Self Organised Criticality, its history and recent developments

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Outline

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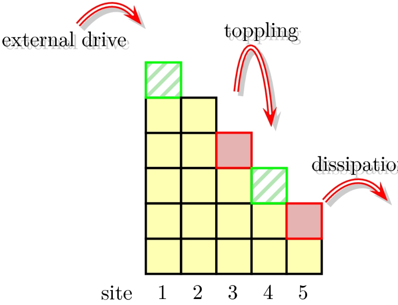
The BTW model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

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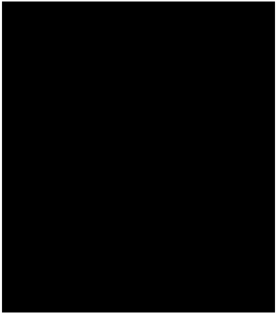

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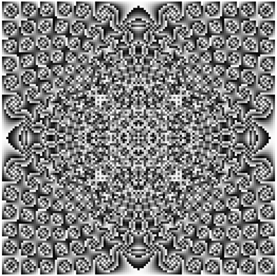

Key ingredients:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.

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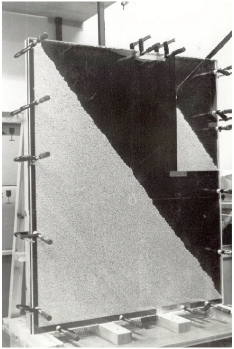
Experiments



Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Pla and Nori (PRL, 1991).
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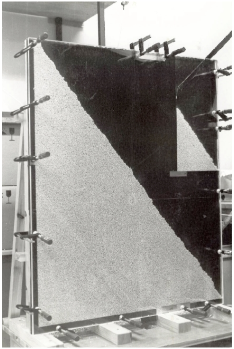

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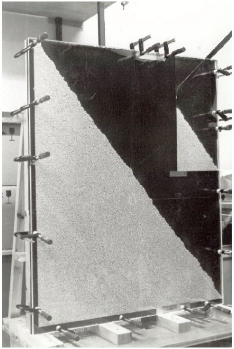

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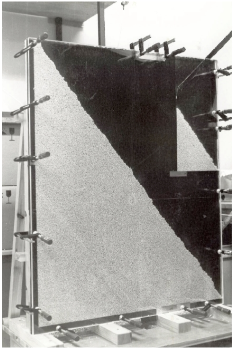

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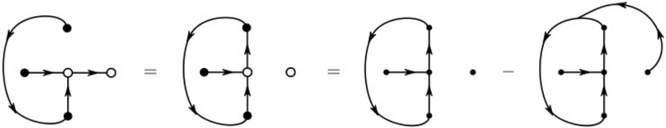
Theory



From Jeng, Piroux, Ruelle (2006)

- Many exact results for the BTW model. No proof of scaling.
  - Most results for Dhar's Abelian version.
  - Mapping to CFT with central charge -2 (spanning trees); $q \to 0$ Potts model
  - Correlation functions known exactly.
  - Wave decomposition.
- Fewer results for other models.
- Other models: Established relation to ordinary critical phenomena.
- No exact solution of non-trivial (long ranged spatio-temporal correlations) model.

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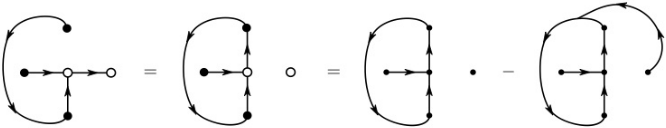
Theory



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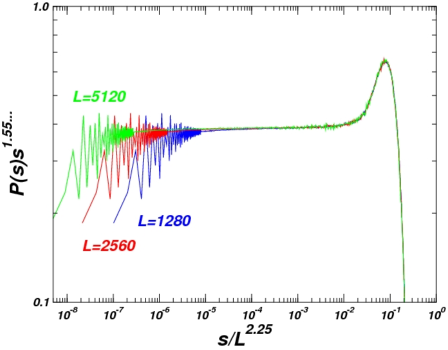
Scaling and Universality



- Only a few models (Manna and Oslo) display solid scaling.
- Robust against (small) changes in the definition $\longrightarrow$ universality.
- Manna and Oslo (apparently) in the same universality class.
- **Is this the only (proper) universality class in SOC?**

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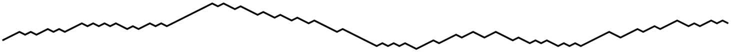
Better Models!

