_n^f depends on at most 10 consecutive quotients $\mathbf{d}_k^f/\mathbf{d}_{k+1}^f$, with $n-10 \leq k < n$. This gives an upper bound of the growth of $\varrho_\infty^f = \limsup \varrho_n^f = \limsup \mathbf{d}_n^f/\mathbf{d}_{n+1}^f$ by

$$\varrho_\infty^f \leq 51200 \cdot \ln^9(\varrho_\infty^f)\cdot(\varrho_\infty^f)^{\frac{31}{\ell}} ,$$

(i.e. $\varrho_\infty^f < 10^{21}$ for large ℓ) and proves the upper bound part of the proposition. For the lower part one can use the estimates from Lemma 3.7, because from (3.17) and from $\ln(\mathbf{d}_{n-4}^f/\mathbf{d}_n^f) > 2/7$ it follows that

$$\frac{\mathbf{d}_n^f}{\mathbf{d}_{n+1}^f} \geq 1 + \frac{1-e^{-2.7}}{\varrho_\infty^f - 1}.$$

□

PROPOSITION 3.13. *There exists $K > 0$ independent of ℓ and n such that for ℓ and n large enough*

$$\mathbf{d}_n^f/\mathbf{u}_n^f > 1 + K.$$

Proof. Consider $f^{S_{n-1}}$ and its interval of monotonicity $t = (z_{n-1}^f, t_{n-1}^f)$ around c^f. Let $l = (z_{n-1}^f, u_n^f)$, $j = (u_n^f, c^f)$ and $r = (c^f, t_{n-1}^f)$. Denote by T, L, J, R the images of t, l, j, r under $f^{S_{n-1}}$. Then

$$\frac{\mathbf{d}_n^f - \mathbf{u}_n^f}{\mathbf{u}_n^f} \geq \frac{\mathbf{z}_{n-1}^f - \mathbf{u}_n^f}{\mathbf{u}_n^f} = \frac{|l|}{|j|} \geq \mathcal{O}_n \frac{|L|}{|J|}\frac{|R|}{|T|} = \mathcal{O}_n \frac{\mathbf{u}_{n-1}^f}{\mathbf{d}_{n-1}^f - \mathbf{u}_{n-1}^f}\frac{\mathbf{d}_{n-5}^f - \mathbf{d}_{n-1}^f}{\mathbf{d}_{n-5}^f}$$

$$\geq \quad \mathcal{O}_n \frac{\mathbf{d}_n^f}{\mathbf{d}_{n-1}^f}\left(1 - \frac{\mathbf{d}_{n-1}^f}{\mathbf{d}_{n-5}^f}\right) \geq \mathcal{O}_n \frac{1-e^{-2.7}}{\varrho_n^f} ,$$

and this is bounded away from 0 uniformly in n. □

The main result of this section is the finally the following theorem.

THEOREM 3.14.

$$\frac{\mathbf{d}_n^f}{\mathbf{u}_n^f} , \ \frac{\mathbf{d}_n^f}{\mathbf{d}_{n+1}^f} \ \textit{and} \ \frac{\mathbf{u}_n^f}{\mathbf{u}_{n+1}^f}$$

are bounded and bounded away from one for all ℓ and n large enough. In particular, there are constants C_1, C_2 for which

$$\frac{C_1}{\ell} \leq \ \frac{|d_n - u_n|}{|u_n - c|} \ , \ \frac{|\mathbf{d}_n - \mathbf{d}_{n+1}|}{|d_n - c|} \ , \ \frac{|\mathbf{u}_n - \mathbf{u}_{n+1}|}{|u_n - c|} \ \leq \frac{C_2}{\ell}.$$

Proof. Follows from the previous two results and the fact that f has a critical point of order ℓ at c. □

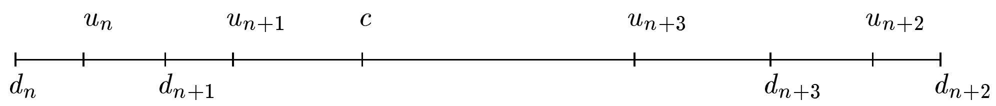

Figure 3.5: The points u_n and d_n are on the same side of c. d_{n+2} and d_n are on opposite sides of c; z_{n-1} is between d_n and u_n.

