Self Organised Criticality

in the third decade after BTW

Gunnar Pruessner

Department of Mathematics
Imperial College London

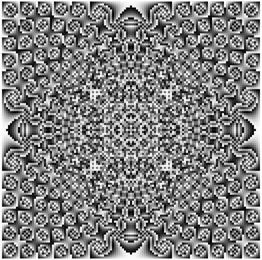
Istanbul, 6 Sep 2011

Imperial College London

Outline

Imperial College London

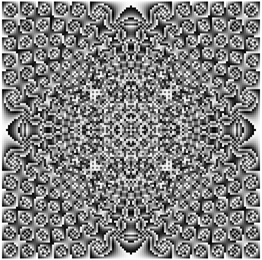
Prelude: The physics of fractals



Question: Where does scale invariant behaviour in nature come from?

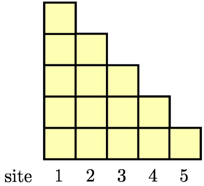
Answer: Due to a phase transition, self-organised to the critical point.

Prelude: The physics of fractals



- Anderson, 1972: *More is different*
  Correlation, cooperation, emergence
- $1/f$ noise "everywhere" (van der Ziel, 1950; Dutta and Horn, 1981)
- Kadanoff, 1986: *Fractals: Where's the Physics?*
- Bak, Tang and Wiesenfeld, 1987: *Self-Organized Criticality: An Explanation of $1/f$ Noise*

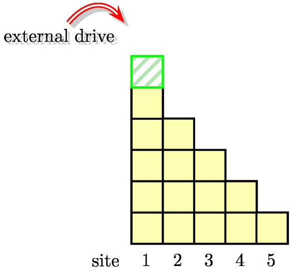
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

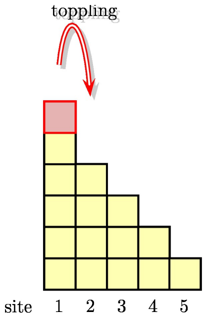
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

The BTW Model

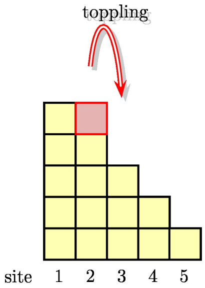


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

Imperial College London

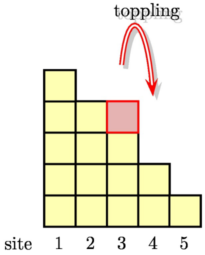
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

The BTW Model

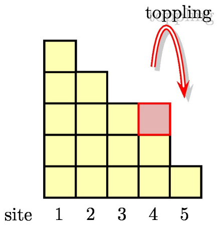


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

Imperial College London

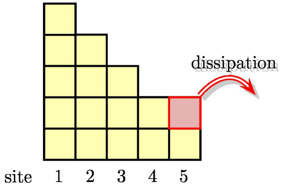
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

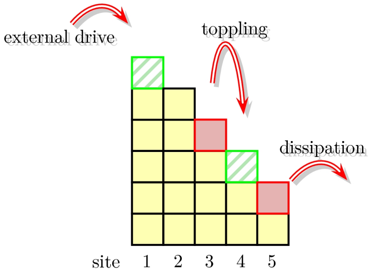
The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

The BTW Model

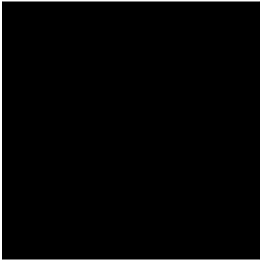


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

Imperial College London

The BTW Model

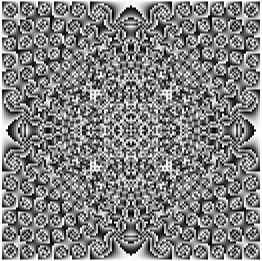


The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

Imperial College London

The BTW Model



Key ingredients for SOC models:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.

