Density of hyperbolicity in dimension one

O. Kozlovski, W. Shen, S. van Strien

August 5, 2004

1 Introduction

In this paper we will solve one of the central problems in dynamical systems:

Theorem 1 (Density of hyperbolicty for real polynomials). *Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.*

Here we say that a real polynomial is *hyperbolic* or *Axiom A*, if the real line is the union of a repelling hyperbolic set, the basin of hyperbolic attracting periodic points and the basin of infinity. We call a C^1 endomorphism of the compact interval (or the circle) *hyperbolic* if it has finitely many periodic attracting points and the complement of the basin of attraction of these points is a hyperbolic set. By a theorem of Mañé, for C^2 maps, this is equivalent to the following conditions: all periodic points are hyperbolic and all critical points converge to periodic attractors. Note that the space of hyperbolic maps is an open subset in the space of real polynomials of fixed degree, and that every hyperbolic map satisfying the mild "no-cycle" condition is structurally stable, [dMvS93].

Theorem 1 solves the 2nd part of Smale's eleventh problem for the 21st century [Sma00]:

Theorem 2 (Density of hyperbolicity in the C^k topology). *Hyperbolic (i.e. Axiom A) maps are dense in the space of C^k maps of the compact interval or the circle, $k = 1, 2, \dots, \infty, \omega$.*

This theorem follows from the previous one. Indeed, one can approximate any smooth (or analytic) map on the interval by polynomial maps, and therefore by Theorem 1 by hyperbolic polynomials. Similarly, one can approximate any map of the circle by trigonometric polynomials. If a circle map does not have periodic points, it is semi conjugate to the rotation and it can be approximated by an Axiom A map (this is a classical result). If a circle map does have a periodic point, then using this periodic point we can construct a piecewise smooth map of an interval conjugate to the circle map.

1.1 History of the hyperbolicity problem

The problem of density of hyperbolicity goes back in some form to Fatou, see §4.1 of [McM94]. Smale gave this problem 'naively' as a thesis problem in the 1960's, see [Sma98]. Back then some people even believed that hyperbolic systems are dense in all dimensions, but this was shown to be false in the late 1960's for diffeomorphisms on manifolds of dimension ≥ 2. The problem whether hyperbolicity is dense in dimension one was studied by many people, and it was solved in the C^1 topology by [Jak71], a partial solution was given in the C^2 topology by [BM00] and C^2 density was finally proved in [She04].

From the 1980's spectacular progress was made in the study of quadratic polynomials. In part, this work was motivated by the survey papers of May (in *Science* and *Nature*) on connections of the quadratic maps $f_a(x) = ax(1-x)$ with population dynamics, and also by popular interest in computer pictures of Julia sets and the Mandelbrot set. Mathematically, the realization that quasi-conformal mappings and the measurable Riemann mapping theorem were natural ingredients, enabled Douady, Hubbard, Sullivan and Shishikura to go far beyond the work of the pioneers Julia and Fatou. Using these quasiconformal rigidity methods, Douady, Hubbard, Milnor, Sullivan and Thurston proved in the early 1980's that bifurcations appear monotonically within the family $f_a\colon [0,1] \to [0,1]$, $a \in [0,4]$. In the early 1990's, as a byproduct of his proof on the Feigenbaum conjectures, Sullivan proved that hyperbolicity of the quadratic family can be reduced to proving that any two conjugate non-hyperbolic quadratic polynomials are quasi-conformally conjugate. In the early 1990's McMullen was able to prove a slightly weaker statement: each real quadratic map can be perturbed to a (possibly complex) hyperbolic quadratic map. A major step was made when, in 1997, Graczyk and Światek (see [GŚ97] and [GŚ98]), and Lyubich (see [Lyu97]) proved independently that hyperbolic maps are dense in the space of real quadratic maps. Both proofs require complex bounds and growth of moduli of certain annuli. The latter part was inspired by Yoccoz's proof that the

Mandelbrot set is locally connected at non-renormalizable parameters, but is heavily based on the fact that $z^2 + c$ has only one quadratic critical point (the statement is otherwise wrong). Using their result, Kozlovski was able to prove hyperbolic maps are dense within the space of smooth unimodal maps in [Koz03].

In 2003, the authors were able to prove density of hyperbolicity for real polynomials with real critical points, see [KSvS03]. The main step in that proof was to obtain estimates for Yoccoz puzzle pieces both from above and below. In the present paper, we solve the original density of hyperbolicity questions completely for real one-dimensional dynamical systems.

1.2 Strategy of the proof and some remarks

The main ingredient for the proof of Theorem 1 is the rigidity result [KSvS03].

The first step in proving Theorem 1 is to prove complex bounds for real maps in full generality. This was done previous in [LvS98], [LY97] and [GŚ96] in the real unimodal case, and in the (real) multimodal minimal case in [She04]. The proof of the remaining case (multimodal non-minimal) will be given in Section 3. As in [KSvS03] one has quasi-conformal rigidity for the box mappings we construct, see Theorem 4.

Next we show (roughly speaking) that if a real analytic family of real analytic maps f_λ has non-constant kneading type, then either f_0 is hyperbolic or f_λ displays already a critical relation for λ arbitrarily close to 0. This will be done in Section 4, using a strategy which is similar to the unimodal situation dealt with in [Koz03], but taking care of the additional combinatorial complexity in the multimodal case and using the existence of box mappings and their quasi-conformal rigidity.

Using this, it is is fairly easy to construct families of polynomial maps f_λ, so that f_λ has more critical relations than f_0 for (some) parameters λ arbitrarily close to 0: approximate an artificial family of C^3 maps by a family of polynomials (of much higher degree). In this way one can approximate the original polynomial by polynomials (of higher degree) so that each critical point either is contained in the basin of attracting periodic points or satisfies a critical relation, i.e., is eventually periodic. From this, and the Straightening Theorem, the main theorem will immediately follow.

Of course it is natural to ask about the Lebesgue measure of parameters for which f_λ is 'good'. At this moment, we are not able to prove the general version of Lyubich's results [Lyu02] that for almost every $c \in \mathbb{R}$, the quadratic map $z \mapsto z^2 + c$ is either hyperbolic or stochastic. (This result was strengthened by Avila and Moreira [AA02], who proved that for almost all real parameters the quadratic map has non-zero Lyapounov exponents.) This would prove the famous Palis conjecture in the real one-dimensional case, see [Pal00]. See, however, [BSvS04].

2 Notation and terminology

Let Z be an open subset of $\mathbb{R}$ or $\mathbb{C}$ and $x \in Z$. The connected component of Z containing x will be denoted as $\mathrm{Comp}_x Z$, or, if it is not misleading, as $Z(x)$.

Let (a, b) be an interval on the real line. For any $\alpha \in (0, \pi)$ we use $D_\alpha(I)$ to denote the set of points $z \in \mathbb{C}$ such that the angle $\angle azb$ is greater than α. $D_\alpha(I)$ is a Poincaré disc: it is equal to the set of points $z \in \mathbb{C}$ with $d_P(z, I) < 2/\sin(\alpha)$ where d_P is the Poincaré metric on $\mathbb{C} \setminus (R \setminus I)$.

Let f be a real C^1 map of a closed interval $X = [-1, 1]$ with a finite number of critical points which are not of inflection type (so any critical point of f is either a local maximum or minimum). The set of critical points of f will be denoted as $\mathrm{Crit}\, f$.

Denote the critical points of f by $c_1 < c_2 < \cdots < c_b$. These critical points divide the interval $[-1, 1]$ into a partition $\mathcal{P}$ which consists of elements $\{[-1, c_1), c_1, (c_1, c_2), c_2, \ldots, (c_b, 1]\}$.

For every point $x \in [-1, 1]$ we can define a sequence $\nu_f(x) = \{i_k\}$, $k = 0, 1, \ldots$, of elements of the partition $\mathcal{P}$ in such a way that $f^k(x) \in i_k$ for all $k \geq 0$. This sequence is called *the itinerary* of x.

We say that $f, \tilde{f}$ are *combinatorially equivalent* if there exists an order preserving bijection h from the postcritical set (i.e., the iterates of the critical points) of f onto the corresponding set for $\tilde{f}$ which conjugates f and $\tilde{f}$. Obviously, the itineraries of the corresponding critical points of f and $\tilde{f}$ are the same.

In many cases we want to control only critical points which do not converge to periodic attractors and for this purpose we introduce the following notion. Two maps f and $\tilde{f}$ are called *essentially combinatorially equivalent* if there exists an order preserving bijection $h : \cup_c \omega(c) \to \cup_{\tilde{c}} \omega(\tilde{c})$, where the union is taken over the set of critical points whose iterates do not converge to a periodic attractor.

Let c be a critical point of f and let $[c]$ denote the collection of critical points $c' \in \omega(c)$ with $\omega(c) = \omega(c')$.

An open set $I \subset X$ is called *nice* if for any $x \in \partial I$ and any $n \geq 1$, $f^n(x) \notin I$. Let c be a critical point of f. An *admissible* neighbourhood of $[c]$ is a nice open set I with the following property:

- I has exactly $\#[c]$ components each of which contains a critical point in $[c]$;
- for each connected component J of the domain of definition of the first return map to I, either J is a component of I or J is compactly contained in I.

Given an admissible neighbourhood I of $[c]$, $\mathrm{Dom}(I)$ will denote the domain of definition of the first entry map to I which intersect the orbit of c. Dom '(I) will denote $\mathrm{Dom}(I) \cup I$, and $\mathbf{D}(I) = \mathrm{Dom}(I) \cap I$. We use $R_I : \mathbf{D}(I) \to I$ to denote the first entry map E_I to I restricted to $\mathbf{D}(I)$. For each admissible neighbourhood I of $[c]$, let

$$\mathrm{C}(I) = \{c' \in [c] : I(c') \subset \mathrm{Dom}(I)\}.$$

3 Induced holomorphic box mappings

In this section we will prove the existence of complex bounds, i.e., the existence of box mappings. There are several definitions of box mappings. Here we will use a definition which is slightly more general than the one given in [KSvS03].