4. The random walk argument

In this section we shall state and prove an abstract result about the evolution of typical points under a (nearly) Markov map with a kind of random walk structure. So let $(X, \mathcal{F}, m)$ be some space with probability measure m and σ-algebra $\mathcal{F}$. Let $\mathcal{A} = \{A_k : k = 0, 1, 2, \ldots\}$ denote a partition of X into $\mathcal{F}$-measurable sets, and let $F: X \to X$ be a $\mathcal{F}$-measurable transformation. We denote $\mathcal{A}_n = \bigvee_{k=0}^{n-1} F^{-k}\mathcal{A}$.

Observe that $\mathcal{A}$ is a Markov partition for F if and only if $F^k H$ is an element of $\mathcal{A}$ for each $H \in \mathcal{A}_{k+1}$ and each $k \geq 0$. In order to make the following proposition most widely applicable we shall not assume that F is strictly Markov but formulate instead some restrictions on iterates of the measure m. Furthermore, even if F is topologically Markov, the nonlinearity of its branches still prevents F to be also measure theoretically Markov. Therefore we do not use in our proof a Markov-like model but instead a more flexible martingale construction. As a general reference to the theory of martingales we give [Sto].

Define $\varphi : X \to \{0,1,2,\dots\}$ by

$$\varphi(x) = n \text{ if } x \in A_n$$

and

$$\Delta\varphi := \varphi \circ F - \varphi \ .$$

THEOREM 4.1. *Assume there are $r_0 \in \mathbb{N}$ and $M > 0$ such that for any $H \in \mathcal{A}_{k+1}$, $k \geq 0$, with $\varphi_{|F^k H} \geq r_0$ holds:*

$$\int_H (\Delta\varphi - 1) \circ F^k \, dm \quad \geq \quad 0 \qquad \textit{and} \tag{4.1}$$

$$\int_H (\Delta\varphi)^2 \circ F^k \, dm \quad \leq \quad M \cdot m(H) \qquad \textit{for all } n \geq 0 \ . \tag{4.2}$$

Then there exists a set $D \in \mathcal{F}$ with $m(D) > 0$ such that for each $x \in D$

$$\liminf_{n\to\infty} \frac{\varphi \circ F^n(x)}{n} \geq 1$$

and such that the trajectory $x, Fx, F^2x, \dots$ visits each set $A_k \in \mathcal{A}$ only finitely often.

Proof. Fix $s > r_0$ and denote by μ the normalized restriction of m to A_s. Let $\mathcal{F}_n$ be the σ-algebra generated by the partition $\mathcal{A}_{n+1}$. Then $\varphi \circ F^n$ is $\mathcal{F}_n$-measurable, i.e.,

$$E_\mu[\varphi \circ F^n | \mathcal{F}_n] = \varphi \circ F^n \ .$$

Define a stopping time $\tau : X \to \mathbb{N} \cup \{\infty\}$ by

$$\tau(x) = \begin{cases} \infty & \text{if } \varphi(F^n x) > r_0 \text{ for all } n \geq 0 \\ \min\{n \geq 0 \colon \varphi(F^n x) \leq r_0\} & \text{otherwise,} \end{cases}$$

and the random variables $(Z_n)_{n\geq 0}$ by

$$Z_n(x) = \begin{cases} \varphi(F^n x) & \text{if } \tau(x) > n \\ \varphi(F^{\tau(x)} x) & \text{if } \tau \leq n. \end{cases}$$

Then also the Z_n are $\mathcal{F}_n$-measurable. So for any $x \in X$ and $n \geq 0$ with $\tau(x) > n$,

$$\begin{aligned} & E_\mu[Z_{n+1}|\mathcal{F}_n](x) - Z_n(x) - 1 \\ = \; & E_\mu[(\Delta\varphi - 1) \circ F^n | \mathcal{F}_n](x) \\ = \; & \sum_{H \in \mathcal{A}_{n+1}} \chi_H(x) \cdot \frac{1}{m(H)} \int_H (\Delta\varphi - 1) \circ F^n \, d\mu \\ \geq \; & 0 \ , \end{aligned} \tag{4.3}$$

where we used (4.1) for the inequality. If $\tau(x) \le n$, then $E_\mu[Z_{n+1}|\mathcal{F}_n](x) - Z_n(x) = 0$. Note that in both cases $E_\mu[Z_{n+1}|\mathcal{F}_n](x) \ge Z_n(x)$, i.e. $(Z_n, \mathcal{F}_n)_{n\ge 0}$ is a submartingale with respect to μ. Now define

$$W_n = Z_0 + \sum_{k=1}^{n} (E_\mu[Z_k|\mathcal{F}_{k-1}] - Z_{k-1}) \quad \text{and} \quad M_n = Z_n - W_n$$

