Self Organised Criticality

25 years of power laws

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University of Warwick, 18 May 2012

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Outline

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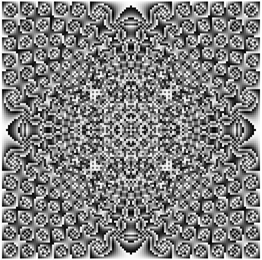
Prelude: The physics of fractals



Question: Where does scale invariant behaviour in nature come from?

Answer: Due to a phase transition, self-organised to the critical point.

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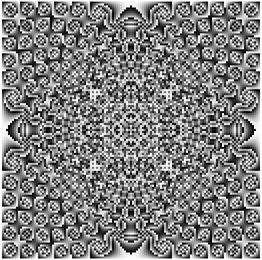
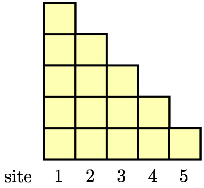


- Anderson, 1972: *More is different*
  Correlation, cooperation, emergence
- $1/f$ noise "everywhere" (van der Ziel, 1950; Dutta and Horn, 1981)
- Kadanoff, 1986: *Fractals: Where's the Physics?*
- Bak, Tang and Wiesenfeld, 1987: *Self-Organized Criticality: An Explanation of $1/f$ Noise*

The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics. (Exact results for correlation functions by Mahieu, Ruelle, Jeng *et al.*)
- **The physics of fractals.**

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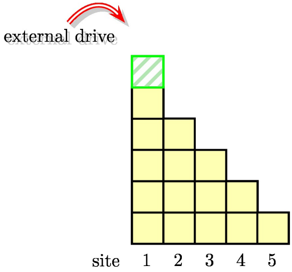

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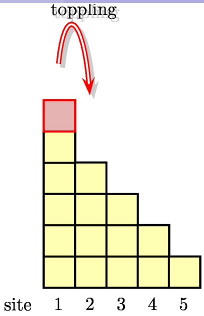

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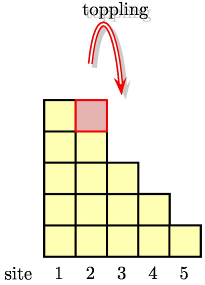

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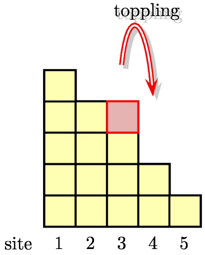

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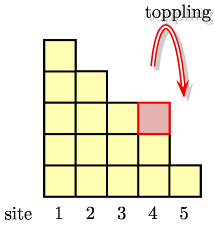

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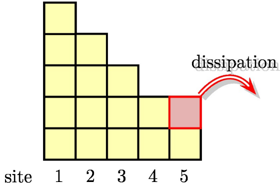

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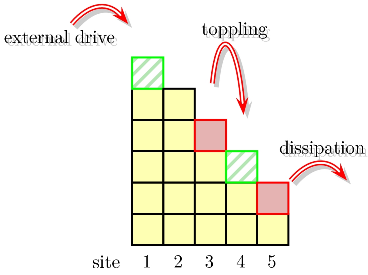

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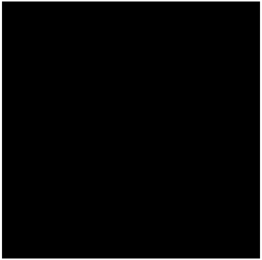

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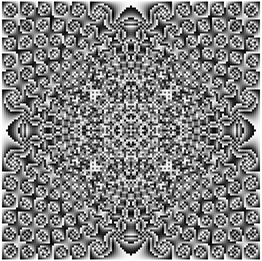

Key ingredients for SOC models:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.

