TOTAL DISCONNECTEDNESS OF JULIA SETS AND ABSENSE OF INVARIANT LINEFIELDS FOR REAL POLYNOMIALS

by

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Abstract. — In this paper we shall consider real polynomials with one (possibly degenerate) non-escaping critical (folding) point. Necessary and sufficient conditions are given for the total disconnectedness of the Julia set of such polynomials. Also we prove that the Julia sets of such polynomials do not carry invariant linefields. In the real case, this generalises the results by Branner and Hubbard [**BH**] for cubic polynomials and by McMullen [**McM**] on absense of invariant linefields.

1. Introduction

In a paper by Branner and Hubbard [**BH**], cubic polynomials were considered, and the problem was solved when the Julia set of such a polynomial is totally disconnected (for the history of this problem see [**BH**], Ch. 5). In the same paper, the question was raised whether this result could be extended to polynomials of higher degrees. The method and results of [**BH**] hold for polynomials P of higher degrees with all but one critical points escaping to infinity, under the condition that the unique non-escaping critical point c is *simple*: $P'(c) = 0, \quad P''(c) \neq 0$, see [**BH**], Ch. 12.

On the other hand, if the non-escaping critical point c is *multiple*, i.e.,

$$P''(c) = \cdots = P^{(\ell-1)}(c) = 0, \quad P^{(\ell)}(c) \neq 0,$$

for some $\ell > 2$, the method of [**BH**] breaks down (see [**Dou1**] for a discussion on this). The positive integer ℓ is called the *multiplicity*, or *local degree* of the critical point c of the polynomial P.

In this paper we shall prove

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Theorem 1.1. — *Let P be a polynomial with real coefficients, such that one (maybe multiple) critical point c of P of even multiplicity ℓ has a bounded orbit, and all other critical points escape to infinity. Then*

- *the filled Julia set of P:*

$$K(P) = \{z : \{P^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}$$

  is totally disconnected if and only if the connected component of the real trace $K(P) \cap \mathbb{R}$ of the filled Julia set containing the critical point c, is equal to a point
- *the Julia set $J(P) = \partial K(P)$ carries no measurable invariant linefields.*

Remark 1.1. — *For the case that the multiplicity is odd, see* [**LS2**].

Remark 1.2. — *The map $P\colon \mathbb{R} \to \mathbb{R}$ does not have a wandering interval (on the real line) with bounded orbit* [**MS**]. *Hence, the condition: "the connected component of the real trace $K(P) \cap \mathbb{R}$ of the filled Julia set which contains the critical point c, is equal to a point" is equivalent to one of the following conditions:*

- *the component of the filled Julia set $K(P)$ containing the non-escaping critical point, is non-periodic;*
- *P does not have an attracting or neutral periodic orbit, and is not renormalizable on the real line (i.e., there is no interval I around $c = 0$, such that $P^i(I) \cap P^j(I) = \emptyset$ for $0 \le i < j \le q-1$ and $P^q(I) \subset I$);*
- *the intersections of the critical puzzle pieces with the real line shrink to the point c (see the next Section).*

Remark 1.3. — *We allow escaping critical points to be non real. Of course, since P is real, the orbit of the unique non-escaping critical point is real. The theorem holds in particular for maps of the form $f(z) = z^\ell + c_1$ when c_1 is real.*

The second part of the Theorem extends the main result of McMullen in [**McM**]. As usual, we say that the Julia set $J(P)$ of P carries a *measurable invariant line field* if there exists a measurable subset E of the Julia set of P and a measurable map which associates to Lebesgue almost every $x \in E$ a line $l(x)$ through x which is P-invariant in the sense that $l(P(x)) = DP(x)\,l(x)$. (So the absence of linefields is obvious if the Julia set has zero Lebesgue measure.) The absence of invariant linefields was proved by McMullen for all maps of the form $P(z) = z^\ell + c_1$, ℓ is even and c_1 is real, which are infinitely renormalizable. If P is quadratic (i.e., $\ell = 2$) and only finitely often renormalizable then this holds because then the corresponding parameter c_1 lies at the boundary of the Mandelbrot set, see [**Y**], [**H**]. (Actually, the result of Yoccoz is much stronger: local connectivity of the boundary of the Mandelbrot set at such points).