BTW constrained by determinism $\longrightarrow$ stochastic sandpiles!



- **Manna model** has a Langevin equation

$$\partial_t \phi(\mathbf{r}, t) = \nu \nabla^2 \phi + \lambda \phi (\phi_0 - \phi) + \omega \phi \rho + \sqrt{\phi} \eta(\mathbf{r}, t)$$

  and

$$\partial_t \rho(\mathbf{r}, t) = \nu_\rho \nabla^2 \phi$$

  similar to **directed percolation (C-DP).**
- **Oslo model** implements **quenched Edwards Wilkinson equation** $\longrightarrow$ interfaces!
- Field theories for both still unclear.
- Mechanism of self-organisation still unclear.
- Link to known universality classes.

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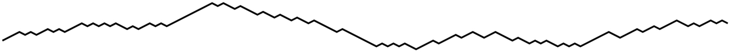

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$$\partial_t \phi(\mathbf{r}, t) = \nu \nabla^2 \phi + \eta(\mathbf{r}, \phi)$$

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- Link to **directed percolation**?

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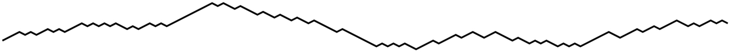

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Quenched Edwards-Wilkinson and Conserved DP

Oslo Model: qEW

$$\partial_t \phi(\mathbf{r},t) = \nu \nabla^2 \phi(\mathbf{r},t) + \eta(\mathbf{r}, \phi(\mathbf{r},t))$$

continuum version of the exact equation of motion of the Oslo model (Pruessner 2003).

- Ordinary phase transition at critical pulling force $F = F_c$ in $\partial_t \phi = \ldots + F$.
- In the Oslo model driving enters as boundary condition $\phi(\mathbf{0},t) = E(t)$.
- Mapping of exponents in interface depinning and SOC (Paczuski, Boettcher 1996).
- Quenched noise term, $\eta(\mathbf{r}, \phi(\mathbf{r},t))$, difficult to handle (Nattermann *et al.* 1992).
- First link between SOC and ordinary critical phenomena as originally envisaged.

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Quenched Edwards-Wilkinson and Conserved DP

Manna Model: C-DP

$$\partial_t \phi(\mathbf{r}, t) = \nu \nabla^2 \phi + \lambda \phi(\phi_0 - \phi) + \omega \phi \rho + \sqrt{\phi}\eta$$

with conserved $\partial_t \rho(\mathbf{r}, t) = \nu_\rho \nabla^2 \phi$ (effective theory — integrate!).

- Describes the Manna Model.
- Link to absorbing state phase transitions and interfaces.
- Link to (tuned) DP (contact process).
- Noise not quenched but Reggeon like.
- Field theory not easy to analyse.
- Same universality class as Oslo model.
- Links qEW and C-DP.

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Outline

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Generic scale invariance I

Constructing a generically scale invariant Langevin equation

Most basic Langevin equation of field $\phi(\mathbf{r}, t)$ (parameterising what?)

$$\partial_t \phi(\mathbf{r}, t) = \nu \nabla^2 \phi(\mathbf{r}, t) - \epsilon \phi(\mathbf{r}, t) + \eta(\mathbf{r}, t)$$

with the usual white, Gaussian noise

$$\langle \eta(\mathbf{r}, t) \eta(\mathbf{r}', t') \rangle = 2D\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

and vanishing mean $\langle \eta(\mathbf{r}, t) \rangle = 0$. Integrate:

$$\begin{aligned} G_1(r, \epsilon) &= \frac{D\pi}{\sqrt{\epsilon\nu}} \exp(-|r|\sqrt{\epsilon/\nu}) \\ G_3(r, \epsilon) &= \frac{D}{2\nu r} \exp(-|r|\sqrt{\epsilon/\nu}) \end{aligned}$$

Scale invariance recovered for $\epsilon = 0$. How?

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Generic scale invariance I

Constructing a generically scale invariant Langevin equation

How? **Conservation!** (Hwa and Kardar 1989):

$$\partial_t \phi(\mathbf{r}, t) = \nu \nabla^2 \phi(\mathbf{r}, t) \cancel{- \epsilon \phi(\mathbf{r}, t)} + \eta(\mathbf{r}, t)$$

But exponents trivial...

