Expansion of Derivatives in One-Dimensional Dynamics

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Abstract

We study the expansion of derivatives along orbits of real and complex one-dimensional maps f, whose Julia set J_f attracts a finite set Crit of non-flat critical points. Assuming that for each $c \in \text{Crit}$, either $|Df^n(f(c))| \to \infty$ (if f is real) or $b_{n-1} \cdot |Df^n(f(c))| \to \infty$ for some summable sequence $\{b_n\}$ (if f is complex; this is equivalent to summability of $|Df^n(f(c))|^{-1}$), we show that for every $x \in J_f \setminus \cup_i f^{-i}(\text{Crit})$, there exists $K'(x) > 0$ and $\ell(x) \leq \max_c \ell(c)$ such that

$$|Df^n(x)|^{\ell(x)} \geq K'(x) \cdot \prod_{i=0}^{s-1} D_{n_i - n_{i+1}}(c_{i+1})$$

for infinitely many n. Here the c_i are the the critical points that $\text{orb}(x)$ shadows for $n_i - n_{i+1}$ iterates, and

$$D_k(c_i) = \begin{cases} \max\Big(\lambda, K \cdot |Df^k(f(c_i))|\Big) & \text{if } f \text{ is real,} \\ \max\Big(\lambda, K \cdot b_{k-1} \cdot |Df^k(f(c_i))|\Big) & \text{if } f \text{ is complex,} \end{cases}$$

for uniform constants $K > 0$ and $\lambda > 1$. If all $c \in \text{Crit}$ have the same critical order, then $K'(x)$ is uniformly bounded away from 0. Several corollaries are derived. The number of ergodic components of the conformal measure is bounded by #Crit, and in the complex case, either $J_f = \hat{\mathbb{C}}$ or J_f has zero Lebesgue measure. Also there exist $\kappa > 0$ such that if n is the smallest integer such that x enters a certain critical neighbourhood, then $|Df^n(x)| \geq \kappa$.

1 Introduction and Statement of Results

The behaviour of (real and complex) one-dimensional systems depends essentially on the behaviour of their critical points. Derivatives along orbits tend to grow at least as fast as the derivatives along the critical orbits, and so the growth along critical orbits plays a central role in the question whether invariant densities exist. This goes back to Collet and Eckmann's result [5] on non-flat S-unimodal maps f. If there exist $C > 0$ and $\lambda > 1$ such that

$$|Df^n(f(c))| \geq C\lambda^n \text{ for all } n \geq 0, \qquad (CE)$$

where c is the critical point, and

$$|Df^n(f(x))| \geq C\lambda^n \text{ where } n \text{ is minimal such that } f^n(x) = c. \qquad (BCE)$$

then f admits an invariant probability measure (acip) that is absolutely continuous with respect to Lebesgue. Nowicki [15] showed that the Collet-Eckmann condition (CE) implies the backward Collet-Eckmann condition (BCE), thus eliminating the need of (BCE). In [16] it was shown that (CE) and (BCE) are equivalent. In the setting of rational maps, Przytycki showed that (CE) together with some technical assumptions imply the existence of an acip (with respect to a non-atomic conformal measure on the Julia set).

Nowicki & van Strien [17] showed that for S-unimodal maps, summability of $|Df^n(f(c))|^{-1/\ell}$ (where ℓ is the order of the critical point c) guarantees the existence of an acip. In fact, without substantially changing the proof, one can verify that this results holds in the multimodal case:

Theorem 1.1 *Let f be a real multimodal map such that $Sf \leq 0$ and all critical points have the same finite order ℓ. If $\sum_n |Df^n(f(c))|^{-1/\ell} < \infty$ for each critical point c, then f admits an acip.*

It is the aim of this paper, using the ideas of [17], to estimate derivatives along arbitrary orbits, and give alternative approaches to (and in some cases improvements of) results by Graczyk & Smirnov [8] and Przytycki [19]. We assume that

$$\begin{cases} |Df^n(f(c))| \to \infty & \text{if } f \text{ is real with } Sf \leq 0, \\ \sum_n |Df^n(f(c))|^{-1} < \infty & \text{if } f \text{ is rational on } \hat{\mathbb{C}}. \end{cases}$$

Except for finiteness, there are no restrictions on the number of critical points and their orders. The complex assumption is the same as used in [8] or in [19, Theorem B].

We will show that $\mathrm{orb}(x)$ can be decomposed into pieces in which it loosely shadows a critical orbit $\mathrm{orb}(c_i)$, and that the derivatives grow accordingly: There exists $K'(x) > 0$ and $\ell(x) \leq \max_c \ell(c)$ such that

$$|Df^n(x)|^{\ell(x)} \geq K'(x) \cdot \prod_{i=0}^{s-1} K_i \cdot |Df^{n_i - n_{i+1}}(f(c_i))|,$$

for infinitely many n. Here $K_i \geq K$ (if f is real) and $K_i \geq K \cdot b_{n_i - n_{i-1} - 1}$ (if f is complex), and $K > 0$ is a uniform constant. See Theorem 1.2 for the precise statement.

The second purpose of this paper is to strengthen the results in [4]. In that paper we established the existence of an acip μ for real multimodal maps under certain summability conditions and we derived strong statistical properties concerning the mixing rate of μ. However, we assumed a property, called *bounded backward contraction* (BBC), which states that $|Df^n(x)|$ is uniformly bounded away from 0 whenever x first enters a critical neighbourhood. Theorem 1.3 below shows that (BBC) holds automatically, both in the real and complex case, provided all critical points have the same order. Examples in Section 5 show that this assumption is essential, and that without it also (CE) no longer implies (BCE).

In future research we hope to strengthen the results of [4] by improving on the existence conditions and mixing rates of absolutely continuous invariant probability measures for multimodal interval maps. Secondly we intend to extend the techniques of [4] to rational maps on the Riemann sphere, especially those that do not satisfy the Collet-Eckmann condition.

In the remainder of this section we give the precise statements of the results and relate them to known results. Let I denote an interval or circle and $\hat{\mathbb{C}}$ the Riemann sphere with spherical metric. For convenience, let us define the Julia set J_f of a real map to be the set of points which are not attracted to ∞ or to a periodic orbit. The set of critical points c such that $\overline{\mathrm{orb}(c)}$ intersects J_f will be denoted by Crit. We assume that $\mathrm{Crit} \neq \emptyset$; otherwise J is hyperbolic and completely understood as far as the result presented here are concerned. The assumptions (1) and (2) below therefore exclude the existence of parabolic points. Each $c_i \in \mathrm{Crit}$ has critical order $\ell_i < \infty$ and we write $\ell_{\max} = \max\{\ell_i; c_i \in \mathrm{Crit}\}$. Let

$$N_\epsilon(\mathrm{Crit}) = \bigcup_i B(c_i, \epsilon^{1/\ell_i}),$$

where $B(x,\delta)$ denotes the open ball of radius δ. We shall consider expansion properties of the following two types of maps:

(i) smooth real one-dimensional maps $f\colon I \to I$ with non-flat critical points, such that the Schwarzian derivative $Sf \leq 0$ and such that

$$|Df^n(f(c))| \to \infty \text{ as } n \to \infty \text{ for each } c \in \mathrm{Crit}. \tag{1}$$

(ii) rational maps $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that there exists a sequence $b_n > 0$ such that

$$\exp(\sum_{n\geq 0} b_n) < 2 \quad \text{and} \quad b_n \cdot |Df^n(f(c))| \to \infty \tag{2}$$

for each $c \in \mathrm{Crit}$. The b_n's feature in the notion of shrinking neighbourhoods from [19]. Condition (2) is equivalent to $\sum_{n\geq 0} 1/|Df^n(f(c))| < \infty$. To see this, choose $m_1, m_2, \ldots$ so that $\sum_{n \geq m_k} 1/|Df^n(f(c))| \leq 1/k^3$. Then choose $d_n = 1$ for $n < m_1$ and define $d_n = k$ for $n = m_k, \ldots, m_{k+1}-1$ and $k = 1, 2, \ldots$ Then $\sum_{n\geq 0} d_n/|Df^n(f(c))| \leq \sum_{n \leq n_1} 1/|Df^n(f(c))| + \sum_{k\geq 1} k/k^3 < \infty$ and so $b_{n-1} = \epsilon \cdot d_n/|Df^n(f(c))|$ satisfies the required properties for some small $\epsilon > 0$. Conversely, $b_{n-1} \cdot |Df^n(f(c))| \to \infty$ implies $|Df^n(f(c))|^{-1} \leq b_{n-1}$ for n sufficiently large.

Given a point x, define for the moment $\ell(x)$ to be the maximal order of all critical points on which x accumulates (if it does not accumulate on any critical points define $\ell(x) = 1$). A more precise definition of $\ell(x)$ will be given in formulas (6) and (32).

Theorem 1.2 *Assume that (1) or (2) holds. Then there exist $\lambda > 1$ and $K > 0$ such that for each $x \in J_f$ such that $f^k(x) \notin$ Crit for all $k \geq 0$, there exists $K'(x) > 0$ and arbitrarily large times n such that (taking $\ell = \ell(x)$),*

$$|Df^n(x)|^\ell \geq K'(x) \cdot \prod_{i=0}^{s-1} D_{n_i - n_{i+1}}(c_{i+1}), \tag{3}$$

where

$$D_k(c_i) = \begin{cases} \max\left\{\lambda, K \cdot |Df^k(f(c_i))|\right\} & \text{if } f \text{ is real,} \\ \max\left\{\lambda, K \cdot b_{k-1} \cdot |Df^k(f(c_i))|\right\} & \text{if } f \text{ is complex,} \end{cases}$$

and $0 = n_s < \ldots < n_0 := n$ *are critical times (defined in later sections). In particular, there exists a sequence of integers* n *so that* $|Df^n(x)| \to \infty$.

This allows the following corollary for periodic points.

Corollary 1.1 *If* f *is an is Theorem 1.2, then the multipliers of all periodic orbits grow essentially as fast as* D_n. *There are no Siegel disks, Herman rings or Cremer points. Furthermore, (CE) implies uniform hyperbolicity on periodic points.*

It is known (adapt the example in [20, Section 5]) that uniform hyperbolicity on periodic points does not imply (CE). Recall the growth condition on Crit immediately rules out the existence of parabolic points. The non-existence of Cremer points, Siegel disks and Herman rings was claimed in [8].

The number $K'(x)$ expresses the small derivatives of close visits of $\mathrm{orb}(x)$ to critical points c_r with order $\ell_r > \ell(x)$. On the set $\{x; \ell(x) = \ell_{\max}\}$, $K'(x)$ is uniformly bounded away from 0. Furthermore, (3) holds for the critical times, which includes any n such that $f^n(x) \in \mathrm{Crit}$. Therefore we have

Corollary 1.2 *If* n *is minimal such that* $f^n(x) = c_i$ *and* $\ell_i = \ell_{\max}$, *then* $|Df^n(x)| \geq K \cdot D_n$. *In particular, (CE) implies (BCE) for all* c_i *with* $\ell_i = \ell_{\max}$.

As mentioned before, the statement that (CE) implies (BCE) was shown in the unimodal case by Nowicki [15]. Graczyk & Smirnov [7, Theorem 1 (i)] proved it for rational maps on the Riemann sphere. The corollary does not hold for critical points c_i with $\ell_i < \ell_{\max}$, as we demonstrate in Section 5.

In the next theorem we prove (BBC). This property was used as assumption in [4].

Theorem 1.3 *(The BBC property) Assume that (1) or (2) holds and assume that all critical points of* f *have the same order. Then there exist* $\kappa > 0$ *so that for any* $\epsilon > 0$ *and any* $x \in J_f$ *and* $n = \min\{k \geq 0; f^k(x) \in N_\epsilon(\mathit{Crit})\}$ *one has*

$$|Df^n(x)| \geq \kappa. \tag{4}$$

More precisely, there exist uniform constants $\lambda > 1$ *and* $K, K' > 0$ *such that*

$$|Df^n(x)|^\ell \geq K' \cdot \prod_{i=0}^{s-1} D_{n_i - n_{i+1}}(c_{i+1}),$$

where $D_k(c_i)$ *is as in the previous theorem.*

The assumption that all critical points have the same order is essential; counterexamples are given in Section 5.

A well-known result that goes back to [14] and [10] is that if f is a C^2 multimodal map and $\sigma > 0$, there exists $C > 0$ and $\lambda > 1$ such that if $\mathrm{dist}(\mathrm{orb}(x), \mathrm{Crit} \cup \{\text{non-repelling periodic orbits}\}) \geq \sigma$, then

$$|Df^n(x)| \geq C\lambda^n \text{ for all } n \geq 0. \tag{5}$$

This determines the hyperbolic subsets of I. In the complex case, Mañé [11] proved a similar result, using the assumption that x does not belong to the omega-limit set of a recurrent critical point. The complex analog of (5) was called *R-expansion* and used as assumption in [7, Theorem 2].