Definition 1 (Complex box mappings). *Let $b \geq 1$ be an integer. We say that a holomorphic map*

$$F : U \to V \tag{1}$$

between open sets in $\mathbb{C}$ is a complex box mapping if the following hold:

- *V is a union of b pairwise disjoint Jordan disks;*
- *every connected component V' of V is either a connected component of U or the intersection of V' and U is a union of Jordan disks with pairwise disjoint closures which are compactly contained in V',*

- *for each component U' of U, $F(U')$ is a component of V and $F|U'$ is a proper map;*
- *in each connected component of V there is a point c which is the unique critical point of the map $F : \mathrm{Comp}_c\, U \to \mathrm{Comp}_{F(c)}\, V$;*
- *all iterates of these critical points by F are in U;*
- *all other branches of F are univalent.*

The filled Julia set of F is defined to be

$$K(F) = \{z \in \mathrm{Dom}(F) : F^n(z) \in \mathrm{Dom}(F) \text{ for any } n \in \mathbb{N}\};$$

and the Julia set is $J(F) = \partial K(F)$.

Such a complex box mapping is called real–symmetric *if F is real, all its critical points are real, and the domains U and V are symmetric with respect to $\mathbb{R}$.*

A real box mapping *is defined similarly: replace "Jordan disks" by "intervals", and "holomorphic" by "real analytic".*

We say that a box mapping F is *induced* by a map f if any branch of F is some iterate of a complex extension of the map $f\colon X \to X$.

This type of box mapping naturally arises in the following setting: let $f : \Delta \to \mathbb{C}$ be a holomorphic map, $f(X) \subset X$, where Δ is some complex neighbourhood of X. Fix some recurrent critical points of f and an appropriate neighbourhood V of these critical points, consider the first entry map $R : U \to V$ of f to V. We will see that if the domain V is carefully chosen, then the map $R : U \to V$ is a complex box mapping.

Theorem 3 (The existence of complex box mappings). *Let $f : \Delta \to \mathbb{C}$ be a real holomorphic map with non-degenerate critical points and let $c_0 \in \mathbb{R}$ be a recurrent critical point of f. Then there exists a real–symmetric complex box mapping $F : U \to V$ such that $U \cap V$ contains $[c_0]$.*

Moreover, if $\omega(c_0)$ is non–minimal and f has no neutral cycles, then for any $K > 0$ one can arrange the box mapping so that it has the following extra properties:

- *Every connected component V' of V is contained in $D_{\pi/4}(V' \cap \mathbb{R})$;*

- *There exists* $\theta_1 > 0$ *such that any connected component* U' *of* U *satisfies*
$$U' \subset D_{\theta_1}(U' \cap \mathbb{R});$$
- *Let* Q *be the closure of* $\partial(U \cap \mathbb{R}) \cup \partial(V \cap \mathbb{R})$. *Then* Q *is a hyperbolic set and there exists a constant* $C > 0$ *such that*
$$\mathrm{dist}_{\mathbb{C}\setminus Q}(\partial U', \partial V') > C \text{ and } \mathrm{dist}_{\mathbb{C}\setminus Q}(\partial U', \partial U'') > C$$
where $\mathrm{dist}_{\mathbb{C}\setminus Q}$ *is the hyperbolic distance in* $\mathbb{C} \setminus Q$, V' *is a connected component of* V *and* $U' \neq U''$ *are connected components of* U;
- *there exists* $\xi \in (0, 1)$ *such that for each* $c' \in [c] - C(I)$ *and each* $x \in \mathrm{Comp}_{c'} V$, *there exists a round disk* W *with* $x \in W \subset \mathrm{Comp}_{c'} V$ *such that* $\mathrm{mod}(\mathrm{Comp}_{c'} V - \overline{W}) \geq 1$, *and*
$$area(W \cap U) \leq \xi area(W);$$
- *the domain* V *can be taken in such a way that*
  - *if* U' *is a connected component of* U *and compactly contained in* V, *then* $\mathrm{Comp}_{U'}(V) \cap \mathbb{R}$ *contains* K*-scaled neighbourhood of* $U' \cap \mathbb{R}$*;*
  - *Moreover,*
$$|f(\mathrm{Comp}_{c_0}(V) \cap \mathbb{R})| > K|f(\mathrm{Comp}_{c_0}(U)) \cap \mathbb{R}|.$$

In the case of minimal $\omega(c_0)$ the existence of the box mapping is proven in [She04], so we only have to prove the non–minimal case. The proof of this theorem will occupy the next two subsections.

3.1 Complex bounds from real bounds

Our goal is to prove that for an appropriate choice of an admissible neighbourhood I of $[c]$, the real box mapping R_I extends to a complex box mapping. To this end, it is convenient to introduce geometric parameters $\mathrm{Len}(I)$, $\mathrm{Space}(I)$, $\mathrm{Gap}(I)$ and $\mathrm{Cen}(I)$ as follows.

If J is an entry domain to a nice open set T with entry time s, and if $\{G_i\}_{i=0}^{s}$ is the chain with G_s equal to the component of T which contains $f^s(J)$ and $G_0 = J$ then we define
$$\mathrm{Len}(J;T) = \sum_{i=0}^{s} |G_i|.$$

The parameter Len(I) is defined to be

$$\mathrm{Len}(T) = \sup_J \mathrm{Len}(J, T),$$

where J runs over all components of $\mathbf{D}(T)$.

For any intervals $j \subset t$, and denoting the components of $t \setminus j$ by l, r, define

$$\mathrm{Gap}(l, r) = \frac{1}{\mathrm{Space}(t, j)} := \frac{|t||j|}{|l||r|}.$$

So if Gap(l, r) is large, then the gap interval j is at least larger than one of the intervals l or r. At the same time, if Space(t, j) is large, than there is large space around the interval j inside t. The parameter Gap(I) is defined as

$$\mathrm{Gap}(I) = \inf_{(J_1, J_2)} \mathrm{Gap}(J_1, J_2),$$

where (J_1, J_2) runs over all distinct pairs of components of Dom '(I).

To introduce the parameter Space(I), let

$$I^* = \bigcup_{c' \in \mathrm{C}(I)} I(c'), \quad I^\sharp = I - I^*. \tag{2}$$

The parameter Space(I) is defined to be

$$\mathrm{Space}(I) = \inf_J \mathrm{Space}(\mathrm{Comp}_J\, I, J),$$

where the infimum is taken over all components J of the domain of R_I which are contained in $I^\sharp$. In the following construction we shall be unable to guarantee that all components of the domain of f are compactly contained in I.

Furthermore, for any $c' \in [c]$, let $\hat{J}(c')$ be the component of Dom '(I) which contains $f(c')$, and define

$$\mathrm{Cen}_1(I) = \max_{c' \in [c] - \mathrm{C}(I)} \frac{|\hat{J}(c')|}{|f(I(c'))|},$$
$$\mathrm{Cen}_2(I) = \max_{c' \in \mathrm{C}(I)} \left(\left| \frac{|\hat{J}(c')|}{|f(I(c'))|} - 2 \right| \right),$$

and Cen$(I) = \max(\mathrm{Cen}_1(I), \mathrm{Cen}_2(I))$.

Proposition 1. *There exists $\epsilon_0 > 0$, $C_0 > 0$ and $\theta_0 \in (0, \pi)$ (depending only on b) with the following property. Let I be an admissible neighbourhood of $[c]$ such that* $\mathrm{Len}(I) < \epsilon_0$, $\mathrm{Cen}(I) < \epsilon_0$, $\mathrm{Space}(I) > C_0$ *and* $\mathrm{Gap}(I) > C_0$. *Assume also that* $\max_{c' \in [c]} |I(c')|$ *is sufficiently small. Then there exists a real–symmetric complex box mapping $F : U \to V$ whose real trace is real box mapping R_I. Moreover, the map F satisfies the properties specified in Theorem 3.*

To prove this proposition we need a few lemmas. Let $\mathcal{U} \subset \mathbb{C}$ be a small neighbourhood of X so that $f \colon X \to X$ extends to a holomorphic function $f : \mathcal{U} \to \mathbb{C}$ which has only critical points in X. Here, as before, $X = [0, 1]$.

Lemma 1. *For any $\theta \in (0, \pi)$ there exists $\eta = \eta(f, \theta) > 0$ such that if $J \subset X$ is an open interval which does not contain a critical point and if $|J| < \eta$, then there exists a Jordan disk Ω with $J \subset \Omega \subset D_{\theta - M|f(J)|}(J)$, such that $f : \Omega \to D_\theta(f(J))$ is a conformal map, where M is a constant depending only on f.*

Proof. This lemma is well-known. In fact, $f(\mathcal{U})$ is an open set in $\mathbb{C}$ which contains a neighbourhood of $f(X)$ and thus contains $D_{\mu|f(J)|}(f(J))$ for constant $\mu > 0$. By analytic continuation, $f^{-1}|f(J)$ extends to a univalent function from $D_{\mu|f(J)|}(f(J))$ into $\mathbb{C}_J$. By Schwarz lemma the lemma follows. $\square$

Lemma 2. *For any $\theta \in (0, \pi)$ there exists $\epsilon_0 = \epsilon_0(f, \theta) > 0$ and $\theta' = \theta'(\theta) \in (0, \theta)$ such that the following holds. Let I be an admissible neighbourhood of $[c]$ with* $\mathrm{Len}(I) < \epsilon_0$ *and* $\mathrm{Cen}_2(I) < \epsilon_0$. *Let J be a component of* $\mathrm{Dom}'(I)$, *let $s \geq 0$ be minimal with $f^s(J) \subset I^\sharp$, and let K be the component of $I^\sharp$ containing $f^s(J)$. Then there exists a Jordan disk U with $J \subset U \subset D_{\theta'}(J)$ such that $f^s : U \to D_\theta(K)$ is a well-defined proper map.*

Proof. First consider the case that $f^s|J$ is a diffeomorphism. Let η and M are as in Lemma 1. Then provided that $\sum_{j=1}^{s} |G_j| < \mathrm{Len}(I)$ is less than $\eta/(2M)$, that lemma implies that we have a sequence of Jordan disks U_j with $U_j \subset D_{\theta/2}(G_j)$, $0 \leq j \leq s$, such that $U_s = D_\theta(K)$ and $f : U_j \to U_{j+1}$ is a conformal map for all $0 \leq j < s$. The lemma follows by taking $U = U_0$.