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$1/f$ noise — a red herring? I

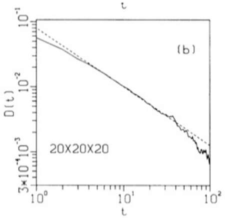

FIG. 3. Distribution of lifetimes corresponding to Fig. 2. (a) For the 50×50 array, the slope $\alpha\approx0.42$, yielding a "$1/f$" noise spectrum $f^{-1.58}$; (b) $20\times20\times20$ array, $\alpha\approx0.90$, yielding an $f^{-1.1}$ spectrum

From: Bak, Tang, Wiesenfeld, 1987

- Power spectrum $P(f) \propto 1/f$, thus correlation function (via Wiener Khinchin) decays "very slowly".

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$1/f$ noise — a red herring? II

- Dimensional analysis:

$$\int \mathrm{d}f\, 1/f^{\alpha} e^{-2\pi\imath f t} = \ldots \propto t^{\alpha-1} = \text{const}$$

- $1/f$ noise suggests long time correlations
- Initially, SOC was intended an explanation of $1/f$ noise.
- Initially the BTW model was thought to display $1/f$ noise.
- Jensen, Christensen and Fogedby: “Not quite.”
- Today: Reduced interest in $1/f$.
- Today: Power laws in other observables.

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Why is SOC important?

SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Emergence!

- Explanation of emergent,
- …cooperative,
- …long time and length scale
- …phenomena,
- …as signalled by power laws.

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Why is SOC important?

SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Universality!

- Understanding and classifying natural phenomena
- …using *Micky Mouse Models*
- …on a small scale (in the lab or on the computer).
- (Triggering critical points?)
- But: Where is the evidence for scale invariance in nature (dirty power laws)?

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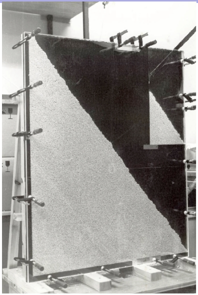
Experiments:

Granular media, superconductors, rain. . .



Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Ling, *et al*. (Physica C, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).
- Precipitation statistics by Peters and Christensen (PRL, 2002).

Where is the evidence?

Lots of “dirty power laws”, but…

- Experiments: Difficult to perform. Result: Mostly no scaling. Few solid results in superconductors, granular media, Gutenberg-Richter, precipitation.
- Numerics: Easy to perform, but require large scales, display slow convergence.
- Analytically: Little support beyond directed and mean-field-like models.

Outline

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More models

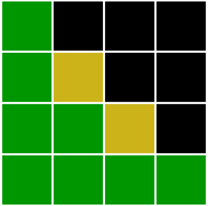
- Initial intention for more models: Expand BTW universality class.
- Later: Provide more evidence for SOC as a whole.
- More models…

More models

The failure of SOC?

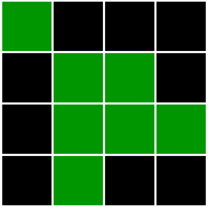
- Zhang Model (1989) [scaling questioned]
- Dhar-Ramaswamy Model (1989) [solved, directed]
- Forest Fire Model (1990, 1992) [no proper scaling]
- Manna Model (1991) [solid!]
- Olami-Feder-Christensen Model (1992) [scaling questioned, $\alpha \approx 0.05$ (localisation), $\alpha = 0.22$ (jump)]
- Bak-Sneppen Model (1993) [scaling questioned]
- Zaitsev Model (1992)
- Sneppen Model (1992)
- Oslo Model (1996) [solid!]
- Directed Models: Exactly solvable (lack of correlations)

The Forest Fire Model



- Originally by Bak, Chen and Tang (1990).
- Intended as a model of turbulence.
- Sites empty, occupied (by tree) or on fire.
- Slow regrowth at rate p.
- Occasional re-lighting.
- Grassberger and Kantz (1991):
  Deterministic pattern, scale given by $1/p$.

The Drossel-Schwabl Forest Fire Model



- Originally by Henley (1989) and independently by Drossel and Schwabl (1992).
- Fires instantaneous, explicit lightning mechanism with θ trees grown between two lightnings attempts.
- Grassberger (2002) and Pruessner and Jensen (2002): Not scale invariant.