Corollary 1.3 *Let f satisfy (1) or (2). There exists $C > 0$ such that for every $\sigma > 0$ there exists $\lambda > 1$ such that: if $x \in J_f$, then dist*$(\mathrm{orb}(x),$ *Crit*$) \geq \sigma$ *implies* $|Df^n(x)| \geq C\sigma^{\ell_{\max}-1}\lambda^n$ *for all n.*

The next result was shown before in [19, Theorem B]. Suppose $m = m_\alpha$ is a non-atomic α-conformal measure on J_f. It is easily seen that m-a.e. orbit accumulates on Crit. The expansion properties shown in Theorem 1.3 enable us to prove that every ergodic component of m has a critical point as point with upper density 1 with respect to m. Therefore

Corollary 1.4 *Suppose f satisfies (1) or (2), then m_α has at most #Crit ergodic components.*

The precise argument is presented in Section 6. In Section 7 we extend the techniques of Section 6 with a random walk argument and prove that Lebesgue-a.e. point reaches "large scale univalently". This results in

Theorem 1.4 *If f satisfies (2), then either $J = \hat{\mathbb{C}}$ or the Lebesgue measure of J is* 0.

2 Notation and Preliminaries

If a map (real or complex) f has a critical point c, then its order ℓ is the number such that $(1/M)|x-c|^\ell \leq |f(c) - f(x)| \leq M|x-c|^\ell$ for all x and a uniform constant $M > 1$. In particular, we assume that in the real case, the critical order of a critical point is the same on either side of c. Obviously, derivatives scale like $(\ell/M)|x-c|^{\ell-1} \leq |Df(x)| \leq \ell M|x-c|^{\ell-1}$. These properties, plus the fact that $\ell < \infty$, will be referred to as *non-flatness.*

Denote by $\nu(\sigma)$ the minimal integer n so that $\mathrm{dist}(f^n(c), c') \leq \sigma$ for some critical points $c, c' \in \mathrm{Crit}$. Since no critical point is mapped to another critical point, $\nu(\sigma) \to \infty$ as $\sigma \to 0$.

Throughout the paper we shall need several Koebe-like estimates:

Lemma 2.1 *(Koebe Lemma) Let $g\colon (a,b) \to \mathbb{R}$ be a diffeomorphism and $Sg < 0$. For each $\tau \in (0,1)$ there exists $KL(\tau)$ such that if $x, y \in (a,b)$ is so that both sides of $g((a,b) \setminus (x,y))$ have size $\geq \tau|g(b) - g(a)|$ then $|Dg(x)|/|Dg(y)| \leq KL(\tau)$.*

Lemma 2.2 *(One-sided Koebe Lemma) Let $g\colon (a,b) \to \mathbb{R}$ be a diffeomorphism and $Sg < 0$. For each $\tau \in (0,1)$ there exists $KL(\tau)$ such that if $x \in (a,b)$ is so that $|g(x)-g(a)| \leq \tau|g(b)-g(a)|$ then $|Dg(x)| \geq KL(\tau) \cdot |Dg(a)|$.*

Lemma 2.3 *(Expansion of Cross-Ratios) Let $g\colon (a,b) \to \mathbb{R}$ be a diffeomorphism and $Sg < 0$. Let $j \subset t$ be subintervals of (a,b) and let l, r be the components of $t \setminus l$. Then*

$$\frac{|g(t)| \cdot |g(j)|}{|g(l)| \cdot |g(r)|} \geq \frac{|t| \cdot |j|}{|l| \cdot |r|}.$$

In the limit $j \to x$ this yields $g'(x) \geq \frac{|t|}{|g(t)|} \cdot \frac{|g(r)| \cdot |g(l)|}{|l| \cdot |r|}$.

Lemma 2.4 *(Koebe Distortion Lemma) Given $\epsilon < \mathrm{diam}(\hat{\mathbb{C}})$ there exists a constant $C = C(\epsilon)$ such that for every $0 < \lambda < 1$, every disc $B(z, \delta)$ with $\delta \in (0, \epsilon)$, every univalent mapping $g\colon B(z,\delta) \to \hat{\mathbb{C}}$ for which* $\mathrm{diam}\, g(B(z,\delta)) \leq \epsilon$, *and very $x, y \in B(z, \lambda\delta)$ one has*

$$\frac{1-\lambda}{C} < \frac{|g'(x)|}{|g'(z)|} < \frac{C}{1-\lambda},$$

and

$$\frac{|g'(x)|}{|g'(y)|} < \frac{C^2}{(1-\lambda)^2}.$$

The proofs of Lemmas 2.1-2.3 can be found in [13]. Lemma 2.4 is a direct consequence of the Koebe $\frac{1}{4}$-Lemma, see [6] or [19].

An interval $U \subset I$ is called *wandering* if $f^n|U$ is monotone for all $n \geq 0$ and U is not attracted to a periodic orbit. It is well-known that C^2 maps do not have wandering intervals, see [13]. This fact is equivalent to the following statement.

Lemma 2.5 *For any $\epsilon > 0$ there exist $\delta > 0$ such that if U is a ball of diameter $\geq \epsilon$ which is not attracted to a periodic orbit, then $f^n(U)$ has diameter $\geq \delta$ for all $n \geq 0$.*

For rational maps on $\hat{\mathbb{C}}$, this lemma is a simple consequence of Montel's Theorem, provided U intersects the Julia set.

3 The Real Case

Throughout this section we shall assume $f\colon I \to I$ is a interval or circle map satisfying (1). Take a point x such that $f^k(x) \notin \text{Crit}$ for all $k \geq 0$. For the moment fix an integer n and choose $\sigma > 0$. Let $T_n(x) \ni x$ be the maximal interval such that $f^n|T_n(x)$ is a diffeomorphism; usually we shall simply write T_n. Let

$$r_n(x) = \text{dist}(\partial f^n(T_n), f^n(x)).$$

We say that the time n is σ-big if $r_n(x) \geq \sigma$. Furthermore, n (or $f^n(x)$) is of type

$$\begin{cases} \text{(AP) } (\textit{almost precritical}) & \text{if } r_n(x) \geq \text{dist}(f^n(x), \text{Crit}), \\ \text{(NAP) } (\textit{not almost precritical}) & \text{if } r_n(x) < \text{dist}(f^n(x), \text{Crit}). \end{cases}$$

Also if n is σ-big, we still distinguish between type (AP) and (NAP). Types (NAP) and (AP) represent the "sliding" and "transfer" case in [17], while σ-big refers to the "regular" case. We call n *critical* if $f^n(T_n)$ contains a critical point. Note that if n is of type (AP), it has to be critical.

Given n, let $0 = n_s < n_{s-1} < \ldots < n_0 := n$ be critical times of x before time n (we do not need to assume that n itself is a critical time) and let c_i the critical point (or the boundary point of the interval) closest to $x_i = f^{n_i}(x)$ for $i = 0, \ldots, s$.

Let $\mathrm{Crit}'(x)$ be the set of critical points which are contained in $f^n(T_n(x))$ for infinitely many times n of type (AP). In Assertion 1 of the proposition below, we shall show that either $\mathrm{Crit}'(x) \neq \emptyset$ or there are infinitely many σ-big times n. Proposition 3.1 below employs an order ℓ depending on the starting point x. Any choice of ℓ works, but for the sharpest asymptotic results we choose

$$\ell = \ell(x) = \begin{cases} \max\{\ell_i; c_i \in \mathrm{Crit}'(x)\} & \text{if } \mathrm{Crit}'(x) \neq \emptyset, \\ 1 & \text{if } \mathrm{Crit}'(x) = \emptyset. \end{cases} \tag{6}$$

For $0 \leq i, j \leq s$ define

$$A_i = \begin{cases} |x_i - f^{n_i - n_{i+1}}(c_{i+1})|^{\ell - \ell_i} & x_i \text{ is (AP)}, \\ |x_i - f^{n_i - n_{i+1}}(c_{i+1})|^{\ell - 1} |f^{n_i - n_{i+1}}(c_{i+1}) - c_i|^{1 - \ell_i} & x_i \text{ is (NAP)}, \end{cases}$$

and

$$B_j = \begin{cases} |x_j - c_j|^{\ell - \ell_j} & x_j \text{ is (AP)}, \\ |x_j - f^{n_j - n_{j+1}}(c_{j+1})|^{\ell - 1} \cdot |x_j - c_j|^{1 - \ell_j} & x_j \text{ is (NAP)}. \end{cases}$$

Proposition 3.1 *Given f such that (1) holds, there exist $\sigma_0 > 0$, $\epsilon_0 > 0$, $K > 0$ and a function $E\colon \mathbb{N} \to \mathbb{R}$ with $E(k) \to \infty$ as $k \to \infty$, such that for each $x \in J_f$ and $n \in \mathbb{N}$ for which $x, \dots, f^{n-1}(x) \notin$ Crit the following properties hold.*

1. *Let $0 = n_s < \dots < n_1$ be the critical times before time $n_0 = n$. Let $\ell = \ell(x)$ and c_i, A_i, B_i be as above. Then for any $0 \leq i < j \leq s$,*

$$|Df^{n_i - n_j}(x_j)|^\ell \geq \frac{A_i}{B_j} \prod_{k=i}^{j-1} \left(K_k \cdot |Df^{n_k - n_{k+1}}(f(c_{k+1}))| \right),$$

where $K_0 = K$ and for each $k = 1, \dots, s-1$,

$$K_k \geq \begin{cases} K \cdot [\mathit{dist}(f^{n_k}(x), \mathit{Crit})]^{\ell_k - \ell} & \text{if both } \ell_k > \ell \text{ and} \\ & \quad n_k \text{ is of type (AP)}, \\ K & \text{otherwise.} \end{cases}$$

By definition of $\mathit{Crit}'(x)$ the former happens only finitely often. Moreover, except for at most a bounded number of i,

$$K \cdot |Df^{n_i - n_{i+1}}(f(c_{i+1}))| \geq 2 \tag{7}$$

or

$$K^2 \cdot |Df^{n_i - n_{i+1}}(f(c_{i+1}))| \cdot |Df^{n_{i+1} - n_{i+2}}(f(c_{i+2}))| \geq 4 \tag{8}$$

or

$$|Df^{n_i - n_j}(x_j)|^{\ell} \geq E(n_i - n_j) \cdot \frac{A_i}{B_j}. \tag{9}$$

2. *There exist arbitrarily large times n for which either $f^n(x)$ is in case (AP) or $r_n(x) \geq \sigma_0$ (so n is a σ_0-big time).*

3. *If $r_n(x) \geq \sigma_0$, then $A_0 \geq \sigma_0^{\ell - \ell_0}$. If $f^n(x)$ is in case (AP) and moreover the critical point nearest to $f^n(x)$ has order $\ell(x) = \ell$ then $A_0 = 1$.*

4. *If $\epsilon \in (0, \epsilon_0)$ and $n \geq 0$ is such that $f^n(x) \in N_\epsilon(Crit)$, but for no critical time $k < n$, $f^k(x) \in N_\epsilon(Crit)$, then $f^n(x)$ is in case (AP).*

Proof of Theorem 1.2 in the real case: By Assertion 2 of Proposition 3.1 there are infinitely many times n which are either of case (AP) or for which $f^n(T_n)$ contains a σ_0-neighbourhood of $f^n(x)$. If the latter holds, taking $n_0 = n$ arbitrarily large in this way, gives by Assertion 3 of Proposition 3.1 that $A_0 \geq \sigma_0^{\ell_{\max} - 1} > 0$ independently of n. If the former holds, then ℓ is defined to be the largest order of a critical which is involved with infinitely many critical times of type (AP). So we can take n_0 arbitrarily large such that $\ell_0 = \ell$. This implies again by Assertion 3 of the Proposition 3.1 that $A_0 = 1$.

Recall that $n_s = 0$. Therefore n_s is of type (AP) by "default" and $B_s = |x - c_s|^{\ell - \ell_s}$ constitutes a single large denominator (if $\ell_s > \ell$), which is represented in the first factor $K'(x)$. Using Assertion 1 of Proposition 3.1, inequality (3) follows.

Let $m_0 \in \mathbb{N}$ be so large that $E(m) > 2$ for all $m \geq m_0$, where E is the function from Proposition 3.1. Then we can take $\lambda := 2^{1/m_0} > 1$, independently of x. If (7) holds, then obviously $D_{n_i - n_{i+1}} \geq 2 > \lambda$. A similar argument works if (8) holds. If (9) holds instead, then take $j = i + m_0$. The case that $i + m_0 > s$ needs obviously be dealt with only once, and in this case we can bring the single constant $E(n_i - n_s)$ (independently of x) into the first factor $K'(x)$. Otherwise

$$\prod_{k=i}^{j-1} |Df^{n_k - n_{k+1}}(x_{k+1})| = |Df^{n_i - n_j}(x_j)| \geq E(m_0) \frac{A_i}{B_j} \geq \lambda^{m_0} \frac{A_i}{B_j}.$$

Thus we can replace each factor $|Df^{n_k-n_{k+1}}(x_k)|$ by $\lambda > 1$ and keep a valid estimate in Theorem 1.2.