(this is, by the way, the Doob-decomposition of $(Z_n, \mathcal{F}_n)_{n\ge 0}$). Then $W_0 = Z_0 = s$ and $M_0 = 0$ μ-a.s., and $(M_n, \mathcal{F}_n)_{n\ge 0}$ is a martingale:

$$M_{n+1} - M_n = Z_{n+1} - Z_n - W_{n+1} + W_n = Z_{n+1} - E_\mu[Z_{n+1}|\mathcal{F}_n]$$

and therefore

$$E_\mu[M_{n+1}|\mathcal{F}_n] = E_\mu[M_n|\mathcal{F}_n] + E_\mu[Z_{n+1}|\mathcal{F}_n] - E_\mu[E_\mu[Z_{n+1}|\mathcal{F}_n]|\mathcal{F}_n] = M_n \ .$$

$(W_n, \mathcal{F}_{n-1})_{n\ge 1}$ is a predictable stochastic sequence with

$$W_{n+1} - W_n = E_\mu[Z_{n+1}|\mathcal{F}_n] - Z_n \ge \chi_{\{\tau > n\}}$$

because of (4.3). It follows that

$$W_n \ge n + s \quad \text{on } \{x \ ; \ \tau(x) \ge n\} \ . \tag{4.4}$$

Next note that on $\{\tau > n\}$ holds

$$\begin{aligned} & E_\mu[(M_{n+1} - M_n)^2|\mathcal{F}_n] = E_\mu[(Z_{n+1} - E_\mu[Z_{n+1}|\mathcal{F}_n])^2|\mathcal{F}_n] \\ \le \ & E_\mu[(Z_{n+1} - Z_n)^2|\mathcal{F}_n] = E_\mu[(\Delta\varphi)^2 \circ F^n|\mathcal{F}_n] \\ \le \ & M \ , \end{aligned}$$

where we used the fact that $E_\mu[(Z_{n+1}-Y)^2|\mathcal{F}_n]$ is minimized by $Y = E_\mu[Z_{n+1}|\mathcal{F}_n]$ for the first inequality and assumption (4.2) for the second one. On $\{\tau \le n\}$ we have

$$M_{n+1} - M_n = Z_{n+1} - E_\mu[Z_{n+1}|\mathcal{F}_n] = Z_{n+1} - Z_n = 0 \ .$$

Both estimates together yield $E_\mu[(M_{n+1} - M_n)^2] \le M$, and we can apply Chow's version of the Hajek-Rényi inequality (see [Sto, Theorem 3.3.7]):

$$\mu\left\{\max_{1\le i\le n} \frac{|M_i|}{s - r_0 + i} \ge 1\right\} \le \sum_{i=1}^{n} \frac{E_\mu[(M_i - M_{i-1})^2]}{(s - r_0 + i)^2} \le M \cdot \sum_{j > s - r_0} \frac{1}{j^2} < \frac{1}{2} \ , \tag{4.5}$$

if $s - r_0$ is large enough. Hence

$$\begin{aligned} \mu\{\tau < \infty\} \ &= \ \mu\left(\bigcup_{n\ge 1} \{\tau = n\}\right) \le \mu\left(\bigcup_{n\ge 1} \{Z_n \le r_0 \text{ and } W_n \ge n + s\}\right) \\ &\le \ \mu\left(\bigcup_{n\ge 1} \{M_n \le r_0 - s - n\}\right) \le \mu\left(\bigcup_{n\ge 1} \{|M_n| \ge s - r_0 + n\}\right) \\ &= \ \sup_{n\ge 1} \mu\left\{\max_{1\le i\le n} \frac{|M_i|}{s - r_0 + i} \ge 1\right\} \le \frac{1}{2} \end{aligned}$$

for such s, i.e. $\mu\{\tau = \infty\} \geq \frac{1}{2}$.