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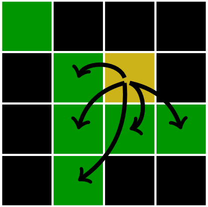

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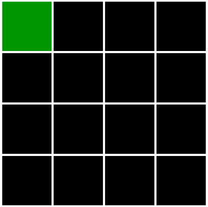

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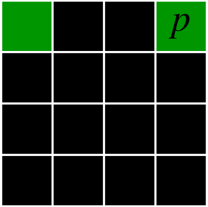

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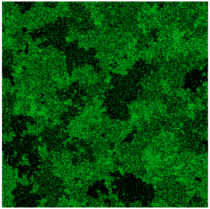

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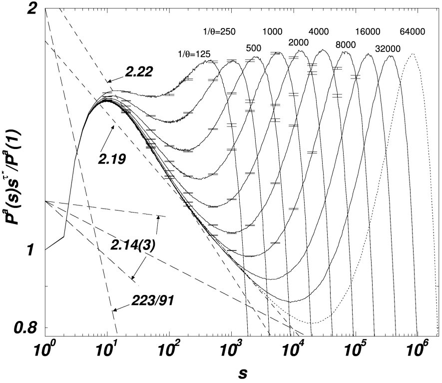
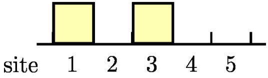
Lack of scaling



- Finite size not the only scale.
- Scale invariance possible only in the limit of $\theta \to \infty$.
- Lower cutoff moves as well.

Better Models: The Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.
- Robust, solid, universal, reproducible.
- Defines a universality class.

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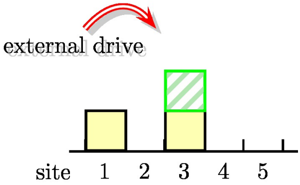

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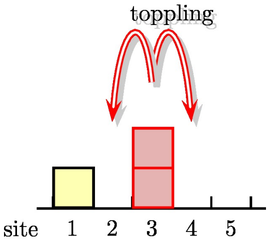

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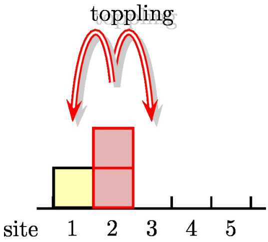

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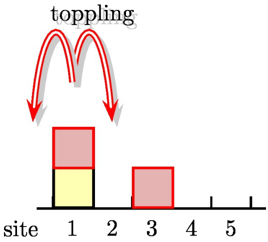

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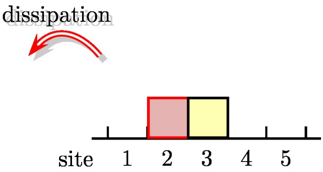

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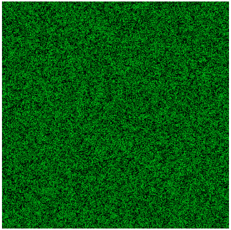

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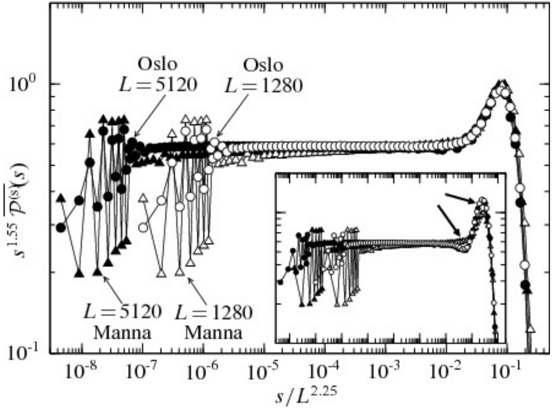
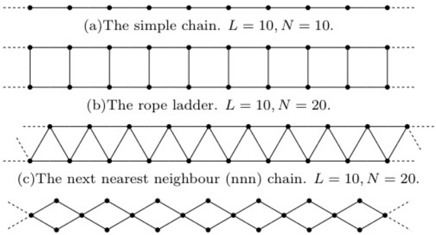
Collapse with Oslo



The Manna Model is in the same universality class as the Oslo model.