Imperial College London

$1/f$ noise — a red herring? I

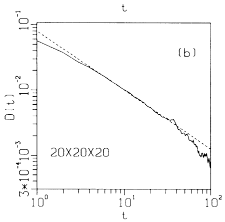


FIG. 3. Distribution of lifetimes corresponding to Fig. 2. (a) For the 50×50 array, the slope $\alpha\approx0.42$, yielding a "$1/f$" noise spectrum $f^{-1.58}$; (b) $20\times20\times20$ array, $\alpha\approx0.90$, yielding an $f^{-1.1}$ spectrum

From: Bak, Tang, Wiesenfeld, 1987

- Power spectrum $P(f) \propto 1/f$, thus correlation function (via Wiener Khinchin)

$1/f$ noise — a red herring? II

- Dimensional analysis:

$$\int \mathrm{d}f\, 1/f^{\alpha} e^{-2\pi\imath f t} = \ldots \propto t^{\alpha-1} = \text{const}$$

- $1/f$ noise suggests long time correlations
- Initially, SOC was intended an explanation of $1/f$ noise.
- Initially the BTW model was thought to display $1/f$ noise.
- Jensen, Christensen and Fogedby: “Not quite.”
- Today: Little interest in $1/f$.
- Today: Power laws in other observables.
- Today: Scaling questioned.

Why is SOC important?

SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Emergence!

- Explanation of emergent,
- . . . cooperative,
- . . . long time and length scale
- . . . phenomena,
- . . . as signalled by power laws.

Imperial College London

Why is SOC important?

SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

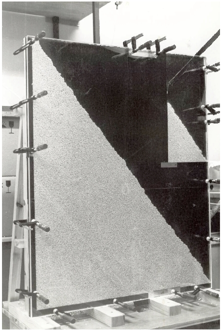
Universality!

- Understanding and classifying natural phenomena
- …using *Micky Mouse Models*
- …on a small scale (in the lab or on the computer).
- (Triggering critical points?)
- But: Where is the evidence for scale invariance in nature (dirty power laws)?

Imperial College London

Experiments:

Sandpiles, ricepiles and superconductors



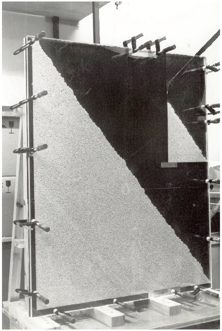
Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Pla and Nori (PRL, 1991).
- Ricepiles experiments by Frette *et al.* (Nature, 1996).

Imperial College London

Experiments:

Sandpiles, ricepiles and superconductors



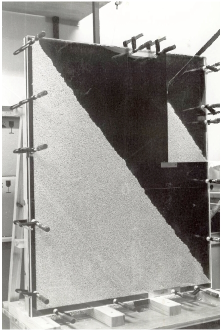
Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Pla and Nori (PRL, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).

Imperial College London

Experiments:

Sandpiles, ricepiles and superconductors



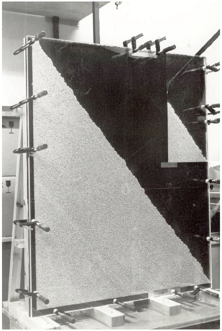
Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Pla and Nori (PRL, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).

Imperial College London

Experiments:

Sandpiles, ricepiles and superconductors



Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
- Sandpile experiments by Jaeger, Liu and Nagel (PRL, 1989).
- Superconductors experiments by Pla and Nori (PRL, 1991).
- Ricepiles experiments by Frette *et al*. (Nature, 1996).

Outline

Imperial College London

More models

- Initial intention for more models: Expand BTW universality class.
- Later: Provide more evidence for SOC as a whole.
- More models...

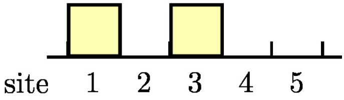
More models

The failure of SOC?

- Zhang Model (1989) [scaling questioned]
- Dhar-Ramaswamy Model (1989) [solved, directed]
- Forest Fire Model (1990, 1992) [no proper scaling]
- Manna Model (1991) [solid!]
- Olami-Feder-Christensen Model (1992) [scaling questioned, $\alpha \approx 0.05$ (localisation), $\alpha = 0.22$ (jump)]
- Bak-Sneppen Model (1993) [scaling questioned]
- Zaitsev Model (1992)
- Sneppen Model (1992)
- Oslo Model (1996) [solid!]
- Directed Models: Exactly solvable (lack of correlations)

Imperial College London

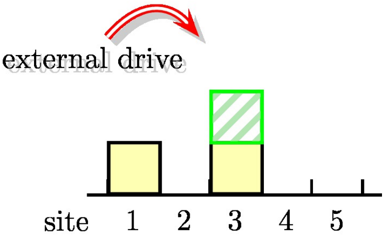
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.

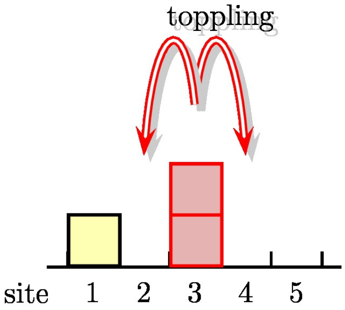
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.