The constant $K'(x)$ results from the finite number of factors when $K_k \geq K$ fails (and so for which $f^n(x)$ is close to a critical point of order $> \ell(x)$). □

Proof of Theorem 1.3 in the real case: From Assertion 4. of Proposition 3.1 and the definition of n, it follows that $f^n(x)$ is in case (AP). From Assertion 3 of Proposition 3.1 and the assumption that all critical points have the same order, it follows that $A_0 = 1$. Moreover, B_s is bounded from above. Hence (4) follows from Assertion 1 of Proposition 3.1. Since by assumption all critical points have the same order, one has $K_k \geq K$ for all k and so Theorem 1.3 follows. □

Proof of Corollary 1.3: From Proposition 3.1, with $\ell = \ell_{\max}$, we get

$$|Df^n(x)|^{\ell_{\max}} \geq \frac{A_0}{B_s} \prod_{i=0}^{s-1} K \cdot D_{n_k-n_{k+1}}(c_{k+1}).$$

Using (7)-(9) and the argument in the proof of Theorem 1.2, we obtain

$$|Df^n(x)|^{\ell_{\max}} \geq C \cdot \frac{A_0}{B_s} \cdot \lambda^s, \tag{10}$$

where C is a constant that is only needed if $n \leq m_0$ with m_0 as in the proof of Theorem 1.2. Due to Lemma 2.5, there exists $\delta = \delta(\sigma)$ such that $|f^n(T_n)| \geq \delta$ for all n. It follows that there exists $L = L(\sigma)$ such that two subsequent critical times are no more than L apart.

We have $B_s = |x - c_s|^{\ell_{\max}-\ell_s} \leq 1$ and, assuming the worst case that n_0 is (NAP), $A_0 \geq |x_0 - f^{n_0-n_1}(c_1)|^{\ell_{\max}-1}$. Since $f^{n_0-n_1}(c_1)$ is the endpoint of $f^n(T_n)$ closest to x_0, we can use One-sided Koebe and nonflatness to obtain

$$\begin{aligned} |x_0 - f^{n_0-n_1}(c_1)| &\geq KL \cdot |Df^{n_0-n_1-1}(f(c_1))| \cdot |f(x_1) - f(c_1)| \\ &\geq (KL/M) \cdot |Df^{n_0-n_1-1}(f(c_1))| \cdot |x_1 - c_1|^{\ell_1} \\ &\geq C_0\, \sigma^{\ell_{\max}}, \end{aligned} \tag{11}$$

for some $C_0 > 0$ independent of x. Using this estimate, inequality (10) becomes

$$|Df^n(x)| \geq C\sigma^{\ell_{\max}-1}\lambda^{s/\ell_{max}} \geq C\sigma^{\ell_{\max}-1}\lambda^{n/L\ell_{\max}}$$

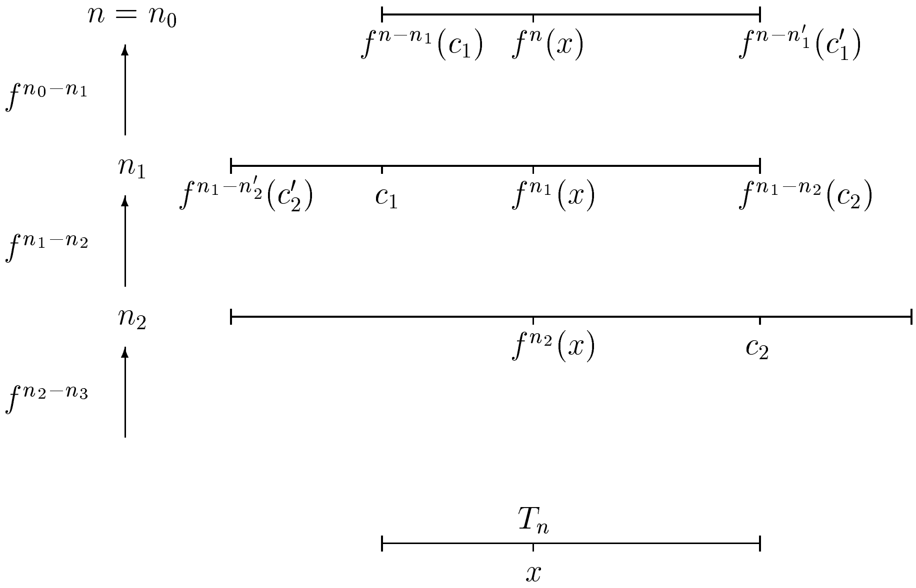

Figure 1: Critical times before n.

for the appropriate $C = C(C_0)$. This proves the corollary.

Note that if f satisfies (CE), then it is not hard to prove that λ can be chosen independently from σ. $\square$

In the remainder of this section we shall prove Proposition 3.1.

Take a point x with critical time n and so that $f^k(x) \notin \text{Crit}$ for $0 \le k \le n$. Let $0 = n_s < n_{s-1} < \ldots < n_0 = n$ be the critical times before time $n_0 = n$ defined as follows: Since T_n is the maximal neighbourhood of x on which f^n is diffeomorphic, there exist $c_1, c'_1 \in \text{Crit}$ and integers $n_1, n'_1 > 0$ such that $f^n(T_n) = (f^{n-n_1}(c_1), f^{n-n'_1}(c'_1))$ and such that $\text{dist}(f^{n-n_1}(c_1), f^n(x)) \le \text{dist}(f^{n-n'_1}(c'_1), f^n(x))$.

Next, for $T_{n_1}(x)$, the maximal neighbourhood of x on which f^{n_1} is diffeomorphic, we take the corresponding critical points $c_2, c'_2 \in \text{Crit}$ and integers $n_2, n'_2 > 0$, such that $\text{dist}(f^{n_1-n_2}(c_2), f^{n_1}(x)) \le \text{dist}(f^{n_1-n'_2}(c'_2), f^{n_1}(x))$, etc. See Figure 1.

Lemma 3.1 *(Transfer Expansion) There exists a universal constant $K > 0$*

such that
$$\frac{|Df^{n_0-n_1}(x_1)|^\ell}{|Df^{n_0-n_1}(f(c_1))|} \geq K \cdot \frac{A_0}{B_1} \tag{12}$$
where
$$A_0 = \begin{cases} |x_0 - f^{n_0-n_1}(c_1)|^{\ell-\ell_0} & x_0 \text{ in case (AP),} \\ |x_0 - f^{n_0-n_1}(c_1)|^{\ell-1}|f^{n_0-n_1}(c_1) - c_0|^{1-\ell_0} & \text{otherwise,} \end{cases}$$
and
$$B_1 = \begin{cases} |x_1 - c_1|^{\ell-\ell_1} & x_1 \text{ in case (AP),} \\ |x_1 - f^{n_1-n_2}(c_2)|^{\ell-1} \cdot |x_1 - c_1|^{1-\ell_1} & \text{otherwise.} \end{cases}$$

Proof: We shall denote the order of the critical point c_i by ℓ_i. We first derive three general estimates, see (13)-(15) below. Since $x_0 = f^n(x)$ is nearer to $f^{n-n_1}(c_1)$ than to the other boundary point of $f^n(T_n)$, we can apply the one-sided Koebe lemma and get
$$|Df^{n_0-n_1-1}(f(x_1))| \geq KL \cdot |Df^{n_0-n_1-1}(f(c_1))| \tag{13}$$
and
$$|Df^{n_0-n_1-1}(f(x_1))| \geq KL \cdot \frac{|f^{n_0-n_1}(x_1) - f^{n_0-n_1}(c_1)|}{|f(x_1) - f(c_1)|}. \tag{14}$$
Moreover, write $(f^{n_1-n_2}(c_2), f^{n_1-n_2'}(c_2')) = f^{n_1}(T_{n_1}) \ni c_1$. Choose either $t \subset (c_1, f^{n_1-n_2}(c_2))$ or $t \subset (c_1, f^{n_1-n_2'}(c_2'))$ maximal so that $f^{n_0-n_1}|t$ is a diffeomorphism and so that $x_1 \in t$. Let l, r be the components of $t \setminus \{x\}$. Note that if x_1 and $f^{n_1-n_2}(c_2)$ lie on the same side of c_1 then we take the former, and otherwise the second. In both cases we have $\min(|l|,|r|) \leq \min\{|x_1 - c_1|, |c_1 - f^{n_1-n_2}(c_2)|\}$ because by construction $f^{n_1-n_2}(c_2)$ lies nearer to x_1 than $f^{n_1-n_2'}(c_2')$ and in the second case c_1 lies between x_1 and $f^{n_1-n_2}(c_2)$. Now we use Lemma 2.3 to derive the following inequality
$$|Df^{n_0-n_1}(x_1)| \geq \frac{|f^{n_0-n_1}(l)| \cdot |f^{n_0-n_1}(r)|}{|f^{n_0-n_1}(t)|} \cdot \frac{|t|}{|l| \cdot |r|}.$$
Since at least one of the two intervals $f^{n_0-n_1}(l), f^{n_0-n_1}(r)$ has length $\geq |f^{n_0-n_1}(t)|/2$ and since $|t| \geq |l|, |r|$, one has that this last expression is at least
$$\geq (1/2) \cdot \min\left\{|f^{n_0-n_1}(l)|, |f^{n_0-n_1}(r)|\right\} \frac{1}{\min\{|l|, |r|\}}.$$

Because $f^{n_0-n_1}(c_1)$ is the endpoint of $f^n(T_n)$ nearest to x_0, and the above estimate, this gives

$$|Df^{n_0-n_1}(x_1)| \geq (1/2) \cdot \frac{|x_0 - f^{n_0-n_1}(c_1)|}{\min\{|x_1 - f^{n_1-n_2}(c_2)|, |x_1 - c_1|\}}. \tag{15}$$

For the remainder of the proof of Lemma 3.1 we merely use the non-flatness at critical points and inequalities (13)-(15). We distinguish two cases:

(AP) case. Let us first assume that $x_0 = f^n(x)$ is in case (AP). In this case, we first use the chain rule and (13) to get that $|Df^{n_0-n_1}(x_1)|^\ell$ is at least

$$KL \cdot |Df(x_1)|^\ell \cdot |Df^{n_0-n_1-1}(f(x_1))|^{\ell-1} \cdot |Df^{n_0-n_1-1}(f(c_1))|.$$

Now using non-flatness and applying (14) to the last but one factor, this is at least

$$K_1 \cdot |x_1 - c_1|^{(\ell_1-1)\ell} \cdot \frac{|f^{n_0-n_1}(c_1) - x_0|^{\ell-1}}{|f(c_1) - f(x_1)|^{\ell-1}} \cdot |Df^{n_0-n_1-1}(f(c_1))|.$$

Since we are in case (AP), this is at least

$$\geq \frac{K_1}{2^{\ell_0-1}} \cdot \frac{|f^{n_0-n_1}(c_1) - x_0|^{\ell-\ell_0} \cdot |f^{n_0-n_1}(c_1) - c_0|^{\ell_0-1}}{|x_1 - c_1|^{(1-\ell_1)\ell} \cdot |f(c_1) - f(x_1)|^{\ell-1}} \cdot |Df^{n_0-n_1-1}(f(c_1))|$$

and by non-flatness this gives

$$\frac{|Df^{n_0-n_1}(x_1)|^\ell}{|Df^{n_0-n_1}(f(c_1))|} \geq K \cdot \frac{|f^{n_0-n_1}(c_1) - x_0|^{\ell-\ell_0}}{|x_1 - c_1|^{\ell-\ell_1}}, \tag{16}$$

which is valid if x_1 is in case (AP) or in (NAP). If x_1 is in case (NAP), i.e. x_1 is closer to $f^{n_0-n_1}(c_2)$ than to c_1, then we can sharpen the above inequality: (15) can be written as

$$|Df^{n_0-n_1}(x_1)|^{\ell-1} \geq (1/2)^{\ell-1} \cdot \frac{|f^{n_0-n_1}(c_1) - x_0|^{\ell-\ell_0} \cdot |f^{n_0-n_1}(c_1) - x_0|^{\ell_0-1}}{|x_1 - f^{n_1-n_2}(c_2)|^{\ell-1}}.$$

Hence, if x_1 is in case (NAP),

$$|Df^{n_0-n_1}(x_1)|^\ell = |Df^{n_0-n_1}(x_1)|^{\ell-1} \cdot |Df^{n_0-n_1-1}(x_1)| \cdot |Df(x_1)|$$

can be estimated by using the previous inequality in the first factor, using (13) in the second factor, and by using that $|Df(f^{n_0-n_1}(c_1))| \approx |f^{n_0-n_1}(c_1) - c_0|^{\ell_0-1} \le 2^{\ell_0-1}|f^{n_0-n_1}(c_1) - x_0|^{\ell_0-1}$ (because of the assumption on x_0). This gives

$$\frac{|Df^{n_0-n_1}(x_1)|^\ell}{|Df^{n_0-n_1}(f(c_1))|} \ge K \cdot \frac{|f^{n_0-n_1}(c_1) - x_0|^{\ell-\ell_0}}{|x_1 - f^{n_1-n_2}(c_2)|^{\ell-1}|x_1 - c_1|^{1-\ell_1}}. \tag{17}$$

(NAP) case. Let us now assume that $x_0 = f^{n_0}(x)$ is in case (NAP). In this case we first use the chain rule, non-flatness, and (13) to get that $|Df^{n_0-n_1}(x_1)|^\ell$ is at least a universal constant times

$$|Df^{n_0-n_1}(x_1)|^{\ell-1} \cdot |Df^{n_0-n_1}(f(c_1))| \cdot \frac{|x_1 - c_1|^{\ell_1-1}}{|f^{n_0-n_1}(c_1) - c_0|^{\ell_0-1}}.$$

Applying (15) to the first factor gives that this is at least a universal constant times

$$\left(\frac{|x_0 - f^{n_0-n_1}(c_1)|}{\min\{|x_1 - f^{n_1-n_2}(c_2)|, |x_1 - c_1|\}}\right)^{\ell-1} \cdot \frac{|x_1 - c_1|^{\ell_1-1}}{|f^{n_0-n_1}(c_1) - c_0|^{\ell_0-1}} \cdot |Df^{n_0-n_1}(f(c_1))|.$$