Manna on different lattices

One and two dimensions



(a)The simple chain. $L = 10, N = 10$.

(b)The rope ladder. $L = 10, N = 20$.

(c)The next nearest neighbour (nnn) chain. $L = 10, N = 20$.

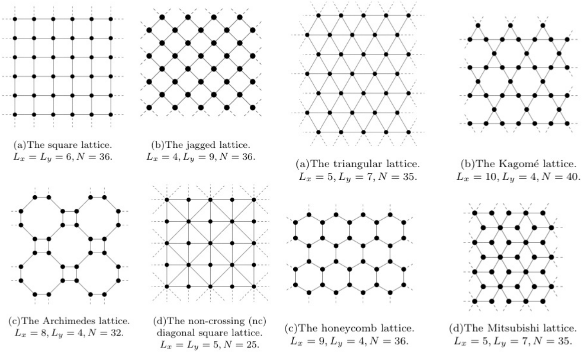
(d)The Futatsubishi lattice. $L = 7, N = 22$.

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Manna on different lattices

One and two dimensions



(a)The square lattice. $L_x = L_y = 6, N = 36$.

(b)The jagged lattice. $L_x = 4, L_y = 9, N = 36$.

(c)The Archimedes lattice. $L_x = 8, L_y = 4, N = 32$.

(d)The non-crossing (nc) diagonal square lattice. $L_x = L_y = 5, N = 25$.

(a)The triangular lattice. $L_x = 5, L_y = 7, N = 35$.

(b)The Kagomé lattice. $L_x = 10, L_y = 4, N = 40$.

(c)The honeycomb lattice. $L_x = 9, L_y = 4, N = 36$.

(d)The Mitsubishi lattice. $L_x = 5, L_y = 7, N = 35$.

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Manna on different lattices

One and two dimensions

lattice	d	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
simple chain	1	2.27(2)	1.117(8)	1.450(12)	1.19(2)	0.998(4)	1.260(13)	2.000(4)	0.27(2)	0.27(3)	0.259(14)
rope ladder	1	2.24(2)	1.108(9)	1.44(2)	1.18(3)	0.998(7)	1.26(2)	1.989(5)	0.24(2)	0.26(5)	0.26(2)
nnn chain	1	2.33(11)	1.14(4)	1.48(11)	1.22(14)	0.997(15)	1.27(5)	1.991(11)	0.33(11)	0.3(2)	0.27(5)
Futatsubishi	1	2.24(3)	1.105(14)	1.43(3)	1.16(6)	0.999(15)	1.24(5)	2.008(11)	0.24(3)	0.23(9)	0.24(5)
square	2	2.748(13)	1.272(3)	1.52(2)	1.48(2)	1.992(8)	1.380(8)	1.9975(11)	0.748(13)	0.73(4)	0.76(2)
jagged	2	2.764(15)	1.276(4)	1.54(2)	1.49(3)	1.995(7)	1.384(8)	2.0007(12)	0.764(15)	0.76(5)	0.77(2)
Archimedes	2	2.76(2)	1.275(6)	1.54(3)	1.50(3)	1.997(10)	1.382(11)	2.001(2)	0.76(2)	0.78(6)	0.76(3)
nc diagonal square	2	2.750(14)	1.273(4)	1.53(2)	1.49(2)	1.992(7)	1.381(8)	2.0005(12)	0.750(14)	0.75(4)	0.76(2)
triangular	2	2.76(2)	1.275(5)	1.51(2)	1.47(3)	2.003(11)	1.388(12)	1.997(2)	0.76(2)	0.71(6)	0.78(3)
Kagomé	2	2.741(13)	1.270(4)	1.53(2)	1.49(2)	1.993(8)	1.381(9)	1.9994(12)	0.741(13)	0.75(5)	0.76(2)
honeycomb	2	2.73(2)	1.268(6)	1.55(4)	1.51(4)	1.990(13)	1.376(14)	2.000(2)	0.73(2)	0.79(8)	0.75(3)
Mitsubishi	2	2.75(2)	1.273(6)	1.54(3)	1.50(4)	1.999(12)	1.387(12)	1.998(2)	0.75(2)	0.77(7)	0.77(3)