Generic scale invariance II

Adding some spice

Adding non-linear terms

$$\partial_t \phi(\mathbf{r},t) = \nu \nabla^2 \phi(\mathbf{r},t) + \frac{\lambda}{2} \partial_\| \phi(\mathbf{r},t)^2 + \eta(\mathbf{r},t)$$

generates non-trivial exponents. In general, diffusion is anisotropic:

$$\partial_t \phi(\mathbf{r},t) = (\nu_\| \partial_\|^2 + \nu_\perp \nabla_\perp^2) \phi(\mathbf{r},t) + \frac{\lambda}{2} \partial_\| \phi(\mathbf{r},t)^2 + \eta(\mathbf{r},t)$$

Problem: non-conservative noise and conservative Langevin equation is “forcing scaling”. Manna and Oslo are conservative in the bulk.

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Generic scale invariance III

Conservative noise – naively

Original equation

$$\partial_t \phi(\mathbf{r},t) = \nu\nabla^2\phi(\mathbf{r},t) - \epsilon\phi(\mathbf{r},t) + \eta(\mathbf{r},t)$$

with conservative noise

$$\langle \eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = -2D\nabla^2\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

gives (Fourier transform)

$$\langle \phi(\mathbf{k},t)\phi(\mathbf{k}',t)\rangle = D\frac{(2\pi)^{d+1}\delta(\mathbf{k}+\mathbf{k}')}{\nu\mathbf{k}^2+\epsilon}\mathbf{k}^2 \ .$$

In the conservative limit ($\epsilon \to 0$) the field $\phi(\mathbf{r},t)$ is δ-correlated in real space.

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Generic scale invariance IV

Conservative noise (Grinstein, Lee, Sachdev 1990)

Anisotropy in diffusion

$$\partial_t \phi(\mathbf{r},t) = (\nu_\parallel \partial_\parallel^2 + \nu_\perp \nabla_\perp^2)\phi(\mathbf{r},t) - \epsilon\phi(\mathbf{r},t) + \eta(\mathbf{r},t)$$

and conservative noise

$$\langle \eta(\mathbf{k},\omega)\eta(\mathbf{k}',\omega')\rangle = -2(D_\parallel \partial_\parallel^2 + D_\perp \partial_\perp^2)\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

gives (Fourier transform)

$$\langle h(\mathbf{k},t)h(\mathbf{k}',t)\rangle = \frac{(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')(D_\parallel k_\parallel^2 + D_\perp k_\perp^2)}{\nu_\parallel k_\parallel^2 + \nu_\perp k_\perp^2 + \epsilon} \; .$$

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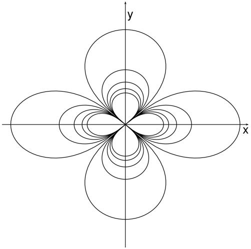
Generic scale invariance IV

Conservative noise (Grinstein, Lee, Sachdev 1990)



After Fourier transforming, a quadrupole-like structure appears:

$$\langle h(\mathbf{x},t)h(\mathbf{x}',t)\rangle = \frac{1}{2}\left(\frac{D_{\|}}{\nu_{\|}} + \frac{D_{\perp}}{\nu_{\perp}}\right)\delta(\mathbf{x}-\mathbf{x}')$$

$$+ \frac{\sqrt{\nu_{\|}\nu_{\perp}}}{2\pi r^2}\left(\frac{D_{\|}}{\nu_{\|}} - \frac{D_{\perp}}{\nu_{\perp}}\right)\frac{\nu_{\|}\sin^2\theta - \nu_{\perp}\cos^2\theta}{\left(\nu_{\|}\sin^2\theta + \nu_{\perp}\cos^2\theta\right)^2}$$

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Generic scale invariance V

Summary

$$\partial_t \phi(\mathbf{r}, t) = (\nu_{\|} \partial_{\|}^2 + \nu_{\perp} \partial_{\perp}^2)\phi + \eta(\mathbf{r}, t)$$

- *Generic* scale invariance (Hwa and Kardar, 1989, and Grinstein, Lee and Sachdev 1990)
- No mass term $-\epsilon\phi$ on the right $\longrightarrow$ conservative dynamics.
- Anisotropy required in the presence of conserved noise.
- Non-trivial exponents in the presence of non-linearities and non-conserved noise.
- Concept abandoned with the arrival of non-conservative models (FFM [1990], OFC [1992], BS [1993]).