Now a convergence theorem of Chow (see [Sto, Theorem 3.3.1]) asserts that

$$\lim_{n\to\infty} M_n/(s - r_0 + n) = 0 \qquad \mu\text{-a.s.}$$

in view of the finiteness of the sum in (4.5). Hence, on $\{\tau = \infty\}$,

$$\liminf_{n\to\infty} \frac{\varphi \circ F^n}{n} = \liminf_{n\to\infty} \frac{Z_n}{n} = \liminf_{n\to\infty} \frac{W_n}{n} + \lim_{n\to\infty} \frac{M_n}{n} \geq 1 \quad \mu\text{-a.s.}$$

in view of (4.4). In particular, for μ-almost every $x \in \{\tau = \infty\}$ the trajectory $x, Fx, F^2x, \ldots$ visits each element $A_k \in \mathcal{A}$ only finitely often. □

5. Proof of the Main Theorem

In this section we shall complete the proof of the Main Theorem. So let f be a C^2 Fibonacci map with a critical point of order ℓ. First we should remark that the complement of the basin of $\omega(c)$ is a residual set. This can be seen as follows. From Theorem A in Chapter IV of [MS] it follows that f has no wandering intervals (a wandering interval is an interval whose forward iterates are all disjoint and which is not in the basin of a periodic attractor). Moreover, f is not renormalizable and has positive topological entropy, see [HK]. It follows that f is semi-conjugate to a tent-map of the form

$$x \mapsto \lambda\,(1 - |2x - 1|)$$

and that the semi-conjugacy only collapses components of basins of periodic attractors. Clearly such components cannot be in the basin of the Cantor set $\omega(c)$. So it suffices to show that there exists a residual set of points x for which $\omega(x)$ (w.r.t. a tent-map) is equal to a cycle of intervals. This fact is well-known, see for example [Mil, page 189]. So the deepest part of the proof consists in showing that $B(\omega(c))$ has positive Lebesgue measure.

Let the points u_k, c_{S_k} and so on be defined as in Section 2 and choose as before $\tilde{u}_{k+1} \in \{u_{k+1}, \hat{u}_{k+1}\}$ so that it is on the same side of c as u_k. As in [KN] consider intervals $I_k = (u_k, \tilde{u}_{k+1})$ and $\hat{I}_k$ (the interval symmetric to I_k) and observe that $[\hat{u}_0, u_0] \setminus \cup_{k\geq 1}(I_k \cup \hat{I}_k)$ is a countable set. Define

$$F : \bigcup_{k\geq 1} (I_k \cup \hat{I}_k) \to [\hat{u}_0, u_0]$$

by

$$F|I_k = f^{S_k}\ .$$

Then $F(I_1) = F(\hat{I}_1) = (\hat{u}_0, u_0)$ and for $k > 1$

$$F(I_k) = F(\hat{I}_k) = (u_{k-2}, u_k)\ .$$

Hence, if we let $A_k = I_k \cup \hat{I}_k$ ($k \geq 1$), then $\mathcal{A} = \{A_k : k = 1, 2, \dots\}$ is a partition of $X = (u_0, \hat{u}_0)$ (modulo a countable set), and F is Markov with respect to $\mathcal{A}$.

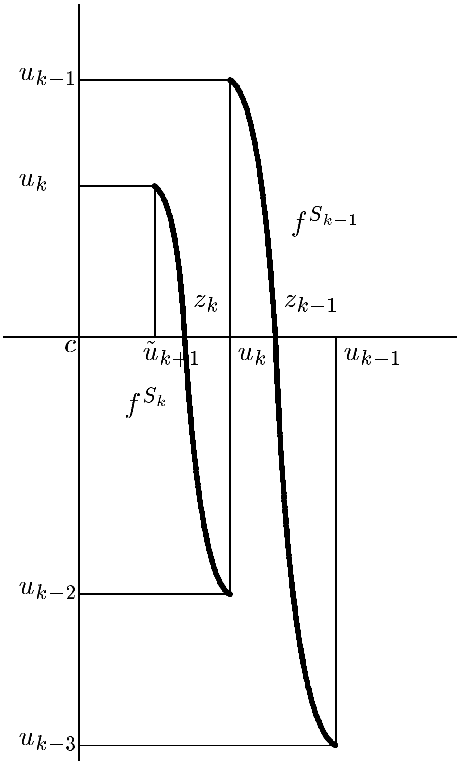

Figure 5.1.

In this section we shall show that F and $\mathcal{A}$ satisfy the assumptions of Theorem 4.1 and thus prove:

THEOREM 5.1. *For all sufficiently large ℓ holds: The set D of all points x for which the trajectory $(F^k x)_{k>0}$ visits each interval I_n and $\hat{I}_n$ at most finitely often, has positive Lebesgue measure.*

Let us first show that this result implies our Main Theorem, which states:

THEOREM 5.2. *$\omega(c)$ is a wild Cantor attractor for f provided ℓ is large enough.*

Proof. First we should remark that f is ergodic with respect to the (non-invariant) Lebesgue measure if its Schwarzian derivative is negative, see [BL]. Take D as in Theorem 5.1. Then $f^{-1}(D) = D$ and D has positive Lebesgue. Hence, if f is ergodic, D has full Lebesgue measure and we are finished. In general f need not be ergodic. For example, it could have a periodic attractor in which case $\omega(c)$ cannot be a 'global' attractor. But even in this case, the argument below will show that it attracts a set of positive Lebesgue measure.

