From random walks to collective phenomena

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Outline

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1. Introduction
   - What is a random walk?
   - Examples: 1D, 2D, boundary conditions
   - Dimensional arguments
2. Properties of random walks
3. The field theory of the Wiener

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Introduction

Random walks are

- ...used everywhere in the natural sciences, finance, sociology *etc.*
- ...a recipe to explore **space** by making independent moves as **time** goes by.
- ...easy to handle mathematically, because there is **no interaction**.
- ...at the basis of more realistic and thus more complicated models.

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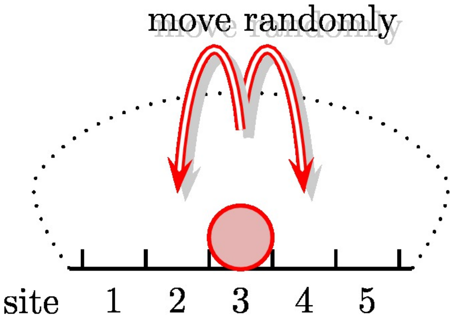
Example: One dimensional random walk with periodic boundaries



- In every **time step**
- a **particle**
- hops with **equal probability** either left or right.

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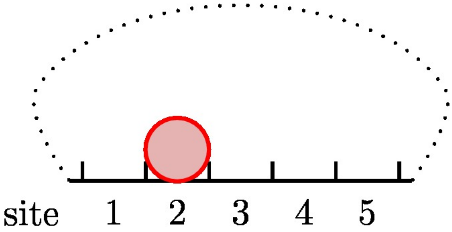

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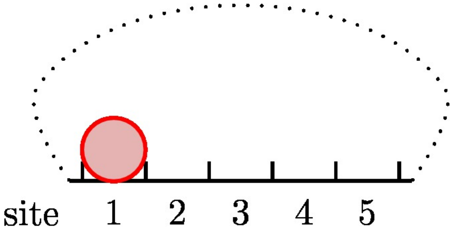

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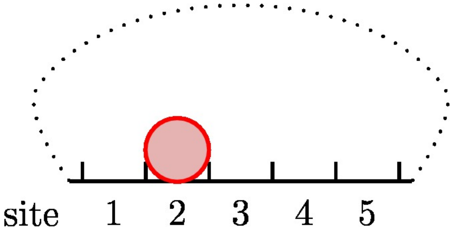

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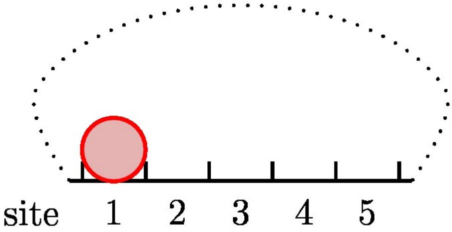

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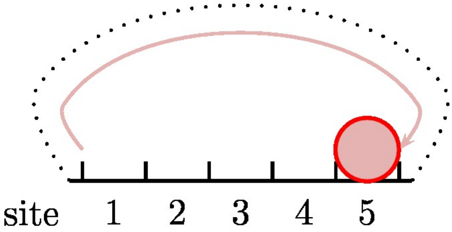

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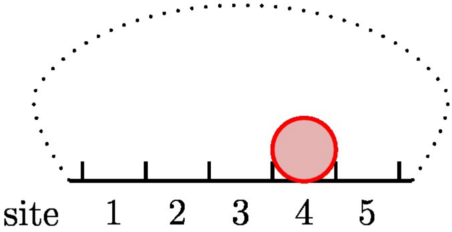

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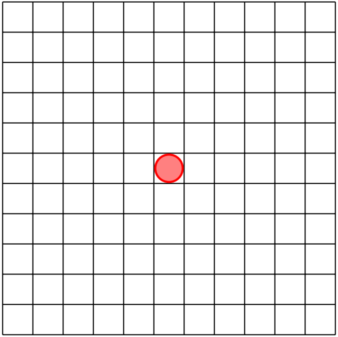
Example: 2D random walk with periodic boundaries



Paradigm shift: Complicated physics behind "kicking" wrapped up in "randomness".
Consequences? **Enormous!!**

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The Maths

Deterministic motion:

$$\dot{x}(t) = v \qquad \text{ballistic motion}$$

Particle with position $x(t)$ takes time $t = L/v$ to explore distance L.
Double the distance, double the time.
Stochastic motion:

$$\dot{x}(t) = \eta(t) \qquad \text{with } \eta \text{ the } \textbf{noise}$$

Particle with position $x(t)$ takes time $t = L^2/D$ to explore distance L.[1]
Double the distance, **quadruple** the time.