The (non-)existence of the invariant linefields is strongly related to the Density of Hyperbolicity Conjecture, see [**MSS**]. It follows from the second part of the Theorem, because of Theorem E of [**MSS**], that for any polynomial P as in the Theorem, there exists another (maybe complex) polynomial Q of the same degree and with the same multiplicity ℓ of the non-escaping critical point $c = 0$, which is as close to P as we wish and such that Q is hyperbolic: every critical point of Q tends either to infinity, or to an attracting periodic orbit. (One can assume that $P(z) = z^\ell \cdot p(z) + t$, where $p(z) = z^m + \cdots + p_m$ is a monic polynomial of the degree $m \geq 0$, and $p_m \neq 0$. Then the polynomial Q as above is of the same form, and P and Q are considered as points of the space $\mathbb{C}^m \times \mathbb{C}$). In fact, for $\ell = 2$ a much stronger statement is true, see for example [**Sw**]: the polynomial Q as above can be chosen real as well.

The proof of the Theorem is postponed until Section 3, and is based on Propositions 2.1- 2.4 of the next section.

Propositions 2.1 and 2.3 give sufficient conditions for the total disconnectedness of Julia set and absence of invariant linefields. They could be applied for complex polynomials as well. Nevertheless, we can only show that this condition is satisfied in the case considered in the theorem: see Proposition 2.4 and Sect.3.

A similar problem exists for the odd multiplicities. If P is a polynomial as in the Theorem, but with ℓ an odd number (say, cubic one), then the Theorem is easy if there are no other critical points (i.e., $P(z) = z^\ell + c_1$, where c_1 is real and ℓ is odd), because then the map is monotone on the real line. On the other hand, if other (escaping) critical points exist, we can still apply Propositions 2.1, but the implementation of it (a statement like Proposition 2.4) uses different methods, see [**Le**] and [**LS2**].

Before giving the proofs, let us make a remark about the non-minimal case (i.e., when the postcritical set $\omega(c)$ contains c, but the system restricted to $\omega(c)$ is not minimal). Such system is relatively simple when it is *real* (see Proposition 3.2 of [**LS1**] and Proposition 2.5 below, or see [**Ly**]). But this is definitely not the case for complex parameters:

Remark 1.4. — *In each of Douady's examples of an infinitely renormalizable quadratic map f with a non-locally connected Julia set, the postcritical set is non-minimal. Indeed, according* [**P-M**] *there exists an invariant Cantor set on which the map is injective. If this Cantor set does not intersect the postcritical set, then according to a well-known result of Mañé* [**Ma**] *the map is expanding on this set. This is impossible since f is injective on this set, see* [**Dou2**].

2. Associated mappings and complex bounds

Let $G\colon \cup_{i=0}^{i_0} \Omega^i \to \Omega$ be an *ℓ-polynomial-like mapping*. As in [**DH**], [**LM**], [**LS1**] this means that all Ω^i are open topological discs with pairwise disjoint closures which are compactly contained in the topological disc Ω, the map $G\colon \Omega^0 \to \Omega$ is ℓ-to-one holomorphic covering with a unique critical point $c = 0 \in \Omega^0$, and that each other map $G\colon \Omega^i \to \Omega$ is a conformal isomorphism. Its *filled Julia set* is defined by

$$K(G) = \{z : G^n(z) \in \cup_{i=0}^{i_0} \Omega^i, n = 0, 1, \dots\}.$$

The boundary $\partial K(G)$ is called the *Julia set* $J(G)$ of G.