We should remark that $\omega(c)$ is not accumulated by periodic attractors, see [MMS] or [MS], so near $\omega(c)$ these periodic attractors are 'invisible'.

Consider a point $x \in X$ for which $(F^k x)_{k>0}$ visits each interval I_n and $\hat{I}_n$ at most finitely often, and denote by $t_1 < t_2 < t_3 < \dots$ the sequence of times for which $F^k x = f^{t_k} x$. We have to show that $\lim_{t\to\infty} \operatorname{dist}(f^t x, \omega_f(c)) = 0$. Along the subsequence t_k this holds as $\lim_{k\to\infty} f^{t_k} x = \lim_{k\to\infty} F^k x = c \in \omega_f(c)$. Consider now $t_k < t < t_{k+1}$ and suppose that $F^k x \in I_n$ (or $F^k x \in \hat{I}_n$). As f^{S_n} is monotone on I_n (and on $\hat{I}_n$) and as $f^{S_n}(I_n) = f^{S_n}(\hat{I}_n)$ is an interval contained in the union of the two central monotonicity intervals of $f^{S_{n-2}}$, the interval $V := f^{t-t_k}(I_n) = f^{t-t_k}(\hat{I}_n)$ is contained in the union of two adjacent monotonicity intervals of $f^{S_{n-2}+t_{k+1}-t}$. Furthermore, $f^t x \in V$, and as $c_{S_{n+1}} \in I_n$, V contains the point $f^{S_{n+1}+t-t_k}(c) \in \omega_f(c)$. Therefore $\operatorname{dist}(f^t x, \omega_f(c)) \le |V| \le 2\delta_{S_{n-2}+t_{k+1}-t} \le 2\delta_{S_{n-2}}$, where δ_k denotes the maximal length of a monotonicity interval of f^k, and $\lim_{k\to\infty} \delta_k = 0$ because f is non-renormalizable. $\square$

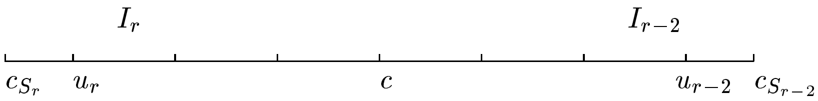

Figure 5.2. The diffeomorphic image of $F^n|H$.

Proof of Theorem 5.1. Let us show that we can apply Theorem 4.1.

Step 1: The first condition of Theorem 4.1 is satisfied. Let m be the Lebesgue measure and F the induced map from above. Take $H \in \mathcal{A}_{n+1}$. We have to show that

$$\int_H (\Delta\phi - 1) \circ F^n \, dm \ge 0 \tag{5.1}$$

where $\Delta\phi(x) = \phi(F(x)) - \phi(x)$ and where $\phi(x)$ is the 'index' of the state x. By definition $F^n(H)$ is equal to I_r or $\tilde{I}_r$ for some $r \ge 1$. Let us consider the former case. Recall that F maps I_r (and also $\hat{I}_r$) diffeomorphically onto

$$\cup_{i=r}^{\infty} A_i \cup \tilde{I}_{r-1} \cup I_{r-2} \ ,$$

where $A_i = I_i \cup \hat{I}_i$. For convenience let $I_i^{\#}$ be the component of A_i which lies on the same side of c as I_{r-2}. Hence

$$\{x \in H \ ; \ \Delta\phi(x) \le 0\} = I_r^{\#} \cup I_{r-1}^{\#} \cup I_{r-2}^{\#} \cup I_r,$$

see Figure 5.2. (Note that I_r and I_{r-2} are on opposite sides of c.) Hence

$$\int_H (\Delta\phi - 1) \circ F^n \, dm \tag{5.2}$$

is equal to

$$\sum_{i=0}^{\infty} i\cdot|H_i| + (-1)\cdot|H^{\#}_{-1}| + (-2)\cdot|H^{\#}_{-2}| - |H| \geq \sum_{i=0}^{\infty} i\cdot|H_i| + (-4)\cdot|H|, \tag{5.3}$$