That is,

$$\frac{|Df^{n_0-n_1}(x_1)|^\ell}{|Df^{n_0-n_1}(f(c_1))|} \ge K \cdot$$

$$\frac{|x_0 - f^{n_0-n_1}(c_1)|^{\ell-1} \cdot |f^{n_0-n_1}(c_1) - c_0|^{1-\ell_0}}{\left[\min\left(|x_1 - f^{n_1-n_2}(c_2)|, |x_1 - c_1|\right)\right]^{\ell-1} \cdot |x_1 - c_1|^{1-\ell_1}}. \tag{18}$$

If x_1 is as in case (AP) then this becomes

$$\frac{|Df^{n_0-n_1}(x_1)|^\ell}{|Df^{n_0-n_1}(f(c_1))|} \ge K \cdot \frac{|x_0 - f^{n_0-n_1}(c_1)|^{\ell-1} \cdot |f^{n_0-n_1}(c_1) - c_0|^{1-\ell_0}}{|x_1 - c_1|^{\ell-\ell_1}}, \tag{19}$$

whereas if x_1 is as in case (NAP) then this becomes

$$\frac{|Df^{n_0-n_1}(x_1)|^\ell}{|Df^{n_0-n_1}(f(c_1))|} \ge K \cdot \frac{|x_0 - f^{n_0-n_1}(c_1)|^{\ell-1} \cdot |f^{n_0-n_1}(c_1) - c_0|^{1-\ell_0}}{|x_1 - f^{n_1-n_2}(c_2)|^{\ell-1} \cdot |x_1 - c_1|^{1-\ell_1}}. \tag{20}$$

This completes the proof of the lemma. □

Proof of part 1 of Proposition 3.1: For $Df^{n_1-n_2}(x_2)$ the analogous statement to (12) holds, replacing $x_0, c_0, c_1, n_0-n_1, x_1$ by $x_1, c_1, c_2, n_1-n_2, x_2$. Let us now analyse what we can say about

$$|Df^{n_0-n_2}(x_2)| = |Df^{n_0-n_1}(x_1)| \cdot |Df^{n_1-n_2}(x_2)| \geq K^2 \frac{A_0}{B_1} \frac{A_1}{B_2}$$

by considering the term A_1/B_1. We distinguish several cases.

If x_1 in case (AP) and $\ell \geq \ell_1$, then

$$\frac{A_1}{B_1} = \frac{|f^{n_1-n_2}(c_2) - x_1|^{\ell-\ell_1}}{|x_1 - c_1|^{\ell-\ell_1}} \geq 1. \tag{21}$$

Also,

x_1 in case (AP) and $\ell < \ell_1$ implies

$$\frac{A_1}{B_1} = \frac{|f^{n_1-n_2}(c_2) - x_1|^{\ell-\ell_1}}{|x_1 - c_1|^{\ell-\ell_1}} \geq [\mathrm{dist}(x_1, \mathrm{Crit})]^{\ell_1-\ell}. \tag{22}$$

Moreover,

x_1 in case (NAP) implies

$$\frac{A_1}{B_1} = \frac{|x_1 - f^{n_1-n_2}(c_2)|^{\ell-1} \cdot |f^{n_1-n_2}(c_2) - c_1|^{1-\ell_1}}{|x_1 - f^{n_1-n_2}(c_2)|^{\ell-1} \cdot |x_1 - c_1|^{1-\ell_1}} \geq 2^{1-\ell_1}. \tag{23}$$

Because of these inequalities and similar ones for A_k/B_k for $k = 2, \ldots, s-1$,

$$|Df^{n_i-n_j}(x_j)|^\ell \geq \frac{A_i}{B_j} \prod_{k=i}^{j-1} K_k \cdot |Df^{n_k-n_{k+1}}(f(c_k))|, \tag{24}$$

where $K_k := K \cdot A_k/B_k$ are bounded below in the way described in the statement of Assertion 1 of the proposition. (So for all but a finite number, $K_k \geq K2^{1-\ell}$ is uniformly bounded away from 0.) Note that we have not used the assumption about the growth rate of derivatives or any assumption on x to obtain this inequality.

Let us now prove the second part of Assertion 1. Take $\rho > 0$ so small that $K \cdot |Df^k(f(c))| \geq 2$ for each $k \geq \nu(\rho)$ and each $c \in \mathrm{Crit}$. Assume that the first alternative fails, that is, assume $K \cdot |Df^{n_i-n_{i+1}}(f(c_{i+1}))| < 2$. Then $n_i - n_{i+1} \leq \nu(\rho)$. Hence $|c_i - f^{n_i-n_{i+1}}(c_{i+1})| \geq \rho$ and so either $|x_i - f^{n_i-n_{i+1}}(c_{i+1})| \geq \rho/2$ or $|x_i - c_i| \geq \rho/2$. If $|x_i - f^{n_i-n_{i+1}}(c_{i+1})| \geq \rho/2$, then $f^{n_i-n_j}(T_{n_i-n_j}(x_j))$ contains a $\rho/2$-neighbourhood of x_i. Let t denote the neighbourhood of x_j for which $f^{n_i-n_j}(t)$ is equal to a $\rho/4$-neighbourhood of x_i. Then using the Koebe Lemma,

$$|Df^{n_i-n_j}(x_j)|^\ell \geq KL \cdot \frac{(\rho/2)^\ell}{|t|^\ell} \geq KL' \cdot \frac{A_i}{|t|^\ell}.$$

Because t is contained in $B(x_j, \mathrm{dist}(x_j, f^{n_j-n_{j+1}}(c_{j+1})))$, this is much larger than A_i/B_j when t is small. Since f has no wandering intervals, t is small when $n_i - n_j$ is large. It follows that there exists a function $E'_\rho \colon \mathbb{N} \to \mathbb{R}$ with $\lim_{k\to\infty} E'_\rho(k) = \infty$ such that

$$|x_i - f^{n_i-n_{i+1}}(c_{i+1})| \geq \rho/2 \text{ implies } |Df^{n_i-n_j}(x_j)|^\ell \geq E'_{\rho/2}(n_i-n_j)\frac{A_i}{B_j}. \quad (25)$$

Let us now consider the case that $|c_i - x_i| \geq \rho/2$ and $|f^{n_i-n_{i+1}}(c_{i+1}) - x_i| < \rho/2$. Therefore there exists $\rho' = \rho'(\rho)$ such that $|x_{i+1} - c_{i+1}| < \rho'$ and $\rho' \to 0$ as $\rho \to 0$. Now if $|f^{n_{i+1}-n_{i+2}}(c_{i+2}) - x_{i+1}| \geq \rho'(\rho)$. then we can repeat the previous argument (replacing $\rho/2$ by ρ') and obtain that $|Df^{n_{i+1}-n_j}(x_j)|^\ell \geq E''_{\rho'}(n_{i+1}-n_j)\frac{A_{i+1}}{B_j}$, for some appropriate function $E''_{\rho'}$. Since x_{i+1} is in case (AP), we find by (21) and Lemma 3.1 that $|Df^{n_i-n_{i+1}}(x_{i+1})|^\ell \geq KA_i/B_{i+1} \geq K$, with at most finitely many exceptions when $\ell_1 > \ell$. If $\ell \geq \ell_{i+1}$, then we get

$$\begin{aligned} &|f^{n_{i+1}-n_{i+2}}(c_{i+2}) - x_{i+1}| \geq \rho' \text{ implies} \\ &\qquad |Df^{n_i-n_j}(x_j)|^\ell \geq K \cdot E''_{\rho'}(n_i-n_j)\frac{A_i}{B_j}. \end{aligned} \quad (26)$$

Together (25) and (26) establish (9) for $E_\rho = \min\{E'_{\rho/2}, K \cdot E''_{\rho'}\}$.

The last case is when $|f^{n_{i+1}-n_{i+2}}(c_{i+2}) - x_{i+1}| < \rho'$, which implies that $|f^{n_{i+1}-n_{i+2}}(c_{i+2}) - c_{i+1}| < 2\rho'$. Therefore $n_{i+1} - n_{i+2} \geq \nu(2\rho')$, and if ρ' is sufficiently small

$$|Df^{n_{i+1}-n_{i+2}}(f(c_{i+2}))| \geq \frac{4}{K^2}|Df^{n_i-n_{i+1}}(f(c_{i+1})|^{-1}.$$

This establishes formula (8), and completes the proof of Assertion 1 of Proposition 3.1. $\square$

Proof of Assertions 2-4 of Proposition 3.1: Let us first prove Assertion 2 by contradiction. Let $n(j)$ be the critical times of x (there are always infinitely many critical times). Assume by contradiction that for each $\sigma_0 > 0$, there exists j_0 so that $r_{n(j)} < \sigma_0$ and $n(j)$ is in case (NAP) for all $j \geq j_0$. Write $L_{n(j)} := (c_{n(j)}, f^{n(j)}(x))$.

Since there are no wandering intervals, we have the following property (Lemma 2.5): Given σ_0, there exists σ_0' such that for each interval J of diameter $\geq \sigma_0'$, the images $|f^n(J)| \geq \sigma_0$ for all $n \geq 0$. Moreover, the smallest such σ_0' can be taken small if σ_0 is. On the other hand, we will always take

$$\frac{\inf\{|Df^n(f(c))|; n \geq 1, c \in \mathrm{Crit}\}}{KL} \cdot \sigma_0' > \sigma_0, \tag{27}$$

where KL is given below; it is a uniform constant, depending only of the non-flatness constant M and a Koebe constant for relative Koebe space 1.

Now if $\mathrm{diam}(L_{n(j)}) > \sigma_0'$, then $\mathrm{diam}(f^{n(j+1)-n(j)}(L_{n(j)})) > \sigma_0$, and since $n(j+1)$ is assumed to be in case (NAP), $\mathrm{diam}(f^{n(j+1)-n(j)}(L_{n(j)}))$ contains the critical point $c_{n(j+1)}$. It follows that if $\mathrm{diam}(L_{n(j)}) > \sigma_0'$ for all $j \geq j_0$, then the component of $f^{n(j_0)}(T_{n(j)} \setminus \{x\})$ that does not contain the critical point is wandering, a contradiction. Therefore there are infinitely many j such that $\mathrm{diam}(L_{n(j)}) \leq \sigma_0'$

Take $j_2 \gg j_1 \geq j_0$ such that $\mathrm{diam}(L_{n(j)}) \leq \sigma_0'$ for both $j = j_1$ and $j = j_2$. Fix $n_0 = n := n(j_2)$ and write $x_0 = f^n(x)$ and let $0 = n_s < \ldots < n_1$ be the critical times before n_0. Let σ_{n_j} be the maximum of the distances between the points $f^{n_j}(x) = x_j$, $f^{n_j - n_{j+1}}(c_{j+1})$ and c_j. By the One-sided Koebe Lemma 2.2, there exists a universal constant KL such that

$$\begin{aligned}
|x_i - c_i|^{\ell_i} &\leq KL \cdot \frac{|x_{i-1} - f^{n_{i-1}-n_i}(c_i)|}{|Df^{n_{i-1}-n_i-1}(f(c_i))|} \\
&\leq KL \cdot \frac{|x_{i-1} - f^{n_{i-1}-n_i}(c_i)|}{|Df^{n_{i-1}-n_i-1}(f(c_i))|} \cdot \frac{|f^{n_{i-1}-n_i}(c_i) - c_{i-1}|^{\ell_{i-1}-1}}{|Df(f^{n_{i-1}-n_i}(c_i))|} \\
&= KL \cdot \frac{|x_{i-1} - f^{n_{i-1}-n_i}(c_i)| \cdot |f^{n_{i-1}-n_i}(c_i) - c_{i-1}|^{\ell_{i-1}-1}}{|Df^{n_{i-1}-n_i}(f(c_i))|}.
\end{aligned}$$

Hence, for any $0 < i \leq s$,

$$|x_i - c_i|^{\ell_i} \leq \frac{|x_{i-1} - f^{n_{i-1}-n_i}(c_i)| \cdot |c_{i-1} - f^{n_{i-1}-n_i}(c_i)|^{\ell_{i-1}-1}}{|Df^{n_{i-1}-n_i}(f(c_i))|/KL} \tag{28}$$

We assume that σ_0 and hence σ_0' is so small that $n_{i-1}-n_i \geq \nu({\sigma_0'}^{1/\ell_{\max}})$ implies that the denominator $|Df^{n_{i-1}-n_i}(f(c_i))|/KL \geq 4^{\ell_{\max}}$. Then , if $|x_i - c_i|^{\ell_i} < \sigma_0'$, and since x_i is in case (NAP), we find

$$\sigma_{n_i} \leq 2|x_i - c_i| \leq (1/2) \cdot \sigma_{n_{i-1}}^{\ell_{i-1}/\ell_i}. \tag{29}$$

If on the other hand $|x_i - c_i|^{\ell_i} \geq \sigma_0'$, then, because $|x_{i-1} - f^{n_{i-1}-n_i}(c_i)| < \sigma_0$,

$$|Df^{n_{i-1}-n_i}(f(c_i))| \cdot \sigma_0' \leq KL \cdot \sigma_0 \cdot \sigma_{i-1}^{\ell_i - 1},$$

which contradicts (27). Therefore (29) holds and

$$\sigma_{n(j_1)} \leq (1/2)^{j_2 - j_1} \cdot \sigma_{n(j_2)}^{\ell_{j_2}/\ell_{j_1}}.$$

But since $\sigma_{n(j_1)}$ is some given positive number, we get a contradiction by choosing $j_2 - j_1$ arbitrarily large.