Now assume that $f^s|J$ is not diffeomorphic, and let $s_1 < s$ be maximal such that G_{s_1} contains a critical point c'. Then as above, we obtain a Jordan disks U_j for all $s_1 < j \leq s$ such that $U_s = D_\theta(K)$, such that

- for all $s_1 < j < s$, $f : U_j \to U_{j+1}$ is a conformal map;

- $U_j \subset D_{\theta/2}(G_j)$.

By minimality of s we have $c' \in \mathrm{C}(I)$ and so by the assumption on $\mathrm{Cen}_2(I)$, $|f(G_{s_1})|/|G_{s_1+1}| = |f(I(c'))|/|\hat{J}(c')|$ is bounded away from zero. Therefore, provided that $|G_{s_1+1}|$ is sufficiently small, we have a Jordan disk U_{s_1} with $G_{s_1} \subset U_{s_1} \subset D_{\theta_1}(G_{s_1})$ such that $f : U_{s_1} \to U_{s_1+1}$ is 2-to-1 proper map, where $\theta_1 \in (0, \pi)$ is a constant depending only on θ. Repeat the argument for the shorter chain $\{G_j\}_{j=0}^{s_1}$ and so on. Since the order of the chain $\{G_j\}_{j=0}^{s}$ is bounded from above by b, the procedure stops within b steps, completing the proof. □

Proof of Proposition 1. Assume that $\mathrm{Len}(I)$ and $\mathrm{Cen}_2(I)$ are both very small. For each $c' \in [c] - \mathrm{C}(I)$, define $V_{c'} = D_{\pi/2}(I(c'))$. By Lemma 2, there exists a constant $\theta_0 \in (0, \pi)$ and for each component J of $\mathrm{Dom}\,'(I)$, there exists a Jordan disk $U(J)$ with $J \subset U(J) \subset D_{\theta_0}(J)$ such that if $s = s(J)$ is the minimal non–negative integer with $f^s(J) \subset I(c')$ for some $c' \in [c] - \mathrm{C}(I)$, then $f^s : U(J) \to V_{c'}$ is a well-defined proper map.

For $c' \in \mathrm{C}(I)$, define $V_{c'} = U(I(c'))$. For each component J of R_I in $I^\sharp$, let $\hat{J}$ be the component of $\mathrm{Dom}\,'(I)$ which contains $f(J)$, and let $U(J)$ be the component of $f^{-1}(U(\hat{J}))$ which contains J. Then $U(J)$ is a Jordan disk with $U(J) \cap \mathbb{R} = J$, and $f : U(J) \to U(\hat{J})$ is a well-defined proper map.

Clearly, for each component J of the domain of R_I, if $c' \in [c]$ is such that $R_I(J) \subset I(c')$, and if $R_I|J = f^s|J$, then $f^s : U(J) \to V_{c'}$ is a well-defined proper map.

Assume now that $\mathrm{Space}(I)$ is very big and and $\mathrm{Cen}_1(I)$ is very small. Then for each $c' \in [c] - \mathrm{C}(I)$ and for each component J of the domain of R_I with $J \subset I(c')$, $\mathrm{mod}(V_{c'} - \overline{U}_J)$ is bounded from below by a large constant. In fact, if $J \not\ni c'$ then by Lemma 1, $U(J) \subset D_{\theta_0/2}(J)$, which implies that $\mathrm{mod}(V_{c'} - \overline{U(J)}) \geq \mathrm{mod}(D_{\pi/2}(I(c')) - \overline{D_{\theta_0/2}(J)})$ is large since $\mathrm{Space}(I, J)$ is large. If $J \ni c'$, then by assumption, $|\hat{J}|/|fI(c')|$ is small, so that $U(J)$ is contained in a round disk centred at c' with radius much smaller than $|I(c')|$, hence $\mathrm{mod}(V_{c'} - \overline{U(J)})$ is again big. Note that provided that $\mathrm{Space}(I)$ is large enough, we also have

$$\bigcup_{J \subset I(c')} U(J) \subset B\left(c', \frac{|I(c')|}{4}\right) \cup D_\alpha(I(c')), \tag{3}$$

where $\alpha \in (0, \pi)$ is a constant close to π.

Next let us assume that $\mathrm{Gap}(I)$ is large and show that there exists $\delta > 0$ such that for any components J_1 and J_2 of the domain of R_I, we have

$$\mathrm{dist}_{\mathbb{C}\setminus Q}(\partial U(J_1), \partial U(J_2)) > \delta, \tag{4}$$

To this end, we may assume that J_1 and J_2 are contained in $I(c')$ for some $c' \in [c] - \mathrm{C}(I)$, and that $|\hat{J}_1| \leq |\hat{J}_2|$. Recall that

$$f(U(J_i)) = U(\hat{J}_i) \subset D_{\theta_0}(\hat{J}_i),\ i = 1, 2. \tag{5}$$

In particular, provided that $\mathrm{Gap}(\hat{J}_1, \hat{J}_2)$ are larger than some number which only depends on θ_0,

$$\overline{U(J_1)} \cap \overline{U(J_2)} = \emptyset. \tag{6}$$

Let us consider the following two cases:

Case 1. $J_1 \ni c'$. Since there exist only finitely many components of $\mathrm{Dom}\,'(I)$ with length not smaller than $|J_1|$, there are only finitely many pairs (J_1, J_2) satisfying the property, and thus (4) follows from (6).

Case 2. $J_1 \not\ni c'$. In this case, (5) implies that $d(\partial U(J_1), \partial U(J_2))/|J_1|$ is big, provided that $\mathrm{Gap}(\hat{J}_1, \hat{J}_2)$ is big enough. Moreover, Lemma 1 implies that $U(J_1) \subset D_{\theta_0/2}(J_1)$. All these imply that the distance between $\mathrm{dist}_{\partial J_1}(\partial U(J_1), \partial U(J_2))$ is large, where $\mathrm{dist}_{\partial J_1}$ denotes the hyperbolic distance in $\mathbb{C} - \partial J_1$. As $\mathrm{dist}_{\partial J_1} \leq \mathrm{dist}_{\mathbb{C}\setminus Q}$, (4) follows.

Now we define a complex box mapping $F\colon U \to V$ by setting $U = \bigcup_J U(J)$, $V = \bigcup_{c' \in [c]} V(c')$ and by defining F so that its real trace is R_I.

The required properties of F easily follow from the construction. $\square$

3.2 Choice of an admissible neighbourhood

We shall prove here

Proposition 2. *Let c be a recurrent critical point of f which has a non-minimal ω-limit set. For any $\epsilon > 0$ and $C > 0$ there exists an admissible neighbourhood I of $[c]$ such that such that* $\mathrm{Len}(I) < \epsilon$, $\mathrm{Gap}(I) > C$, $\mathrm{Space}(I) > C$, *and* $\mathrm{Cen}(I) < \epsilon$.

Given an admissible neighbourhood T of $[c]$, let us define a new admissible neighbourhood $\mathcal{A}(T)$ as follows. Recall that $\mathrm{C}(T)$ is the subset of $[c]$ consisting of points $c' \in [c]$ so that $T(c') \subset \mathrm{Dom}(T)$, and that $T^* = \bigcup_{c' \in \mathrm{C}(T)} T(c')$,

$T^\sharp = T - T^*$. Let $t(T) = \# \mathrm{C}(T)$. For any $c' \in [c]$, there exists a minimal non–negative integer $k(c')$ such that $R_T^{k(c')}(c') \in T^\sharp$. (So $k(c') = 0$ if $c' \in [c] - \mathrm{C}(T)$.) The set $\mathcal{A}(T)$ is defined so that $\mathcal{A}(T)(c')$ is the maximal interval containing c' such that $R_T^{k(c')}(\mathcal{A}(T)(c')) \subset \mathrm{Dom}(T)$. Clearly $\mathrm{C}(\mathcal{A}(T)) \subset \mathrm{C}(T)$.

Lemma 3. *Assume that* $\mathrm{C}(\mathcal{A}(T)) = \mathrm{C}(T)$. *Then*

1. *for each* $c' \in \mathrm{C}(T)$, $E_T(\mathcal{A}(T)(c')) \subset \mathcal{A}(T)$*;*
2. *for each* $c' \in [c]$ *and* $x \in T(c') - \mathcal{A}(T)(c')$*, there exists an interval* $J(x)$ *with* $x \in J(x) \subset T(c') - \mathcal{A}(T)(c')$ *such that* $E_T^{k(c')+1}$ *maps* $J(x)$ *onto a component of* T *diffeomorphically;*
3. *for each landing domain* J *of* $\mathcal{A}(T)$*, there exists an interval* $\hat{J}$ *with* $J \subset \hat{J} \subset \mathrm{Dom}'(T)$ *such that if* s *is the landing time of* J *into* $\mathcal{A}(T)$*, then* f^s *maps* $\hat{J}$ *diffeomorphically onto a component of* T*.*

Proof. Let us prove the first statement by contradiction. It is enough to prove that $E_T(c') \in \mathcal{A}(T)$, so assume that this is not the case. For $0 \le i \le k(c')$, let $c'_i \in [c]$ be such that $E_T^i(c') \in T(c'_i)$. Let $m \le k(c') - 1$ be maximal so that $E_T(c'_m) \notin \mathcal{A}(T)(c'_{m+1})$. Let $p \in \mathbb{N}$ be minimal such that $E_T^p(c'_m) \in \mathcal{A}(T)$. By the maximality of m, we obtain $E_T(\mathcal{A}(T)(c_i)) \subset \mathcal{A}(T)(c_{i+1})$ for $i = m+1, \dots, k-1$. Hence $E_T^i(c'_m) \notin \mathcal{A}(T)$ for all $1 \le i \le k - m$, and so $p > k - m$. But $E_T^{k-m}(\partial \mathcal{A}(T)(c'_m))$ is contained in $\partial \,\mathrm{Dom}(T)$, which implies that $E_T^p(\partial \mathcal{A}(T)(c'_m)) \notin T$. Since $\mathcal{A}(T) \Subset T$ the minimality of p gives that $c'_m \notin \mathrm{C}(\mathcal{A}(T))$. However, since $c'_m \in \mathrm{C}(T) = \mathrm{C}(\mathcal{A}(T))$ this gives a contradiction.