where

$$H_i = \{x\in H\ ;\ F^{n+1}(x)\in A_{r+i}\}, i=0,1,2,\ldots$$

and

$$H_i^{\#} = \{x\in H\ ;\ F^{n+1}(x)\in I^{\#}_{r+i}\},\quad i=-2,-1.$$

We shall show below that there exists a universal constant $C>0$ such that for $i=1,2,\ldots,$

$$C\cdot|H_{i-1}| \leq |H_i| \leq \frac{1}{C}|H_{i-1}| \tag{5.4}$$

and for $1\leq i\leq \ell$,

$$|H_i| \geq \frac{C}{i^2}|H_0|. \tag{5.5}$$

Moreover, for $i=-1,0$,

$$C\cdot|H^{\#}_{i-1}| \leq |H^{\#}_i| \leq \frac{1}{C}|H^{\#}_{i-1}|. \tag{5.6}$$

Before proving these inequalities let us show that they suffice. Indeed, using these inequalities, we can estimate (5.3) from below by:

$$\sum_{1\leq i\frac{1}{\ell}\ell} i\cdot\frac{C}{i^2}\cdot|H_0| - 4|H| \geq (C/2)\log(\ell)|H_0| - 4|H|. \tag{5.7}$$

Hence if

$$\frac{|H_0|}{|H|} \geq \frac{8}{C\log(\ell)} \tag{5.8}$$

then we get that (5.7) is non-negative and we are done. So assume that (5.8) is not satisfied. In this case, we use (5.4) and (5.6) and we have that there exists a universal constant $C_1<\infty$ and $\lambda\in(1,\infty)$ (which one can choose to be $1/C$) so that for $k\geq 1$

$$|H^{\#}_{-2}\cup H^{\#}_{-1}\cup H_0\cup\ldots\cup H_{k-1}| \leq C_1\lambda^k|H_0|.$$

Hence

$$|\cup_{i\geq k} H_i| \geq |H| - C_1\lambda^k|H_0| = \left(1 - C_1\lambda^k\frac{|H_0|}{|H|}\right)|H|,$$

and therefore choosing $k=8$ we can also bound (5.3) from below by

$$\sum_{i=8}^{\infty} i\cdot|H_i| - 4|H| \geq 8\cdot|\cup_{i\geq 8} H_i| - 4|H| \geq \left(8\cdot\left[1 - C_1\lambda^8\frac{|H_0|}{|H|}\right] - 4\right)|H|. \tag{5.9}$$

Since we have assumed that (5.8) fails, we can make sure that $C_1\lambda^8\frac{|H_0|}{|H|}$ is at most $1/2$ by taking ℓ sufficiently large and therefore (5.9) becomes non-negative in this case.

So we need to prove (5.4), (5.5) and (5.6). By the estimates from Theorem 3.14 there exist constants $C_1, C_2 \in (0,\infty)$ such that for large ℓ and large j,

$$1+\frac{C_1}{\ell} \le \frac{|u_j - c|}{|\tilde{u}_{j+1} - c|} \le 1 + \frac{C_2}{\ell}\ . \tag{5.10}$$

In particular, since $(1+C/\ell)^i \le e^C$ for $i \le \ell$, there exist universal constants $C_3, C_4 \in (0,\infty)$ such that

$$C_3|I_r| \le |I_{r+i}| \le C_4|I_r| \tag{5.11}$$

for each integer i between -2 and ℓ (and the same holds for $\tilde{I}_r$ and A_r because of the symmetry of f near c).

It follows from Proposition 2.1 that F satisfies the following extension properties:

- $F^n_{|H}$ is of the form f^m (in fact, f^m is a composition of maps of the form f^{S_i}) and therefore $F \circ F^n = f^{S_r+m}$;
- there exists an interval $T \supset H$ which is mapped by f^{S_r+m} diffeomorphically onto $(c_{S_{r-2}}, c_{S_r})$.

Now note that we have the following Koebe inequality: if $j \subset T$ is an interval such that $F^{n+1}(T) = (c_{S_{r-2}}, c_{S_r})$ contains a τ-scaled neighbourhood of $F^{n+1}(j)$ then

$$\frac{|DF^{n+1}(x)|}{|DF^{n+1}(y)|} = \frac{|Df^m(x)|}{|Df^m(y)|} \le \mathcal{O}\left[\frac{1+\tau}{\tau}\right]^2$$

where $\mathcal{O} < \infty$ is some universal number. (We shall refer to τ as the *Koebe space*.) If $Sf < 0$ this follows immediately from the Koebe Principle, see Proposition 3.2. In the general case it follows by combining the Koebe Principle 3.4 with Proposition 3.6. Indeed, the first assumption of this proposition holds by definition of F; assumptions 2) and 3) follow from the above extension properties and from the bounds from Theorem 3.14 (where τ is a constant which is independent of ℓ); finally in assumption 4) the constant K can be taken as the intersection multiplicity 3 from Proposition 2.1.