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

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Manna on different lattices

Three dimensions

Lattice	$\overline{q}$	$\overline{q^{(v)}}$	$\langle z \rangle$	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
SC	6	1	[0.622325(1)]	3.38(2)	1.408(3)	1.779(7)	1.784(9)	3.04(5)	1.45(4)	2.0057(5)	1.38(2)	1.395(16)	1.36(13)
BCC	8	4	[0.600620(2)]	3.36(2)	1.404(4)	1.777(8)	1.78(1)	2.99(2)	1.444(18)	2.0030(5)	1.36(2)	1.390(19)	1.33(6)
BCCN	14	5	[0.581502(1)]	3.38(3)	1.408(4)	1.776(9)	1.783(11)	3.01(3)	1.44(3)	2.0041(6)	1.38(3)	1.39(2)	1.32(7)
FCC	12	4	[0.589187(3)]	3.35(4)	1.402(8)	1.765(16)	1.78(2)	3.1(2)	1.48(14)	2.0035(11)	1.35(4)	1.37(4)	1.5(5)
FCCN	18	5	[0.566307(3)]	3.38(4)	1.408(7)	1.781(14)	1.787(18)	3.00(4)	1.44(3)	2.0051(8)	1.38(4)	1.40(3)	1.32(9)
Overall				3.370(11)	1.407(2)	1.777(4)	1.783(5)	3.003(14)	1.442(12)	2.0042(3)		1.380(13)	

From: Huynh, G P, 2012

The Manna Model has been investigated numerically in great detail.

Outline

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Why are so many so tired to hear about power laws?

(1) **"Power laws are trivial."** — simple physics is power law based, *e.g.* exponent -2 in Newton's gravitation

$$F = G\frac{m_1 m_2}{r^2}$$

(2) **"Power laws are not any different from any other functional dependence."** — What is the physical significance of scaling?

(3) **"There is no significance in non-integer (weird) exponents."** — what makes an exponent of, say, 2.24 any different from, say, the exponent of -2 in Newton's law of gravitation?

(4) **"Power laws are wrong."** — Nature is different and/or more complicated; see the "fractal discussion" by Avnir, Biham, Lidar, Malcai, 1998.

(5) **"Power laws are irrelevant."** or **"Physicists get excited about power laws, biologist do not."** — see Stumpf and Porter, 2012.

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(1): Power laws are trivial.

> *"Why get excited? Lots of basic physics is based on power laws!"*

Indeed. This is universal physics. The fact that Newton's law of gravitation goes like r^{-2} *on every*[1] *scale* makes it *universal*[2].

Masslessness of the graviton (Gravitation) and the photon (Coulomb interaction) vs. finite range for other fundamental forces.

Note: Power law of observables vs. PDF.

[1] every scale = enormous, intermdediate scale; GR!
[2] Until the next level of physics kicks in.

(1): Power laws are trivial.

"Why get excited? Lots of basic physics is based on power laws!"

Willinger *et al.*, 2004, Stumpf and Porter, 2012: All it takes for a power law distribution is a power law distribution!

- Willinger *et al.*: Power law distributions are stable under some operations.
- Power law distributions are limiting distrubtions for suitably normalised sums/extreme values drawn from heavy tailed (asymptotically heavy-tailed) distributions.
- Where do they come from?
- Underlying and resulting distributions have finite support (finite size scaling).
- Agenda? HOT?

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(1): Power laws are trivial.

> *"Why get excited? Lots of basic physics is based on power laws!"*

Mechanisms producing non-trivial power law distributions require (by definition) non-trivial, non-linear interaction.

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(2): Power law or not makes no physical difference.