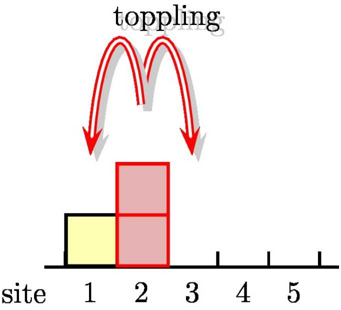
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.

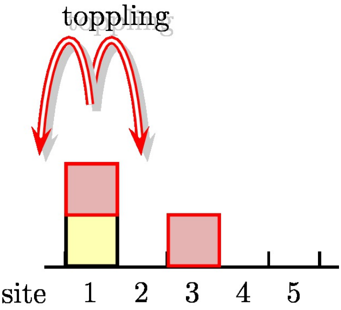
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.

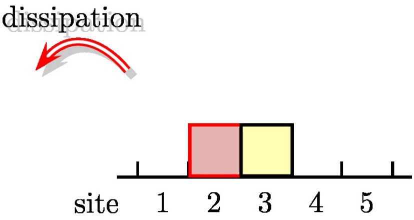
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.

Manna Model

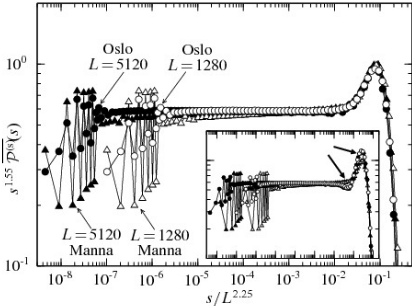


Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.

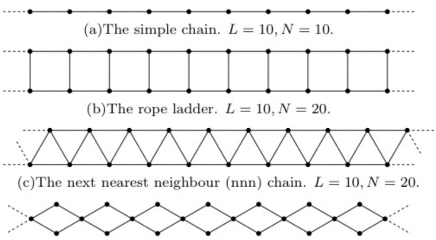
Imperial College London

Collapse with Oslo



The Manna Model is in the same universality class as the Oslo model.

Manna on different lattices



(a)The simple chain. $L = 10, N = 10$.

(b)The rope ladder. $L = 10, N = 20$.

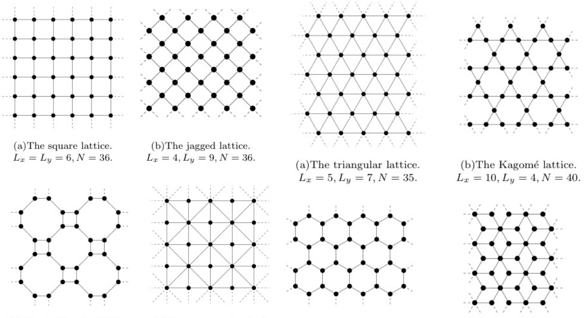
(c)The next nearest neighbour (nnn) chain. $L = 10, N = 20$.

(d)The Futatsubishi lattice. $L = 7, N = 22$.

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Manna on different lattices


(a)The square lattice. $L_x = L_y = 6, N = 36$.
(b)The jagged lattice. $L_x = 4, L_y = 9, N = 36$.
(c)The Archimedes lattice. $L_x = 8, L_y = 4, N = 32$.
(d)The non-crossing (nc) diagonal square lattice. $L_x = L_y = 5, N = 25$.

(a)The triangular lattice. $L_x = 5, L_y = 7, N = 35$.
(b)The Kagomé lattice. $L_x = 10, L_y = 4, N = 40$.
(c)The honeycomb lattice. $L_x = 9, L_y = 4, N = 36$.
(d)The Mitsubishi lattice. $L_x = 5, L_y = 7, N = 35$.