The Absorbing State Mechanism

Dickman, Vespignani, Zapperi 1998

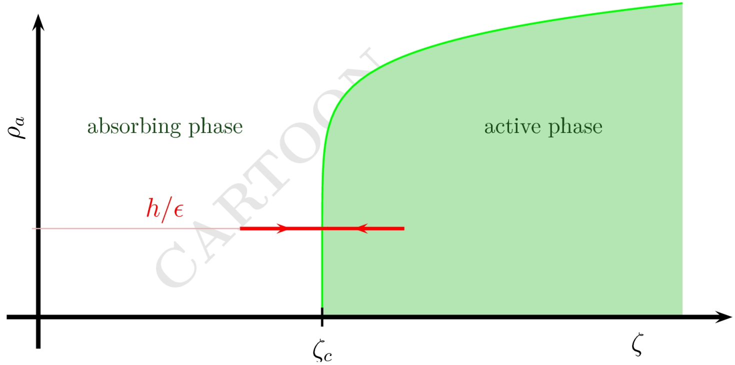
- SOC model: activity ρ_a leads to dissipation
- dissipation reduces particle density ζ
- density is reduced until system is inactive
  $\longrightarrow$ absorbing phase
- external drive increases particle density
  $\longrightarrow$ back to active phase

An SOC model can be seen as an AS model that drives itself into the inactive phase by dissipation ϵ and is pushed back into the active phase by external drive h.

$$\dot{\zeta} = h - \epsilon \rho_a \xrightarrow{\text{stationarity}} \rho_a = h/\epsilon$$

The Absorbing State Mechanism

Dickman, Vespignanim Zapperi 1998 and Pruessner, Peters 2006



Idea: SOC drives $h/\epsilon = \rho_a$ to 0 as $L \to \infty$

Leading orders: $h(L) = h_0 L^{-\omega}$ and $\epsilon(L) = \epsilon_0 L^{-\kappa}$

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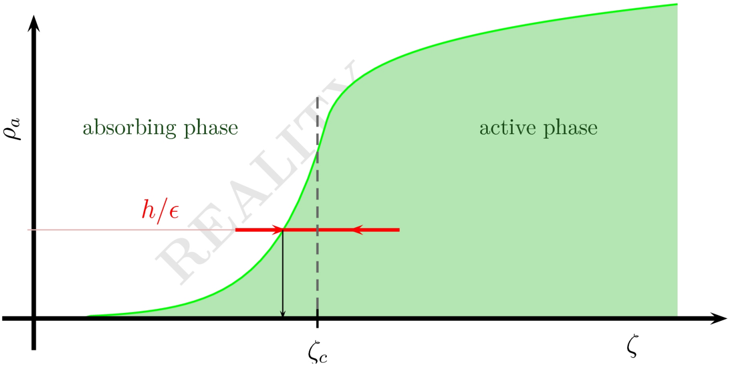

Analysis based on real scaling function.

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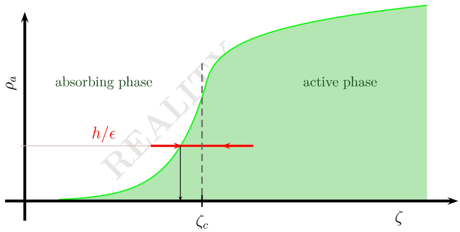

What is the resulting scaling of other observables if the order parameter is forced to scale like $\rho_a \propto L^{\kappa-\omega}$?
$\longrightarrow$ New exponent $\mu > \nu$ replaces ν,
so that $\langle\rho\rangle \propto L^{\beta/\mu}$, $L^d\sigma^2(\rho)L^{\gamma/\mu}$ etc.

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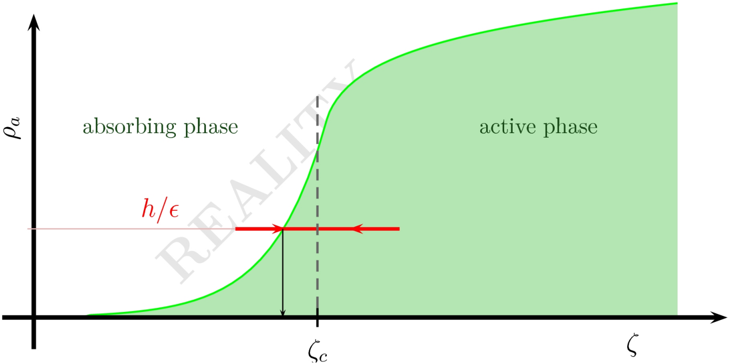

Problem: SOC exponents would be affected by the way how driving and dissipation are implemented $\longrightarrow$ no universality.

Fey, Levine and Wilson suggest that critical point is not reached.

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

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