Let us now pass to the proof of the second statement. By the first statement, for each $c' \in [c]$, $E_T^{k(c')}$ maps each component of $T(c') - \mathcal{A}(T)(c')$ onto a component of $T - \mathcal{A}(T)$ in a diffeomorphic way. For each $x \in T(c') - \mathcal{A}(T)(c')$, we take $J(x)$ to be the maximal interval so that $E_T^{k(c')}(J(x))$ is contained in (a component of) $\mathrm{Dom}(T)$. Clearly, $E_T^{k(c')+1}$ maps $J(x)$ onto a component of T in a diffeomorphic way.

The third statement follows from the observation that any branch of the first landing map to $\mathcal{A}(T)$ can be written as the composition of the first landing map to T with finitely many maps of the form $E_T^{k(c')+1}|J(x)$, $x \in T(c') - \mathcal{A}(T)(c')$. $\square$

Let us say that a sequence of open intervals $\{G_i\}_{i=0}^s$ is a *chain* if G_i is a component of $f^{-1}(G_{i+1})$ for each $i = 0, \ldots, s-1$. The *order* of this chain is the number of G_i's which contain a critical point.

The following lemma is usually referred to as the *Koebe principle*. See [vSV00] for a proof.

Lemma 4. *Assume that $\{G_i\}_{i=0}^s$ is a chain such that G_s is contained in a small neighbourhood of a non-periodic and recurrent critical point.*

1. *For each $N > 0$ and $C > 0$ there exists $C' > 0$ such that if the order of the chain $\{G_i\}_{i=0}^s$ is at most N and $\{J_i\}_{i=0}^s$ is a chain with $J_i \subset G_i$, $i = 0, \ldots, s$ then if $\mathrm{Space}(G_s, J_s) \geq C'$ then $\mathrm{Space}(G_s, J_s) > C$.*

2. *For each $C > 0$ there exists $K > 1$ such that if $f^s|G_0$ is a diffeomorphism, does not contain any non-hyperbolic periodic points and $\mathrm{Space}(G_s, J_s) \geq C$ then $|Df^s(x)|/|Df^s(y)| \leq K$ for each $x, y \in J_0$. Moreover, $K \to 1$ as $C \to \infty$.*

Lemma 5. *Let $c_1, c_2 \in [c]$, let $x \in \mathcal{A}(T)(c_1)$ be such that $E_{\mathcal{A}(T)}(x) \in \mathcal{A}(T)(c_2)$, and let s be such that $E_{\mathcal{A}(T)} = f^s$ near x. Consider the chain $\{G_i\}_{i=0}^s$ with $G_s = T(c_2)$ and $G_0 \ni x$. Then the order of the chain is not greater than $\#[c]+1$. Moreover, if $c_1 \notin \mathrm{C}(\mathcal{A}(T))$, then $G_0 \subset \mathcal{A}(T)(c_1)$.*

Proof. First observe that $\mathcal{A}(T)(c') \supset \mathrm{Comp}_{c'} \mathrm{Dom}(T(c'))$ for all $c' \in [c]$. It follows that for each $c' \in [c]$, there can be at most one i with $0 < i \leq s$ such that $G_i \ni c'$. Thus the order of the chain $\{G_i\}_{i=0}^s$ is at most $\#[c]+1$.

Now let us assume that $c_1 \notin \mathrm{C}(\mathcal{A}(T))$ and show that $G_0 \subset \mathcal{A}(T)(c_1)$. Let k be the minimal positive integer such that $E_T^k = f^s$ near x. Since $G_0 \subset \mathrm{Comp}_{c_1} \mathrm{Dom}(T)$, we may assume that $c_1 \in \mathrm{C}(T)$. If $G_0 \Subset T(c_1)$, then $k > k(c_1)$, and $E_T^{k(c_1)}(G_0)$ is contained in a component of $\mathrm{Dom}(T)$ so that $G_0 \subset \mathcal{A}(T)(c_1)$. Therefore we may assume that $G_0 = T(c_1)$. Then $k \leq k(c_1)$, so $f^i(x) \notin T^\sharp$ for all $1 \leq i \leq s-1$. It follows that

$$E_T^k(\mathcal{A}(T)(c_1)) \subset \mathrm{Comp}_{f^s(x)} \mathrm{Dom}\,'(T^\sharp \cap \mathrm{Dom}(T)) = \mathcal{A}(T)(c_2),$$

which implies that $c_1 \in \mathrm{C}(\mathcal{A}(T))$. The contradiction completes the proof. □

Lemma 6. *For any $\epsilon > 0$ there exists $\epsilon' > 0$ such that if $\mathrm{Space}(T) > 1/\epsilon'$ then $\mathrm{Space}(\mathcal{A}(T)) > 1/\epsilon$ and $\mathrm{Cen}_1(\mathcal{A}(T)) < \epsilon$.*

Proof. By the previous lemma and the above Koebe principle, it suffices to show that $|\mathcal{A}(T)(c')|/|T(c')|$ is small for every $c' \in [c]$, provided that $\mathrm{Space}(T)$ is sufficiently large. To this end, let s be such that $E_T^{k(c')} = f^s$ on $T(c')$, and consider the chain $\{G_i\}_{i=0}^s$ with $G_s = T(f^s(c'))$ and $G_0 = T(c')$. The order of this chain is bounded from above by $k(c') \leq \#[c]$. Since $f^s(\mathcal{A}(T)(c'))$ is contained in a component of $\mathrm{Dom}(T)$ which is deep inside G_s, again by the above Koebe principle, we obtain the desired estimate. □

Lemma 7. *For any $\epsilon > 0$ and $C > 0$ there exists $C' > 0$ with the following property. Assume that* $\mathrm{C}(\mathcal{A}(T)) = \mathrm{C}(T)$ *and* $\mathrm{Space}(T) > C'$*. Then* $\mathrm{Len}(\mathcal{A}(T)) < \epsilon$*, and* $\mathrm{Gap}(\mathcal{A}(T)) > C$*. Moreover, if* $\mathrm{C}(\mathcal{A}^2(T)) = \mathrm{C}(T)$*, then* $\mathrm{Cen}(\mathcal{A}(T)) < \epsilon$*.*

Proof. Assume that $\mathrm{Space}(T)$ is large. Then by Lemma 6, for each $c' \in [c]$, $\mathcal{A}(T)(c')$ is deep inside $T(c')$.

Let us first show that $\mathrm{Gap}(\mathcal{A}(T))$ is big. To this end, let J_1 and J_2 be distinct components of $\mathrm{Dom}\,'(\mathcal{A}(T))$ and let s_1, s_2 be their landing times to $\mathcal{A}(T)$. Without loss of generality, assume $s_1 \leq s_2$. It is enough to show that the gap between J_1 and J_2 is much bigger than J_2. Let $\hat{J}_i$, $i = 1, 2$, be as in Lemma 3 (3). By the Koebe principle, J_i is deep insider $\hat{J}_i$, so it suffices to show that $J_1 \cap \hat{J}_2 = \emptyset$. Let us prove this by contradiction. Assume that $J_1 \cap \hat{J}_2 \neq \emptyset$. Since both J_1 and $\hat{J}_2$ are pull backs of the nice set T, so either $J_1 \supset \hat{J}_2$ or $J_1 \subset \hat{J}_2$. Since $J_1 \cap J_2 = \emptyset$, the first alternative cannot happen. Therefore, $J_1 \subset \hat{J}_2$. It follows that for all $0 \leq i \leq s_2$, $f^i(J_1) \subset f^i(\hat{J}_2) - f^i(J_2)$, hence $f^i(J_1) \cap [c] = \emptyset$. But $f^{s_1}(J_1)$ is a component of $\mathcal{A}(T)$, a contradiction.

Now let us prove that $\mathrm{Len}(\mathcal{A}(T))$ is small. Let J be a landing domain to $\mathcal{A}(T)$ with landing time s. Let s_1 be the minimal non–negative integer such that $K_1 := f^{s_1}(J) \subset T(c_1)$ for some $c_1 \in [c]$. If $s_1 < s$, then $K_1 \cap \mathcal{A}(T)(c_1) = \emptyset$. Moreover, if $k(c_1) > 0$ then $c_1 \in \mathrm{C}(T)$, then $E_T^i(K_1) \cap \mathcal{A}(T) = \emptyset$ for all $i \leq k(c_1)$. So if we let $s_2 > s_1$ be such that $E_T^{k(c_1)+1} = f^{s_2 - s_1}$ on $f^{s_1}(J)$ then $s_2 \leq s$. Let $c_2 \in [c]$ be such that $f^{s_2}(J) \subset T(c_2)$, and let $\hat{K}_1 \supset f^{s_1}(J)$ be the interval determined as in Lemma 3 (2), i.e., $\hat{K}_1$ is the interval containing $f^{s_1}(J)$ such that $E_T^{k(c_1)+1} : \hat{K}_1 \to T(c_2)$ is a diffeomorphism. If $s_2 < s$ then we define $\hat{K}_2$ and s_3, and so on. In this way we obtain a sequence of integers $0 \leq s_1 < s_2 < \cdots < s_n = s$ such that for each $1 \leq i < n$, there exist $c_i \in [c]$ and an interval $\hat{K}_i$ with

- $f^{s_i}(J) \subset \hat{K}_i \subset T(c_i) - \mathcal{A}(T)(c_i)$.

- $f^{s_{i+1}-s_i}|\hat{K}_i = E_T^{k(c_i)+1}|\hat{K}_i$ is a diffeomorphism from $\hat{K}_i$ onto $T(c_{i+1})$.