The *puzzle* (see [**BH**]) of the map G is said to be the set of connected components of all preimages $G^{-k}(\Omega)$, $k = 0, 1, \dots$. A *piece* of level $k \geq 0$ is a connected component of $G^{-k}(\Omega)$. A piece is *critical* if it contains the critical point c of G. We call another ℓ-polynomial-like mapping $G'\colon \cup_{i=0}^{i_0'} \Omega'^i \to \Omega'$ *associated to* or *induced by* G if G' restricted to each Ω'^i is some iterate $G^{j(i)}$ of G. We also call G *real* iff all topological discs Ω^i, Ω are symmetric w.r.t. the real axis, and $\overline{G(z)} = G(\overline{z})$, for any $z \in \cup_{i=0}^{i_0}\Omega^i$. In particular, this implies that the postcritical set of the unique critical point $c = 0 \in \Omega_0$ is real.

Proposition 2.1. — *Fix a real ℓ-polynomial-like mapping G. Assume there exists an infinite sequence $G(j)\colon \cup_i \Omega^i(j) \to \Omega(j)$ of real ℓ-polynomial-like mappings associated to the mapping G with $\omega(c)$ minimal such that the critical point $c = 0 \in \Omega^0(j)$ of $G(j)$ does not escape the domain of $G(j)$ under iterations of $G(j)$. Assume moreover that*

1. *each $\Omega^i(j) \cap \mathbb{R}$ coincides with the intersection of some piece of G with the real line*
2. *when $G^i(c) \in \Omega^0(j)$ then $G^i(c)$ is an iterate of c under $G(j)$ (we call this* **the first return condition for G on $\Omega^0(j)$***);*
3. *the modulus of the annuli $\Omega(j) \setminus \Omega^0(j)$ is uniformly bounded away from zero;*
4. *the diameter of $\Omega(j)$ tends to zero as $j \to \infty$.*

Then the filled Julia set of G is totally disconnected.

Remark 2.1. — *Conditions 1. and 4. obviously imply the third condition of Remark 1.2: the traces of the critical pieces on the real line shrink to the point.*

Remark 2.2. — *The proposition also holds for a complex (i.e., not real) map G, if one replaces 1. by the following condition:*

for each j, one can find another ℓ-polynomial-like mapping $R(j)$ having as its range a critical piece $P(j)$ of G, so that the two mappings $G(j)$ and $R(j)$ satisfy the conditions of Proposition 2.1 from **[LS1]**.

Proof. — Let us first observe that the first return condition also holds for the first return map of $G(j)$ to $\Omega^0(j)$. Indeed, consider the first return map of $G(j)$ to $\Omega^0(j)$ along the iterates of the critical point. This is again a real ℓ-polynomial-like mapping $\tilde{G}(j)\colon \cup_i \tilde{\Omega}^i(j) \to \Omega^0(j)$. Obviously, if $G^i(c) \in \Omega^0(j)$ then $G^i(c)$ is an iterate of c under $\tilde{G}(j)$. In addition, the modulus of $\Omega^0(j) \setminus \tilde{\Omega}^0(j)$ is greater or equal to $\frac{1}{\ell}$ mod $(\Omega(j) \setminus \Omega^0(j))$. So we may replace $G(j)$ by the first return map to $\Omega^0(j)$ and therefore in the remainder of the proof we can and will assume that the first return condition even holds on $\Omega(j)$ (by renaming everything). Note, that this condition is crucial in the proof of *Claim 1* below.

Fix j and let $P(j)$ be the open piece of G based on $\Omega(j) \cap \mathbb{R}$. Let $P^i(j)$ be a piece of G based on $\Omega^i(j) \cap \mathbb{R}$. Since $G(j)$ is associated to G, each restriction of $G(j)$ to $\Omega^i(j) \cap \mathbb{R}$ is an iterate of G. This gives another real ℓ-polynomial-like mapping $G'(j)\colon \cup_i P^i(j) \to P(j)$, which is also associated with G and coincides with $G(j)$ on the real line.

Now we are going to use the following statement from **[LS1]** (called Proposition 2.3 in that paper).