> *"Why is a power law any different from any other functional dependence? What is the physical significance of scaling?"*

Full scaling[1] — pure power law: No scale from within.
Example:

- Exponential correlations, $C(r) = \exp(-x/\xi)$. Correlation length[2] = distance over which correlations decay by e^{-1}.

$$C(r+\xi) = C(r)/e$$

- Power law, $C(r) = ar^{-2}$: Correlations decay by the same factor at every multiple:

$$C(r\sqrt{e}) = C(r)/e$$

[1] As opposed to finite size scaling with intermediate power law scaling.
[2] In general, this holds only asymptotically.

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(3): There is no significance in non-integer exponents.

> *"What's the difference between an exponents of, say, 2.24 and, say, the exponent of -2 in Newton's law of gravitation?"*

Dimensional consistency usually requires other scales to be present — to fix the dimension, yet, not to *govern* the behaviour:

$$\mathcal{P}(E) = a^{\tau-1}E^{-\tau}$$

rather than[1]

$$\mathcal{P}(E) = a^{-1}e^{-E/a}$$

Other scales are present without destroying the scaling.

There is an *arbitrarily wide*, intermediate range of power law scaling.†

[1] Below can be cast in the form above with $\tau = 1$.

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(3): There is no significance in non-integer exponents.

“What's the difference between an exponents of, say, 2.24 and, say, the exponent of -2 in Newton's law of gravitation?”

Dimensional consistency usually requires other scales to be present — to fix the dimension, yet, not to govern the behaviour:

$$\mathcal{P}(E) = a^{\tau-1} E^{-\tau} \mathcal{G}\left(\frac{E}{E_c}\right) \quad \text{for } E > E_0$$

rather than $\mathcal{P}(E) = a^{-1} e^{-E/a}$.

Other scales are present without destroying the scaling.

There is an *arbitrarily wide*, intermediate range of power law scaling.†

†**Terms and conditions:**

I have ignored a couple of points here. Let's retrace the naive argument and what happens with it:
Naively one might think that $\mathcal{P}(E) = E^{-\tau}$ is the sort of power law we are after. However, this is dimensionally inconsistent. So, we require an additional scale a, so that $\mathcal{P}(E) = a^{\tau-1}E^{-\tau}$.
So, an additional scale a is not only allowed, it is necessary. And yet, it does not dominate the large scale behaviour of $\mathcal{P}(E)$, as it does in $\mathcal{P}(E) = a^{-1}\exp(-E/a)$.
If the presence of an additional scale is not the criterion to distinguish scaling and non-scaling, one might be tempted to dismiss $\mathcal{P}(E) = a^{-1}\exp(-E/a)$ on the basis that it contains a “modulating” function, whose effect is parameterised by the additional scale a, *i.e.* it is not a pure power law. However, in finite systems, one has to allow for such scaling functions even where standard scaling is found, $\mathcal{P}(E) = a^{\tau-1}E^{-\tau}\mathcal{G}\left(\frac{E}{E_c}\right)$, with upper cutoff E_c.

So, what is the difference between scaling and non-scaling? Both may be modulated by additional functions and both incorporate additional scales. And while a does (apparently — why?) not dominate the large scale in the scaling case, E_c does. It gets worse: Finite size scaling usually requires an additional lower cutoff. Scaling breaks down below a certain lower cutoff, not least to guarantee normalistion of $\mathcal{P}(E)$.

The physics is in the ruler! $\mathcal{P}(E) = a^{\tau-1}E^{-\tau}\mathcal{G}\left(\frac{E}{E_c}\right)$ should be regarded a scaling symmetry, the physics of which becomes visible if there is a way to reach an intermediate asymptotic regime, $E_0 \ll E \ll E_c$, where $\mathcal{P}(E)$ a approximated arbitrarily well by a multiple of a pure power law $E^{-\tau}$. In SOC, E_c diverges with the system size and this is the *only* scale that enters.

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(3): There is no significance in non-integer exponents.