The same argument also proves Assertion 4. Indeed assume that $n = n_0$ is the smallest critical time with $f^n(x) \in N_\epsilon(\mathrm{Crit})$ and assume by contradiction that $x_0 = f^n(x)$ is in case (NAP). Then (28) implies that

$$|x_1 - c_1|^{\ell_1} \cdot \frac{|Df^{n_0-n_1}(f(c_1))|}{KL} \leq |x_0 - f^{n_0-n_1}(c_1)| \cdot |c_0 - f^{n_0-n_1}(c_1)|^{\ell_0 - 1}.$$

Either $|x_1 - c_1|^{\ell_1} < \sigma_0'$ and $|x_1 - c_1|^{\ell_1} \leq 2^{\ell_0 - 1 - \ell_{\max}}|x_0 - c_0|^{\ell_0}$, contradicting that x_0 is the first entry in $N_\epsilon(\mathrm{Crit})$. Or $|x_1 - c_1|^{\ell_1} > \sigma_0'$, but then $|x_0 - c_0| > \sigma_0$. If we take $\epsilon < \sigma_0^{\ell_{\max}}$, then x_0 does not belong to $N_\epsilon(\mathrm{Crit})$.

If n is not a critical time, then $f^n(T_n(x)) = (f^{n-n_1}(c_1), f^{n-n_1'}(c_1'))$ and $|f^{n+1}(x) - f^{n-n_1+1}(c_1)| < 2^{\ell_{\max}}|f^{n+1}(x) - f(c_0)| < 2^{\ell_{\max}}M\epsilon$. On the other hand, $|f^{n_1+1}(x) - f(c_1)| > \epsilon/M$ and the One-sided Koebe Lemma gives

$$|Df^{n-n_1}(f(c_1))| \leq KL \cdot \frac{|f^{n+1}(x) - f^{n-n_1+1}(c_1)|}{|f^{n_1+1}(x) - f(c_1)|} \leq KL \cdot M^2.$$

As before, this is impossible if $n - n_1$ is too large, but if $n_0 - n_1$ is small then $|f^{n-n_1}(c_1) - c_0| > 2\epsilon^{1/\ell_0}$ for ϵ sufficiently small. So this is impossible too.

Assertion 3. of Proposition 3.1 follows from the formulas for A_0. $\square$

4 The Complex Case

In this section we assume that $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map, and that there exists a sequence $b_n > 0$ such that

$$\exp(\sum_{n \geq 0} b_n) < 2 \quad \text{and} \quad b_n \cdot |Df^n(f(c))| \to \infty \tag{30}$$

for each $c \in \text{Crit}$. The reason that we need a stronger assumption than the corresponding assumption (1) in the real case is that the one-sided Koebe Lemma 2.2 does not hold in the complex case.

Let $N_\epsilon(\text{Crit})$ be defined as before. Take a point x such that $f^k(x) \notin \text{Crit}$ for all $k \geq 0$, fix a large integer n and let $x_0 = f^n(x)$. Our aim is to estimate $Df^n(x)$ or at least $Df^m(x)$ for some large $m < n$. To do this, we replace the maximal intervals of monotonicity $T_n(x)$ and $f^n(T_n(x))$ from the previous section by suitable disks (shrinking neighbourhoods in the terminology of [19]) using the sequence b_n from (30). For $m \leq n$, let $s_m > 0$ be maximal so that the inverse branch of $f^{-(n-m)}$ from $B(x_0, s_m)$ to a neighbourhood of $f^m(x)$ is univalent. Maximality of s_m means that that for some $k = 1, 2, \dots, n-m$ the component W_{n-k} of $f^{-k}(B(x_0, s_m))$ containing $f^{n-k}(x)$ contains a critical point c in its boundary. In this case $s_m = \text{dist}(x_0, f^k(c))$. Of course,

$$s_n \geq s_{n-1} \geq \dots \geq s_0.$$

Let $\tilde{n}_t < \tilde{n}_{t-1} < \dots < \tilde{n}_0 := n$ be the collection of integers $\tilde{n}_j \in \{0, \dots, n\}$ for which $s_{\tilde{n}_j} < s_{\tilde{n}_j+1}$ when $j > 0$. The set $W_{\tilde{n}_j+1}$ contains a critical value of f in its interior, whereas $f(W_{\tilde{n}_j}) \subset W_{\tilde{n}_j+1}$ contains this critical value only in its boundary.

Consider all the integers $\tilde{n}_j < n$ as above such that

$$\frac{s_{\tilde{n}_{j-1}}}{s_{\tilde{n}_j}} \geq \exp\left(b_{n-\tilde{n}_j}\right). \tag{31}$$

Let $n_1 < n$ be the smallest such integer $\tilde{n}_j$ (i.e. with the largest index j) and define $r_{n_0} = s_{n_1}$. By construction f^{n-n_1} maps a neighbourhood W of $f^{n_1}(x)$ univalently onto $B_{n_0} := B(x_0, r_{n_0})$. Moreover, W contains a critical point in its boundary, and $f(W)$ is contained in a set W' which is mapped univalently by f^{n-n_1-1} onto the disc $B(x_0, \exp(b_{n-n_1-1}) \cdot r_{n_0})$.

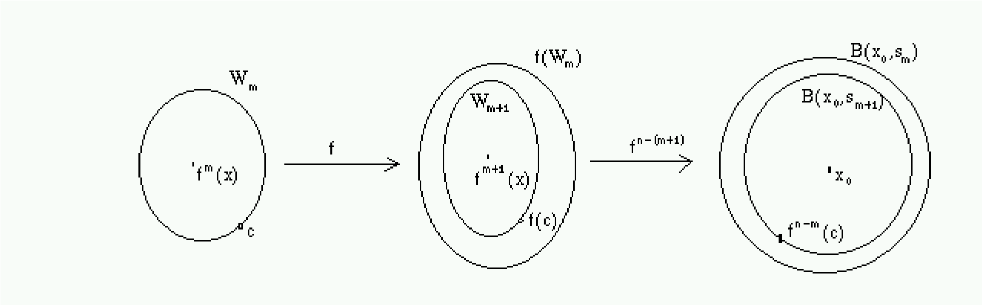

Figure 2: Shrinking neighbourhoods and their preimages.

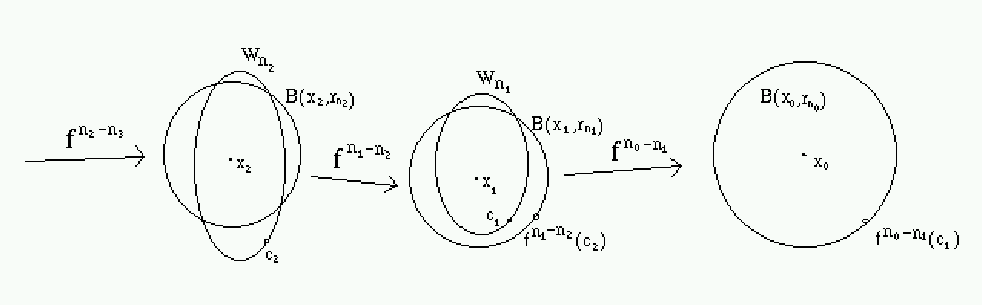

Figure 3: Construction of critical times and the discs $B_{n_j} = B(x_j, r_{n_j})$.

Claim 1: the branch of f^{-n} mapping $B(x_0, r_{n_0})$ to a neighbourhood of x is univalent on $B(x_0, r_{n_0}/2)$.

Indeed, by minimality of $n_1 = \tilde{n}_j$ one has $s_{\tilde{n}_i}/s_{\tilde{n}_{i+1}} \leq \exp(b_{n-\tilde{n}_{i+1}})$, for all $i \geq j$. But since $s_0 = s_{\tilde{n}_t}$, this implies that $s_{n_1}/s_0 = s_{\tilde{n}_j}/s_{\tilde{n}_s} \leq \exp(\sum b_i) \leq 2$. So $s_0 \geq s_{n_1}/2 = r_{n_0}/2$ and since by definition f^{-n} is univalent on $B(x_0, s_0)$ the claim follows.

We say that n_1 is a *critical time* before time $n = n_0$. Now let $0 = n_s < n_{s-1} < n_1 < n_0 = n$ where for $j = 0, \ldots, s$ the integer n_j is the critical time before n_{j-1} and for each $x_j := f^{n_j}(x)$, let $B_{n_j} = B(x_j, r_{n_j})$ be the disc defined above. The component W of $f^{-(n_j - n_{j+1})}(B_{n_j})$ containing $f^{n_{j+1}}(x)$ contains a critical point in its boundary. Call this critical point c_{j+1}. By construction we have the following properties

1. $f^{n_j - n_{j+1}}$ maps a neighbourhood $W_{n_{j+1}}$ of $f^{n_{j+1}}(x)$ univalently onto $B_{n_j} = B(f^{n_j}(x), r_{n_j})$;

2. $W_{n_{j+1}}$ contains a critical point c_{j+1} in its boundary,

3. $f(W_{n_{j+1}})$ is contained in a set W' which is mapped univalently by $f^{n_j - n_{j+1} - 1}$ onto the disc $B(f^{n_j}(x), \exp(b_{n_j - n_{j+1} - 1}) \cdot r_{n_j})$.

4. either $f^{n_j - n_{j+1}}(c_{j+1}) \in \partial B_{n_j}$ and then $n_j - n_{j+1} \geq \nu(\sigma)$, or $n_{j+1} = 0$ and then B_{n_j} is a σ-neighbourhood of x_j.

Note that there are some differences with the real case: $B_{n_{j+1}}$ need not be contained in $W_{n_{j+1}} \subset f^{-(n_j - n_{j+1})}(B_{n_j})$, whereas in the real case $f^{n_{j+1}}(T_{n_{j+1}}) \subset f^{n_{j+1}}(T_{n_j})$. Moreover, B_{n_j} need not contain a critical point, whereas in the real case $f^{n_j}(T_{n_j})$ always contains a critical point (except possibly for n_0). A consequence of this is that the critical times $n_s < n_{s-1} < \ldots < n_0$ are only defined once the integer $n_0 = n$ is given.

We say that n_j is of type

$$\begin{cases} \text{(AP) } (\textit{almost precritical}) & \text{if } r_{n_j}(x) \geq \operatorname{dist}(x_j, \operatorname{Crit}), \\ \text{(NAP) } (\textit{not almost precritical}) & \text{if } r_{n_j}(x) < \operatorname{dist}(x_j, \operatorname{Crit}). \end{cases}$$

Types (NAP) and (AP) represent the "sliding" and "transfer" case in [17]. Let $\operatorname{Crit}'(x)$ be the set of critical points for which there are infinitely many integers n_j of type (AP). In Assertion 2 of Proposition 4.1 below, we shall show that either $\operatorname{Crit}'(x) \neq \emptyset$ or there are infinitely many σ-big times n. Let

$$\ell = \ell(x) = \begin{cases} \max\{\ell_i; c_i \in \operatorname{Crit}'(x)\} & \text{if } \operatorname{Crit}'(x) \neq \emptyset, \\ 1 & \text{if } \operatorname{Crit}'(x) = \emptyset. \end{cases} \tag{32}$$

Lemma 4.1 *(Transfer Expansion in the Complex Case)*

$$\frac{|Df^{n_0 - n_1}(x_1)|^\ell}{|Df^{n_0 - n_1}(f(c_1))|} \geq K \cdot b_{n_0 - n_1 - 1} \cdot \frac{A_0}{B_1} \tag{33}$$

where A_0, B_1 are as in Lemma 3.1.

Proof: The proof of Lemma 4.1 goes exactly as that of Lemma 3.1: it suffices to prove the analogues of inequalities (13)-(15). Let us first consider (13). Let g be the inverse branch of $f^{n_0 - n_1 - 1}\colon f(W_{n_1}) \to B_{n_0}$. Because of properties 1-3 above, we can apply Lemma 2.4 and obtain

$$\frac{|Dg(x_0)|}{|Dg(f^{n_0 - n_1}(c_1))|} \leq C \cdot (1 - \exp(-b_{n_0 - n_1 - 1}))^{-1} \leq \frac{C'}{b_{n_0 - n_1 - 1}}. \tag{34}$$

This gives that the constant K in (13) should be replaced by $K \cdot b_{n_0-n_1-1}$.

Next we show that the constant KL in (14) is still universal (so it does not depend on n_0 and n_1). Indeed, let B'_{n_0} be the disc centered at x_0 with half the radius of that of B_{n_0} and g the inverse branch from above. By the Koebe $\frac{1}{4}$-Lemma,

$$|Dg(x_0)| \leq K \cdot \frac{\operatorname{diam}(g(B'_{n_0}))}{r/2},$$

where K is universal, and r is the radius of B_{n_0}. Moreover, $g(B'_{n_0})$ is almost round. Since $g(B_{n_0}))$ contains $f(c_1)$ in its boundary, since $g(B'_{n_0}) \subset g(B_{n_0})$, and since $r = \operatorname{dist}(x_0, f^{n_0-n_1}(c_1))$, it follows that $\operatorname{diam}(g(B'_n))$ is at most a universal constant times $\operatorname{dist}(f(x_1), f(c_1))$ and therefore that

$$|Dg(x_0)| \leq K \cdot \frac{\operatorname{dist}(f(x_1), f(c_1))}{\operatorname{dist}(x_0, f^{n_0-n_1}(c_1))}.$$

It follows that (14) also holds in the complex case for a universal choice of constant KL.