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Manna on different lattices

lattice	d	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
simple chain	1	2.27(2)	1.117(8)	1.450(12)	1.19(2)	0.998(4)	1.260(13)	2.000(4)	0.27(2)	0.27(3)	0.259(14)
rope ladder	1	2.24(2)	1.108(9)	1.44(2)	1.18(3)	0.998(7)	1.26(2)	1.989(5)	0.24(2)	0.26(5)	0.26(2)
nnn chain	1	2.33(11)	1.14(4)	1.48(11)	1.22(14)	0.997(15)	1.27(5)	1.991(11)	0.33(11)	0.3(2)	0.27(5)
Futatsubishi	1	2.24(3)	1.105(14)	1.43(3)	1.16(6)	0.999(15)	1.24(5)	2.008(11)	0.24(3)	0.23(9)	0.24(5)
square	2	2.748(13)	1.272(3)	1.52(2)	1.48(2)	1.992(8)	1.380(8)	1.9975(11)	0.748(13)	0.73(4)	0.76(2)
jagged	2	2.764(15)	1.276(4)	1.54(2)	1.49(3)	1.995(7)	1.384(8)	2.0007(12)	0.764(15)	0.76(5)	0.77(2)
Archimedes	2	2.76(2)	1.275(6)	1.54(3)	1.50(3)	1.997(10)	1.382(11)	2.001(2)	0.76(2)	0.78(6)	0.76(3)
nc diagonal square	2	2.750(14)	1.273(4)	1.53(2)	1.49(2)	1.992(7)	1.381(8)	2.0005(12)	0.750(14)	0.75(4)	0.76(2)
triangular	2	2.76(2)	1.275(5)	1.51(2)	1.47(3)	2.003(11)	1.388(12)	1.997(2)	0.76(2)	0.71(6)	0.78(3)
Kagomé	2	2.741(13)	1.270(4)	1.53(2)	1.49(2)	1.993(8)	1.381(9)	1.9994(12)	0.741(13)	0.75(5)	0.76(2)
honeycomb	2	2.73(2)	1.268(6)	1.55(4)	1.51(4)	1.990(13)	1.376(14)	2.000(2)	0.73(2)	0.79(8)	0.75(3)
Mitsubishi	2	2.75(2)	1.273(6)	1.54(3)	1.50(4)	1.999(12)	1.387(12)	1.998(2)	0.75(2)	0.77(7)	0.77(3)

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Imperial College London

Outline

1. SOC: The early programme
2. More models
3. Tools in SOC
   - The Absorbing State Mechanism
4. Some new ideas
5. Any Answers?

Imperial College London

Tools in SOC

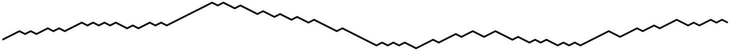
- (Extensive) numerics (BTW, FFM, BS, Manna, Oslo).
- Analytical tools:
  - Exact solutions (so far: directed models only).
  - Mappings to known (understood?) phenomena.
  - Growth processes and field theories.

Imperial College London

Link to growth phenomena

Generic scale invariance

Stochastic evolution of sandpile surface.



$$\partial_t \phi(\mathbf{r}, t) = (\nu_{\|}\partial_{\|}^2 + \nu_{\perp}\partial_{\perp}^2)\phi + \eta(\mathbf{r}, t)$$

- *Generic* scale invariance (Hwa and Kardar, 1989, and Grinstein, Lee and Sachdev 1990)
- No mass term $-\epsilon\phi$ on the right $\longrightarrow$ conservative dynamics (finiteness generates ϵ).
- Anisotropy (boundaries?) required in the presence of conserved noise.
- Non-trivial exponents in the presence of non-linearities and non-conserved noise.

Effect of a mass term

Mass term

$$\partial_t \phi = \nu \nabla^2 \phi - \epsilon \phi + \ldots + \eta \quad (1)$$

represents disspation

$$\partial_t \int_V \mathrm{d}^d x\, \phi = \text{surface terms} - \epsilon \int_V \mathrm{d}^d x\, \phi \quad (2)$$

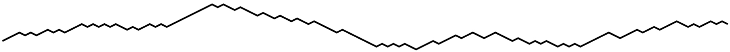
and correlation length

$$\phi = \ldots e^{-|x|\sqrt{\epsilon/\nu}} \,. \quad (3)$$

But: How can a renormalised $\epsilon = 0$ be maintained without trivialising the phenomenon?

Field theories for Manna and Oslo

Number of charges interpreted as an interface.



- **Manna model** has a Langevin equation

$$\partial_t \phi(\mathbf{r}, t) = \nu \nabla^2 \phi - \mu \phi + \lambda \phi^2 + \omega \rho \phi + \sqrt{2\Gamma^2 \phi}\, \eta(\mathbf{r}, t)$$

  and

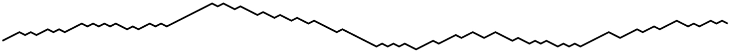
$$\partial_t \rho(\mathbf{r}, t) = \nu_\rho \nabla^2 \phi$$

  similar to **directed percolation (C-DP).**
- **Oslo model** implements **quenched Edwards Wilkinson equation** $\longrightarrow$ interfaces!
- Field theories for both still unclear.
- Mechanism of self-organisation still unclear.

Imperial College London

Field theories for Manna and Oslo

Number of charges interpreted as an interface.