Proposition 2.2. — *Let*

$$G_1 : \Omega_1^0 \bigcup \Omega_1^1 \bigcup \cdots \bigcup \Omega_1^r \to \Omega_1,$$

$$G_2\colon \Omega_2^0 \bigcup \Omega_2^1 \bigcup \cdots \bigcup \Omega_2^r \to \Omega_2$$

be two real ℓ-polynomial-like mappings, with the common critical point $c \in \Omega_1^0 \cap \Omega_2^0$, and assume that the following conditions hold:

1. *$G_1(z) = G_2(z)$ whenever $z \in \cup_{i=0}^r \Omega_1^i \cap \Omega_2^i$.*
2. *Denoting $I_k = \Omega_k \cap \mathbb{R}$ and $I_k^i = \Omega_k^i \cap \mathbb{R}$, one has $I_2 \subsetneq I_1$, $I_2^i \subsetneq I_1^i$.*

Under these conditions, the Julia sets of G_1 and G_2 coincide. If, additionally, the point c lies in the Julia set of G_1 (and, hence of G_2), then there exists a component of a preimage $G_1^{-n}(\Omega_1)$, which contains c and is contained in Ω_2.

Applying this proposition, we conclude that the Julia sets of $G(j)$ and $G'(j)$ coincide for every j. On the other hand, the Julia set is the intersection of the full preimages of the range. Hence, there exists a large integer N such that the full preimage $G'(j)^{-N}(P(j))$ is inside the domain of definition $\cup_i \Omega^i(j)$ of $G(j)$. Note that $G'(j)^{-N}(P(j))$ consists of finitely many (open) pieces of G. In particular, since the critical point c does not escape under the map $G(j)$, we obtain, that there is a critical piece of G inside the domain $\Omega(j)$. As the diameters of $\Omega(j)$ tends to zero, the intersections of the critical pieces is the point.

Let us consider the pieces of $G'(j)^{-N}(P(j))$ inside the central domain $\Omega^0(j)$, i.e.,

$$P'(j) = G'(j)^{-N}(P(j)) \cap \Omega^0(j).$$

If z is in the Julia set but $\omega(z)$ does not hit some critical piece then $\{z\}$ is a component of $J(G)$ by [**BH**]. So choose a point z from the Julia set of G so that the forward orbit of z hits every critical piece. Then there exists a *minimal* $K = K(j)$ such that $G^K(z) \in P'(j)$. In particular, the point $G^K(z)$ belongs to one of the pieces inside $\Omega^0(j)$. Let B_j be the branch of G^{-K} which maps a neighbourhood of $G^K(z)$ to a neighbourhood of z.

Claim 1. The map B_j extends to a holomorphic map on $\Omega(j)$.

Proof of the claim. Assume the contrary. We then get for some minimal $r < K$ that $G^{-r}(j)(\Omega(j))$ (along the same orbit) meets the critical value $c_1 = G(c)$. This means that the branch G^{-r} follows the points $c_{r+1} = G^r(c_1) \in \Omega$, $c_r = G^{r-1}(c_1), \ldots, c_2 = f(c_1), c_1$. Among these iterates of c_1, let us mark all those $c_{j_1}, c_{j_2}, \ldots, c_{j_m}$, where $j_1 < j_2 < \cdots < j_m$, which hit the domain $\Omega(j)$. Because of *the first return condition* there exist integers $k(1) < k(2) < \ldots$ such that $c_{j_1} = G^{k(1)}(c)$, $c_{j_2} = G^{k(2)-k(1)}(c_{j_1}) = G^{k(2)}(c), \ldots, c_{j_m} = G^{k(m)}(c)$. It follows, that $G^{-r} = G^{-(s-1)} \circ G^{-(k(m)-1)}$, where $G^{-(s-1)}$ is the branch corresponding to the restriction of $G(j)$ on $\Omega^0(j)$ (so $G(j)|\Omega^0(j) = G^{s-1} \circ G$). Hence, $G^{-(r+1)}(\Omega(j)) \subset G^{-k(m)}(j)(\Omega(j)) \subset \Omega^0(j)$ and $G^{K-r-1}(z) \in G^{-(r+1)}(P'(j)) = (G(j)|_{\Omega^0(j)})^{-1} \circ G(j)^{-k(m)+1}(P'(j)) \subset P'(j)$. This contradicts the minimality of K and proves the claim.