> *"What's the difference between an exponents of, say, 2.24 and, say, the exponent of −2 in Newton's law of gravitation?"*

$$\mathcal{P}(E) = a^{\tau-1} E^{-\tau} \mathcal{G}\left(\frac{E}{E_c}\right) \quad \text{for } E \gg E_0$$

rather than

$$\mathcal{P}(E) = a^{-1} e^{-E/a}$$

Other scales are present without destroying the scaling.

There is an *arbitrarily wide*, intermediate range of power law scaling.† Different physics kicks in below and above a certain scales.

In between: The same physics throughout.

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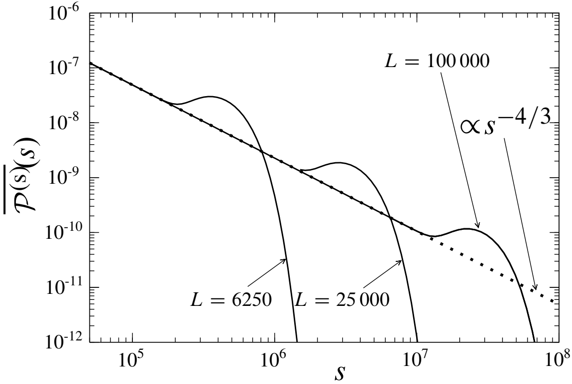
(3): There is no significance in non-integer exponents.

Other scales are present without destroying the scaling.

There is an *arbitrarily wide*, intermediate range of power law scaling.†



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(4): Power laws are wrong.

> *"Nature is different and more complicated."*
> *(e.g. Avnir, Biham, Lidar, Malcai, 1998)*

Perfect power laws are much less common than alleged.
A year in the lab is often not enough to extract the allegedly *ubiquituous* power law.

Nature is full of *dirty* power laws, "almost scaling".

Problem: Publication bias and self-selection.

(4): Power laws are wrong.

"Nature is different and more complicated."
(e.g. Avnir, Biham, Lidar, Malcai, 1998)

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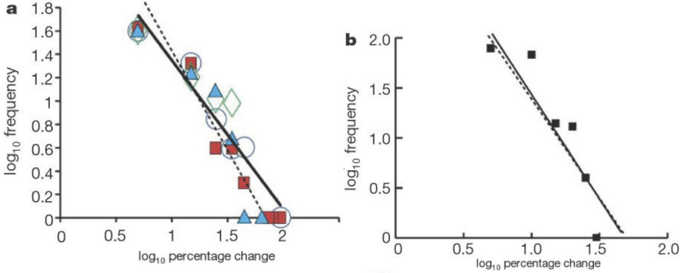

(Freckleton and Sutherland, 2001)

Problem: Publication bias and self-selection.

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(4): Power laws are wrong.

Power laws are misunderstood!
Powerlaws do NOT indicated unpredictability and/or optimisation

- Predictability: Power law correlated events are predictable (Gutenberg and Richter law).
- Optimisation: Large susceptibility is an optimum of what? (HOT? COLD? TEPID?)

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(5): Power laws are irrelevant.

> *"Physicists get excited about power laws, biologists do not."*
> *(see Stumpf and Porter, 2012)*

Suppose a power law has been identified. What does it mean?

- Exponents: Actual values can play a rôle in engineering (predicting observables).
- Exponents: Determine universality class.
- Scaling suggests emergence & universality $\Rightarrow$ underlying physics.
- Scaling provides a mechanism (not the other way around).
- Scaling: Same physics on different scales (simple models).
- Scaling: Usually characterises asymptote (large upper cutoff).

Why CLT, $N^{-1/2} \sum_i^N x_i$?

Summary: Why bother?

Narrative: **If** power laws are observed in a PDF (or other observable) on an arbitrarily large but intermediate range:

- ...they are (likely to be) caused by power law correlations.
- ...they indicate the absence of an intrinsic scale.
- ...they are the signature of emergence, collective behaviour, "more is different" (Anderson, 1972), extreme events(?).
- Exponents identify universality classes.
- Exponents characterise observables ("summary" of a PDF).

Power laws are not an end in itself.

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