Since $|c_{S_{r-2}} - u_{r-2}|$ is of the same order as the size of $I^{\#}_{r+i}$ for $i = -2, -1, 0$ by Theorem 3.14, F^{n+1} maps some neighbourhood of each component of (the closure of) $H_i \cup H_{i+1}$ diffeomorphically to a definite neighbourhood of one component of (the closure of) $A_{r+i} \cup A_{r+i+1}$ for $i \ge 0$. Hence this map has uniformly bounded distortion on each component of (the closure of) $H_i \cup H_{i+1}$. The images of H_i and H_{i+1} are equal to the union of A_{r+i} and A_{r+i+1}. Since each of the components of these sets has roughly equal size, see (5.11), the components of the sets H_i and H_{i+1} on one side of c have roughly the same size. This proves (5.4). Similarly, one has that $H^{\#}_{-2}$, $H^{\#}_{-1}$ and $H^{\#}_0$ have roughly equal length and thus we get (5.6).

Moreover, all the components of H_i, $1 \le i \le \ell$, and also the Koebe space are of the same order because of (5.11). Hence the Koebe space around the F^{n+1}-image of a component of (the closure of) $H_0 \cup H_1 \cup \ldots \cup H_i$ is of order $1/i$ provided $1 \le i \le \ell$. It follows by the above Koebe inequality (and the Mean Value Theorem) that the ratio of the two expressions

$$\frac{|A_0|}{|H_0|} = \frac{|F^{n+1}(H_0)|}{|H_0|} \quad \text{and} \quad \frac{|A_i|}{|H_i|} = \frac{|F^{n+1}(H_i)|}{|H_i|}$$

is bounded from above by i^2/C_6 where $C_6 > 0$ is uniformly bounded away from zero (and $1 \le i \le \ell$). By (5.11) this implies (5.5).

Step 2: The second condition of Theorem 4.1 is satisfied. We need to show that there exists M such that for any $n \in \mathbb{N}$ and any $H \in \mathcal{A}_{n+1}$,

$$\int_H (\Delta\phi)^2 \circ F^n \, dm \le M \cdot m(H)\,.$$

Since $(1 - 1/\ell)^\ell \approx e^{-1}$ and by (5.10), $|\cup_{i>\ell} I_{r+i}|$ is of the same order as $|\cup_{0\le i\le \ell} I_{r+i}|$. Therefore by the Koebe Principle and by (5.10) there exist constant $\lambda = 1 + C/\ell$ and K (where C and K are universal) so that

$$|H_i| \le K \cdot \lambda^{\ell-i} \cdot |H_{[\ell]}| \le K\lambda^{\ell-i} \cdot |H|$$

for $i \ge \ell$. Hence there exists M which is independ of H and n (but does depend on ℓ) so that

$$\int_H (\Delta\phi)^2 \circ F^n \, dm \le \ell^2|H| + \sum_{i\ge\ell} i^2|H_i| \le M \cdot |H|.$$

Step 3: Conclusion of the proof of theorem. Because we have checked both condition of Theorem 4.1 the proof is complete. □

Univ. of Delft, the Netherlands
Univ. of Erlangen, Germany
Univ. of Warsaw, Poland
Univ. of Amsterdam, the Netherlands

References

[AKYZ] J.C. Alexander, I. Kan, J.A. Yorke and Zhiping You, *Riddled basins*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **2**, 795-813, (1992).

[BC] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Ann. Math. **133**, 73–169, (1991).

[Bow] R. Bowen, *A horseshoe with positive measure*, Invent. Math. **28**, 203-204, (1975).

[BH] B. Branner and J.H.Hubbard, *The iteration of cubic polynomials I*, Acta Math. **160**, (1988), 143-206, *The iteration of cubic polynomials II*, Acta Math. **169**, (1992), 229-325.

[BL] A.M. Blokh and M.Yu. Lyubich, *Measurable dynamics of S-unimodal maps of the interval*, Ann. Sc. E.N.S. 4e série, **24**, 545–573, (1991).

[Br] H. Bruin, *Induced maps, Markov extensions and invariant measures in one-dimensional dynamics*, Preprint Delft, (1993).