Let us now prove the analogue (15). Let $\hat{g}$ be the inverse branch of $f^{n_0-n_1}\colon W_{n_1} \to B_{n_0}$ and B'_{n_0} be the disc from above. Note that W_{n_1} contains c_1 in its boundary. Again we have by the Koebe $\frac{1}{4}$-Lemma,

$$|D\hat{g}(x_0)| \leq K \cdot \frac{\operatorname{diam}(\hat{g}(B'_{n_0}))}{r/2}. \tag{35}$$

By Claim 1, $f^{n_0-n_2}(c_2)$ is outside B'_{n_0}. This implies that $f^{n_1-n_2}(c_2)$ is outside $\hat{g}(B'_{n_0})$. But this, (35) and the definition of r implies that

$$|D\hat{g}(x_0)| \leq K \cdot \frac{|x_1 - f^{n_1-n_2}(c_2)|}{|x_0 - f^{n_0-n_1}(c_1)|}.$$

Again using that $\hat{g}(B'_{n_0})$ is almost round and contained in the set $\hat{g}(B_{n_0}) = W_{n_1}$ which contains c_1 in its boundary also

$$|D\hat{g}(x_0)| \leq K \cdot \frac{|x_1 - c_1|}{|x_0 - f^{n_0-n_1}(c_1)|}$$

also holds. Combined with $D\hat{g}(x_0) = 1/Df^{n_0-n_1}(x_1)$ this shows that (15) holds. From these three estimates Lemma 4.1 follows. Here we use that only in one of the ℓ factors of $|Df^{n_0-n_1}(x_1)|^\ell$ one uses (13), and so the spoiling constant $b_{n_0-n_1-1}$ appears only once. $\square$

Proposition 4.1 *Given f such that (2) holds, there exist $\sigma_0 > 0$, $\epsilon_0 > 0$, $K > 0$ and a function $E\colon \mathbb{N} \to \mathbb{R}$ with $E(k) \to \infty$ as $k \to \infty$ such that for each $x \in J_f$ and $n \in \mathbb{N}$, for which $x, \dots, f^{n-1}(x) \notin Crit$ the following properties hold.*

1. *Let $0 = n_s < \ldots < n_1$ be the critical times before time $n_0 = n$. Let $\ell = \ell(x)$ and c_i be as above and let A_i, B_i be as in Proposition 3.1. Then for any $0 \le i < j \le s$,*

$$|Df^{n_i - n_j}(x_j)|^\ell \geq \frac{A_i}{B_j} \prod_{k=i}^{j-1} \left(K_k \cdot b_{n_k - n_{k+1} - 1} \cdot |Df^{n_k - n_{k+1}}(f(c_{k+1}))| \right),$$

where $K_0 = K$ and for each $k = 1, \dots, s-1$,

$$K_k \geq \begin{cases} K \cdot [dist(f^{n_k}(x), Crit)]^{\ell_k - \ell} & \text{if both } \ell_k > \ell \text{ and } \\ & \qquad n_k \text{ is of type (AP)}, \\ K & \text{otherwise.} \end{cases}$$

By definition of $Crit'(x)$ the former happens only finitely often. Moreover, with a bounded number of exceptions,

$$K \cdot b_{n_i - n_{i+1} - 1} \cdot |Df^{n_i - n_{i+1}}(f(c_{i+1}))| \geq 2, \tag{36}$$

or

$$\begin{aligned} K \cdot b_{n_i - n_{i+1} - 1} \cdot |Df^{n_i - n_{i+1}}(f(c_{i+1}))| \cdot \\ K \cdot b_{n_{i+1} - n_{i+2} - 1} \cdot |Df^{n_{i+1} - n_{i+2}}(f(c_{i+2}))| \geq 4, \end{aligned} \tag{37}$$

or

$$|Df^{n_i - n_j}(x_j)|^\ell \geq E(n_i - n_j) \cdot \frac{A_i}{B_j}. \tag{38}$$

2. *There exist arbitrarily large times n for which either $f^n(x)$ is in case (AP) or $f^n(T_n)$ contains a σ_0 neighbourhood of $f^n(x)$ (so n is a σ_0-big time).*

3. *If $f^n(T_n)$ contains a σ_0 neighbourhood of $f^n(x) = x_0$ then $A_0 \geq \sigma_0^{\ell - \ell_0}$. If $f^n(x)$ is in case (AP) and moreover the critical point nearest to $f^n(x)$ has order $\ell(x) = \ell$, then $A_0 = 1$.*

4. *If $\epsilon \in (0, \epsilon_0)$ and $n \geq 0$ is the smallest critical time so that $f^n(x) \in N_\epsilon(Crit)$, then $f^n(x)$ is in case (AP).*

Proof of Assertion 1 of Proposition 4.1: The proof of Assertion 1 of this proposition is analogous to the proof of Proposition 3.1. The first part is identical. Let us now prove analogues of (25) and (26). The proof is a bit different because now B_{n_i} does not necessarily contain W_{n_i}. Choose $\rho > 0$ so that $K \cdot b_{k-1} \cdot |Df^k(f(c))| \geq 2$ for all $k \geq \nu(\rho)$ and $c \in \text{Crit}$. (We may assume that $K \in (0,1)$.) Assume that $K \cdot b_{n_i - n_{i+1} - 1} \cdot |Df^{n_i - n_{i+1}}(f(c))| < 2$. Then $n_i - n_{i+1} \leq \nu(\rho)$ and so $|c_i - f^{n_i - n_{i+1}}(c_{i+1})| \geq \rho$ and either $|x_i - f^{n_i - n_{i+1}}(c_{i+1})| \geq \rho/2$ or $|x_i - c_i| \geq \rho/2$. If $|x_i - f^{n_i - n_{i+1}}(c_{i+1})| \geq \rho/2$, then by Claim 1, $f^{n_i - n_j}(W_{n_i - n_j}(x_j))$ contains a $\rho/4$-neighbourhood of x_i for all $j > i$. Hence, if we denote by t the neighbourhood of x_j for which $f^{n_i - n_j}(t)$ is equal to a $\rho/16$-neighbourhood of x_i, then using the Koebe Lemma 2.4,

$$|Df^{n_i - n_j}(x_j)|^\ell \geq KL \cdot \frac{(\rho/16)^\ell}{\text{diam}(t)^\ell} \geq KL' \cdot \frac{A_i}{\text{diam}(t)^\ell}.$$

Since by the Koebe $\frac{1}{4}$-Lemma, t is contained in $B(x_j, \text{dist}(x_j, f^{n_j - n_{j+1}}(c_{j+1})))$, this is much larger than $\frac{A_i}{B_j}$ when the diameter of t is small. Since $x \in J_f$, the set t has small diameter when $n_i - n_j$ is large. Because t maps with bounded distortion to a disc, t is almost a disc. It follows that there exists a function $E'_\rho \colon \mathbb{N} \to \mathbb{R}$ with $\lim_{k\to\infty} E'_\rho(k) = \infty$ such that

$$|x_i - f^{n_i - n_{i+1}}(c_{i+1})| \geq \rho/2 \text{ implies } |Df^{n_i - n_j}(x_j)|^\ell \geq E_{\rho/2}(n_i - n_j)\frac{A_i}{B_j}. \quad (39)$$

Let us now consider the case that $|c_i - x_i| \geq \rho/2$ and $|f^{n_i - n_{i+1}}(c_{i+1}) - x_i| < \rho/2$. This implies that the component $W_{n_{i+1}}$ of $f^{-(n_i - n_{i+1})}(B_{n_i})$ which contains x_{n_i} has diameter $\leq \rho' = \rho'(\rho)$ where $\rho' > 0$ is small when $\rho > 0$ is small. If $\text{dist}(c_{i+1}, x_{i+1}) \geq \rho'$, then we can repeat the above argument with ρ' instead of $\rho/2$. Hence for some appropriate function $E''_{\rho'}$ with $\lim_k E''_{\rho'}(k) = \infty$,

$$|Df^{n_{i+1} - n_j}(x_j)|^\ell \geq E''_{\rho'}(n_{i+1} - n_j)\frac{A_{i+1}}{B_j}.$$

Noting that x_{i+1} is in case (AP), we get with at most finitely many exceptions that $Df^{n_i - n_{i+1}}(x_{i+1}) \geq K$. Therefore

$$|Df^{n_i - n_j}(x_j)|^\ell \geq K \cdot E''_{\rho'}(n_{i+1} - n_j) \cdot \frac{A_{i+1}}{B_j}.$$

This proves (38) with $E_\rho = \min\{E'_{\rho/2}, K \cdot E''_{\rho'}\}$.

Finally, if $\mathrm{dist}(c_{i+1}, x_{i+1}) < \rho'$, then $|f^{n_{i+1}-n_{i+2}}(c_{i+2}) - c_{i+1}| < 2\rho'$. Therefore $n_{i+1} - n_{i+2} \geq \nu(2\rho')$, and if ρ' is sufficiently small

$$b_{n_{i+1}-n_{i+2}-1} \cdot |Df^{n_{i+1}-n_{i+2}}(f(c_{i+2}))| \geq \frac{4}{K^2 \cdot b_{n_i-n_{i+1}-1} \cdot |Df^{n_i-n_{i+1}}(f(c_{i+1})|}.$$

This establishes formula (37) and proves Assertion 1 of Proposition 4.1.

Proof of Assertions 2-4 of Proposition 4.1: The proof of Assertion 2 is slightly different from the real case: Equation (28) becomes

$$|x_i - c_i|^{\ell_i} \leq \frac{|x_{i-1} - f^{n_{i-1}-n_i}(c_i)| \cdot |f^{n_{i-1}-n_i}(c_i) - c_{i-1}|^{\ell_{i-1}-1}}{b_{n_{i-1}-n_i-1} \cdot |Df^{n_{i-1}-n_i}(f(c_i))|}. \tag{40}$$

Because we have assumed (2), we still get (29) and the remainder of the proof of Assertion 2 still holds. The proof of Assertion 3 of Proposition 4.1 is trivial (as before), and the proof of Assertion 4 goes as in the real case too. □

Proof of Theorems 1.2 and 1.3 in the complex case: The proofs of these propositions and lemma goes exactly as in the real case, except that one has to use Proposition 4.1. □

Proof of Corollary 1.3 in the complex case This is basically the same as in the real case, except that we cannot use the One-sided Koebe Lemma as in (11). Instead, $f(W_{n_1})$ is contained in a disk W' that is mapped univalently onto $B(x_0, r_{n_0} \cdot \exp(b_{n_0-n_1-1}))$. It follows

$$\begin{aligned} |x_0 - f^{n_0-n_1}(c_1)| &\geq KL \cdot b_{n_0-n_1-1} \cdot |Df^{n_0-n_1-1}(f(c_1))| \cdot |f(x_1) - f(c_1)| \\ &\geq (KL/M) \cdot b_{n_0-n_1-1} \cdot |Df^{n_0-n_1-1}(f(c_1))| \cdot |x_1 - c_1|^{\ell_1} \\ &\geq C_0\, \sigma^{\ell_{\max}}, \end{aligned}$$

for some $C_0 > 0$. This gives the missing estimate. □

5 Some Counterexamples

In this section we will show that a multimodal Collet-Eckmann map need not satisfy (BCE) or (BBC) if there are critical points with different orders.

Theorem 5.1 *For each of the following three statements, there is a bimodal Collet-Eckmann polynomial f satisfying it:*

1. *f fails both (BCE) and (BBC).*
2. *f fails (BCE), but satisfies (BBC).*
3. *f satisfies (BCE), but fails (BBC).*

Proof: To prove the first statement, we construct a bimodal map satisfying (CE) on both critical points, but for which one of them, c_1, has a sequence of preimages $y_i \in f^{-n_i}(c_1)$ such that

$$|Df^{n_i}(y_i)| \le Ce^{-\alpha n_i}$$

for some $C, \alpha > 0$ and all $i \ge 1$. This shows that (BCE) and (BBC) fail.

Let $f : [0,1] \to [0,1]$ be a bimodal map, say a fifth order polynomial, with two critical points $0 < c_1 < c_2 < 1$ such that the corresponding critical orders $\ell_1 = 2$ and $\ell_2 = 4$. Furthermore, suppose that $f(0) = 0$, $f(1) = f(c_1) = 1$. The map assumes a local minimum $f(c_2) > 0$ at c_2; its precise value will be determined in the below construction.