Let us now prove that for all $1 \le i \le n-1$,

$$\mathrm{Space}(T(c_i), f^{s_i}(J)) \ge 2^{n-i}\,\mathrm{Space}(T(c_n), f^{s_n}(J)). \tag{7}$$

To this end, we first notice that $\mathrm{Space}(T(c_i), \hat{K}_i)$ is large for all $1 \le i \le n-1$. In fact, $k(c_i) < b$ and $E_T^{k(c_i)}(\hat{K}_i)$ is contained in a component of $\mathrm{Dom}(T) \cap T(c_{i+1})$ which is deep inside $T(c_{i+1})$ by assumption, so the statement follows by the Koebe principle. Hence, for any $A > 0$ one can find $C' > 0$ such that $\mathrm{Space}(T(c_i), f^{s_i}(J)) \ge A\,\mathrm{Space}(\hat{K}_i, f^{s_i}(J))$ provided that $\mathrm{Space}(T) > C'$. Since $f^{s_{i+1}-s_i} : (\hat{K}_i, f^{s_i}(J)) \to (T(c_{i+1}), f^{s_{i+1}}(J))$ is a diffeomorphism, the Koebe principle mentioned above gives a constant K such that $\mathrm{Space}(\hat{K}_i, f^{s_i}(J)) \ge K\,\mathrm{Space}(T(c_{i+1}), f^{s_{i+1}}(J))$. Combined this gives

$$\mathrm{Space}(T(c_i), f^{s_i}(J)) \ge 2\,\mathrm{Space}(T(c_{i+1}), f^{s_{i+1}}(J)).$$

The equation (7) follows.

Let us now prove that $\sum_{j=0}^{s} |f^j(J)|$ is small. Let $\Delta := \mathrm{Space}(T(c_n), f^{s_n}(J))$. For each $1 \le i \le n-1$, and for any $1 \le j \le s_{i+1} - s_i$, we have

$$\mathrm{Space}(f^j(\hat{K}_i), f^{s_i+j}(J)) \ge \frac{\mathrm{Space}(T(c_{i+1}), f^{s_{i+1}}(J))}{K} \ge \frac{2^{n-i}}{K}\Delta,$$

since $f^{s_{i+1}-s_i} : \hat{K}_i \to T(c_{i+1})$ is a diffeomorphism. In particular, $|f^{s_i+j}(J)| \le \frac{K}{2^{n-i}\Delta}|f^j(\hat{K}_i)|$. On the other hand, the intersection multiplicity of the chain $\{f^j(\hat{K}_i)\}_{j=1}^{s_{i+1}-s_i}$ is at most $k(c_i)+1$, so

$$\sum_{j=1}^{s_{i+1}-s_i} |f^j(\hat{K}_i)| \le (k(c_i)+1)|X| \le 2b$$

where X is the dynamical interval. Thus

$$\begin{aligned}
\sum_{j=s_1+1}^{s} |f^j(J)| &= \sum_{i=1}^{n-1} \sum_{j=1}^{s_{i+1}-s_i} |f^{s_i+j} J| \\
&\le \sum_{i=1}^{n-1} \frac{K}{2^{n-i}\Delta} \sum_{j=1}^{s_{i+1}-s_i} |f^j(\hat{K}_i)| \\
&\le \frac{2Kb}{\Delta}
\end{aligned}$$

is small (because Δ is large). To show that $\sum_{j=0}^{s_1} |f^j(J)|$ is small, we use the fact that $f^{s_1}|J$ extends to a diffeomorphism onto $T(c_1)$ and argue similarly.

Finally, let us assume also that $\mathrm{C}(\mathcal{A}^2(T)) = \mathrm{C}(\mathcal{A}(T))$, and show that $\mathrm{Cen}(\mathcal{A}(T))$ is small. In Lemma 6, we have already shown that that $\mathrm{Cen}_1(\mathcal{A}(T))$ is small. So it remains to show that $\mathrm{Cen}_2(T)$ is small. To this end, take $c' \in \mathrm{C}(T)$ and let $c'' \in [c]$ be such that $E_T(c') \in T(c'')$. By assumption we have $E_T(c') \in \mathcal{A}^2(T)(c'')$. Since $|\mathcal{A}^2(T)(c'')|/|\mathcal{A}(T)(c'')|$ is small, the components of $\mathcal{A}(T)(c'') - \{E_T(c')\}$ have almost the same length. If $J \ni f(c')$ is the landing domain to $\mathcal{A}(T)$ and if s is the landing time, then $f^s : J \to f^s(J)$ extends to a diffeomorphism onto $T(c'')$ which implies by the Koebe principle that $f^s|J$ is almost linear. Thus the components of $J - \{f(c')\}$ have almost the same length. $\square$

Proof of Proposition 2. Since $\omega(c)$ is non-minimal, we can apply Theorem 1.2 in [She03a]. Hence, for any $K > 0$ there exists an arbitrarily small K-nice neighbourhood Q of c. In the present terminology this means that for any $C > 0$ there exists an admissible neighbourhood

$$T_0 := \bigcup_{c' \in [c]} \mathrm{Comp}_{c'} \mathrm{Dom}\,'(Q)$$

with $\mathrm{Space}(T_0) > C$. For $n \geq 0$, define inductively $T_{n+1} = \mathcal{A}(T_n)$. Then, since $\mathrm{C}(T_n) \supset \mathrm{C}(T_{n+1})$ there exists $N \leq 2b$ such that

$$\mathrm{C}(T_{N-1}) = \mathrm{C}(T_N) = \mathrm{C}(T_{N+1}).$$

By Lemmas 6 and 7, defining $I = T_N$ completes the proof. $\square$

3.3 Rigidity of box mappings

The following theorem is the direct analogue of the Rigidity theorem in [KSvS03] for the box mappings defined in the previous section. The proof is the same.

Theorem 4 (Rigidity theorem for box mappings). *Let $f : U \to V$ and $\tilde{f} : \tilde{U} \to \tilde{V}$ be two combinatorially equivalent real-symmetric complex box mappings without neutral cycles. Moreover, suppose that there exists a q.c. homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that h conjugates f and $\tilde{f}$ on the boundaries of their domains of definition.*

Then there exists a q.c. homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ which conjugates f and $\tilde{f}$ on their domains and such that $\phi = h$ outside U.

4 Instantaneous change of combinatorics in analytic families

In this section we shall use the two theorems from the previous section in order to prove that under certain conditions the only structurally stable maps within analytic families of analytic maps are hyperbolic maps. The main condition we put on such families is that all the maps in the family are regular (see the definition below). This condition was introduced in [Koz03] in the context similar to present. It seems conceivable that this condition is superfluous, however we not know how to prove the theorem below without it.

Definition 2. *A C^1 interval map $f : I \to I$ is called* regular *if each of its neutral periodic points contains a non preperiodic critical point in the interior of its attraction basin and each of its critical points is contained in the interior of I. A family of interval maps are called* regular *if all maps in this family are regular.*

Real polynomial maps with only real critical points are regular.

Definition 3. *A critical point c of a map f is called* prime *if from $c' \in \omega(c)$, where c' is another critical point of f, follows that $c \in \omega(c')$.*

Prime critical points always exist provided the map has at least one critical point. Indeed, we can introduce a partial order on the set of critical points by setting $c_1 \preceq c_2$ if $c_1 \in \omega(c_2)$. The set of minimal elements with respect to this partial order is the set of prime critical points (we could call these points minimal, but this could introduce a confusion with critical points whose ω–limit sets are minimal).

Theorem 5. *Let f_λ, $\lambda \in (-1, 1)$, be a regular analytic family of real–analytic maps of the interval. Suppose that*

- *for any λ, each real critical point $c(\lambda)$ of f_λ is non-degenerate (and so depends real analytically on λ);*
- *the map f_0 has no neutral cycles;*
- *the map f_0 has a prime critical point c_0 which is not in the closure of the immediate basin of periodic attractors of f_0 such that the itinerary $\nu_{f_\lambda}(c_0(\lambda))$ is non-constant as λ varies in $[0, 1)$;*

- *attracting cycles of f_0 do not bifurcate as $\lambda \in (-1,1)$ varies;*

- *if an iterate of some real critical point is mapped onto another (or the same) critical point under f_0, then this critical relation persists for all $\lambda \in (-1,1)$;*

- *if iterates of some real critical point $\hat{c}(0)$ converge to a periodic attractor under f_0, then the iterates of $\hat{c}(\lambda)$ converge to a periodic attractor for every $\lambda \in (-1,1)$.*

Then in any neighbourhood of $\lambda = 0$, there exists a non-periodic critical point $\tilde{c}(0)$ such that the itinerary $\nu_{f_\lambda}(\tilde{c}(\lambda))$ is not constant.

In the unimodal case this theorem was proved previously in [Koz03]. The proof of the above theorem follows the same strategy, except that we need to deal with the possibility of more general types of critical relations (compared to the unimodal case). Moreover, we use a method of [ALdM03] to construct a holomorphic motion of the boundary of the box mappings (although one could also proceed as in [Koz03] or [LvS00]).

One can extend the above theorem to multi-parameter families easily (see [Koz03]).

Before proving the above theorem, we prove a simple proposition.

Proposition 3. *Suppose $\{f_n\}$, $n = 1, 2, \dots$ is a sequence of C^3 maps of the interval satisfying the following properties.*

- *The sequence f_n converges to some map f_0 in the C^1 topology.*

- *All f_n, $n = 0, 1, \dots$, have the same critical points and these are not of the inflection type.*

- *All f_n, $n = 1, 2, \dots$, have the same number of attractors and the periods of attractors do not change with n; if c is a critical point such that the iterates of c converge to some periodic attractor under f_k for some $k \geq 1$, then the iterates of c converge to a periodic attractor under all f_n, $n \geq 0$; the set of these critical points will be denoted by $\mathcal{C}$.*

- *If $c \notin \mathcal{C}$, then the itinerary $\nu_{f_n}(c)$ does not change with n, i.e. $\nu_{f_1}(c) = \nu_{f_n}(c)$ for any $n \geq 1$.*

Then if $c \notin \mathcal{C}$, then $\nu_{f_0}(c) = \nu_{f_1}(c)$.

$\lhd$ Suppose that the conclusion of the lemma does not hold for some critical point $c \notin \mathcal{C}$. Let $\nu_{f_0}(c) = \{j_k\}$ and $\nu_{f_1}(c) = \{i_k\}$, $k = 0, 1, \ldots$. By continuity it is easy to see that if $i_k \neq j_k$ for some k, then i_k is some interval and j_k is a critical point on the boundary of i_k. This implies that there exists $m > 0$ such that $i_l = j_l$ for all $l > m$. Indeed, if it does not hold, the sequence $\{j_l\}$ would have infinitely many critical points in it, and since there are just finitely many critical points the map f_0 would have a super attractive critical periodic point and some iterate of c would be mapped onto this point by f_0. This means that c would be in the basin of some periodic attractor for large values of n. This contradicts the third assumption of the lemma.