- **Manna model** has a Langevin equation similar to **directed percolation (C-DP).**
- **Oslo model** implements **quenched Edwards Wilkinson equation** $\longrightarrow$ interfaces!

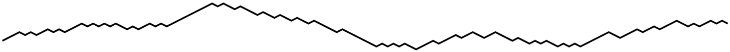
$$\partial_t \phi(\mathbf{r}, t) = \nu \nabla^2 \phi + \sqrt{2\Gamma^2}\, \eta(\mathbf{r}, \phi)$$

- Field theories for both still unclear.
- Mechanism of self-organisation still unclear.
- Link to known universality classes.
- Link to **directed percolation**?

Imperial College London

Field theories for Manna and Oslo

Number of charges interpreted as an interface.



- **Manna model** has a Langevin equation similar to **directed percolation (C-DP).**
- **Oslo model** implements **quenched Edwards Wilkinson equation** $\longrightarrow$ interfaces!
- Field theories for both still unclear.
- Mechanism of self-organisation still unclear.
- Link to known universality classes.
- Link to **directed percolation**?

The Absorbing State Mechanism

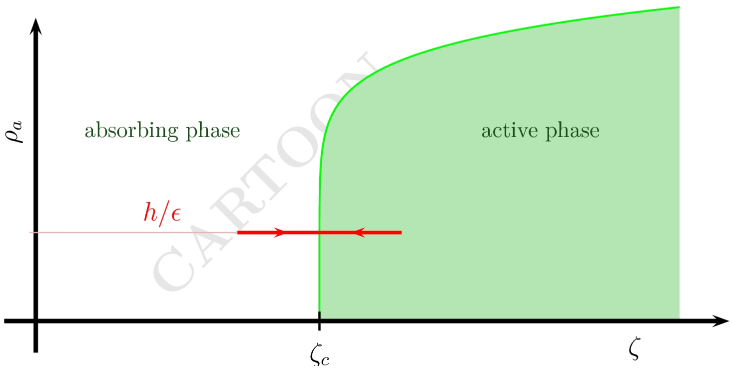
Dickman, Vespignani, Zapperi 1998

- SOC model: activity ρ_a leads to dissipation
- dissipation reduces particle density ζ
- density is reduced until system is inactive
  $\longrightarrow$ absorbing phase
- external drive increases particle density
  $\longrightarrow$ back to active phase

An SOC model can be seen as an AS model that drives itself into the inactive phase by dissipation ϵ and is pushed back into the active phase by external drive h.

$$\dot{\zeta} = h - \epsilon\rho_a \xrightarrow{\text{stationarity}} \rho_a = h/\epsilon$$

The Absorbing State Mechanism

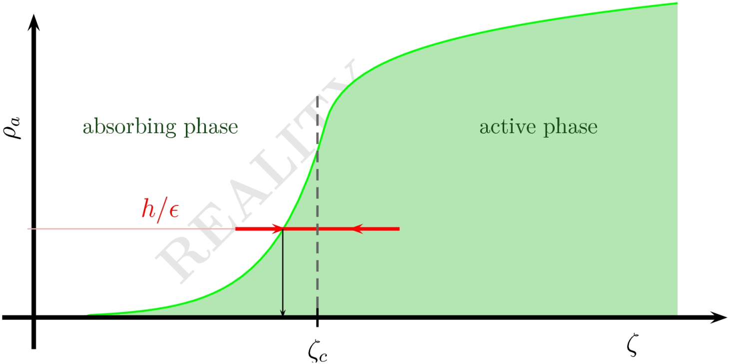


Idea: SOC drives $h/\epsilon = \rho_a$ to 0 as $L \to \infty$

Leading orders: $h(L) = h_0 L^{-\omega}$ and $\epsilon(L) = \epsilon_0 L^{-\kappa}$

Imperial College London

The Absorbing State Mechanism



Problem: SOC exponents would be affected by the way how driving and dissipation are implemented $\longrightarrow$ no universality.

Fey, Levine and Wilson suggest that critical point is not reached.

Outline

Imperial College London

Some new ideas

- The exponents of the Manna universality class seem to have an $\epsilon = 4 - d$ expansion.
- There *must* be a field theory!
- Take the reaction-diffusion route. Issues: Fermionic, surfaces.
- Symmetries (Ward identities) will maintain the system at the critical point.
- Mapping to ordinary critical phenomena should be straight forward.
- All features must be visible at tree level.
- Note that mean field theories so far were effective theories.

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London

Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Possibly, superconductors and granular media.
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Maybe, AS Mechanism suggested, but has problems.
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

Imperial College London