Assume that $f^i(c_2)$ stays out of a neighbourhood of the critical set for $1 \le i < n$, such that $Df^{n-1}(f(c_2)) \sim \lambda^{n-1}$ for some $\lambda > 1$. For example, this holds if c_2 spends most of the iterates $i < n$ very close to the fixed point $p \in (c_1, c_2)$, and this fixed point has multiplier $|Df(p)| = \lambda$. Next assume that $\epsilon := |f^n(c_2) - c_1| \approx \lambda^{-\beta n}$ for some $\beta \in (\frac{1}{3}, 1)$. Then $Df^n(f(c_2)) \approx \lambda^{(1-\beta)n}$, so the Collet-Eckmann condition is not violated here. In addition, assume that f^n maps a neighbourhood of c_2 with one fold onto a neighbourhood of c_1. Then there is a point y, say $y < c_2$, such that $f^n(y) = c_1$ and

$$|y - c_2| \approx |f(y) - f(c_2)|^{1/4} \approx (\epsilon \cdot \lambda^{-n})^{1/4} \approx \lambda^{-(1+\beta)n/4}.$$

(Notice that $|y - c_2| \gg \epsilon$ if $\beta > \frac{1}{3}$.) This gives

$$|Df^n(y)| \approx \lambda^n \cdot |y - c_2|^3 \approx \lambda^{(1-3\beta)n/4},$$

which is exponentially small if $\beta > \frac{1}{3}$. The idea is now to construct a map f exhibiting a cascade of the above events, so for which there exists a sequence n_i such that

$$|Df^{n_i-1}(f(c_2))| \approx \lambda^{n_i} \text{ and } |f^{n_i}(c_2) - c_1| \approx \lambda^{-\beta n_i}$$

for all i, and such that f^{n_i} maps a neighbourhood of c_2 with one fold onto a neighbourhood of c_1. Then we get a sequence of points $y_i \to c_2$ such that

$$f^{n_i}(y_i) = c_1 \text{ and } |Df^{n_i}(y_i)| \approx \lambda^{-\alpha n_i}$$

for $\alpha = (3\beta - 1)/4 > 0$. There are no (combinatorial) restrictions to such a cascade construction.

Let us now change the construction a little to prove the other two statements. For the second statement, choose $\epsilon := |f^n(c_2) - c_1| \approx \lambda^{-\beta n}$ for $\beta = \frac{1}{3}$. The previous computations then give $|Df^{n_i}(y_i)| \approx 1$.

For the third statement we argue as follows: Instead of having f^n map a neighbourhood of c_2 onto a neighbourhood of c_1, let $f^n(c_2)$ be very close to c_1 such that c_2 assumes a local minimum of $|f^n(x) - c_1|$. Therefore no neighbourhood of c_2 on which f^n has only one fold contains c_1 in its f^n-image. This adjustment is compatible with *long-branchedness* of f, i.e. there exists $\gamma > 0$ such that for every i and every maximal monotonicity interval J of f^i satisfies $|f^i(J)| > \gamma$.

By use of a cascade of the above construction, we arrive at a map for which there are sequences $\{n_i\} \subset \mathbb{N}$ and $\{y_i\} \subset [0,1]$ such that $y_i \to c_2$, $f^{n_i}(y_i) \to c_1$ and $|Df^{n_i}(y_i)| \leq Ce^{-n_i}$. Moreover, $|f^{n_i}(y_i) - c_1| < |f^n(y_i) - c_1|$ for all $n < n_i$. It follows that (BBC) fails. To ensure that (BCE) holds, we prove the following lemma.

Lemma 5.1 *Let f be a long-branched map with non-flat critical points and $Sf \leq 0$. Then (CE) implies (BCE).*

Proof: Take any critical point c and let U be a neighbourhood of c such that $|U| < \gamma/2$ and also $|f(U')| < \gamma/2$ for each component U' of $U \setminus \{c\}$. Here the γ comes from the definition of long-branchedness. Let $X = \cup_{n \geq 0} f^n(U)$. Obviously X is forward invariant.

By Proposition 3.1 applied to X, every critical point $\tilde{c} \in X$ of maximal critical order satisfies (BCE). Assume by contradiction that (BCE) fails in c; say there are sequences $\{n_i\} \subset \mathbb{N}$ and $\{y_i\} \subset [0,1]$ such that $f^{n_i}(y_i) = c$ and $\limsup_i \frac{1}{n_i} \log |Df^{n_i}(y_i)| \leq 0$.

Recall that T_{n_i} is the maximal neighbourhood of y_i on which f^{n_i} is diffeomorphic. By long-branchedness, n_i is a γ-big time of type (NAP). By the

Koebe Principle There exist N such that $\tilde{c} \in f^N(U)$. Take $x_i \in T_{n_i}$ such that $f^{n_i}(x_i) \in U$ and $f^{N+n_i}(x_i) = \tilde{c}$. Then, due to the Koebe Lemma 2.1,

$$|Df^{N+n_i}(x_i)| \leq KL \cdot \sup\{|Df^N(z)|; z \in U\} \cdot |Df^{n_i}(y_i)|.$$

This contradicts that (BCE) holds in $\tilde{c}$. □

This proves Theorem 5.1 □

6 Ergodic Components

In this section we prove that if f satisfies (1) or (2), then a non-atomic conformal measure $m = m_\alpha$ on the Julia set has no more that #Crit ergodic components. If in the real case m is just Lebesgue measure, then it is known that m decomposes into finitely many ergodic components on J_f, see [1, 2, 12, 13]. In the complex case, an α-conformal probability measure exists for some $0 < \alpha \leq 2$, due to Sullivan's result [21]. The finiteness of ergodic components under the summability condition has been proved by Przytycki [19]. Prado [18] proves (not using summability) that m is ergodic for certain unicritical polynomials on the complex plane.

In general, m can be atomic, but only if m gives positive mass to neutral periodic points or critical points. We will exclude these cases by assumption. So in this section let m be a non-atomic α-conformal measure, i.e. $m(f(U)) = \int_U |Df(x)|^\alpha dm$ whenever U is measurable and $f : U \to f(U)$ is one-to-one. Naturally $\mathrm{supp}(m) \subset J_f$. We start with an easy (well-known) lemma.

Lemma 6.1 *Let $J' = \{x \in J_f; \overline{\mathrm{orb}(x)} \cap Crit \neq \emptyset\}$. Then $m(J') = 1$.*

Proof: Let X_k be the set of points x in J_f such that $\mathrm{dist}(\mathrm{orb}(x), \mathrm{Crit}) \geq 1/k$. By Proposition 4.1, $x \in X_k$ has infinitely many times that are either σ_0-big or of type (AP). But as $x \in X_k$, the (AP) times are also $(1/k)$-big. Take $\sigma = \min\{\sigma_0, 1/k\}$. Therefore each $x \in X_k$ has arbitrarily small neighbourhoods U and $n \in \mathbb{N}$ such that f^n maps U onto $B(f^n(x), \sigma/2)$ with distortion depending only on σ. A definite proportion of $B(f^n(x), \sigma/2)$ is mapped into a $1/k$-neighbourhood of Crit and is therefore disjoint from X_k. It follows that X_k cannot have any density points. Since $J' = J_f \setminus (\cup_k X_k$, the lemma follows. □

Proposition 6.1 *Suppose f satisfies (1) or (2), then m has at most #Crit ergodic components.*

We will carry out the proof for the complex case; the real case goes likewise. For real f this is obviously sharp, because each critical point can be contained in a separate forward invariant cycle of intervals. In the complex case it is not clear whether #Crit is the sharp bound.

Proof: Let x be a point whose orbit accumulates on Crit, but which is not precritical, i.e. $x \in J' \setminus \cup_{n\geq 0} f^{-n}(\mathrm{Crit})$. By Lemma 6.1, this applies to m-a.e. $x \in J_f$. Let $\epsilon > 0$ be small compared to σ_0 from Propositions 4.1, let $\rho \in (0,1)$ and $\epsilon_i = \epsilon\rho^i$. Write N_i for $N_{\epsilon_i}(\mathrm{Crit})$ and $N_i(c)$ for the component of N_i containing c. Let

$$m_0 = m_0(x) = \min\{n > 0; n \text{ is critical for } x \text{ and } f^n(x) \in N_0\}.$$

Since x is not precritical, $f_0^m(x) \notin N_{t_0}$ for some minimal $t_0 = t_0(x)$. Recursively define

$$m_i = m_i(x) = \min\{n > m_{i-1}(x); n \text{ is critical for } x \text{ and } f^n(x) \in N_{t_{i-1}}\},$$

and $t_i = t_i(x)$ is minimal such that $f^{m_i}(x) \notin N_{t_i}$. By Assertion 4 of Proposition 4.1, all the $m_i(x)$ are of type (AP). Let m_i' be the critical time before m_i. By (40), we have the estimate:

$$\begin{aligned}|f^{m_i'}(x) - c_{m_i'}|^{\ell_{m_i'}} \cdot \frac{b_{m_i-m_i'-1} \cdot |Df^{m_i-m_i'}(f(c_{m_i'}))|}{KL} \\ \leq |f^{m_i}(x) - c_{m_i}| \cdot |c_{m_i} - f^{m_i-m_i'}(c_{m_i'})|^{\ell_{m_i}-1}.\end{aligned}$$

If $|f^{m_i'}(x) - c_{m_i'}|^{\ell_{m_i'}} > \sigma_0'$, then $m_i - m_i'$ can be small, but $|c_{m_i} - f^{m_i-m_i'}(c_{m_i'})| \gg \epsilon^{1/\ell_{m_i}}$. If $|f^{m_i'}(x) - c_{m_i'}|^{\ell_{m_i'}} \leq \sigma_0'$, then $m_i - m_i'$ is large and

$$b_{m_i-m_i'-1} \cdot \frac{|Df^{m_i-m_i'}(f(c_{m_i'}))}{KL} \geq \rho^{-5\ell_{\max}}.$$

Since also $|f^{m_i-m_i'}(c_{m_i}) - c_{m_i'}|^{\ell} \geq |f^{m_i'}(x) - c_{m_i}|^{\ell}$, we obtain

$$\rho^{-5} \cdot |f^{m_i'}(x) - c_{m_i'}| \leq |c_{m_i} - f^{m_i-m_i'}(c_{m_i'})|.$$

Thus $f^{m_i}(T_{m_i}(x))$ either covers $N_{t_{i-1}+5}(c_{m_i})$ for $i \geq 5$. Therefore

$$f^{m_i} : T_{m_i}(x) \cap f^{-m_i}(N_{t_{i-1}+5}(c_{m_i})) \to N_{t_{i-1}+5}(c_{m_i}) \text{ is onto,} \tag{41}$$

with uniformly bounded distortion. For each $c \in$ Crit, let $H_x(c)$ be the integers h such that $f^{m_i}(T_{m_i}(x))$ covers N_h with uniform distortion constant K. We have just proved that $\cup_c H_x(c)$ is co-finite.

Suppose by contradiction that m allows for #Crit + 1 pairwise disjoint ergodic components, say Y_j, $1 \leq j \leq \#\text{Crit} + 1$. Let $y_j \in Y_j \cap J'$ be density points of Y_j. Take N so large that for all $i \geq N$ and all $1 \leq j \leq \#\text{Crit} + 1$,

$$m(T_{m_i}(y_j) \setminus Y_j) \leq \frac{1}{10K^{1/\alpha}} m(T_{m_i}(y_j)).$$

By the pigeon hole principle, there exists j and j' and $c \in$ Crit such that $h \in H_y(c) \cap H_{y'}(c)$ for infinitely many h. Here $y = y_j$ and $y' = y_{j'}$. Take such h; assume it is larger than $t_N(y)$ and $t_N(y')$. Then by boundedness of distortion,

$$m(N_h \setminus Y_j) \leq \frac{1}{10} m(N_h) \text{ and } m(N_h \setminus Y_{j'}) \leq \frac{1}{10} m(N_h).$$

It follows that Y_j and $Y_{j'}$ are not disjoint. □

Next let us prove Corollary 1.1

Proof of Corollary 1.1: Most statements follow immediately from Theorem 1.2 applied to the periodic points in J_f. In the real case, any neutral periodic point would attract a critical point (by negative Schwarzian derivative), which is impossible under the growth condition (1). In the complex case there are no neutral periodic points in J_f either. This rules out Cremer points.

Suppose now that D is a Siegel disk. If Crit $\cap$ orb$(\partial D) = \emptyset$, then Lemma 6.1 applies to each $x \in \partial D$. Therefore there are arbitrarily small neighbourhoods U of x that map with bounded distortion onto $B(f^n(x), \sigma)$ for some fixed $\sigma > 0$. It follows that there are infinitely many $n \in \mathbb{N}$ and $y_n \in D$ such that $\text{dist}(f^n(y_n), \partial D) > \sigma'(\sigma) > 0$ and $|Df^n(y_n)| \to \infty$.

If Crit $\cap$ orb$(\partial D) \neq \emptyset$, then each $x \in \partial D$ accumulates on Crit. One can easily adapt the proof of Proposition 6.1 so that (41) holds with 5 replaced by 5#Crit. It follows that there exists $y \in \text{orb}(x) \cap \partial D$ and an iterate m

such that f^m maps $B(y;\epsilon) \subset T_m(y)$ univalently onto a neighbourhood of $B(y,\epsilon)$ for some $\epsilon = \epsilon(m) > 0$. Restricting f^m to $B(y,\epsilon) \cap D$, we find that D contains a repelling periodic point. This contradiction shows that there cannot be any Siegel disks.

The proof for Herman rings is the same. $\square$

7 Lebesgue measure of Julia sets

Let $J = J_f$ denote the Julia set or (in the real case) the set of points that is not attracted to a stable or neutral periodic orbit. In this appendix, m stands for normalized Lebesgue measure, either on the Riemann sphere or the interval (or circle).