Diffusion constant

[1] ... on average ... something like that ...
What does it mean? Explore? On average?

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The diffusion constant

- Degree (strength) of randomness: Parameterised Diffusion constant D.
- Units? Square length per time (square meters per seconds).
- Rule of thumb: Where ballistic motion uses velocity ($t = L/v$), random motion uses the diffusion constant ($t = L^2/D$).

Looks like a random walker is very inefficient. It's not! Not at all!

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1. Introduction
2. Properties of random walks
   - Escape time
   - Trajectory of a random walker
   - Volume of a Wiener
3. The field theory of the Wiener

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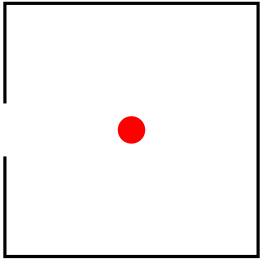
Escape time

Application: Time for a drunkard to leave the pub.



How does the escape time depend on the size of the door and the system size?

Hardly at all (logarithmically, $\ln(d/L)$)!

Walker explores area linearly in time.

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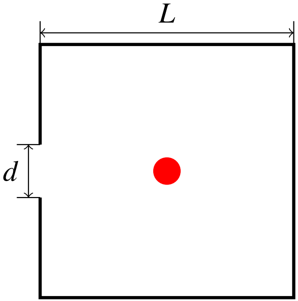

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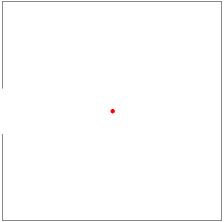

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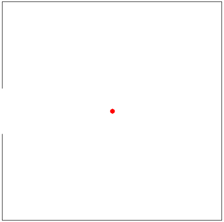

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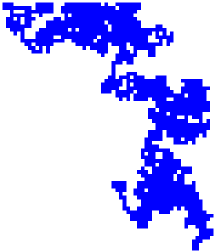
The trajectory of a random walker is self-similar



A random walkers explores the plane

Random walkers are very efficient at exploring two dimensions.
Random walkers are very inefficient at exploring one dimension.

Inefficiency due to returning where they have been before.

Question: How many **distinct** sites are visited per time (in one dimension, two dimensions, . . .)?
Volume of a Wiener!

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Which Wiener?

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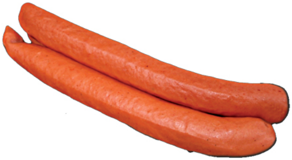

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Wiener process

(named after Norbert Wiener)

Consider a random walker on a 2D lattice:

Think of the random walker (red dot) as the tip of a pen, spilling ink. What is the area covered in blue (volume of a “Wiener sausage”, traced out in one, two, three dimensions)?

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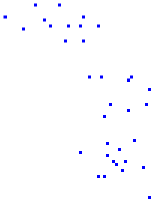
Determine the volume of the Wiener using Statistical Field Theory

Keeping track of a walker's trace is hard.
Easy (-ier): Walker spatters ink as it walks.



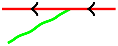
On the large scale, spatter becomes continuous trace.

Determine the volume of the Wiener using Statistical Field Theory

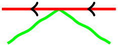
- Walker walks:

$$= \frac{1}{-\imath\omega + D\mathbf{k}^2}$$

- …and leaves behind a trace in the form of branched-off particles

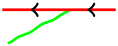


- No deposition if a particle is there already

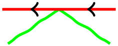


Details of the diagrams

Deposition is suppressed in the presence of deposits.
Without that, deposits could be found all along the walker's trajectory (multiple deposits at revisited sites):



This diagrams probes the lattice for deposits (and suppresses further deposition):

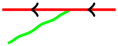


Interaction of the walker with its past trace.