Let $P_j(z) = B_j(P'(j))$. We want to show that the Euclidean diameters of $P_j(z)$ tends to zero as $j \to \infty$. For this, let us consider the domain $M_j' \subset \Omega(j)$ bounded by the core curve of the annulus $\Omega(j) \backslash \Omega^0(j)$. Then $\max_{y \in \partial M_j'} |G^K(z) - y| / \min_{y \in \partial M_j'} |G^K(z) - y| \le C$ for all $j = 1, 2, \ldots$ where C only depends on the uniform bound for the moduli of $\Omega(j) \setminus \Omega^0(j)$ (see e.g. [**McM**]). Define $E_j = B_j(M_j')$. Since the modulus of the annulus $\Omega(j) \setminus M_j'$ is half the modulus of $\Omega(j) \setminus \Omega^0(j)$, by the Koebe Distortion Theorem,

$\max_{y\in\partial E_j}|z-y|/\min_{y\in\partial E_j}|z-y|\le C_1$, $j=1,2,\ldots$. If we assume by contradiction that $\mathrm{diam}(P_j(z))\ge d>0$ for $j=1,2,\ldots$, then $\min_{y\in\partial E_j}|z-y|\ge d/2C_1=r>0$, i.e., the disc $D_z(r)\subset E_j$. Hence, $G^K(D_z(r))\subset M'_j$, for $j\to\infty$ and $K=K(j)\to\infty$. This contradicts the assumption that z is in the boundary of the filled Julia set of G. Thus, $\bigcap_{j>0}P_j(z)=\{z\}$ and so $\{z\}$ is a component of $J(G)$. □

We shall derive the absence of invariant linefields from the following statement, which is very similar to the previous one.

Proposition 2.3. — *Under the conditions of Proposition 2.1, the Julia set of G carries no invariant linefield.*

Remark 2.3. — *In fact, the proof of this statement is a purely complex one, and, therefore, holds for every (complex) map G, such that $\omega(c)$ is minimal and conditions 2-4 of Proposition 2.1 are satisfied.*

Proof of Proposition 2.3. — We will follow the main idea of Theorem 10.3 in [**McM**] (absence of invariant line field for infinitely renormalizable quadratic polynomial with complex apriori bounds). Let us first outline the differences with the proof in [**McM**].

First, our renormalizations are ℓ-(i.e., generalised) polynomial-like maps; hence their Julia sets are not connected, and the number of the components in the domain of definition of these maps can increase. To overcome this, we consider the dynamics only on the central domains. The second problem is that the central domain $\Omega_0(j)$ can become smaller and smaller compared to the range $\Omega(j)$, so that the range of the limit dynamics can be the whole plane (after rescaling $\Omega_0(j)$ to a definite size). To avoid this, we shall rescale $\Omega_0(j)$ and $\Omega(j)$ by different factors. The third difference is that in our setting the critical value of the 'limit dynamics' can escape to the boundary of the range. For this, we extend the dynamics passing to the first return maps. Fourthly, in [**McM**] a contradiction against the existence of a measurable invariant linefield (defined on a set E) is obtained through a univalent map from the range of the renormalization to a neighbourhood of a point of density of the set E. In our setting we cannot argue like that, and instead we use two consecutive first return maps. So let us prove the proposition:

1. Fix for a moment a mapping $G=G(j)$, and consider the first return to its central domain. We obtain in this way another mapping $G'=G'(j)$ (we drop the index j). Its central branch $G'_0\colon\Omega'_0\to\Omega_0$ extends to an ℓ-covering $G'_0\colon\widetilde{\Omega}'_0\to\Omega$ (we keep the same notation for the extension), where $\widetilde{\Omega}'_0\subset\Omega_0$, and

$$\mathrm{mod}\ (\Omega_0\setminus\Omega'_0)>m_0=\frac{m}{\ell}.$$

Also, as in the proof of the previous proposition, the condition 2 holds for the new map. Now, let us *replace the initial sequence of mappings $G(j)$ by the sequence of the first return maps $G'(j)$ replacing the notations as well* (so forget about the initial sequence).