The same argument as above shows that for any $c \notin \mathcal{C}$ there exists m such that j_k is not a critical point for all $k > m$, where $\{j_k\} = \nu_{f_0}(c)$ and that for any $c_1, c_2 \notin \mathcal{C}$ if c_2 is an element in $\nu_{f_0}(c_1)$, then c_1 is not an element in $\nu_{f_0}(c_2)$. The last property allows us to introduce a partial ordering on the set of critical points outside of $\mathcal{C}$: $c_1 \succ c_2$ if c_2 is an element of $\nu_{f_0}(c_1)$.

Take a minimal element in this ordering for which the conclusion of the lemma does not hold. Denote it as c and let $\nu_{f_0}(c) = \{j_k\}$. Let m be maximal such that j_m is a critical point. From the discussion above we know that such m is finite and greater than 0. From the minimality of c we know that $j_m \in \mathcal{C}$, hence its iterates converge to some periodic attractor. Thus the iterates of c converge to an attractor under f_0 and, therefore, under f_n for large values of n as well. This is a contradiction. $\rhd$

Proof of Theorem 5. Maps from the family f_λ do not have degenerate critical points, therefore the critical points in this family do not bifurcate and for any critical point c of f_0 there is an analytic function $\lambda \mapsto c(\lambda)$ such that $c(0) = c$ and $c(\lambda)$ is a critical point of f_λ. Often we will suppress the dependence of c on λ if it does not lead to a confusion.

Suppose that the assertion of the theorem does not hold. Then there exists $\lambda_0 > 0$ such that for all maps corresponding to a parameter in $[0, \lambda_0)$, the critical points which do not converge to periodic attractors do not change their itineraries for $\lambda \in [0, \lambda_0)$. Due to Proposition 3 we know that in this case the itineraries of critical points of f_{λ_0} whose iterates do not converge to periodic attractors are the same as for any map f_λ, $\lambda \in [0, \lambda_0]$. We can choose λ_0 be maximal with this property. Then for the critical point c_0, $\nu_{f_\lambda}(c_0) = \nu_{f_0}(c_0)$ for all $\lambda \in [0, \lambda_0]$, and there are parameters $\lambda \in (\lambda_0, 1)$ arbitrarily close to λ_0 such that $\nu_{f_\lambda}(c_0) \neq \nu_{f_0}(c_0)$.

We claim that the map f_{λ_0} cannot have neutral cycles. Arguing by contradiction, assume that that f_{λ_0} has a neutral periodic point p of period n. Since f_{λ_0} is regular, there exists a (real) non pre-periodic critical point c of f_{λ_0} in the interior of the attracting basin of the orbit of p. The itinerary of this critical point is preperiodic with eventual period n under iterations of f_{λ_0}, and hence under iterations of f_λ for $\lambda \in [0, \lambda_0]$ as well. By an easy continuity argument, the assumption that f_0 has no neutral cycle implies that there exists $\lambda_1 \in (0, \lambda_0)$ such that f_λ has no neutral cycle of period $\leq n$ for all $\lambda \in [0, \lambda_1)$. We have the following two cases.

Case 1. For some $\lambda_2 \in [0, \lambda_1)$, c converges to a hyperbolic attracting cycle O_λ of period n.

In this case, using the fact that f_λ has no neutral cycle for all $\lambda \in [0, \lambda_1)$, we conclude easily that c converges to the corresponding hyperbolic attracting cycle O_0 under iterations of f_0. But then the assumption of the theorem implies that c is contained in the attracting basin of a hyperbolic attracting cycle of f_{λ_0}, which is a contradiction.

Case 2. For each $\lambda \in [0, \lambda_1)$, there exists $k(\lambda)$ such that $f_\lambda^{k(\lambda)}(c)$ is a repelling periodic point of f_λ of period n.

For each $k \geq 0$, let $J_k = \{\lambda \in (-1, 1) : f_\lambda^{k+n}(c) = f_\lambda^k(c)\}$. Then $J_0 \subset J_1 \subset \cdots$ and $\bigcup_k J_k \supset [0, \lambda_1)$. By Baire's category theorem, for some k, J_k has an accumulation point in $(-1, 1)$. By analytic continuation it follows that $J_k = (-1, 1)$. In particular, c is preperiodic under iteration of f_{λ_0}, which is a contradiction again.

The map f_{λ_0} satisfies the same assumptions of the theorem as the map f_0. We rename f_{λ_0} by f_0. So for small negative values of λ the itineraries of the critical points which are not in the basin of attractors are the same as for $\lambda = 0$ and there are small positive values of λ such that the itinerary of c_0 for f_λ is different from its itinerary for f_0.

Let us first assume that the critical point c_0 is recurrent. According to the theorem in the previous section we can then construct a box mapping for f_0. More precisely, there is a complex box mapping $F_0 : U \to V$ such that the orbit of c_0 is contained in U. Moreover, we can choose F_0 in such a way that the forward iterates of $\partial(U \cap \mathbb{R})$ under f_0 do not contain any critical point of f_0.

Before continuing the proof of the theorem we need a few lemmas.

Lemma 8. *There exists a neighbourhood $\Lambda \subset \mathbb{C}$ of* 0 *and a normalised holomorphic motion $h_\lambda : \mathbb{C} \to \mathbb{C}$, $\lambda \in \Lambda$, such that h_λ is $\mathbb{R}$–symmetric for real values of λ and $F_\lambda : h_\lambda(U) \to h_\lambda(V)$ is a box mapping induced by f_λ.*

Moreover,

$$h_\lambda \circ F_0(x) = F_\lambda \circ h_\lambda(x)$$

for all $x \in \partial U$ *and* $\lambda \in \Lambda$.

$\triangleleft$ Here we use method given in [ALdM03]. The point c_0 is prime, hence for any $c \in \mathrm{Crit}(f_0) \setminus [c_0]$ we have $c \notin \omega(c_0)$. Let N be a neighbourhood of $\mathrm{Crit}(f_0) \setminus [c_0]$ disconnected from $\omega(c_0)$. Decrease N if necessary in such a way that the forward iterates of points from the set $\partial(U \cap \mathbb{R})$ do not enter N. Denote the set of real points which are outside of the basins of periodic attractors and whose iterates avoid $\cup_{c\in \mathrm{Crit}(f_0)} \mathrm{Comp}_c U \cup N$ by Q. Obviously, Q is a hyperbolic set and $\partial(U \cap \mathbb{R}) \subset Q$. This set persists under small (complex) perturbations and due to λ–lemma there exists a neighbourhood $\Lambda \subset \mathbb{C}$ of zero, a neighbourhood $W \subset \mathbb{C}$ of Q and a holomorphic motion $h_\lambda : \mathbb{C} \to \mathbb{C}$, $\lambda \in \mathbb{C}$, such that $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ on W. Shrinking W if necessary we can assume that W does not contain critical points of F_0.

Every connected component U' of U is mapped onto some connected component V' of V after several, say $n(U')$ iterations, so $f_0^{n(U')}(U') = V'$. Then the map $h_{\lambda,U'} : Q \cup \partial U' \to \mathbb{C}$ defined by $h_{\lambda,U'}(x) = f_\lambda^{-n(U')} \circ h_\lambda \circ f_0^{n(U')}(x)$ for $x \in \partial U'$, and $h_{\lambda,U'}(x) = h_\lambda(x)$ for $x \in Q$, $\lambda \in \Lambda$, is a holomorphic motion (We might shrink Λ first in such a way that for all $\lambda \in \Lambda$ the map f_λ has no critical points in $h_\lambda(W) \bigcup h_\lambda(V)$, and in $f_\lambda^{-m} \circ h_\lambda \circ f_0^{n(U')}(\partial U')$, where $m = 1, \dots, n(U')$ and U' is not contained in W; there are finitely many such domains, so we can always shrink Λ in such a way).

Fact 1 (Lemma 2.3 in [ALdM03]). *For any* $M > m > 0$ *there exists* $\delta > 0$ *with the following property. Let* $S, \tilde{S} \subset \mathbb{C}$ *be two hyperbolic Riemann surfaces and* h_1, $h_2 : S \to \tilde{S}$ *be* $(1+\delta)$*–q.c. maps homotopic rel boundary. Let* X *and* Y *be subsets of* S*. If* $\mathrm{dist}_S(X, Y) > M$*, then* $\mathrm{dist}_{\tilde{S}}(X, Y) > m$*.*

Due to Theorem 3 we have $\mathrm{dist}_{\mathbb{C}\setminus Q}(\partial U', \partial U'') > C$ for $U' \neq U''$ and $\mathrm{dist}_{\mathbb{C}\setminus Q}(\partial U', \partial V) > C$ if U' is not a connected component of V. Shrinking Λ further we can insure that qc dilatation of h_λ and of all $h_{\lambda,U'}$, $\lambda \in \Lambda$, is smaller then $1+\delta$ for any beforehand given $\delta > 0$. Using the fact above we conclude that the sets $h_{\lambda,U'}(\partial U')$ and $h_{\lambda,U''}(\partial U'')$ and the sets $h_{\lambda,U'}(\partial U')$ and $h_\lambda(\partial V)$ never intersect if λ is small enough. Applying the λ–lemma we can construct a homotopic motion of the complex plane with the required properties, completing the proof of Lemma 8. $\triangleright$

So, we have constructed a box mapping for every sufficiently small (complex) λ together with a holomorphic motion h_λ. Denote the Beltrami coefficient of h_λ by μ_λ. Define now $\hat{\nu}_\lambda$ to be zero outside of V and on the filled Julia set of the map $F_0 : U \to V$, and everywhere else define it as the pullback of μ_λ by F_0. Obviously, the map $\lambda \mapsto \hat{\nu}_\lambda(x)$ is holomorphic for fixed values of $x \in \mathbb{C}$ and there exists a normalised holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ whose Beltrami coefficient is $\hat{\nu}_\lambda$. Since the map F_0 preserves the Beltrami coefficient $\hat{\nu}_\lambda$, the map

$$G_\lambda = H_\lambda \circ F_0 \circ H_\lambda^{-1} : H_\lambda(U) \to H_\lambda(V)$$

is a complex box mapping. It also depends holomorphically on λ.