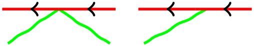
Field theory of the Wiener sausage

Interaction diagrams

Calculate features of the Wiener sausage using **renormalisation**.
Deposit along the trajectory



…is reduced by suppressed deposition



Loop = interaction = signature of collective phenomenon

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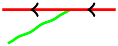

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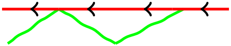

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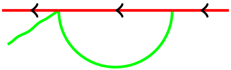
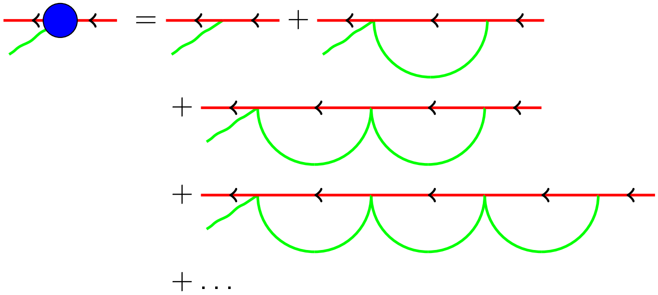


Loop = interaction = signature of collective phenomenon

Renormalisation

At the heart of the theory is the **renormalisation** of the following process:



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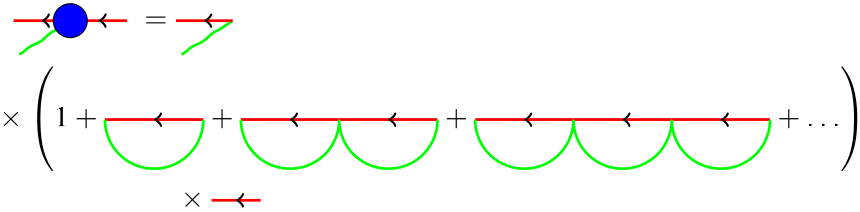

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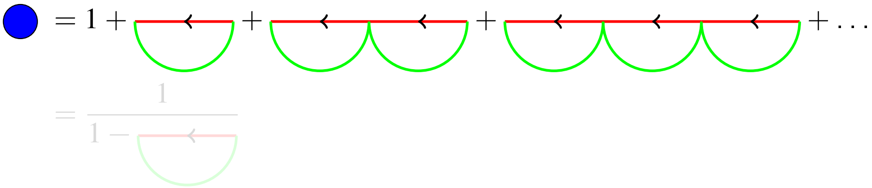

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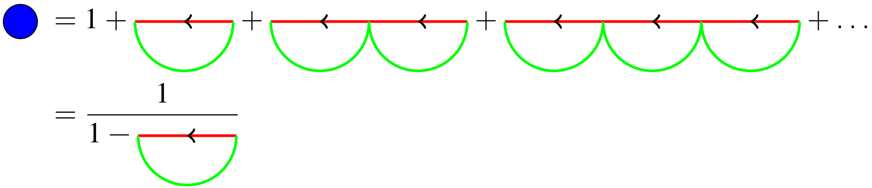

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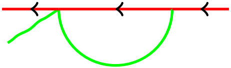
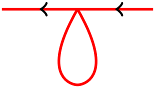
Renormalisation

What are the loops?

What physical process do the loops



correspond to? Trajectory intersecting itself:



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Renormalisation

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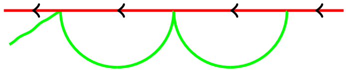
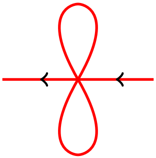


correspond to? Trajectory intersecting itself twice:



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Volume of a Wiener by field theory

Results

- In **one** dimensions: Length covered proportional to square root of time, $\sqrt{t}$ (inefficient).
- In **two** dimensions: Area covered linear in time, t (efficient!).
- Area grows as length — so no return?
- It does, nevertheless (infinitely often to every point. . .).
- In three dimensions and higher: Volume linear in time, t.
- . . . random walker may never return.
- Well known results (Leontovich and Kolmogorov, Berezhkovskii, Makhnovskii and Suris). . .
- . . . but, hey, what a nice playground for field theory (fermionicity, renormalisation, calculating moments easily . . . sort of).

Thank you!

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