2. Let us assume by contradiction that G admits an invariant line field, i.e., there is a G-invariant Beltrami differential μ supported on $J(G)$. Then we can fix a point $x\in J(G)$ of almost continuity of μ. We may assume from the beginning that $\omega_G(x)$

contains c since the set of the points of $J(G)$ without this property has Lebesgue measure zero (this follows from the well-known fact that for almost all point x one has that $\omega(x) \subset \omega(c)$, see **[Ly2]**, **[McM]**, and from the minimality of $\omega(c)$).

3. Let us consider a mapping $G(j)$ so that the point x is outside of the range $\Omega(j)$. Then there is minimal $k = k(j) > 0$ so that $y(j) = G^k(x) \in \Omega_0(j)$. Due to the condition 2 of Proposition 2.1, we can apply *Claim 1* from the proof of the previous proposition: there exists a branch F_j of G^{-k} univalent in the range $\Omega(j)$ of the map $G(j)$ and such that $F_j(y(j)) = x$. Let us consider a domain M_j (containing $\Omega_0(j)$ and contained in $\Omega(j)$) bounded by the core curve of the annulus $\Omega(j) \setminus \Omega_0(j)$, so that $\operatorname{mod}(\Omega(j) \setminus M_j) = \operatorname{mod}(M_j \setminus \Omega_0(j)) > m_0/2$. By the Koebe Distortion Theorem, the image $F_j(M_j)$ is roughly a disc around x (it means that it contains a disc centred at x of radius r and it is contained in a disc centred at x of radius R so that R/r is less than a constant depending of m_0 only). Since x belongs to the Julia set, the domains $F_j(M_j)$ shrink to the point x as $j \to \infty$.

4. Let us consider the first return map $G'(j)$ of the mapping $G(j)$ to its central domain $\Omega_0(j)$ (as we did in Step 1 with respect to the old $G(j)$). By Step 1, the central branch $G'_0(j)\colon \Omega'_0(j) \to \Omega_0(j)$ extends to an ℓ-covering onto $\Omega(j)$, and again any iterate of c by G entering the $\Omega_0(j)$ is an iterate of c under the first return map $G'(j)$. Let $m = m(j) > 0$ be minimal so that $z(j) = G^m(x) \in \Omega'_0(j)$. By Step 3, there exists a branch F'_j of G^{-m} univalent in the domain $\Omega_0(j)$ and such that $F'_j(z(j)) = x$. Define a domain M'_j as the preimage of M_j under $G'_0(j)$. Then $\operatorname{mod}(\Omega_0(j) \setminus M'_j) = \operatorname{mod}(M'_j \setminus \Omega'_0(j)) > m_0/2\ell$. Repeating the argument from Step 3, we get that the domains $F'_j(M'_j)$ are roughly discs around x and shrink to this point.

5. Finally, let us rescale the dynamical system $G'_0(j)\colon M'_j \to M_j$ by the maps $A_j(z) = (z - z(j))/\operatorname{diam}(M'_j)$ and $B_j(z) = (z - y(j))/\operatorname{diam}(M_j)$ in the domain of the definition and in the range of $G'_0(j)$ respectively. Denote the new system by

$$g_j\colon D'_j \to D_j, \text{ where}$$

$$D'_j = A_j(M'_j) \text{ and } D_j = B_j(M'_j)$$

are approximately Euclidean discs centred at zero with diameter 1 and $g_j = B_j \circ G'_0(j) \circ A_j^{-1}$ is an ℓ-covering with a unique critical point $c_j = A_j(c) \in U_j = A_j(\Omega'_0(j))$ and critical value $g_j(c_j) \in V_j = B_j(\Omega_0(j))$.

The map g_j takes the line field $\mu'_j = (F'_j \circ A_j^{-1})^*(\mu)$ in D'_j onto the line field $\mu_j = (F_j \circ B_j^{-1})^*(\mu)$ in D_j (we use here that all maps $F'_j, F_j, G'_0(j)$ are iterates of G or inverses of iterates).