Lemma 9. *Take $\lambda \in \Lambda \cap \mathbb{R}$. Then maps $F_0 : U \to V$ and $F_\lambda : h_\lambda(U) \to h_\lambda(U)$ are combinatorially equivalent if and only if $F_\lambda = G_\lambda$ where defined.*

$\lhd$ It is obvious that $F_\lambda = G_\lambda$ implies combinatorial equivalence.

From the Rigidity Theorem for box mappings we know that there exists a q.c. homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ which conjugates F_0 and F_λ on their postcritical set and equals to h_λ on ∂U.

Construct a new qc homeomorphism ψ^0 which is equal to H_λ outside U and equal to ϕ on $U \cap \mathbb{R}$. Define inductively a sequence $\psi^k(x) = F_\lambda^{-1} \circ \psi^{k-1} \circ F_0(x)$, $x \in U$. The Beltrami coefficient of $\{\psi^k\}$ is uniformly bounded with respect to k, so we can take a convergent subsequence. Denote the limit by Ψ. The Beltrami coefficient of Ψ is equal to $\hat{\nu}_\lambda$. Indeed, outside of the Julia set of $F_0 : U \to V$ it follows from the construction and on the Julia set the Beltrami coefficient is zero because of the absence of the invariant line field on the Julia set. For the case of the minimal ω–limit set of the critical point it is proved in [She03b]. If the ω–limit set is not minimal, then even a stronger statement holds: the Julia set of F_λ has zero Lebesgue measure set of the. This directly follows from the forth property of the box mapping F in Theorem 3. Thus, Ψ is equal to H_λ and $F_\lambda = G_\lambda$. $\rhd$

This lemma implies that $F_\lambda = G_\lambda$ for $\lambda \in [-\epsilon, 0]$ for some small $\epsilon > 0$. By analyticity of these families, therefore $F_\lambda = G_\lambda$ for all $\lambda \in \Lambda$. This contradicts the choice of λ_0, and completes the proof of Theorem 5 in the recurrent case.

Now we have to consider the Misiurewicz case in which the critical point c_0 is non-recurrent. Since c_0 is prime there are no critical points in $\omega(c_0)$. Take a neighbourhood U of $\omega(c_0)$ containing no critical points and neutral or attracting periodic points. The set of points in U which never leaves U under iterates of f_0 is hyperbolic. Moreover, for some n we have $f_0^n(c_0) \in U$. This hyperbolic set persists for small values of $|\lambda|$, and we have a holomorphic motion h_λ of this set. The maps f_0 and f_λ are combinatorially equivalent if and only if we have

$$h_\lambda(f_0(c_0)) = f_\lambda(c_0(\lambda)).$$

This equation is analytic in λ, and we can argue as before, completing the proof of Theorem 5. □

There exist two important versions of the theorem above where the regularity condition is not needed:

Theorem 6. *Let f_λ, $\lambda \in (-1, 1)$, be an analytic family of real–symmetric complex box mappings. Suppose that*

- *for any λ all real critical points of f_λ are non-degenerate;*
- *the map f_0 has no neutral or attracting cycles;*
- *there is a recurrent real critical point c_0 of f_0 and its itinerary is not constant for all $\lambda \in (-1, 1)$;*
- *if an iterate of some critical point is mapped onto another critical point, then it is so for all λ.*

Then there exist a critical point c and arbitrarily small λ such that the itineraries $\nu_{f_0}(c)$ and $\nu_{f_\lambda}(c)$ are different.

Theorem 7. *Let f_λ, $\lambda \in (-1, 1)$, be an analytic family of real–analytic maps of interval without degenerate critical points. Suppose that*

- *the map f_0 has no neutral cycles;*
- *there is a critical point c_0 of f_0 such that the itinerary $\nu_{f_\lambda}(c_0(\lambda))$ changes with λ and ω–limit set of c_0 under f_0 is a minimal set;*
- *attracting cycles of f_0 do not bifurcate as $\lambda \in (-1, 1)$ varies;*
- *if an iterate of some real critical point $\hat{c}$ is mapped onto another (or the same) critical point under f_0, then this iterate is mapped onto the critical point for all λ;*

- *if iterates of some real critical point $\hat{c}$ converge to a periodic attractor under f_0, then the iterates of $\hat{c}$ converge to a periodic attractor for all λ.*

Then there exists a non-periodic critical point c which changes its itinerary $\nu_{f_\lambda}(c)$ in any arbitrarily small neighbourhood of $\lambda = 0$.

The proof of these versions of the Theorem 5 is essentially identical to the proof given above. The rigidity condition is automatically satisfied in the case of Theorem 6 and it is not needed in Theorem 7 because in that case the complex box mapping for f_0 has just finitely many branches.

5 Perturbations with more critical relations

Let f be a real polynomial. We want to find hyperbolic polynomials of the same degree arbitrarily close to f.

We may assume (see Lemma 10 below) that all critical points of f (including complex ones) are non-degenerate and that f has no neutral periodic points (again including complex). Such polynomials we will call *admissible*.

Now we will describe an inductive procedure which will allow us to obtain a hyperbolic polynomial from the given polynomial in finitely many steps. First we introduce a few definitions.

By a *critical relation* for f we mean a triple (n, c_i, c_j) such that c_i, c_j are critical points of f, $f^n(c_i) = c_j$ and $n > 0$. If the iterates of a real critical point c of f converge to some periodic attractor or some iterate of c lands on a critical point of f, then the critical point c is called *controlled*.

Proposition 4. *Suppose f is an admissible real polynomial with K controlled critical points and suppose that K is less than the number of real critical points of f. Then arbitrarily close to f, one can find an admissible real polynomial of the same degree with $K + 1$ controlled critical points.*

This proposition clearly implies the main theorem (density of hyperbolicity). Indeed, in a few steps we obtain an admissible polynomial with all real critical points controlled, which means it is Axiom A.

In the rest of this section we will prove this proposition.

The proof of the proposition will be carried out in three steps. In Step I we construct a C^3 perturbation of the map, however this perturbation can still be included in an analytic family of complex box mapping. This step can be skipped if there is a prime critical point with minimal ω–limit set. In Step II we show that one can construct a non-trivial polynomial family passing through f to which one of the Theorems 5, 6, or 7 applies. In the last step we will show that there are polynomials of the same degree as f, close to f and conjugate to certain maps from the family constructed in the previous step.

Consider prime critical points of f. Obviously, if all these points are controlled, then all critical points are controlled. So, there is a prime non controlled critical point c_0. If this critical point is non recurrent, the situation is rather simple and can be done by simplifications of arguments below. So, we will assume that c_0 is recurrent.

Step I. Here we construct a C^3 perturbation of f (in the same way as in [Koz03]). Suppose $\omega(c_0)$ is non–minimal (minimal case will be considered in the next step). Due to Theorem 3, there exists a box mappings $F : U \to V$ for the map f such that $c_0 \in U$. and there are universal constants $\theta_1 \in (0, \pi)$, $C_1 > 0$ such that for any connected component U' of U, we have that $f(U')$ is contained in $D_{\theta_1}(f(U') \cap \mathbb{R})$ and moreover, if $U' \subset \mathrm{Comp}_{c_0}(V)$ then the C_1–scaled neighbourhood of $U' \cap \mathbb{R}$ is contained in V.

Let a be a real boundary point of the domain $\mathrm{Comp}_{c_0} V$. Consider the following perturbation of the map f:

$$f_\lambda(x) = \begin{cases} f(x) & , \quad x \notin \mathrm{Comp}_{c_0} V \\ f(x) + \lambda \frac{(f(x)-f(a))^4}{(f(c_0)-f(a))^3} & , \quad x \in \mathrm{Comp}_{c_0} V \end{cases}$$

Notice that for all λ the map f_λ is C^3.

For constants θ_1 and C_1 there exists $\lambda_1 > 0$ such that for any $\lambda \in [-\lambda_1, \lambda_1]$ and given complex box mapping induced by f_0 the map f_λ induces a box mapping F_λ with the same domain V as for the map f_0 and a deformed domain U^λ.

By choosing the complex box mapping F appropriately, we can assume that $|f(\mathrm{Comp}_{c_0} U) \cap \mathbb{R}|/|f(\mathrm{Comp}_{c_0} V) \cap \mathbb{R}|$ is very small, so that the critical value $f_{\lambda_1}(c_0)$ is not in $f(\mathrm{Comp}_{c_0} U)$. This implies that the map f_{λ_1} is not essentially combinatorially equivalent to f, so we obtain a family satisfying the conditions of Theorem 6. Note also that provided that V is small enough, all controlled critical points of f are still controlled for all maps f_λ with $\lambda \in [-\lambda_1, \lambda_1]$.

The map f is admissible and, hence, regular, therefore it has a C^3 neighbourhood W consisting of regular maps (the proof of this statement for multimodal maps is the same as in [Koz03], Lemma 4.6, where instead of the results for the negative Schwarzian condition of [Koz00], one uses its generalisation [vSV00]). By shrinking this neighbourhood if necessary, we may assume that the following hold:

- all (hyperbolic) periodic attractors of f persist in this neighbourhood
- if a real critical point of f is contained in the attracting basin of some periodic attractor, then for all $g \in W$, the corresponding critical point is contained in the attracting basin of the corresponding periodic attractor.

Combining this observation and Theorem 6 we get that there exists $\lambda_2 \in (-\lambda_1, \lambda_1)$ such that f_{λ_2} is not essentially combinatorially equivalent to f and such that f_{λ_2} is in W.

Step II. Construction of a family satisfying conditions of Theorem 5 or 7. *Case 1:* the set $\omega(c_0)$ is non–minimal.

We can approximate f_{λ_2} by a real polynomial g which is still contained in W, is not essentially combinatorially equivalent to f, and has the same real critical points as f (the degree of g can be greater then the degree of f). Moreover, we can join f and g by a polynomial family of maps g_λ all of which are in W and, thus, this family satisfies all conditions of Theorem 5.

Case 2: the set $\omega(c_0)$ is minimal.

In this case we do not have to care about the regularity of the family. We can construct a real–analytic family of polynomials g_λ satisfying the following:

- $g_0 = f$;
- All real critical points of g_λ are the same;
- The controlled critical points of f are also controlled critical points of g_λ;
- The degree of g_λ is bounded by a constant independent of λ.