Theorem 7.1 *If f satisfies the expansion assumptions (1) or (2), then there exists $\delta > 0$ and $K > 0$ such that for m-a.e. $x \in J$ there exists $t_i \to \infty$ and neighbourhoods $U_i(x)$ of x such that*

$$f^{t_i} : U_i(x) \to B(c, \delta^{1/\ell(c)}), \tag{42}$$

is univalent and has distortion bounded by K. Here $c \in Crit$ is the critical point closest to $f^{t_i}(x)$.

Points x that have arbitrarily small neighbourhoods that map univalently to large scale are sometimes called *conical* points. Hence Theorem 7.1 says that m-a.e. point is conical. Before proving this theorem, we give some easy corollaries.

Corollary 7.1 *If f satisfies (2) and $J \neq \hat{\mathbb{C}}$, then $m(J) = 0$.*

Proof: Suppose x is a density point of J that satisfies (42). Then there exists $c \in J$ such that f^{t_i} maps $U_i(x)$ with bounded distortion onto $B(c, \delta^{1/\ell(c)})$ for arbitrarily small neighbourhoods $U_i(x)$ and corresponding iterates t_i. It follows that $m(B(c, \delta^{1/\ell(c)}) \cap J) = m(B(c, \delta^{1/\ell(c)}))$. Now $J = \hat{\mathbb{C}}$ is immediate. $\square$

Corollary 7.2 *If f satisfies (2) and $J = \hat{\mathbb{C}}$, then m is ergodic, conservative and exact.*

Proof: Suppose X is any forward invariant set of positive measure. As in the previous corollary, there exists c such that $m(B(c,\delta^{1/\ell(c)}) \cap X) = m(B(c,\delta^{1/\ell(c)}))$, and therefore X has full measure in $\hat{\mathbb{C}}$. Hence m is ergodic and conservative. Moreover, $f^n(X) \cap f^{n+1}(X) \neq \emptyset$. It follows that m is exact as well, see e.g. [3, Proposition 2.1]. □

Corollary 7.3 *If f satisfies (1) then for m-a.e. $x \in I$, either $\omega(x)$ is a cycle of intervals or $\omega(x)$ is a periodic orbit.*

Hence, there are no solenoidal or wild attractors.

Proof: Suppose that $m(J) > 0$, otherwise there is nothing to prove. Since J is closed and invariant, J contains an (at least one-sided) interval $B(c,\delta^{1/\ell(c)})$ for some $c \in J$. Consequently, J is the union of cycles of intervals. If there exists an open interval U in any of these cycles, call it J_0, such that $m(\{x \in J_0; \omega(x) \cap U = \emptyset\}) > 0$, then some interval of the form $B(c,\delta^{1/\ell(c)})$ contains a preimage V of U. This means m-a.e. $x \in J_0$ has arbitrarily small neighbourhoods $U_i(x)$ of which a definite proportion belongs to $\cup_n f^{-n}(U)$. Therefore $\{x \in J_0; \omega(x) \cap U = \emptyset\}$ has no density points, a contradiction. □

Corollary 7.4 *If f satisfies (2) then either $\omega(x)$ is an attracting periodic orbit m-a.e., or $\omega(x) = \hat{\mathbb{C}}$ m-a.e.*

Proof: Similar to the proof of Corollary 7.3 □

Proof of Theorem 7.1: We will carry out the proof for the complex case. The real case is proven in the same way.

Let $N_i(c) = B(c,\epsilon_i^{1/\ell(c)})$, where $\epsilon_i = \epsilon\rho^i$ as in Proposition 6.1. The numbers $\epsilon > 0$ and $\rho \in (0,1)$ will be chosen in the course of the proof. By convention, $N_{-1} = \hat{\mathbb{C}}$. For $i \geq 0$, we denote the annuli $N_i(c) \setminus N_{i+1}(c)$ by $A_i(c)$. Let $J' = \{z \in J; \overline{\mathrm{orb}(z)} \cap \mathrm{Crit} \neq \emptyset\} \setminus \cup_n f^{-n}(\mathrm{Crit})$. We know by Lemma 6.1 that $m(J \setminus J') = 0$.

For any $\beta \in \mathbb{N}$ (for simplicity we take β even), there exists $\epsilon > 0$ such that the following holds: Given $x \in J' \cap N_i$, there exists a minimal $m = m(x) > 0$ such that $f^m(x) \in N_{i+1} \cap N_{\beta+1}$ and for this m there exists a neighbourhood $U(x)$ such that f^m maps $U(x)$ univalently and with bounded distortion onto $N_{i-\beta}(c)$, where c is the critical point closest to $f^m(x)$.

The distortion bound is due to the fact that there is a larger neighbourhood $T(x)$ such that f^m maps $U(x)$ univalently onto $N_{i-\beta-1}(c)$. Therefore the distortion tends to 1 as $\rho \to 0$. This property was shown in Proposition 6.1. Note that each m is of type (AP), and therefore the expansion formula of Theorem 1.2. Hence, if ϵ is small, $|Df^m(x)|$ is large, so we can assume that $T(x) \subset N_{i-1}$, provided $x \in A_i$, $i \geq 0$.

Let $V(x) \subset U(x)$ be such that $f^{m(x)}(V(x)) = N_{i-\beta+1}(c)$. Define

$$\tau(x) = \tau_1(x) = \left\{ \begin{array}{ll} m(y); & [x \in V(y)] \text{ or } [x \in U(y) \text{ and} \\ & U(y) \supset U(y') \text{ for all } y' \text{ such that } x \in V(y')] \end{array} \right\}$$

and

$$\Phi(x) = \{f^t(x); t \in \tau(x)\}.$$

Note that in general Φ is a multivalued function; this is intentional.

Lemma 7.1 *If ρ is sufficiently small, then $V(y) \cap V(y') \neq \emptyset$ implies $U(y) \subset U(y')$ or $U(y') \subset U(y)$.*

Proof: As ρ tends to 0, the distortion of $f^{m(y)}|U(y)$ tends to 1, and at the same time $\mathrm{diam}(V(y)) \to 0$ and $\mathrm{dist}(\partial U(y), \partial V(y))/\mathrm{diam}(U(y))$ is bounded away from 0. Assume by contradiction that $U(y) \neq U(y')$. Assume without loss of generality that $m(y) < m(y')$. If $V(y)$ intersects $V(y')$ and $U(y') \supset V(y)$, then $f^{m(y)}(U(y')) \ni c$ for some critical point, so $f^{m(y')}|U(y')$ cannot be univalent. Therefore $U(y') \not\supset V(y)$. This means that $\mathrm{diam}(U(y')) \approx \mathrm{diam}(V(y)) \ll \mathrm{diam}(U(y))$, and $U(y') \subset U(y)$. Note that we proved at the same time that the transfer time of the larger set is smaller. $\square$

Define

$$\tau_2(x) = \left\{ \begin{array}{ll} m(y) + t; & t \in \tau_1(x), [f^t(x) \in V(y)] \text{ or } [f^t(x) \in U(y) \text{ and} \\ & U(y) \supset U(y') \text{ for all } y' \text{ such that } f^t(x) \in V(y)] \end{array} \right\}$$

and

$$\Phi^2(x) = \{f^t(x); t \in \tau_2(x)\}.$$

Assume that z is such that $x \in U(z) \cap N_i$, $m(z) = t$, and $f^t(x) \in A_j(c) \cap U(y)$ for some $j \geq 1$ and $y \in \hat{\mathbb{C}}$. Because $T(y) \subset N_{j-1}(c)$ and $f^t(T(z)) = N_{j-\beta-1}(c)$, there is a neighbourhood $T \ni x$ such that $f^{t+m(y)}$ maps T univalently on $N_{j-\beta-1}(c')$, where c' is the critical point closest to $f^{t+m(y)}(x)$.

Similarly we can define τ_n and Φ^n for every $x \in J'$ and $n \in \mathbb{N}$. The above properties hold for every iterate and every branch of Φ^n. If $x \notin J$, then we can define τ_n as above as long as there are $y \in J$ such that $f^t(x) \in U(y)$ for some $t \in \tau_{n-1}(x)$. As soon as this is no longer the case simply set $\tau_n = \tau_{n-1} + 1$. For example, if x belongs to the Fatou set, then under the assumption (2), x is attracted to a stable periodic orbit. We do not need any distortion results for these points.

We want to consider the action of Φ as a random walk. Therefore define χ by $\chi(x) = k$ when $x \in A_k$.

Lemma 7.2 *For ρ sufficiently small, the following holds: If $V(y) \cap N_k \neq \emptyset$ for some $k \geq \beta$, then for every neighbourhood U such that $f^t(U) = V(y)$ for some branch of Φ^n,*

$$E_m(\chi \circ f^{m(y)+t} \,|\, U) := \frac{1}{m(U)} \int_U \chi \circ f^{m(y)+t}(x) dm < k - \beta/2.$$

If $V(y) \cap N_k \neq \emptyset$ for some $k < \beta$, then for every neighbourhood U such that $f^t(U) = V(y)$ for some branch of Φ^n,

$$E_m(\chi \circ f^{m(y)+t} \,|\, U) := \frac{1}{m(U)} \int_U \chi \circ f^{m(y)+t}(x) dm \leq \beta/2.$$

Proof: This is a straightforward computation. Since ρ is assumed small, the distortion of $f^{t+m(y)}$ is close to 1 and we have

$$m(\{x \in U; f^{m(y)+t}(x) \in A_{k-\beta+j}\}) \approx m(U) \cdot (1 - \rho^d) \cdot \rho^{jd},$$

where $d = \dim/\ell(c)$ and $\dim \in \{1, 2\}$ according to whether f is real or complex. Thus $E_m(\chi \circ f^{m(y)+t}|U) \approx k + \sum_{j \geq 1}(j - \beta)\rho^{jd} < k - \beta/2$.

The second statement is proved similarly. Note that the same estimate holds when $f^t(U) = U(y)$ instead of $V(y)$. $\square$

Define $\chi_n(x) = \min\{\chi \circ f^t(x); t \in \tau_n(x)\}$. We claim:

$$\text{For all } n \in \mathbb{N} \text{ holds } E_m(\chi_n) \leq \frac{\beta}{2}. \tag{43}$$

Proof of the Claim: Clearly (43) is true for $n = 0$, because N_0 is only a small part of $\hat{\mathbb{C}}$ and $m(N_i) \to 0$ exponentially fast.

Assume (43) holds for $n - 1$. For each $x \in \hat{\mathbb{C}}$, let $s(x) = \min\{t; t \in \tau_{n-1}(x)\}$ and $\tilde{s}(x) = \min\{t; t \in \tau(f^{s(x)}(x))\}$. Obviously $E_m(\chi_n) \leq \int \chi \circ f^{s+\tilde{s}} dm$. We divide $\hat{\mathbb{C}}$ into $Y_0 \cup Y_1 \cup Y_2$, where the Y_i are defined as follows:

- $Y_0 = \{x; f^{s(x)+\tilde{s}(x)}(x) \notin N_0\}$. For example, if x belongs to the Fatou set, then x belongs to Y_0 for n sufficiently large.
- $Y_1 = \{x \notin Y_0; \tilde{s}(x) = m(y)$ and $f^s(x) \in U(y) \backslash V(y)$ for some $y\}$. There is a countable set of points y such that $Y_1 = \cup_y Y_{1,y}$ where $Y_{1,y} = \{x \notin Y_0; \tilde{s}(x) = m(y)$ and $f^s(x) \in U(y) \setminus V(y)\}$. The points in $Y_{1,y}$ satisfy $\chi_n(x) \leq \max\{\chi(y) - \beta, 0\}$.
- $Y_2 = \{x \notin Y_0; \tilde{s}(x) = m(y)$ and $f^s(x) \in V(y)$ for some $y\}$. There is a countable set of points y such that $Y_2 = \cup_y Y_{2,y}$, where $Y_{2,y} = \{x \notin Y_0; \tilde{s}(x) = m(y)$ and $f^s(x) \in V(y)\}$. Due to Lemma 7.1, all the sets $V(y)$ are disjoint. We can use Lemma 7.2 to compute the expectations on the sets $V(y)$.

We obtain

$$\begin{aligned}
E_m(\chi_n) &\leq m(Y_0)E_m(\chi_n|Y_0) + m(Y_1)E_m(\chi_n|Y_1) + m(Y_2)E_m(\chi_n|Y_2) \\
&\leq 0 + \sum_y m(Y_{1,y}) \max\{(E_m(\chi_{n-1}|Y_{1,y}) - \beta), 0\} + \\
&\qquad \sum_y m(Y_{2,y}) \max\{(E_m(\chi_{n-1}|Y_{2,y}) - \frac{\beta}{2}), \frac{\beta}{2}\} \\
&\leq m(Y_1 \cup Y_2) \max\{E_m(\chi_{n-1} - \frac{\beta}{2} \,|\, Y_1 \cup Y_2), \frac{\beta}{2}\} \\
&\leq \max\{E_m(\chi_{n-1}), \frac{\beta}{2}\} \leq \frac{\beta}{2}.
\end{aligned}$$

This shows that $E_m(\chi_n) \leq \beta/2$ for all n. □

From Proposition 6.1 we know that f has only finitely many ergodic components. Let X be such an ergodic component, then the above arguments show that also $E_m(\chi_n|X) \leq \beta/2$, or at least for n sufficiently large. It follows that for m-a.e. $x \in X$, $\liminf_n \inf\{t; t \in \tau_n(x)\} \leq \beta/2$. This proves Theorem 7.1 with $\delta = \epsilon_{\beta/2}$. □

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