Now we are in a position to apply general theorems on sequences of invariant line fields and covering maps as in **[McM]**. By Theorem 5.2 in **[McM]** the sequences $(D'_j, 0)$ and $(D_j, 0)$ are pre-compact in the Caratheodory topology. Since their diameters are 1, we find two limit domains D' and D respectively, which are roughly discs around 0. Since the critical value $g_j(c_j) \in V_j$, and $\operatorname{mod}(D_j \setminus V_j) > m_0/2\ell$, by Theorem 5.6(3) **[McM]**, there is a limit map $g\colon D' \to D$, which is a branched degree ℓ-covering with a unique critical point $q \in D'$. On the other hand, by our construction and Theorem 5.16 **[McM]**, some subsequences of the line fields μ'_j and μ_j converge in measure to univalent line fields μ'_* and μ_* on D' and D respectively, and g takes μ'_* to μ_*. (A linefield is said to be univalent if it is a univalent pullback of the horizontal linefield). Since g has a critical point, this is a contradiction. □

In order to use Propositions 2.1 and 2.3, we need to construct a sequence of ℓ-polynomial-like mappings as in the propositions. This is the content of the following statement, in which we assume that all critical points of a real polynomial are real.

Proposition 2.4. — *(*[**GS**], [**Ly1**], [**LS1**]*[Theorem C]). Let f be a polynomial with the real coefficients and so that all critical points of f are real. Moreover, assume that all critical points escape to infinity, except for the critical point c of an even multiplicity ℓ. Assume that f is not renormalizable on the real axis and $\omega(c)$ is minimal. Then there exists a sequence of topological discs $\Omega_n \ni c$ such that $\Omega_n \cap \mathbb{R}$ is equal to the real trace of a puzzle piece, and such that $diam(\Omega_n) \to 0$ so that the first return map to Ω_n along the points of the set $\omega(c) \cap \Omega_n$ is an ℓ-polynomial-like map $R_n : \cup_i \Omega_n^i \to \Omega_n$ and so that the modulus of the annulus between the range Ω_n and the central domain Ω_n^0 is bounded away from zero by a constant which only depends on f.*

This result was proved in [**Ly1**] and [**GS**] for the real quadratic polynomials, and adapted in [**LS1**][Theorem C] and in [**GS1**] for real unimodal polynomials. For the polynomials as in the proposition still minor modifications of the proofs are needed, see the Remark after Theorem C in [**LS1**], and also the next Section.

The following (in fact, well known) proposition settles the case when $\omega(c)$ is not minimal.

Proposition 2.5. — *Assume that G is a real ℓ-polynomial-like mapping, the map G restricted to the real line has no attracting or neutral periodic orbit, is non-renormalizable and that $\omega(c)$ is not minimal. Then $J(G)$ is totally disconnected and has zero Lebesgue measure.*

Proof. — If $\omega(c)$ is not minimal then it contains a point x whose forward orbit avoids some critical piece P_N. Here we use that the traces of the critical pieces on the real line tend to zero in diameter, because the real map G has no wandering interval. (In general, it is possible that a point remains outside a neighbourhood of c and still visits every critical piece (if they do not shrink to zero in diameter)). In particular, this forward orbit lies in a hyperbolic set. Therefore the puzzle-pieces $P_n(x)$ containing x shrink down in diameter to zero. The puzzle-pieces $P_n(x)$ are mapped by some iterates of G onto a fixed critical piece P_N: there is a fixed critical piece P_N and a sequence of critical pieces P_{n_k} with $n_k \to \infty$ so that each map $f^{n_k - N} : P_{n_k} \to P_N$ is ℓ-covering. Since there are no points of the postcritical set in a neighbourhood of the boundary of P_N, this proves the proposition. The statement that the Lebesgue measure is zero, follows from this as well. □