The construction of such a family can be easily done in the following way. Suppose that all real critical points of f are in the interval $(-1, 1)$ and that iterates of all real points outside of this interval are attracted to infinity. Let U be a neighbourhood of c_0 containing no other critical points and such that the iterates of the controlled critical points never visit $U \setminus c_0$. Let $\hat{g}$ be a C^∞ function equal to f outside of U, having the same real critical points as f, and $\hat{g}(c_0) \notin (-1, 1)$, so the itinerary of c_0 of maps f and $\hat{g}$ are different. Fix some neighbourhood V of all periodic attractors of f outside of U which is properly contained in their immediate basins of attraction. There is a neighbourhood W of $\hat{g}$ in the space of C^2 maps such that for all maps in W iterates of all points in V are attracted to some periodic attractors in V. For any controlled critical point c of f some of its iterate must be either in V or coincide with another critical point. Fix a piece of trajectory of c until it gets to V or is mapped on another critical point.

Now we can approximate $\hat{g}$ on $(-1, 1)$ by a polynomial g_1 in C^2 topology in such a way that

- g_1 is in W;
- All real critical points of $\hat{g}$ are also critical points of g_1;
- The values of g_1 on the fixed pieces of trajectories of the controlled critical points coincide with the corresponding values for $\hat{g}$ (and, hence, f);
- The second derivative of g_1 is so close to the second derivative of $\hat{g}$, that the map g_1 does not have new critical points (recall that all critical points of f are non degenerate).

There exists a small $\epsilon > 0$ such that the function $f - \lambda g_1$ for $\lambda \in [0, \epsilon]$ has only non-degenerate critical points. The family $g_\lambda = (1 - \lambda)f + \lambda g_1$, $\lambda \in [-\epsilon, 1]$, is the required family satisfying all conditions of Theorem 7.

Step III.

So far, we have obtained a polynomial family g_λ going through $f = g_0$ and satisfying conditions of either Theorem 5 or Theorem 7. In either case we conclude that there is neighbourhood L of zero in the parameter space such that the combinatorial types of maps in $L \setminus 0$ and f are different.

Lemma 10. *Any real polynomial g can be approximated by an admissible real polynomial $\hat{g}$ of the same degree in such a way that the number of controlled critical points of $\hat{g}$ is larger or equal than the number of controlled critical points of g.*

$\triangleleft$ We will prove the existence of $\tilde{g} \in W$ without neutral cycles. The perturbation to a polynomial without degenerate critical points is rather trivial.

Let us fix some neighbourhood of all periodic attractors of g such that iterates of any point in this neighbourhood converge to some attractor cycle. Let W be a so small neighbourhood of g that for maps in W all points in the fixed neighbourhood of attractors still converge to periodic cycles. For each controlled critical point of g we can fix some finite piece of its orbit which we have to control: if some iterate of a controlled critical point lands on a critical point, we shall control this piece of orbit (all iterates between the controlled critical point and the point it is eventually mapped on); otherwise we we shall control the orbit of the controlled critical until its iterate is mapped in the fixed neighbourhood of the attractors. We can construct a perturbation family of g in such a way that the values at the points of the controlled pieces of orbits of the controlled points are fixed and all neutral cycles of g become attracting cycles as it is done, for example, in the proof of Theorem VI.1.2 in [CG93]. If a polynomial has a neutral cycle of some period n, its coefficients satisfy some polynomial equation. This implies that either in this family all maps have a neutral cycle of the same period or the set of parameters whose corresponding maps have a neutral cycle is countable. In the latter case we can get a map arbitrarily close to g without neutral cycles and the required number of controlled critical points.

So, suppose we are in the former case. Take a map g_1 in the family close to g. All neutral cycles of g are attracting cycles of g_1 and g_1 has some extra neutral cycles. Apply to g_1 the same perturbation procedure as in the beginning of the proof of this lemma. The maps in the new family close to g_1 have attracting cycles inherited from g_1 plus attracting cycles converted from the neutral cycles of g_1. Again, if all maps in this new family have neutral cycles, we take a map g_2 close to g_1 and continue the procedure. Since all maps g, g_1, g_2, ... are polynomials of the same degree, and g_{k+1} has strictly more attracting cycles than g_k, this procedure will stop in finitely many steps. $\triangleright$

Theorems 5, 7 imply that the combinatorial type of the map f changes with arbitrarily small change of the parameter λ in the family g_0. Any change of the combinatorial type corresponds to the creation of a new controlled critical point. So, we get a sequence of maps $\{g_{\lambda_i}\}$ converging to f and having strictly more controlled critical points than f. Due to the lemma above, if some map g_{λ_i} is not admissible, we can approximate it by admissible polynomial $\tilde{g}_i$ of the same degree as g_{λ_i} and such that $\tilde{g}_i$ has the same controlled critical points as g_{λ_i}.

Now we can complete the proof of Proposition 4.

Lemma 11. *For any polynomial f and the neighbourhood W of this polynomial (in the space of polynomials of the same degree) there exist $R > 0$ and $\delta > 0$ such that the following holds.*

Let D be a disk of radius R centred at 0 and let $g : D \to \mathbb{C}$ be a holomorphic map such that $\|g|_D - f|_D\| < \delta$. Then there exists a polynomial $\tilde{f} \in W$ conjugate to g in D.

$\lhd$ The proof of this lemma is the same as the proof of the Straightening Theorem (one should notice that in the case of the lemma above it is possible to construct a q.c. conjugating homeomorphism with an arbitrarily small dilatation). $\rhd$

From the previous section, for any fixed bounded domain $\Omega \subset \mathbb{C}$ we have a uniformly convergent sequence of hyperbolic polynomials $\tilde{g}_i \to f$. Using the lemma above we can construct a sequence of polynomials of the same degree as f which converge to f and are conjugate to $\tilde{g}_i$. These polynomials are admissible and have more controlled critical points than f. This completes the proof of Proposition 4 and of the Main Theorem.

References

[AA02] C. G. Moreira A. Avila, *Statistical properties of unimodal maps: the quadratic family*, preprint, to appear in the Annals, 2002.

[ALdM03] Artur Avila, Mikhail Lyubich, and Welington de Melo, *Regular or stochastic dynamics in real analytic families of unimodal maps*, Invent. Math. **154** (2003), no. 3, 451–550. MR 2 018 784

[BM00] Alexander Blokh and Michał Misiurewicz, *Typical limit sets of critical points for smooth interval maps*, Ergodic Theory Dynam. Systems **20** (2000), no. 1, 15–45. MR 2001d:37041

[BSvS04] Henk Bruin, Weixiao Shen, and Sebastian van Strien, *Existence of unique srb-measures is typical for unimodal families*, preprint, 2004.

[CG93] Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR 94h:30033

[dMvS93] Welington de Melo and Sebastian van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993. MR 95a:58035

[GŚ96] Jacek Graczyk and Grzegorz Świątek, *Polynomial-like property for real quadratic polynomials*, Topology Proc. **21** (1996), 33–112. MR 99k:58156

[GŚ97] Jacek Graczyk and Grzegorz Światek, *Generic hyperbolicity in the logistic family*, Ann. of Math. (2) **146** (1997), no. 1, 1–52. MR 99b:58079

[GŚ98] ———, *The real Fatou conjecture*, Annals of Mathematics Studies, vol. 144, Princeton University Press, Princeton, NJ, 1998. MR 2000a:37020

[Jak71] M. V. Jakobson, *Smooth mappings of the circle into itself*, Mat. Sb. (N.S.) **85 (127)** (1971), 163–188. MR 44 #7587

[Koz00] O. S. Kozlovski, *Getting rid of the negative Schwarzian derivative condition*, Ann. of Math. (2) **152** (2000), no. 3, 743–762. MR 2002e:37050

[Koz03] ———, *Axiom A maps are dense in the space of unimodal maps in the C^k topology*, Ann. of Math. (2) **157** (2003), no. 1, 1–43. MR 2004b:37052

[KSvS03] O.S. Kozlovski, W. Shen, and S. van Strien, *Rigidity for real polynomials*, preprint, 2003.

[LvS98] Genadi Levin and Sebastian van Strien, *Local connectivity of the Julia set of real polynomials*, Ann. of Math. (2) **147** (1998), no. 3, 471–541. MR 99e:58143

[LvS00] ———, *Bounds for maps of an interval with one critical point of inflection type. II*, Invent. Math. **141** (2000), no. 2, 399–465. MR 2001i:37061

[LY97] M. Lyubich and M. Yampolsky, *Dynamics of quadratic polynomials: complex bounds for real maps*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 4, 1219–1255. MR 98m:58113

[Lyu97] Mikhail Lyubich, *Dynamics of quadratic polynomials. I, II*, Acta Math. **178** (1997), no. 2, 185–247, 247–297. MR 98e:58145

[Lyu02] ———, *Almost every real quadratic map is either regular or stochastic*, Ann. of Math. (2) **156** (2002), no. 1, 1–78. MR 2003i:37032

[McM94] Curtis T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies, vol. 135, Princeton University Press, Princeton, NJ, 1994. MR 96b:58097

[Pal00] Jacob Palis, *A global view of dynamics and a conjecture on the denseness of finitude of attractors*, Astérisque (2000), no. 261, xiii–xiv, 335–347, Géométrie complexe et systèmes dynamiques (Orsay, 1995). MR 2001d:37025

[She03a] Weixiao Shen, *Bounds for one-dimensional maps without inflection critical points*, J. Math. Sci. Univ. Tokyo **10** (2003), no. 1, 41–88. MR 1 963 798

[She03b] ———, *On the measurable dynamics of real rational functions*, Ergodic Theory Dynam. Systems **23** (2003), no. 3, 957–983. MR 2004e:37069

[She04] ———, *On the metric properties of multimodal interval maps and c^2 density of axiom a*, Invent. Math **156** (2004), no. 2, 301–403.

[Sma98] Steve Smale, *The work of Curtis T. McMullen*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), no. Extra Vol. I, 1998, pp. 127–132 (electronic). MR 99k:58001

[Sma00] ———, *Mathematical problems for the next century*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 271–294. MR 2001i:00003

[vSV00] S. van Strien and E. Vargas, *Real bounds, ergodicity and negative schwarzian for multimodal maps*, accepted by Journal of AMS, 2000.