[GST] J.-M. Gambaudo, S. van Strien and C. Tresser, *Hénon-like maps with strange attractors: there exist C^∞ Kupka-Smale diffeomorphisms on S^2 with neither sinks nor sources*, Nonlinearity **2**, 287–304, (1989).

[Gu] J. Guckenheimer, *Sensitive dependence on initial conditions for one dimensional maps*, Commun. Math. Phys **70**, 133–160 (1979).

[GJ] J. Guckenheimer and S. Johnson, *Distortion of S-unimodal maps*, Annals of Math., **132**, 71-130, (1990).

[HK] F. Hofbauer and G. Keller, *Some remarks on recent results about S-unimodal maps*, Ann. Insitut Henri Poincaré, **53**, 413-425, (1990).

[JS1] M.V. Jacobson and G. Światek, *Metric properties of non-renormalizable S-unimodal maps, I Induced expansion and invariant measures.* Preprint I.H.E.S.

[JS2] M.V. Jacobson and G. Światek, *Quasisymmetric conjugacies between unimodal maps*, Preprint IMS at Stony Brook #1991/16.

[Kan] I. Kan, *Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin*, Bull. A.M.S. **31**, 68-74 (1994).

[Ke] G. Keller, *Exponents, attractors, and Hopf decompositions for interval maps*, Ergod. Th. & Dynam. Sys. **10**, 717-744 (1990).

[KN] G. Keller and T. Nowicki, *Fibonacci maps re(aℓ)visited*, to appear in Ergod. Th. & Dynam. Sys.

[L1] M.Yu. Lyubich, *On the typical behaviour of trajectories of the exponent*, Russian Math. Surveys **41**, 207-208, (1986)

[L2] M.Yu. Lyubich, *The measurable dynamics of the exponential map*, Siberian J. of Math. **28**, 111-127, (1987)

[L3] M.Yu. Lyubich, *Ergodic theory for smooth one-dimensional dynamical systems.* Preprint StonyBrook, (1991).

[L4] M.Yu. Lyubich, *On the Lebesgue measure of the Julia set of a quadratic polynomial.* Preprint StonyBrook, (1991).

[L5] M.Yu. Lyubich, *Combinatorics, geometry and attractors of quasi-quadratic maps.* Preprint StonyBrook, (1992).

[L6] M.Yu. Lyubich, *Milnor's attractors, persistent recurrence and renormalization.* In: "Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday", (1992).

[LM] M. Lyubich and J. Milnor, *The unimodal Fibonacci map*, Journal of the A.M.S. **6**, 425-457, (1993).

[Mar] M. Martens, thesis, Delft, (1990).

[MMS] M. Martens, W. de Melo and S. van Strien, *Julia-Fatou-Sullivan theory for real dynamics*, Acta Math. **168**, 273-318, (1992).

[Mil] J. Milnor, *On the concept of attractor*, Commun. Math. Phys. **99**, 177-195, (1985).

[Mi1] M. Misiurewicz, *Absolutely continuous invariant measures for certain maps of an interval*, Publ. Math. I.H.E.S. **53**, 17-51, (1981).

[Mi2] M. Misiurewicz, *On iterates of e^z*, Ergod. Th. & Dynam. Sys., **1**, 103-106, (1981).

[MS] W. de Melo and S. van Strien, *One-dimensional dynamics.* Ergebnisse Series **25**, Springer Verlag, (1993).

[NS] T. Nowicki and S. van Strien, *Absolutely continuous invariant measures under a summability condition.* Invent. Math. **105**, 123-136, (1991).

[Re] M. Rees, *The exponential map is not recurrent.* Math. Z., **191**, 593-598, (1986).

[Sto] W.F. Stout, *Almost Sure Convergence*, Academic Press, New York-San Francisco-London (1974)

[SN] S. van Strien and T. Nowicki, *Polynomial maps with a Julia set of positive Lebesgue measure: Fibonacci maps*, Preprint StonyBrook ims94-4

[Str] S. van Strien, *Hyperbolicity and invariant measures for general C^2 intervals maps satisfying the Misiurewicz condition*, Commun. Math Phys. **128**, 437-496, (1990).

[Str2] S. van Strien, *Polynomial and rational maps which are transitive but not ergodic.* Manuscript.

[Sw1] G. Światek, *One-dimensional maps and Poincaré metric*, Nonlinearity. **5**, 81-108 (1991)

[Sw2] G. Światek, *Hyperbolicity is dense in the real quadratic family*, Preprint StonyBrook, (1992)

Received ??.
Revised ??.