3. Proof of the Theorem

Proof of the Theorem. — Let again $\ell \geq 2$ be the multiplicity (i.e., the local degree) of the critical point c of the polynomial P, and assume ℓ is even. Let us first prove the total disconnectedness result. If $\ell = 2$ (the critical point c is simple), then it is proved in [**BH**] (even for the complex P). Let $\ell > 2$. If the critical point is not recurrent, then the proof is already given in [**BH**]. When the limit set $\omega(c)$ of the critical point c is not minimal, then the proof is also easy (though we use the reality of the maps: see Proposition 2.5). So assume $\omega(c)$ is minimal. We may assume $c = 0$. As a first step, we construct an initial ℓ-polynomial-like mapping as follows. Fix a level curve Γ of the Green function of the polynomial P, such that Γ is connected and not critical, i.e., all preimages $P^{-n}(\Gamma)$ are smooth curves. By the condition of the theorem, one can pick up a full preimage $\gamma = P^{-k}(\Gamma)$ such that γ bounds finitely many topological discs $V_0, V_1, \dots$ such that $P : V_0 \to P(V_0)$ is ℓ-covering, and all other $P : V_i \to P(V_i)$ are one-to-one (this is still not a polynomial-like mapping because the images $P(V_i)$ could be different). Denote $V_0 = \Omega$ (the only domain containing the critical point $c = 0$ of P). The desired ℓ-polynomial-like mapping $G\colon \cup_{i=0}^{i_0} \Omega^i \to \Omega$ will be the *first return map* of the points of the set $\omega(c) \cap \Omega$ to Ω. Since $\omega(c)$ is minimal, the definition makes sense.

G obeys (by construction) the following *first return property*:

$$\text{if } P^n(c) \in \Omega, \text{ then } P^n(c) \text{ is an iterate of } G.$$

Moreover, the map G is real since it is the first return of a real map. We are going to apply Proposition 2.4. By the Straightening Theorem for polynomial-like maps [**DH**], [**LM**], [**LS1**], $G\colon \cup_{i=0}^{i_0} \Omega^i \to \Omega$ can be quasi-conformally conjugated to a real polynomial f with all critical points real. (After all, we only need to move the escaping critical points.) To this end, let us consider the restriction $G|_{\mathbb{R}}$ of the map $G\colon \cup_{i=0}^{i_0} \Omega^i \to \Omega$ to the real axis. Then $G|_{\mathbb{R}}$ is unimodal on the central interval $T_0 = \Omega^0 \cap \mathbb{R} \ni c$, and maps each other interval $T_i = \Omega^i \cap \mathbb{R}$, $i = 1, \dots, i_0$ diffeomorphically onto T. Let us now add (finitely many) components to the domain of definition of G in such a way, that the new map $\hat{G}$ is again a real ℓ-polynomial-like map, with an advantage that the graph of the new map on the real axis "looks like" a graph of a polynomial with all critical points being real, i.e., the monotone increasing and monotone decreasing branches alternate each other. Now we can use the Straightening Theorem to conjugate the polynomial-like map $\hat{G}$ with a polynomial f, so that all critical points of f are real. The existence of the sequence of maps induced by f follows now from Proposition 2.4. Moreover, the quasi-conformal conjugacy between f and $\hat{G}$ transfers any polynomial-like structure induced by f to a one induced by $\hat{G}$. On the other hand, since the induced maps we consider are the first returns along the postcritical set, and since the $\hat{G}$-orbit of the critical point visits only the branches of the original map G, the maps induced by $\hat{G}$ are, in fact, the ones, induced by G. Thus the statement of the theorem follows from Proposition 2.1.

To prove that no invariant linefields exist we consider two complementary cases: P is not renormalizable on the real line, or P is renormalizable. In the first case, we apply Propositions 2.3 and 2.4, if $\omega(c)$ is minimal, and Proposition 2.5 otherwise. In the second case, some iterate of P restricted to an appropriate neighbourhood of the critical point c is quasi-conformally conjugate to a polynomial f of the form $f(z) = z^\ell + c_1$, where c_1 is real, and the critical point $c = 0$ of f does not escape to infinity under the iterates of f. Again, there are two possibilities. If f is finitely many times renormalizable, we consider the last renormalization and again apply Propositions 2.3 and 2.4, if $\omega_f(c)$ is minimal, or Proposition 3.2 of [**LS1**] (similar to Proposition 2.5) otherwise. On the other hand, if f is infinitely renormalizable, the result is proved already in [**McM**]. $\square$

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